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Regularizing Effect and Local Existence for the Non-Cutoff Boltzmann Equation

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Abstract

The Boltzmann equation without Grad's angular cutoff assumption is believed to have a regularizing effect on the solutions because of the non-integrable angular singularity of the cross-section. However, even though this has been justified satisfactorily for the spatially homogeneous Boltzmann equation, it is still basically unsolved for the spatially inhomogeneous Boltzmann equation. In this paper, by sharpening the coercivity and upper bound estimates for the collision operator, establishing the hypo-ellipticity of the Boltzmann operator based on a generalized version of the uncertainty principle, and analyzing the commutators between the collision operator and some weighted pseudo-differential operators, we prove the regularizing effect in all (time, space and velocity) variables on the solutions when some mild regularity is imposed on these solutions. For completeness, we also show that when the initial data has this mild regularity and a Maxwellian type decay in the velocity variable, there exists a unique local solution with the same regularity, so that this solution acquires the C^∞ regularity for any positive time.

Contents

1. Introduction

Consider the Boltzmann equation,

$$f_t + v \cdot \nabla_x f = Q(f, f), \tag{1.1}$$

where $f = f(t, x, v)$ is the density distribution function of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time t. The right-hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B\,(v - v_*, \sigma)\,\big\{g(v'_*) f(v') - g(v_*) f(v)\big\}\, \mathrm{d}\sigma \mathrm{d}v_*,$$

which is well-defined for suitable functions f and g specified later. Notice that the collision operator $Q(\cdot, \cdot)$ acts only on the velocity variable $v \in \mathbb{R}^3$. In the following discussion, we will use the σ−representation, that is, for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma,$$

which give the relations between the post- and pre-collisional velocities.

It is well known that the Boltzmann equation is a fundamental equation in statistical physics. For the mathematical theories on this equation, we refer the readers to [16,17,27,31,46], and the references therein, also, for the physics background.

In addition to the special bilinear structure of the collision operator, the cross-section $B(v - v_*, \sigma)$ varies with different physical assumptions on the particle interactions, and it plays an important role in the well-posedness theory for the Boltzmann equation. In fact, except for the hard sphere model, for most of the other molecular interaction potentials, such as the inverse power laws, the cross section $B(v - v_*, \sigma)$ has a non-integrable angular singularity. For example, if the interaction potential obeys the inverse power law $r^{-(p-1)}$ for $2 < p < \infty$, where r denotes the distance between two interacting molecules, the cross-section behaves like

$$B(|v - v_*|, \cos\theta) \sim |v - v_*|^{\gamma} \theta^{-2-2s}, \quad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leqq \theta \leqq \frac{\pi}{2},$$

with

$$-3 < \gamma = \frac{p-5}{p-1} < 1, \quad 0 < s = \frac{1}{p-1} < 1.$$

As usual, the hard and soft potentials correspond to $2 < p < 5$ and $p > 5$, respectively, and the Maxwellian potential corresponds to $p = 5$. The fact that the singularity θ^{-2-2s} is not integrable on the unit sphere leads to the conjecture that the nonlinear collision operator should behave like a fractional Laplacian in the variable v. That is,

$$Q(f, f) \sim -(-\Delta_v)^s f + \text{lower order terms}.$$

Indeed, consider the Kolmogorov type equation

$$f_t + v \cdot \nabla_x f = -(-\Delta_v)^s f.$$

Straightforward calculation by Fourier transformation shows that the solution is in the Gevrey class when $0 < s \leqq \frac{1}{2}$ and is ultra-analytic if $\frac{1}{2} < s < 1$ for initial data only in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ if it admits a unique solution (see [40] for a more general study). However, for the Boltzmann equation, establishing Gevrey regularity of solutions is a long lasting open problem which has only been proved, so far, in the linear and spatially homogeneous setting, see [38].

The mathematical study of inverse power law potentials can be traced back to work by Pao [42] in the 1970s. In the early 1980s, Arkeryd, in [12], proved the existence of weak solutions to the spatially homogeneous Boltzmann equation when $0 < s < \frac{1}{2}$, while Ukai in [43] applied an abstract Cauchy–Kovalevskaya theorem to obtain local solutions in the functional space of functions which are analytic in x and Gevrey in v. However, the smoothing effect of the collision operator was not studied at that time.

Since then, this problem has attracted increasing interest in the area of kinetic theory, and much progress has been made on the existence and regularity theories. More precisely, that the long-range interactions have smoothing effects on the solutions to the Boltzmann equation was first proved by Desvillettes for some simplified models, compare [21,22]. This is in contrast with the hard sphere model and the potentials of Grad's angular cutoff assumption. In fact, for the hard sphere model, the cross-section has the form (in the σ representation)

$$B(|v - v_*|, \cos\theta) = q_0 |v - v_*|,$$

where q_0 is the surface area of a hard sphere. For singular cross-sections, Grad [31] introduced the idea to cut off the singularity at $\theta = 0$ so that $B(|v - v_*|, \cos\theta) \in L^1(\mathbb{S}^2)$. This assumption, now called Grad's angular cutoff assumption, has been widely accepted and has influenced a few decades of mathematical studies on the Boltzmann equation. Under this angular cutoff assumption, the solution has the same regularity, at least in the Sobolev space, as the initial data. In fact, it was shown in [28], that the solution has the form

$$f(t, x, v) = a(t, x, v) f(0, x - vt, v) + b(t, x, v),$$

when the initial data $f(0, x, v)$ is in some weighted $L^p_{x,v}$ space. Here, $a(t, x, v)$ and $b(t, x, v)$ are in the Sobolev space $H^\delta_{t,x,v}$ for some $\delta > 0$. And the term $f(0, x - vt, v)$ simply represents the free transport so that it is clear that $f(t, x, v)$ and $f(0, x, v)$ have the same regularity.

One of the main features of the Boltzmann equation is the celebrated Boltzmann's H-theorem saying that the H-functional

$$H(t)=\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f\,\mathrm{d}x\,\mathrm{d}v,$$

satisfies

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t}+D(t)=0,$$

where

$$D(t)=-\int_{\mathbb{R}^3\times\mathbb{R}^3} Q(f,f)\log f\,\mathrm{d}x\,\mathrm{d}v\geqq 0,$$

which is called the entropy dissipation rate. Notice that $D(t)$ is non-negative and vanishes only when f is a Maxwellian. The non-negativity of D indicates that the Boltzmann equation is a dissipative equation. This fact is a basic ingredient in the L^1 theory of the Boltzmann equation, see for example [27].

By using the entropy dissipation rate D and the "Q^+ smoothing property", the formal smoothing estimate was derived by LIONS [34] (see the complete references in [4])

$$\left\|\sqrt{f}(\sqrt{f}*\langle v\rangle^{-m})\right\|^2_{B^{\delta,2}_{\infty}}\leqq C\|f\|^{1-\theta}_{L^1}\left(\|f\|_{L^1}+D(f)^{\frac{1}{2}}\right)^{\theta},\ \delta=\frac{s}{1+s},\ \theta=\frac{1}{1+s},$$

for any constant $m>3$. Notice that the above regularity estimate is on $\sqrt{f}$, not f itself. Later, some almost optimal estimates together with some extremely useful results, such as the cancellation lemma, were obtained in the work by ALEXANDRE et al. [4]. By using these analytic tools, the mathematical theory regarding the regularizing effect for the spatially homogeneous problems may now be considered as quite satisfactory, see [8,9,23,24,26,32,38,45], and the references therein.

However, for spatially inhomogeneous equations, there are far fewer results. The main difficulty comes from the coupling of the transport operator with the collision operator, and the commutators of the differential (pseudo-differential) operators with the collision operator. Two inroads have been achieved so far. One regards the local existence of solutions between two moving MAXWELLIANS in [2], obtained by constructing upper and lower solutions. The other regards the global existence of renormalized solutions with defect measures constructed in [10], which become weak solutions if the defect measures vanish. Some results on similar but linear kinetic equations were also given in [7] and [15]. In particular, a generalized uncertainty principle à la FEFFERMAN [29] (see also [35–37]) was introduced in [7] in order to prove smoothing effects of the linearized and spatially inhomogeneous Boltzmann equation with non-cutoff cross sections, and to get partial smoothing effects for the nonlinear Boltzmann equation. In the following analysis, this partial regularity result, together with its proof, will also be used.

This paper can be viewed as a continuation of our recent work [7]. Under some mild regularity assumptions on the initial data, we will prove the existence of solutions and their C^∞ regularity with respect to all (time, space and velocity) variables.

Even though, on the initial data, it is still not known whether only some natural bounds, such as total mass, energy and entropy, can lead to the C? regularizing effect, as far as we know, the results shown in this paper are the first to justify the C^∞ regularizing effect for the nonlinear and spatially inhomogeneous Boltzmann equation without Grad's angular cutoff assumption.

In order to state our theorems, let us first introduce the notations and assumptions used in this paper.

The non-negative cross-section $B(z, \sigma)$ (for a monatomic gas, which is the case considered herein) depends only on $|z|$ and the scalar product $\langle \frac{z}{|z|}, \sigma \rangle$. In most cases, the collision kernel cannot be expressed explicitly, but to capture the essential properties, it can be assumed to have the form

$$B(|v - v_*|, \cos\theta) = \Phi(|v - v_*|)b(\cos\theta), \quad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \ \ 0 \leqq \theta \leqq \frac{\pi}{2}.$$

Furthermore, to keep the presentation as simple as possible, and in particular to avoid the difficulty coming from the vanishing of the cross-section at zero relative velocity, we assume that the kinetic factor Φ in the cross-section is modified as

$$\Phi(|v - v_*|) = \left(1 + |v - v_*|^2\right)^{\frac{\gamma}{2}}, \quad \gamma \in \mathbb{R}. \tag{1.2}$$

This point can certainly be removed using our whole calculus, at the expense of technical and more complicated details.

Moreover, the angular factor is assumed to have the following singular behavior

$$\sin\theta b(\cos\theta) \approx K\theta^{-1-2s}, \quad \text{when } \theta \to 0+, \tag{1.3}$$

where $0 < s < 1$ and K is a positive constant. In fact $\gamma = 0$ corresponds to the Maxwellian molecule, $\gamma < 0$ corresponds to the modified soft potential, and $\gamma > 0$ corresponds to the modified hard potential. The singularity will be called mild for $0 < s < \frac{1}{2}$ and strong for $\frac{1}{2} \leqq s < 1$. The case $s = \frac{1}{2}$ is critical in the sense that different computations are required in many parts of our proofs for mild and strong singularities, as will be seen below. This is similar to the known fractional Laplacian studies.

It is now well known from the work [4] that the singular behavior of the collision kernel (1.3) implies a sub-elliptic estimate in the velocity variable v. In the following analysis, we shall need a somewhat precise weighted sub-elliptic estimate in the velocity variable. We shall show that for $\gamma \in \mathbb{R}$ and $0 < s < 1$, if $f \geqq 0, \not\equiv 0,\ f \in L^1_2 \bigcap L \log L(\mathbb{R}^3_v)$, there exists a constant $C > 0$ such that for any function $g \in H^1(\mathbb{R}^3_v)$ we have

$$C^{-1} \|\Lambda^s_v W_{\gamma/2} g\|^2_{L^2(\mathbb{R}^3_v)} \leqq (-Q(f, g),\ g)_{L^2(\mathbb{R}^3_v)} + C\|f\|_{L^1_{\tilde{\gamma}}(\mathbb{R}^3)} \|g\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)},$$

where $\tilde{\gamma} = \max(\gamma^+, 2 - \gamma^+), \gamma^+ = \max\{\gamma, 0\}$. Here $W_l = W_l(v) = (1 + |v|^2)^{l/2} = \langle v \rangle^l$, $l \in \mathbb{R}$, is the weight function in the variable $v \in \mathbb{R}^3$.

Similar sub-elliptic estimates, first proved in [4] and then developed in many other works, such as [41], in a linearized context, have been used crucially at least for the following two aspects:

i) the proof of the regularizing effect on the solutions to the spatially homogeneous Boltzmann equation, see [8,9,26,32,38];
ii) the proof of existence of solutions to the nonlinear and spatially inhomogeneous Boltzmann equation [2,10,46].

In this paper, we will apply this tool in order to study the complete smoothing effect for the spatially inhomogeneous and nonlinear Boltzmann equation.

It is now well understood (see [11] and references therein) that the Landau equation corresponds to the grazing limit of the Boltzmann equation. However, while the Landau operator involves conventional partial differential operators, it should be kept in mind that fractional differential operators appear in the Boltzmann case, see [1,3]. Therefore, the analysis on the Boltzmann equation appears much more involved because it requires the unavoidable use of harmonic analysis. In particular, we shall use a generalized uncertainty principle which was introduced in [7], and the estimation of commutators used in the work [39] for the study of hypo-elliptic properties.

Throughout this paper, we shall use the following standard weighted (with respect to the velocity variable $v \in \mathbb{R}^3$) Sobolev spaces. For $m,\ l \in \mathbb{R}$, set $\mathbb{R}^7 = \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ and

$$H^m_l(\mathbb{R}^7) = \left\{ f \in \mathcal{S}'(\mathbb{R}^7);\ W_l(v) f \in H^m(\mathbb{R}^7) \right\},$$

which is a Hilbert space. Here H^m is the usual Sobolev space. We shall also use the functional spaces $H^k_l(\mathbb{R}^6_{x,\,v})$ and $H^k_l(\mathbb{R}^3_v)$, specifying the variables, the weight being always taken with respect to $v \in \mathbb{R}^3$.

Since the regularity property to be proved here is local in space and time, for convenience, we define the following local version of weighted Sobolev space. For $-\infty \leqq T_1 < T_2 \leqq +\infty$, and any given open domain $\Omega \subset \mathbb{R}^3_x$, define

$$\begin{aligned} &\mathcal{H}^m_l \left( ]T_1,\ T_2[\times \Omega \times \mathbb{R}^3_v \right) \\ &\quad = \Big\{ f \in \mathcal{D}'(]T_1,\ T_2[\times\Omega \times \mathbb{R}^3_v);\ \varphi(t)\psi(x) f \in H^m_l(\mathbb{R}^7), \\ &\qquad\qquad \forall\, \varphi \in C^\infty_0(]T_1,\ T_2[),\ \ \psi \in C^\infty_0(\Omega) \Big\}. \end{aligned}$$

Our first main result is about the smoothing effect on the solution and can be stated as follows

Theorem 1.1 (Regularizing effect on solutions). *Assume that* $0 < s < 1,\ \ \gamma \in \mathbb{R}$, $-\infty \leqq T_1 < T_2 \leqq +\infty$ *and let* $\Omega \subset \mathbb{R}^3_x$ *be an open domain. Let* f *be a non-negative function belonging to* $\mathcal{H}^5_l(]T_1,\ T_2[\times\Omega \times \mathbb{R}^3_v)$ *for all* $l \in \mathbb{N}$ *and solving the Boltzmann equation* (1.1) *in the domain* $]T_1,\ T_2[\times\Omega \times \mathbb{R}^3_v$ *in the classical sense. Furthermore, if* f *satisfies the non-vacuum condition*

$$\| f(t, x, \cdot)\|_{L^1(\mathbb{R}^3_v)} > 0,$$

for all $(t, x) \in]T_1,\ T_2[\times\Omega$*, then we have*

$$f \in \mathcal{H}_l^{+\infty}\left(]T_1,\ T_2[\times\Omega \times \mathbb{R}_v^3\right),$$

for any $l \in \mathbb{N}$*, and hence*

$$f \in C^\infty\left(]T_1,\ T_2[\times\Omega;\ \mathcal{S}(\mathbb{R}_v^3)\right).$$

With this result in mind, a natural question is whether the Boltzmann equation has solutions satisfying the assumptions imposed above. Let us recall that solutions constructed in [2,10] do not work for our purpose because of the lack of the weighted regularity $\mathcal{H}_l^5$. For Gevrey class solutions from [43], there is, of course, nothing to prove.

Thus, our second main result is about the local existence and uniqueness of solutions for the Cauchy problem of the non-cutoff Boltzmann equation.

We consider the solution in the functional space with Maxwellian type exponential decay in the velocity variable. More precisely, for $m \in \mathbb{R}$, set

$$\mathcal{E}_0^m(\mathbb{R}^6) = \left\{g \in \mathcal{D}'(\mathbb{R}_{x,v}^6);\ \exists\, \rho_0 > 0\ s.t.\ e^{\rho_0 <v>^2} g \in H^m(\mathbb{R}_{x,v}^6)\right\},$$

and for $T > 0$

$$\mathcal{E}^m\left([0, T] \times \mathbb{R}_{x,v}^6\right) = \Big\{f \in C^0\left([0, T]; \mathcal{D}'(\mathbb{R}_{x,v}^6)\right); \exists \rho > 0$$
$$s.t. e^{\rho\langle v\rangle^2} f \in C^0\left([0, T]; H^m(\mathbb{R}_{x,v}^6)\right)\Big\}.$$

Theorem 1.2. *Assume that* $0 < s < 1/2$ *and* $\gamma + 2s < 1$*. Let* $f_0 \geqq 0$ *and* $f_0 \in \mathcal{E}_0^{k_0}(\mathbb{R}^6)$ *for some* $4 \leqq k_0 \in \mathbb{N}$*. Then, there exists* $T_* > 0$ *such that the Cauchy problem*

$$\begin{cases} f_t + v \cdot \nabla_x f = Q(f, f), \\ f|_{t=0} = f_0, \end{cases} \tag{1.4}$$

admits a non-negative and unique solution in the functional space $\mathcal{E}^{k_0}([0, T_*]\times\mathbb{R}^6)$*.*

Furthermore, if we assume that the initial data f_0 *is in* $\mathcal{E}_0^5(\mathbb{R}^6)$ *and does not vanish on a compact set* $K \subset \mathbb{R}_x^3$*, that is,*

$$\|f_0(x, \cdot)\|_{L^1(\mathbb{R}_v^3)} > 0, \quad \forall x \in K,$$

then we have the regularizing effect on the above solution, that is, there exist $0 < \tilde{T}_0 \leqq T_*$ *and a neighborhood* V_0 *of* K *in* $\mathbb{R}_x^3$ *such that*

$$f \in C^\infty\left(]0, \tilde{T}_0[\times V_0;\ \mathcal{S}(\mathbb{R}_v^3)\right).$$

Moreover, if $\gamma \leqq 0$, the non-negative solution of the Cauchy problem (1.4) *is unique in the functional space $C^0([0, T_*]; H^m_p(\mathbb{R}^6))$ for $m > 3/2 + 2s$, $p > 3/2 + 4s$.*

Remark 1.3. For the inverse power law potential $r^{-(p-1)}$, the condition $0 < s < 1/2$, $\gamma + 2s < 1$ corresponds to $3 < p < \infty$ which includes both soft and hard potentials.

At the moment, it is not clear whether we can relax the regularity assumption initially made on the solutions. Note that, for example, the condition that $f \in L^1 \cap L^\infty(\mathbb{R}^7)$ is enough to give a meaningful sense to a weak formulation for the spatially inhomogeneous Boltzmann equation. However, the analysis used here cannot be applied to this case, and so further study is needed. On the other hand, the above two theorems give an answer to a long lasting conjecture on the regularizing effect of non-cutoff cross-sections for the spatially inhomogeneous Boltzmann equation.

To conclude the introduction, let us review some related works on the regularizing effect and the existence of solutions for the Landau equation. The regularizing effect from the Landau collision operator has been rather well studied. See [11,18,25] for the spatially homogeneous case. For the spatially inhomogeneous problem, a regularizing result was obtained in [20], where the H^8 regularity is assumed on the solutions to start with. Similar results were also recently proved for the Vlasov–Maxwell–Landau and the Vlasov–Poisson–Landau systems, compare [19] and the references therein. As for the existence of solutions, see [25] where unique weak solutions for the spatially homogeneous case have been constructed with rather general initial data, and see [30] where the classical solutions for the spatially inhomogeneous case have been constructed in a periodic box with small initial data.

The rest of the paper will be organized as follows. First of all, in the next section, we will use pseudo-differential calculus to study the upper bounds on the collision operator. We shall give a precise coercivity estimate linked to the singularity in the cross-section, and estimate the commutators between some pseudo-differential operators and the nonlinear collision operators. In Section 3, the regularizing effect will be proved under the initial regularity assumption on the solution. The strategy of the proof is as follows. We first choose some suitable mollifiers such that the mollified solutions can work as test functions for the weak formulation of the problem. We then establish a small gain of the regularity in the velocity variable, by using the coercivity estimate coming from the singularity of the cross section. On account of the generalized uncertainty principle, a small gain of the regularity in the space and time variables can be derived. The $H^{+\infty}$ regularity will follow from an induction argument. Finally, in Section 4, local solutions to the non-cutoff Boltzmann equation which meet the initialization condition of Theorem 1.1 are constructed, using a family of cutoff Boltzmann equations with time local uniform bounds independent of cutoff parameter in some weighted Sobolev space. In particular, the uniform bounds are established with the help of time dependent Maxwellian type weight functions which were introduced in [43,44]. The convergence of the approximate solutions follows from compactness argument , while the uniqueness of the solutions can also be proved by using our sharp upper bounds on the collision operator.

2. Pseudo-differential calculus

Under the non-cutoff cross section assumption, the Boltzmann collision operator is a (nonlinear) singular integral operator with respect to $v \in \mathbb{R}^3_v$. In the linearized case, it behaves like a pseudo-differential operator.

In this section, we study the pseudo-differential calculus on the Boltzmann operator, which is one of the key analytic tools for proving the regularizing effect of the non-cutoff Boltzmann equation. Even though the regularity proved in this paper is local in space and time variables, note that the collision operator is non-local in the space of the v variable. Moreover, since the kinetic factor in the cross-section is of the form $\langle v \rangle^\gamma$, which might be unbounded, we need to consider multiplication by the weight function $W_l(v)$ of the pseudo-differential operators. Hence, they are not the standard pseudo-differential operators of order 0 on the usual Sobolev space. In other words, we shall consider pseudo-differential operators with unbounded coefficients on the weighted Sobolev space $H^m_l(\mathbb{R}^3_v)$. The variables (t, x) are considered as parameters for the collision operators in this section.

2.1. Upper bound estimates

We shall need some functional estimates on the Boltzmann collision operator in the existence and regularization proofs below. The first one regards boundedness of the collision operator in some weighted Sobolev spaces, see also [3,5,32].

Theorem 2.1. *Let* $0 < s < 1$ *and* $\gamma \in \mathbb{R}$. *Then for any* $m, \alpha \in \mathbb{R}$, *there exists* $C > 0$ *such that*

$$\|Q(f, g)\|_{H^m_\alpha(\mathbb{R}^3_v)} \leqq C\|f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}(\mathbb{R}^3_v)}\|g\|_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)} \tag{2.1.1}$$

for all $f \in L^1_{\alpha^+ + (\gamma+2s)^+}(\mathbb{R}^3_v)$ *and* $g \in H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)$.

Remark 2.2. (1) The collision operator $Q(f, g)$ behaves differently with respect to f and g: (2.1.1) shows that, in some sense, it is linear with respect to the second factor in the velocity variable v because the action of differentiation of $Q(f, g)$ with respect to v goes only on g when considered in the Sobolev space. This is clear for the Landau operator which is the grazing limit of the Boltzmann operator.

(2) The estimate (2.1.1) is in some sense optimal with respect to the order of differentiation (exact order of $2s$) and also with respect to the order of the weight in v coming from the cross-section. In [32], the cases of both the modified hard potential and Maxwellian molecule type cross-sections corresponding to $0 \leqq \gamma < 1$ are discussed. Let us also mention that a similar estimate was given in [6], but it is not optimal in terms of weight and differentiation. However, its

proof is more straightforward as it uses only the Fourier transformation of the collision operator (Bobylev's type formula [13] and see also the Appendix of [4]). For our purpose, the full precise estimate (2.1.1) will be needed.

Proof of Theorem 2.1. Firstly, we consider the case when $\alpha = 0$. To prove (2.1.1) in this case, it suffices to show that for any $m \in \mathbb{R}$

$$\left|(Q(f,g),h)_{L^2(\mathbb{R}^3_v)}\right| \leqq C||f||_{L^1_{(\gamma+2s)^+}(\mathbb{R}^3_v)}||g||_{H^{m+2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v)}||h||_{H^{-m}(\mathbb{R}^3_v)}. \quad (2.1.2)$$

The proof needs some harmonic analysis tools based on the dyadic decomposition. It is similar to the proof in [32], where the hard potential case $\gamma \geqq 0$ was studied. Interested readers may refer to the papers [3,4,32] for more details, though we will keep this paper self-contained.

Recall that

$$(Q(f,g),h)_{L^2(\mathbb{R}^3_v)} = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)\Phi(|v-v_*|)g(v) \times\{h(v')-h(v)\}\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v,$$

where $\Phi(|v-v_*|) = \Phi(|v'-v'_*|) = \langle v'-v'_*\rangle^\gamma$. Set

$$F(v,v_*) = \Phi(|v-v_*|)g(v),$$

and write

$$\begin{aligned}(Q(f,\ g),\ h)_{L^2(\mathbb{R}^3_v)} &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)F(v,v_*)\{h(v')-h(v)\}\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\ &= \int_{\mathbb{R}^3} f(v_*)(U_1-U_2)\mathrm{d}v_*.\end{aligned}$$

Then we have (formally) by inverse Fourier formula,

$$\begin{aligned}U_1 &\equiv \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)F(v,v_*)h(v')\mathrm{d}\sigma\mathrm{d}v\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} H(\xi,\eta,v_*)\hat{F}(\xi,v_*)\overline{\hat{h}(\eta)}\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

where (also formally)

$$\begin{aligned}H(\xi,\eta,v_*) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(k\cdot\sigma)e^{iv\cdot\xi - iv'\cdot\eta}\mathrm{d}\sigma\,\mathrm{d}v\\ &= \int_{\mathbb{R}^3} e^{iv\cdot\xi - i\frac{v+v_*}{2}\cdot\eta}\left[\int_{\mathbb{S}^2} b(k\cdot\sigma)e^{-i\frac{|v-v*|}{2}\sigma\cdot\eta}\mathrm{d}\sigma\right]\mathrm{d}v\\ &= \int_{\mathbb{R}^3} e^{iv\cdot\xi - i\frac{v+v_*}{2}\cdot\eta}\left[\int_{\mathbb{S}^2} b(\tilde{\eta}\cdot\sigma)e^{-i\frac{|v-v*|}{2}|\eta|\sigma\cdot k}\mathrm{d}\sigma\right]\mathrm{d}v, \quad (\tilde{\eta} = \eta/|\eta|)\\ &= \int_{\mathbb{R}^3} e^{iv\cdot\xi - i\frac{v+v_*}{2}\cdot\eta}\left[\int_{\mathbb{S}^2} b(\tilde{\eta}\cdot\sigma)e^{-i|\eta|\frac{v-v*}{2}\cdot\sigma}\mathrm{d}\sigma\right]\mathrm{d}v\\ &= \int_{\mathbb{S}^2} b(\tilde{\eta}\cdot\sigma)e^{-iv_*\cdot\eta^-}\left[\int_{\mathbb{R}^3} e^{iv\cdot(\xi-\eta^+)}\mathrm{d}v\right]\mathrm{d}\sigma\\ &= \int_{\mathbb{S}^2} b(\tilde{\eta}\cdot\sigma)e^{-iv_*\cdot\eta^-}\mathrm{d}\sigma\ \delta(\xi-\eta^+),\end{aligned}$$

with

$$\eta^- = \frac{1}{2}(\eta - |\eta|\sigma), \quad \eta^+ = \frac{1}{2}(\eta + |\eta|\sigma),$$

so that

$$U_1 = \int_{\mathbb{R}^3} \left[\int_{\mathbb{S}^2} b(\tilde{\eta} \cdot \sigma) e^{-iv_* \cdot \eta^-} \mathrm{d}\sigma \right] \hat{F}(\eta^+, v_*) \overline{\hat{h}(\eta)} \mathrm{d}\eta.$$

On the other hand,

$$\begin{aligned} U_2 &\equiv \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta) F(v, v_*) h(v) \mathrm{d}\sigma \mathrm{d}v \\ &= \left[\int_{\mathbb{S}^2} b(\cos\theta) \mathrm{d}\sigma \right] \int_{\mathbb{R}^3} \hat{F}(\eta, v_*) \overline{\hat{h}(\eta)} \mathrm{d}\eta \\ &= \int_{\mathbb{R}^3} \left[\int_{\mathbb{S}^2} b(\tilde{\eta} \cdot \sigma) \mathrm{d}\sigma \right] \hat{F}(\eta, v_*) \overline{\hat{h}(\eta)} \mathrm{d}\eta, \end{aligned}$$

because (formally) we have

$$\int_{\mathbb{S}^2} b(\cos\theta) \mathrm{d}\sigma = \int_{\mathbb{S}^2} b(\tilde{\eta} \cdot \sigma) \mathrm{d}\sigma = \text{const.}$$

Therefore, we have obtained the following generalized Bobylev formula

$$\begin{aligned} &(Q(f, g), h)_{L^2(\mathbb{R}^3_v)} \\ &\quad = \int_{\mathbb{R}^3} f(v_*) \Bigg[\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\tilde{\eta} \cdot \sigma) \left\{ e^{-iv_* \cdot \eta^-} \hat{F}(\eta^+, v_*) - \hat{F}(\eta, v_*) \right\} \\ &\qquad\qquad \times \overline{\hat{h}(\eta)} \mathrm{d}\eta \mathrm{d}\sigma \Bigg] \mathrm{d}v_* \\ &= \int_{\mathbb{R}^3} f(v_*) \Bigg[\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\tilde{\eta} \cdot \sigma) \left\{ e^{iv_* \cdot \eta^+} \hat{F}(\eta^+, v_*) - e^{iv_* \cdot \eta} \hat{F}(\eta, v_*) \right\} \\ &\qquad\qquad \times \overline{e^{iv_* \cdot \eta} \hat{h}(\eta)} \mathrm{d}\eta \mathrm{d}\sigma \Bigg] \mathrm{d}v_*. \end{aligned} \tag{2.1.3}$$

Notice that the above derivation is formal only for non-cutoff cross-sections because we can not split the gain and loss terms in this case. However, the derivation can be easily justified as a limit process of cutoff cross-sections when combining the gain term and loss term together.

We now introduce a dyadic decomposition in $\mathbb{R}^3_v$ as follows:

$$\sum_{k=0}^{\infty} \phi_k(v) = 1, \quad \phi_k(v) = \phi(2^{-k} v), \quad k \geqq 1,$$

with $0 \leqq \phi_0, \phi \in C_0^\infty(\mathbb{R}^3)$, and

$$\text{supp}\, \phi_0 \subset \{|v| < 2\}, \quad \text{supp}\, \phi \subset \{1 < |v| < 3\}.$$

Take also $\tilde\phi_0$ and $\tilde\phi \in C_0^\infty$ such that

$$\tilde\phi_0 = 1 \quad \text{on } \{|v| \leqq 2\}, \quad \text{supp}\, \tilde\phi_0 \subset \{|v| < 3\},$$
$$\tilde\phi = 1 \quad \text{on } \{1/2 \leqq |v| \leqq 3\}, \quad \text{supp}\, \tilde\phi \subset \{1/3 < |v| < 4\}.$$

We assume that all these functions are radial. It follows from $|v' - v_*| \leqq |v - v_*| \leqq \sqrt{2}|v' - v_*|$, that

$$\tilde\phi_k(v' - v_*)\phi_k(v - v_*) = \phi_k(v - v_*) = \tilde\phi_k(v - v_*)\phi_k(v - v_*), \quad k \geqq 0,$$

and thus we get

$$(Q(f,g),h)_{L^2(\mathbb{R}^3_v)} = \sum_{k=0}^\infty \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*) F_k(v, v_*) \times \{h_k(v', v_*) - h_k(v, v_*)\} d\sigma dv_* dv,$$

where

$$F_k(v, v_*) = \phi_k(v - v_*)\Phi(|v - v_*|)g(v), \quad h_k(v, v_*) = \tilde\phi_k(v - v_*)h(v). \tag{2.1.4}$$

Similarly to (2.1.3), we also obtain

$$\begin{aligned}(Q(f,g),h)_{L^2(\mathbb{R}^3_v)} &= \sum_{k=0}^\infty \int_{\mathbb{R}^3} f(v_*) \Big[\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\tilde\eta\cdot\sigma)\left\{e^{iv_*\cdot\eta^+}\hat F_k(\eta^+, v_*) - e^{iv_*\cdot\eta}\hat F_k(\eta, v_*)\right\} \\ &\quad \times \overline{e^{iv_*\cdot\eta}\hat h_k(\eta, v_*)} d\eta d\sigma\Big] dv_* \\ &= \int_{\mathbb{R}^3} f(v_*) \sum_{k=0}^\infty K^k(v_*) dv_*.\end{aligned}$$

In the following, we will estimate $\sum_{k=0}^\infty |K^k(v_*)|$, regarding v_* as a parameter.

By setting

$$\Omega_k = \left\{\sigma \in \mathbb{S}^2;\ \tilde\eta\cdot\sigma \geqq 1 - 2^{1-2k}\langle\eta\rangle^{-2}\right\},$$

and

$$\tilde{\hat F}_k(\eta, v_*) = e^{iv_*\cdot\eta}\hat F_k(\eta, v_*), \quad \tilde{\hat h}_k(\eta, v_*) = e^{iv_*\cdot\eta}\hat h_k(\eta, v_*),$$

we split $K^k(v_*)$ into

$$\begin{aligned}K^k(v_*) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k} b(\tilde\eta\cdot\sigma)\left\{\tilde{\hat F}_k(\eta^+, v_*) - \tilde{\hat F}_k(\eta, v_*)\right\}\overline{\tilde{\hat h}_k(\eta, v_*)} d\eta d\sigma \\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b(\tilde\eta\cdot\sigma)\left\{\tilde{\hat F}_k(\eta^+, v_*) - \tilde{\hat F}_k(\eta, v_*)\right\}\overline{\tilde{\hat h}_k(\eta, v_*)} d\eta d\sigma \\ &= K_1^k(v_*) + K_2^k(v_*).\end{aligned}$$

Note that

$$\int_{\mathbb{S}^2\cap\Omega_k} \theta^2\ b(\cos\theta)\mathrm{d}\sigma$$
$$= 2\pi \int_{\{\theta\in[0,\pi/2];\ \sin(\theta/2)\leqq 2^{-k}\langle\eta\rangle^{-1}\}} \sin\theta\ b(\cos\theta)\theta^2\mathrm{d}\theta$$
$$\leqq C\langle\eta\rangle^{2s-2}2^{k(2s-2)}, \quad \text{if } 0<s<1, \tag{2.1.5}$$
$$\int_{\mathbb{S}^2\cap\Omega_k^c} b(\cos\theta)\mathrm{d}\sigma$$
$$= 2\pi \int_{\{\theta\in[0,\pi/2];\ \sin(\theta/2)\geqq 2^{-k}\langle\eta\rangle^{-1}\}} \sin\theta\ b(\cos\theta)\mathrm{d}\theta$$
$$\leqq C\langle\eta\rangle^{2s}2^{2ks}, \quad \text{for any } s>0. \tag{2.1.6}$$

It follows from (2.1.6) that

$$|K_2^k(v_*)| \leqq \int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b(\tilde\eta\cdot\sigma)\left|\widetilde{\hat F}_k(\eta^+,v_*)-\widetilde{\hat F}_k(\eta,v_*)\right|\left|\widetilde{\hat h}_k(\eta,v_*)\right|\mathrm{d}\eta\mathrm{d}\sigma$$
$$\leqq \left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b\langle\eta\rangle^{2m+2s}\left(\left|\widetilde{\hat F}_k(\eta^+,v_*)\right|^2+\left|\widetilde{\hat F}_k(\eta,v_*)\right|^2\right)\mathrm{d}\eta\mathrm{d}\sigma\right)^{1/2}$$
$$\times\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b\langle\eta\rangle^{-2m-2s}\left|\widetilde{\hat h}_k(\eta,v_*)\right|^2 d\eta\mathrm{d}\sigma\right)^{1/2}$$
$$\leqq C2^{2ks}\left\|\langle D_v\rangle^{m+2s}F_k(v,v_*)\}\right\|_{L^2}\left\|\langle D_v\rangle^{-m}h_k(v,v_*)\right\|_{L^2}. \tag{2.1.7}$$

Here, we have used the change of variables $\eta\to\eta^+$, which is regular because the Jacobian can be computed, with $\tilde\eta=\eta/|\eta|$, as

$$\left|\frac{\partial(\eta^+)}{\partial(\eta)}\right| = \left|\frac12 I+\frac12\sigma\otimes\tilde\eta\right| = \frac18(1+\sigma\cdot\tilde\eta) = \frac14\cos^2\frac{\theta}{2}.$$

It should be noted that after this change of variable, θ no longer plays the role of the polar angle, because the "pole" $\tilde\eta$ now moves with σ and hence the measure $d\sigma$ is no longer given by $\sin\theta\mathrm{d}\theta\mathrm{d}\phi$. However, the situation is rather good because if we take $\tilde\eta^+=\eta^+/|\eta^+|$ as a new pole which is independent of σ, then the new polar angle ψ defined by $\cos\psi=\tilde\eta^+\cdot\sigma$ satisfies

$$\psi=\frac{\theta}{2},\quad d\sigma=\sin\psi\mathrm{d}\psi\mathrm{d}\phi,\quad \psi\in\left[0,\frac{\pi}{4}\right],$$

and thus θ almost works as the polar angle. Therefore, since $\langle\eta\rangle\leqq 2\langle\eta^+\rangle\leqq 2\langle\eta\rangle$ we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b(\tilde\eta\cdot\sigma)\langle\eta\rangle^{2m+2s}\left|\widetilde{\hat F}_k(\eta^+,v_*)\right|^2\mathrm{d}\eta\mathrm{d}\sigma$$
$$\leqq C\int_{\mathbb{R}^3_{\eta^+}} D_0(\eta^+)\left|\widetilde{\hat F}_k(\eta^+,v_*)\right|^2\mathrm{d}\eta^+$$

with

$$
\begin{aligned}
D_0(\eta^+) &= \int_{\mathbb{S}^2\cap\Omega_k^c} b(\tilde\eta\cdot\sigma)\langle\,\eta(\eta^+,\sigma)\,\rangle^{2m+2s}\mathrm{d}\sigma \\
&\leqq C\int_{\mathbb{S}^2\cap\Omega_k^c}\langle\,\eta(\eta^+,\sigma)\,\rangle^{2m+2s}\theta^{-2-2s}\mathrm{d}\sigma \\
&\leqq C\langle\eta^+\rangle^{2m+2s}\int_{2^{-k}\langle\eta^+\rangle^{-1}}^{\pi/4}\psi^{-2-2s}\sin\psi\,\mathrm{d}\psi \leqq 2^{2ks}\langle\eta^+\rangle^{2m+4s},
\end{aligned}
$$

which implies (2.1.7). Notice that for $p = 0, 1, 2$,

$$
\begin{aligned}
\left|\frac{2^{k(2s-p)}|v-v_*|^p\phi_k(v-v_*)\Phi(|v-v_*|)}{\langle v_*\rangle^{(\gamma+2s)^+}}\right| &\leqq C\frac{\langle v-v_*\rangle^{\gamma+2s}}{\langle v_*\rangle^{(\gamma+2s)^+}}\phi_k(v-v_*) \\
&\leqq C\langle v\rangle^{(\gamma+2s)^+}\phi_k(v-v_*). \qquad (2.1.8)
\end{aligned}
$$

Then, recalling (2.1.4) and using (2.1.8) with $p = 0$ we have

$$
\begin{aligned}
|K_2^k(v_*)| \;&\leqq\; C\langle v_*\rangle^{(\gamma+2s)^+}\left\|\frac{\langle D_v\rangle^{m+2s}}{\langle v_*\rangle^{(\gamma+2s)^+}}\{2^{2ks}F_k(v,v_*)\}\right\|_{L^2}||\langle D_v\rangle^{-m}h_k(v,v_*)||_{L^2} \\
&\leqq\; C\langle v_*\rangle^{(\gamma+2s)^+}\Big(\|\tilde\phi_k(v-v_*)\langle D_v\rangle^{m+2s}g\|^2_{L^2_{(\gamma+2s)^+}} \\
&\quad + 2^{-k}\|\langle D_v\rangle^{m+2s}g\|^2_{L^2_{(\gamma+2s)^+}}\Big)^{1/2} \\
&\quad \times\Big(\|\tilde\phi_k(v-v_*)\langle D_v\rangle^{-m}h\|^2_{L^2} + 2^{-k}\|\langle D_v\rangle^{-m}h\|^2_{L^2}\Big)^{1/2} \\
&:= C\,\Gamma_k(v_*),
\end{aligned}
$$

where $\Gamma_k(v_*)$ stands for the quantity defined by this right-hand side up to a constant multiple.

