

BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF IN THE WHOLE SPACE: III, QUALITATIVE PROPERTIES OF SOLUTIONS

R. ALEXANDRE, Y. MORIMOTO, S. UKAI, C.-J. XU, AND T. YANG

ABSTRACT. This is a continuation of our series of works for the inhomogeneous Boltzmann equation. We study qualitative properties of classical solutions, precisely, the full regularization in all variables, uniqueness, non-negativity and convergence rate to the equilibrium. Together with the results of Parts I and II about the well posedness of the Cauchy problem around Maxwellian, we conclude this series with a satisfactory mathematical theory for Boltzmann equation without angular cutoff.

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1. INTRODUCTION

Following our series of works [9, 10], extending results from [7, 8], this Part III is concerned with qualitative properties associated with solutions to the Cauchy problem for the inhomogeneous Boltzmann equation

$$(1.1) \quad f_t + v \cdot \nabla_x f = Q(f, f), \quad f|_{t=0} = f_0.$$

We refer the reader for the complete framework, definitions and bibliography, to our previous papers [9, 10]. General details about Boltzmann equation for non cutoff cross sections can be found in [1, 13, 37]. Let us just recall herein that the Boltzmann bilinear collision operator is given by

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g'_* f' - g_* f\} d\sigma dv_*,$$

Date: 30 October 2010.

2000 Mathematics Subject Classification. 35A02, 35B40, 35B65, 35H10, 35S05, 35Q20, 82B40.

Key words and phrases. Boltzmann equation, non-cutoff cross sections, hypoellipticity, uniqueness, non-negativity, convergence rate.

where $f'_* = f(t, x, v'_*)$, $f' = f(t, x, v')$, $f_* = f(t, x, v_*)$, $f = f(t, x, v)$, and for $\sigma \in \mathbb{S}^2$, the pre- and post-collisional velocities are linked by the relations

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

The non-negative cross section $B(z, \sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|} \cdot \sigma$. As in the previous parts, we assume that it takes the form

$$B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where

$$(1.2) \quad \Phi(|z|) = \Phi_\gamma(|z|) = |z|^\gamma, \quad b(\cos \theta) \approx \theta^{-2-2s} \text{ when } \theta \rightarrow 0+,$$

for some $\gamma > -3$ and $0 < s < 1$.

In the present work, we are concerned with qualitative properties of classical solutions to the Boltzmann equation, under the previous assumptions. By qualitative properties, we mean specifically regularization properties, positivity, uniqueness of solutions and asymptotic trend to global equilibrium.

Let us recall that in a close to equilibrium framework, the existence of such classical solutions was proven in our series of papers [9, 10] and using a different method, by Gressmann and Strain [22, 23, 24]. We refer also to [11] for bounded local solutions.

The first qualitative property which will be addressed here is concerned with regularization properties of classical solutions, that is, the immediate smoothing effect on the solution. For the homogeneous Boltzmann equation, after the works of Desvillettes [16, 17, 18], this issue has now a long history [3, 4, 14, 19, 26, 28, 29, 34]. All these works deal with smoothed type kinetic part for the cross sections, which therefore rules out the more physical assumption above, that is, including the singular behavior for relative velocity near 0. We refer the reader to our forthcoming work [12] for this issue.

Regularization effect for the inhomogeneous Boltzmann equation was studied in our previous works [6, 8], but for Maxwellian type molecules or smoothed kinetic parts for the cross section. Nevertheless, we have introduced many technical tools, some of which are helpful for tackling the singular assumption above. In particular, by improving the pseudo-differential calculus and functional estimates from [6, 8], we shall be able to prove our regularity result.

We shall use the following standard weighted Sobolev space defined, for $k, \ell \in \mathbb{R}$, as

$$H_\ell^k = H_\ell^k(\mathbb{R}_v^3) = \{f \in \mathcal{S}'(\mathbb{R}_v^3); W_\ell f \in H^k(\mathbb{R}_v^3)\}$$

and for any open set $\Omega \subset \mathbb{R}_x^3$

$$H_\ell^k(\Omega \times \mathbb{R}_v^3) = \{f \in \mathcal{D}'(\Omega \times \mathbb{R}_v^3); W_\ell f \in H^k(\Omega \times \mathbb{R}_v^3)\}$$

where $W_\ell(v) = \langle v \rangle^\ell = (1 + |v|^2)^{\ell/2}$ is always the weight for v variables. Herein, $(\cdot, \cdot)_{L^2} = (\cdot, \cdot)_{L^2(\mathbb{R}_v^3)}$ denotes the usual scalar product in $L^2 = L^2(\mathbb{R}^3)$ for v variables. Recall that $L_\ell^2 = H_\ell^0$.

Theorem 1.1. *Assume (1.2) holds true, with $0 < s < 1$, $\gamma > \max\{-3, -3/2 - 2s\}$, $0 < T \leq +\infty$. Let Ω be an open domain of \mathbb{R}_x^3 . Let $f \in L^\infty([0, T]; H_\ell^5(\Omega \times \mathbb{R}^3))$, for any $\ell \in \mathbb{N}$, be a solution of Cauchy problem (1.1). Moreover, assume that f satisfies the following local coercivity estimate : for any compact $K \subset \Omega$ and $0 < T_1 < T_2 < T$, there exist two constants $\eta_0 > 0$, $C_0 > 0$ such that*

$$(1.3) \quad -(Q(f, h), h)_{L^2(\mathbb{R}^7)} \geq \eta_0 \|h\|_{H_{\gamma/2}^s(\mathbb{R}^7)}^2 - C_0 \|h\|_{L_{\gamma/2+s}^2(\mathbb{R}^7)}^2$$

for any $h \in C_0^1([T_1, T_2]; C_0^\infty(K; H_\ell^{+\infty}(\mathbb{R}^3)))$. Then we have

$$f \in C^\infty([0, T] \times \Omega; \mathcal{S}(\mathbb{R}^3)).$$

Classical solutions satisfying such a local coercivity estimate do exist [9, 10], see Corollary 2.15 in next section.

Our next result is related to uniqueness of solutions. We shall consider function spaces with exponential decay in the velocity variable, for $m \in \mathbb{R}$

$$\tilde{\mathcal{E}}_0^m(\mathbb{R}^6) = \left\{ g \in \mathcal{D}'(\mathbb{R}_{x,v}^6); \exists \rho > 0 \text{ s.t. } e^{\rho \langle v \rangle^2} g \in L^\infty(\mathbb{R}_x^3; H^m(\mathbb{R}_v^3)) \right\},$$

and for $T > 0$

$$\begin{aligned} \tilde{\mathcal{E}}^m([0, T] \times \mathbb{R}_{x,v}^6) &= \{f \in C^0([0, T]; \mathcal{D}'(\mathbb{R}_{x,v}^6)); \exists \rho > 0 \\ &\text{s.t. } e^{\rho(v)^2} f \in L^\infty([0, T] \times \mathbb{R}_x^3; H^m(\mathbb{R}_v^3))\}. \end{aligned}$$

Theorem 1.2. *Assume that $0 < s < 1$ and $\max\{-3, -3/2 - 2s\} < \gamma < 2 - 2s$. Let $f_0 \geq 0$ and $f_0 \in \tilde{\mathcal{E}}_0^0(\mathbb{R}^6)$. Let $0 < T < +\infty$ and suppose that $f \in \tilde{\mathcal{E}}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$ is a non-negative solution to the Cauchy problem (1.1). Then any solution in the function space $\tilde{\mathcal{E}}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$ coincides with f .*

Remark.

1) Note that the solutions considered above are not necessarily classical ones. Moreover, Theorem 1.2 does not require the coercivity. On the other hand, if we suppose the coercivity, then we can get the uniqueness in the function space $\tilde{\mathcal{E}}^s([0, T] \times \mathbb{R}_{x,v}^6)$, without the non-negativity assumption, see precisely Theorem 4.1 in Section 4.

2) We can also remove the restriction $\gamma + 2s < 2$, if we consider the small perturbation around Maxwellian, see precisely Theorem 4.3 in Section 4.

3) Finally, in the soft potential case $\gamma + 2s \leq 0$, we can refine the above uniqueness results which can be applied to the solution of Theorem 1.4 of [9], see precisely Theorem 4.4 in Section 4.

Our next issue is about the non-negativity of solutions. We shall use the following modified weighted Sobolev spaces: For $k \in \mathbb{N}$, $\ell \in \mathbb{R}$

$$\tilde{\mathcal{H}}_\ell^k(\mathbb{R}^6) = \left\{ f \in \mathcal{S}'(\mathbb{R}_{x,v}^6); \|f\|_{\tilde{\mathcal{H}}_\ell^k(\mathbb{R}^6)}^2 = \sum_{|\alpha|+|\beta| \leq N} \|\tilde{W}_{\ell-|\beta|} \partial_\beta^\alpha f\|_{L^2(\mathbb{R}^6)}^2 < +\infty \right\},$$

where $\tilde{W}_\ell = (1 + |v|^2)^{s+\gamma/2+\ell/2}$.

Combining with the existence results of [9, 10] and the above Theorem 1.2, one has

Theorem 1.3. *Let $0 < s < 1$, $\gamma > \max\{-3, -3/2 - 2s\}$, $k \geq 6$. There exist $\varepsilon_0 > 0$ and ℓ_0 such that the Cauchy problem (1.1) admits a unique global solution $f = \mu + \mu^{1/2} g$ for initial datum $f_0 = \mu + \mu^{1/2} g_0$ satisfying*

1) $g \in L^\infty([0, +\infty[; H_{\ell_0}^k(\mathbb{R}^6))$, if $\gamma + 2s > 0$ and $\|g_0\|_{H_{\ell_0}^k(\mathbb{R}^6)} \leq \varepsilon_0$.

2) $g \in L^\infty([0, +\infty[; \tilde{\mathcal{H}}_{\ell_0}^k(\mathbb{R}^6))$, if $\gamma + 2s \leq 0$ and $\|g_0\|_{\tilde{\mathcal{H}}_{\ell_0}^k(\mathbb{R}^6)} \leq \varepsilon_0$.

If $f_0 = \mu + \mu^{1/2} g_0 \geq 0$, then the above solution $f = \mu + \mu^{1/2} g \geq 0$.

Remark. *The existence of global solution was proved in [9, 10], while the uniqueness follows from Theorem 1.2, more precisely Theorem 4.3, in Section 4.*

One of the basic issues in the mathematical theory for Boltzmann equation theory is about the convergence of solutions to equilibrium. This topic has been recently renewed and complemented by proofs of optimal convergence rates in the whole space, see for example [20, 21, 27, 37, 38] and references therein. This is closely related to the study of the hypocoercivity theory that is about the interplay of a conservative operator and a degenerate diffusive operator which gives the convergence to the equilibrium. Note that this kind of interplay also gives the full regularization.

For later use, denote

$$\mathcal{N} = \text{span}\{\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}} v_i, \mu^{\frac{1}{2}} |v|^2, \quad i = 1, 2, 3\},$$

as the null space of the linearized Boltzmann collision operator, and \mathbf{P} the projection operator to \mathcal{N} in $L^2(\mathbb{R}_v^3)$.

For the problem considered in this paper, we have the following convergence rate estimates.