On the other hand, in order to estimate $K_1^k(v_*)$, write

$$
\begin{aligned}
&\Big\{\widetilde{\hat F}_k(\eta^+,v_*) - \widetilde{\hat F}_k(\eta,v_*)\Big\}\overline{\widetilde{\hat h}_k(\eta,v_*)} \\
&\quad = \Big\{\widetilde{\hat F}_k(\eta^+,v_*) - \widetilde{\hat F}_k(\eta,v_*)\Big\}\Big\{\overline{\widetilde{\hat h}_k(\eta,v_*)} - \overline{\widetilde{\hat h}_k(\eta^+,v_*)}\Big\} \\
&\quad\quad - \eta^-\cdot\nabla\widetilde{\hat F}_k(\eta^+,v_*)\overline{\widetilde{\hat h}_k(\eta^+,v_*)} - \int_0^1\Big\{\nabla\widetilde{\hat F}_k(\eta^+ + \tau(\eta-\eta^+),v_*) \\
&\quad\quad - \nabla\widetilde{\hat F}_k(\eta^+,v_*))\mathrm{d}\tau\Big\}(\eta-\eta^+)\overline{\widetilde{\hat h}_k(\eta^+,v_*)}.
\end{aligned}
$$

Correspondingly, we decompose $K_1^k(v_*)$ into

$$
K_1^k(v_*) = K_1^{k,1}(v_*) + K_1^{k,2}(v_*) + K_1^{k,3}(v_*).
$$

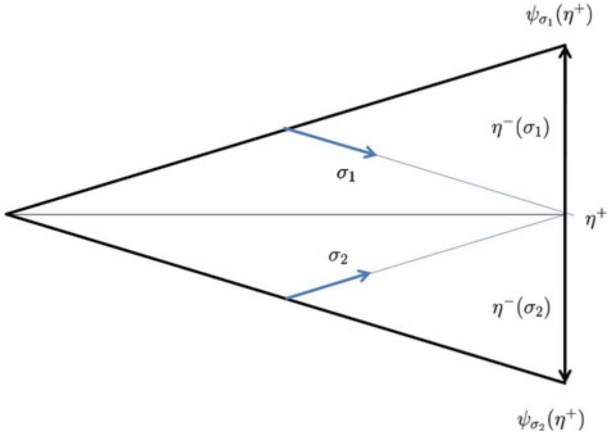

Fig. 1. Symmetry of σ_1 and σ_2 in Fourier variable

For the variable transformation $\eta \longrightarrow \eta^+ = \frac{1}{2}(\eta + |\eta|\sigma)$, we denote its inverse transformation $\eta^+ \longrightarrow \eta$ by $\psi_\sigma(\eta^+)$. Then

$$\begin{aligned} K_1^{k,2}(v_*) = &-\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k} b\left(\frac{\psi_\sigma(\eta^+)}{|\psi_\sigma(\eta^+)|}\cdot\sigma\right)\left|\frac{\partial(\psi_\sigma(\eta^+))}{\partial(\eta^+)}\right| \\ &\times\eta^-(\sigma)\cdot\left(\nabla\widetilde{\hat{F}_k}\right)(\eta^+,v_*)\overline{\widetilde{\hat{h}_k}(\eta^+,v_*)}\mathrm{d}\eta^+\mathrm{d}\sigma \\ = &\ 0,\ \ \text{with } \eta^-(\sigma) = \psi_\sigma(\eta^+) - \eta^+, \end{aligned}$$

because $\sigma_1, \sigma_2 \in \mathbb{S}^2 \cap \Omega_k$ are symmetric with respect to each other in the sense that, compare Fig. 1,

$$\eta^-(\sigma_1) = \psi_{\sigma_1}(\eta^+) - \eta^+ = -(\psi_{\sigma_2}(\eta^+) - \eta^+) = -\eta^-(\sigma_2).$$

Write $K_1^{k,1}(v_*)$ into

$$\begin{aligned} &K_1^{k,1}(v_*) \\ &= -\int_0^1\int_0^1\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k} b(\tilde{\eta}\cdot\sigma)\left\{\left(\nabla\widetilde{\hat{F}_k}\right)(\eta^+ + \tau(\eta-\eta^+), v_*)\cdot(\eta-\eta^+)\right\}\right. \\ &\quad\left.\times\overline{\left\{\left(\nabla\widetilde{\hat{h}_k}\right)(\eta^+ + s(\eta-\eta^+), v_*)\cdot(\eta-\eta^+)\right\}}\,\mathrm{d}\eta\,\mathrm{d}\sigma\right)\mathrm{d}\tau\,\mathrm{d}s. \end{aligned}$$

Since $|\eta-\eta^+|^2 = |\eta^-|^2 = |\eta|^2\sin^2(\theta/2)$ and the change of variable

$$\eta \to \eta^+ + \tau(\eta - \eta^+)$$

is also regular (see Page 2044 of [7]), (2.1.5) implies

$$
\begin{aligned}
&|K_1^{k,1}(v_*)| \\
&\leqq C\int_0^1\int_0^1\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k}\theta^2\ b(\tilde\eta\cdot\sigma)\langle\eta\rangle^2\left|\left(\nabla\widetilde{\hat F_k}\right)(\eta^+ +\tau(\eta-\eta^+),v_*)\right|\right.\\
&\quad\left.\times\left|\left(\nabla\widetilde{\hat h_k}\right)(\eta^+ +s(\eta-\eta^+),v_*)\right|\mathrm{d}\eta\mathrm{d}\sigma\right)\mathrm{d}\tau\mathrm{d}s\\
&\leqq C\int_0^1\int_0^1\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k}\theta^2\ b(\tilde\eta\cdot\sigma)\langle\eta\rangle^{2+2s+2m}\right.\\
&\quad\left.\times\left|\left(\nabla\widetilde{\hat F_k}\right)(\eta^+ +\tau(\eta-\eta^+),v_*)\right|\mathrm{d}\eta\mathrm{d}\sigma\right)^{1/2}\\
&\quad\times\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k}\theta^2\ b(\tilde\eta\cdot\sigma)\langle\eta\rangle^{2-2s-2m}\right.\\
&\quad\left.\times\left|\left(\nabla\widetilde{\hat h_k}\right)(\eta^+ +s(\eta-\eta^+),v_*)\right|\mathrm{d}\eta\mathrm{d}\sigma\right)^{1/2}\mathrm{d}\tau\mathrm{d}s\\
&\leqq C2^{k(2s-2)}\left\|\langle\eta\rangle^{2s+m}\left(\nabla\widetilde{\hat F_k}\right)\right\|_{L^2(\mathbb{R}^3)}\left\|\langle\eta\rangle^{-m}\left(\nabla\widetilde{\hat h_k}\right)\right\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Hence, we have obtained, by using (2.1.8) with $p=1$

$$
\begin{aligned}
&\left|K_1^{k,1}(v_*)\right|\\
&\leqq C\langle v_*\rangle^{(\gamma+2s)^+}\left\|\frac{\langle D_v\rangle^{m+2s}}{\langle v_*\rangle^{(\gamma+2s)^+}}\left\{2^{k(2s-1)}(v-v_*)F_k(v,v_*)\right\}\right\|_{L^2}\\
&\quad\times\left\|2^{-k}(v-v_*)h_k(v,v_*)\right\|_{H^{-m}}\\
&\leqq C\langle v_*\rangle^{(\gamma+2s)^+}\left(\left\|\tilde\phi_k(v-v_*)\langle D_v\rangle^{m+2s}g\right\|^2_{L^2_{(\gamma+2s)^+}}+2^{-k}\left\|\langle D_v\rangle^{m+2s}g\right\|^2_{L^2_{(\gamma+2s)^+}}\right)^{1/2}\\
&\quad\times\left(\left\|\tilde\phi_k(v-v_*)\langle D_v\rangle^{-m}h\right\|^2_{L^2}+2^{-k}\left\|\langle D_v\rangle^{-m}h\right\|^2_{L^2}\right)^{1/2},
\end{aligned}
$$

which has the same bound $\Gamma_k(v_*)$ as in the previous case, up to a constant factor. Finally, we consider

$$
\begin{aligned}
K_1^{k,3}(v_*)=-\int_0^1\int_0^1&\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k}b(\tilde\eta\cdot\sigma)\right.\\
&\times\left\{\left(\nabla^2\widetilde{\hat F_k}\right)(\eta^+ +\tau s(\eta-\eta^+),v_*)\tau(\eta-\eta^+)^2\right\}\\
&\left.\times\left\{\overline{\widetilde{\hat h}_k(\eta^+,v_*)}\right\}\mathrm{d}\eta\mathrm{d}\sigma\right)\mathrm{d}\tau\mathrm{d}s.
\end{aligned}
$$

Then, by using (2.1.8) with $p = 2$, we have

$$\begin{aligned}\left|K_1^{k,3}(v_*)\right| &\leqq C\langle v_*\rangle^{(\gamma+2s)^+}\left\|\frac{\langle D_v\rangle^{m+2s}}{\langle v_*\rangle^{(\gamma+2s)^+}}\left\{2^{k(2s-2)}(v-v_*)^2F_k(v,v_*)\right\}\right\|_{L^2}\\&\quad\times\|h_k(v,v_*)\|_{H^{-m}}\\&\leqq C\Gamma_k(v_*).\end{aligned}$$

Therefore, it follows from Schwarz's inequality that

$$\begin{aligned}&\left|(Q(f,g),h)_{L^2(\mathbb{R}^3_v)}\right|\\&\quad\leqq C\|f\|_{L^1_{(\gamma+2s)^+}}\\&\quad\quad\times\left(\sum_{k=0}^{\infty}\left\{\|\tilde\phi_k(v-v_*)\langle D_v\rangle^{m+2s}g\|^2_{L^2_{(\gamma+2s)^+}}+2^{-k}\|\langle D_v\rangle^{m+2s}g\|^2_{L^2_{(\gamma+2s)^+}}\right\}\right)^{1/2}\\&\quad\quad\times\left(\sum_{k=0}^{\infty}\left\{\|\tilde\phi_k(v-v_*)\langle D_v\rangle^{-m}h\|^2_{L^2}+2^{-k}\|\langle D_v\rangle^{-m}h\|^2_{L^2}\right\}\right)^{1/2}\\&\quad\leqq C\|f\|_{L^1_{(\gamma+2s)^+}}\|g\|_{H^{m+2s}_{(\gamma+2s)^+}}\|h\|_{H^{-m}},\end{aligned}$$

which yields (2.1.2). Now the proof of Theorem 2.1 is complete for the case $\alpha = 0$. □

To prove (2.1.1) for the case $\alpha \neq 0$, it suffices to show that

$$\begin{aligned}&\left|\left(Q(f,\,g),\langle v\rangle^{\alpha}h\right)_{L^2(\mathbb{R}^3_v)}\right|\\&\quad\leqq C||f||_{L^1_{\alpha^++(\gamma+2s)^+}(\mathbb{R}^3_v)}||g||_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)}||h||_{H^{-m}(\mathbb{R}^3_v)}.\end{aligned}\tag{2.1.9}$$

The argument is similar to the one for $\alpha = 0$, up to the estimation on $h_k(v, v_*)$ in (2.1.4) which must be replaced by

$$\tilde\phi_k(v-v_*)\langle v\rangle^{\alpha}h(v)=\langle v\rangle^{\alpha}h_k(v,v_*).$$

We can write

$$\langle v\rangle^{\alpha}h_k(v,v_*)=\left(\langle v_*\rangle^{\alpha}+2^{k\alpha}\right)\psi_k(v,v_*)h_k(v,v_*),\quad\text{if }\alpha>0,\tag{2.1.10}$$

$$\langle v\rangle^{\alpha}h_k(v,v_*)=\left(\frac{\langle v_*\rangle}{2^k}\right)^{\min\{(\gamma+2s)^+,-\alpha\}}\psi_k(v,v_*)h_k(v,v_*),\quad\text{if }\alpha<0.\tag{2.1.11}$$

with a suitable $\psi_k(v, v_*)$ belonging to $C_b^{\infty}(\mathbb{R}^3_v)$, uniformly with respect to k, v_*. For $p = 0, 1, 2$, we have

$$\begin{aligned}&\left|\frac{2^{k(\alpha+2s-p)}|v-v_*|^p\phi_k(v-v_*)\Phi(|v-v_*|)}{\langle v_*\rangle^{(\alpha+\gamma+2s)^+}}\right|\\&\quad\leqq C\frac{\langle v-v_*\rangle^{\alpha+\gamma+2s}}{\langle v_*\rangle^{(\alpha+\gamma+2s)^+}}\phi_k(v-v_*)\\&\quad\leqq C\langle v\rangle^{(\alpha+\gamma+2s)^+}\phi_k(v-v_*),\end{aligned}\tag{2.1.12}$$

which is similar to (2.1.8). We first consider the case $\alpha > 0$. It follows from (2.1.6) that

$$
\begin{aligned}
&\left|K_2^k(v_*)\right| \\
&\leqq \left(\langle v_*\rangle^\alpha + 2^{k\alpha}\right)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b(\tilde\eta\cdot\sigma)\left|\tilde{\hat{F}}_k(\eta^+, v_*) - \tilde{\hat{F}}_k(\eta, v_*)\right|\left|\widetilde{\widehat{\psi_k h_k}}(\eta, v_*)\right| \mathrm{d}\eta\mathrm{d}\sigma \\
&\leqq \left(\langle v_*\rangle^\alpha + 2^{k\alpha}\right)\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b\langle\eta\rangle^{2m+2s}\left(\left|\tilde{\hat{F}}_k(\eta^+, v_*)\right|^2 + \left|\tilde{\hat{F}}_k(\eta, v_*)\right|^2\right)\mathrm{d}\eta\mathrm{d}\sigma\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2\cap\Omega_k^c} b\langle\eta\rangle^{-2m-2s}\left|\widetilde{\widehat{\psi_k h_k}}(\eta, v_*)\right|^2 \mathrm{d}\eta\mathrm{d}\sigma\right)^{1/2} \\
&\leqq C2^{2ks}\left(\langle v_*\rangle^\alpha + 2^{k\alpha}\right)\left\|\langle D_v\rangle^{m+2s}F_k(v, v_*)\right\|_{L^2}\|\langle D_v\rangle^{-m}h_k(v, v_*)\|_{L^2}.
\end{aligned}
$$

Then, recalling (2.1.4), and using (2.1.8) and (2.1.12) with $p = 0$, we have

$$
\begin{aligned}
|K_2^k(v_*)| \leqq\ & C\left\{\langle v_*\rangle^{\alpha+(\gamma+2s)^+}\left\|\frac{\langle D_v\rangle^{m+2s}}{\langle v_*\rangle^{(\gamma+2s)^+}}\left\{2^{2ks}F_k(v, v_*)\right\}\right\|_{L^2}\right. \\
&\left. + \langle v_*\rangle^{(\alpha+\gamma+2s)^+}\left\|\frac{\langle D_v\rangle^{m+2s}}{\langle v_*\rangle^{(\alpha+\gamma+2s)^+}}\left\{2^{k(\alpha+2s)}F_k(v, v_*)\right\}\right\|_{L^2}\right\} \\
&\times||\langle D_v\rangle^{-m}h_k(v, v_*)||_{L^2} \\
\leqq\ & C\langle v_*\rangle^{\alpha+(\gamma+2s)^+} \\
&\times\left(\|\tilde\phi_k(v - v_*)\langle D_v\rangle^{m+2s}g\|^2_{L^2_{(\alpha+\gamma+2s)^+}}\right. \\
&\left. + 2^{-k}\|\langle D_v\rangle^{m+2s}g\|^2_{L^2_{(\alpha+\gamma+2s)^+}}\right)^{1/2} \\
&\times\left(\|\tilde\phi_k(v - v_*)\langle D_v\rangle^{-m}h\|^2_{L^2} + 2^{-k}\|\langle D_v\rangle^{-m}h\|^2_{L^2}\right)^{1/2} \\
:=\ & C\,\Gamma_k^\alpha(v_*),
\end{aligned}
$$

where $\Gamma_k^\alpha(v_*)$ stands for the quantity defined by this right-hand side up to a constant multiple.

Performing the same computation as above for $K_2^k(v_*)$, it follows from (2.1.5) that

$$
\left|K_1^{k,1}(v_*)\right| + \left|K_1^{k,3}(v_*)\right| \leqq C\Gamma_k^\alpha(v_*),
$$

so that (2.1.9) holds in this case.

The estimation on the case $\alpha < 0$ is also similar by using (2.1.11) if one considers the cases $\gamma + 2s \leqq 0$, $0 < \gamma + 2s \leqq -\alpha$ and $\gamma + 2s \geqq -\alpha$ separately. Details are omitted. And this completes the proof of Theorem 2.1.

In the following, we also need estimates on the commutator between the collision operator Q and the weight W_l. For this purpose, estimations on $|W_l - W_l'|$ are needed.

Lemma 2.3. *Let $l \in \mathbb{N}$. There exists $C > 0$ depending only on l such that*

$$\left|W_l - W_l'\right| \leqq C \sin\left(\frac{\theta}{2}\right)\left(W_l' + W_{l,*}'\right) \leqq C \sin\left(\frac{\theta}{2}\right) W_l' W_{l,*}', \quad (2.1.13)$$

and

$$\left|W_l - W_l'\right| \leqq C \sin\left(\frac{\theta}{2}\right)\left(W_l' + W_{l-1}' W_{1,*}' + \sin^{l-1}\left(\frac{\theta}{2}\right) W_{l,*}'\right). \quad (2.1.14)$$

Proof. It follows from $|v - v_*| = |v' - v_*'|$ and $|v|^2 + |v_*|^2 = |v'|^2 + |v_*'|^2$ that, for any $\lambda > 0$

$$|v|^2 \leqq |v'|^2 + |v_*'|^2, \quad W_\lambda \leqq 2^\lambda\left(W_\lambda' + W_{\lambda,*}'\right).$$

On the other hand

$$|v - v'|^2 = \sin^2\left(\frac{\theta}{2}\right)|v - v_*|^2,$$

where $0 \leqq \theta \leqq \pi/2$. The Taylor formula yields

$$\begin{aligned}
\left|W_l - W_l'\right| &\leqq C|v - v'|\left(W_{l-1} + W_{l-1}'\right)\\
&\leqq C \sin\left(\frac{\theta}{2}\right)|v - v_*|\left(W_{l-1}' + \left(1 + |v - v' + v'|^2\right)^{\frac{(l-1)}{2}}\right)\\
&\leqq C \sin\left(\frac{\theta}{2}\right)|v' - v_*'|\left(W_{l-1}' + |v - v'|^{l-1}\right)\\
&\leqq C \sin\left(\frac{\theta}{2}\right)\left(\left(W_1' + W_{1,*}'\right) W_{l-1}' + \sin^{l-1}\left(\frac{\theta}{2}\right)|v' - v_*'|^l\right)\\
&\leqq C \sin\left(\frac{\theta}{2}\right)\left(W_l' + W_{l-1}' W_{1,*}' + \sin^{l-1}\left(\frac{\theta}{2}\right) W_{l,*}'\right),
\end{aligned}$$

which gives (2.1.14). (2.1.13) follows from this by the interpolation inequality. And this completes the proof of the lemma. □

Lemma 2.4. *Let $l \in \mathbb{N}$, $m \in \mathbb{R}$.*

(1) *If $0 < s < 1/2$, there exists $C > 0$ such that*

$$\begin{aligned}
&\left|\left(\left(W_l Q(f, g) - Q(f, W_l g)\right), h\right)_{L^2(\mathbb{R}_v^3)}\right|\\
&\quad \leqq C\|f\|_{L^1_{l+\gamma^+}(\mathbb{R}_v^3)}\|g\|_{L^2_{l+\gamma^+}(\mathbb{R}_v^3)}\|h\|_{L^2(\mathbb{R}_v^3)}. \quad (2.1.15)
\end{aligned}$$

Moreover, if $l \geqq 3$ (actually, we need only $l > \frac{3}{2} + 2s$), then

$$\begin{aligned}
&\left|\left(\left(W_l Q(f, g) - Q(f, W_l g)\right), h\right)_{L^2(\mathbb{R}^3)}\right|\\
&\quad \leqq C\|f\|_{L^2_{l+\gamma^+}(\mathbb{R}_v^3))}\|g\|_{L^2_{l+\gamma^+}(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)}. \quad (2.1.16)
\end{aligned}$$

(2) *If* $1/2 < s < 1$*, then for any* $\varepsilon > 0$*, there is a constant* $C_\varepsilon > 0$ *such that*

$$\left|((W_l Q(f,g) - Q(f, W_l g)), h)_{L^2(\mathbb{R}^3_v)}\right| \leqq C_\varepsilon \|f\|_{L^1_{l+2s-1+\gamma^+}(\mathbb{R}^3_v)} \|g\|_{H^{2s-1+\varepsilon}_{l+2s-1+\gamma^+}(\mathbb{R}^3_v)} \|h\|_{L^2(\mathbb{R}^3_v)}, \tag{2.1.17}$$

and

$$\left|((W_l Q(f,g) - Q(f, W_l g)), h)_{L^2(\mathbb{R}^3_v)}\right| \leqq C_\varepsilon \|f\|_{L^1_{l+2s-1+\gamma^+}(\mathbb{R}^3_v)} \|g\|_{L^2_{l+2s-1+\gamma^+}(\mathbb{R}^3_v)} \|h\|_{H^{2s-1+\varepsilon}_l(\mathbb{R}^3_v)}. \tag{2.1.18}$$

(3) *When* $s = 1/2$*, we have the same estimates as* (2) *with* $2s-1$ *replaced by any small* $\kappa > 0$*.*

With Lemma 2.2, we immediately have the following improved upper bound estimate with respect to the weight.

Corollary 2.5. (1) *When* $0 < s < 1/2$*, we have*

$$\|Q(f,g)\|_{H^m_l(\mathbb{R}^3_v)} \leqq C\|f\|_{L^1_{\max\{l+\gamma^+,\,(\gamma+2s)^+\}}(\mathbb{R}^3_v)} \|g\|_{H^{m+2s}_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)},$$

provided that $m \leqq 0$ *and* $0 \leqq m + 2s$*.*

(2) *When* $1/2 < s < 1$*, we have*

$$\|Q(f,g)\|_{H^m_l(\mathbb{R}^3_v)} \leqq C\|f\|_{L^1_{\max\{l+2s-1+\gamma^+,(2s+\gamma)^+\}}(\mathbb{R}^3_v)} \|g\|_{H^{m+2s}_{l+\max\{2s-1+\gamma^+,(2s+\gamma)^+\}}(\mathbb{R}^3_v)}, \tag{2.1.19}$$

provided that $-1 < m \leqq 0$*.*

(3) *When* $s = 1/2$*, we have the same form of estimate as* (2.1.19) *with* $2s-1$ *replaced by any small* $\kappa > 0$*.*

In fact, this corollary is a direct consequence of Theorem 2.1 and Lemma 2.4.

Proof of Lemma 2.4. (1) **the case** $0 < s < 1/2$**.** By using $\Phi(|v'-v'_*|) \leqq \langle v'\rangle^{\gamma^+}\langle v'_*\rangle^{\gamma^+}$, we have

$$\begin{aligned}
&\left|((W_l Q(f,g) - Q(f, W_l g)), h)_{L^2(\mathbb{R}^3_v)}\right| \\
&= \left|\iiint b\,\Phi\, f'_* g'(W'_l - W) h \,\mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma\right| \\
&\leqq C \iiint b|\theta||(W_{l+\gamma^+} f)'_*|\; |(W_{l+\gamma^+} g)'||h| \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\
&= C \iiint b\,|\theta||(W_{l+\gamma^+} f)_*||(W_{l+\gamma^+} g)|\; |h'| \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\
&\leqq C \left(\iiint b|\theta|\; |(W_{l+\gamma^+} f)_*|\; |(W_{l+\gamma^+} g)|^2 \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma\right)^{1/2} \\
&\quad \times \left(\iiint b\,|\theta|\; |(W_{l+\gamma^+} f)_*||h'|^2 \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma\right)^{1/2} \\
&= C J_1 \times J_2.
\end{aligned}$$

Clearly, one has

$$J_1^2 \leqq C\|f\|_{L^1_{l+\gamma^+}} \|g\|^2_{L^2_{l+\gamma^+}} \int_{\mathbb{S}^2} b(\cos\theta)\,|\theta|\mathrm{d}\sigma \leqq C\|f\|_{L^1_{l+\gamma^+}} \|g\|^2_{L^2_{l+\gamma^+}}.$$

Next, by the regular change of variables $v \to v'$, compare [4,10], we have

$$J_2^2 = \iint D_0(v_*, v')|(W_{l+\gamma^+} f)_*||h'|^2 \mathrm{d}v_*\, \mathrm{d}v',$$

where

$$\begin{aligned} D_0(v, v') &= 2\int_{\mathbb{S}^2} \frac{\theta(v_*, v', \sigma)}{\cos^2(\theta(v_*, v', \sigma)/2)} b(\cos\theta(v_*, v', \sigma))\mathrm{d}\sigma \\ &\leqq C\int_0^{\pi/4} \psi^{-1-2s} \sin\psi \mathrm{d}\psi, \end{aligned}$$

and

$$\cos\psi = \frac{v' - v_*}{|v' - v_*|}\cdot\sigma, \quad \psi = \theta/2, \qquad \mathrm{d}\sigma = \sin\psi\mathrm{d}\psi\, \mathrm{d}\phi.$$

Thus,

$$J_2^2 \leqq C\|f\|_{L^1_{l+\gamma^+}} \|h\|^2_{L^2},$$

and this, together with the estimate on J_1, gives (2.1.15).

We now prove (2.1.16) by using (2.1.14) instead of (2.1.13). We have

$$\begin{aligned} &\left|((W_l Q(f, g) - Q(f, W_l g))\,, h)_{L^2(\mathbb{R}^3_v)}\right| \\ &\leqq C\left\{\iiint b|\theta|^l |(W_{l+\gamma_+} f)'_*||(W_{\gamma_+} g)'||h|\mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \right.\\ &\qquad + \iiint b\,|\theta|\, |(W_{1+\gamma^+} f)'_*|\, |(W_{l-1+\gamma_+} g)'|\, |h|\mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\ &\qquad \left. + \iiint b\,|\theta|\, |(W_{\gamma_+} f)'_*||(W_{l+\gamma_+} g)'|\, |h|\mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \right\} \\ &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3. \end{aligned}$$

$\mathcal{M}_2, \mathcal{M}_3$ can be estimated similarly to (2.1.15), and we have

$$\begin{aligned} \mathcal{M}_2 &\leqq C\|f\|_{L^1_{1+\gamma^+}} \|g\|_{L^2_{l-1+\gamma^+}} \|h\|_{L^2}, \\ \mathcal{M}_3 &\leqq C\|f\|_{L^1_{\gamma^+}} \|g\|_{L^2_{l+\gamma^+}} \|h\|_{L^2}. \end{aligned}$$

$\mathcal{M}_1$ can be estimated as follows. Firstly, we have

$$\begin{aligned} \mathcal{M}_1^2 &= C^2\left(\iiint b\,|\theta|^l |(W_{l+\gamma_+} f)_*||(W_{\gamma_+} g)|\, |h'|\mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma\right)^2 \\ &\leqq C^2 \iiint b\,|\theta|^{l-\frac{3}{2}} |(W_{\gamma_+} g)||(W_{l+\gamma_+} f)_*|^2 \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\ &\quad \times \iiint b|\theta|^{l+\frac{3}{2}} |(W_{\gamma_+} g)||h'|^2 \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\ &= \mathcal{M}_{1,1} \times \mathcal{M}_{1,2}. \end{aligned}$$

Then, if $l - \frac{3}{2} - 2s - 1 > -1$, that is, $l > 2s + \frac{3}{2}$, we have

$$\mathcal{M}_{1,1} \leqq C\|g\|_{L^1_{\gamma+}} \|f\|^2_{L^2_{l+\gamma+}}.$$

On the other hand, for $\mathcal{M}_{1,2}$ we need to apply the singular change of variables $v_* \to v'$. The Jacobian of this transform is, with $\mathbf{k} = (v - v_*)/|v - v_*|$,

$$\left|\frac{\partial v_*}{\partial v'}\right| = \frac{8}{|I - \mathbf{k}\otimes\sigma|} = \frac{8}{|1 - \mathbf{k}\cdot\sigma|} = \frac{4}{\sin^2(\theta/2)} \leqq 16\theta^{-2}, \quad \theta \in [0, \pi/2]. \tag{2.1.20}$$

Notice that this gives rise to an additional singularity in the angle θ around 0. Actually, the situation is even worse in the following sense. Recall that θ is no longer a legitimate polar angle. In this case, the best choice of the pole is $\mathbf{k}'' = (v' - v)/|v' - v|$ for which polar angle ψ defined by $\cos\psi = \mathbf{k}''\cdot\sigma$ satisfies (compare [4, Fig. 1])

$$\psi = \frac{\pi - \theta}{2}, \quad \mathrm{d}\sigma = \sin\psi \mathrm{d}\psi\, \mathrm{d}\phi, \quad \psi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

This measure does not cancel any of the singularity of $b(\cos\theta)$, unlike the case in the usual polar coordinates. Nevertheless, this singular change of variables yields

$$\begin{aligned}\mathcal{M}_{1,2} &= C\iiint b\,|\theta|^{l+\frac{3}{2}}|(W_{\gamma_+}g)|\ |h'|^2 \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\\ &\leqq C\iint D_1(v, v')|(W_{\gamma_+}g)|\ |h'|^2\mathrm{d}v\,\mathrm{d}v',\end{aligned}$$

when $l > \frac{3}{2} + 2s$ because

$$D_1(v, v') = \int_{\mathbb{S}^2} \theta^{l+\frac{3}{2}-2} b(\cos\theta)\mathrm{d}\sigma \leqq C\int_{\pi/4}^{\pi/2}\left(\frac{\pi}{2} - \psi\right)^{-2-2s+l+\frac{3}{2}-2} \mathrm{d}\psi \leqq C.$$

Therefore,

$$\mathcal{M}_{1,2} \leqq C\|g\|_{L^1_{\gamma+}}\|h\|^2_{L^2}.$$

Now the proof of (2.1.16) is completed by embedding the estimate for $l > \frac{3}{2}$,

$$\|g\|_{L^1_{\gamma+}} \leqq C\|g\|_{L^2_{l+\gamma+}}.$$

(2) **the case** $1/2 \leqq s < 1$**.** Since we look for an upper bound estimate and $\varepsilon > 0$, it is sufficient to assume $s > 1/2$ for our purpose. Write

$$\begin{aligned}((W_l Q(f, g) - Q(f, W_l g)), h)_{L^2(\mathbb{R}^3_v)} &= \iiint B\ f'_* g'(W'_l - W_l)\, h\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\\ &= \iiint B\ f_* g(W_l - W'_l)\, h'\,\mathrm{d}v\mathrm{d}v_*\,\mathrm{d}\sigma\end{aligned}$$

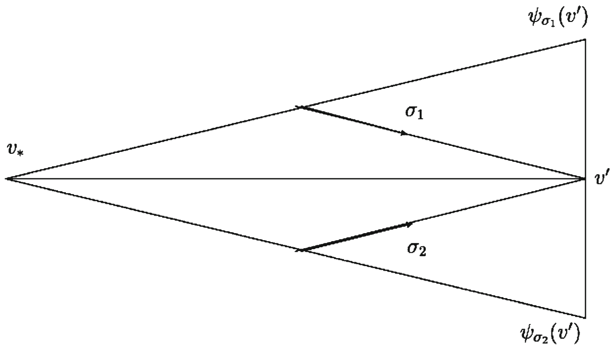

Fig. 2. Symmetry of σ_1 and σ_2 in velocity variable

$$\begin{aligned}&= \iiint Bf_* g'(W_l - W_l')\, h' \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\ &\quad + \iiint B\, f_*(g-g')(W_l - W_l')\, h'\, \mathrm{d}v\, \mathrm{d}v_* \mathrm{d}\sigma \\ &= I_1 + I_2.\end{aligned}$$

The Taylor expansion gives

$$W_l - W_l' = \nabla W_l(v')(v - v') - \int_0^1 (1-\tau)\nabla^2 W_l(v' + \tau(v - v'))\mathrm{d}\tau (v - v')^2,$$

so that

$$I_1 = -\int_0^1 (1-\tau) \iiint Bf_* \left\{\nabla^2 W_l(v' + \tau(v - v'))\right\} (v - v')^2 g' h' \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}\tau.$$

By using the symmetry property shown in Fig. 1 (see also Fig. 2, and §3 in [32]), the first order term in the Taylor expansion vanishes, that is,

$$\begin{aligned}&\iiint Bf_* g' \nabla W_l(v')(v - v') h' \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\ &= \iint f_* \left\{ \int_{\mathbb{S}^2} b\left(\frac{\psi_\sigma(v') - v_*}{|\psi_\sigma(v') - v_*|} \cdot \sigma\right) \Phi(|\psi_\sigma(v') - v_*|) \right. \\ &\qquad \left. \times \left|\frac{\partial(\psi_\sigma(v'))}{\partial(v')}\right| (\psi_\sigma(v') - v')\mathrm{d}\sigma \right\} \cdot \quad \nabla W_l(v')\, g' h' \mathrm{d}v'\, \mathrm{d}v_* = 0.\end{aligned}$$

Here, we have used the notation that for a transformation $v \to v'$, its inverse transformation is denoted by $v' \to \psi_\sigma(v') = v$. And σ_1, σ_2 are symmetric with respect to each other, in the sense that $\psi_{\sigma_1}(v') - v' = -(\psi_{\sigma_2}(v') - v')$.

Furthermore, since

$$\begin{aligned}&\left|\left\{\nabla^2 W_l(v' + \tau(v - v'))\right\} (v - v')^2\right| \\ &\leqq C\theta^2 |v_* - v'|^2 \{W_{l-2}(v_*) + W_{l-2}(v' + \tau(v - v') - v_*)\} \\ &\leqq C\theta^2 \{W_l(v_*) + W_l(v')\} \leqq C\theta^2 W_l(v_*) W_l(v')\end{aligned}$$

and $\Phi(|v-v_*|) \leqq (\sqrt{2}\langle v'-v_*\rangle)^{\gamma^+} \leqq \sqrt{2}^{\gamma^+}\langle v_*\rangle^{\gamma^+}\langle v'\rangle^{\gamma^+}$, we get by the regular change of variables $v \to v'$ and the Schwartz inequality

$$|I_1| \leqq C||f||_{L^1_{l+\gamma^+}(\mathbb{R}^3_v)}||g||_{L^2_{l+\gamma^+}(\mathbb{R}^3_v)}||h||_{L^2(\mathbb{R}^3_v)}. \tag{2.1.21}$$

In order to estimate I_2, we shall apply the Littlewood–Paley decomposition $\{\triangle_j\}_{j=0}^{\infty}$, which is a dyadic decomposition in the Fourier variable (see also [3,14,47]),

$$\triangle_j g(v) = \mathcal{F}^{-1}\left(\phi_j(\eta)\hat{g}(\eta)\right), \quad g = \sum_0^{\infty}\triangle_j g,$$

and for $m \in \mathbb{R}$,

$$\|\triangle_j g\|_{H^m} \approx 2^{jm}\|\triangle_j g\|_{L^2}, \quad \|g\|^2_{H^m} \approx \sum 2^{2jm}\|\triangle_j g\|^2_{L^2}.$$

Then we have the following decomposition

$$\begin{aligned}
I_2 = &\sum_{j=0}^{\infty}\int_0^1\left(\int_{\mathbb{R}^6}\left\{\int_{\Omega_j} B\, f_*\nabla_v(\triangle_j g)(v'+\tau(v-v'))\cdot(v-v')\right.\right.\\
&\qquad\left.\left.\times(W_l-W_l')\,h'\mathrm{d}\sigma\right\}\mathrm{d}v\,\mathrm{d}v_*\right)\mathrm{d}\tau\\
&+\sum_{j=0}^{\infty}\int_{\mathbb{R}^6}\left\{\int_{\Omega_j^c} B\, f_*\left\{(\triangle_j g)(v)-(\triangle_j g)(v')\right\}(W_l-W_l')\,h'\mathrm{d}\sigma\right\}\mathrm{d}v\,\mathrm{d}v_*\\
=&\sum_{j=0}^{\infty}\left(I^1_{2,j}+I^2_{2,j}\right),
\end{aligned}$$

where

$$\Omega_j = \Omega_j(v,v_*) = \left\{\sigma \in \mathbb{S}^2\,;\ \frac{v-v_*}{|v-v_*|}\cdot\sigma \geqq 1-2^{1-2j}\langle v-v_*\rangle^{-2}\right\}.$$

Note that if $1/2 < s < 1$, then

$$\begin{aligned}
\int_{\Omega_j} b(\cos\theta)\,\theta^2\mathrm{d}\sigma &= 2\pi\int_{\{\theta\in[0,\pi/2];\ \sin(\theta/2)\leqq 2^{-j}\langle v-v_*\rangle^{-1}\}}\sin\theta\ b(\cos\theta)\theta^2\mathrm{d}\theta\\
&\leqq C2^{j(2s-2)}\langle v-v_*\rangle^{2s-2},
\end{aligned} \tag{2.1.22}$$

and

$$\begin{aligned}
\int_{\Omega_j^c} b(\cos\theta)\,\theta\mathrm{d}\sigma &= 2\pi\int_{\{\theta\in[0,\pi/2];\ \sin(\theta/2)\geqq 2^{-j}\langle v-v_*\rangle^{-1}\}}\sin\theta\ b(\cos\theta)\,\theta\mathrm{d}\theta\\
&\leqq C2^{j(2s-1)}\langle v-v_*\rangle^{2s-1}.
\end{aligned} \tag{2.1.23}$$

To estimate $I^1_{2,j}$, we need the change of variables

$$v \to z = v' + \tau(v - v') = \frac{1+\tau}{2}v + \frac{1-\tau}{2}(|v - v_*|\sigma + v_*). \tag{2.1.24}$$

The Jacobian of this transform is bounded from below uniformly in v_*, σ, τ, because

$$\begin{aligned}
\left|\frac{\partial(z)}{\partial(v)}\right| &= \left|\det\left(\frac{1+\tau}{2}I + \frac{1-\tau}{2}\sigma \otimes \mathbf{k}\right)\right| \qquad \left(\mathbf{k} = \frac{v - v_*}{|v - v_*|}\right)\\
&= \frac{(1+\tau)^3}{2^3}\left|1 + \frac{1-\tau}{1+\tau}\mathbf{k}\cdot\sigma\right| = \frac{(1+\tau)^3}{2^3}\left|\frac{2\tau}{1+\tau} + 2\frac{1-\tau}{1+\tau}\cos^2\frac{\theta}{2}\right|\\
&\geqq \frac{(1+\tau)^3}{2^3}\left|\frac{2\tau}{1+\tau} + \frac{1-\tau}{1+\tau}\right| = \frac{(1+\tau)^3}{2^3} \geqq \frac{1}{2^3}.
\end{aligned}$$

Recall, compare [4], that the cross-section $B(v - v_*, \theta)$ is assumed to be supported in $0 \leqq \theta \leqq \pi/4$. Furthermore, we have

$$\begin{aligned}
|z - v_*| &= \left|\frac{1+\tau}{2}(v - v_*) + \frac{1-\tau}{2}|v - v_*|\sigma\right|\\
&= |v - v_*|\left|\left(\frac{1+\tau}{2}\right)^2 + \left(\frac{1-\tau}{2}\right)^2 + \frac{1-\tau^2}{2}\mathbf{k}\cdot\sigma\right|^{1/2}\\
&= |v - v_*|\left|\tau^2 + (1-\tau^2)\cos^2\frac{\theta}{2}\right|^{1/2}\\
&\geqq \frac{1}{\sqrt{2}}|v - v_*|,
\end{aligned} \tag{2.1.25}$$

which implies $\langle v - v_*\rangle^{2s}\Phi(|v - v_*|) \leqq C\langle z\rangle^{2s+\gamma_+}\langle v_*\rangle^{2s+\gamma_+}$. We have from (2.1.22) that for any $\varepsilon > 0$

$$\begin{aligned}
\left|I^1_{2,j}\right| &\leqq C\int_0^1 \iiint_{\Omega_j} b\theta^2|(W_{l-1+2s+\gamma_+}f)_*|\\
&\qquad \times\left|\left(W_{l-1+2s+\gamma_+}(\nabla_v\triangle_j g)\right)(z)\right|\langle v - v_*\rangle^{2-2s}|h'|\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\tau\\
&\leqq C\left[\int_0^1\left(\iint\left(\int_{\Omega_j} b\theta^2\left|(W_{l-1+2s+\gamma_+}f)_*\right|\left|\left(W_{l-1+2s+\gamma_+}(\nabla_v\triangle_j g)\right)(z)\right|^2\right.\right.\right.\\
&\qquad\left.\left.\left.\times\langle v - v_*\rangle^{2-2s}|\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2}\mathrm{d}\tau\right]\\
&\qquad \times\left[\iint\left(\int_{\Omega_j} b\theta^2|(W_{l-1+2s+\gamma_+}f)_*|\langle v - v_*\rangle^{2-2s}|h'|^2\mathrm{d}\sigma\right)\mathrm{d}v\mathrm{d}v_*\right]^{1/2}\\
&\leqq C2^{-\varepsilon j}\|f\|_{L^1_{l+2s-1+\gamma_+}(\mathbb{R}^3_v)}\|g\|_{H^{2s-1+\varepsilon}_{l+2s-1+\gamma_+}(\mathbb{R}^3_v)}\|h\|_{L^2(\mathbb{R}^3_v)},
\end{aligned}$$

where we have used

$$
\begin{aligned}
|(v-v')(W_l - W_l')| &\leqq C\theta^2 |v-v_*|^2 (W_{l-1}(z) + W_{*l-1}) \\
&\leqq C\theta^2 |v-v_*|^2 W_{l-1}(z) W_{*l-1},
\end{aligned}
$$

the regular change of variables $v \to z$ defined by (2.1.24) and the regular change of variables $v \to v'$. The estimate (2.1.23) yields the same bound for $I_{2,j}^2$. Therefore, we obtain

$$
|I_2| \leqq C||f||_{L^1_{l+2s-1}(\mathbb{R}^3_v)} ||g||_{H^{2s-1+\varepsilon}_{l+2s-1}(\mathbb{R}^3_v)} ||h||_{L^2(\mathbb{R}^3_v)}. \tag{2.1.26}
$$

Estimates (2.1.21) and (2.1.26) together give the desired estimate (2.1.17). □

For the convenience of the readers, we postpone the proof of (2.1.18) to the end of Section 2.3. This completes the proof of Lemma 2.2 because (3) comes from (2) for the case $s = 1/2 + \kappa$.

2.2. *Coercivity estimates*

We establish coercivity estimates of the Boltzmann collision operator. We will show that the angular singularity in the cross-section yields sub-elliptic estimates which are lower bounds of the collision operator, see [4]. Notice that we need precise weighted sub-elliptic estimates as given in the following theorem. For more detailed explanations and notations, interested readers can refer to [3,32].

Theorem 2.6. *Assume that* $\gamma \in \mathbb{R}$, $0 < s < 1$. *Let* $g \in L^1_{\max\{\gamma^+, 2-\gamma^+\}} \cap L\log L(\mathbb{R}^3_v)$, $g \geqq 0, \not\equiv 0$. *Then there exists a constant* $C_g > 0$ *depending only on* $B(v - v_*, \theta)$, $\|g\|_{L^1_{\max\{\gamma^+, 2-\gamma^+\}}}$ *and* $\|g\|_{L\log L}$, *and* $C > 0$ *depending on* $B(v - v_*, \theta)$ *such that for any smooth function* $f \in H^1_{\gamma/2}(\mathbb{R}^3_v) \cap L^2_{\gamma^+/2}(\mathbb{R}^3_v)$, *we have*

$$
\begin{aligned}
-(Q(g,f), f)_{L^2(\mathbb{R}^3_v)} &\geqq C_g \|W_{\gamma/2} f\|^2_{H^s(\mathbb{R}^3_v)} \\
&\quad - C||g||_{L^1_{\max\{\gamma^+, 2-\gamma^+\}}(\mathbb{R}^3_v)} \|f\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)}.
\end{aligned} \tag{2.2.1}
$$

Remark 2.7. From the proof of the theorem, the constant C_g is seen to be an increasing function of $\|\tilde{g}\|_{L^1}$, $\|\tilde{g}\|^{-1}_{L^1_1}$ and $\|\tilde{g}\|^{-1}_{L\log L}$ where $\tilde{g} = \langle v\rangle^{-|\gamma|} g$. If the function g depends continuously on a parameter $\tau \in \Xi$, then the constant C_g depends on $\inf_{\tau\in\Xi} \|\langle v\rangle^{-|\gamma|} g_\tau\|_{L^1}$, $\sup_{\tau\in\Xi} \|g_\tau\|_{L\log L}$ and $\sup_{\tau\in\Xi} \|g\|_{L^1_{\max\{\gamma^+, 2-\gamma^+\}}}$. In the later application, we take $\tau = (t, x)$.

Proof. Firstly, we have

$$
\begin{aligned}
(Q(g,f), f) &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \Phi(|v-v_*|) b(\cos\theta) g(v_*) f(v) \{f(v') - f(v)\} \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v \\
&= \frac{1}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \Phi(|v-v_*|) b(\cos\theta) g(v_*) \{f(v')^2 - f(v)^2\} \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} \Phi(|v-v_*|) b(\cos\theta) g(v_*) \{f(v') - f(v)\}^2 \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v \\
&= \mathcal{R}_1 - \mathcal{R}_2.
\end{aligned}
$$

For $\mathcal{R}_1$, according to the cancellation lemma, Corollary 2 of [4], we have

$$\begin{aligned}\mathcal{R}_1 &= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\Phi(|v-v_*|)b(\cos\theta)g(v_*)\{f(v')^2-f(v)^2\}\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\
&= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\left\{\Phi\left(\frac{|v-v_*|}{\cos\frac{\theta}{2}}\right)\frac{1}{\cos^3\frac{\theta}{2}}-\Phi(|v-v_*|)\right\}bg(v_*)f(v)^2dv\mathrm{d}\sigma\,\mathrm{d}v_*\\
&= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\Phi\left(\frac{|v-v_*|}{\cos\frac{\theta}{2}}\right)\left\{\frac{1}{\cos^3\frac{\theta}{2}}-1\right\}bg(v_*)f(v)^2\mathrm{d}v\,\mathrm{d}\sigma\,\mathrm{d}v_*\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\left\{\Phi\left(\frac{|v-v_*|}{\cos\frac{\theta}{2}}\right)-\Phi(|v-v_*|)\right\}bg(v_*)f(v)^2\mathrm{d}v\,\mathrm{d}\sigma\,\mathrm{d}v_*\\
&= \mathcal{R}_{11}+\mathcal{R}_{12}.\end{aligned}$$

For the first term $\mathcal{R}_{11}$, from $1-\cos^3\frac{\theta}{2}\leqq 3(1-\cos\frac{\theta}{2})=6\sin^2\frac{\theta}{4}$, it follows that

$$\mathcal{R}_{11}\leqq C\|g\|_{L^1_{\gamma^+}}\|f\|^2_{L^2_{\gamma^+/2}},$$

because $\Phi\leqq 1$ when $\gamma<0$. For the second term $\mathcal{R}_{12}$, we first note that the mean value theorem gives

$$\begin{aligned}&\Phi\left(\frac{|v-v_*|}{\cos\frac{\theta}{2}}\right)-\Phi(|v-v_*|)\\
&\quad= -\left(\frac{1}{\cos\frac{\theta}{2}}-1\right)|v-v_*|^2\left(1+\left(\frac{|v-v_*|}{a}\right)^2\right)^{\frac{\gamma}{2}-1}\frac{2}{a^3}\\
&\quad\leqq C\left(\frac{1}{\cos\frac{\theta}{2}}-1\right)\Phi(|v-v_*|),\end{aligned}$$

where $\frac{\sqrt{2}}{2}\leqq\cos\frac{\theta}{2}<a<1$. Similar to $\mathcal{R}_{11}$, we can obtain

$$\mathcal{R}_{12}\leqq C\|g\|_{L^1_{\gamma^+}}\|f\|^2_{L^2_{\gamma^+/2}}.$$

For the term $\mathcal{R}_2$, noting that

$$\Phi(|v-v_*|)=\left(1+|v-v_*|^2\right)^{\frac{\gamma}{2}}\geqq\frac{\langle v\rangle^\gamma}{\langle v_*\rangle^{|\gamma|}},$$

we have

$$\begin{aligned}\mathcal{R}_2 &= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\Phi(|v-v_*|)b(\cos\theta)g(v_*)\left\{f(v')-f(v)\right\}^2\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\
&\geqq C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}b(\cos\theta)\frac{g(v_*)}{\langle v_*\rangle^{|\gamma|}}\langle v\rangle^\gamma\left\{f(v')-f(v)\right\}^2\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\
&= C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}b(\cos\theta)\frac{g(v_*)}{\langle v_*\rangle^{|\gamma|}}\left\{\langle v\rangle^{\frac{\gamma}{2}}f(v')-\langle v\rangle^{\frac{\gamma}{2}}f(v)\right\}^2\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\end{aligned}$$

$$= C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\frac{g(v_*)}{\langle v_*\rangle^{|\gamma|}}\Big\{\langle v'\rangle^{\frac{\gamma}{2}} f(v') - \langle v\rangle^{\frac{\gamma}{2}} f(v) + \langle v\rangle^{\frac{\gamma}{2}} f(v') - \langle v'\rangle^{\frac{\gamma}{2}} f(v')\Big\}^2 \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v,$$

and then, by using the fact that $(a-b)^2 \geqq a^2/2 - b^2$, we have

$$\mathcal{R}_2 \geqq C_1\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\frac{g(v_*)}{\langle v_*\rangle^{|\gamma|}}\{\langle v'\rangle^{\frac{\gamma}{2}} f(v') - \langle v\rangle^{\frac{\gamma}{2}} f(v)\}^2 \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v$$
$$-C_2\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\frac{g(v_*)}{\langle v_*\rangle^{|\gamma|}}\{\langle v\rangle^{\frac{\gamma}{2}} f(v') - \langle v'\rangle^{\frac{\gamma}{2}} f(v')\}^2 \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v = \mathcal{R}_{21} - \mathcal{R}_{22}.$$

For the first term $\mathcal{R}_{21}$, by using Corollary 3 and Proposition 2 of [4], we have

$$\begin{aligned}\mathcal{R}_{21} &= C_1\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\frac{g(v_*)}{\langle v_*\rangle^{|\gamma|}}\Big\{\langle v'\rangle^{\frac{\gamma}{2}} f(v') - \langle v\rangle^{\frac{\gamma}{2}} f(v)\Big\}^2 \mathrm{d}\sigma \mathrm{d}v_*\, \mathrm{d}v\\ &\geqq C\int_{\mathbb{R}^3} |\mathcal{F}(W_{\gamma/2}f)(\xi)|^2\left\{\int_{\mathbb{S}^2} b(\tilde{\xi}\cdot\sigma)\left(\mathcal{F}(\tilde{g})(0) - \left|\mathcal{F}(\tilde{g})(\xi^-)\right|\right)\right\}\mathrm{d}\xi\\ &\geqq \tilde{C}_g\|W_{\gamma/2}f\|^2_{H^s} - C\|\tilde{g}\|_{L^1}\|f\|^2_{L^2_{\gamma^+/2}},\end{aligned}$$

where $\tilde{g} = \langle v\rangle^{-|\gamma|}g$. Here $\tilde{C}_g$ is an increasing function with respect to $\|\tilde{g}\|_{L^1}$, $\|\tilde{g}\|^{-1}_{L^1_1}$ and $\|\tilde{g}\|^{-1}_{L\log L}$, according to the proof in the last part of [4] (see also Lemma 2.1 of [38]).

For the second term $\mathcal{R}_{22}$, note that for some $\tau \in (0,1)$, we have

$$\begin{aligned}\frac{\langle v\rangle^{\frac{\gamma}{2}} - \langle v'\rangle^{\frac{\gamma}{2}}}{\langle v_*\rangle^{\frac{|\gamma|}{2}}} &\leqq C\frac{\langle v' + \tau(v-v')\rangle^{\frac{\gamma-2}{2}}}{\langle v_*\rangle^{\frac{|\gamma|}{2}}}|v-v'|\\ &\leqq C\frac{\langle v_*\rangle^{\frac{|2-\gamma|}{2}-\frac{|\gamma|}{2}}}{\langle v' + \tau(v-v') - v_*\rangle^{\frac{2-\gamma}{2}}}|v'-v_*|\tan(\theta/2)\\ &\leqq C\langle v_*\rangle^{\frac{|2-\gamma|}{2}}\langle v'-v_*\rangle^{\frac{\gamma}{2}}\tan(\theta/2)\\ &\leqq C\begin{cases}\langle v_*\rangle^{\frac{|2-\gamma|}{2}}\langle v'\rangle^{\frac{\gamma}{2}}\tan(\theta/2), & \text{if } \gamma \geqq 0,\\ \langle v_*\rangle\tan(\theta/2), & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, we get

$$\begin{aligned}\mathcal{R}_{22} &= C_2\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)g(v_*)\left\{\frac{\langle v\rangle^{\frac{\gamma}{2}} - \langle v'\rangle^{\frac{\gamma}{2}}}{\langle v_*\rangle^{\frac{|\gamma|}{2}}}\right\}^2 f(v')^2 \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v\\ &\leqq C_2\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\tan^2(\theta/2)\langle v_*\rangle^{|2-\gamma^+|}g(v_*)\left\{\langle v'\rangle^{\frac{\gamma^+}{2}} f(v')\right\}^2 \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v\\ &\leqq C_2||g||_{L^1_{|2-\gamma^+|}}\|f\|^2_{L^2_{\gamma^+/2}}.\end{aligned}$$

This completes the proof of Theorem 2.6. □

In the following analysis, we shall also need the following interpolation inequality between weighted Sobolev spaces in v, see for instance [26,32].

Lemma 2.8. *For any $k \in \mathbb{R}$, $p \in \mathbb{R}_+$, $\delta > 0$,*

$$\|f\|^2_{H^k_p(\mathbb{R}^3_v)} \leqq C_\delta \|f\|_{H^{k-\delta}_{2p}(\mathbb{R}^3_v)} \|f\|_{H^{k+\delta}_0(\mathbb{R}^3_v)}.$$

2.3. Commutator estimates

We are now going to study the commutators of a family of pseudo-differential operators with the Boltzmann collision operator. This is a key step in the regularity analysis of weak solutions because it requires mollifiers defined by pseudo-differential operators. Below, we denote $(\cdot,\cdot)_{L^2(\mathbb{R}^3_v)}$ by $(\cdot,\cdot)$ for simplicity of notation, without any confusion.

Proposition 2.9. *Let $\lambda \in \mathbb{R}$ and $M(\xi)$ be a positive symbol of pseudo-differential operator in $S^\lambda_{1,0}$ of the form of $M(\xi) = \tilde{M}(|\xi|^2)$. Assume that for any $c > 0$ there exists a constant $C > 0$ such that for any $s, \tau > 0$*

$$c^{-1} \leqq \frac{s}{\tau} \leqq c \quad \textit{implies} \quad C^{-1} \leqq \frac{\tilde{M}(s)}{\tilde{M}(\tau)} \leqq C. \tag{2.3.1}$$

Furthermore assume that $M(\xi)$ satisfies

$$|M^{(\alpha)}(\xi)| = |\partial^\alpha_\xi M(\xi)| \leqq C_\alpha M(\xi)\langle\xi\rangle^{-|\alpha|}, \tag{2.3.2}$$

for any $\alpha \in \mathbb{N}^3$. Then the followings hold.

(1) *If $0 < s < 1/2$, for any $N > 0$ there exists a $C_N > 0$ such that*

$$\begin{aligned}&\left|(M(D_v)Q(f,g) - Q(f, M(D_v)g), h)_{L^2(\mathbb{R}^3_v)}\right| \\ &\quad\leqq C_N \|f\|_{L^1_{\gamma^+}(\mathbb{R}^3_v))} \left(\|Mg\|_{L^2_{\gamma^+}(\mathbb{R}^3_v)} + \|g\|_{H^{\lambda-N}_{\gamma^+}(\mathbb{R}^3_v))}\right) \|h\|_{L^2(\mathbb{R}^3_v)}. \end{aligned}\tag{2.3.3}$$

(2) *If $1/2 < s < 1$, for any $N > 0$ and any $\varepsilon > 0$ there exists a $C_{N,\varepsilon} > 0$ such that*

$$\begin{aligned}&\left|(M(D_v)Q(f,\ g) - Q(f,\ M(D_v)g), h)_{L^2(\mathbb{R}^3_v)}\right| \\ &\quad\leqq C_{N,\varepsilon} \|f\|_{L^1_{(2s+\gamma-1)^+}(\mathbb{R}^3_v))} \\ &\qquad\times \left(\|Mg\|_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)^+}(\mathbb{R}^3_v))} + \|g\|_{H^{\lambda-N}_{\gamma^+}(\mathbb{R}^3_v))}\right) \|h\|_{L^2(\mathbb{R}^3_v)}. \end{aligned}\tag{2.3.4}$$

(3) *If $s = 1/2$, we have the same estimate as (2.3.4) with $(2s+\gamma-1)$ replaced by $(\gamma+\kappa)$ for any small $\kappa > 0$.*

Proof. Firstly, set $\Phi_*(v) = \Phi(|v - v_*|)$ and write

$$
\begin{aligned}
&(M(D_v)Q(f,g),h) - (Q(f,\ M(D_v)g),h) \\
&= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma) f(v_*)g(v)\left(\overline{(M\,h)\,(v')} - \overline{(M\,h)\,(v)}\right) \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\quad - \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma) f(v_*)\,(M\,g)\,(v)\left(\overline{h(v')} - \overline{h(v)}\right) \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)\Big[(\Phi_* g)(v)\overline{(Mh)(v')} - \{M(\Phi_* g)\}\,(v)\overline{h(v')}\Big] \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)\,\{M(\Phi_* g)\}\,(v)\left(\overline{h(v')} - \overline{h(v)}\right) \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\quad - \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)\,\{\Phi_*\,(M\,g)\}\,(v)\left(\overline{h(v')} - \overline{h(v)}\right) \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)\Big[(\Phi_* g)(v)\overline{(Mh)(v')} - \{M(\Phi_* g)\}\,(v)\overline{h(v')}\Big] \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta) f(v_*)\,([M,\ \Phi_*]\,g)\,(v)\left(\overline{h(v')} - \overline{h(v)}\right) \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

The above computation is justified with cutoff approximation, see the remark given after (2.1.3) and also [32]. The first term $\mathcal{I}_1$ can be rewritten by using the Bobylev formula (see for example [4]) as

$$
\begin{aligned}
\mathcal{I}_1 = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) f(v_*)\left(M(\xi) - M(\xi^+)\right) \\
\times \mathcal{F}(\Phi_* g)(\xi^+) e^{-i v_* \xi^-} \mathrm{d}\sigma\,\mathrm{d}v_*\,\overline{\hat{h}(\xi)}\mathrm{d}\xi,
\end{aligned}
$$

where

$$
\xi^{\pm} = \frac{\xi \pm |\xi|\sigma}{2}.
$$

Notice that in the case of a Maxwellian molecule type cross section with $\gamma = 0$, that is $\Phi(|v - v_*|) = 1$, $\mathcal{I}_2 \equiv 0$.

Since $\tilde{M}'(|\xi|^2) = 2\xi\cdot\nabla M(\xi)/|\xi|^2$ and $|\xi^+| \leqq |\xi| \leqq 2|\xi^+|$, it follows from (2.3.1) and (2.3.2) that

$$
|M(\xi) - M(\xi^+)| \leqq C\left|\sin\frac{\theta}{2}\right|^2 M(\xi^+), \tag{2.3.5}
$$

and

$$
\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left|\sin\frac{\theta}{2}\right|^2 \mathrm{d}\sigma \leqq C < +\infty.
$$

Thus,

$$
\begin{aligned}
|\mathcal{I}_1| &\leqq C \int_{\mathbb{R}^3} \langle v_* \rangle^{\gamma_+} |f(v_*)| \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \sin^2\frac{\theta}{2} M(\xi^+) \\
&\qquad \times |\mathcal{F}(\Phi_* \langle v_*\rangle^{-\gamma_+} g)(\xi^+)| \; |\overline{\hat{h}(\xi)}| \mathrm{d}\xi \, \mathrm{d}\sigma \, \mathrm{d}v_* \\
&\leqq C \Bigg(\int_{\mathbb{R}^3} |\langle v_*\rangle^{\gamma_+} f(v_*)| \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \sin^2\frac{\theta}{2} \\
&\qquad \times |M(\xi^+) \mathcal{F}(\Phi_* \langle v_*\rangle^{-\gamma_+} g)(\xi^+)|^2 \mathrm{d}\xi \, \mathrm{d}\sigma \, \mathrm{d}v_* \Bigg)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{R}^3} |\langle v_*\rangle^{\gamma_+} f(v_*)| \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \sin^2\frac{\theta}{2} |\overline{\hat{h}(\xi)}|^2 \mathrm{d}\xi \, \mathrm{d}\sigma \, \mathrm{d}v_* \right)^{1/2} \\
&\leqq C \|f\|_{L^1_{\gamma_+}} \left(\sup_{v_*} \|M(D_v)\Phi_* \langle v_*\rangle^{-\gamma_+} g(v)\|_{L^2_{\gamma_+}} \right) \|h\|_{L^2},
\end{aligned}
$$

where we have used Plancherel's equality, the change of variables $\xi \to \xi^+$ for which $d\xi \sim d\xi^+$ uniformly with respect to σ, the estimate $\Phi_* \langle v_*\rangle^{-\gamma_+} \leqq \langle v\rangle^{\gamma_+}$. Then by using the expansion formula of the pseudo-differential calculus

$$
[M(D_v), \Phi_*(v)]\, g = \sum_{1 \leqq |\alpha| < N_1} \frac{1}{\alpha!} \Phi_{*(\alpha)} M^{(\alpha)}(D_v) g + r_{N_1}(v, D_v; v_*) g, \tag{2.3.6}
$$

with $N_1 > \lambda$, and the condition (2.3.2), we obtain

$$
\sup_{v_*} \left\| M(D_v)\Phi_* \langle v_*\rangle^{-\gamma_+} g(v) \right\|_{L^2_{\gamma_+}} \leqq C \left(\|Mg\|_{L^2_{\gamma_+}} + ||g||_{H^{\lambda - N_1}_{\gamma^+}} \right). \tag{2.3.7}
$$

Hence,

$$
|\mathcal{I}_1| \leqq C \|f\|_{L^1_{\gamma_+}} \left(\|Mg\|_{L^2_{\gamma_+}} + ||g||_{H^{\lambda - N_1}_{\gamma^+}} \right) \|h\|_{L^2}. \tag{2.3.8}
$$

We now turn to the term $\mathcal{I}_2$. Firstly, set

$$
F(v, v_*) = [M, \Phi_*] g(v),
$$

and decompose

$$
\begin{aligned}
\mathcal{I}_2 &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) f(v_*) \left\{ F(v', v_*) h(v') - F(v, v_*) h(v) \right\} \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\
&\quad + \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) f(v_*) \left(F(v, v_*) - F(v', v_*) \right) h(v') \mathrm{d}v_* \, \mathrm{d}v \, \mathrm{d}\sigma \\
&= J_1 + J_2.
\end{aligned}
$$

According to the cancellation lemma [4], we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta) \left\{ F(v', \, v_*) h(v') - F(v, v_*) h(v) \right\} \mathrm{d}\sigma \mathrm{d}v \\
&\quad = \left(S * \{ F(\cdot, v_*) h \} \right)(v_*),
\end{aligned}
$$

where the convolution product is in $v \in \mathbb{R}^3$, and in this case,

$$S = 2\pi \int_0^{\pi/2} \sin\theta b(\cos\theta) \left[\frac{1}{\cos^3(\theta/2)} - 1\right] \mathrm{d}\theta$$

is a constant function. Consequently,

$$J_1 = \int_{\mathbb{R}^3} f(v_*) \left(S * \{F(\cdot, v_*)h\}\right)(v_*)\mathrm{d}v_* = S\int_{\mathbb{R}^6} f(v_*)F(v, v_*)h(v)\mathrm{d}v\,\mathrm{d}v_*.$$

By (2.3.6) and (2.3.7), we get

$$\begin{aligned} |J_1| &\leqq C\int_{\mathbb{R}^3} |f(v_*)| \|F(\cdot, v_*)\|_{L^2} \|h\|_{L^2} \mathrm{d}v_* \\ &\leqq C\|f\|_{L^1_{\gamma+}} \left(\|Mg\|_{L^2_{\gamma+}} + ||g||_{H^{\lambda-N_1}_{\gamma+}}\right) \|h\|_{L^2}. \end{aligned} \tag{2.3.9}$$

To estimate the term J_2, we need to consider the following two cases.
Case 1: $0 < s < 1/2$**.** Since the mean value theorem yields

$$F(v, v_*) - F(v', v_*) = (v - v') \cdot \int_0^1 \nabla_v (F(v' + \tau(v - v'), v_*)\mathrm{d}\tau,$$

by noticing that

$$|v' - v| = |v - v_*| \sin(\theta/2) = |v' - v_*| \tan(\theta/2),$$

we have

$$\begin{aligned} |J_2| \leqq &\int_0^1 \left(\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|v' - v||f(v_*)||h(v')| \right.\\ &\left.\times|(\nabla_v F)(v' + \tau(v - v'), v_*)|\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right) \mathrm{d}\tau. \\ \leqq &\, C\left(\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|\theta|\langle v_*\rangle^{\gamma+}|f(v_*)||h(v')|^2 \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right)^{1/2} \\ &\times \int_0^1 \left(\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|\theta|\langle v_*\rangle^{\gamma+}|f(v_*)| \right.\\ &\left.\times\left|\frac{|v - v_*|}{\langle v_*\rangle^{\gamma+}}(\nabla_v F)(v' + \tau(v - v'), v_*)\right|^2 \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right)^{1/2} \mathrm{d}\tau \\ = &\, CJ_{21} \times J_{22}. \end{aligned}$$

By the change of variables $v \to v'$ for which $dv \sim dv'$ uniformly in $v_* \in \mathbb{R}^3$, $\sigma \in \mathbb{S}^2$ (see [4]), we get

$$J_{21}^2 \leqq C\|f\|_{L^1_{\gamma+}} \|h\|_{L^2}^2. \tag{2.3.10}$$

To estimate J_{22}, we apply the change of variables (2.1.24) and use (2.1.25). Setting

$$\psi^*(v) = \frac{\langle v - v_* \rangle}{\langle v_* \rangle^{\gamma_+}},$$

we get

$$\begin{aligned} J_{22}^2 &\leqq C\int_0^1 \left[\int_{\mathbb{R}^{2n}\times\mathbb{S}^2} b(\cos\theta)|\theta|\langle v_*\rangle^{\gamma_+}|f(v_*)|\left|\psi^*(z)(\nabla_v F)(z,v_*)\right|^2 \mathrm{d}z\,\mathrm{d}v_*\,\mathrm{d}\sigma\right]\mathrm{d}\tau \\ &\leqq C\|f\|_{L^1_{\gamma+}} \sup_{v_*}\|\psi^*(\cdot)(\nabla_v F)(\cdot, v_*)\|^2_{L^2}. \end{aligned}$$

On the other hand, it follows from the expansion formula of pseudo-differential operators that, with $\Phi_*(v) = (1+|v-v_*|^2)^{\gamma/2}$ we have for any $N_1 \in \mathbb{N}$

$$\begin{aligned} &(\nabla_v F)(v, v_*) \\ &= \nabla_v[M,\ \Phi_*]g(v) \\ &= \sum_{1\leqq|\alpha|<N_1} \frac{1}{\alpha!}\Big\{(\nabla\Phi_{*(\alpha)})M^{(\alpha)}(D_v)g + \Phi_{*(\alpha)}M^{(\alpha)}(D_v)\nabla_v g\Big\} + \tilde{r}_{N_1}(v, D_v; v_*)g \\ &= F_{N_1}(v, D_v; v_*)g(v) + \tilde{r}_{N_1}(v, D_v; v_*)g(v), \end{aligned} \tag{2.3.11}$$

where $\tilde{r}_{N_1}$ is a pseudo-differential operator with symbol belonging to $S^{\lambda-N_1}_{1,0}$ uniformly with respect to $v_* \in \mathbb{R}^3$ (compare [33]). Since

$$\left|\psi^*\Phi_{*(\alpha)}\right| \leqq C_\alpha \frac{\langle v - v_*\rangle}{\langle v_*\rangle^{\gamma_+}} \langle v - v_*\rangle^{\gamma-|\alpha|} \leqq C_\alpha \langle v\rangle^{\gamma_+},$$

by (2.3.2), we have for $\alpha \neq 0$ that,

$$|M^{(\alpha)}(\xi)\ \xi| \leqq C_\alpha M(\xi)\langle\xi\rangle^{-|\alpha|+1} \leqq C_\alpha M(\xi).$$

Hence

$$J_{22}^2 \leqq C\|f\|_{L^1_{\gamma+}}\left(\|Mg\|^2_{L^2_{\gamma+}} + ||g||^2_{H^{\lambda-N_1}_{\gamma+}}\right). \tag{2.3.12}$$

Now, it follows from (2.3.9), (2.3.10), and (2.3.12) that

$$|\mathcal{I}_2| \leqq C\|f\|_{L^1_{\gamma+}}\left(\|Mg\|_{L^2_{\gamma+}} + ||g||_{H^{\lambda-N_1}_{\gamma+}}\right)||h||_{L^2}., \tag{2.3.13}$$

holds when $0 < s < 1/2$.

Case 2: $1/2 < s < 1$**.** We now decompose J_2 as follows:

$$
\begin{aligned}
J_2 &= \int_0^1 \Bigg(\int_{\mathbb{R}^6 \times \mathbb{S}^2} b(\cos\theta) f(v_*) h(v') \\
&\qquad \times (v - v') \cdot (\nabla_v F)(v' + \tau(v - v'), v_*) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma \Bigg) \mathrm{d}\tau \\
&= \int_{\mathbb{R}^6 \times \mathbb{S}^2} b(\cos\theta) f(v_*) h(v')(v - v') \cdot (\nabla_v F)(v', v_*) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma \\
&\quad + \int_0^1 \Bigg(\int_{\mathbb{R}^6 \times \mathbb{S}^2} b(\cos\theta) f(v_*) h(v') \\
&\qquad \times (v - v') \cdot \Big\{ (\nabla_v F)(v' + \tau(v - v'), v_*) - (\nabla_v F)(v', v_*) \Big\} \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma \Bigg) \mathrm{d}\tau \\
&= J_2^0 + J_2^1.
\end{aligned}
$$

The essential feature of this decomposition is that J_2^0 vanishes by symmetry as in the proof of Lemma 2.4. Indeed, we have

$$
\begin{aligned}
J_2^0 = \int_{\mathbb{R}^6} f(v_*) h(v') &\left\{ \int_{\mathbb{S}^2} b\left(\frac{\psi_\sigma(v') - v_*}{|\psi_\sigma(v') - v_*|} \cdot \sigma \right) \left| \frac{\partial(\psi_\sigma(v'))}{\partial(v')} \right| (\psi_\sigma(v') - v') \mathrm{d}\sigma \right\} \\
&\cdot (\nabla_v F)(v', v_*) \mathrm{d}v' \, \mathrm{d}v_* = 0,
\end{aligned}
$$

because of the symmetry in σ_1 and σ_2 in the sense that $\psi_{\sigma_1}(v') - v' = -(\psi_{\sigma_2}(v') - v')$, compare Fig. 2.