Theorem 1.4. *Let $0 < s < 1$ and $f = \mu + \mu^{1/2} g$ be a global solution of the Cauchy problem (1.1) with initial datum $f_0 = \mu + \mu^{1/2} g_0$. We have the following two cases:*

1) Let $\gamma + 2s > 0$, $N \geq 6$, $\ell > 3/2 + 2s + \gamma$. There exists $\varepsilon_0 > 0$ such that if $\|g_0\|_{L^1(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}^2 + \|g_0\|_{H_\ell^N(\mathbb{R}^6)}^2 \leq \varepsilon_0$ and $g \in L^\infty([0, +\infty[; H_\ell^N(\mathbb{R}^6))$, then we have for all $t > 0$,

$$\|g(t)\|_{L^2(\mathbb{R}^6)}^2 = \|\mathbf{P}g(t)\|_{L^2(\mathbb{R}^6)}^2 + \|(\mathbf{I} - \mathbf{P})g(t)\|_{L^2(\mathbb{R}^6)}^2 \lesssim (1+t)^{-3/2},$$

and

$$\sum_{1 \leq |\alpha| \leq N} \|\partial^\alpha \mathbf{P}g(t)\|_{L^2(\mathbb{R}^6)}^2 + \sum_{|\alpha| \leq N} \|\partial^\alpha (\mathbf{I} - \mathbf{P})g(t)\|_{L^2(\mathbb{R}^6)}^2 \lesssim (1+t)^{-5/2}.$$

2) Let $\max\{-3, -\frac{3}{2} - 2s\} < \gamma \leq -2s, N \geq 6, \ell \geq N + 1$. There exists $\varepsilon_0 > 0$ such that if $\|g_0\|_{\tilde{H}_\ell^N(\mathbb{R}^6)}^2 \leq \varepsilon_0$ and $g \in L^\infty([0, +\infty[; \tilde{H}_\ell^N(\mathbb{R}^6))$, then we have for all $t > 0$,

$$\sup_{x \in \mathbb{R}^3} \|g(t)\|_{H^{N-3}(\mathbb{R}_v^3)}^2 \lesssim (1+t)^{-1}.$$

We emphasize that the above convergence rate for the hard potential case is optimal in the sense that it is the same for the linearized problem through either spectrum analysis in [32], or direct Fourier transform using the compensating function introduced in [27]. However, the convergence rate for soft potential is not optimal. In fact, how to obtain an optimal convergence rate even for the cutoff soft potential is still an unsolved problem [33, 36].

We also would like to mention that the above convergence rate is for the whole space setting. If the problem is instead considered on the torus with small perturbation, then the exponential decay for hard potential can be obtained, and this point is a direct consequence of the energy estimates given in [10] by using Poincaré inequality (this is for example the case considered in [23]).

Before presenting the plan of the paper we want to give some comments on our proofs. First of all, our proof of regularization property applies to the classical solutions obtained in [9, 10]. Note that from those existence theorems, one can show that if the initial data satisfying $\|g_0\|_{H_\ell^k} \leq \varepsilon_k$ for $k \geq 6$ and $\ell \geq \ell_0$ for some ℓ_0 , the solution is also in H^k when ε_k is small. However, the current existence theory does not yield that $g \in H^{k+N}$, under the condition that $g_0 \in H^{k+N}$ for $N > 0$ if $\|g_0\|_{H^{k+N}}$ is not small. Therefore, we can not just mollify the initial data to study the full regularity by working formally on the smooth solution. Instead, we need analytic tools from pseudo-differential theory and harmonic analysis to study the gain of regularity rigorously. In fact, it is a standard technic for the hypoellipticity of linear differential operators [25, 30, 31]. The same comments apply for the uniqueness and positivity issues for which we give also rigorous proofs.

The paper is organized as follows. In Section 2, we give the functional analysis of the collision operator, including upper bounds, commutators estimates and coercivity. In Section 3, we prove Theorem 1.1 giving the regularization of solutions. Section 4 is devoted to precise versions of uniqueness results related to Theorem 1.2, while Section 5 proves the non-negativity of solutions. Finally the last Section proves Theorem 1.4 about the convergence of solutions to equilibrium.

Notations: Herein, letters f, g, \dots stand for various suitable functions, while C, c, \dots stand for various numerical constants, independent from functions f, g, \dots and which may vary from line to line. Notation $A \lesssim B$ means that there exists a constant C such that $A \leq CB$, and similarly for $A \gtrsim B$. While $A \sim B$ means that there exist two generic constants $C_1, C_2 > 0$ such that

$$C_1 A \leq B \leq C_2 A.$$

2. FUNCTIONAL ANALYSIS OF THE COLLISION OPERATOR

In this section, we study the upper bound and commutators estimates for the collision operator $Q(\cdot, \cdot)$. Since it is only an operator with respect to velocity variable, in this section, our analysis is on \mathbb{R}_v^3 , forgetting variable x . In what follows, we denote $\tilde{\Phi}_\gamma$ by $\tilde{\Phi}_\gamma(z) = (1 + |z|^2)^{\gamma/2}$. $Q_{\tilde{\Phi}_\gamma}$ will denote the collision operator defined with the modified kinetic factor $\tilde{\Phi}_\gamma$.

2.1. Upper bound estimate. For $0 < s < 1, \gamma \in \mathbb{R}$, we proved the following upper bounded estimate (Theorem 2.1 of [6])

$$(2.1) \quad |(Q_{\tilde{\Phi}_\gamma}(f, g), h)| \lesssim \|f\|_{L_{\ell^+ + (\gamma+2s)^+}^1} \|g\|_{H_{(\ell+\gamma+2s)^+}^{m+s}} \|h\|_{H_{-\ell}^{s-m}},$$

for any $m, \ell \in \mathbb{R}$, and the estimate of commutators with weight (Lemma 2.4 of [6])

$$(2.2) \quad \left| (W_\ell Q_{\tilde{\Phi}_\gamma}(f, g) - Q_{\tilde{\Phi}_\gamma}(f, W_\ell g), h) \right| \lesssim \|f\|_{L_{\ell+(2s-1)^+ + \gamma^+}^1} \|g\|_{H_{\ell+(2s-1)^+ + \gamma^+}^{(2s-1+\varepsilon)^+}} \|h\|_{L^2},$$

for any $\ell \in \mathbb{R}$.

For the singular type of kinetic factors considered herein $|v - v_*|^\gamma$, we need to take into account the singular behavior close to 0. Therefore, we decompose the kinetic factor in two parts. Let $0 \leq \phi(z) \leq 1$ be a smooth radial function with value 1 for z close to 0, and 0 for large values of z . Set

$$\Phi_\gamma(z) = \Phi_\gamma(z)\phi(z) + \Phi_\gamma(z)(1 - \phi(z)) = \Phi_c(z) + \Phi_c(z).$$

And then correspondingly we can write

$$Q(f, g) = Q_c(f, g) + Q_{\bar{c}}(f, g),$$

where the kinetic factor in the collision operator is defined according to the decomposition respectively. Since $\Phi_{\bar{c}}(z)$ is smooth, and $\Phi_{\bar{c}}(z) \leq \tilde{\Phi}_\gamma(z)$, $Q_{\bar{c}}(f, g)$ has similar properties as for $Q_{\tilde{\Phi}_\gamma}(f, g)$ as regards upper bounds and commutators estimations, which means that (2.1) and (2.2) hold true for $Q_{\bar{c}}(f, g)$.

From now on, we concentrate on the study the singular part $Q_c(f, g)$, referring for the smooth part $Q_{\bar{c}}(f, g)$ to [6]. Note that in [9], the same decomposition was also used, but for the modified operator $\Gamma(f, g)$. Here, the absence of the gaussian factor slightly adds some more difficulties.

Proposition 2.1. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$ and $m \in [s - 1, s]$. Then we have*

$$|(Q_c(f, g), h)| \lesssim \|f\|_{L^2} \|g\|_{H^{s+m}} \|h\|_{H^{s-m}}.$$

Remark 2.2. *As will be clearer from the proof below, the following precise estimates are also available: if $\gamma + 2s > 0$, we have*

$$|(Q_c(f, g), h)| \lesssim \|f\|_{L^1} \|g\|_{H^{s+m}} \|h\|_{H^{s-m}}.$$

and moreover if $\gamma + 2s > -1$, we have

$$|(Q_c(f, g); h)| \lesssim \|f\|_{L^{3/2}} \|g\|_{H^{s+m}} \|h\|_{H^{s-m}}.$$

For the proof of Proposition 2.1, we shall follow some of the arguments form [9]. First of all, by using the formula from the Appendix of [2], and as in [9], one has

$$\begin{aligned} (Q_c(f, g), h) &= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma. \\ &= \iiint_{|\xi^-| \leq \frac{1}{2} \langle \xi_* \rangle} \dots d\xi d\xi_* d\sigma + \iiint_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \dots d\xi d\xi_* d\sigma \\ &= A_1(f, g, h) + A_2(f, g, h). \end{aligned}$$

Then, we write $A_2(f, g, h)$ as

$$\begin{aligned} A_2 &= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \hat{\Phi}_c(\xi_* - \xi^-) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma. \\ &\quad - \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \hat{\Phi}_c(\xi_*) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma \\ &= A_{2,1}(f, g, h) - A_{2,2}(f, g, h). \end{aligned}$$

While for A_1 , we use the Taylor expansion of $\hat{\Phi}_c$ at order 2 to have

$$A_1 = A_{1,1}(f, g, h) + A_{1,2}(f, g, h)$$

where

$$A_{1,1} = \iiint b \xi^- \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2} \langle \xi_* \rangle} \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma,$$

and $A_{1,2}(F, G, H)$ is the remaining term corresponding to the second order term in the Taylor expansion of $\hat{\Phi}_c$. The $A_{i,j}$ with $i, j = 1, 2$ are estimated by the following lemmas.

Lemma 2.3. *We have*

$$|A_{1,1}| + |A_{1,2}| \lesssim \|f\|_{L^2} \|f\|_{H^{s+m}} \|h\|_{H^{s-m}}.$$

Proof. Considering firstly $A_{1,1}$, by writing

$$\xi^- = \frac{|\xi|}{2} \left(\left(\frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\xi}{|\xi|} - \sigma \right) + \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2},$$

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on SS^2 . Hence, we have

$$A_{1,1} = \iint_{\mathbb{R}^6} K(\xi, \xi_*) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_*,$$

where

$$K(\xi, \xi_*) = \int_{S^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left(1 - \left(\frac{\xi}{|\xi|} \cdot \sigma\right)\right) \frac{\xi}{2} \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} d\sigma.$$

Note that $|\nabla \hat{\Phi}_c(\xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+1}}$, from the Appendix of [9]. If $\sqrt{2}|\xi| \leq \langle \xi_* \rangle$, then $|\xi^-| \leq \langle \xi_* \rangle/2$ and this imply the fact that $0 \leq \theta \leq \pi/2$, and we have

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi/2} \theta^{1-2s} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle}\right).$$

On the other hand, if $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$, then

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi\langle \xi_* \rangle/(2|\xi|)} \theta^{1-2s} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle}\right)^{2s-1}.$$

Hence we obtain

$$(2.3) \quad |K(\xi, \xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle}\right) \mathbf{1}_{\sqrt{2}|\xi| \leq \langle \xi_* \rangle} + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} \geq |\xi|/2} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle}\right)^{2s} \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \right\}.$$

Notice that

$$(2.4) \quad \begin{cases} \langle \xi \rangle \lesssim \langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle & \text{on supp } \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \\ \langle \xi \rangle \sim \langle \xi - \xi_* \rangle & \text{on supp } \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \\ \langle \xi \rangle \sim \langle \xi_* \rangle \gtrsim \langle \xi - \xi_* \rangle & \text{on supp } \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} \geq |\xi|/2}. \end{cases}$$

Replacing the factors $\langle \xi \rangle/\langle \xi_* \rangle$ and $(\langle \xi \rangle/\langle \xi_* \rangle)^{2s}$ on the right hand side of (2.3) by

$$\left(\frac{\langle \xi - \xi_* \rangle}{\langle \xi_* \rangle}\right)^{s+m} \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle}\right)^{s-m} \quad \text{and} \quad \frac{\langle \xi \rangle^{s+m} \langle \xi - \xi_* \rangle^{s-m}}{\langle \xi_* \rangle^{2s}},$$

respectively, we obtain

$$(2.5) \quad |K(\xi, \xi_*)| \lesssim \frac{\langle \xi \rangle^{s-m} \langle \xi - \xi_* \rangle^{s+m}}{\langle \xi_* \rangle^{3+\gamma+2s}} + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi_* \rangle^{3+\gamma+s-m} \langle \xi - \xi_* \rangle^{s+m}} \langle \xi \rangle^{s-m} \langle \xi - \xi_* \rangle^{s+m}.$$