Now, by the change of variable $v \to z = v' + \tau(v - v')$ defined by (2.1.24), we consider

$$
J_2^1(\tau) = \int_{\mathbb{R}^6 \times \mathbb{S}^2} b f(v_*) h(v')(v - v') \cdot \{ (\nabla_v F)(z, v_*) - (\nabla_v F)(v', v_*) \} \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma.
$$

By recalling the expansion formula (2.3.11) of $(\nabla_v F)(v, v_*)$, we first consider

$$
\begin{aligned}
J_2^1(\tau, \alpha) &= \int_{\mathbb{R}^6 \times \mathbb{S}^2} b f(v_*) h(v') \\
&\qquad \times (v - v') \cdot \Big\{ \Phi_{*(\alpha)} M^{(\alpha)} \nabla_v g(z) - \Phi_{*(\alpha)} M^{(\alpha)} \nabla_v g(v') \Big\} \mathrm{d}\, v \mathrm{d}v_* \, \mathrm{d}\sigma \\
&= \int_{\mathbb{R}^6 \times \mathbb{S}^2} b f(v_*) h(v') \left\{ \Phi_{*(\alpha)}(z) - \Phi_{*(\alpha)}(v') \right\} \\
&\qquad \times (v - v') \cdot M^{(\alpha)} \nabla_v g(z) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma + \int_{\mathbb{R}^6 \times \mathbb{S}^2} b f(v_*) h(v') \Phi_{*(\alpha)}(v') \\
&\qquad \times (v - v') \cdot \{ M^{(\alpha)} \nabla_v g(z) - M^{(\alpha)} \nabla_v g(v') \} \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma \\
&= J_2^{1,0}(\tau, \alpha) + \tilde{J}_2^1(\tau, \alpha).
\end{aligned}
$$

Notice that the case in which $|\alpha| = 1$ is the most difficult one, in the sense that $M^{(\alpha)}(D_v)\, \nabla_v$ is a pseudo-differential operator of order λ with symbol bounded by

$CM(\xi)$ due to the assumption (2.3.2). By writing (1) instead of (α) when $|\alpha| = 1$, we have

$$\left|\left\{\Phi_{*(1)}(z) - \Phi_{*(1)}(v')\right\}|v - v'|\right| \leqq C\langle z - v_*\rangle^{\gamma}\theta^2,$$

which gives

$$\begin{aligned}
&\left|J_2^{1,0}(\tau, (1))\right| \\
&\leqq \left(\int_{\mathbb{R}^6\times\mathbb{S}^2} b\theta^2|\langle v_*\rangle^{\gamma_+} f(v_*)||h(v')|^2 \mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}^6\times\mathbb{S}^2} b\theta^2|\langle v_*\rangle^{\gamma_+} f(v_*)|\left|\frac{\langle z - v_*\rangle^{\gamma}}{\langle v_*\rangle^{\gamma_+}} M^{(\alpha)}\,\nabla_v g(z)\right|^2 \mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2} \\
&\leqq C||f||_{L^1_{\gamma_+}}||M\,g||_{L^2_{\gamma_+}}||h||_{L^2}. \qquad (2.3.14)
\end{aligned}$$

In order to evaluate the term $\tilde{J}_2^1(\tau, (1))$, we take the same Littlewood–Paley partition of unity $\{\psi_j(\xi)\}$ as in the proof of Lemma 2.4 and write

$$\begin{aligned}
&\tilde{J}_2^1(\tau, (1)) \\
&= \int_{\mathbb{R}^6\times\mathbb{S}^2} bf(v_*)h(v')\Phi_{*(1)}(v')(v-v')\cdot\left\{M^{(1)}\nabla_v g(z) - M^{(1)}\nabla_v g(v')\right\}\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\
&= \sum_{j=0}^{\infty}\int_{\mathbb{R}^6\times\mathbb{S}^2} bf(v_*)h(v')\Phi_{*(1)}(v')(v - v')\cdot\left(g_j(z) - g_j(v')\right)\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\
&= \sum_{j=0}^{\infty}\tilde{J}_{2,j}^1(\tau),
\end{aligned}$$

where $g_j(v) = \psi_j(D_v)M^{(1)}(D_v)\nabla_v g(v)$. For each j we apply the following decomposition by using Ω_j introduced in the proof of Lemma 2.4 to have

$$\begin{aligned}
\tilde{J}_{2,j}^1(\tau) &= \int_0^1\left(\int_{\mathbb{R}^6}\left(\int_{\Omega_j} bf(v_*)h(v')\Phi_{*(1)}(v')(v - v')\right.\right. \\
&\qquad\left.\left.\cdot(z - v')\nabla g_j(v' + s(z - v'))\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_*\right)\mathrm{d}s \\
&\qquad + \int_{\mathbb{R}^6}\left(\int_{\Omega_j^c} bf(v_*)h(v')\Phi_{*(1)}(v')(v-v')\cdot\left(g_j(z)-g_j(v')\right)\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_* \\
&= \tilde{J}_{2,j}^{1,1}(\tau) + \tilde{J}_{2,j}^{1,2}(\tau).
\end{aligned}$$

By setting $v'_{\tau,s} = v' + s(z - v')$, we have

$$
\begin{aligned}
|\tilde{J}^{1,1}_{2,j}(\tau)| &\leqq \int_0^1 \left(\int_{\mathbb{R}^6} \left(\int_{\Omega_j} b(\cos\theta)|v' - v|^2 |f(v_*)||h(v')| \right.\right.\\
&\qquad \left.\left. \times |\Phi_{*(1)}(v')||\nabla g_j(v'_{\tau,s})| d\sigma \right) \mathrm{d}v\, \mathrm{d}v_* \right) \mathrm{d}s \\
&\leqq C \int_0^1 \left(\int_{\mathbb{R}^6} \left(\int_{\Omega_j} b(\cos\theta)\theta^2 \langle v - v_*\rangle^{2-2s} \langle v_*\rangle^{(2s+\gamma-1)_+} |f(v_*)| \right.\right.\\
&\qquad \left.\left. \times |h(v')| \left| \frac{\langle v'_{\tau,s} - v_*\rangle^{2s+\gamma-1}}{\langle v_*\rangle^{(2s+\gamma-1)_+}} \nabla g_j(v'_{\tau,s}) \right| d\sigma \right) \mathrm{d}v\, \mathrm{d}v_* \right) \mathrm{d}s \\
&\leqq C 2^{-\varepsilon j} \left(\int_{\mathbb{R}^6} \left(\int_{\Omega_j} b(\cos\theta)\theta^2 2^{j(2-2s)} \langle v - v_*\rangle^{2-2s} \right.\right.\\
&\qquad \left.\left. \times \langle v_*\rangle^{(2s+\gamma-1)_+} |f(v_*)||h(v')|^2 \mathrm{d}\sigma \right) \mathrm{d}v\, \mathrm{d}v_* \right)^{1/2} \\
&\qquad \times \int_0^1 \left(\left(\int_{\mathbb{R}^6} \left(\int_{\Omega_j} b(\cos\theta)\theta^2 2^{j(2-2s)} \langle v - v_*\rangle^{2-2s} \langle v_*\rangle^{(2s+\gamma-1)_+} |f(v_*)| \right.\right.\right.\\
&\qquad \left.\left.\left. \times \left| \langle v'_{\tau,s}\rangle^{(2s+\gamma-1)_+} 2^{j(2s-2+\varepsilon)} \nabla g_j(v'_{\tau,s}) \right|^2 \mathrm{d}\sigma \right) \mathrm{d}v\, \mathrm{d}v_* \right)^{1/2} \right) \mathrm{d}s \\
&= C 2^{-\varepsilon j} \tilde{J}^{1,1}_{2,j,1}(\tau) \times \tilde{J}^{1,1}_{2,j,2}(\tau).
\end{aligned}
$$

By using the same change of variables as for $J_{2,1}$ in the previous case, it follows from (2.1.22) that

$$
\tilde{J}^{1,1}_{2,j,1}(\tau)^2 \leqq C \|f\|_{L^1_{(2s+\gamma-1)_+}} \|h\|^2_{L^2}. \tag{2.3.15}
$$

Similarly, by taking the change of variables $v \to v'_{\tau,s}$, again as in the previous case, (2.1.22) leads to

$$
\tilde{J}^{1,1}_{2,j,2}(\tau)^2 \leqq C ||f||_{L^1_{(2s+\gamma-1)_+}} \left(||M\, g||^2_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}} + ||g||^2_{H^{\lambda-N_1+2s-1+\varepsilon}_{(2s+\gamma-1)_+}} \right), \tag{2.3.16}
$$

where we have used

$$
\left\| 2^{j(2s-2-\varepsilon)} \nabla g_j(v) \right\|^2_{L^2_{(2s+\gamma-1)_+}} \leqq C \left(||M\, g||^2_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}} + ||g||^2_{H^{\lambda-N_1+2s-1+\varepsilon}_{(2s+\gamma-1)_+}} \right).
$$

Hence, it follows from (2.3.15) and (2.3.16) that, for $N_1 > \lambda + 2s - 1 + \varepsilon$, we have

$$
\left| \tilde{J}^{1,1}_{2,j}(\tau) \right| \leqq C 2^{-\varepsilon j} ||f||_{L^1_{(2s+\gamma-1)_+}} \left(||M\, g||_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}} + ||g||_{H^{\lambda-N}_{(2s+\gamma-1)_+}} \right) |h||_{L^2}. \tag{2.3.17}
$$

On the other hand, for $\tilde{J}^{1,2}_{2,j}(\tau)$, note that

$$\tilde{J}^{1,2}_{2,j}(\tau) = \int_{\mathbb{R}^6}\left(\int_{\Omega_j^c} bf(v_*)h(v')\Phi_{*(1)}(v')(v-v')\cdot g_j(z)\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_*,$$

by the symmetry in Ω_j^c. We have

$$\begin{aligned}|\tilde{J}^{1,2}_{2,j}(\tau)| &\leqq C\int_{\mathbb{R}^6}\left(\int_{\Omega_j^c} b(\cos\theta)\theta\langle v-v_*\rangle^{1-2s}\langle v_*\rangle^{(2s+\gamma-1)_+}|f(v_*)|\right.\\ &\qquad\left.\times|h(v')|\left|\frac{\langle z-v_*\rangle^{2s+\gamma-1}}{\langle v_*\rangle^{(2s+\gamma-1)_+}}g_j(z)\right|\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_*\\ &\leqq C2^{-\varepsilon j}\left(\int_{\mathbb{R}^6}\left(\int_{\Omega_j^c} b(\cos\theta)\theta 2^{j(1-2s)}\langle v-v_*\rangle^{1-2s}\langle v_*\rangle^{(2s+\gamma-1)_+}\right.\right.\\ &\qquad\left.\left.\times|f(v_*)||h(v')|^2\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2}\\ &\qquad\times\left(\int_{\mathbb{R}^6}\left(\int_{\Omega_j^c} b(\cos\theta)\theta 2^{j(1-2s)}\langle v-v_*\rangle^{1-2s}\langle v_*\rangle^{(2s+\gamma-1)_+}|f(v_*)|\right.\right.\\ &\qquad\left.\left.\times\left|\langle z\rangle^{(2s+\gamma-1)_+}2^{j(2s-1+\varepsilon)}g_j(z)\right|^2\mathrm{d}\sigma\right)\mathrm{d}v\,\mathrm{d}v_*\right)^{1/2}\\ &\leqq C2^{-\varepsilon j}||f||_{L^1_{(2s+\gamma-1)_+}}|h||_{L^2}\left(||M\,g||_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}}+||g||_{H^{\lambda-N}_{(2s+\gamma-1)_+}}\right),\end{aligned}$$

because of (2.1.23). This together with (2.3.14) and (2.3.17) yield

$$\left|J^1_2(\tau,(1))\right| \leqq C\|f\|_{L^1_{(2s+\gamma-1)_+}}\left(\|M\,g\|_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}}+\|g\|_{H^{\lambda-N}_{(2s+\gamma-1)_+}}\right)|h||_{L^2}.$$

It is easy to see that all other terms coming from $F_{N_1}(v, D_v; v_*)g(v)$ in (2.3.11) have the same upper bound estimates. All the terms coming from $\tilde{r}_{N_1}(v, D_v; v_*)g(v)$ can be estimated by

$$C\|f\|_{L^1_{(2s+\gamma-1)_+}}||g||_{H^{\lambda-N}_{(2s+\gamma-1)_+}}|h||_{L^2}.$$

Therefore, we finally obtain

$$|J_2| = |J^1_2| \leqq C||f||_{L^1_{(2s+\gamma-1)_+}}\left(||M\,g||_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}}+||g||_{H^{\lambda-N}_{(2s+\gamma-1)_+}}\right)|h||_{L^2}.$$

In summary, when $1/2 < s < 1$ we obtain, instead of (2.3.13), that

$$|\mathcal{I}_2| \leqq C||f||_{L^1_{(2s+\gamma-1)_+}}\left(||M\,g||_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)_+}}+||g||_{H^{\lambda-N}_{(2s+\gamma-1)_+}}\right)|h||_{L^2}. \quad (2.3.18)$$

By combining (2.3.8), (2.3.13) and (2.3.18), the proof of Proposition 2.9 is completed. □

The rest of this section is devoted to the proof (2.1.18) of Lemma 2.4.

Proof of (2.1.18) of Lemma 2.4. For $m = 2s - 1 + \varepsilon > 0$, we have with $\Lambda = (1 - \Delta_v)^{1/2}$

$$\begin{aligned}
&(W_l Q(f, g) - Q(f, W_l g), h)\\
&= \big((\Lambda^{-m} Q(f, g) - Q(f, \Lambda^{-m} g)), W_l \Lambda^m h\big)\\
&\quad + \big((W_l Q(f, \Lambda^{-m} g) - Q(f, W_l \Lambda^{-m} g)), \Lambda^m h\big)\\
&\quad + \big((Q(f, \Lambda^{-m} W_l g) - \Lambda^{-m} Q(f, W_l g)), \Lambda^m h\big)\\
&\quad + \big(([\Lambda^{-m}, W_l] Q(f, g) - Q(f, [\Lambda^{-m}, W_l] g)), \Lambda^m h\big)\\
&= (1) + (2) + (3) + (4).
\end{aligned}$$

It follows from (2.3.4) with $M(\xi) = \Lambda^{-m}$ that

$$\begin{aligned}
|(1)| &\leqq C\|f\|_{L^1_{(2s+\gamma-1)^+}} \|g\|_{L^2_{(2s+\gamma-1)^+}} \|h\|_{H^m_l},\\
|(3)| &\leqq C\|f\|_{L^1_{(2s+\gamma-1)^+}} \|W_l g\|_{L^2_{(2s+\gamma-1)^+}} \|h\|_{H^m}.
\end{aligned}$$

By means of (2.1.17), we have

$$|(2)| \leqq C\|f\|_{L^1_{l+2s-1+\gamma^+}} \|g\|_{L^2_{l+2s-1+\gamma^+}} \|h\|_{H^m}.$$

To estimate (4), we first note that

$$[\Lambda^{-m}, W_l] = \sum_{|\alpha|=1} (W_l)_{(\alpha)} \left(\Lambda^{-m}\right)^{(\alpha)} + W_{l-1} R(v, D_v),$$

where R is a pseudo-differential operator which belongs to $S^{-m-2}_{1,0}$. Write

$$\begin{aligned}
(4) &= \sum_{|\alpha|=1} \left(\left\{\left(\Lambda^{-m}\right)^{(\alpha)} Q(f, g) - Q(f, \left(\Lambda^{-m}\right)^{(\alpha)} g)\right\}, (W_l)_{(\alpha)} \Lambda^m h\right)\\
&\quad + \sum_{|\alpha|=1} \left(\left\{(W_l)_{(\alpha)} Q(f, \left(\Lambda^{-m}\right)^{(\alpha)} g) - Q(f, (W_l)_{(\alpha)} \left(\Lambda^{-m}\right)^{(\alpha)} g)\right\}, \Lambda^m h\right)\\
&\quad + \big(R(v, D_v) Q(f, g), W_{l-1} \Lambda^m h\big) + \big(Q(f, W_{l-1} R(v, D_v) g), \Lambda^m h\big)\\
&= (a) + (b) + (c) + (d).
\end{aligned}$$

It follows from (2.1.1) that

$$\begin{aligned}
|(c)| &\leqq C\|Q(f, g)\|_{H^{-2}} \|h\|_{H^m_{l-1}} \leqq C\|f\|_{L^1_{(\gamma+2s)^+}} \|g\|_{L^2_{(\gamma+2s)^+}} \|h\|_{H^m_{l-1}},\\
|(d)| &\leqq C\|Q(f, W_{l-1} R g)\|_{L^2} \|h\|_{H^m} \leqq C\|f\|_{L^1_{(\gamma+2s)^+}} \|g\|_{L^2_{l-1+(\gamma+2s)^+}} \|h\|_{H^m}.
\end{aligned}$$

By exactly the same method as the one for (2.1.17), namely, by replacing W_l by $(W_l)^{(\alpha)}$ which is bounded by $W_{l-|\alpha|}$, we have

$$\begin{aligned}
|(b)| &\leqq C\|f\|_{L^1_{l-2+2s+\gamma^+}} \|\left(\Lambda^{-m}\right)^{(\alpha)} g\|_{H^m_{l+2s-2+\gamma^+}} \|h\|_{H^m}\\
&\leqq C\|f\|_{L^1_{l-2+2s+\gamma^+}} \|g\|_{L^2_{l+2s-2+\gamma^+}} \|h\|_{H^m}.
\end{aligned}$$

The estimation on (a) is the same as the argument in Proposition 2.9 by replacing $M(D)$ by $(\Lambda^{-m})^{(\alpha)}$, except for the term corresponding to $\mathcal{I}$. Notice that $D_\xi^\alpha(\langle\xi\rangle^{-m}) := M^{(\alpha)}(\xi)$ is no longer a function of $|\xi|^2$. Instead of (2.3.5), we have only

$$\left|M^{(\alpha)}(\xi) - M^{(\alpha)}(\xi^+)\right| \leqq C\left|\sin\frac{\theta}{2}\right|\langle\xi^+\rangle^{-m-1}. \tag{2.3.19}$$

Thus, we need to use the symmetry property as in the proof of Theorem 2.1. The term corresponding to $\mathcal{I}$ is

$$\begin{aligned}\mathcal{I}^\alpha = &\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)f(v_*)\\ &\times\left(M^{(\alpha)}(\xi) - M^{(\alpha)}(\xi^+)\right)\mathcal{F}(\Phi_* g)(\xi^+)e^{-iv_*\xi^-}\mathrm{d}\sigma\,\mathrm{d}v_*\overline{\hat{h}_0(\xi)}\mathrm{d}\xi,\end{aligned}$$

where $h_0 = (W_l)_{(\alpha)}\Lambda^m h$. By letting

$$F(v, v_*) = \frac{\Phi(|v - v_*|)}{\langle v_*\rangle^{\gamma^+}}g(v), \quad h(v, v_*) = \frac{h_0(v)}{\langle v_*\rangle},$$

we write

$$\begin{aligned}\mathcal{I}^\alpha &= \int_{\mathbb{R}^3}\langle v_*\rangle^{1+\gamma^+}f(v_*)\left\{\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)\left(M^{(\alpha)}(\xi) - M^{(\alpha)}(\xi^+)\right)\right.\\ &\qquad\left.\times e^{iv_*\xi^+}\hat{F}(\xi^+, v_*)\overline{e^{iv_*\xi}\hat{h}(\xi, v_*)}\mathrm{d}\sigma\,\mathrm{d}\xi\right\}\mathrm{d}v_*\\ &= \int_{\mathbb{R}^3}\langle v_*\rangle^{1+\gamma^+}f(v_*)\mathcal{L}(v_*)\mathrm{d}v_*.\end{aligned}$$

Set

$$\tilde{\hat{F}}(\xi, v_*) = e^{iv_*\xi}\hat{F}(\xi, v_*), \quad \tilde{\hat{h}}(\xi, v_*) = e^{iv_*\xi}\hat{h}(\xi, v_*),$$

and write

$$\begin{aligned}\mathcal{L}(v_*) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(M^{(\alpha)}(\xi) - M^{(\alpha)}(\xi^+)\right)\tilde{\hat{F}}(\xi^+, v_*)\\ &\quad\times\overline{\left(\tilde{\hat{h}}(\xi, v_*) - \tilde{\hat{h}}(\xi^+, v_*)\right)}\mathrm{d}\sigma\,\mathrm{d}\xi + \frac{1}{2}\int_0^1(1-\tau)\left\{\int_{\mathbb{R}_n}\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\right.\\ &\quad\left.\times\left(\nabla_\xi^2 M^{(\alpha)}\right)\left(\xi^+ + \tau(\xi - \xi^+)\right)(\xi^-)^2\tilde{\hat{F}}(\xi^+, v_*)\overline{\tilde{\hat{h}}(\xi^+, v_*)}\mathrm{d}\sigma\,\mathrm{d}\xi\right\}\mathrm{d}\tau\\ &= \mathcal{L}^1(v_*) + \mathcal{L}^2(v_*).\end{aligned}$$

By the same symmetry property as shown in Fig. 1 in the proof of Theorem 2.1, we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)(\nabla_\xi M^{(\alpha)})(\xi^+)\cdot\xi^-(\sigma)\tilde{\hat{F}}(\xi^+, v_*)\overline{\tilde{\hat{h}}(\xi^+, v_*)}\mathrm{d}\sigma\,\mathrm{d}\xi = 0.$$

Then it follows from (2.3.19) that

$$\sup_{v_*} |\mathcal{L}^1(v_*)| \leqq C\|g\|_{L^2_{\gamma+}} \|h_0\|_{L^2_1} \leqq C\|g\|_{L^2_{\gamma+}} \|h\|_{H^m_l},$$

and

$$\sup_{v_*} |\mathcal{L}^2(v_*)| \leqq C\|g\|_{L^2_{\gamma+}} \|h_0\|_{L^2} \leqq C\|g\|_{L^2_{\gamma+}} \|h\|_{H^m_{l-1}},$$

whence we obtain

$$|\mathcal{I}^\alpha| \leqq C\|f\|_{L^1_{1+\gamma^+}} \|g\|_{L^2_{\gamma+}} \|h\|_{H^m_l}.$$

In summary, we obtained the desired estimate (2.1.18). □

3. Regularizing effect

In this section, we will prove the regularizing effect on solutions to the non-cutoff Boltzmann equation starting from $f \in \mathcal{H}^5_l(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v))$. This will actually be proved by using an induction argument in the subsections which follow. In the first step, we will show the gain of regularity in the variable v mainly by using singularity in the cross-section, that is, the coercivity property in (3.1.3). In the second step, we will apply the hypo-elliptic estimate obtained by a generalized version of the uncertainty principle to show the gain of regularity in the (x, t) variables. Then an induction argument will lead to at least one order higher regularity in (x, t) variables. By using the equation and an induction argument again, at least one order higher regularity can be obtained in the v variable. Therefore, the solution is shown to be in $\mathcal{H}^6_l(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v)$ which by induction leads to $\mathcal{H}^\infty_l(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v)$.

Let $f \in \mathcal{H}^5_l(]T_1, T_2[\times\Omega\times\mathbb{R}^3_v))$, for all $l \in \mathbb{N}$, be a (classical) solution of the Boltzmann equation (1.1). We now want to prove the full regularity of $\varphi(t)\psi(x)f$ for any smooth cutoff functions $\varphi \in C^\infty_0(]T_1, T_2[)$, $\psi \in C^\infty_0(\Omega)$.

3.1. Initialization

Here and below, ϕ denotes a cutoff function satisfying $\phi \in C^\infty_0$ and $0 \leqq \phi \leqq 1$. The notation $\phi_1 \subset\subset \phi_2$ stands for two cutoff functions such that $\phi_2 = 1$ on the support of ϕ_1.

Take the smooth cutoff functions $\varphi, \varphi_2, \varphi_3 \in C^\infty_0(]T_1, T_2[)$ and $\psi, \psi_2, \psi_3 \in C^\infty_0(\Omega)$ such that $\varphi \subset\subset \varphi_2 \subset\subset \varphi_3$ and $\psi \subset\subset \psi_2 \subset\subset \psi_3$. Set $f_1 = \varphi(t)\psi(x)f$, $f_2 = \varphi_2(t)\psi_2(x)f$ and $f_3 = \varphi_3(t)\psi_3(x)f$. For $\alpha \in \mathbb{N}^7$, $|\alpha| \leqq 5$, define

$$g = \partial^\alpha(\varphi(t)\psi(x)f) = \partial^\alpha_{t,x,v}(\varphi(t)\psi(x)f) \in L^2_l(\mathbb{R}^7).$$

Firstly, the translation invariance of the collision operator with respect to the variable v implies that (see [26,31,43]), for the translation operation τ_h in v by h, we have

$$\tau_h G(f, g) = Q(\tau_h f, \tau_h g).$$

Then the Leibniz formula with respect to the t, x variables yields the following equation in a weak sense

$$g_t + v \cdot \partial_x g = Q(f_2, g) + G,\ (t, x, v) \in \mathbb{R}^7, \tag{3.1.1}$$

where

$$\begin{aligned} G = &\sum_{\alpha_1+\alpha_2=\alpha, 1\leqq|\alpha_1|} C^{\alpha_1}_{\alpha_2} Q\left(\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1\right) \\ &+\partial^{\alpha}\left(\varphi_t \psi(x) f + v\cdot\psi_x(x)\varphi(t) f\right) + [\partial^{\alpha},\ v\cdot\partial_x](\varphi(t)\psi(x) f) \\ \equiv\ &(A) + (B) + (C). \end{aligned} \tag{3.1.2}$$

To prove the regularity of $g = \partial^{\alpha}(\varphi(t)\psi(x)f)$, the natural idea would be to use g as a test function for equation (3.1.1). But at this point, g belongs only to $L^2_l(\mathbb{R}^7)$ so that it is only a weak solution to equation (3.1.1). By using the upper bound estimate on Q, we have $Q(f_2, g) \in L^2(\mathbb{R}^4_{t,x};\ H^{-2s}(\mathbb{R}^3_v))$. Thus, we need to choose the test functions at least in the space $L^2(\mathbb{R}^4_{t,x};\ H^{2s}(\mathbb{R}^3_v))$. For this, we will use a mollification of g with respect to the variables $(x,\ v)$ as a test function.

For this purpose, let $S \in C^{\infty}_0(\mathbb{R})$ satisfy $0 \leqq S \leqq 1$ and

$$S(\tau) = 1, \quad |\tau| \leqq 1; \qquad S(\tau) = 0, \quad |\tau| \geqq 2.$$

Then

$$S_N(D_x)S_N(D_v) = S(2^{-2N}|D_x|^2)S(2^{-2N}|D_v|^2)\ :\ H^{-\infty}_l(\mathbb{R}^6) \to H^{+\infty}_l(\mathbb{R}^6),$$

is a regularization operator such that

$$\|(S_N(D_x)S_N(D_v)f) - f\|_{L^2_l(\mathbb{R}^6)} \to 0, \quad \text{as } N \to \infty.$$

Choose another cutoff function $\psi \subset\subset \psi_1 \subset\subset \psi_2$ and set

$$P_{N,l} = \psi_1(x)S_N(D_x)\, W_l\ S_N(D_v).$$

Then we can take

$$\tilde{g} = P^{\star}_{N,l}(P_{N,l}g) \in C^1(\mathbb{R}; H^{+\infty}(\mathbb{R}^6))$$

as a test function for the equation (3.1.1).

It follows by integration by parts on $\mathbb{R}^7 = \mathbb{R}^1_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ that

$$\begin{aligned} &\left([S_N(D_v), v]\cdot\ \nabla_x S_N(D_x)g, \psi_1(x)W_l P_{N,l}g\right)_{L^2(\mathbb{R}^7)} \\ &\quad= \left(P_{N,l}\ Q(f_2, g),\ P_{N,l}\ g\right)_{L^2(\mathbb{R}^7)} + (G,\ \tilde{g})_{L^2(\mathbb{R}^7)}, \end{aligned}$$

which implies that

$$\begin{aligned} &-\left(Q(f_2, P_{N,l}g), P_{N,l}g\right)_{L^2(\mathbb{R}^7)} \\ &\quad= -\left([S_N(D_v),\ v]\cdot\nabla_x S_N(D_x)g, \psi_1(x)W_l P_{N,l}\ g\right)_{L^2(\mathbb{R}^7)} \\ &\qquad+\left(P_{N,l}\ Q(f_2, g) - Q(f_2, P_{N,l}\, g),\ P_{N,l}g\right)_{L^2(\mathbb{R}^7)} + (G,\ \tilde{g})_{L^2(\mathbb{R}^7)}. \end{aligned} \tag{3.1.3}$$

By using (3.1.3), we can deduce the regularity of g from the coercivity property of the collision operator on the left-hand side and the upper bound estimate on the right-hand side. The detailed argument will be given in the next subsection.

3.2. Gain of regularity in v

In this subsection, we will prove a partial smoothing effect of the cross-section on the weak solution g in the velocity variable v.

Proposition 3.1. *Assume that* $0 < s < 1, \gamma \in \mathbb{R}$. *Let* $f \in \mathcal{H}_l^5(]T_1, T_2[\times\Omega \times \mathbb{R}_v^3)$ *be a solution of the equation* (1.1) *for all* $l \in \mathbb{N}$. *Assume furthermore that*

$$f(t, x, v) \geqq 0 \quad \textit{and} \quad \|f(t, x, \cdot)\|_{L^1(\mathbb{R}_v^3)} > 0, \tag{3.2.1}$$

for all $(t, x, v) \in]T_1,\ T_2[\ \times\Omega \times \mathbb{R}_v^3$. *Then one has,*

$$\Lambda_v^s f_1 \in H_l^5(\mathbb{R}^7), \tag{3.2.2}$$

for any $l \in \mathbb{N}$, *where* $f_1 = \varphi(t)\psi(x) f$ *with* $\varphi \in C_0^\infty(]T_1, T_2[), \psi \in C^\infty(\Omega)$.

Proof. Firstly, the local positive lower bound assumption (3.2.1) implies that

$$\inf_{(t,x)\in \text{supp } \varphi\times \text{supp } \psi_1} \|f_2(t, x, \cdot)\|_{L^1(\mathbb{R}_v^3)} = c_0 > 0.$$

Thus, the coercivity estimate (2.2.1) in Theorem 2.6 gives that for any $\gamma \in \mathbb{R},\ 0 < s < 1$,

$$\begin{aligned}
&-\big(Q(f_2,\ P_{N,l}\, g),\ \ P_{N,l}\, g\big)_{L^2(\mathbb{R}^7)} \\
&\quad = -\int_{t\in\text{supp}\,\varphi}\int_{x\in\text{supp}\,\psi_1} \big(Q(f_2,\ P_{N,l}\, g), P_{N,l}\, g\big)_{L^2(\mathbb{R}_v^3)}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad \geqq \int_{\mathbb{R}_t}\int_{\mathbb{R}_x^3} \Big(C_0\|W_{\gamma/2} P_{N,l}\, g(t, x, \cdot)\|^2_{H^s(\mathbb{R}_v^3)} \\
&\qquad - C\|f_2(t, x, \cdot)\|_{L^1_{\max\{\gamma^+,\, 2-\gamma^+\}}(\mathbb{R}_v^3)}\|P_{N,l}\, g(t, x, \cdot)\|^2_{L^2_{\gamma^+/2}(\mathbb{R}_v^3)}\Big)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad \geqq C_0\|\Lambda_v^s W_{\gamma/2} P_{N,l}\, g\|^2_{L^2(\mathbb{R}^7)} \\
&\qquad - C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x};\ L^1_{\max\{\gamma^+,\, 2-\gamma^+\}}(\mathbb{R}_v^3))}\|W_l g\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^7)},
\end{aligned} \tag{3.2.3}$$

where C_0 depends on c_0, $\sup_{t,x}\|f_2(t, x, \cdot)\|_{L^1_1(\mathbb{R}_v^3)}$ and $\sup_{t,x}\|f_2(t, x, \cdot)\|_{L\log L(\mathbb{R}_v^3)}$, see Remark 2.7.

For the terms in (3.1.3), note first of all that

$$\begin{aligned}
&[S_N(D_v), v]\cdot\ \nabla_x\ S_N(D_x) \\
&\quad = 2^{-2N}\left(S'\right)_N(D_v) D_v\cdot\nabla_x\ S_N(D_x)\ :\ L^2(\mathbb{R}^6_{x,v})\ \to\ L^2(\mathbb{R}^6_{x,v}),
\end{aligned} \tag{3.2.4}$$

is a uniformly bounded operator so that

$$\Big|\big([S_N(D_v), v]\cdot\nabla_x\ S_N(D_x) g,\ \ \psi_1(x) W_l P_{N,l} g\big)_{L^2(\mathbb{R}^7)}\Big| \leqq C\|f_1\|^2_{H_l^5(\mathbb{R}^7)}.$$

Hence, by using (3.1.3), we get, for $l > 3/2 + 2$,

$$\begin{aligned}
&\left\| \Lambda_v^s W_{\gamma/2} P_{N,l}\, g \right\|^2_{L^2(\mathbb{R}^7)} \\
&\leqq C \left\{ \left(1 + \| f_2 \|_{H^{2+\delta}_{l+\gamma^+}(\mathbb{R}^6)} \right) \| f_1 \|^2_{H^5_l(\mathbb{R}^7)} \right. \\
&\quad \left. + \left| (G, \tilde{g})_{L^2(\mathbb{R}^7)} \right| + \left| \left(P_{N,l} Q(f_2, g) - Q(f_2,\ P_{N,l}\, g), P_{N,l}\, g \right)_{L^2(\mathbb{R}^7)} \right| \right\}.
\end{aligned} \tag{3.2.5}$$

The above constants $C > 0$ are independent of N.

We complete the proof of Proposition 3.1 by estimating the last two terms in (3.2.5) through the following three Lemmas. □

Lemma 3.2. *Assume* $0 < s < 1, \gamma \in \mathbb{R}$. *Let* $f \in \mathcal{H}^5_l(]T_1,\ T_2[\times\Omega\times\mathbb{R}^3_v)$, $l \geqq 3/2 + 2$. *Then, for any* $\alpha \in \mathbb{N}^7$, $|\alpha| \leqq 5$, *we have, for any* $\varepsilon > 0$,

$$\left| (G,\ \tilde{g})_{L^2(\mathbb{R}^7)} \right| \leqq C_\varepsilon \| f_3 \|^4_{H^5_{l+4+|\gamma|}(\mathbb{R}^7)} + \varepsilon \| \Lambda_v^s W_{\gamma/2} P_{N,l}\, g \|^2_{L^2(\mathbb{R}^7_{t,x,v})}. \tag{3.2.6}$$

Proof. Firstly, we prove that

$$G \in L^2 \left(\mathbb{R}^4_{t,x};\ H_l^{-(2s-1+\delta)^+}(\mathbb{R}^3_v) \right), \tag{3.2.7}$$

for any $l \in \mathbb{N}$, where $(2s - 1 + \delta)^+ = \max\{2s - 1 + \delta, 0\}$ and $\delta > 0$ satisfying $2s - 1 + \delta < s$. By using the decomposition in (3.1.2), it is obvious that

$$(B) = \partial^\alpha \left(\varphi_t \psi(x) f + v \cdot \psi_x(x) \varphi(t) f \right) \in L^2_l(\mathbb{R}^7),$$

and

$$\| (B) \|_{L^2_l(\mathbb{R}^7)} \leqq C \| f_2 \|_{H^5_{l+1}(\mathbb{R}^7)}.$$

Since $[\partial^\alpha, v \cdot \partial_x]$ is a differential operator of order $|\alpha|$, we have

$$\| (C) \|_{L^2_l(\mathbb{R}^7)} \leqq C \| f_2 \|_{H^5_l(\mathbb{R}^7))}.$$

For the term (A), recall that $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| \leqq 5$ and $|\alpha_2| < 5$. In the following, we will apply Theorem 2.1 with $m = 1 - \delta - 2s$. We separate the discussion into two cases.

Case 1. If $|\alpha_1| = 1, 2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_t} \int_{\mathbb{R}^3_x} \| Q(\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1)(t, x, \cdot) \|^2_{H_l^{1-\delta-2s}(\mathbb{R}^3_v)} \mathrm{d}x\, \mathrm{d}t \\
&\leqq C \int_{\mathbb{R}_t} \int_{\mathbb{R}^3_x} \| \partial^{\alpha_1} f_2(t, x, \cdot) \|^2_{L^1_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)} \| \partial^{\alpha_2} f_1(t, x, \cdot) \|^2_{H^{1-\delta}_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)} \mathrm{d}x\, \mathrm{d}t \\
&\leqq C \| \partial^{\alpha_1} f_2 \|^2_{L^\infty(\mathbb{R}^4_{t,x};\ L^1_{l+(2s+\gamma)^+}(\mathbb{R}^3_v))} \int_{\mathbb{R}_t} \int_{\mathbb{R}^3_x} \| \partial^{\alpha_2} f_1(t, x, \cdot) \|^2_{H^1_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)} \mathrm{d}x\, \mathrm{d}t \\
&\leqq C \| f_2 \|^2_{H^{2+4/2+\delta}_{l+3/2+\delta+(2s+\gamma)^+}(\mathbb{R}^7)} \| f_1 \|^2_{H^5_{l+(2s+\gamma)^+}(\mathbb{R}^7)}.
\end{aligned}$$

Case 2. If $|\alpha_1| \geqq 3$, then $|\alpha_2| \leqq 2$, it follows that

$$\int_{\mathbb{R}_t} \int_{\mathbb{R}^3_x} \|\partial^{\alpha_1} f_2(t,x,\cdot)\|^2_{L^1_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)} \|\partial^{\alpha_2} f_1(t,x,\cdot)\|^2_{H^{1-\delta}_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)} \mathrm{d}x\,\mathrm{d}t$$
$$\leqq C\|\partial^{\alpha_2} f_1\|^2_{L^\infty(\mathbb{R}^4_{t,x};\, H^{1-\delta}_{l+(2s+\gamma)^+}(\mathbb{R}^3_v))} \int_{\mathbb{R}_t} \int_{\mathbb{R}^3_x} \|\partial^{\alpha_1} f_2(t,x,\cdot)\|^2_{L^2_{l+3/2+\delta+(2s+\gamma)^+}(\mathbb{R}^3_v)} \mathrm{d}x\,\mathrm{d}t$$
$$\leqq C\|f_1\|^2_{H^{2+1-\delta+4/2+\delta/2}_{l+(2s+\gamma)^+}(\mathbb{R}^6)} \|f_2\|^2_{H^5_{l+3/2+\delta+(2s+\gamma)^+}(\mathbb{R}^7)}.$$

By combining these two cases, we have proved (3.2.7).

Now if $2s-1<0$, then (3.2.7) implies that

$$\left|(G,\tilde{g})_{L^2(\mathbb{R}^7)}\right| \leqq C\|f_3\|^3_{H^5_{l+4+\gamma^+}(\mathbb{R}^7)}.$$

On the other hand, if $0 \leqq 2s-1$ and $\gamma<0$ (the case $\gamma>0$ is easier), then (3.2.7) implies that

$$\left|(G,\tilde{g})_{L^2(\mathbb{R}^7)}\right| \leqq \|G\|_{L^2(\mathbb{R}^4_{t,x}; H^{1-2s-\delta}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \|W_{-|\gamma|/2} P_{N,l}\, g\|_{L^2(\mathbb{R}^4_{t,x};\, H^{2s-1+\delta}(\mathbb{R}^3_v))}$$
$$\leqq C\|f_3\|^2_{H^5_{l+4+|\gamma|}(\mathbb{R}^7)} \|W_{-|\gamma|/2} P_{N,l}\, g\|_{L^2(\mathbb{R}^4_{t,x};\, H^{2s-1+\delta}(\mathbb{R}^3_v))},$$

because $2s-1+\delta<s$. Therefore, the proof of Lemma 3.2 is completed. □

We now turn to the estimates of commutators between the mollification operators and the collision operator, which are given in the following two lemmas.