Putting $\tilde{g}(\xi) = \langle \xi \rangle^{s+m} \hat{g}(\xi)$, $\tilde{h}(\xi) = \langle \xi \rangle^{s-m} \hat{h}(\xi)$, we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |A_{1,1}|^2 &\lesssim \int_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+2s}} |\tilde{g}(\xi - \xi_*)|^2 d\xi d\xi_* \int_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+2s}} |\tilde{h}(\xi)|^2 d\xi d\xi_* \\ &+ \int_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{6+2\gamma+2s-2m}} \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2s+2m}} d\xi d\xi_* \int_{\mathbb{R}^6} |\tilde{g}(\xi - \xi_*)|^2 |\tilde{h}(\xi)|^2 d\xi d\xi_* \\ &= \mathcal{A} + \mathcal{D}. \end{aligned}$$

Since $\langle \xi_* \rangle^{-(3+\gamma+2s)} \in L^2$, the Cauchy-Schwarz inequality again shows

$$\mathcal{A} \lesssim \left(\int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+2s}} d\xi_* \right) \|g\|_{H^{s+m}} \lesssim \|f\|_{L^2} \|g\|_{H^{s+m}}, \quad \mathcal{B} \lesssim \|f\|_{L^2} \|h\|_{H^{s-m}}.$$

Note that

$$\int \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2s+2m}} d\xi \lesssim \begin{cases} \frac{1}{\langle \xi_* \rangle^{-3+2s+2m}} & \text{if } s+m < 3/2 \\ \log \langle \xi_* \rangle & \text{if } s+m \geq 3/2. \end{cases}$$

Since $3 + 2(\gamma + 2s) > 0$ and $6 + 2\gamma + 2(s - m) > 0$, we get $\mathcal{D} \leq \|f\|_{L^2}^2$, which concludes the desired bound for $A_{1,1}$.

Remark that if $\gamma + 2s > 0$ then we obtain $|A_{1,1}| \lesssim \|F\|_{L^1} \|G\|_{H^{s+m}} \|H\|_{H^{s-m}}$ because $\|\hat{F}\|_{L^\infty} \leq \|F\|_{L^1}$. If $0 \geq \gamma + 2s > -3/2$ then we can just estimate $|A_{1,1}| \lesssim \|F\|_{L^2} \|G\|_{H^{s+m}} \|H\|_{H^{s-m}}$. If $0 \geq \gamma + 2s > -1$ then $|A_{1,1}| \lesssim \|F\|_{L^{3/2}} \|G\|_{H^s} \|H\|_{H^s}$. Those follow from the Hölder inequality and $\|\hat{F}\|_{L^p} \leq \|F\|_{L^q}$ with $1/p + 1/q = 1$.

Now we consider $A_{1,2}(f, g, h)$, which comes from the second order term of the Taylor expansion. Note that

$$A_{1,2} = \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \int_0^1 d\tau (\nabla^2 \hat{\Phi}_c)(\xi_* - \tau\xi^-) \cdot \xi^- \cdot \xi^- \hat{F}(\xi_*) \hat{G}(\xi - \xi_*) \hat{H}(\xi) d\sigma d\xi d\xi_*.$$

Again from the Appendix of [9], we have

$$|(\nabla^2 \hat{\Phi}_c)(\xi_* - \tau \xi^-)| \lesssim \frac{1}{\langle \xi_* - \tau \xi^- \rangle^{3+\gamma+2}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+2}},$$

because $|\xi^-| \leq \langle \xi_* \rangle / 2$. Similar to $A_{1,1}$, we can obtain

$$|A_{1,2}| \lesssim \iint_{\mathbb{R}^6} \tilde{K}(\xi, \xi_*) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{h}(\xi) d\xi d\xi_*,$$

where $\tilde{K}(\xi, \xi_*)$ has the following upper bound

$$(2.6) \quad \begin{aligned} \tilde{K}(\xi, \xi_*) &\lesssim \int_0^{\min(\pi/2, \pi \langle \xi_* \rangle / (2|\xi|))} \theta^{1-2s} d\theta \frac{\langle \xi \rangle^2}{\langle \xi_* \rangle^{3+\gamma+2}} \\ &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^2 \mathbf{1}_{\sqrt{2}|\xi| \leq \langle \xi_* \rangle} + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s} \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \right\}, \end{aligned}$$

from which we obtain the same inequality as (2.5) for $\tilde{K}(\xi, \xi_*)$. Hence we obtain the desired bound for $A_{1,2}$. And this completes the proof of the lemma. \square

Lemma 2.4. *We have also*

$$|A_{2,1}| + |A_{2,2}| \lesssim \|f\|_{L^2} \|f\|_{H^{s+m}} \|h\|_{H^{s-m}}.$$

Proof. In view of the definition of $A_{2,2}$, the fact that $|\xi| \sin(\theta/2) = |\xi^-| \geq \langle \xi_* \rangle / 2$ and $\theta \in [0, \pi/2]$ imply $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$. We can then directly compute the spherical integral appearing inside $A_{2,2}$ together with Φ as follows:

$$(2.7) \quad \begin{aligned} \int b \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \Phi(\xi_*) \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} d\sigma &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \frac{\langle \xi \rangle^{2s}}{\langle \xi_* \rangle^{2s}} \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} \\ &\lesssim \frac{\langle \xi \rangle^{s-m} \langle \xi - \xi_* \rangle^{s+m}}{\langle \xi_* \rangle^{3+\gamma+2s}} + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi_* \rangle^{3+\gamma+s-m} \langle \xi - \xi_* \rangle^{s+m}} \langle \xi \rangle^{s-m} \langle \xi - \xi_* \rangle^{s+m}, \end{aligned}$$

which yields the desired estimate for $A_{2,2}$.

We now turn to

$$A_{2,1} = \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \hat{\Phi}_c(\xi_* - \xi^-) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{h}(\xi) d\sigma d\xi d\xi_*.$$

Firstly, note that we can work on the set $|\xi_* \cdot \xi^-| \geq \frac{1}{2} |\xi^-|^2$. In fact, on the complementary of this set, we have $|\xi_* \cdot \xi^-| \leq \frac{1}{2} |\xi^-|^2$ so that $|\xi_* - \xi^-| \geq |\xi_*|$, and in this case, we can proceed in the same way as for $A_{2,2}$. Therefore, it suffices to estimate

$$A_{2,1,p} = \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \mathbf{1}_{|\xi_* \cdot \xi^-| \geq \frac{1}{2} |\xi^-|^2} \hat{\Phi}_c(\xi_* - \xi^-) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\sigma d\xi d\xi_*.$$

By

$$\mathbf{1} = \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle > \langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2}$$

we decompose

$$A_{2,1,p} = A_{2,1,p}^{(1)} + A_{2,1,p}^{(2)} + A_{2,1,p}^{(3)}.$$

On the sets for above integrals, we have $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi_* \rangle$, because $|\xi^-| \lesssim |\xi_*|$ that follows from $|\xi^-|^2 \leq 2|\xi_* \cdot \xi^-| \lesssim |\xi^-| |\xi_*|$. Furthermore, on the sets for $A_{2,1,p}^{(1)}$ and $A_{2,1,p}^{(2)}$ we have $\langle \xi \rangle \sim \langle \xi_* \rangle$, so that $\sup \left(b \mathbf{1}_{|\xi^-| \geq \frac{1}{2} \langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \right) \lesssim \mathbf{1}_{|\xi^-| \leq |\xi|/\sqrt{2}}$ and $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi \rangle$. Hence we have, in view of $s - m \geq 0$,

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^2 |\hat{f}(\xi_*)|^2 \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle}}{\langle \xi_* - \xi^- \rangle^{2s-2m} \langle \xi - \xi_* \rangle^{2s+2m}} d\xi d\xi_* d\sigma \\ &\quad \times \iiint |\langle \xi - \xi_* \rangle^{s+m} \hat{g}(\xi - \xi_*)|^2 |\langle \xi \rangle^{s-m} \hat{h}(\xi)|^2 d\sigma d\xi d\xi_* . \end{aligned}$$

If $\gamma + 2s > 0$ then by the change of variables $\xi_* - \xi^- \rightarrow u$ we have

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \|\hat{F}\|_{L^\infty}^2 \int \langle u \rangle^{-(6+2\gamma+2s-2m)} \int \frac{\mathbf{1}_{\langle w \rangle \leq \langle u \rangle}}{\langle w \rangle^{2s+2m}} dw du \|G\|_{H^{s+m}}^2 \|H\|_{H^{s-m}}^2 \\ &\lesssim \|F\|_{L^1}^2 \|G\|_{H^{s+m}}^2 \|H\|_{H^{s-m}}^2. \end{aligned}$$

If $\gamma + 2s > -3/2$ then with $u = \xi_* - \xi^-$ we have

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \int |\hat{f}(\xi_*)|^2 \left\{ \sup_u \langle u \rangle^{-(6+2\gamma+2s-2m)} \int \frac{\mathbf{1}_{\langle \xi^+ - u \rangle \leq \langle u \rangle}}{\langle \xi^+ - u \rangle^{2s+2m}} d\xi^+ \right\} d\xi_* \|g\|_{H^{s+m}}^2 \|h\|_{H^{s-m}}^2 \\ &\lesssim \|f\|_{L^2}^2 \|g\|_{H^{s+m}}^2 \|h\|_{H^{s-m}}^2, \end{aligned}$$

because $d\xi \sim d\xi^+$ on the support of $\mathbf{1}_{|\xi^-| \leq |\xi|/\sqrt{2}}$. In the case $\gamma + 2s > -1$, by the Hölder inequality and the change of variables $u = \xi_* - \xi^-$ we have

$$\begin{aligned} |A_{2,1,p}^{(1)}|^2 &\lesssim \left(\int |\hat{f}(\xi_*)|^3 d\xi_* \right)^{2/3} \\ &\quad \times \left(\int \left(\langle u \rangle^{-(6+2\gamma+2s-2m)} \int \frac{\mathbf{1}_{\langle \xi^+ - u \rangle \leq \langle u \rangle}}{\langle \xi^+ - u \rangle^{2s+2m}} d\xi^+ \right) du \right)^{1/3} \|g\|_{H^{s+m}}^2 \|h\|_{H^{s-m}}^2 \\ &\lesssim \|f\|_{L^{3/2}}^2 \|g\|_{H^{s+m}}^2 \|h\|_{H^{s-m}}^2. \end{aligned}$$

As for $A_{2,1,p}^{(2)}$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |A_{2,1,p}^{(2)}|^2 &\lesssim \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^{2s}} |\langle \xi - \xi_* \rangle^{s+m} \hat{g}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^{2s}} |\langle \xi \rangle^{s-m} \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

Since we have

$$\iint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^{2s}} d\xi_* d\sigma \lesssim \begin{cases} \|f\|_{L^1} & \text{if } \gamma + 2s > 0 \\ \|f\|_{L^2} & \text{if } \gamma + 2s > -3/2, \\ \|f\|_{L^{3/2}} & \text{if } \gamma + 2s > -1, \end{cases}$$

we have the desired estimates for $A_{2,1,p}^{(2)}$.