Lemma 3.3. *For any $\gamma \in \mathbb{R}$, we have*

(1) *If $0<s<1/2$, then for any suitable functions f and g with the following norms well defined, one has*

$$\|S_N(D_v)Q(f,g) - Q(f, S_N(D_v)g)\|_{L^2(\mathbb{R}^3_v)} \leqq C\|f\|_{L^1_{\gamma^+}(\mathbb{R}^3_v)} \|g\|_{L^2_{\gamma^+}(\mathbb{R}^3_v)}, \tag{3.2.8}$$

for some constant C independent of N.

(2) *If $1/2<s<1$, then for any $\delta>0$ there exists a constant $C_\delta>0$ such that*

$$\|S_N(D_v)Q(f,g) - Q(f,\, S_N(D_v)g)\|_{L^2(\mathbb{R}^3_v)} \leqq C_\delta \|f\|_{L^1_{(2s+\gamma-1)^+}(\mathbb{R}^3_v)} \|g\|_{H^{2s-1+\delta}_{(2s+\gamma-1)^+}(\mathbb{R}^3_v)}, \tag{3.2.9}$$

and

$$\|S_N(D_v)Q(f,g) - Q(f,\, S_N(D_v)g)\|_{H^{1-2s-\delta}(\mathbb{R}^3)} \leqq C_\delta \|f\|_{L^1_{(2s+\gamma-1)^+}(\mathbb{R}^3_v)} \|g\|_{L^2_{(2s+\gamma-1)^+}(\mathbb{R}^3_v)}. \tag{3.2.10}$$

(3) *When $s=1/2$, we have the same form of estimate as (3.2.9) with $(2s+\gamma-1)$ replaced by $(\gamma+\kappa)$ for any small $\kappa>0$.*

Before giving the proof of this lemma, we notice that when $\gamma = 0$ in the Maxwellian molecule case, the following proof of Lemma 3.3 is similar to Lemma 3.1 in [38] (see also Lemma 5.1 in [6]) by using the Fourier transformation of the collision operator. However, here we consider the case for $\gamma \in \mathbb{R}$.

Proof of Lemma 3.3. The proof is a slight modification of the proof for Proposition 2.9. Set

$$M(|\xi|) = S_N(|\xi|) = S(2^{-2N}|\xi|^2).$$

Then $S_N \in S^0_{1,0}$ uniformly. Even though it does not satisfy (2.3.2), we have

$$|\partial^\alpha S_N(|\xi|)| \leqq C_\alpha S_{N+1}(|\xi|) < \xi >^{-|\alpha|}$$

with C_α independent of $N \in \mathbb{N}$. Thus, (2.3.3) implies (3.2.8) and (2.3.4) implies (3.2.9) respectively.

For (3.2.10), note that with $m = 2s - 1 + \delta$ we have

$$\begin{aligned}(S_N Q(f,g) - Q(f, S_N g), h) &= \big((\Lambda^{-m} Q(f,g) - Q(f, \Lambda^{-m} g)), \Lambda^m S_N h\big)\\ &\quad + \big((S_N Q(f, \Lambda^{-m} g) - Q(f, \Lambda^{-m} S_N g)), \Lambda^m h\big)\\ &\quad + \big((Q(f, S_N \Lambda^{-m} g) - \Lambda^{-m} Q(f, S_N g)), \Lambda^m h\big)\\ &= (I_1) + (I_2) + (I_3).\end{aligned}$$

By applying (2.3.4) with $M(\xi) = \langle \xi \rangle^{-m}$ to (I_1) and (I_3), we obtain

$$|(I_1)| + |(I_3)| \leqq C\|f\|_{L^1_{(2s+\gamma-1)^+}} \|g\|_{L^2_{(2s+\gamma-1)^+}} \|h\|_{H^m},$$

because $S_N \in S^0_{1,0}$ uniformly. The same bound on (I_2) follows from (3.2.9).

Notice that the case of $s = 1/2$ follows from the case of $s = 1/2 + \kappa$ for any positive κ because the main concern here is the upper bound. And this completes the proof of the lemma. □

The following lemma is on the commutator of the collision operator with a mollifier in the x variable.

Lemma 3.4. *Let $0 < s < 1$ and $\gamma, m \in \mathbb{R}$. For any suitable functions f and h with the following norms well defined, one has*

$$\begin{aligned}&\|S_N(D_x) Q(f,h) - Q(f, S_N(D_x) h)\|_{L^2(\mathbb{R}^4_{t,x}, H^{m-2s}(\mathbb{R}^3_v))}\\ &\leqq C 2^{-N} \|\nabla_x f\|_{L^\infty(\mathbb{R}^4_{t,x}, L^1_{(2s+\gamma)^+}(\mathbb{R}^3_v))} \|h\|_{L^2(\mathbb{R}^4_{t,x}, H^m_{(2s+\gamma)^+}(\mathbb{R}^3_v))}.\end{aligned} \tag{3.2.11}$$

for a constant C independent of N.

Proof. Let us introduce $\tilde{K}_N(z) = 2^{3N}\hat{S}(2^N z)2^N z$. Note that $\tilde{K}_N \in L^1(\mathbb{R}^3)$ uniformly with respect to N. Then for any smooth function $\tilde{h}$, one has

$$\Big((S_N(D_x)Q(f,h) - Q(f, S_N(D_x)h))\,,\tilde{h}\Big)_{L^2(\mathbb{R}^7)}$$
$$= \int_0^1 \Bigg\{\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x\times\mathbb{R}^3_y} \tilde{K}_N(x-y)$$
$$\times\Big(Q\Big(\nabla_x f(t, y+\tau(x-y),\cdot), 2^{-N}h(t,y,\cdot)\Big), \tilde{h}(t,x,\cdot)\Big)_{L^2(\mathbb{R}^3_v)} \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\Bigg\}\,\mathrm{d}\tau.$$

By applying Theorem 2.1 with $m-2s$, the right-hand side of this equality can be estimated from above by

$$C\left\{\sup_{t,x}||\nabla_x f(t,x,\cdot)||_{L^1_{(2s+\gamma)^+}(\mathbb{R}^3_v)}\right\}\int_{\mathbb{R}_t}\int_{\mathbb{R}^3_x}\Big(|\tilde{K}_N| * ||2^{-N}h(t,\cdot)||_{H^m_{(2s+\gamma)_+}(\mathbb{R}^3_v)}\Big)(x)$$
$$\times||\tilde{h}(t,x,\cdot)||_{H^{2s-m}(\mathbb{R}^3_v)}\mathrm{d}x\,\mathrm{d}t$$
$$\leqq C2^{-N}\|\nabla_x f\|_{L^\infty(\mathbb{R}^4_{t,x};L^1_{(2s+\gamma)^+}(\mathbb{R}^3_v))}\|h\|_{L^2(\mathbb{R}^4_{t,x};\,H^m_{(2s+\gamma)^+}(\mathbb{R}^3_v))}||\tilde{h}||_{L^2(\mathbb{R}^4_{t,x};H^{2s-m}(\mathbb{R}^3_v))},$$

which completes the proof of the lemma. □

We now apply (3.2.11) with $h = S_N(D_v)g$ and $m = 1$, we get

$$\|S_N(D_x)Q(f,\ S_N(D_v)g) - Q(f, S_N(D_x)S_N(D_v)g)\|_{L^2(\mathbb{R}^4_{t,x},\,H^{1-2s}(\mathbb{R}^3_v))}$$
$$\leqq C\|\nabla_x f\|_{L^\infty(\mathbb{R}^4_{t,x},L^1_{(2s+\gamma)^+}(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^4_{t,x},L^2_{(2s+\gamma)^+}(\mathbb{R}^3_v))}. \tag{3.2.12}$$

Here, we have used the fact that a mollification operator $S_N(D_v)$ in the v variable has the property that

$$\|2^{-N}S_N(D_v)g(t,x,\cdot)\|_{H^1_{(2s+\gamma)^+}(\mathbb{R}^3_v)} \leqq C\|g(t,x,\cdot)\|_{L^2_{(2s+\gamma)^+}(\mathbb{R}^3_v)},$$

where C is a constant independent on N.

Now we are ready to complete the proof of Proposition 3.1.

Completion of proof of Proposition 3.1. We now study the commutator terms in (3.2.5). For this purpose, note that

$$\big(P_{N,l}Q(f_2,g) - Q(f_2,\ P_{N,l}\,g),\, P_{N,l}\,g\big)_{L^2(\mathbb{R}^7)}$$
$$= \big(S_N(D_v)\,Q(f_2,g) - Q(f_2,\ S_N(D_v)\,g),\, S^\star_N(D_x)\psi_1(x)W_l P_{N,l}\,g\big)_{L^2(\mathbb{R}^7)}$$
$$+\big(S_N(D_x)Q(f_2, S_N(D_v)g) - Q(f_2, S_N(D_x)S_N(D_v)g), \psi_1(x)W_l P_{N,l}g\big)_{L^2(\mathbb{R}^7)}$$
$$+\big(\psi_1(x)W_l\,Q(f_2, S_N(D_x)S_N(D_v)\,g) - Q(f_2,\ P_{N,l}\,g),\ \ P_{N,l}\,g\big)_{L^2(\mathbb{R}^7)}.$$
$$= (1) + (2) + (3). \tag{3.2.13}$$

Note that $\Lambda^s_v[\psi_1(x),\ \ S_N(D_x)]S_N(D_v)$ is an L^2 uniformly bounded operator with respect to the parameter N for $0 \leqq s \leqq 1$, and that $[W_l,\ \ S_N(D_v)]$ is also a uniformly bounded operator from L^2 to L^2_{l-1} with respect to the parameter N. The discussion on (3.2.13) can be divided into the following two cases.

Case 1. $0 < s < 1/2$**.** In this case, for $l > \max\{4, (\gamma + 2s)^+\}$ Lemma 3.3 implies that,

$$|(1)| \leqq C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x},\ L^1_{\gamma^+ + 2s}(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^7)}\|g\|_{L^2_{2l}(\mathbb{R}^7)} \leqq C\|f_3\|^3_{H^5_{2l}(\mathbb{R}^7)}.$$

And Lemma 3.4 implies that

$$|(2)| \leqq C\|\nabla_x f_2\|_{L^\infty(\mathbb{R}^4_{t,x}, L^1_{\gamma^+ + 2s}(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^4_{t,x}, L^2_{\gamma^+}(\mathbb{R}^3_v))}\|g\|_{L^2_{2l}(\mathbb{R}^7)}$$
$$\leqq C\|f_3\|^3_{H^5_{2l}(\mathbb{R}^7)}.$$

As for the term (3), we use Lemma 2.4 to have

$$|(3)| \leqq C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x}, L^1_{l+\gamma^+ + 2s}(\mathbb{R}^3_v))}\|g\|_{L^2_{l+\gamma^+ + 2s}(\mathbb{R}^7)}\|P_{N,l}\, g\|_{L^2(\mathbb{R}^7)}$$
$$\leqq C\|f_3\|^3_{H^5_{2l}(\mathbb{R}^7)}.$$

Case 2. $1/2 \leqq s < 1$**.** By using (3.2.10), we have

$$|(1)| \leqq C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x}, L^1_{l+\gamma^+ + 2s-1}(\mathbb{R}^3_v))}\|g\|_{L^2_{l+\gamma^+ + 2s-1}(\mathbb{R}^7)}$$
$$\times \|W_{\gamma/2} P_{N,l}\, g\|_{L^2(\mathbb{R}^4_{t,x}, H^{2s-1+\delta}(\mathbb{R}^3_v))}$$
$$\leqq \varepsilon\|\Lambda^s_v W_{\gamma/2} P_{N,l}\, g\|^2_{L^2(\mathbb{R}^7)} + C_\varepsilon\|f_3\|^4_{H^5_{l+4+\gamma^+}(\mathbb{R}^7)}.$$

We can use (3.2.12) to show that

$$|(2)| \leqq C\|\nabla_x f_2\|_{L^\infty(\mathbb{R}^4_{t,x}, L^1_{\gamma^+ + 2s}(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^4_{t,x}, L^2_{\gamma^+ + 2s}(\mathbb{R}^3_v))}$$
$$\times \|W_l P_{N,l}\, g\|_{L^2(\mathbb{R}^4_{t,x}, H^{2s-1}(\mathbb{R}^3_v))}$$
$$\leqq \varepsilon\|\Lambda^s_v W_{\gamma/2} P_{N,l} g\|^2_{L^2(\mathbb{R}^7_{t,x,v})} + C_\varepsilon\|f_3\|^{\frac{4}{\theta}+2}_{H^5_{kl}(\mathbb{R}^7)}.$$

Then, (2.1.17) implies that

$$|(3)| \leqq C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x}, L^1_{2s+l-1+\gamma_+}(\mathbb{R}^3_v))}$$
$$\times \|\psi_1(x) S_N(D_x) S_N(D_v)\, g\|_{L^2(\mathbb{R}^4_{t,x}, H^{2s-1+\delta}_{2s+l-1+\gamma_+}(\mathbb{R}^3_v))}\|P_{N,l}\, g\|_{L^2(\mathbb{R}^7)}$$
$$\leqq \varepsilon\|\Lambda^s_v W_{\gamma/2} P_{N,l} g\|^2_{L^2(\mathbb{R}^7_{t,x,v})} + C_\varepsilon\|f_3\|^{\frac{4}{\theta}+2}_{H^5_{kl}(\mathbb{R}^7)}.$$

In summary, we have obtained the following estimate for the second term on the right-hand side of (3.2.5).

$$\Big|\big(P_{N,l}\, Q(f_2, g) - Q(f_2,\ P_{N,l} g),\ P_{N,l} g\big)_{L^2(\mathbb{R}^7)}\Big|$$
$$\leqq \varepsilon\|\Lambda^s_v W_{\gamma/2} P_{N,l}\, g\|^2_{L^2(\mathbb{R}^7_{t,x,v})} + C_\varepsilon\|f_3\|^{2k'}_{H^5_{kl}(\mathbb{R}^7)}.$$

Finally, it holds that

$$\|\Lambda^s_v W_{\gamma/2} P_{N,l}\, g\|^2_{L^2(\mathbb{R}^7)} \leqq C\|f_3\|^{2k'}_{H^5_{kl}(\mathbb{R}^7)}, \tag{3.2.14}$$

where the constants C, k, and k' are independent of N. Therefore, Proposition 3.1 is proved by taking the limit $N \to \infty$. □

3.3. Gain of regularity in (t, x)

First of all, let us consider a transport equation in the form of

$$f_t + v \cdot \nabla_x f = g \in D'(\mathbb{R}^{2n+1}), \tag{3.3.1}$$

where $(t, x, v) \in \mathbb{R}^{1+n+n} = \mathbb{R}^{2n+1}$. In [7], by using a generalized uncertainty principle, we proved the following hypo-elliptic estimate.

Lemma 3.5. *Assume that $g \in H^{-s'}(\mathbb{R}^{2n+1})$, for some $0 \leqq s' < 1$. Let $f \in L^2(\mathbb{R}^{2n+1})$ be a weak solution of the transport equation (3.3.1) such that $\Lambda_v^s f \in L^2(\mathbb{R}^{2n+1})$ for some $0 < s \leqq 1$. Then it follows that*

$$\Lambda_x^{s(1-s')/(s+1)} f \in L^2_{-\frac{ss'}{s+1}}(\mathbb{R}^{2n+1}), \quad \Lambda_t^{s(1-s')/(s+1)} f \in L^2_{-\frac{s}{s+1}}(\mathbb{R}^{2n+1}),$$

where $\Lambda_\bullet = (1 + |D_\bullet|^2)^{1/2}$.

As mentioned earlier, this hypo-elliptic estimate, together with Proposition 3.1, is used to obtain the partial regularity in the variable (t, x). With this partial regularity in (t, x), by applying a Leibniz type formula for fractional derivatives, we will show some improved regularity in all variables, v and (t, x). Then the hypo-elliptic estimate can be used again to get higher regularity in the variable (t, x). This procedure can be continued to obtain at least one order higher regularity in (t, x) variable.

For the details, we first recall a Leibniz type formula for fractional derivatives with respect to variable (t, x).

Lemma 3.6. *Let $0 < \lambda < 1$. Then there exists a positive constant $C_\lambda \neq 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^n)$, one has*

$$|D_y|^\lambda f(y) = \mathcal{F}^{-1}\left(|\xi|^\lambda \hat{f}(\xi)\right) = C_\lambda \int_{\mathbb{R}^n} \frac{f(y) - f(y+h)}{|h|^{n+\lambda}} \mathrm{d}h. \tag{3.3.2}$$

Indeed, note that

$$\int_{\mathbb{R}^n} \frac{f(y) - f(y+h)}{|h|^{n+\lambda}} \mathrm{d}h = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{iy\cdot\xi} \int_{\mathbb{R}^n} \frac{1 - e^{ih\cdot\xi}}{|h|^{n+\lambda}} \mathrm{d}h\, \mathrm{d}\xi,$$

while

$$\int_{\mathbb{R}^n} \frac{1 - e^{-i\,h\cdot\xi}}{|h|^{n+\lambda}}\, \mathrm{d}h = |\xi|^\lambda \int_{\mathbb{R}^n} \frac{1 - e^{-i\,u\cdot\frac{\xi}{|\xi|}}}{|u|^{n+\lambda}}\, \mathrm{d}u,$$

so that (3.3.2) follows from

$$\int_{\mathbb{R}^n} \frac{1 - e^{-i\,u\cdot\frac{\xi}{|\xi|}}}{|u|^{n+\lambda}} \mathrm{d}u \neq 0,$$

which is a positive constant depending only on λ and the dimension n, but independent from ξ.

Using this Lemma, we have the following Leibniz type formula,

$$\begin{aligned}|D_y|^\lambda \left(f(y)g(y)\right) &= C_\lambda \int_{\mathbb{R}^n} \frac{f(y)g(y) - f(y+h)g(y+h)}{|h|^{n+\lambda}} \mathrm{d}h \\ &= g(y)|D_y|^\lambda f(y) + f(y)|D_y|^\lambda g(y) \\ &\quad + C_\lambda \int_{\mathbb{R}^n} \frac{\left(f(y) - f(y+h)\right)\left(g(y+h) - g(y)\right)}{|h|^{n+\lambda}} \mathrm{d}h.\end{aligned} \tag{3.3.3}$$

We now turn to the analysis of the fractional derivative with respect to (t, x) of the nonlinear collision operator. Denote the difference with respect to (t, x) by

$$f_h(t, x, v) = f(t, x, v) - f((t, x) + h, v), \quad h \in \mathbb{R}^4_{t,x}.$$

It follows that for the collision operator (where $n = 1 + 3$),

$$\begin{aligned}|D_{t,x}|^\lambda Q(f, g) &= Q\left(|D_{t,x}|^\lambda f, g\right) \\ &\quad + Q\left(f, |D_{t,x}|^\lambda g\right) + C_\lambda \int_{\mathbb{R}^4} |h|^{-4-\lambda} Q(f_h, g_h)\, \mathrm{d}h.\end{aligned} \tag{3.3.4}$$

This kind of decomposition will be used extensively below in order to obtain partial regularity with respect to the (t, x) variable.

First of all, we have the following proposition on the gain of regularity in the variable (t, x) through the uncertainty principle.

Proposition 3.7. *Under the hypothesis of Theorem 1.1, one has*

$$\Lambda^{s_0}_{t,x} f_1 \in H^5_l(\mathbb{R}^7), \tag{3.3.5}$$

for any $l \in \mathbb{N}$ *and* $0 < s_0 = \frac{s(1-s)}{(s+1)}$.

Proof. In fact, for any $l \in \mathbb{N}$, it follows from Proposition 3.1 that

$$\Lambda^s_v W_l g \in L^2(\mathbb{R}^7).$$

Then the upper bound estimation given by Corollary 2.5 with $m = -s$ implies that

$$W_l Q(f_2, g) \in L^2\left(\mathbb{R}^4_{t,x}; H^{-s}(\mathbb{R}^3_v)\right).$$

On the other hand, Proposition 3.2 and (3.2.7) gives

$$W_l G \in L^2(\mathbb{R}^4_{t,x}; H^{-(2s-1+\delta)}(\mathbb{R}^3_v)).$$

By using (3.1.1), it follows that

$$\partial_t(W_l g) + v \cdot \partial_x(W_l g) = W_l Q(f_2,\ g) + W_l G \in H^{-s}(\mathbb{R}^7). \tag{3.3.6}$$

Finally, by using Lemma 3.5 with $s' = s$, we can conclude (3.3.5) and this completes the proof of the proposition. □

Therefore, under the hypothesis $f \in \mathcal{H}^5_l(]T_1, T_2[\times\Omega \times \mathbb{R}^3_v)$ for all $l \in \mathbb{N}$, it follows that for any $l \in \mathbb{N}$ we have

$$\Lambda^s_v(\varphi(t)\psi(x)f) \in H^5_l(\mathbb{R}^7), \quad \Lambda^{s_0}_{t,x}(\varphi(t)\psi(x)f) \in H^5_l(\mathbb{R}^7). \tag{3.3.7}$$

We now improve this partial regularity in the (t, x) variable.

Proposition 3.8. *Let $0 < \lambda < 1$. Suppose that $f \in \mathcal{H}^5_l(]T_1, T_2[\times\Omega \times \mathbb{R}^3_v)$ is a solution of the equation (1.1) for all $l \in \mathbb{N}$. Furthermore, assume that for any cutoff functions φ, ψ,*

$$\Lambda^s_v(\varphi(t)\psi(x)f) \in H^5_l(\mathbb{R}^7), \quad \Lambda^{\lambda}_{t,x}(\varphi(t)\psi(x)f) \in H^5_l(\mathbb{R}^7). \tag{3.3.8}$$

Then, one has

$$\Lambda^s_v\Lambda^{\lambda}_{t,x}(\varphi(t)\psi(x)f) \in H^5_l(\mathbb{R}^7), \tag{3.3.9}$$

for any $l \in \mathbb{N}$ and any cutoff functions φ, ψ.

Proof. Set

$$g_{N,l} = P_{N,l}\,g = \psi_1(x)S_N(D_x)\,W_l S_N(D_v)\partial^\alpha(\varphi(t)\psi(x)f),$$

where $\alpha \in \mathbb{N}^7$, $|\alpha| \leqq 5$ and $l \in \mathbb{N}$. Then (3.3.8) yields

$$\|\Lambda^s_v g_{N,l}\|_{L^2(\mathbb{R}^7)} \leqq C\|\Lambda^s_v\partial^\alpha(\varphi(t)\psi(x)f)\|_{L^2_l(\mathbb{R}^7)},$$

and

$$\|\Lambda^{\lambda}_{t,x} g_{N,l}\|_{L^2(\mathbb{R}^7)} \leqq C\|\Lambda^{\lambda}_{t,x}\partial^\alpha(\varphi(t)\psi(x)f)\|_{L^2_l(\mathbb{R}^7)},$$

where the constant C is independent of N.

It follows that $g_{N,l}$ satisfies the equation

$$\partial_t(g_{N,l}) + v\cdot\partial_x(g_{N,l}) = Q(f_2,\ g_{N,l}) + G_{N,l}, \tag{3.3.10}$$

where $G_{N,l}$ is given by

$$\begin{aligned} G_{N,l} &= \psi_1(x)W_l\,[S_N(D_v), v]\cdot\nabla_x S_N(D_x)g \\ &\quad + \left(P_{N,l}Q\,(f_2, g) - Q\left(f_2, P_{N,l}g\right)\right) \\ &\quad + ((v\cdot\nabla_x)\psi_1(x))\,W_l\,S_N(D_x)S_N(D_v)g + P_{N,l}G, \end{aligned}$$

with G defined in (3.1.2).

We now choose $|D_{t,x}|^\lambda\psi_2^2(x)|D_{t,x}|^\lambda g_{N,l}$ as a test function for equation (3.3.10). It follows that

$$\begin{aligned} &\left(v\cdot\ (\partial_x\psi_2)\,|D_{t,x}|^\lambda g_{N,l}, \psi_2(x)|D_{t,x}|^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)} \\ &\quad = \left(\psi_2(x)|D_{t,x}|^\lambda\left\{Q(f_2,\ g_{N,l}) + G_{N,l}\right\}, \psi_2(x)|D_{t,x}|^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)}. \end{aligned} \tag{3.3.11}$$

It is sufficient to prove that, for any $l \in \mathbb{N}$,

$$\Lambda^s_v\Lambda^{\lambda}_{t,x}P_{N,l}\,g \in L^2(\mathbb{R}^7),$$

and is uniformly bounded with respect to N. In the rest of the proof, we use C to denote a constant independent of N.

We first consider the linear terms in (3.3.11). On the left-hand side of (3.3.11), the hypothesis (3.3.8) implies that

$$\left\| v\cdot(\partial_x\psi_2)\,|D_{t,x}|^\lambda g_{N,l}\right\|_{L^2(\mathbb{R}^7)}\leqq C\|\ |\Lambda_{t,x}|^\lambda\partial^\alpha(\varphi(t)\psi(x)f)\|_{L^2_{l+1}(\mathbb{R}^7)}.$$

For the linear terms in $G_{N,l}$, by using (3.2.4), one has

$$\begin{aligned}&\left\|\psi_2(x)|D_{t,x}|^\lambda\,\{\psi_1(x)W_l\ [S_N(D_v),\ v]\cdot\nabla_x S_N(D_x)g\}\right\|_{L^2(\mathbb{R}^7)}\\&\quad\leqq C\||\Lambda_{t,x}|^\lambda\partial^\alpha(\varphi(t)\psi(x)f)\|_{L^2_l(\mathbb{R}^7)},\end{aligned}$$

and

$$\begin{aligned}&\left\|\psi_2(x)|D_{t,x}|^\lambda\,(v\cdot(\nabla_x\,\psi_1)(x))\,W_l\,S_N(D_x)S_N(D_v)g\right\|_{L^2(\mathbb{R}^7)}\\&\quad\leqq C\||\Lambda_{t,x}|^\lambda\partial^\alpha(\varphi(t)\psi(x)f)\|_{L^2_{l+1}(\mathbb{R}^7)}.\end{aligned}$$

Similarly, concerning the linear terms (B) and (C) in G, we have

$$\left\|\psi_2(x)|D_{t,x}|^\lambda P_{N,l}\,((B)+(C))\right\|_{L^2(\mathbb{R}^7)}\leqq C\||\Lambda_{t,x}|^\lambda\partial^\alpha(\varphi(t)\psi(x)f)\|_{L^2_{l+1}(\mathbb{R}^7)}.$$

For the nonlinear terms in (3.3.11), we shall use the formula (3.3.4). First of all, the coercivity estimate (2.2.1) gives, as in (3.2.3), that

$$\begin{aligned}&-\left(Q(f_2,\ \psi_1(x)|D_{t,x}|^\lambda g_{N,l}),\,\psi_1(x)|D_{t,x}|^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)}\\&\quad\geqq C_0\|\Lambda_v^s W_{\gamma/2}\psi_1(x)|D_{t,x}|^\lambda g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\&\quad\quad -C\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x};L^1_{\max\{\gamma^+,\,2-\gamma^+\}}(\mathbb{R}^3_v))}\|\psi_1(x)|D_{t,x}|^\lambda g_{N,l}\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^7)}.\end{aligned}\qquad(3.3.12)$$

On the other hand, the upper estimate of Theorem 2.1 with $m=-s$ and $\alpha=-\gamma/2>0$ (the case $\gamma>0$ is easier) gives,

$$\begin{aligned}&\left|\left(Q(|D_{t,x}|^\lambda f_2,\ \psi_1(x)g_{N,l}),\,\psi_1(x)|D_{t,x}|^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)}\right|\\&\quad\leqq C\||D_{t,x}|^\lambda f_2\|_{L^\infty(\mathbb{R}^4_{t,x},L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\psi_1(x)\Lambda_v^s g_{N,l}\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\\&\quad\quad\times\|\psi_1(x)|D_{t,x}|^\lambda\Lambda_v^s W_{\gamma/2}g_{N,l}\|_{L^2(\mathbb{R}^7)}\\&\quad\leqq\varepsilon\|\psi_1(x)|D_{t,x}|^\lambda\Lambda_v^s W_{\gamma/2}g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\&\quad\quad+C_\varepsilon\||D_{t,x}|^\lambda f_2\|^2_{L^\infty(\mathbb{R}^4_{t,x},\,L^2_{|\gamma|/2+\gamma^++2s+4}(\mathbb{R}^3_v))}\|\Lambda_v^s g\|^2_{L^2_{|\gamma|/2+\gamma^++2s+l}(\mathbb{R}^7)}.\end{aligned}$$

For the cross term coming from the decomposition (3.3.4), by again using estimate (2.1.1) with $m=-s$ and $\alpha=|\gamma|/2$, we get

$$\begin{aligned}&\int_{\mathbb{R}^4}|h|^{-4-\lambda}\left|\left(Q((f_2)_h,\ (g_{N,l})_h),\,\psi_1^2(x)|D_x|^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)}\mathrm{d}h\right|\\&\quad\leqq|C_\lambda|\|\psi_1(x)|D_x|^\lambda\Lambda_v^s W_{\gamma/2}g_{N,,l}\|_{L^2(\mathbb{R}^7)}\\&\quad\quad\times\int_{\mathbb{R}^4}|h|^{-4-\lambda}\|(f_2)_h\|_{L^\infty(\mathbb{R}^4_{t,x},\,L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda_v^s(g_{N,l})_h\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\mathrm{d}h.\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{\mathbb{R}^4} |h|^{-4-\lambda}\|(f_2)_h\|_{L^\infty(\mathbb{R}^4_{t,x},\, L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda^s_v(g_{N,l})_h\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\mathrm{d}h\\
&\leqq \int_{|h|<1} |h|^{-4-\lambda}\|(f_2)_h\|_{L^\infty(\mathbb{R}^4_{t,x},\, L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda^s_v(g_{N,l})_h\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\mathrm{d}h\\
&\quad +4\tilde{C}_\lambda\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x},L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda^s_v g_{N,l}\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\\
&\leqq 2\int_{|h|<1} |h|^{-4-\lambda+1}\|\nabla_{t,x} f_2\|_{L^\infty(\mathbb{R}^4_{t,x},\, L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda^s_v g_{N,l}\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\mathrm{d}h\\
&\quad +4\tilde{C}_\lambda\|f_2\|_{L^\infty(\mathbb{R}^4_{t,x},\, L^1_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda^s_v g_{N,l}\|_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^4} |h|^{-4-\lambda}\left|\Big(Q((f_2)_h,\ (g_{N,l})_h),\ \psi_1^2(x)|D_x|^\lambda W_{\gamma/2}g_{N,l}\Big)_{L^2(\mathbb{R}^7)}\mathrm{d}h\right|\\
&\quad\leqq \varepsilon\|\psi_1(x)|D_x|^\lambda\Lambda^s_v g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\
&\qquad +C_\varepsilon\|\Lambda^1_{t,x} f_2\|^2_{L^\infty(\mathbb{R}^4_{t,x};L^2_{|\gamma|/2+\gamma^++2s+4}(\mathbb{R}^3_v))}\|\Lambda^s_v g_{N,l}\|^2_{L^2_{|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}.
\end{aligned}$$

Hence, the formula (3.3.4) yields

$$\begin{aligned}
&\left|\Big(|D_{t,x}|^\lambda Q(f_2,\psi_1(x)g_{N,l})-Q(|D_{t,x}|^\lambda f_2,\psi_1(x)g_{N,l}),\psi_1(x)|D_{t,x}|^\lambda g_{N,l}\Big)_{L^2(\mathbb{R}^7)}\right|\\
&\quad\leqq \varepsilon\|\psi_1(x)|D_{t,x}|^\lambda\Lambda^s_v W_{\gamma/2}g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\
&\qquad +C_\varepsilon\|\Lambda^1_{t,x} f_2\|^2_{L^\infty(\mathbb{R}^4_{t,x},\, L^2_{|\gamma|/2+\gamma^++2s+4}(\mathbb{R}^3_v))}\|\Lambda^s_v g\|^2_{L^2_{|\gamma|/2+\gamma^++2s+l}(\mathbb{R}^7)}.
\end{aligned}$$

In conclusion, we get from the coercivity property (3.3.12) that

$$\begin{aligned}
&\|\Lambda^s_v W_{\gamma/2}\psi_1(x)|D_{t,x}|^\lambda g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\
&\quad\leqq C\|\Lambda^1_{t,x} f_2\|^2_{L^\infty(\mathbb{R}^4_{t,x},\, L^2_{|\gamma|/2+\gamma^++2s+4}(\mathbb{R}^3_v))}\\
&\qquad\times\left(\|\ |D_{t,x}|^\lambda g\|^2_{L^2_{l+|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}+\|\Lambda^s_v g\|^2_{L^2_{l+|\gamma|/2+\gamma^++2s}(\mathbb{R}^7)}\right)\\
&\qquad+\left|\Big(|D_{t,x}|^\lambda\left(P_{N,l}\,Q\left(f_2,\ g\right)-Q\left(f_2,\ P_{N,l}\,g\right)\right),\ \psi_2^2(x)|D_{t,x}|^\lambda g_{N,l}\Big)_{L^2(\mathbb{R}^7)}\right|\\
&\qquad+\left|\Big(|D_{t,x}|^\lambda P_{N,l}\ (A),\ \psi_2^2(x)|D_{t,x}|^\lambda\, g_{N,l}\Big)_{L^2(\mathbb{R}^7)}\right|\\
&\quad=(\mathrm{I})+(\mathrm{II})+(\mathrm{III}).
\end{aligned}\tag{3.3.13}$$

For the term (II), since $[|D_{t,x}|^\lambda,\ \psi_1(x)]$ is a bounded operator, we can replace $P_{N,l}$ by $\tilde{P}_{N,l}=W_l\ S_N(D_x)S_N(D_v)$. Again, the formula (3.3.4) yields

$$
\begin{aligned}
&\left(|D_{t,x}|^{\lambda}\left(\tilde{P}_{N,l}Q\left(f_2,\ g\right)-Q\left(f_2,\ \tilde{P}_{N,l}g\right)\right),\ \psi_2^2(x)|D_{t,x}|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\\
&=\left(\left(\tilde{P}_{N,l}Q\left(|D_{t,x}|^{\lambda}f_2,\ g\right)-Q\left(|D_{t,x}|^{\lambda}f_2,\ \tilde{P}_{N,l}g\right)\right),\ \psi_2^2(x)|D_{t,x}|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\\
&\quad+\left(\left(\tilde{P}_{N,l}Q\left(f_2,\ |D_{t,x}|^{\lambda}g\right)-Q\left(f_2,\ \tilde{P}_{N,l}|D_x|^{\lambda}g\right)\right),\ \psi_2^2(x)|D_{t,x}|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\\
&\quad+C_\lambda\int_{\mathbb{R}^4}|h|^{-4-\lambda}\left(\left(\tilde{P}_{N,l}Q\left((f_2)_h,\ g_h\right)\right.\right.\\
&\quad\left.\left.-Q\left((f_2)_h,\ \tilde{P}_{N,l}g_h\right)\right),\ \psi_2^2(x)|D_{t,x}|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\mathrm{d}h.
\end{aligned}
$$

As for (3.2.14), in the case when $1/2 \leqq s < 1$ (the other case when $0 < s < 1/2$ is similar and easier to handle), by applying Lemmas 2.4, 3.3 and 3.4, we have

$$
\begin{aligned}
&\left|\left(\left(\tilde{P}_{N,l}Q\left(|D_{t,x}|^{\lambda}f_2,\ g\right)-Q\left(|D_{t,x}|^{\lambda}f_2,\tilde{P}_{N,l}g\right)\right),\psi_2^2(x)|D_{t,x}|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\right|\\
&\leqq C\|\Lambda_{t,x}^{1+\lambda}f_2\|_{L^\infty(\mathbb{R}^4_{t,x},L^1_{(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^4_{t,x},H^{2s-1+\delta}_{(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\|\,|D_{t,x}|^{\lambda}g\|_{L^2_{2l}(\mathbb{R}^7)}.
\end{aligned}
$$

By using (3.2.10) of Lemma 3.3, we can get, for $2s-1+\delta < s$,

$$
\begin{aligned}
&\left|\left(\left(\tilde{P}_{N,l}Q\left(f_2,|D_{t,x}|^{\lambda}g\right)-Q\left(f_2,\tilde{P}_{N,l}|D_{t,x}|^{\lambda}g\right)\right),\psi_2^2(x)|D_{t,x}|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\right|\\
&\leqq C\|\Lambda_{t,x}f_2\|_{L^\infty(\mathbb{R}^4_{t,x},L^1_{(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\||D_{t,x}|^{\lambda}g\|_{L^2(\mathbb{R}^4_{t,x},L^2_{l+(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\\
&\quad\times\|\,|D_{t,x}|^{\lambda}\psi_1 g_{N,l}\|_{L^2(\mathbb{R}^4_{t,x},H^{2s-1+\delta}_{l+(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\\
&\leqq \varepsilon\|\Lambda_v^s W_{\gamma/2}\psi_1(x)|D_{t,x}|^{\lambda}g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\
&\quad+C_\varepsilon\|\Lambda^1_{t,x}f_2\|^{2k'}_{L^\infty(\mathbb{R}^4_{t,x},L^2_{l+3/2+\delta+(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\||D_{t,x}|^{\lambda}g\|^{2k'}_{L^2(\mathbb{R}^4_{t,x},L^2_{kl}(\mathbb{R}^3_v))},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int|h|^{-4-\lambda}\left(\left(\tilde{P}_{N,l}Q\left(f_{2,h},g_h\right)-Q\left(f_{2,h},\tilde{P}_{N,l}g_h\right)\right),\psi_2^2(x)|D_x|^{\lambda}g_{N,l}\right)_{L^2(\mathbb{R}^7)}\mathrm{d}h\right|\\
&\leqq C\|\Lambda_{t,x}\Lambda_x f_2\|_{L^\infty(\mathbb{R}^4_{t,x};L^1_{l+(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\|g\|_{L^2(\mathbb{R}^4_{t,x},H^{2s-1+\delta}_{l+(\gamma+2s-1)^+}(\mathbb{R}^3_v))}\||D_{t,x}|^{\lambda}g\|_{L^2_l(\mathbb{R}^7)}\\
&\leqq C\|f_2\|_{H^{2+4/2+\delta}_{l+3/2+\delta+(\gamma+2s-1)^+}(\mathbb{R}^7)}\|\Lambda_v^s g\|_{L^2_{l+(\gamma+2s-1)^+}(\mathbb{R}^7)}\|\,|D_{t,x}|^{\lambda}g\|_{L^2_l(\mathbb{R}^7)}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\text{(II)} &\leqq \varepsilon\|\Lambda_v^s W_{\gamma/2}\psi_1(x)|D_{t,x}|^{\lambda}g_{N,l}\|^2_{L^2(\mathbb{R}^7)}\\
&\quad+C_\varepsilon\|f_2\|^{2k'}_{H^2_{l+\gamma^++2s+4}(\mathbb{R}^3_v))}\left(\||D_{t,x}|^{\lambda}g\|^{2k'}_{L^2_{kl+\gamma^++2s}(\mathbb{R}^7)}+\|\Lambda_v^s g\|^2_{L^2_{l+\gamma^++2s}(\mathbb{R}^7)}\right).
\end{aligned}
$$