On the set $A_{2,1,p}^{(3)}$ we have $\langle \xi \rangle \sim \langle \xi - \xi_* \rangle$. Hence

$$\begin{aligned} |A_{2,1,p}^{(3)}|^2 &\lesssim \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi \rangle^{2s}} |\langle \xi - \xi_* \rangle^{s+m} \hat{g}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi \rangle^{2s}} |\langle \xi \rangle^{s-m} \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

We use the change of variables in ξ_* , $u = \xi_* - \xi^-$. Note that $|\xi^-| \geq \frac{1}{2}\langle u + \xi^- \rangle$ implies $|\xi^-| \geq \langle u \rangle / \sqrt{10}$. If $\gamma + 2s > 0$ then we have

$$\begin{aligned} \iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\hat{f}(\xi_*)|}{\langle \xi \rangle^{2s}} d\sigma d\xi_* &\lesssim \|\hat{f}\|_{L^\infty} \int \left(\frac{|\xi|}{\langle u \rangle} \right)^{2s} \langle u \rangle^{-(3+\gamma)} \langle \xi \rangle^{-2s} du \\ &\lesssim \|\hat{f}\|_{L^\infty}. \end{aligned}$$

On the other hand, if $\gamma + 2s > -3/2$ (or $0 \geq \gamma + 2s > -1$) then this integral is upper bounded by

$$\begin{aligned} & \iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| \langle \xi_* \rangle^{2s/q} |\hat{f}(\xi_*)|}{\langle \xi \rangle^{2s/p} \langle \xi_* - \xi^- \rangle^{2s/q}} d\sigma d\xi_* \\ & \leq \left(\iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^p}{\langle \xi \rangle^{2s} \langle \xi_* - \xi^- \rangle^{2sp/q}} d\sigma d\xi_* \right)^{1/p} \left(\iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{\langle \xi_* \rangle^{2s} |\hat{f}(\xi_*)|^q}{\langle \xi \rangle^{2s}} d\sigma d\xi_* \right)^{1/q} \\ & \leq \left(\iint b \mathbf{1}_{|\xi^-| \geq \langle u \rangle} \frac{|\hat{\Phi}_c(u)|^p}{\langle \xi \rangle^{2s} \langle u \rangle^{2sp/q}} d\sigma du \right)^{1/p} \|f\|_{L^q} \lesssim \int \frac{du}{\langle u \rangle^{p(3+\gamma+2s)}} \|f\|^p, \end{aligned}$$

where $1/p + 1/q = 1$, $p = 2$ (or $p = 3/2$). Hence we also obtain the desired estimates for $A_{2,1,p}^{(3)}$. The proof of the lemma is complete \square

Proposition 2.1 is then a direct consequence of Lemmas 2.3 and 2.4, while the statements of Remark 2.2 are mentioned in the proof of the two previous lemmas.

2.2. Estimate of commutators with weights. The following estimation on commutators will now be proved. Because of the weight loss related to the Boltzmann equation, test functions involve these weights, and therefore, this estimation is quite necessary.

Proposition 2.5. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$. For any $\ell, \beta, \delta \in \mathbb{R}$*

$$\left| (W_\ell Q_c(f, g) - Q_c(f, W_\ell g), h) \right| \lesssim \|f\|_{L_{\ell-1-\beta-\delta}^2} \|g\|_{H_\beta^{(2s-1+\epsilon)^+}} \|h\|_{L_\delta^2}.$$

The next two lemmas are a preparation for the complete proof of this Proposition.

Lemma 2.6. *If $\lambda < 3/2$ then*

$$(2.8) \quad \iint_{|v-v_*| \leq 1} |f(v_*)| \frac{|g(v)|^2}{|v-v_*|^\lambda} dv dv_* \lesssim \|f\|_{L^2} \|g\|_{L^2}^2.$$

If $3/2 < \lambda < 3$ then

$$\iint_{|v-v_*| \leq 1} |f(v_*)| \frac{|g(v)|^2}{|v-v_*|^\lambda} dv dv_* \lesssim \|f\|_{L^2} \|g\|_{H^{\frac{\lambda}{2} - \frac{3}{4}}}.$$

Proof. Since $|v_*|^{-\lambda} \mathbf{1}_{|v_*| \leq 1} \in L^2$ for $\lambda < 3/2$, it follows from the Cauchy-Schwarz inequality that if $\lambda < 3/2$ then

$$\begin{aligned} \iint_{|v-v_*| \leq 1} |f_*| \frac{|g|^2}{|v-v_*|^\lambda} dv dv_* & \leq \int |g|^2 \left(\int_{|v-v_*| \leq 1} |v-v_*|^{-2\lambda} dv_* \right)^{1/2} \left(\int |f_*|^2 dv_* \right)^{1/2} dv \\ & \lesssim \|f\|_{L^2} \|g\|_{L^2}^2. \end{aligned}$$

It follows from the Hardy-Littlewood-Sobolev inequality that if $3/2 < \lambda < 3$ then

$$\begin{aligned} \iint_{|v-v_*| \leq 1} |f_*| \frac{|g|^2}{|v-v_*|^\lambda} dv dv_* & \lesssim \|f\|_{L^2} \|g\|_{L^p}^2 \quad \text{with } \frac{1}{p} = \frac{3}{2} - \frac{\lambda}{3} < 1 \\ & \lesssim \|f\|_{L^2} \|g\|_{H^{\frac{\lambda}{2} - \frac{3}{4}}}^2 \end{aligned}$$

because of the Sobolev embedding theorem. \square

Lemma 2.7. *Let $0 < s < 1$ and $\gamma > \max\{-3, -2s - 3/2\}$. Then*

$$\iiint b \Phi_c^\gamma |f(v_*)| |g(v) - g(v')|^2 dv dv_* d\sigma \lesssim \|f\|_{L^2} \|g\|_{H^s}^2.$$

Proof. Note that

$$\begin{aligned} (Q_c^{\Phi_\gamma}(|f|, g), g) & = -\frac{1}{2} \iiint b \Phi_c^\gamma |f_*| |g - g'|^2 dv dv_* d\sigma \\ & \quad + \frac{1}{2} \iiint b \Phi_c^\gamma |f_*| (g'^2 - g^2) dv dv_* d\sigma. \end{aligned}$$

Since Proposition 2.1 with $m = 0$ is applicable to the left hand side, it suffices to consider the second term of the right hand side. It follows from the cancellation lemma of [2] (more precisely the formula (29) there) that

$$\iiint b\Phi_c^\gamma |f_*| (g'^2 - g^2) dv dv_* d\sigma = \int |f(v_*)| S(v - v_*) g(v)^2 dv dv_*,$$

where

$$\begin{aligned} S(v - v_*) &= \Phi_\gamma(v - v_*) \phi(v - v_*) \left(2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta \left(\frac{1}{\cos^{3+\gamma+1}(\theta/2)} - 1 \right) d\theta \right) \\ &\quad + \Phi_\gamma(v - v_*) \left(2\pi \int_0^{\pi/2} \frac{b(\cos \theta) \sin \theta}{\cos^{3+\gamma+1}(\theta/2)} \left(\phi\left(\frac{v - v_*}{\cos(\theta/2)}\right) - \phi(v - v_*) \right) d\theta \right). \end{aligned}$$

The integral of the second term on the right hand side can be written as $\tilde{\phi}(v - v_*)$ whose support is contained in $\{0 < |v - v_*| \leq 1\}$. Since $s > -\gamma/2 - 3/4$, the estimation for the first term just follows from Lemma 2.6 because the case $\gamma = -3/2$ can be treated as $\gamma - \varepsilon$ for any small $\varepsilon > 0$. \square

Proof of Proposition 2.5. We write

$$\begin{aligned} (W_\ell Q_c(f, g) - Q_c(f, W_\ell g), h) &= \iiint b\Phi_c f_*' g'(W_\ell(v) - W_\ell(v')) h dv dv_* d\sigma \\ &= \iiint b\Phi_c (W_\ell(v') - W_\ell(v)) f_* g' h' dv dv_* d\sigma \\ &\quad + \iiint b\Phi_c (W_\ell(v') - W_\ell(v)) f_* (g - g') h' dv dv_* d\sigma \\ &= J_1 + J_2. \end{aligned}$$

Set $v_\tau = v + \tau(v' - v)$ for $\tau \in [0, 1]$ and notice that

$$|W_\ell(v') - W_\ell(v)| \lesssim \int_0^1 W_{\ell-1}(v_\tau) d\tau |v - v_*| \sin(\theta/2).$$

On the support of $\phi(v - v_*)$ we have for a large $C > 0$

$$\begin{aligned} \langle v_* \rangle &\lesssim \frac{1}{2} + \frac{1}{C} (|v_*| - |v - v_*|) \leq \frac{1}{2} + \frac{1}{C} (|v_*| - |v_\tau - v_*|) \leq \langle v_\tau \rangle \\ &\leq 1 + |v_*| + |v_\tau - v_*| \leq 1 + |v_*| + |v - v_*| \lesssim \langle v_* \rangle, \end{aligned}$$

so that $\langle v_\tau \rangle \sim \langle v_* \rangle \sim \langle v \rangle \sim \langle v' \rangle$. The Cauchy-Schwarz inequality shows

$$\begin{aligned} |J_2|^2 &\lesssim \iiint b(\cos \theta) \sin^{2s+2\varepsilon}(\theta/2) \Phi_c^{\gamma+2s} |\langle v_* \rangle^{\ell-1-\beta-\delta} f_*| |\langle v' \rangle^\delta h'|^2 dv dv_* d\sigma \\ &\quad \times \left(\iiint b(\cos \theta) \sin^{2-2s-2\varepsilon}(\theta/2) \Phi_c^{\gamma+2-2s} |\langle v_* \rangle^{\ell-1-\beta-\delta} f_*| |\langle v \rangle^\beta g - \langle v \rangle^\beta g'|^2 dv dv_* d\sigma \right. \\ &\quad \left. + \iiint b(\cos \theta) \sin^{2-2s-2\varepsilon}(\theta/2) \Phi_c^{\gamma+2-2s} |\langle v_* \rangle^{\ell-1-\beta-\delta} f_*| (\langle v \rangle^\beta - \langle v' \rangle^\beta)^2 |g'|^2 dv dv_* d\sigma \right) \\ &= J_{2,1} \times (J_{2,2}^{(1)} + J_{2,2}^{(2)}). \end{aligned}$$

Take the change of variables $v \rightarrow v'$ for $J_{2,1}$. Since $-(\gamma + 2s) < 3/2$, it follows from (2.8) that

$$J_{2,1} \lesssim \iint_{|v' - v_*| \leq 1} |\langle v_* \rangle^{\ell-1-\beta-\delta} f_*| \frac{|\langle v' \rangle^\delta h'|^2}{|v' - v_*|^{-\gamma-2s}} dv' dv_* \lesssim \|f\|_{L_{\ell-1-\beta-\delta}^2} \|h\|_{L_\delta^2}$$

Apply Lemma 2.7 with $s = (2s - 1 + \varepsilon)^+$ and $\gamma = \gamma + 2 - 2s$ to $J_{2,2}^{(1)}$. Then

$$J_{2,2}^{(1)} \lesssim \|f\|_{L_{\ell-1-\beta-\delta}^2} \|g\|_{H_\beta^{(2s-1+\varepsilon)^+}}$$

because $\max\{(2s - 1 + \varepsilon)^+, -\frac{\gamma}{2} + s - 1 - \frac{3}{4}\} = (2s - 1 + \varepsilon)^+$. Since (2.8) also implies

$$J_{2,2}^{(2)} \lesssim \iint_{|v' - v_*| \leq 1} |v' - v_*|^{\gamma+4-2s} |\langle v_* \rangle^{\ell-1-\beta-\delta-2} f_*| |\langle v' \rangle^\beta g'|^2 dv' dv_* \lesssim \|f\|_{L_{\ell-3-\beta-\delta}^2} \|g\|_{L_\beta^2},$$

we obtain the desired bound for J_2 . As for J_1 we use the Taylor expansion

$$\langle v' \rangle^\ell - \langle v \rangle^\ell = \nabla(\langle v' \rangle^\ell) \cdot (v' - v) + \frac{1}{2} \int_0^1 \nabla^2(\langle v_\tau \rangle^\ell) d\tau (v' - v)^2.$$

Then, it follows from the symmetry that the integral corresponding to the first term vanishes, so that we have

$$\begin{aligned} |J_1| &\lesssim \left| \iiint b(\cos \theta) \Phi_c \sin^2(\theta/2) |v' - v_*|^2 \nabla^2(\langle v_\tau \rangle^\ell) |f_* g' h'| dv dv_* d\sigma \right| \\ &\lesssim \iint_{|v' - v_*| \leq 1} |v' - v_*|^{\gamma+2} \langle v_* \rangle^{\ell-2-\beta-\delta} f_* |(\langle v \rangle^\beta g)'| \langle v \rangle^\delta h' |dv' dv_* \\ &\lesssim \|f\|_{L^2_{\ell-2-\beta-\delta}} \|g\|_{L^2_\beta}^2 \|h\|_{L^2_\delta}, \end{aligned}$$

which completes the proof of the of Proposition 2.5.

Now using (2.2) with $Q_{\bar{c}}(f, g)$ and the Proposition 2.5, we get

Proposition 2.8. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$. For any $\ell \in \mathbb{R}$,*

$$(2.9) \quad \left| (W_\ell Q(f, g) - Q(f, W_\ell g), h) \right| \lesssim \|f\|_{L^2_{\ell+3/2+(2s-1)^++\gamma^++\epsilon}} \|g\|_{H^{\max\{s+m, (2s-1+\epsilon)^+\}_{\ell+(2s-1)^++\gamma^+}}} \|h\|_{L^2}.$$

We can now prove the upper bound estimate with weights.

Proposition 2.9. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$. Then we have, for any $\ell \in \mathbb{R}$ and $m \in [s - 1, s]$,*

$$\left| (Q(f, g), h) \right| \lesssim (\|f\|_{L^1_{\ell^++(\gamma+2s)^+}} + \|f\|_{L^2}) \|g\|_{H^{\max\{s+m, (2s-1+\epsilon)^+\}_{\ell+(\gamma+2s)^+}}} \|h\|_{H^s_{-\ell}}.$$

Proof. Using (2.1), for any $m, \ell \in \mathbb{R}$,

$$\left| (Q_{\bar{c}}(f, g), h) \right| \lesssim \|f\|_{L^1_{\ell^++(\gamma+2s)^+}} \|g\|_{H^{m+s}_{\ell+(\gamma+2s)^+}} \|h\|_{H^s_{-\ell}}.$$

On the other hand, for any $\ell \in \mathbb{R}$ we have

$$\left| (Q_c(f, g), h) \right| \leq \left| (Q_c(f, W_\ell g), W_{-\ell} h) \right| + \left| (W_\ell Q_c(f, g) - Q_c(f, W_\ell g), W_{-\ell} h) \right|,$$

then Proposition implies, for $m \in [s - 1, s]$

$$\left| (Q_c(f, W_\ell g), W_{-\ell} h) \right| \lesssim \|f\|_{L^2} \|g\|_{H^{s+m}} \|h\|_{H^s_{-\ell}}$$

and Proposition 2.5, for any $\ell, \beta, \delta \in \mathbb{R}$

$$\left| (W_\ell Q_c(f, g) - Q_c(f, W_\ell g), W_{-\ell} h) \right| \lesssim \|f\|_{L^2_{\ell-1-\beta-\delta}} \|g\|_{H^{(2s-1+\epsilon)^+}_\beta} \|W_{-\ell} h\|_{L^2_\delta}.$$

We choose $\delta = 0, \beta = \ell$, since for $m \in [s - 1, s]$, $s - m \geq 0$, ending the proof of Proposition. \square

2.3. Coercivity of collision operators. We study now the coercivity estimate for a small perturbation of μ . For any $0 < s < 1$ and $\gamma > -3$, we recall the non-isotropic norm associated with the cross-section $B(v - v_*, \sigma)$ introduced in [9]

$$(2.10) \quad \|g\|_{\Phi_\gamma}^2 = \int B(v - v_*, \sigma) \mu_* (g' - g)^2 + \int B(v - v_*, \sigma) g_*^2 (\sqrt{\mu'} - \sqrt{\mu})^2 = J_1(g) + J_2(g)$$

where the integration is over $\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times S S^2_\sigma$. The following link with weighted Sobolev norm was shown previously

$$(2.11) \quad C_1 \left\{ \|g\|_{H^s_{\gamma/2}(\mathbb{R}_v^3)}^2 + \|g\|_{L^2_{s+\gamma/2}(\mathbb{R}_v^3)}^2 \right\} \leq \|g\|_{\Phi_\gamma}^2 \leq C_2 \|g\|_{H^s_{s+\gamma/2}(\mathbb{R}_v^3)}^2,$$

where $C_1, C_2 > 0$ are two generic constants. Recall that in the definition of the non-isotropic norm, we obtain an equivalent norm if we replace μ by any positive power of μ .

The coercivity of the linearized operator $-Q(\mu, h)$ is given by the next result

Proposition 2.10. *There exists $C > 0$ such that*

$$-(Q(\mu, h), h)_{L^2(\mathbb{R}_v^3)} \geq \frac{1}{2} \|h\|_{\Phi_\gamma}^2 - C \|h\|_{L^2_{\gamma/2+s}}^2.$$

Proof. Though the statement follows from [9], we give a proof for the convenience of the reader. By definition,

$$\begin{aligned}
-(Q(\mu, h), h)_{L^2(\mathbb{R}_v^3)} &= - \int B(v - v_*, \sigma) (\mu_*' h' - \mu_* h) h d\sigma dv_* dv \\
&= - \int B(v - v_*, \sigma) \mu_* h (h' - h) d\sigma dv_* dv \\
&= \frac{1}{2} \int B(v - v_*, \sigma) \mu_* (h' - h)^2 d\sigma dv_* dv \\
&\quad + \frac{1}{2} \int B(v - v_*, \sigma) \mu_* (h^2 - h'^2) d\sigma dv_* dv \\
&= \frac{1}{2} J_1(h) + I = \frac{1}{2} \|h\|_{\Phi_\gamma}^2 - \frac{1}{2} J_2(h) + I.
\end{aligned}$$

Then we have from [9]

$$|J_2(h)| \leq C_1 \|h\|_{L_{\gamma/2+s}^2},$$

and the cancellation Lemma [2] implies

$$|I| \leq C \|h\|_{L_{\gamma/2+s}^2},$$

thus proving proposition 2.10. \square

Let us note than another proof is also possible by using instead the Appendix.

Lemma 2.11. *Let $0 < s < 1$ and $\gamma > \max\{-3, -2s - 3/2\}$. If we put*

$$\mathcal{D}(\sqrt{\mu} f, g) = \int B(\sqrt{\mu} f)_* (g - g')^2 dv dv_* d\sigma,$$

then there exists a $C > 0$ such that

$$(2.12) \quad \left| (Q(\sqrt{\mu} f, g), g)_{L^2(\mathbb{R}^3)} + \frac{1}{2} \mathcal{D}(\sqrt{\mu} f, g) \right| \leq C \begin{cases} \|f\|_{L^2} \|g\|_{L_{\gamma/2}^2}^2 & \text{if } \gamma > -3/2 \\ \|f\|_{L^2} \|g\|_{H_{\gamma/2}^{s'}}^2 & \text{if } -3/2 \geq \gamma \\ \|f\|_{H^{2s'}} \|g\|_{L_{\gamma/2+s'}^2}^2 & \text{if } -3/2 \geq \gamma \end{cases}$$

for any $s' \in]0, s[$ satisfying $\gamma + 2s' > -3/2$ and $s' < 3/4$.

Proof. The left hand side of (2.12) equals

$$\frac{1}{2} \int B \sqrt{\mu}_* f_* (g^2 - (g')^2) d\sigma dv dv_*,$$

and by the cancellation lemma of [2]

$$\lesssim \int \sqrt{\mu}_* |f_*| \int |v - v_*|^\gamma g^2 dv dv_* = J(f, g).$$

Divide the integral to $\{|v - v_*| \leq 1\}$ and another region, if necessary. Then it follows from Lemma 2.6 that we obtain the first two estimates. The third estimate is a direct consequence of Pitt's inequality,

$$\begin{aligned}
J(f, g) &\lesssim \int \left(\int |v - v_*|^{2(\gamma+2s')} \mu_* dv_* \right)^{1/2} \left(\int \frac{f_*^2}{|v - v_*|^{4s'}} dv_* \right)^{1/2} |g|^2 dv \\
&\lesssim \|f\|_{H^{2s'}} \|g\|_{L_{\gamma/2+s'}^2}^2,
\end{aligned}$$

where we choose $0 < 2s' < 3/2$. \square

Lemma 2.12. *Let $0 < s < 1$ and $\gamma > \max\{-3, -3/2 - 2s\}$. Then for any $N \in \mathbb{N}$ we have*

$$(2.13) \quad \mathcal{D}(\sqrt{\mu} |f|, g) \lesssim \|f\|_{L_{-N}^2} \|g\|_{\Phi_\gamma}^2.$$

Proof. Put $F = \sqrt{\mu}|f|$. Then (2.13) in the case $\gamma \geq 0$ follows from Lemma 3.2 of [9] with $f^2 = F$. Suppose that $\gamma < 0$. In view of $\Phi_\gamma = \Phi_c + \Phi_{\bar{c}}$, we write

$$\mathcal{D}(F, g) = \int b(\Phi_c + \Phi_{\bar{c}})F_*(g - g')^2 dv dv_* d\sigma = \mathcal{D}_c(F, g) + \mathcal{D}_{\bar{c}}(F, g).$$

We have

$$\begin{aligned} \mathcal{D}_c(F, g) &\lesssim \int b\Phi_c(\langle v_* \rangle^{|\gamma|} F_*) (\langle v \rangle^{\gamma/2} g - \langle v' \rangle^{\gamma/2} g')^2 dv dv_* d\sigma \\ &\quad + \int b\Phi_c(\langle v_* \rangle^{|\gamma|} F_*) (\langle v' \rangle^{\gamma/2} - \langle v \rangle^{\gamma/2})^2 g^2 dv dv_* d\sigma \\ &= \mathcal{D}_c^{(1)}(F, g) + \mathcal{D}_c^{(2)}(F, g), \end{aligned}$$

because $\langle v_* \rangle \sim \langle v + \tau(v' - v) \rangle$ for any $\tau \in [0, 1]$ on the support of $\phi(v - v_*)$, as stated in the proof of Proposition 2.5. Since

$$\left| \langle v' \rangle^{\gamma/2} - \langle v \rangle^{\gamma/2} \right| \lesssim \int_0^1 \langle v + \tau(v' - v) \rangle^{\gamma/2-1} d\tau |v - v_*| \sin \theta / 2 \lesssim \langle v \rangle^{\gamma/2} |v - v_*| \sin \theta / 2$$

on the support of $\phi(v - v_*)$, it follows from the Cauchy-Schwarz inequality that

$$\mathcal{D}_c^{(2)}(F, g) \lesssim \int \left(\int_{|v-v_*| \leq 1} |v - v_*|^{\gamma+2} \langle v_* \rangle^{|\gamma|} F_* dv_* \right) (\langle v \rangle^{\gamma/2} g)^2 dv \lesssim \|F\|_{L^2_{|\gamma|}} \|g\|_{L^2_{\gamma/2}}^2.$$

By means of Lemma 2.7 we have $\mathcal{D}_c^{(1)}(F, g) \lesssim \|F\|_{L^2_{|\gamma|}} \|g\|_{H^2_{\gamma/2}}^2$. On the other hand, noting that $\Phi_{\bar{c}} \lesssim \langle v - v_* \rangle^\gamma \lesssim \langle v_* \rangle^{|\gamma|} \langle v \rangle^\gamma$ we have

$$\begin{aligned} \mathcal{D}_{\bar{c}}(F, g) &\lesssim \int b(\langle v_* \rangle^{|\gamma|} F_*) (\langle v \rangle^{\gamma/2} g - \langle v' \rangle^{\gamma/2} g')^2 dv dv_* d\sigma \\ &\quad + \int b(\langle v_* \rangle^{|\gamma|} F_*) (\langle v' \rangle^{\gamma/2} - \langle v \rangle^{\gamma/2})^2 g^2 dv dv_* d\sigma \\ &= \mathcal{D}_{\bar{c}}^{(1)}(F, g) + \mathcal{D}_{\bar{c}}^{(2)}(F, g). \end{aligned}$$