We now consider the last term (III) of (3.3.13). Recall that (A) stands for the nonlinear terms from G given in (3.1.2). Precisely

$$
(A)=\sum_{\alpha_1+\alpha_2=\alpha\alpha_1\neq 0}C^{\alpha_1}_{\alpha_2}Q\left(\partial^{\alpha_1}f_2,\partial^{\alpha_2}f_1\right).
$$

By using (2.1.1) (we also consider only the case $1/2 \leqq s < 1$) and formula (3.3.4), we have

$$\begin{aligned}
&\left|\left(|D_{t,x}|^{\lambda}\ \left(Q\left(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1\right)\right),\ P_{N,l}\psi_2^2(x)|D_{t,x}|^{\lambda}\, g_{N,l}\right)_{L^2(\mathbb{R}^7)}\right| \\
&\quad\leqq C\|\Lambda_v^{-m} W_{\gamma/2}\psi_1(x)|D_{t,x}|^{\lambda} g_{N,l}\|_{L^2(\mathbb{R}^7)} \\
&\qquad\times\Big\{\|Q\left(|D_{t,x}|^{\lambda}\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1\right)\|_{L^2(\mathbb{R}^4_{t,x}; H^m_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\qquad+\|Q\left(\partial^{\alpha_1} f_2,\ |D_{t,x}|^{\lambda}\partial^{\alpha_2} f_1\right)\|_{L^2(\mathbb{R}^4_{t,x}; H^m_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\qquad+\left\|\int h^{-4-\lambda} Q\left(\partial^{\alpha_1}(f_2)_h, \partial^{\alpha_2}(f_1)_h\right) \mathrm{d}h\right\|_{L^2(\mathbb{R}^4_{t,x}; H^m_{l+|\gamma|/2}(\mathbb{R}^3_v))}\Big\}.
\end{aligned}$$

We divide the discussion into two cases.
Case 1. $|\alpha_1| = 1, 2$. Take $m = -s$. We have

$$\begin{aligned}
&\|Q\left(|D_{t,x}|^{\lambda}\partial^{\alpha_1} f_2,\ \partial^{\alpha_2} f_1\right)\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\qquad+\|Q\left(\partial^{\alpha_1} f_2,\ |D_{t,x}|^{\lambda}\partial^{\alpha_2} f_1\right)\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\quad\leqq C\|\Lambda^{\lambda}_{t,x}\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda_v^s\Lambda_x^{\lambda}\partial^{\alpha_2} f_1\|_{L^2_{l+\gamma^++2s}(\mathbb{R}^7)} \\
&\quad\leqq C\|f_2\|_{H^{\lambda+2+4/2+\delta}_{l+3/2+\delta+\gamma^++2s}(\mathbb{R}^7)}\|\Lambda_v^s f_1\|_{H^{4+\lambda}_{l+\gamma^++2s}(\mathbb{R}^7)},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^4} h^{-4-\lambda} Q\left(\partial^{\alpha_1}(f_2)_h,\ \partial^{\alpha_2}(f_1)_h\right) \mathrm{d}h\right\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\quad\leqq C\int |h|^{-4-\lambda}\|\partial^{\alpha_1}(f_2)_h\|_{L^\infty(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda_v^s\partial^{\alpha_2}(f_1)_h\|_{L^2_{l+\gamma^++2s}(\mathbb{R}^7)}\mathrm{d}h \\
&\quad\leqq C\|\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda_v^s\partial^{\alpha_2}\nabla_{t,x} f_1)\|_{L^2_{l+\gamma^++2s}(\mathbb{R}^7)} \\
&\quad\leqq C\|f_2\|_{H^{2+4/2+\delta}_{l+3/2+\delta+\gamma^++2s}(\mathbb{R}^7)}\|\Lambda_v^s f_1\|_{H^5_{l+\gamma^++2s}(\mathbb{R}^7)}.
\end{aligned}$$

Case 2. $|\alpha_1| \geqq 3$. By the same argument as above, one has

$$\begin{aligned}
&\|Q\left(|D_{t,x}|^{\lambda}\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1\right)\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\qquad+\|Q\left(\partial^{\alpha_1} f_2, |D_{t,x}|^{\lambda}\partial^{\alpha_2} f_1\right)\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\quad\leqq C\|\Lambda^{\lambda}_{t,x}\partial^{\alpha_1} f_2\|_{L^2(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^++2s}(\mathbb{R}^3_v))}\|\Lambda_v^s\Lambda^{\lambda}_{t,x}\partial^{\alpha_2} f_1\|_{L^\infty(\mathbb{R}^4_{t,x}; L^2_{l+\gamma^++2s}(\mathbb{R}^3_v))} \\
&\quad\leqq C\|\Lambda^{\lambda}_{t,x} f_2\|_{H^5_{l+3/2+\delta+\gamma^++2s}(\mathbb{R}^7)}\|\Lambda_v^s f_1\|_{H^{2+4/2+\lambda+\delta}_{l+\gamma^++2s}(\mathbb{R}^7)}.
\end{aligned}$$

When $|\alpha_1| = 3, 4$, we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^4} h^{-4-\lambda} Q\left(\partial^{\alpha_1}(f_2)_h, \partial^{\alpha_2}(f_1)_h\right) \mathrm{d}h \right\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\leqq C \int |h|^{-4-\lambda} \| \partial^{\alpha_1}(f_2)_h \|_{L^2(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \\
&\quad \times \| \Lambda^s_v \partial^{\alpha_2}(f_1)_h \|_{L^\infty(\mathbb{R}^4_{t,x}; L^2_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \mathrm{d}h \\
&\leqq C \| \nabla_{t,x} \partial^{\alpha_1} f_2 \|_{L^2(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \| \Lambda^s_v \partial^{\alpha_2} f_1) \|_{L^\infty(\mathbb{R}^4_{t,x}; L^2_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \\
&\leqq C \| f_2 \|_{H^5_{l+3/2+\delta+\gamma^+ +2s}(\mathbb{R}^7)} \| f_1 \|_{H^{2+4/2+s+\delta}_{l+\gamma^+ +2s}(\mathbb{R}^7)},
\end{aligned}$$

while when $|\alpha_1| = |\alpha| = 5$, we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^4} h^{-4-\lambda} Q\left(\partial^{\alpha}(f_2)_h, (f_1)_h\right) \mathrm{d}h \right\|_{L^2(\mathbb{R}^4_{t,x}; H^{-s}_{l+|\gamma|/2}(\mathbb{R}^3_v))} \\
&\leqq C \int |h|^{-4-\lambda} \| \partial^{\alpha}(f_2)_h \|_{L^2(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \\
&\quad \times \| \Lambda^s_v (f_1)_h \|_{L^\infty(\mathbb{R}^4_{t,x}; L^2_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \mathrm{d}h \\
&\leqq C \| \partial^{\alpha} f_2 \|_{L^2(\mathbb{R}^4_{t,x}; L^1_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \| \Lambda^s_v \nabla_{t,x} f_1) \|_{L^\infty(\mathbb{R}^4_{t,x}; L^2_{l+\gamma^+ +2s}(\mathbb{R}^3_v))} \\
&\leqq C \| f_2 \|_{H^5_{l+3/2+\delta+\gamma^+ +2s}(\mathbb{R}^7)} \| f_1 \|_{H^{1+4/2+s+\delta}_{l+\gamma^+ +2s}(\mathbb{R}^7)},
\end{aligned}$$

Thus, by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\text{(III)} &\leqq \varepsilon \| \Lambda^s_v W_{\gamma/2} \psi_1(x) |D_{t,x}|^\lambda g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \\
&\quad + C_\varepsilon \left(\| \Lambda^\lambda_{t,x} f_3 \|^4_{H^5_{2l+\gamma^+ +7}(\mathbb{R}^7)} + \| \Lambda^s f_3 \|^4_{H^5_{2l+\gamma^+ +7}(\mathbb{R}^7)} \right).
\end{aligned}$$

Finally, we get from (3.3.13) that

$$\begin{aligned}
&\| \Lambda^s_v W_{\gamma/2} \psi_1(x) |D_{t,x}|^\lambda g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \\
&\quad \leqq C \left(\| \Lambda^\lambda_{t,x} f_3 \|^{k'4}_{H^5_{kl+\gamma^+ +7}(\mathbb{R}^7)} + \| \Lambda^s_v f_3 \|^4_{H^5_{2l+\gamma^+ +7}(\mathbb{R}^7)} \right).
\end{aligned}$$

Therefore, we complete the proof for Proposition 3.8. □

We are now ready to prove the following regularity result on the solution with respect to the (t, x) variable.

Proposition 3.9. *Under the hypothesis of Theorem 1.1, one has*

$$\Lambda^{1+\varepsilon}_{t,x} (\varphi(t)\psi(x) f) \in H^5_l(\mathbb{R}^7), \tag{3.3.14}$$

for any $l \in \mathbb{N}$ and some $\varepsilon > 0$.

Proof. Fix $s_0 = \frac{s(1-s)}{(s+1)}$. Then (3.3.7) and Proposition 3.8 with $\lambda = s_0$ imply

$$\Lambda_v^s \Lambda_{t,x}^{s_0} g \in H_l^5(\mathbb{R}^7).$$

It follows that,

$$(\Lambda_{t,x}^{s_0} g)_t + v \cdot \partial_x (\Lambda_{t,x}^{s_0} g) = \Lambda_{t,x}^{s_0} Q(f_2,\ g) + \Lambda_{t,x}^{s_0} G \in H_l^{-s}(\mathbb{R}^7).$$

By applying Lemma 3.5 with $s' = s$, we can deduce that

$$\Lambda_{t,x}^{s_0+s_0}(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7),$$

for any $l \in \mathbb{N}$. If $2s_0 < 1$, by using Proposition 3.8 with $\lambda = 2s_0$ and Lemma 3.5 with $s' = s$, we have

$$\Lambda_v^s(\varphi(t)\psi(x)f),\ \Lambda_{t,x}^{2s_0}(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7) \ \Rightarrow\ \Lambda_{t,x}^{3s_0}(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7).$$

Choose $k_0 \in \mathbb{N}$ such that

$$k_0 s_0 < 1, \quad (k_0+1)s_0 = 1 + \varepsilon > 1.$$

Finally, (3.3.14) follows from (3.3.5) and Proposition 3.8 with $\lambda = k_0 s_0$ by induction. And this completes the proof of the proposition. □

3.4. Proof of Theorem 1.1

In this subsection, we give the proof of Theorem 1.1 with the above preparations. The proof is also based on an induction argument.

From Propositions 3.1 and 3.9, it follows that for any $l \in \mathbb{N}$,

$$\Lambda_v^s(\varphi(t)\psi(x)f), \quad \nabla_{t,x}(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7).$$

These facts will be used to get the high order regularity with respect to the variable v.

Proposition 3.10. *Let $0 < \lambda < 1$. Suppose that, for any $\varphi \in C_0^\infty(]T_1,\ T_2[)$, $\psi \in C_0^\infty(\Omega)$ and all $l \in \mathbb{N}$,*

$$\Lambda_v^\lambda(\varphi(t)\psi(x)f), \quad \nabla_x(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7). \tag{3.4.1}$$

Then, for any cutoff function and any $l \in \mathbb{N}$,

$$\Lambda_v^{\lambda+s}(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7).$$

Proof. Recall that $g = \partial^\alpha(\varphi(t)\psi(x)f)$ with $|\alpha| \leqq 5$ and

$$g_{N,l} = P_{N,l}\, g = \psi_1(x) S_N(D_x)\, W_l S_N(D_v) g.$$

Choose $\Lambda_v^{2\lambda} g_{N,l}$ as a test function for equation (3.3.10). Then, one has

$$\left([\Lambda_v^\lambda, v] \cdot \partial_x g_{N,l}, \Lambda_v^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)} = \left(\Lambda_v^\lambda \left\{Q(f_2, g_{N,l}) + G_{N,l}\right\}, \Lambda_v^\lambda g_{N,l}\right)_{L^2(\mathbb{R}^7)}.$$

Since

$$\left[\Lambda_v^{\lambda}, v\right] \cdot \partial_x = \lambda \Lambda_v^{\lambda-2} \partial_v \cdot \partial_x,$$

and $\Lambda_v^{\lambda-2}\partial_v$ are bounded operators in L^2, for any $0 < \lambda < 1$, we have, by using the hypothesis (3.4.1) that

$$\left| \left(\left[\Lambda_v^{\lambda}, v\right] \cdot \partial_x g_{N,l}, \Lambda_v^{\lambda} g_{N,l} \right)_{L^2(\mathbb{R}^7)} \right| \leqq C \| \Lambda_v^{\lambda} g \|_{L^2_l(\mathbb{R}^7)} \| \nabla_x g \|_{L^2_l(\mathbb{R}^7)},$$

and when $1/2 \leqq s < 1$,

$$\begin{aligned}
&\left| \left(\Lambda_v^{\lambda} G_{N,l}, \Lambda_v^{\lambda} g_{N,l} \right)_{L^2(\mathbb{R}^7)} \right| \\
&\quad \leqq C \| f_2 \|_{H^5_7(\mathbb{R}^7)} \| \Lambda_v^{\lambda} g \|_{L^2_{l+\gamma^+ +2s}(\mathbb{R}^7)} \| \Lambda_v^{\lambda+2s-1+\delta} g_{N,l} \|_{L^2(\mathbb{R}^7)} \\
&\quad \leqq \varepsilon \| \Lambda_v^s W_{\gamma/2} \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^7)} + C_\varepsilon \| f_2 \|^2_{H^5_7(\mathbb{R}^7)} \| \Lambda_v^{\lambda} g \|^{2k}_{L^2_{k'l}(\mathbb{R}^7)}.
\end{aligned}$$

By setting $M = \Lambda_v^{\lambda}$ in Proposition 2.9, we have

$$\begin{aligned}
&\left| \left(\Lambda_v^{\lambda} Q(\tilde{f}, g_{N,l}) - Q(\tilde{f}, \Lambda_v^{\lambda} g_{N,l}), \Lambda_v^{\lambda} g_{N,l} \right)_{L^2(\mathbb{R}^7)} \right| \\
&\quad \leqq C \| f_2 \|_{L^\infty(\mathbb{R}^4_{t,x}; L^1_{\gamma+}(\mathbb{R}^3_v))} \\
&\qquad \times \left(\| \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^4_{t,x}; L^2_{\gamma+}(\mathbb{R}^3_v))} + \| g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \right) \| \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \\
&\quad \leqq C \| f_3 \|_{H^5_7(\mathbb{R}^7)} \| \Lambda_v^{\lambda} g \|^2_{L^2_{l+1}(\mathbb{R}^7)},
\end{aligned}$$

when $0 < s < 1/2$. Moreover when $1/2 \leqq s < 1$, we have

$$\begin{aligned}
&\left| \left(\Lambda_v^{\lambda} Q(f_2, g_{N,l}) - Q(f_2, \Lambda_v^{\lambda} g_{N,l}), \Lambda_v^{\lambda} g_{N,l} \right)_{L^2(\mathbb{R}^7)} \right| \\
&\quad \leqq C \| f_2 \|_{L^\infty(\mathbb{R}^4_{t,x}; L^1_{(2s+\gamma-1)_+}(\mathbb{R}^3_v))} \\
&\qquad \times \left(\| \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^4_{t,x}; L^2_{(2s+\gamma-1)_+}(\mathbb{R}^3_v))} + \| g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \right) \| \Lambda_v^{\lambda+2s-1+\delta} g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \\
&\quad \leqq \varepsilon \| \Lambda_v^s W_{\gamma/2} \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^7)} + C_\varepsilon \| f_3 \|^2_{H^5_7(\mathbb{R}^7)} \| \Lambda_v^{\lambda} g \|^{2k}_{L^2_{k'l}(\mathbb{R}^7)}.
\end{aligned}$$

Now the coercivity estimate (2.2.1) gives,

$$\begin{aligned}
-\left(Q(f_2, \Lambda_v^{\lambda} g_{N,l}), \Lambda_v^{\lambda} g_{N,l} \right)_{L^2(\mathbb{R}^7)} &\geqq C_0 \| \Lambda_v^s W_{\gamma/2} \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \\
&\quad - C \| f_2 \|_{L^\infty(\mathbb{R}^4_{t,x}; L^1_{\max\{\gamma^+, 2-\gamma^+\}}(\mathbb{R}^3_v))} \| \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2_{\gamma^+/2}(\mathbb{R}^7)}.
\end{aligned}$$

Thus, Proposition 3.10 is proved by the following estimate

$$\| \Lambda_v^s W_{\gamma/2} \Lambda_v^{\lambda} g_{N,l} \|^2_{L^2(\mathbb{R}^7)} \leqq C \left(\| f_3 \|^2_{H^5_7(\mathbb{R}^7)} + \| \Lambda_v^{\lambda} g \|^{2k}_{L^2_{k'l}(\mathbb{R}^7)} \right), \tag{3.4.2}$$

where C is independent on N. □

We can now conclude the following regularity result with respect to the variable v.

Proposition 3.11. *Under the hypothesis of Theorem 1.1, one has*

$$\Lambda_v^{1+\varepsilon}(\varphi(t)\psi(x)f) \in H_l^5(\mathbb{R}^7), \tag{3.4.3}$$

for any $l \in \mathbb{N}$ *and some* $\varepsilon > 0$.

Again, this result follows by induction. Indeed, notice that there exists $k_0 \in \mathbb{N}$ such that

$$k_0 s < 1, \quad (k_0+1)s = 1+\varepsilon > 1.$$

Then we get (3.4.3) from (3.2.2), Proposition 3.10 with $\lambda = k_0 s$ and (3.4.2), by induction.

High order regularity by iterations From Proposition 3.9 (more precisely (3.3.14)) and Proposition 3.11, we can now deduce that, for any $l \in \mathbb{N}$, and any cutoff functions $\varphi(t)$ and $\psi(x)$,

$$\varphi(t)\psi(x)f \in H_l^6(\mathbb{R}^7).$$

The proof of Theorem 1.1 is then completed by induction.

Indeed, if f is a solution of the Boltzmann equation satisfying the assumptions of Theorem 1.1, then, when $m \geqq 5$, we have

$$f \in \mathcal{H}_l^m(]T_1, T_2[\times\Omega\times\mathbb{R}_v^3), \forall l \in \mathbb{N} \Longrightarrow f \in \mathcal{H}_l^{m+1}(]T_1, T_2[\times\Omega\times\mathbb{R}_v^3), \forall l \in \mathbb{N}.$$

Thus, the full regularity of Theorem 1.1 is obtained by induction from $m = 5$.

4. Existence and uniqueness of local solutions

The local existence of solutions to the spatially inhomogeneous Boltzmann equation without angular cutoff has not been well studied to date. The strategy of proving the existence in this section is to approximate the non-cutoff cross-section by a family of cutoff cross-sections and to approximate the Boltzmann equation by a sequence of iterative linear equations. Then by proving the existence of solutions to these approximate linear equations and by obtaining a uniform estimate on these solutions with respect to the cutoff parameter in some suitable weighted Sobolev space, the compactness will lead to the convergence of the approximate solutions to the desired solution for the original problem. One of the techniques used here is to introduce a transformation defined by the time dependent Maxwellian developed previously in [44]. The purpose of this transformation is to get an extra gain of one order higher weight in the velocity variable at the expense of the loss of the decay in the time dependent Maxwellian. Moreover, the uniqueness of the solution is also proved in some function space.

4.1. Modified Cauchy Problem

By taking $\kappa, \rho > 0$, we set, for $0 \leqq t \leqq T_0 = \rho/(2\kappa)$,

$$\mu_\kappa(t) = \mu(t, v) = e^{-(\rho-\kappa t)(1+|v|^2)},$$

and

$$f = \mu_\kappa(t) g, \quad \Gamma^t(g, g) = \mu_\kappa(t)^{-1} Q(\mu_\kappa(t) g,\ \mu_\kappa(t) g).$$

Then the Cauchy problem (1.4) is reduced to

$$\begin{cases} g_t + v \cdot \nabla_x g \ + \kappa(1+|v|^2) g = \Gamma^t(g, g), \\ g|_{t=0} = g_0. \end{cases} \tag{4.1.1}$$

Our existence theorem can be stated as follows

Theorem 4.1. *Assume that* $0 < s < 1/2$, $\gamma + 2s < 1$ *and* $\kappa, \rho > 0$. *Let* $g_0 \in H^k_l(\mathbb{R}^6)$, $g_0 \geqq 0$ *for some* $l \geqq 3$ *and* $k \geqq 4$. *Then there exists* $T_* \in]0, T_0]$ *such that the Cauchy problem (4.1.1) admits a unique non-negative solution*

$$g \in C^0\left([0, T_*]; H^k_l(\mathbb{R}^6)\right) \bigcap L^2\left(]0, T_*[; H^k_{l+1}(\mathbb{R}^6)\right).$$

We shall prove Theorem 4.1 by cutoff approximations. For simplicity of notation, we will denote $\mu_\kappa(t)$ by $\mu(t)$ without any confusion.

Recall that the cross-section is of the form of $B(|v - v_*|, \cos\theta) = \Phi(|v - v_*|) b(\cos\theta)$ which satisfies (1.2) and (1.3). For $0 < \varepsilon << 1$, we approximate (cut off) the cross-section by

$$b_\varepsilon(\cos\theta) = \begin{cases} b(\cos\theta), & \text{if } |\theta| \geqq 2\varepsilon, \\ b(\cos\varepsilon), & \text{if } |\theta| \leqq 2\varepsilon. \end{cases}$$

Denote by $\Gamma^t_\varepsilon(g,\ g)$ the collision operator corresponding to the above cutoff cross-section $B_\varepsilon = \Phi(v - v_*) b_\varepsilon(\cos\theta)$.

By using the collisional energy conservation,

$$|v'_*|^2 + |v'|^2 = |v_*|^2 + |v|^2,$$

we have $\mu_*(t) = \mu^{-1}(t)\, \mu'_*(t)\, \mu'(t)$. Then for some suitable functions U, V, it holds that

$$\begin{aligned}
&\Gamma^t_\varepsilon(U,\ V)(v) \\
&= \mu^{-1}(t, v) \iint_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma} B_\varepsilon(v - v_*,\ \sigma) \left(\mu'_*(t) U'_* \mu'(t) V' - \mu_*(t) U_* \mu(t) V\right) \mathrm{d}v_*\, \mathrm{d}\sigma \\
&= \iint_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma} B_\varepsilon(v - v_*,\ \sigma) \mu_*(t) \ \left(U'_* V' - U_* V\right) \mathrm{d}v_*\, \mathrm{d}\sigma \\
&= \mathcal{T}_\varepsilon(U,\ V,\ \mu(t)) \\
&= Q_\varepsilon(\mu(t) U,\ V) + \iint_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma} B_\varepsilon(v - v_*,\ \sigma)(\mu_*(t) - \mu'_*(t)) U'_* V' \mathrm{d}v_*\, \mathrm{d}\sigma.
\end{aligned} \tag{4.1.2}$$

Then we have the following formula coming from the Leibniz formula in the x variable and the translation invariance property in the v variable. For any $\alpha, \beta \in \mathbb{N}^3$,

$$\begin{aligned}
&\partial_x^\alpha \partial_v^\beta \Gamma_\varepsilon^t(U, \ V) \\
&= \sum_{\alpha_1+\alpha_2=\alpha;\beta_1+\beta_2+\beta_3=\beta} C_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3} \mathcal{T}_\varepsilon(\partial_x^{\alpha_1}\partial_v^{\beta_1} U, \ \partial_x^{\alpha_2}\partial_v^{\beta_2} V, \partial_v^{\beta_3}\mu(t)) \\
&= Q_\varepsilon(\mu(t)U, \partial_x^\alpha \partial_v^\beta V) \\
&\quad + \iint_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*, \sigma)(\mu_*(t) - \mu'_*(t))U'_*(\partial_x^\alpha\partial_v^\beta V)' dv_* d\sigma \\
&\quad + \sum_{|\alpha_2|+|\beta_2|\leqq|\alpha+\beta|-1} C_{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3} \mathcal{T}_\varepsilon\left(\partial_x^{\alpha_1}\partial_v^{\beta_1} U, \partial_x^{\alpha_2}\partial_v^{\beta_2} V, \partial_v^{\beta_3}\mu(t)\right) \\
&= A_1 + A_2 + A_3. \qquad (4.1.3)
\end{aligned}$$

Firstly, we give the following upper weighted estimate on the nonlinear collision operator with cutoff.

Lemma 4.2. *Let $\gamma \in \mathbb{R}$. Then for any $\varepsilon > 0$, $k \geqq 4$, $l \geqq 0$, there exists $C > 0$ depending on ε, k, l such that for any U, V belonging to $H^k_l(\mathbb{R}^6)$*

$$\|\Gamma_\varepsilon^t(U, V)\|_{H^k_l(\mathbb{R}^6)} \leqq C\|U\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}\|V\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \quad 0 \leqq t \leqq T_0 = \frac{\rho}{2\kappa}. \qquad (4.1.4)$$

Proof. To prove (4.1.4), put

$$g_1 = \partial_x^{\alpha_1}\partial_v^{\beta_1} U, \quad h_2 = \partial_x^{\alpha_2}\partial_v^{\beta_2} V, \quad \mu_3(t) = \partial_v^{\beta_3}\mu(t),$$
$$\mathcal{T}_\varepsilon(g_1, h_2, \mu_3(t)) = \mathcal{T}_\varepsilon^+ - \mathcal{T}_\varepsilon^-.$$

Throughout this section, the estimates

$$\mu(t, v), \quad |\mu_3(t)| = |\partial_v^{\beta_3}\mu(t, v)| \leqq C_{\rho,k} e^{-\rho\langle v\rangle^2/4}, \quad t \in [0, T_0], \quad v \in \mathbb{R}^3,$$

will often be used.

Firstly, we compute $\mathcal{T}_\varepsilon^+$ as follows.

$$\begin{aligned}
|W_l \mathcal{T}_\varepsilon^+| &\leqq C \iint \langle |v - v_*|\rangle^\gamma |\mu_3(t, v_*)| \frac{W_l}{(W_l)'_*(W_l)'} |(W_l g_1)'_*||(W_l h_2)'| \mathrm{d}v_* \,\mathrm{d}\sigma \\
&\leqq C \left[\iint \left|\mu_3(t, v_*)\frac{W_l}{(W_l)'_*(W_l)'}\right|^2 \mathrm{d}v_* \,\mathrm{d}\sigma\right]^{1/2} \\
&\quad \times \left[\iint \langle v' - v'_*\rangle^{2\gamma} |(W_l g_1)'_*(W_l h_2)'|^2 \mathrm{d}v_* \,\mathrm{d}\sigma\right]^{1/2} \\
&\leqq C_\varepsilon \left[\iint |(W_{l+\gamma^+} g_1)'_*(W_{l+\gamma^+} h_2)'|^2 \mathrm{d}v_* \,\mathrm{d}\sigma\right]^{1/2},
\end{aligned}$$

where we have used $|v - v_*| = |v' - v'_*|$ and $\frac{W_l}{(W_l)'_*(W_l)'} \leqq 1$. Since the change of variables

$$(v, v_*, \sigma) \to (v', v'_*, \sigma'), \quad \sigma' = (v - v_*)/|v - v_*|, \tag{4.1.5}$$

has a unit Jacobian, we get

$$\begin{aligned}\|W_l \mathcal{T}_\varepsilon^+\|^2_{L^2(\mathbb{R}^6)} &\leqq C \iiiint |(W_{l+\gamma^+} g_1)'_*(W_{l+\gamma^+} h_2)'|^2 \mathrm{d}v_* \,\mathrm{d}\sigma \,\mathrm{d}v \,\mathrm{d}x \\ &= C \iiiint |(W_{l+\gamma^+} g_1)'_*(W_{l+\gamma^+} h_2)'|^2 \mathrm{d}v'_* \,\mathrm{d}\sigma' \,\mathrm{d}v' \,\mathrm{d}x \\ &\leqq C \int \|(W_{l+\gamma^+} g_1)\|^2_{L^2(\mathbb{R}^3_v)} \|(W_{l+\gamma^+} h_2)\|^2_{L^2(\mathbb{R}^3_v)} \mathrm{d}x.\end{aligned}$$

If $|\alpha_1 + \beta_1| \leqq k/2$, then we have

$$\begin{aligned}\|W_l \mathcal{T}_\varepsilon^+\|_{L^2(\mathbb{R}^6)} &\leqq C \|(W_{l+\gamma^+} g_1)\|_{L^\infty(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))} \|(W_{l+\gamma^+} h_2)\|_{L^2(\mathbb{R}^6_{x,v})} \\ &\leqq C \|U\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)} \|V\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)},\end{aligned}$$

because of the Sobolev embedding theorem and the fact $k/2 + 3/2 < k$ when $k \geqq 4$. When $|\alpha_2 + \beta_2| \leqq k/2$, the proof is similar. This completes the proof of the lemma. □

4.2. Cutoff approximations

We now study the following Cauchy problem for the cutoff Boltzmann equation

$$\begin{cases} g_t + v \cdot \nabla_x g + \kappa \langle v \rangle^2 g = \Gamma^t_\varepsilon(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{4.2.1}$$

for which we shall obtain uniform estimates in weighted Sobolev spaces.

We first prove the existence of weak solutions to this cutoff Boltzmann equation.

Theorem 4.3. *Assume that $\gamma \leqq 1$. Let $k \geqq 4$, $l \geqq 0$, $\varepsilon > 0$ and $D_0 > 0$. Then, there exists $T_\varepsilon \in]0, T_0]$ such that for any non-negative initial data g_0 satisfying*

$$g_0 \in H^k_l(\mathbb{R}^6), \quad \|g_0\|_{H^k_l(\mathbb{R}^6)} \leqq D_0,$$

the Cauchy problem (4.2.1) admits a unique non-negative solution g^ε having the property

$$g^\varepsilon \in C^0(]0, T_\varepsilon[; H^k_l(\mathbb{R}^6)), \quad \|g^\varepsilon\|_{L^\infty(]0,T_\varepsilon[; H^k_l(\mathbb{R}^6))} \leqq 2D_0.$$

Moreover, this solution enjoys a moment gain in the sense that

$$g^\varepsilon \in L^2\left(]0, T_\varepsilon[; H^k_{l+1}(\mathbb{R}^6)\right). \tag{4.2.2}$$

Remark 4.4. (1) Notice that we do not assume $g_0 \in H^k_{l+1}(\mathbb{R}^6)$ and the gain of the moment will be essentially used below in the proof of uniform estimates to compensate for the singularity in the cross-section.

(2) The regularity of g^ε with respect to the t variable follows directly from the equation (4.2.1).
(3) Fix γ, k, l as in the theorem. Then T_ε is a function of ε and D_0. In the following, when we need to emphasize this dependency, we shall write

$$T = T_\varepsilon(D_0).$$

(4) If $\gamma \leqq 0$, we may take $\kappa = 0$. In this case, we do not have the moment gain (4.2.2), which is not needed anyway.

Proof of Theorem 4.3. We prove the existence of non-negative solutions by successive approximation that preserves non-negativity, which is defined by using the usual splitting of the collision operator (4.1.2) into the gain (+) and loss (−) terms,

$$\begin{aligned}\Gamma_\varepsilon^{t,+}(g,h) &= \iint_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*,\sigma)\mu_*(t)g'_*h'\mathrm{d}v_*\,\mathrm{d}\sigma,\\ \Gamma_\varepsilon^{t,-}(g,h) &= hL_\varepsilon(g),\\ L_\varepsilon(g) &= \iint_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*,\sigma)\mu(t,v_*)g_*\mathrm{d}v_*\,\mathrm{d}\sigma.\end{aligned}$$

Evidently, Lemma 4.2 applies to $\Gamma_\varepsilon^{t,\pm}$, and in view of (1.2), the linear operator L_ε satisfies

$$|\partial_x^\alpha\partial_v^\beta L_\varepsilon(g)(t,x,v)| \leqq C\langle v\rangle^{\gamma-|\beta|}\|\partial_x^\alpha g\|_{L^2(\mathbb{R}^3_v)},\quad t\in[0,T_0], \tag{4.2.3}$$

for a constant $C>0$ depending on ε, because $|\mu(t,v_*)\partial_v^\beta\langle v-v_*\rangle^\gamma| \leqq C\langle v\rangle^{\gamma-|\beta|}$.

We now define a sequence of approximate solutions $\{g^n\}_{n\in\mathbb{N}}$ by

$$\begin{cases} g^0 = g_0\,;\\ \partial_t g^{n+1} + v\cdot\nabla_x g^{n+1} + \kappa\langle|v|\rangle^2 g^{n+1}\\ \qquad\qquad = \Gamma_\varepsilon^{t,+}(g^n,\ g^n) - \Gamma_\varepsilon^{t,-}(g^n,\ g^{n+1}),\\ g^{n+1}|_{t=0} = g_0.\end{cases} \tag{4.2.4}$$

Actually, in view of (4.2.3) we consider the mild form

$$\begin{aligned} g^{n+1}(t,x,v) &= e^{-\kappa\langle|v|\rangle^2 t - V^n(t,0)}g_0(x-tv,v)\\ &\quad + \int_0^t e^{-\kappa\langle|v|\rangle^2(t-s)-V^n(t,s)}\Gamma_\varepsilon^{s,+}(g^n,\ g^n)(s,x-(t-s)v,v)\mathrm{d}s,\end{aligned} \tag{4.2.5}$$

where

$$V^n(t,s) = \int_s^t L_\varepsilon(g^n)(s,x-(t-s)v,v)\mathrm{d}s.$$

First, we note from Lemma 4.2 that for any $T\in]0,T_0]$, $T_0 = \rho/(2\kappa)$, $g_0 \geqq 0$, and

$$g^n \in L^\infty\Big(]0,T[;\,H_l^k(\mathbb{R}^6)\Big),\quad g^n \geqq 0,$$

the mild form (4.2.5) determines g^{n+1} in the function class

$$g^{n+1} \in L^\infty\left(]0,T[;\, H^k_{l-\gamma^+}(\mathbb{R}^6)\right), \quad g^{n+1} \geqq 0, \tag{4.2.6}$$

and solves (4.2.4). Thus g^{n+1} exists and is non-negative, but appears to have a loss of weight in the velocity variable. We shall now show that the term $\kappa\langle v\rangle^2 g^{n+1}$ in (4.2.4) not only recovers this weight loss but also creates a higher moment. More precisely, we have the following lemma. Introduce the space and norm by

$$X = L^\infty\left(]0,T[;\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0,T[;\, H^k_{l+1}(\mathbb{R}^6)\right),$$
$$|||g|||^2 = \|g\|^2_{L^\infty(]0,T[;\,H^k_l(\mathbb{R}^6))} + \kappa\|g\|^2_{L^2(]0,T[;\,H^k_{l+1}(\mathbb{R}^6))}.$$

This norm depends on k, l, T, κ, but we omit this dependence in the notation for simplicity. □

Lemma 4.5. *Assume that $\gamma \leqq 1$ and let $k \geqq 4, l \geqq 0, \varepsilon > 0$. Then, there exist positive numbers C_1, C_2 such that if $\rho > 0,\ \kappa > 0$ and if*

$$g_0 \in H^k_l(\mathbb{R}^6),\ g^n \in L^\infty(]0,T[;\, H^k_l(\mathbb{R}^6)), \tag{4.2.7}$$

with some $T \leqq T_0$, the function g^{n+1} given by (4.2.5) enjoys the properties

$$g^{n+1} \in X,$$
$$|||g^{n+1}|||^2 \leqq e^{C_1 K_n T}\left(\|g_0\|^2_{H^k_l(\mathbb{R}^6)} + \frac{C_2}{\kappa}||g^n||^4_{L^4(]0,T[;\ H^k_l(\mathbb{R}^6))}\right), \tag{4.2.8}$$

where K_n is a positive constant depending on $\|g^n\|_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))}$ and κ.

Proof. Put

$$h^n = h^n_\alpha = \partial^\alpha g^n.$$

Differentiation of equation (4.2.4) yields

$$\partial_t h^{n+1} + v\cdot\nabla_x h^{n+1} + \kappa\langle v\rangle^2 h^{n+1} = G_1^+ - G_1^- + G_2 + G_3,$$
$$G_1^+ = \partial^\alpha \Gamma^{t,+}_\varepsilon(g^n, g^n), \qquad G_1^- = \partial^\alpha \Gamma^{t,-}_\varepsilon(g^n, g^{n+1}),$$
$$G_2 = -[\partial^\alpha, v\cdot\nabla_x]g^{n+1},$$
$$G_3 = -\kappa \sum_{|\tilde\beta|=1,2} C_{\tilde\beta}\partial_v^{\tilde\beta}\langle v\rangle^2 \partial^{\alpha-(0,\tilde\beta)} g^{n+1}.$$

Let $\chi_j \in C_0^\infty(\mathbb{R}^3)$, $j \in \mathbb{N}$, be the cutoff function

$$\chi_j(v) = \begin{cases} 1, & |v| \leqq j, \\ 0, & |v| \geqq j+1. \end{cases}$$

We remark that (4.2.6) does not necessarily imply $W_{l+1}h^{n+1}(t) \in L^2(\mathbb{R}^6)$, but $\chi_j W_{l+1}h^{n+1}(t) \in L^2(\mathbb{R}^6)$ for all $j \in \mathbb{N}$. Hence, we can use $\chi_j^2 W_l^2 S_N^2(D_x)h^{n+1}$ as a test function to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|S_N(D_x)\chi_j W_l h^{n+1}\|^2 + \kappa\|S_N(D_x)\chi_j W_{l+1}h^{n+1}\|^2 \\ = \left(G_1^+ - G_1^- + G_2 + G_3, S_N(D_x)^2\chi_j^2 W_l^2 h^{n+1}\right). \qquad (4.2.9)\end{aligned}$$

Here, and in what follows, the norm $\| \ \|$ and inner product $(\ , \ )$ are those of $L^2(\mathbb{R}^6_{x,v})$ unless otherwise stated. We shall evaluate the inner products on the right-hand side. Observe that Lemma 4.2 gives, for $t \in [0, T]$,

$$\begin{aligned}\left|(G_1^+, S_N^2\chi_j^2 W_l^2 h^{n+1})\right| &= \left|\left(S_N\chi_j W_{l-1}G_1^+, S_N\chi_j W_{l+1}h^{n+1}\right)\right| \\ &\leqq C\|W_{l-1}G_1^+\|\,\|S_N\chi_j W_{l+1}h^{n+1}\| \\ &\leqq C\|\Gamma_\varepsilon^{t,+}(g^n, g^n)\|_{H^k_{l-1}(\mathbb{R}^6)}\|S_N\chi_j W_{l+1}h^{n+1}\| \\ &\leqq C\|g^n\|^2_{H^k_l(\mathbb{R}^6)}\|\,\|S_N\chi_j W_{l+1}h^{n+1}\| \\ &\leqq \frac{C}{\kappa}\|g^n\|^4_{H^k_l(\mathbb{R}^6)} + \frac{\kappa}{4}\|S_N\chi_j W_{l+1}h^{n+1}\|^2.\end{aligned}$$

On the other hand, Lemma 4.2 is not enough to evaluate G_1^- because G_1^- contains g^{n+1} which is not known, at this point, to have moments required by Lemma 4.2. However, this obstacle is only superficial. Observe that

$$G_1^- = \sum_{(\alpha_1,\beta_1)+\alpha_2=\alpha} C_{\alpha_1,\beta_1,\alpha_2}\left(\partial^{\alpha_2}g^{n+1}\right)\left(\partial_v^{\beta_1}L(\partial_x^{\alpha_1}g^n)\right).$$

Define,

$$H_{j,l}(g) = \sum_{|\alpha|\leqq k}\|\chi_j W_l\partial^\alpha g\|^2,$$

and write $H^n_{j,l} = H^n_{j,l}(t) = H_{j,l}(g_n(t))$. By recalling (4.2.3), we get

$$\begin{aligned}\left|\left(G_1^-, S_N^2\chi_j^2 W_l^2 h^{n+1}\right)\right| &\leqq \sum_{(\alpha_1,\beta_1)+\alpha_2=\alpha} C_{\alpha_1,\beta_1,\alpha_2}\|\chi_j\langle v\rangle^{\gamma-|\beta_1|}W_{l-1}\partial^{\alpha_2}g^{n+1}\| \\ &\quad\times\|\partial_x^{\alpha_1}g^n\|\|S_N\chi_j W_{l+1}h^{n+1}\| \\ &\leqq C\|g^n\|_{H^k_l(\mathbb{R}^6)}\|(H^{n+1}_{j,l})^{1/2}\|S_N\chi_j W_{l+1}h^{n+1}\| \\ &\leqq \frac{C'}{\kappa}\|g^n\|^2_{H^k_l(\mathbb{R}^6)}H^{n+1}_{j,l} + \frac{\kappa}{4}\|S_N\chi_j W_{l+1}h^{n+1}\|^2.\end{aligned}$$

Here C, C' are positive constants independent of κ.

The estimate on the remaining two inner products are more straightforward and can be given as follows.

$$\begin{aligned}\left|(G_2+G_3,\ S_N^2\chi_j^2W_l^2h^{n+1})\right| &\leqq C\|\chi_jW_{l-1}(G_2+G_3)\|\,\|S_N\chi_jW_{l+1}h^{n+1}\|\\ &\leqq C(\kappa+1)\left(H_{j,l}^{n+1}\right)^{1/2}\|S_N\chi_jW_{l+1}h^{n+1}\|\\ &\leqq C''\frac{(\kappa+1)^2}{\kappa}H_{j,l}^{n+1}+\frac{\kappa}{4}\|S_N\chi_jW_{l+1}h^{n+1}\|^2.\end{aligned}$$

The constants C, C'' are independent of ε and κ.

Putting together all the estimates obtained above in (4.2.9) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|S_N\chi_jW_lh^{n+1}\|^2+\frac{\kappa}{4}\|S_N\chi_jW_{l+1}h^{n+1}\|^2\\ &\quad\leqq C'''\left\{\kappa+\frac{1}{\kappa}\left(1+\|g^n\|^2_{H_l^k(\mathbb{R}^6)}\right)\right\}H_{j,l}^{n+1}+\frac{C}{\kappa}\|g^n\|^4_{H_l^k(\mathbb{R}^6)}.\end{aligned}$$

Summing up estimates for $h^{n+1}=h_\alpha^{n+1}$ over $|\alpha|\leqq k$ then yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{j,l}(S_Ng^{n+1})+\kappa H_{j,l+1}(S_Ng^{n+1})\leqq C_1K_nH_{j,l}(g^{n+1})+\frac{C_2}{\kappa}\|g^n\|^4_{H_l^k(\mathbb{R}^6)},$$

where

$$K_n=\kappa+\frac{1}{\kappa}\left(\|g^n\|^2_{L^\infty(]0,T[;H_l^k(\mathbb{R}^6))}+1\right),$$

and $C_1>0$ is a constant independent of ε,κ while C_2 is independent of κ but depends on ε. By integrating the above estimate over $[0,t]$ and taking the limit $N\to\infty$, we get

$$\begin{aligned}&H_{j,l}^{n+1}(t)+\kappa\int_0^tH_{j,l+1}^{n+1}(\tau)\mathrm{d}\tau\\ &\quad\leqq H_{j,l}^{n+1}(0)+C_1K_n\int_0^tH_{j,l}^{n+1}(\tau)d\tau+\frac{C_2}{\kappa}\int_0^t\|g^n(\tau)\|^4_{H_l^k(\mathbb{R}^6)}\mathrm{d}\tau,\quad t\in[0,T],\end{aligned}$$

which gives a Gronwall type inequality

$$\begin{aligned}&H_{j,l}^{n+1}(t)+\kappa\int_0^te^{C_1K_n(t-\tau)}H_{j,l+1}^{n+1}(\tau)\mathrm{d}\tau\\ &\quad\leqq e^{C_1K_nt}H_{j,l}^{n+1}(0)+\frac{C_2}{\kappa}\int_0^te^{C_1K_n(t-\tau)}\|g^n(\tau)\|^4_{H_l^k(\mathbb{R}^6)}\mathrm{d}\tau,\end{aligned}\tag{4.2.10}$$

for all $t\in[0,T]$ and $j\in\mathbb{N}$. Since

$$H_{j,l}^{n+1}(0)\leqq\|g_0\|^2_{H_l^k},$$

and $1\leqq e^{C_1K_n(t-\tau)}\leqq e^{C_1K_nt}$, (4.2.10) gives, for $t\in[0,T]$,

$$H_{j,l}^{n+1}(t)+\kappa\int_0^tH_{j,l+1}^{n+1}(\tau)\mathrm{d}\tau\leqq e^{C_1K_nt}\left\{\|g_0\|^2_{H_l^k}+\frac{C_2}{\kappa}\int_0^t\|g^n(\tau)\|^4_{H_l^k(\mathbb{R}^6)}\mathrm{d}\tau\right\}.$$

Since the right-hand side is independent of j, $\{\chi_j \partial^\alpha g^{n+1}\}_{j\in\mathbb{N}}$, $|\alpha| \leqq k$ is weakly* compact in $L^\infty(]0, T[; L^2_l(\mathbb{R}^6))$ and weakly compact in $L^2(]0, T[; L^2_{l+1}(\mathbb{R}^6))$. Take a convergent subsequence. Apparently, its limit is $h^{n+1}(t)$. This is true for all $|\alpha| \leqq k$ so that we can now conclude that

$$g^{n+1} \in X = L^\infty\left(]0, T[; H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T[; H^k_{l+1}(\mathbb{R}^6)\right),$$

and by Fatou's theorem,

$$\begin{aligned}|||g^{n+1}|||^2 &\leqq \liminf_{j\to\infty} \|H^{n+1}_{j,l}\|_{L^\infty(]0,T[)} + \kappa \liminf_{j\to\infty} \|H^{n+1}_{j,l+1}\|_{L^1(]0,T[)} \\ &\leqq e^{C_1 K_n T}\left(\|g_0\|^2_{H^k_l} + \frac{C_2}{\kappa}\|g^n\|^4_{L^4(]0,T[;H^k_l(\mathbb{R}^6))}\right).\end{aligned}$$

Now the proof of Lemma 4.5 is completed. □

We are now ready to prove the convergence of $\{g^n\}_{n\in\mathbb{N}}$. Fix $\kappa > 0$, let D_0, g_0 be as in Theorem 4.3 and introduce an induction hypothesis

$$\|g^n\|_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))} \leqq 2D_0. \tag{4.2.11}$$

for some $T \in]0, T_0]$. Notice that the factor 2 can be any number > 1.

(4.2.11) is true for $n = 0$ due to (4.2.7). Suppose that this is true for some $n > 0$. We shall determine T independent of n. A possible choice is given by

$$e^{C_1 K_0 T} = 2, \quad \frac{2^4 C_2}{\kappa} T D_0^2 = 1 \quad \text{where } K_0 = \kappa + \frac{2D_0 + 1}{\kappa}, \tag{4.2.12}$$

or

$$T = \min\left\{\frac{\log 2}{C_1 K_0}, \frac{\kappa}{2^4 C_2 D_0^2}\right\}.$$

In fact, (4.2.8) and (4.2.11) yield that $g^{n+1} \in X$ and

$$\begin{aligned}|||g^{n+1}|||^2 &\leqq e^{C_1 K_0 T}\left(\|g_0\|^2_{H^k_l(\mathbb{R}^6)} + \frac{C_2}{\kappa} T ||g^n||^4_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))}\right) \\ &\leqq e^{C_1 K_0 T}\left(D_0^2 + \frac{C_2}{\kappa} T 2^4 D_0^4\right) \leqq 4D_0^2.\end{aligned}$$

That is, the induction hypothesis (4.2.11) is fulfilled for $n + 1$, and hence holds for all n.

For the convergence, set $w^n = g^n(t) - g^{n-1}(t)$, for which (4.2.4) leads to

$$\begin{cases} \partial_t w^{n+1} + v \cdot \nabla_x w^{n+1} + \kappa \langle |v| \rangle^2 w^{n+1} = \Gamma^{t,+}_\varepsilon(w^n, g^n) + \Gamma^{t,+}_\varepsilon(g^{n-1}, w^n) \\ \quad -\Gamma^{t,-}_\varepsilon(w^n, g^{n+1}) - \Gamma^{t,-}_\varepsilon(g^{n-1}, w^{n+1}), \\ w^{n+1}|_{t=0} = 0. \end{cases}$$

By the same computation as used for (4.2.9), but more directly since we can now use test functions as $S_N(D_x)^2 W_l^2 \partial^\alpha w^{n+1}$, we get

$$|||w^{n+1}|||^2 \leqq \frac{1}{2} C_2 e^{C_1 K_0 T} \frac{1}{\kappa} T \left\{ \|g^{n+1}\|^2_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))} + \|g^n\|^2_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))} + \|g^{n-1}\|^2_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))} \right\} \|w^n\|^2_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))},$$

with the same constants C_1, C_2 and K_0 as above. Then, (4.2.11) and (4.2.12) give

$$|||g^{n+1} - g^n|||^2 \leqq 2^4 C_2 D_0^2 \kappa^{-1} T \|g^n - g^{n-1}\|^2_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))}.$$

Finally, choose T smaller,if necessary, so that

$$2^4 C_2 D_0^2 \kappa^{-1} T \leqq \frac{1}{4}.$$

Then, we have proved that for any $n \geqq 1$,

$$|||g^{n+1} - g^n||| \leqq \frac{1}{2}\ |||g^n - g^{n-1}|||. \tag{4.2.13}$$

Consequently, $\{g^n\}$ is a convergence sequence in X, and the limit

$$g^\varepsilon \in X,$$

is therefore a non-negative solution of the Cauchy problem (4.2.1). The estimate (4.2.13) also implies the uniqueness of solutions.

By means of the mild form (4.2.5), it can be proved also that for each n,

$$g^n \in C^0\left([0, T];\ H^k_l(\mathbb{R}^6)\right)$$

and hence so is the limit g^ε. The non-negativity of g^ε follows because $g^n \geqq 0$. Now the proof of Theorem 4.3 is completed.

4.3. Uniform estimate

We now prove the existence of solutions for the Cauchy problem (4.1.1) by the convergence of the approximation sequence $\{g^\varepsilon\}$ as $\varepsilon \to 0$. The first step is to prove the uniform boundedness of this approximation sequence. Below, the constant C represents various constants independent of $\varepsilon > 0$.

Theorem 4.6. *Assume that* $0 < s < 1/2,\ \gamma + 2s < 1$. *Let* $g_0 \in H^k_l(\mathbb{R}^6)$, $g_0 \geqq 0$ *for some* $k \geqq 4,\ \ l \geqq 3$. *Then there exists* $T_* \in]0, T_0]$ *depending only on* $\|g_0\|_{H^k_l}$ *and independent of* ε *satisfying the following property: If for some* $0 < T \leqq T_0$,

$$g^\varepsilon \in C^0\left(]0, T];\ H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T[;\ H^k_{l+1}(\mathbb{R}^6)\right), \tag{4.3.1}$$

is a non-negative solution of the Cauchy problem (4.2.1) and if $T_{**} = \min\{T, T_*\}$, *then it holds that*

$$\|g^\varepsilon\|_{L^\infty(]0,T_{**}[;H^k_l(\mathbb{R}^6))} \leqq 2\|g_0\|_{H^k_l(\mathbb{R}^6)}. \tag{4.3.2}$$

Remark 4.7. The case $T_* \leqq T$ gives a uniform estimate of local solutions on the fixed time interval $[0, T_*]$ while the case $T < T_*$ gives an a priori estimate on the existence time interval $[0, T]$ of local solutions. The latter is used for the continuation argument of local solutions, in Section 4.4.

In the following, $\rho > 0, \kappa > 0$ are fixed. Furthermore, recall $T_0 = \rho/(2\kappa)$. We start with a solution g^ε subject to (4.3.1) for some $T \in]0, T_0]$. For $\alpha \in \mathbb{N}^6$, $|\alpha| \leqq k$, the differentiation of the equation (4.2.1) implies

$$\begin{aligned} &\partial_t(\partial^\alpha g^\varepsilon) + v \cdot \nabla_x(\partial^\alpha g^\varepsilon) + \kappa \langle v\rangle^2(\partial^\alpha g^\varepsilon) \\ &\quad = \partial^\alpha \Gamma^t_\varepsilon(g^\varepsilon, g^\varepsilon) - [\partial^\alpha, v\cdot\nabla_x]g^\varepsilon - \kappa[\partial^\alpha, \langle v\rangle^2]g^\varepsilon. \end{aligned} \tag{4.3.3}$$

Since $\partial^\alpha g^\varepsilon$ only belongs to L^2_l, now as in Section 3, we take,

$$P^\star_{N,l} P_{N,l}(\partial^\alpha g^\varepsilon)$$

as a test function in (4.3.3), where $l \geqq 3$ and $P_{N,l} = S_N(D_x)S_N(D_v)W_l$ (we do not need the cutoff functions φ, ψ here). Then we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|P_{N,l}(\partial^\alpha g^\varepsilon)(t)\|^2_{L^2(\mathbb{R}^6)} + \kappa\|W_1\, P_{N,l}(\partial^\alpha g^\varepsilon)(t)\|^2_{L^2(\mathbb{R}^6)} \\ &\qquad + \kappa\Big([S_N(D_v), \langle v\rangle^2]W_l\,(\partial^\alpha g^\varepsilon), S_N(D_x)P_{N,l}(\partial^\alpha g^\varepsilon)\Big)_{L^2(\mathbb{R}^6)} \\ &\quad = \big(A_1 + A_2 + A_3 + A_4 + A_5, P^\star_{N,l}P_{N,l}(\partial^\alpha g^\varepsilon)\big)_{L^2(\mathbb{R}^6)}, \end{aligned} \tag{4.3.4}$$

where A_1, A_2, A_3 are defined in (4.1.3) with $U = V = g$ and

$$A_4 = -[\partial^\alpha, v\cdot\nabla_x]g^\varepsilon, \quad A_5 = -\kappa \sum_{|\tilde\beta|=1,2} C_{\tilde\beta}\partial_v^{\tilde\beta}\langle v\rangle^2 \partial^{\alpha-(0,\tilde\beta)} g^\varepsilon.$$

We have firstly,

$$\Big|\big(A_4,\ P^\star_{N,l}P_{N,l}(\partial^\alpha g^\varepsilon)\big)_{L^2(\mathbb{R}^6)}\Big| \leqq C\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)}, \tag{4.3.5}$$

and

$$\Big|\big(A_5,\ P^\star_{N,l}P_{N,l}(\partial^\alpha g^\varepsilon)\big)_{L^2(\mathbb{R}^6)}\Big| \leqq C\kappa\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} + \frac{\kappa}{4}\|g^\varepsilon(t)\|^2_{H^k_{l+1}(\mathbb{R}^6)}. \tag{4.3.6}$$

We also have

$$\begin{aligned} &\left|\kappa\Big([S_N(D_v),\ \langle v\rangle^2]W_l\,(\partial^\alpha g^\varepsilon),\ S_N(D_x)P_{N,l}(\partial^\alpha g^\varepsilon)\Big)_{L^2(\mathbb{R}^6)}\right| \\ &\quad \leqq C\kappa\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} + \frac{\kappa}{4}\|g^\varepsilon(t)\|^2_{H^k_{l+1}(\mathbb{R}^6)}. \end{aligned} \tag{4.3.7}$$

We now study the term A_1 by using the non-negativity of g^ε and the coercivity of collision operators.

Proposition 4.8. *Assume that* $0 < s < 1/2$, $\gamma \in \mathbb{R}$. *There exists* $C > 0$ *independent of* ε *such that for any* $\alpha \in \mathbb{N}^6$, $|\alpha| \leqq k$, $k \geqq 4$, $l \geqq 3$,

$$\left(A_1,\ P^{\star}_{N,l} P_{N,l}(\partial^\alpha g^\varepsilon)\right)_{L^2(\mathbb{R}^6)} \leqq C\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} \|g^\varepsilon(t)\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \tag{4.3.8}$$

for any $0 \leqq t \leqq T \leqq T_0$.

Proof. By setting $h = \partial^\alpha g^\varepsilon$, we have,

$$\begin{aligned}
&\left(A_1,\, P^{\star}_{N,l} P_{N,l} h\right)_{L^2(\mathbb{R}^6)}\\
&\quad = \left(P_{N,l} Q_\varepsilon(\mu g^\varepsilon,\ h),\ \ (P_{N,l} h)\right)_{L^2(\mathbb{R}^6)}\\
&\quad = \left(Q_\varepsilon\left(\mu(t) g^\varepsilon,\ (P_{N,l} h)\right),\, (P_{N,l} h)\right)_{L^2(\mathbb{R}^6)}\\
&\qquad + \left(P_{N,l} Q_\varepsilon\left(\mu(t) g^\varepsilon, h\right) - Q_\varepsilon\left(\mu(t) g^\varepsilon, (P_{N,l} h)\right),\, (P_{N,l} h)\right)_{L^2(\mathbb{R}^6)}\\
&\quad = B_1 + B_2.
\end{aligned}$$

Since $\mu(t)\, g^\varepsilon(t,\, x,\, v) \geqq 0$, we have, in the same way as Theorem 2.6 with the cancellation lemma,

$$\begin{aligned}
B_1 &= -\frac{1}{2} \iiiint_{\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma} B_\varepsilon\left(\mu(t)\, g^\varepsilon\right)_* \left((P_{N,l} h)' - (P_{N,l} h)\right)^2 \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x\\
&\quad + \frac{1}{2} \iiiint_{\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma} B_\varepsilon\left(\mu(t)\, g^\varepsilon\right)_* \left\{\left((P_{N,l} h)'\right)^2 - \left(P_{N,l} h\right)^2\right\} \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x\\
&\leqq \frac{1}{2} \iiiint_{\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma} B_\varepsilon\left(\mu(t)\, g^\varepsilon\right)_* \left\{\left((P_{N,l} h)'\right)^2 - \left(P_{N,l} h\right)^2\right\} \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x\\
&\leqq C \iiint_{\mathbb{R}^3_x \times \mathbb{R}^3_{v_*} \times \mathbb{R}^3_v} \left(\mu(t)\, g^\varepsilon\right)_* \langle v - v_*\rangle^{\gamma^+} (P_{N,l} h)^2 \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}x\\
&\leqq C\|\mu W_{\gamma^+} g^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3_x; L^1(\mathbb{R}^3_v))} \|W_l h(t)\|_{L^2(\mathbb{R}^6_{x,v})} \|W_{l+\gamma^+} h(t)\|_{L^2(\mathbb{R}^6_{x,v})}\\
&\leqq C\|g^\varepsilon(t)\|_{H^{3/2+\delta}(\mathbb{R}^6_{x,v})} \|g^\varepsilon(t)\|_{H^k_l(\mathbb{R}^6_{x,v})} \|g^\varepsilon(t)\|_{H^k_{l+\gamma^+}(\mathbb{R}^6_{x,v})}, \quad t \in [0,\ T],
\end{aligned}$$

where $B_\varepsilon = B_\varepsilon(v - v_*,\ \sigma)$ and we used the fact that $b_\varepsilon(\cos\theta) \leqq b(\cos\theta)$.

By putting $S_N = S_N(D_x)$ $\tilde{S}_N = S_N(D_v)$, we decompose

$$\begin{aligned}
B_2 &= \Big(S_N \tilde{S}_N \left\{W_l Q_\varepsilon\left(\mu(t) g^\varepsilon,\ h\right) - Q_\varepsilon\left(\mu(t) g^\varepsilon,\ (W_l h)\right)\right\},\ \ (P_{N,l} h)\Big)_{L^2(\mathbb{R}^6)}\\
&\quad + \Big(S_N \Big\{\tilde{S}_N Q_\varepsilon\left(\mu(t) g^\varepsilon,\ (W_l h)\right) - Q_\varepsilon\Big(\mu(t) g^\varepsilon,\ \tilde{S}_N(W_l h)\Big)\Big\},\ \ (P_{N,l} h)\Big)_{L^2(\mathbb{R}^6)}\\
&\quad + \Big(S_N Q_\varepsilon\Big(\mu(t) g^\varepsilon, (\tilde{S}_N W_l h)\Big) - Q_\varepsilon\Big(\mu(t) g^\varepsilon,\ S_N(\tilde{S}_N W_l h)\Big),\ \ (P_{N,l} h)\Big)_{L^2(\mathbb{R}^6)}\\
&= B_{21} + B_{22} + B_{23}.
\end{aligned}$$

By Lemma 2.4, we get

$$
\begin{aligned}
|B_{21}| &= \left| \Big(\big\{ W_l Q_\varepsilon \left(\mu(t) g^\varepsilon,\ h\right) - Q_\varepsilon \left(\mu(t) g^\varepsilon,\ (W_l h)\right) \big\},\ (\tilde{S}_N S_N P_{N,\,l} h) \Big)_{L^2(\mathbb{R}^6)} \right| \\
&\leqq C \| \mu(t) g^\varepsilon(t) \|_{L^\infty(\mathbb{R}^3_x; L^1_{l+\gamma^+}(\mathbb{R}^3_v))} \int_{\mathbb{R}^3_x} \| W_{l+\gamma^+} h \|_{L^2(\mathbb{R}^3_v)} \| P_{N,\,l} h \|_{L^2(\mathbb{R}^3_v)} \mathrm{d}x \\
&\leqq C \| g^\varepsilon(t) \|_{L^\infty(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))} \| g^\varepsilon(t) \|_{H^k_l(\mathbb{R}^6)} \| g^\varepsilon(t) \|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \\
&\leqq C \| g^\varepsilon(t) \|^2_{H^k_l(\mathbb{R}^6)} \| g^\varepsilon(t) \|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \quad t \in [0,\ T].
\end{aligned}
$$

It follows from Lemma 3.3 that

$$
\begin{aligned}
|B_{22}| &\leqq \left(\int_{\mathbb{R}^3_x} \| \tilde{S}_N Q_\varepsilon(\mu(t) g^\varepsilon, (W_l h)) - Q_\varepsilon(\mu(t) g^\varepsilon,\ \tilde{S}_N (W_l h)) \|^2_{L^2(\mathbb{R}^3_v)} dx \right)^{1/2} \\
&\qquad \times \| P_{N,\,l} h \|_{L^2(\mathbb{R}^6)} \\
&\leqq C \| \mu(t) g^\varepsilon(t) \|_{L^\infty(\mathbb{R}^3_x; L^1_{\gamma^+}(\mathbb{R}^3_v))} \| W_{l+\gamma^+} h \|_{L^2(\mathbb{R}^6_v)} \| g^\varepsilon(t) \|_{H^k_l(\mathbb{R}^6)} \\
&\leqq C \| g^\varepsilon(t) \|^2_{H^k_l(\mathbb{R}^6)} \| g^\varepsilon(t) \|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \quad t \in [0,\ T].
\end{aligned}
$$

Lemma 3.4 with $m = 2s$ yields

$$
\begin{aligned}
|B_{23}| &\leqq C \| S_N Q(\mu(t) g^\varepsilon,\ (\tilde{S}_N W_l h)) - Q(\mu(t) g^\varepsilon,\ S_N\, (\tilde{S}_N W_l h)) \|_{L^2(\mathbb{R}^3_x,\, L^2(\mathbb{R}^3_v))} \\
&\quad \times \| P_{N,\,l} h \|_{L^2(\mathbb{R}^6)} \\
&\leqq C \| \mu(t) \nabla_x g^\varepsilon \|_{L^\infty(\mathbb{R}^3_x, L^1_{(2s+\gamma)^+}(\mathbb{R}^3_v))} \| (2^{-N} \tilde{S}_N (W_l h) \|_{L^2(\mathbb{R}^3_x, H^{2s}_{(2s+\gamma)^+}(\mathbb{R}^3_v))} \\
&\quad \times \| P_{N,\,l} h \|_{L^2(\mathbb{R}^6)} \\
&\leqq C \| g^\varepsilon(t) \|^2_{H^k_l(\mathbb{R}^6)} \| g^\varepsilon(t) \|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \quad t \in [0,\ T].
\end{aligned}
$$

Combining the above estimates proves Proposition 4.8. □

For the term A_2 and A_3, we prove the following proposition.

Proposition 4.9. *Assume that* $0 < s < 1/2$, $\gamma + 2s < 1$. *Then, for any* $\delta > 0$, *there exists* $C > 0$ *independent of* $\varepsilon > 0$ *such that for any* $\alpha \in \mathbb{N}^6$, $|\alpha| \leqq k$, $k \geqq 4$, $l \geqq 3$,

$$
\left(A_2 + A_3,\ P^\star_{N,\,l} P_{N,l} (\partial^\alpha g^\varepsilon) \right)_{L^2(\mathbb{R}^6)} \leqq C \| g^\varepsilon(t) \|^2_{H^k_l(\mathbb{R}^6)} \| g^\varepsilon(t) \|_{H^k_{l+(\gamma+2s+\delta)^+}(\mathbb{R}^6)}, \tag{4.3.9}
$$

for $t \in [0,\ T]$.

Proof. By putting $h = \partial^\alpha g^\varepsilon$ and $\tilde{h} = W_l^{-2} P_{N,l}^\star P_{N,l}(\partial^\alpha g^\varepsilon)$, we get

$$
\begin{aligned}
&\left|\left(A_2,\ W_l^2\tilde{h}\right)_{L^2(\mathbb{R}^6)}\right| \\
&= \left|\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(\mu_*(t)-\mu'_*(t))(g^\varepsilon)'_* h'(W_l^2\tilde{h})\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\right| \\
&\leqq \iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B|\mu_*(t)-\mu'_*(t)|\,|(g^\varepsilon)'_*|\,|(W_l h)'(W_l\tilde{h})|\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x \\
&\quad+\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B|(\mu_*(t)-\mu'_*(t))|\,|(g^\varepsilon)'_*|\,\left|W_l-W'_l\right| \\
&\quad\times|h'(W_l\tilde{h})|\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x \\
&= I_1+I_2.
\end{aligned}
$$

To estimate I_1, we notice that for $\lambda\in[0,1]$, $t\in[0,T_0]$,

$$
|\mu(t,v_*)-\mu(t,v'_*)| \leqq C|v_*-v'_*|^\lambda \leqq C\theta^\lambda|v-v_*|^\lambda \leqq C\theta^\lambda|v'-v'_*|^\lambda,
$$

which is elementary for $\lambda = 0, 1$ and is obtained for general $\lambda\in(0,1)$ by interpolation. Since $\gamma+2s<1$ is assumed in the proposition, there is $\lambda\in(0,1)$ such that $\lambda>2s$, $\gamma+\lambda\leqq 1$. By the manipulation on the primed and non-primed variables (see (4.1.5)) we have

$$
\begin{aligned}
I_1 &\leqq C\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\langle v'-v'_*\rangle^{(\gamma+\lambda)^+}\theta^{-2-2s+\lambda} \\
&\qquad\times|(g^\varepsilon)'_*|\ |(W_l h)'|\ |(W_l\tilde{h})|\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x \\
&\leqq C\iiint_{\mathbb{R}^3_x\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\theta^{-2-2s+\lambda}|(W_{(\gamma+\lambda)^+}g^\varepsilon)_*| \\
&\qquad\times\left\{\int_{\mathbb{R}^3_v}|(W_{l+(\gamma+\lambda)^+}h)(W_l\tilde{h})'|\mathrm{d}v\right\}\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}x \\
&\leqq C\|g^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3_x;L^1_{(\gamma+\lambda)^+}(\mathbb{R}^3_v))}\|g^\varepsilon(t)\|_{H^k_l(\mathbb{R}^6)}\|g^\varepsilon(t)\|_{H^k_{l+(\gamma+\lambda)^+}(\mathbb{R}^6)} \\
&\leqq C\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)}\|g^\varepsilon(t)\|_{H^k_{l+(\gamma+\lambda)^+}(\mathbb{R}^6)},
\end{aligned}
$$

for $l>(\gamma+\lambda)^+ + 3/2$. In the third inequality we have again used the fact that the Jacobian of changing of variable $v\to v'$ is bounded.

Using (2.1.13) gives

$$
\begin{aligned}
I_2 &\leqq C\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\langle v'-v'_*\rangle^\gamma\theta^{-1-2s}(\mu_*(t)+\mu'_*(t)) \\
&\qquad\times|(g^\varepsilon)'_*\left(W'_l W'_{l,*}\right)h'(W_l\tilde{h})|\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x = C(J_1+J_2).
\end{aligned}
$$

By the Cauchy–Schwarz inequality and the Sobolev inclusion, we have

$$
\begin{aligned}
J_1 &\leqq C\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} \theta^{-1-2s}\mu_*(t)|(W_{l+\gamma^+}g^\varepsilon)'_*(W_{l+\gamma^+}h)'(W_l\tilde{h})|\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\leqq C\int_{\mathbb{R}^3_x}\left(\iiint_{\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\theta^{-1-2s}\mu_*(t)^2|(W_l\tilde{h})|^2\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right)^{1/2}\\
&\qquad\times\left(\iiint_{\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\theta^{-1-2s}|(W_{l+\gamma^+}g^\varepsilon)'_*(W_{l+\gamma^+}h)'|^2\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right)^{1/2}\mathrm{d}x\\
&\leqq C\|\mu\|_{L^2(\mathbb{R}^3_v)}\int_{\mathbb{R}^3_x}\|W_l\tilde{h}(x)\|_{L^2(\mathbb{R}^3_v)}\|W_{l+\gamma^+}g^\varepsilon(x)\|_{L^2(\mathbb{R}^3_v)}\|W_{l+\gamma^+}h(x)\|_{L^2(\mathbb{R}^3_v)}\mathrm{d}x\\
&\leqq C\|g^\varepsilon\|_{L^\infty(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))}\|W_{l+\gamma^+}h\|_{L^2(\mathbb{R}^6)}\|W_l\tilde{h}\|_{L^2(\mathbb{R}^6)}\\
&\leqq C\|g^\varepsilon\|^2_{H^k_{l+\gamma^+}(\mathbb{R}^6)}\|g^\varepsilon\|_{H^k_l(\mathbb{R}^6)}.
\end{aligned}
$$

On the other hand, again by the manipulation on the primed and non-primed variables,

$$
\begin{aligned}
J_2 &\leqq C\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\theta^{-1-2s}|(\mu(t)W_{l+\gamma^+}g^\varepsilon)'_*(W_{l+\gamma^+}h)'(W_l\tilde{h})|\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\leqq C\iiint_{\mathbb{R}^3_x\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}\theta^{-1-2s}|(\mu(t)W_{l+\gamma^+}g^\varepsilon)_*|\left\{\int_{\mathbb{R}^3_v}|W_{l+\gamma^+}h||(W_l\tilde{h})'|\mathrm{d}v\right\}\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}x\\
&\leqq C\|\mu(t)W_{l+\gamma^+}g^\varepsilon\|_{L^\infty(\mathbb{R}^3_x;L^1(\mathbb{R}^3_v))}\|W_{l+\gamma^+}h\|_{L^2(\mathbb{R}^6)}\|W_l\tilde{h}\|_{L^2(\mathbb{R}^6)}\\
&\leqq C\|g^\varepsilon\|^2_{H^k_l(\mathbb{R}^6)}\|g^\varepsilon\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)},
\end{aligned}
$$

where we have used $W_{l+\gamma^+}\mu^{1/2}(t)\leqq C$.

We consider now the term A_3. For any $\alpha\in\mathbb{N}^6$, $|\alpha|\leqq k$, $k\geqq 4$, $l\geqq 3$, denote

$$h_1=\partial^{\alpha_1}g^\varepsilon,\quad h_2=\partial^{\alpha_2}g^\varepsilon,$$

where

$$\alpha_1+\alpha_2\leqq\alpha,\quad \alpha_2<\alpha.$$

We shall compute

$$
\begin{aligned}
&\left(\mathcal{T}_\varepsilon(h_1,h_2,\tilde{\mu}),\,W_l^2\tilde{h}\right)_{L^2(\mathbb{R}^6)}\\
&\quad=\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}B_\varepsilon(\tilde{\mu}_*(t)-\tilde{\mu}'_*(t))(h_1)'_*h'_2(W_l^2\tilde{h})\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad+\iiiint_{\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma}B_\varepsilon\,(\tilde{\mu}h_1)'_*\,\left(W_l-W'_l\right)\,h'_2(W_l\tilde{h})\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad+\left(Q_\varepsilon(\tilde{\mu}h_1,\,(W_lh_2)),\,W_l\tilde{h}\right)_{L^2(\mathbb{R}^6)}.
\end{aligned}
$$

For the last term, by Theorem 2.1 with $m = 2s < 1$, there exists $C > 0$ independent of ε such that for $|\alpha_2| \leqq |\alpha| - 1$ and $\delta > 0$,

$$\begin{aligned}
&\|Q_\varepsilon(\tilde{\mu}h_1, (W_l h_2))\|^2_{L^2(\mathbb{R}^6)} \\
&\leqq C \int_{\mathbb{R}^3} \|\tilde{\mu}h_1(t, x, \cdot)\|^2_{L^1_{(\gamma+2s)^+}(\mathbb{R}^3_v)} \|(W_l h_2)(t, x, \cdot)\|^2_{H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v)} \mathrm{d}x \\
&\leqq \begin{cases} C\|\tilde{\mu}h_1(t)\|^2_{L^\infty(\mathbb{R}^3_x, L^2_{3/2+(\gamma+2s)^++\delta}(\mathbb{R}^3_v))} \|(W_l h_2)(t)\|^2_{L^2(\mathbb{R}^3_x; H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v))}, & |\alpha_1| \leqq 2, \\ C\|\tilde{\mu}h_1(t)\|^2_{L^2(\mathbb{R}^3_x; L^2_{3/2+(\gamma+2s)^++\delta}(\mathbb{R}^3_v))} \|(W_l h_2)(t)\|^2_{L^\infty(\mathbb{R}^3_x, H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v))}, & |\alpha_1| > 2, \end{cases} \\
&\leqq \begin{cases} C\|h_1(t)\|^2_{H^{3/2+\delta}(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))} \|(W_l h_2)(t)\|^2_{L^2(\mathbb{R}^3_x; H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v))}, & |\alpha_1| \leqq 2, \\ C\|h_1(t)\|^2_{L^2(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))} \|(W_l h_2)(t)\|^2_{H^{3/2+\delta}(\mathbb{R}^3_x; H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v))}, & |\alpha_1| > 2, \end{cases} \\
&\leqq C\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} \|g^\varepsilon(t)\|^2_{H^k_{l+(\gamma+2s)^+}(\mathbb{R}^6)}, \quad k \geqq 4 > 3 + 2s, \ l > (\gamma + 2s)^+ + 3/2.
\end{aligned}$$

The estimation on the first term is similar to $(A_2, W_l^2\tilde{h})_{L^2(\mathbb{R}^6)}$ by taking into account the same manipulation concerning α_2. The estimation for the second term is also similar to the part J_2 of I_2 as above. Hence, we have obtained

$$\left|\left(A_3, \ W_l^2\tilde{h}\right)_{L^2(\mathbb{R}^6)}\right| \leqq C\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} \|g^\varepsilon(t)\|_{H^k_{l+\gamma+2s}(\mathbb{R}^6)}.$$

This completes the proof of Proposition 4.9. □

If (4.3.5), (4.3.6), (4.3.7), (4.3.8) and (4.3.9) are combined, then it follows from (4.3.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|P_{N,l}(\partial^\alpha g^\varepsilon)(t)\|^2_{L^2(\mathbb{R}^6)} + \kappa\|W_1 \, P_{N,l}(\partial^\alpha g^\varepsilon)(t)\|^2_{L^2(\mathbb{R}^6)} - \frac{\kappa}{2}\|g^\varepsilon(t)\|^2_{H^k_{l+1}(\mathbb{R}^6)} \\
&\leqq C\left(\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} + C_2\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)}\|g^\varepsilon(t)\|_{H^k_{l+(\gamma+2s+\delta)^+}(\mathbb{R}^6)}\right).
\end{aligned}$$

Take the sum over $|\alpha| \leqq k$, integrate from 0 to $t \in [0, T]$ and make $N \to \infty$. Then there exists $C_1, C_2 > 0$ independent of $\varepsilon > 0$ such that , for any $\delta > 0$ and $t \in [0, T]$,

$$\begin{aligned}
&\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)} + \kappa\int_0^t \|g^\varepsilon(\tau)\|^2_{H^k_{l+1}(\mathbb{R}^6)}\mathrm{d}\tau \\
&\quad\leqq \|g^\varepsilon(0)\|^2_{H^k_l(\mathbb{R}^6)} + C_1\int_0^t \|g^\varepsilon(\tau)\|^2_{H^k_l(\mathbb{R}^6)}\mathrm{d}\tau \\
&\quad\quad + C_2\int_0^t \|g^\varepsilon(\tau)\|^2_{H^k_l(\mathbb{R}^6)}\|g^\varepsilon(\tau)\|_{H^k_{l+(\gamma+2s+\delta)^+}(\mathbb{R}^6)}\mathrm{d}\tau. \qquad (4.3.10)
\end{aligned}$$

Remark 4.10. We give here some technical reasons regarding the choice of the time dependent distribution $\mu(t)$ as moment control in the equation (4.1.1). If we take $\kappa = 0$ in the definition of Maxwellian distribution $\mu(t)$, the above computation gives also (4.3.10) without the second term on the left-hand side because $\kappa = 0$.