It follows from Lemma 3.2 of [9] with $\gamma = 0$ together with Proposition 2.4 of [9] that

$$\mathcal{D}_{\bar{c}}^{(1)}(F, g) \lesssim \|\langle v \rangle^{|\gamma|} F\|_{L^1_{2s}} \|\langle v \rangle^{\gamma/2} g\|_{\Phi_0}^2 \lesssim \|F\|_{L^1_{|\gamma|+2s}} \|g\|_{\Phi_\gamma}^2.$$

Since with $v_\tau = v + \tau(v' - v)$ we have

$$\begin{aligned} \left| \langle v' \rangle^{\gamma/2} - \langle v \rangle^{\gamma/2} \right| &\lesssim \int_0^1 \langle v_\tau \rangle^{\gamma/2-1} d\tau |v - v_*| \sin \theta / 2 \\ &\lesssim \langle v_* \rangle^{|\gamma|/2+1} \int_0^1 \langle v_\tau - v_* \rangle^{\gamma/2-1} |v - v_*| \sin \theta / 2 \\ &\lesssim \langle v_* \rangle^{|\gamma|/2+1} \langle v - v_* \rangle^{\gamma/2-1} |v - v_*| \sin \theta / 2 \lesssim \langle v_* \rangle^{|\gamma|+1} \langle v \rangle^{\gamma/2} \sin \theta / 2 \end{aligned}$$

we see that

$$\mathcal{D}_{\bar{c}}^{(2)}(F, g) \lesssim \int \langle v_* \rangle^{3|\gamma|+2} F_* (\langle v \rangle^{\gamma/2} g)^2 dv dv_* \lesssim \|F\|_{L^1_{3|\gamma|+2}} \|g\|_{L^2_{\gamma/2}}^2.$$

Summing up above four estimates, in view of (2.11) we also obtain the desired estimate (2.13) in the case $\gamma < 0$. \square

By means of Lemma 2.11 and Lemma 2.12, in view of (2.11) we get the following upper bounded estimate, which is needed in order to prove the non linear coercivity for small perturbative solution.

Proposition 2.13. *Let $0 < s < 1$ and $\gamma > \max\{-3, -3/2 - 2s\}$. Then we have*

$$\left| (Q(\sqrt{\mu}g, h), h)_{L^2(\mathbb{R}_v^3)} \right| \lesssim \|g\|_{L^2} \|h\|_{\Phi_\gamma}^2.$$

Remark 2.14. *If we proceed as in the proof of Proposition 3.1 from [9], we can prove*

$$\left| (Q(\sqrt{\mu}g, f), h)_{L^2(\mathbb{R}_v^3)} \right| \lesssim \left\{ \|g\|_{\Phi_\gamma} + \|g\|_{W^{1,\infty}} \right\} \|f\|_{\Phi_\gamma} \|h\|_{\Phi_\gamma}$$

for any $\gamma > -3$.

From Proposition 2.10, Proposition 2.13 and (2.11), we can deduce the following non linear coercivity for the small perturbation g .

Corollary 2.15. *Let $0 < s < 1$ and $\gamma > \max\{-3, -3/2 - 2s\}$. There exist $\eta_0 > 0, \epsilon_0 > 0$ and $C > 0$ such that if $\|g\|_{L^2(\mathbb{R}_+^3)} \leq \epsilon_0$, then we have*

$$\left(-Q(\mu + \sqrt{\mu}g, h), h \right)_{L^2(\mathbb{R}_+^3)} \geq \eta_0 \|h\|_{\Phi_\gamma}^2 - C \|h\|_{L_{\gamma/2+s}^2}^2 \gtrsim \eta_0 \|h\|_{H_{\gamma/2}^s}^2 - C \|h\|_{L_{\gamma/2+s}^2}^2.$$

2.4. Estimate of commutators with pseudo-differential operators. We study now the commutators with pseudo-differential operators: again in the next Sections, these will be used as a rigorous replacement of formal derivatives, and when the operator is a smoothed one, as completely justified test functions.

Proposition 2.16. *Let $M_\lambda(\xi) = \langle \xi \rangle^\lambda$ for $\lambda \geq 0$. Assume that $0 < s < 1$ and $\gamma > \max\{-3, -\frac{3}{2} - 2s\}$. Let $\alpha, \beta, \rho \geq 0$ satisfy*

$$(2.14) \quad 3 + \gamma + \alpha + \beta + \rho > 3/2,$$

$$(2.15) \quad \alpha + \beta \geq 2s - 1,$$

$$(2.16) \quad \beta \leq 1,$$

$$(2.17) \quad \alpha + \beta + \rho \geq s.$$

If $\alpha + \lambda < 3/2$ then we have

$$\left| \left(M_\lambda(D) Q_c(f, g) - Q_c(f, M_\lambda(D)g), h \right) \right| \lesssim \|f\|_{H^\rho} \|M_\lambda(D)g\|_{H^\alpha} \|h\|_{H^\beta}.$$

If $\alpha + \lambda \geq 3/2$ then $\rho = (\lambda - \beta)^+$ satisfies (2.14) and we have

$$\left| \left(M_\lambda(D) Q_c(f, g) - Q_c(f, M_\lambda(D)g), h \right) \right| \lesssim \|f\|_{H^{(\lambda-\beta)^+}} \|M_\lambda(D)g\|_{H^\alpha} \|h\|_{H^\beta}.$$

Proof. We recall (2.4), that is,

$$\begin{cases} \langle \xi \rangle \lesssim \langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle & \text{on } \text{supp } \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \\ \langle \xi \rangle \sim \langle \xi - \xi_* \rangle & \text{on } \text{supp } \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2} \\ \langle \xi \rangle \sim \langle \xi_* \rangle \gtrsim \langle \xi - \xi_* \rangle & \text{on } \text{supp } \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2}. \end{cases}$$

Since $M_\lambda(\xi)$ is increasing function of $|\xi|$, we have

$$(2.18) \quad \begin{aligned} \left| M_\lambda(\xi) - M_\lambda(\xi - \xi_*) \right| &\lesssim M_\lambda(\xi - \xi_*) \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} + \frac{\langle \xi_* \rangle}{\langle \xi \rangle} M_\lambda(\xi - \xi_*) \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2} \\ &\quad + M_\lambda(\xi - \xi_*) \frac{M_\lambda(\xi_*)}{\langle \xi - \xi_* \rangle^\lambda} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}, \end{aligned}$$

where we have used the mean value theorem to gain the second term of the right hand side. Since we have

$$(Q_c(f, g), h) = \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma,$$

it follows that

$$\begin{aligned} &\left(M_\lambda(D) Q_c(f, g) - Q_c(f, M_\lambda(D)g), h \right) \\ &= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)) (M_\lambda(\xi) - M_\lambda(\xi - \xi_*)) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma. \\ &= \iiint_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma + \iiint_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma \\ &= B_1(f, g, h) + B_2(f, g, h). \end{aligned}$$

The estimations for $B_1(f, g, h)$ and $B_2(f, g, h)$ are almost similar as those for $A_1(f, g, h)$ and $A_2(f, g, h)$ in the proof of Lemma 2.2, by adding the extra factor $M_\lambda(\xi) - M_\lambda(\xi - \xi_*)$. Indeed, for $B_{1,j}(f, g, h)$ corresponding to

$A_{1,j}(f, g, h)$, ($j = 1, 2$), note that it follows from (2.3), (2.6) and (2.18) that

$$\begin{aligned} & \left| (|K(\xi, \xi_*)| + |\tilde{K}(\xi, \xi_*)|)(M_\lambda(\xi) - M_\lambda(\xi - \xi_*)) \right| \\ & \lesssim \frac{\langle \xi_* \rangle^\rho \langle \xi - \xi_* \rangle^\alpha \langle \xi \rangle^\beta}{\langle \xi_* \rangle^{3+\gamma+\alpha+\beta+\rho}} M_\lambda(\xi - \xi_*) \left(\mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} + \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{1+\alpha+\beta-2s} \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2} \right) \\ & \quad + \frac{\langle \xi_* \rangle^\rho \langle \xi - \xi_* \rangle^\alpha \langle \xi \rangle^\beta}{\langle \xi_* \rangle^{3+\gamma+\beta+\rho}} M_\lambda(\xi - \xi_*) \left(\frac{M_\lambda(\xi_*)}{\langle \xi - \xi_* \rangle^{\alpha+\lambda}} + \frac{1}{\langle \xi - \xi_* \rangle^\alpha} \right) \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}, \end{aligned}$$

where we have estimated the factor $\langle \xi \rangle / \langle \xi_* \rangle$ of K on the supp $\mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|}$ by $(\langle \xi \rangle / \langle \xi_* \rangle)^\beta$ because of (2.16). Noting (2.15) we have

$$\begin{aligned} & \left| (|K(\xi, \xi_*)| + |\tilde{K}(\xi, \xi_*)|)(M_\lambda(\xi) - M_\lambda(\xi - \xi_*)) \right| \\ & \lesssim \frac{\langle \xi_* \rangle^\rho \langle \xi - \xi_* \rangle^\alpha \langle \xi \rangle^\beta}{\langle \xi_* \rangle^{3+\gamma+\alpha+\beta+\rho}} M_\lambda(\xi - \xi_*) \left\{ \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{1+\alpha+\beta-2s} + \left(\frac{M_\lambda(\xi_*)}{\langle \xi - \xi_* \rangle^{\alpha+\lambda}} + \frac{1}{\langle \xi - \xi_* \rangle^\alpha} \right) \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle} \right\}. \end{aligned}$$

The Cauchy-Schwarz inequality shows

$$\begin{aligned} & |B_{1,1}|^2 + |B_{1,2}|^2 \\ & \lesssim \left(\iint_{\mathbb{R}^6} (|K(\xi, \xi_*)| + |\tilde{K}(\xi, \xi_*)|)(M_\lambda(\xi) - M_\lambda(\xi - \xi_*)) \left| \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} \right| d\xi d\xi_* \right)^2 \\ & \lesssim \int_{\mathbb{R}^3} \frac{|\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+\alpha+\beta+\rho}} d\xi_* \|M_\lambda(D)g\|_{H^\alpha}^2 \int_{\mathbb{R}^3} \frac{\langle \xi_* \rangle^\rho |\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma+\alpha+\beta+\rho}} d\xi_* \|h\|_{H^\beta}^2 \\ & \quad + \int_{\mathbb{R}^6} \frac{|\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{6+2(\gamma+\beta+\rho)}} \left(\frac{M_\lambda(\xi_*)^2 \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2(\alpha+\lambda)}} + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2\alpha}} \right) d\xi d\xi_* \|M_\lambda(D)g\|_{H^\alpha}^2 \|h\|_{H^\beta}^2. \end{aligned}$$

Since it follows from (2.14) that $\langle \xi_* \rangle^{-(3+\gamma+\alpha+\beta+\rho)} \in L^2$, the first term has the upper bound $\|f\|_{H^\rho}^2 \|M_\lambda(D)g\|_{H^\alpha}^2 \|h\|_{H^\beta}^2$. If $\alpha + \lambda < 3/2$ then

$$\int \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2(\alpha+\lambda)}} d\xi \lesssim \frac{1}{\langle \xi_* \rangle^{-3+2\lambda+2\alpha}},$$

which gives the same upper bound for the second term. If $\alpha + \lambda \geq 3/2$ then the condition $\gamma > -3$ implies

$$\int_{\mathbb{R}^6} \frac{|\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{6+2(\gamma+\beta+\rho)}} \left(\frac{M_\lambda(\xi_*)^2 \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2(\alpha+\lambda)}} + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}}{\langle \xi - \xi_* \rangle^{2\alpha}} \right) d\xi d\xi_* \lesssim \|\langle \xi_* \rangle^{\lambda-\beta} \hat{f}(\xi_*)\|_{L^2}^2.$$

Thus $B_{1,j}$ ($j = 1, 2$) have the desired upper bound.