But the upper bound estimate, by using Theorem 2.1, always gives the last term in (4.3.10) with the factor $\|g^\varepsilon(t)\|_{H^k_{l+(\gamma+2s)^+ +\delta}(\mathbb{R}^6)}$. If $\gamma+2s<0$, there is no loss of moment, we can get (4.3.11) with $\kappa=0$. If $0\leqq\gamma+2s<1$, we choose δ such that $\gamma+2s+\delta\leqq 1$ so the second term on the left-hand side absorbs the last term in (4.3.10) because

$$\|g^\varepsilon(t)\|_{H^k_{l+(\gamma+2s+\delta)^+}(\mathbb{R}^6)}\leqq\|g^\varepsilon(t)\|_{H^k_{l+1}(\mathbb{R}^6)}.$$

In conclusion, the choice of $\mu(t)$ is mainly for the hard potential.

Completion of proof of Theorem 4.6. Set $X(t)=\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6)}$ and $F(t)=\int_0^t X(\tau)(1+X(\tau))\mathrm{d}\tau$. Since $\gamma+2s<1$, by (4.3.10) there exists a $C>0$ independent of $\varepsilon>0$ such that

$$X(t)+\frac{\kappa}{2}\int_0^t\|g^\varepsilon(\tau)\|^2_{H^k_{l+1}(\mathbb{R}^6)}\mathrm{d}\tau\leqq X(0)+CF(t). \tag{4.3.11}$$

Noticing that $F'(t)\leqq(X(0)+CF(t))\,(1+X(0)+CF(t))$, we have

$$\|g^\varepsilon(t)\|^2_{H^k_l(\mathbb{R}^6))}\leqq\frac{\|g_0\|^2_{H^k_l(\mathbb{R}^6)}e^{Ct}}{1-\left(e^{Ct}-1\right)\|g_0\|^2_{H^k_l(\mathbb{R}^6)}},$$

as long as the denominator remains positive. We choose $T_*>0$ small enough such that

$$\frac{e^{CT_*}}{1-\left(e^{CT_*}-1\right)\|g_0\|^2_{H^k_l(\mathbb{R}^6)}}=4.$$

Then

$$T_*=\frac{1}{C}\log\left(1+\frac{3}{1+4\|g_0\|^2_{H^k_l(\mathbb{R}^6)}}\right),$$

is independent of $\varepsilon>0$, but depends on $\|g_0\|_{H^k_l(\mathbb{R}^6)}$ and the constant C which depends on ρ,κ,k and l. Now we have (4.3.2) for $T_{**}=\min(T,T_*)$.

From (4.3.2) and (4.3.11), we get also, for $\kappa>0$,

$$\kappa\|g^\varepsilon\|^2_{L^2(]0,T_{**}[;\,H^k_{l+1}(\mathbb{R}^6))}\leqq 2\|g_0\|^2_{H^k_l(\mathbb{R}^6)}\left(1+2CT_*\left(1+2\|g_0\|^2_{H^k_l(\mathbb{R}^6)}\right)\right). \tag{4.3.12}$$

We have proved Theorem 4.6. □

4.4. Convergence and uniqueness

The second step is to prove that, for any $0 < \varepsilon < 1$, we can extend the approximation solution g^ε, obtained by Theorem 4.2.1, to a fixed interval $]0, T_*[$ with $T_* > 0$ determined in Theorem 4.6 which is independent on $\varepsilon > 0$. This sequence is then convergent.

Theorem 4.11. *Assume that* $0 < s < 1/2, \gamma + 2s < 1$, $g_0 \geqq 0$, $g_0 \in H^k_l(\mathbb{R}^6)$ *for some* $k \geqq 4, l \geqq 3$. *Let* $T_* > 0$ *be given in Theorem 4.6. Then the Cauchy problem (4.2.1) admits a unique non-negative solution up to* T_* *satisfying*

$$g^\varepsilon \in L^\infty\left(]0, T_*[;\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T_*[;\, H^k_{l+1}(\mathbb{R}^6)\right).$$

Proof. We recall the notation $T = T_\varepsilon(D_0)$ from Remark 4.4. Then Theorem 4.3 asserts that the Cauchy problem (4.2.1) with initial data g_0 admits a unique non-negative solution

$$g^\varepsilon_1 \in C^0\left([0, 2T_{1,\varepsilon}];\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, 2T_{1,\varepsilon}[;\, H^k_{l+1}(\mathbb{R}^6)\right), \quad T_{1,\varepsilon} = \frac{1}{2} T_\varepsilon(\|g_0\|_{H^k_l(\mathbb{R}^6)}).$$

If $T_{1,\varepsilon} \geqq T_*$, then the proof is completed. If $T_{1,\varepsilon} < T_*$, then Theorem 4.6 implies

$$\|g^\varepsilon_1(T_{1,\varepsilon})\|_{H^k_l(\mathbb{R}^6)} \leqq 2\,\|g_0\|_{H^k_l(\mathbb{R}^6)}.$$

We now consider the Cauchy problem (4.2.1) with initial data $g^\varepsilon(T_{1,\varepsilon})$. Again Theorem 4.3 asserts that there exists

$$T_{2,\varepsilon} = \frac{1}{2} T_\varepsilon(2\|g_0\|_{H^k_l(\mathbb{R}^6)}),$$

such that the Cauchy problem (4.2.1) admits a unique non-negative solution

$$g^\varepsilon_2 \in C^0\left([T_{1,\varepsilon},\ T_{1,\varepsilon} + 2T_{2,\varepsilon}];\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]T_{1,\varepsilon}, T_{1,\varepsilon} + 2T_{2,\varepsilon}[;\, H^k_{l+1}(\mathbb{R}^6)\right).$$

By uniqueness of solution, we obtain a non-negative solution of the Cauchy problem (4.2.1),

$$g^\varepsilon \in C^0\left([0, T_{1,\varepsilon} + 2T_{2,\varepsilon}];\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T_{1,\varepsilon} + 2T_{2,\varepsilon}[;\, H^k_{l+1}(\mathbb{R}^6)\right).$$

If $T_{1,\varepsilon} + 2T_{2,\varepsilon} \geqq T_*$, we finish the proof. If $T_{1,\varepsilon} + 2T_{2,\varepsilon} < T_*$, we again consider the Cauchy problem (4.2.1) with initial date $g^\varepsilon(T_{1,\varepsilon} + T_{2,\varepsilon})$. Since Theorem 4.6 gives again

$$\|g^\varepsilon(T_{1,\varepsilon} + T_{2,\varepsilon})\|_{H^k_l(\mathbb{R}^6)} \leqq 2\,\|g_0\|_{H^k_l(\mathbb{R}^6)},$$

the interval of the existence of the solution is the same, that is, $2T_{2,\varepsilon}$, so that we can extend the solution to

$$g^\varepsilon \in L^\infty\left(]0, T_{1,\varepsilon} + 3T_{2,\varepsilon}[;\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T_{1,\varepsilon} + 3T_{2,\varepsilon}[;\, H^k_{l+1}(\mathbb{R}^6)\right).$$

By iteration, there exists $m \in \mathbb{N}$ such that

$$T_{1,\varepsilon} + mT_{2,\varepsilon} < T_*, \quad T_{1,\varepsilon} + (m+1)T_{2,\varepsilon} \geqq T_*,$$

and we extend the solution up to

$$g^\varepsilon \in C^0\left([0, T_{1,\varepsilon}+(m+1)T_{2,\varepsilon}];\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T_{1,\varepsilon}+(m+1)T_{2,\varepsilon}[;\, H^k_{l+1}(\mathbb{R}^6)\right).$$

We have proved Theorem 4.11. □

Theorem 4.11 asserts the existence of an approximation solution sequence

$$\{g^\varepsilon\}_{\varepsilon>0} \subset C^0\left([0, T_*];\, H^k_l(\mathbb{R}^6)\right) \cap L^2\left(]0, T_*;\, H^k_{l+1}(\mathbb{R}^6)\right),$$

and

$$\|g^\varepsilon\|_{L^\infty(]0,T_*[;\, H^k_l(\mathbb{R}^6))} \leqq 2\,\|g_0\|_{H^k_l(\mathbb{R}^6)}.$$

This implies that it is a weakly* compact set of $L^\infty(]0, T_*[;\, H^k_l(\mathbb{R}^6))$. Let

$$g \in L^\infty\left(]0, T_*[;\, H^k_l(\mathbb{R}^6)\right),$$

be a limit of a subsequence of $\{g^\varepsilon\}_{\varepsilon>0}$.

On the other hand, by using the equation (4.2.1) and Theorem 2.1, we obtain

$$\begin{aligned}\|\partial_t g^\varepsilon\|_{L^\infty(]0,T_*[;\, H^{k-1}_{l-1}(\mathbb{R}^6))} &\leqq C\left(\|g^\varepsilon\|_{L^\infty(]0,T_*[;\, H^k_l(\mathbb{R}^6))} + \|g^\varepsilon\|^2_{L^\infty(]0,T_*[;\, H^k_l(\mathbb{R}^6))}\right)\\ &\leqq 2C\left(1 + 2\|g_0\|_{H^k_l(\mathbb{R}^6)}\right)\|g_0\|_{H^k_l(\mathbb{R}^6)}.\end{aligned}$$

Thus, $\{g^\varepsilon\}_{\varepsilon>0}$ is a compact subset in

$$C^{1-\delta}\left(]0, T_*[;\, H^{k-1-\delta}_{l-1}(\Omega \times \mathbb{R}^3_v)\right),$$

for any compact bounded open set $\Omega \subset \mathbb{R}^3_x$ and for any $\delta > 0$. For the variable v, we have the weight W_{l-1} with $l - 1 > 3/2$. We can then take the limit in the equation (4.2.1) and also in the mild form (4.2.5). It is then the case that g is a solution of the Cauchy problem (4.1.1). The limit g belongs to $L^2(]0, T_*[;\ H^k_{l+1}(\mathbb{R}^6))$ deduced from (4.3.12). Now if $g_0 \geqq 0$, Theorem 4.3 implies that $g^\varepsilon \geqq 0$, so that the limit g is also non-negative on $]0, T_*[$. We have completed the proof for the local existence of solutions stated in Theorem 4.1.

It remains to prove the uniqueness of solutions in Theorem 4.1. We state it more precisely as follows.

Proposition 4.12. *Assume that* $0 < s < 1/2, \gamma + 2s < 1, 0 < T \leqq T_0, m > 3$ *and* $g_0 \geqq 0, g_0 \in H^m_3(\mathbb{R}^6)$. *Suppose that the Cauchy problem* (4.1.1) *admits two (non-negative) solutions*

$$g_1, g_2 \in C^0\left([0, T];\, H^m_4(\mathbb{R}^6)\right).$$

Then $g_1 \equiv g_2$.

Set $f = g_1 - g_2$, by using (4.1.1), we have

$$\begin{cases} f_t + v \cdot \nabla_x f \ + \kappa(1+|v|^2) f = \Gamma^t(g_1, f) + \Gamma^t(f, g_2), \\ f|_{t=0} = 0. \end{cases}$$

We can now take $W_3 f$ as a test function to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6)} + \kappa \|W_4 f(t)\|^2_{L^2(\mathbb{R}^6)} = \left(W_3\Gamma^t(g_1,\ f) + W_3\Gamma^t(f,\ g_2), W_3 f\right)_{L^2(\mathbb{R}^6)}. \tag{4.4.1}$$

Recall that

$$\Gamma^t(g,h) = Q(\mu(t)g, h) + \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} B\left(\mu(t)_* - \mu(t)'_*\right) g'_* h' \mathrm{d}v_* \mathrm{d}\sigma.$$

We estimate the last two terms of (4.4.1) in the following lemma.

Lemma 4.13. *Assume that* $g_1 \geqq 0$*. Then for any* $\varepsilon > 0$*, there exist constants* $C_\varepsilon > 0$ *and* $K(\varepsilon, \|g_2\|_{L^\infty(]0,T[;H^m_4(\mathbb{R}^6))}) > 0$ *such that*

$$\begin{aligned}&\left(W_3\Gamma^t(g_1, f), W_3 f\right)_{L^2(\mathbb{R}^6)} \\ &\quad\leqq \varepsilon\|W_4 f(t)\|^2_{L^2(\mathbb{R}^6)} + C_\varepsilon\|g_1\|^2_{L^\infty(]0,T[;H^m_4(\mathbb{R}^6))}\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6)},\end{aligned} \tag{4.4.2}$$

$$\begin{aligned}&\left|\left(W_3\Gamma^t(f,\ g_2), W_3 f\right)_{L^2(\mathbb{R}^6)}\right| \\ &\quad\leqq \varepsilon\|W_4 f(t)\|^2_{L^2(\mathbb{R}^6)} + K(\varepsilon, \|g_2\|_{L^\infty(]0,T[;H^m_3(\mathbb{R}^6))})\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6)}.\end{aligned} \tag{4.4.3}$$

Notice that by using the above lemma with $\varepsilon = \kappa/4$ and (4.4.1), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6)} \\ &\leqq \left(C\|g_1\|^2_{L^\infty(]0,T[;H^m_4(\mathbb{R}^6))} + K(\varepsilon, \|g_2\|_{L^\infty(]0,T[;H^m_4(\mathbb{R}^6))})\right)\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6)}.\end{aligned}$$

Then $\|W_3 f(0)\|_{L^2(\mathbb{R}^6)} = 0$ implies $\|W_3 f(t)\|_{L^2(\mathbb{R}^6)} = 0$ for all $0 \leqq t \leqq T$ which gives Proposition 4.12.

Proof of Lemma 4.13. As for (4.4.2), we have

$$\begin{aligned}&\left(W_3\Gamma^t(g_1,\ f), W_3 f\right)_{L^2(\mathbb{R}^6)} \\ &= (W_3 Q(\mu(t)g_1,\ f), W_3 f)_{L^2(\mathbb{R}^6)} \\ &\quad + \iiiint B(\mu(t)_* - \mu(t)'_*)g'_{1*} f' W_3^2 f \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\ &= (Q(\mu(t)g_1,\ W_3 f), W_3 f)_{L^2(\mathbb{R}^6)} \\ &\quad + (W_3 Q(\mu(t)g_1,\ f) - Q(\mu(t)g_1,\ W_3 f), W_3 f)_{L^2(\mathbb{R}^6)} \\ &\quad + \iiiint B(\mu(t)_* - \mu(t)'_*)g'_{1*}\,(W_3 f)'\, W_3 f \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\ &\quad + \iiiint B(\mu(t)_* - \mu(t)'_*)g'_{1*}\left(W_3 - W'_3\right) f' W_3 f \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\ &= D_1 + D_2 + D_3 + D_4.\end{aligned}$$

The term D_1 is similar to B_1 in the proof of Proposition 4.8. By using $\mu(t)g_1 \geqq 0$, we have

$$D_1 \leqq C\|g_1(t)\|_{H^{3/2+\delta}(\mathbb{R}^6_{x,\,v})}\|f(t)\|_{L^2_3(\mathbb{R}^6_{x,\,v})}\|f(t)\|_{L^2_{3+\gamma^+}(\mathbb{R}^6_{x,\,v})},$$

for some small $\delta > 0$. The term D_2 is similar to B_2 and we can obtain

$$|D_2| \leqq C\|g_1(t)\|_{H^{3/2+\delta}(\mathbb{R}^6_{x,v})}\|f(t)\|_{L^2_3(\mathbb{R}^6_{x,v})}\|f(t)\|_{L^2_{3+\gamma^+}(\mathbb{R}^6_{x,v})}.$$

The terms D_3, D_4 are similar to I_1, I_2 in the proof of Proposition 4.9. Namely

$$|D_3| + |D_4| \leqq C\|g_1(t)\|_{H^{6/2+\delta}_{3+(\gamma+2s+\delta)^+}(\mathbb{R}^6_{x,\,v})}\|f(t)\|_{L^2_3(\mathbb{R}^6_{x,\,v})}\|f(t)\|_{L^2_{3+(\gamma+2s+\delta)^+}(\mathbb{R}^6_{x,\,v})}.$$

Thus, for any $0 < t \leqq T$ and $m > 3$, we have

$$\begin{aligned}
&\big(W_3\Gamma^t(g_1,\ f),\, W_3 f\big)_{L^2(\mathbb{R}^6)}\\
&\quad \leqq C\|g_1\|_{L^\infty(]0,T[;\,H^m_4(\mathbb{R}^6_{x,\,v}))}\|W_3 f(t)\|_{L^2(\mathbb{R}^6_{x,\,v})}\|W_4 f(t)\|_{L^2(\mathbb{R}^6_{x,\,v})},
\end{aligned}$$

which implies (4.4.2). The left-hand side of (4.4.3) can be written as

$$\begin{aligned}
&\big(W_3\,\Gamma^t(f,\ g_2),\, W_3\ f\big)_{L^2(\mathbb{R}^6)}\\
&\quad = (W_3 Q(\mu(t)f,\ g_2),\, W_3 f)_{L^2(\mathbb{R}^6)}\\
&\qquad + \iiiint B\ (\mu(t)_* - \mu(t)'_*) f'_* g'_2 W_3^2 f\,\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = (W_3 Q(\mu(t)f,\ g_2),\, W_3 f)_{L^2(\mathbb{R}^6)}\\
&\qquad + \iiiint B\ (\mu(t)_* - \mu(t)'_*) f'_*\,(W_3 g_2)'\,W_3 f\,\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad + \iiiint B\ (\mu(t)_* - \mu(t)'_*) f'_*\,\big(W_3 - W'_3\big)\,g'_2 W_3 f\,\mathrm{d}v_*\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = E_1 + E_2 + E_3.
\end{aligned}$$

Using Corollary 2.5 with $m = 0$, $l = 3$ gives

$$\begin{aligned}
|E_1| &\leqq \int_{\mathbb{R}^3_x}\|W_3 Q(\mu(t)f, g_2)\|_{L^2(\mathbb{R}^3_v)}\|W_3 f\|_{L^2(\mathbb{R}^3_v)}\mathrm{d}x\\
&\leqq C\int_{\mathbb{R}^3_x}\|\mu(t)f\|_{L^1_{3+(\gamma+2s)^+}(\mathbb{R}^3_v)}\|g_2\|_{H^{2s}_{3+(\gamma+2s)^+}(\mathbb{R}^3_v)}\|W_3 f\|_{L^2(\mathbb{R}^3_v)}\mathrm{d}x\\
&\leqq C\|g_2\|_{L^\infty(]0,T[\times\mathbb{R}^3_x;\,H^{2s}_{3+(\gamma+2s+\delta)^+}(\mathbb{R}^3_v))}\|f(t)\|_{L^2(\mathbb{R}^6)}\|W_3 f(t)\|_{L^2(\mathbb{R}^6)}\\
&\leqq C\|g_2\|_{L^\infty(]0,T[;\,H^{3/2+2s+\delta}_{3+(\gamma+2s+\delta)^+}(\mathbb{R}^3_v))}\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6)}.
\end{aligned}$$

The term E_2 is similar to D_3, and we have

$$\begin{aligned}
|E_2| &\leqq C\|f(t)\|_{L^2(\mathbb{R}^3_x;\,L^1_{(\gamma+2s+\delta)^+}(\mathbb{R}^3_v))}\|g_2\|_{L^\infty(]0,T[\times\mathbb{R}^3_x;\,L^2_{3+\gamma^+}(\mathbb{R}^3_v))}\|W_3 f(t)\|_{L^2(\mathbb{R}^6)}\\
&\leqq C\|f(t)\|_{L^2_{3/2+\delta+(\gamma+2s+\delta)^+}(\mathbb{R}^6))}\|g_2\|_{L^\infty(]0,T[;\,H^{3/2+\delta}_{3+\gamma^+}(\mathbb{R}^6))}\|W_3 f(t)\|_{L^2(\mathbb{R}^6)}\\
&\leqq C\|g_2\|_{L^\infty(]0,T[;\,H^{3/2+\delta}_4(\mathbb{R}^6))}\|W_3 f(t)\|^2_{L^2(\mathbb{R}^6))}.
\end{aligned}$$

For the term E_3, we can use (2.1.14) with $l = 3$. Then

$$
\begin{aligned}
|E_3| &\leqq \iiiint b(\cos\theta)\ \langle v - v_*\rangle^\gamma\ |\mu(t)_* - \mu(t)'_*|\,|f'_*| \\
&\qquad \times \left|W_3 - W'_3\right|\ |g'_2|\,|W_3 f| \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\
&\leqq C \iiiint \sin\left(\frac{\theta}{2}\right) b(\cos\theta) |(W_{1+\gamma^+} f)'_*|\,|(W_{3+\gamma^+} g_2)'|\,|W_3 f|\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\
&\quad + C \iiiint \sin^3\left(\frac{\theta}{2}\right) b(\cos\theta) \mu'_*(t) |(W_{3+\gamma^+} f)'_*|\,|(W_{\gamma^+} g_2)'|\,|W_3 f|\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\
&\quad + C \iiiint \sin^3\left(\frac{\theta}{2}\right) b(\cos\theta)\ \mu_*(t) |(W_{3+\gamma^+} f)'_*|\,|(W_{\gamma^+} g_2)'|\,|W_3 f|\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\
&= E_{3,1} + E_{3,2} + E_{3,3}.
\end{aligned}
$$

Since $0 < 2s < 1$ is assumed, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
|E_{3,1}| &\leqq C \int_{\mathbb{R}^3_x} \|f(t,\, x,\, \cdot)\|_{L^1_{1+\gamma^+}(\mathbb{R}^3_v)}\ \|g_2(t,\, x,\, \cdot)\|_{L^2_4(\mathbb{R}^3_v)}\ \|f(t,\, x,\, \cdot)\|_{L^2_3(\mathbb{R}^3_v)} \mathrm{d}x \\
&\leqq C \|f(t)\|_{L^2(\mathbb{R}^3_x;\, L^1_{1+\gamma^+}(\mathbb{R}^3_v))}\ \|g_2\|_{L^\infty(]0,T[\times\mathbb{R}^3_x;\, L^2_4(\mathbb{R}^3_v))}\ \|f(t)\|_{L^2_3(\mathbb{R}^6_{x,v})} \\
&\leqq C \|f(t)\|_{L^2_{3/2+\delta+1+\gamma^+}(\mathbb{R}^6_{x,v})}\ \|g_2\|_{L^\infty(]0,T[;\, H^{3/2+\delta}_4(\mathbb{R}^6_{x,v}))}\ \|f(t)\|_{L^2_3(\mathbb{R}^6_{x,v})} \\
&\leqq \left(\varepsilon \|W_4 f(t)\|^2_{L^2(\mathbb{R}^6_{x,v})} + C_\varepsilon \|W_3 f(t)\|^2_{L^2(\mathbb{R}^6_{x,v})}\right) \|g_2\|_{L^\infty(]0,T[;\, H^{3/2+\delta}_4(\mathbb{R}^6_{x,v}))}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
|E_{3,2}| &\leqq C \int_{\mathbb{R}^3_x} \|\mu(t) f(t,\, x,\, \cdot)\|_{L^1_{1+\gamma^+}(\mathbb{R}^3_v)}\ \|g_2(t,\, x,\, \cdot)\|_{L^2_{\gamma^+}(\mathbb{R}^3_v)}\ \|f(t,\, x,\, \cdot)\|_{L^2_3(\mathbb{R}^3_v)} \mathrm{d}x \\
&\leqq C \|g_2\|_{L^\infty(]0,T[;\, H^{3/2+\delta}_4(\mathbb{R}^6_{x,v}))} \|f(t)\|_{L^2(\mathbb{R}^6_{x,v})} \|W_3 f(t)\|_{L^2(\mathbb{R}^6_{x,v})}.
\end{aligned}
$$

Since $3/2 + (3 + \gamma^+) > 4$, we cannot estimate $E_{3,3}$ in the same way as for $E_{3,2}$. Instead, we have

$$
\begin{aligned}
|E_{3,3}| &\leqq C \|W_{\gamma^+} g_2\|_{L^\infty(]0,T[\times\mathbb{R}^6_{x,v})} \\
&\qquad \times \iiiint \theta^3 b(\cos\theta)\ \mu_*(t) |(W_{3+\gamma^+} f)'_*|\,|W_3 f|\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}x \\
&\leqq C \|g_2\|_{L^\infty(]0,T[;\, H^{3+\delta}_3(\mathbb{R}^6_{x,v}))} \int_{\mathbb{R}^3_x} \left(\iiint \theta^1 b(\cos\theta)\ \mu_*(t) |W_3 f|^2\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\right)^{\frac{1}{2}} \\
&\qquad \times \left(\iiint \theta^5 b(\cos\theta)\ \mu_*(t) |(W_{3+\gamma^+} f)'_*|^2\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v\right)^{\frac{1}{2}} \mathrm{d}x.
\end{aligned}
$$

We now take the singular change of variables $v'_* \to v$. The Jacobian is computed in (2.1.20) which is of the order of θ^{-2}. Then this singular change of variables yields

$$
\begin{aligned}
&\iiint \theta^5 b(\cos\theta) \mu_*(t) |(W_{3+\gamma^+} f)'_*|^2\, \mathrm{d}v_*\, \mathrm{d}\sigma\, \mathrm{d}v \\
&\qquad \leqq C \iint D_1(v_*,\, v'_*) \mu_*(t) |(W_{3+\gamma^+} f)'_*|^2\, \mathrm{d}v_*\, \mathrm{d}v'_*,
\end{aligned}
$$

with $D_1(v_*, v'_*) = \int_{S^2} \theta^{5-2} b(\cos\theta) \mathrm{d}\sigma \leqq C \int_{\pi/4}^{\pi/2} (\frac{\pi}{2} - \psi)^{-2-2s+5-2} \mathrm{d}\psi \leqq C$. Hence

$$\iiint \theta^5 b(\cos\theta) \mu_*(t) |(W_{3+\gamma^+} f)'_*|^2 \, |\, \mathrm{d}v_* \, \mathrm{d}\sigma \, \mathrm{d}v$$
$$\leqq C \|\mu(t)\|_{L^1(\mathbb{R}^3_v)} \|W_{3+\gamma^+} f(t, x, \cdot)\|^2_{L^2(\mathbb{R}^3_v)}.$$

Therefore,

$$|E_{3,3}| \leqq C \|g_2\|_{L^\infty(]0,T[;\, H^{3+\delta}_4(\mathbb{R}^6_{x,v}))} \|W_3 f\|_{L^2(\mathbb{R}^6_{x,v})} \|W_{3+\gamma^+} f\|_{L^2(\mathbb{R}^6_{x,v})}.$$

By combining the estimates on E_1, E_2, E_3, we have proved (4.4.3). Now the proof of Lemma 4.13 is complete. □

4.5. Proof of Theorem 1.2

Assume that $f_0 \in \mathcal{E}^{k_0}_0(\mathbb{R}^5)$. Then there exists $\rho_0 > 0$ such that $e^{\rho_0 \langle v \rangle^2} f_0 \in H^{k_0}(\mathbb{R}^6)$. Choose $0 < \rho < \rho_0$ and $\kappa > 0$ small enough. By setting $g_0 = e^{\rho \langle v \rangle^2} f_0$, then $g_0 \in H^{k_0}_l(\mathbb{R}^6)$ for all $l \in \mathbb{N}$. Theorem 4.1 asserts that the Cauchy problem (4.1.1) with the initial datum g_0 admits a non-negative local solution

$$g \in C^0\left([0, T_*];\, H^{k_0}_l(\mathbb{R}^6)\right) \bigcap L^2\left(]0, T_*[;\, H^{k_0}_{l+1}(\mathbb{R}^6)\right), \quad \forall l \in \mathbb{N},$$

with $T_* \in]0, T_0]$ $(T_0 = \frac{\rho}{2\kappa})$. Then

$$f(t, x, v) = e^{-(\rho - \kappa t)\langle v \rangle^2} g(t, x, v) \in C^0\left([0, T_*];\, H^{k_0}_l(\mathbb{R}^6)\right)$$
$$\bigcap L^2\left(]0, T_*[;\, H^{k_0}_l(\mathbb{R}^6)\right), \quad \forall l \in \mathbb{N},$$

is a non-negative solution of the Cauchy problem (1.4). Since for $0 \leqq t \leqq T_* \leqq T_0$,

$$e^{\frac{\rho}{2}\langle v \rangle^2} f \in C^0\left([0, T_*];\, H^{k_0}(\mathbb{R}^6)\right), \tag{4.5.1}$$

we can conclude $f \in \mathcal{E}^{k_0}([0, T_*] \times \mathbb{R}^6_{x,v})$, which leads to the local existence stated in Theorem 1.2.

Suppose now for some $f_0 \in \mathcal{E}^4_0(\mathbb{R}^5)$, the Cauchy problem (1.4) admits two solutions $f_1 \in \mathcal{E}^4([0, T_1] \times \mathbb{R}^6_{x,v})$ and $f_2 \in \mathcal{E}^4([0, T_2] \times \mathbb{R}^6_{x,v})$. This implies that there exist $\rho_0, \rho_1, \rho_2 > 0$ such that

$$e^{\rho_0 \langle v \rangle^2} f_0 \in H^4(\mathbb{R}^6),$$

and

$$e^{\rho_1 \langle v \rangle^2} f_1 \in C^0\left([0, T_1];\, H^4(\mathbb{R}^6)\right), \quad e^{\rho_2 \langle v \rangle^2} f_2 \in C^0\left([0, T_2];\; H^4(\mathbb{R}^6)\right).$$

Take $0 < \rho < \min\{\rho_0,\ \rho_1,\ \rho_2\}$ and $\kappa > 0$ sufficiently small such that $\frac{\rho}{2\kappa} > T_{**} = \min\{T_1, T_2\}$. Then we have

$$g_0 = e^{\rho \langle v \rangle^2} f_0 \in H^4_l(\mathbb{R}^6),$$

for any $l \in \mathbb{N}$, and

$$g_1 = e^{(\rho-\kappa t)\langle v\rangle^2} f_1 \in C^0\left([0, T_{**}]; H_l^4(\mathbb{R}^6)\right),$$
$$g_2 = e^{(\rho-\kappa t)\langle v\rangle^2} f_2 \in C^0\left([0, T_{**}]; H_l^4(\mathbb{R}^6)\right),$$

are two solutions of the Cauchy problem (4.1.1) with the common initial datum g_0. Then Proposition 4.12 gives $g_1 = g_2$, so that $f_1 = f_2$ for $t \in [0, T_{**}]$. Now the uniqueness of solutions stated in Theorem 1.2 is obvious since $T_1 = T_2 = T_{**}$.

On the other hand, in view of (4.5.1), $\|f(t, x, \cdot)\|_{L^1}$ is continuous for $(t, x) \in [0, T_*] \times \mathbb{R}^3_x$. Therefore, if for a compact $K \subset \mathbb{R}^3$, we have

$$\inf_{x\in K} \|f_0(x, \cdot)\|_{L^1} = c_0 > 0,$$

then there exist $0 < \tilde{T}_0 \leqq T_*$ and a closed neighborhood of K denoted by V_0 in $\mathbb{R}^3_x$ such that

$$\inf_{(t,x)\in[0,\tilde{T}_0]\times V_0} \|f(t, x, \cdot)\|_{L^1} \geqq \frac{c_0}{2}.$$

Now Theorem 1.1 implies that

$$f \in \bigcap_{l\in\mathbb{N}} \mathcal{H}_l^{+\infty}\left(]0, \tilde{T}_0[\times V_0 \times \mathbb{R}^3_v\right) \subset C^\infty\left(]0, \tilde{T}_0[\times V_0; \mathcal{S}(\mathbb{R}^3_v)\right).$$

It remains to prove the uniqueness of solutions of Theorem 1.2 in the soft potential case $\gamma \leqq 0$. In this case, uniqueness of solution can be proved in a larger functional space. We state it as follows.

Proposition 4.14. *Assume that* $0 < s < 1/2, \gamma \leqq 0, 0 < T \leqq +\infty$ *and* $m > 2s + 3/2, l > 2s + 3/2$. *Let* $f_0 \geqq 0$, $f_0 \in H^m_{l+2s}(\mathbb{R}^6)$. *Suppose that the Cauchy problem* (1.4) *admits two non-negative solutions*

$$f_1,\ f_2 \in L^\infty(]0, T[;\ H^m_{l+2s}(\mathbb{R}^6)).$$

Then $f_1 \equiv f_2$.

Proof. The proof is similar to the one for Proposition 4.12. Set $F = f_1 - f_2$, by using (1.4), we have

$$\begin{cases} F_t + v \cdot \nabla_x F = Q(f_1,\ F) + Q(F,\ f_2), \\ F|_{t=0} = 0. \end{cases}$$

We can now take $W_l F$ as a test function to have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|F(t)\|^2_{L^2_l(\mathbb{R}^6)} = (W_l\, Q(f_1,\ F) + W_l\, Q(F,\ f_2), W_l F)_{L^2(\mathbb{R}^6)}.$$

Since $f_1 \geqq 0$ and $\gamma \leqq 0$, similar to the analysis on B_1 in the proof of Proposition 4.8, we have

$$(Q(f_1,\ W_l F), W_l F)_{L^2(\mathbb{R}^6)} \leqq C\|f_1(t)\|_{L^\infty(\mathbb{R}^3_x;\, L^1(\mathbb{R}^3_v))}\|F(t)\|^2_{L^2_l(\mathbb{R}^6_{x,\, v})}.$$

Using (2.1.16) with $\gamma^+ = 0$ gives

$$\left|(W_l Q(f_1,\ F) - Q(f_1, W_l F), W_l F)_{L^2(\mathbb{R}^6)}\right| \\ \leqq C\|f_1(t)\|_{L^\infty(\mathbb{R}^3_x;\, L^2_l(\mathbb{R}^3_v))}\|F(t)\|^2_{L^2_l(\mathbb{R}^6_{x,v})},$$

and

$$\left|(W_l Q(F,\ f_2) - Q(F,\ W_l f_2), W_l F)_{L^2(\mathbb{R}^6)}\right| \\ \leqq C\|F(t)\|_{L^2_l(\mathbb{R}^6_{x,\,v})}\|f_2(t)\|_{L^\infty(\mathbb{R}^3_x;\, L^2_l(\mathbb{R}^3_v))}\|F(t)\|_{L^2_l(\mathbb{R}^6_{x,\,v})}.$$

Finally, for $l > 3/2 + 2s$, we have

$$\begin{aligned}\left|(Q(F,\ W_l f_2), W_l F)_{L^2(\mathbb{R}^6)}\right| &\leqq C\|Q(F,\ W_l f_2)\|_{L^2(\mathbb{R}^6)}\|F(t)\|_{L^2_l(\mathbb{R}^6)} \\ &\leqq \|F(t)\|_{L^2_l(\mathbb{R}^6)}\left(\int_{\mathbb{R}^3_x}\|F(t,x,\cdot)\|^2_{L^1_{2s}(\mathbb{R}^3_v)}\|f_2(t,x,\cdot)\|^2_{H^{2s}_{l+2s}(\mathbb{R}^3_v)}\right)^{1/2} \\ &\leqq C\|F(t)\|^2_{L^2_l(\mathbb{R}^6)}\|f_2(t)\|_{L^\infty(\mathbb{R}^3_x;\ H^{2s}_{l+2s}(\mathbb{R}^3_v))}.\end{aligned}$$

Thus, we have, for any $0 < t < T$ and $\delta > 0$ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F(t)\|^2_{L^2_l(\mathbb{R}^6)} \\ \leqq C\left(\|f_1\|_{L^\infty(]0,T[;\,H^{3/2+\delta}_l(\mathbb{R}^6_{x,v}))} + \|f_2\|_{L^\infty(]0,T[;\,H^{3/2+\delta+2s}_{l+2s}(\mathbb{R}^6_{x,v}))}\right)\|F(t)\|^2_{L^2_l(\mathbb{R}^6)}.$$

Therefore, $\|F(0)\|_{L^2_l(\mathbb{R}^6)} = 0$ implies $\|F(t)\|_{L^2_l(\mathbb{R}^6)} = 0$ for all $t \in [0, T[$. □

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