The estimation for $B_{2,2}(f, g, h)$ are almost the same as above, in view of (2.7). As for $B_{2,1}$, it remains only to estimate

$$\begin{aligned} B_{2,1,p} & = \iiint b \mathbf{1}_{\langle \xi_* \rangle \leq \sqrt{2}|\xi|} \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2} \hat{\Phi}_c(\xi_* - \xi^-) (M_\lambda(\xi) - M_\lambda(\xi - \xi_*)) \\ & \quad \times \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\sigma d\xi d\xi_*. \end{aligned}$$

By

$$\mathbf{1} = \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle > \langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2}$$

we decompose

$$B_{2,1,p} = B_{2,1,p}^{(1)} + B_{2,1,p}^{(2)} + B_{2,1,p}^{(3)}.$$

On the sets for above integrals, we have $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi_* \rangle$, because $|\xi^-| \lesssim |\xi_*|$ that follows from $|\xi^-|^2 \leq 2|\xi_* \cdot \xi^-| \lesssim |\xi^-| |\xi_*|$. Furthermore, on the sets for $B_{2,1,p}^{(1)}$ and $B_{2,1,p}^{(2)}$ we have $\langle \xi \rangle \sim \langle \xi_* \rangle$, so that $\sup \left(b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \right) \lesssim$

$\mathbf{1}_{|\xi^-| \leq |\xi|/\sqrt{2}}$ and $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi \rangle$. Hence we have

$$\begin{aligned} & |B_{2,1,p}^{(1)}|^2 \\ & \lesssim \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|^2 \langle \xi_* \rangle^\rho \hat{f}(\xi_*)^2}{\langle \xi_* - \xi^- \rangle^{2\beta+2\rho}} \left(\frac{M_\lambda(\xi_*)^2 \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{2\alpha+2\lambda}} + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{2\alpha}} \right) d\xi d\xi_* d\sigma \\ & \times \iiint |\langle \xi - \xi_* \rangle^\alpha M_\lambda(\xi - \xi_*) \hat{g}(\xi - \xi_*)|^2 |\langle \xi \rangle^\beta \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

Note that $\langle \xi_* \rangle \sim \langle \xi \rangle \sim \langle \xi^+ \rangle \lesssim \langle \xi^+ - u \rangle + \langle u \rangle$ with $u = \xi_* - \xi^-$. Since $3 + 2(\gamma + \alpha + \beta + \rho) > 0$ then if $\alpha + \lambda < 3/2$ we have

$$\begin{aligned} & |B_{2,1,p}^{(1)}|^2 \lesssim \int |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|^2 \\ & \times \left\{ \sup_u \langle u \rangle^{-(6+2\gamma+2\beta+2\rho)} \int \frac{\mathbf{1}_{\langle \xi^+ - u \rangle \leq \langle u \rangle} (\langle \xi^+ - u \rangle^{2\lambda} + \langle u \rangle^{2\lambda})}{\langle \xi^+ - u \rangle^{2(\alpha+\lambda)}} d\xi^+ \right\} d\xi_* \|M_\lambda(D)g\|_{H^\alpha}^2 \|h\|_{H^\beta}^2 \\ & \lesssim \|f\|_{H^\rho}^2 \|M_\lambda(D)g\|_{H^\alpha}^2 \|h\|_{H^\beta}^2 \end{aligned}$$

because $d\xi \sim d\xi^+$ on the support of $\mathbf{1}_{|\xi^-| \leq |\xi|/\sqrt{2}}$. The case $\alpha + \lambda \geq 3/2$ can be considered by the same arguments as above. As for $B_{2,1,p}^{(2)}$, we first note that $\xi^+ = \xi - \xi_* + u$ implies

$$(M_\lambda(\xi_*) \sim) M_\lambda(\xi^+) \lesssim \langle \xi - \xi_* \rangle^\lambda + \langle u \rangle^\lambda \lesssim \langle \xi - \xi_* \rangle^\lambda$$

on the integral set, and hence we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |B_{2,1,p}^{(2)}|^2 & \lesssim \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^{\alpha+\beta+\rho}} |\langle \xi - \xi_* \rangle^\alpha M_\lambda(\xi - \xi_*) \hat{g}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ & \times \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\langle \xi \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi_* - \xi^- \rangle^{\alpha+\beta+\rho}} |\langle \xi \rangle^\beta \hat{h}(\xi)|^2 d\sigma d\xi d\xi_* \\ & \lesssim \|f\|_{H^\rho}^2 \|M_\lambda(D)g\|_{H^\alpha}^2 \|h\|_{H^\beta}^2 \end{aligned}$$

because $\hat{\Phi}_c(u) \langle u \rangle^{-(\alpha+\beta+\rho)} \in L^2$.

On the integral set of $B_{2,1,p}^{(3)}$ we have $\langle \xi \rangle \sim \langle \xi - \xi_* \rangle$ and

$$|M_\lambda(\xi) - M_\lambda(\xi - \xi_*)| \lesssim \frac{\langle \xi_* \rangle}{\langle \xi \rangle} M_\lambda(\xi - \xi_*),$$

so that

$$\begin{aligned} |B_{2,1,p}^{(3)}|^2 & \lesssim \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi \rangle^{\alpha+\beta+1} \langle \xi_* \rangle^{\rho-1}} |\langle \xi - \xi_* \rangle^\alpha M_\lambda(\xi - \xi_*) \hat{g}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ & \times \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi \rangle^{\alpha+\beta+1} \langle \xi_* \rangle^{\rho-1}} |\langle \xi \rangle^\beta \hat{h}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

We use the change of variables in ξ_* , $u = \xi_* - \xi^-$. Note that $|\xi^-| \geq \frac{1}{2}\langle u + \xi^- \rangle$ implies $|\xi^-| \geq \langle u \rangle / \sqrt{10}$. Since $\langle \xi_* - \xi^- \rangle + \langle \xi_* \rangle \lesssim \langle \xi \rangle$, in view of (2.15) and (2.17) we have

$$\begin{aligned} & \iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi \rangle^{\alpha+\beta+1} \langle \xi_* \rangle^{\rho-1}} d\sigma d\xi_* \\ & = \iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\langle \xi_* - \xi^- \rangle^s \hat{\Phi}_c(\xi_* - \xi^-) \langle \xi_* \rangle^s |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|}{\langle \xi \rangle^s \langle \xi_* - \xi^- \rangle^{\alpha+\beta+\rho} \langle \xi \rangle^s} \\ & \quad \times \left(\frac{\langle \xi_* - \xi^- \rangle^{\alpha+\beta+\rho-s} \langle \xi_* \rangle^{1-s-\rho}}{\langle \xi \rangle^{\alpha+\beta+1-2s}} \right) d\sigma d\xi_* \\ & \lesssim \left(\iint b \mathbf{1}_{|\xi^-| \geq \langle u \rangle} \frac{\langle u \rangle^{2s} |\hat{\Phi}_c(u)|^2}{\langle \xi \rangle^{2s} \langle u \rangle^{2(\alpha+\beta+\rho)}} d\sigma du \right)^{1/2} \left(\iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{\langle \xi_* \rangle^{2s} |\langle \xi_* \rangle^\rho \hat{f}(\xi_*)|^2}{\langle \xi \rangle^{2s}} d\sigma d\xi_* \right)^{1/2} \\ & \lesssim \|f\|_{H^\rho}, \end{aligned}$$

from which we also can obtain the desired bound for $B_{2,1,p}^{(3)}$. The proof of the proposition is complete. \square

We give an application of Proposition 2.16. Let $S \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq S \leq 1$ and

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$

Set $\Lambda_N^\lambda(D_v) = M_\lambda(D_v)S_N(D_v) = \langle D_v \rangle^\lambda S_N(D_v)$.

Corollary 2.17. *Assume that $0 < s < 1$ and $\gamma > \max\{-3, -\frac{3}{2} - 2s\}$. If $0 \leq \lambda < 3/2$ then for any $\max\{2s - 1, s/2\} \leq s' < s$ satisfying $\gamma + 2s' > -3/2$ we have*

$$\left| \left(\Lambda_N^\lambda \mathcal{Q}(f, g) - \mathcal{Q}(f, \Lambda_N^\lambda g), h \right) \right| \lesssim \|f\|_{H_{3/2+(2s-1)^++\gamma^+}^{s'}} \|g\|_{H_{(2s-1)^++\gamma^+}^\lambda} \|h\|_{H^{s'}}$$

and moreover if $\lambda \geq 3/2$ then we have the same estimate with $\|f\|_{H^{s'}}$ replaced by $\|f\|_{H^{\lambda-s'}}$.

Proof. In the proof of Proposition 2.16, instead of (2.18) we use

$$\begin{aligned} \left| \Lambda_N^\lambda(\xi) - \Lambda_N^\lambda(\xi - \xi_*) \right| &\lesssim M_\lambda(\xi - \xi_*) \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} + \frac{\langle \xi_* \rangle}{\langle \xi \rangle} M_\lambda(\xi - \xi_*) \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2} \\ &+ M_\lambda(\xi - \xi_*) \frac{M_\lambda(\xi_*)}{\langle \xi - \xi_* \rangle^\lambda} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq \langle \xi_* \rangle}, \end{aligned}$$

and choose

$$(\alpha, \beta, \rho) = (0, s', s') \text{ or } (0, s', \lambda - s'),$$

we can get,

$$\left| \left(\Lambda_N^\lambda \mathcal{Q}_c(f, g) - \mathcal{Q}_c(f, \Lambda_N^\lambda g), h \right) \right| \lesssim \|f\|_{H^{s'}} \|M_\lambda(D_v)g\|_{L^2} \|h\|_{H^{s'}}.$$

On the other hand, using Proposition 2.9 of [6],

$$\begin{aligned} \left| \left(\Lambda_N^\lambda \mathcal{Q}_{\bar{c}}(f, g) - \mathcal{Q}_{\bar{c}}(f, \Lambda_N^\lambda g), h \right) \right| &= \left| \left(\Lambda^{-s'} (\Lambda_N^\lambda \mathcal{Q}_{\bar{c}}(f, g) - \mathcal{Q}_{\bar{c}}(f, \Lambda_N^\lambda g)), \Lambda^{s'} h \right) \right| \\ &\leq \left| \left(\Lambda_N^{\lambda-s'} \mathcal{Q}_{\bar{c}}(f, g) - \mathcal{Q}_{\bar{c}}(f, \Lambda_N^{\lambda-s'} g), \Lambda^{s'} h \right) \right| \\ &+ \left| \left(\mathcal{Q}_{\bar{c}}(f, \Lambda_N^{\lambda-s'} g) - \Lambda_N^{-s'} \mathcal{Q}_{\bar{c}}(f, \Lambda_N^\lambda g), \Lambda^{s'} h \right) \right| \\ &\lesssim \|f\|_{L_{3/2+(2s-1)^++\gamma^+}^2} \|g\|_{H_{(2s-1)^++\gamma^+}^{\lambda-s'+(2s-1)^+}} \|h\|_{H^{s'}}, \end{aligned}$$

which completes the proof of Corollary. \square

3. FULL REGULARITY OF SOLUTIONS

Let $f \in L^\infty([0, T[; H_\ell^5(\Omega \times \mathbb{R}^3))$, for any $\ell \in \mathbb{N}$ be a solution of Cauchy problem (1.1). The regularity of f will now be considered. First of all, note that $f \in C^1([0, T[; H_\ell^1(\Omega \times \mathbb{R}^3))$ by using the equation.

For $\alpha \in \mathbb{N}^6$, we recall the Leibniz formula

$$\partial^\alpha \mathcal{Q}(g, f) = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_2}^{\alpha_1} \mathcal{Q}(\partial^{\alpha_1} g, \partial^{\alpha_2} f).$$

Here and below, ϕ denotes a cutoff function satisfying $\phi \in C_0^\infty$ and $0 \leq \phi \leq 1$. Notation $\phi_1 \ll \phi_2$ stands for two cutoff functions such that $\phi_2 = 1$ on the support of ϕ_1 .

Take some smooth cutoff functions $\varphi, \varphi_2, \varphi_3 \in C_0^\infty([T_1, T_2])$ and $\psi, \psi_2, \psi_3 \in C_0^\infty(K)$ such that $\varphi \ll \varphi_2 \ll \varphi_3$ and $\psi \ll \psi_2 \ll \psi_3$. Set $f_1 = \varphi(t)\psi(x)f$, $f_2 = \varphi_2(t)\psi_2(x)f$ and $f_3 = \varphi_3(t)\psi_3(x)f$, so here we can suppose that $0 < T < \infty$. For $\alpha \in \mathbb{N}^6$, $|\alpha| \leq 5$, denote

$$F = \partial_{x,v}^\alpha(\varphi(t)\psi(x)f) \in L^\infty([T_1, T_2[; L_\ell^2(\mathbb{R}^6)).$$

Then the Leibniz formula yields the following equation :

$$(3.1) \quad F_t + v \cdot \partial_x F - \mathcal{Q}(f, F) = G, \quad (t, x, v) \in \mathbb{R}^7,$$

where

$$(3.2) \quad \begin{aligned} G &= \sum_{\alpha_1 + \alpha_2 = \alpha, 1 \leq |\alpha_1|} C_{\alpha_2}^{\alpha_1} \mathcal{Q}(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) + \partial^\alpha(\varphi_t \psi(x)f + v \cdot \psi_x(x)\varphi(t)f) \\ &+ [\partial^\alpha, v \cdot \partial_x](\varphi(t)\psi(x)f) \equiv (A) + (B) + (C). \end{aligned}$$

Note carefully that a priori F is not regular enough, and therefore at that point, taking it as test function in the equation of (3.1) is not allowed. This is one of the main difficulties alluded to in the Introduction. Therefore, as in [6], we need to mollify F . This mollification process of course complicates the analysis below, but is necessary if we want to avoid formal proofs. The previous set of tools related to commutators estimations will then be used. For this purpose, let $S \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq S \leq 1$ and

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$

Then

$$S_N(D_x)S_N(D_v) = S(2^{-2N}|D_x|^2)S(2^{-2N}|D_v|^2) : H_\ell^{-\infty}(\mathbb{R}^6) \rightarrow H_\ell^{+\infty}(\mathbb{R}^6),$$

is a regularization operator such that

$$\|(S_N(D_x)S_N(D_v)f) - f\|_{L_t^2(\mathbb{R}^6)} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Set

$$P_{N,\ell} = \psi_2(x)W_\ell S_N(D_x)S_N(D_v).$$

Then

$$P_{N,\ell} F \in C_0^1([T_1, T_2]; C_0^\infty(K; H^{+\infty}(\mathbb{R}^3))),$$

and we can take

$$h = P_{N,\ell}^* (P_{N,\ell} F) \in C^1(\mathbb{R}; H^{+\infty}(\mathbb{R}^6))$$

as a test function for equation (3.1).

It follows by integration by parts on $\mathbb{R}^7 = \mathbb{R}_t^1 \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ that

$$\begin{aligned} & \left([S_N(D_v), v] \cdot \nabla_x S_N(D_x)F, W_\ell P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} \\ & - \left(P_{N,\ell} Q(f_2, F), P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} = (G, h)_{L^2(\mathbb{R}^7)}, \end{aligned}$$

where we used the fact

$$\left((\partial_t + v \cdot \nabla_x) P_{N,\ell} F, P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} = 0.$$

We get then

$$\begin{aligned} & - \left(Q(f, P_{N,\ell} F), P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} = - \left([S_N(D_v), v] \cdot \nabla_x S_N(D_x)F, W_\ell P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} \\ & + \left(P_{N,\ell} Q(f_2, F) - Q(f_2, P_{N,\ell} F), P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} + (G, h)_{L^2(\mathbb{R}^7)}. \end{aligned}$$

Next, we follow the main steps in our previous works [6], but need to be careful due to the singular behavior of the relative velocity part of the kernel.

3.1. Gain of regularity in v . In this subsection, we will prove a partial smoothing effect on the weak solution F in the velocity variable v .

Proposition 3.1. *Assume that $0 < s < 1, \gamma > \max\{-3, -2s - 3/2\}$. Let $f \in L^\infty([0, T]; H_\ell^s(\Omega_x \times \mathbb{R}_v^3))$, for any $\ell \in \mathbb{N}$ be a solution of the equation (1.1) satisfying the coercivity condition (1.3). Then one has*

$$(3.3) \quad \Lambda_v^s(\varphi(t)\psi(x)f) \in L^2(\mathbb{R}_t; H_\ell^s(\mathbb{R}^6)),$$

for any big $\ell > 0$ and any cut off function $\varphi \in C_0^\infty([T_1, T_2])$, $\psi \in C_0^\infty(K)$.

Similarly to [6],

$$\left| \left([S_N(D_v), v] \cdot \nabla_x S_N(D_x)F, W_\ell P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} \right| \leq C \|f_1\|_{L^2([0, T]; H_\ell^s(\mathbb{R}^6))}^2.$$

Then the coercivity assumption (1.3) implies

$$(3.4) \quad \begin{aligned} \eta_0 \|\Lambda_v^s W_{\gamma/2} P_{N,\ell} F\|_{L^2(\mathbb{R}^7)}^2 & \leq C \|f_1\|_{L^2(\mathbb{R}; H_{\ell+(\gamma/2+s)^+}^s(\mathbb{R}^6))}^2 + \left| (G, h)_{L^2(\mathbb{R}^7)} \right| \\ & + \left| \left(P_{N,\ell} Q(f_2, F) - Q(f_2, P_{N,\ell} F), P_{N,\ell} F \right)_{L^2(\mathbb{R}^7)} \right|. \end{aligned}$$

The proof of Proposition 3.1 will be completed by estimating the last two terms in (3.4) through the following three Lemmas.

Lemma 3.2. *Let $f_1 \in L^\infty([0, T]; H_\ell^5(\mathbb{R}^6))$, $\ell \geq 0$. Then, we have, for any $\varepsilon > 0$,*

$$\left| (G, h)_{L^2(\mathbb{R}^7)} \right| \leq C_\varepsilon \|f_2\|_{L^\infty([0, T]; H_{\ell+2+(\gamma+2s)^+}^5(\mathbb{R}^6))}^4 + \varepsilon \|P_{N, \ell} F\|_{L^2(\mathbb{R}_{t,x}^4, H_{\gamma/2}^s(\mathbb{R}_v^3))}^2.$$

Proof. By using the decomposition in (3.2), it is obvious that

$$(B) = \partial^\alpha (\varphi_t \psi(x) f + v \cdot \psi_x(x) \varphi(t) f) \in L_\ell^2(\mathbb{R}^7),$$

and

$$\|(B)\|_{L_\ell^2(\mathbb{R}^7)} \leq C \|f_2\|_{L^\infty(\mathbb{R}, H_{\ell+1}^5(\mathbb{R}^6))}.$$

Note that $[\partial^\alpha, v \cdot \partial_x]$ is a differential operator of order $|\alpha|$ so that we have

$$\|(C)\|_{L_\ell^2(\mathbb{R}^7)} \leq C \|f_2\|_{L^\infty(\mathbb{R}, H_\ell^5(\mathbb{R}^6))}.$$

For the term (A), recall that $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| \leq 5$ and $|\alpha_2| \leq 4$. Here we use the following upper bounded estimate from Proposition 2.9

$$(3.5) \quad \left| (Q(f, g), h)_{L^2(\mathbb{R}_v^3)} \right| \leq C \|f\|_{L_{\ell-\gamma/2+2+(\gamma+2s)^+}^2(\mathbb{R}_v^3)} \|g\|_{H_{\ell-\gamma/2+(\gamma+2s)^+}^s(\mathbb{R}_v^3)} \|h\|_{H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3)}.$$

Then, by separating the cases $|\alpha_1| \leq 3$ and $|\alpha_1| > 3$, we get

$$\begin{aligned} & \left| (Q(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1), P_{N, \ell}^* P_{N, \ell} F)_{L^2(\mathbb{R}^7)} \right| \\ & \leq C \int \|\partial^{\alpha_1} f_2\|_{L_{\ell-\gamma/2+2+(\gamma+2s)^+}^2(\mathbb{R}_v^3)} \|\partial^{\alpha_2} f_1\|_{H_{\ell-\gamma/2+(\gamma+2s)^+}^s(\mathbb{R}_v^3)} \|P_{N, \ell}^* P_{N, \ell} F\|_{H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3)} dx dt \\ & \leq C \|f_2\|_{L^\infty(\mathbb{R}, H_{\ell+2+(\gamma+2s)^+}^5(\mathbb{R}^6))} \|f_1\|_{L^2(\mathbb{R}, H_{\ell+2+(\gamma+2s)^+}^{4+s}(\mathbb{R}^6))} \|P_{N, \ell} F\|_{L^2(\mathbb{R}_{t,x}^4, H_{\gamma/2}^s(\mathbb{R}_v^3))}. \end{aligned}$$

Here we used the fact that $W_{-\ell} P_{N, \ell}^*$ is a uniformly (with respect to N, ℓ) bounded operator. This ends the proof of Lemma 3.2 by Cauchy Schwarz inequality. \square

We turn now to estimating the commutators of the regularization operator with the collision operator which are given in the following two Lemmas.

The next lemma is about the commutator of the collision operator with a mollification w.r.t. x variable.

Lemma 3.3. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$. For any suitable functions f and h with the following norms well defined, one has*

$$(3.6) \quad \left| (S_N(D_x) Q(f, g) - Q(f, S_N(D_x) g), h)_{L^2(\mathbb{R}^7)} \right| \leq C 2^{-N} \|\nabla_x f\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|g\|_{L^2(\mathbb{R}_{t,x}^4, H_{\ell-\gamma/2+(2s+\gamma)^+}^s(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}_{t,x}^4, H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3))}.$$

for a constant C independent of N .

Proof. Let us introduce $\tilde{K}_N(z) = 2^{3N} \hat{S}(2^N z) 2^N z$. Note that $\tilde{K}_N \in L^1(\mathbb{R}^3)$ uniformly with respect to N . Then for any smooth function \tilde{h} , one has

$$\begin{aligned} & ((S_N(D_x) Q(f, g) - Q(f, S_N(D_x) g)), h)_{L^2(\mathbb{R}^7)} = \int_0^1 \left\{ \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} \tilde{K}_N(x-y) \right. \\ & \quad \left. \times (Q(\nabla_x f(t, +\tau(x-y), \cdot), 2^{-N} g(t, y, \cdot)), h(t, x, \cdot))_{L^2(\mathbb{R}_v^3)} dt dx dy \right\} d\tau. \end{aligned}$$

By applying (3.5), the right hand side of this equality can be estimated from above by

$$\begin{aligned} & C \left\{ \sup_{t,x} \|\nabla_x f(t, x, \cdot)\|_{L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3)} \right\} \times \\ & \int_{\mathbb{R}_t} \int_{\mathbb{R}_x^3} (|\tilde{K}_N| * \|2^{-N} g(t, \cdot)\|_{H_{\ell-\gamma/2+(2s+\gamma)^+}^s(\mathbb{R}_v^3)})(x) \|h(t, x, \cdot)\|_{H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3)} dx dt \\ & \leq C 2^{-N} \|\nabla_x f\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|g\|_{L^2(\mathbb{R}_{t,x}^4, H_{\ell-\gamma/2+(2s+\gamma)^+}^s(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}_{t,x}^4, H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3))}, \end{aligned}$$

which completes the proof of the lemma. \square

We now apply (3.6) with $g \sim S_N(D_v)g$, and use the fact that a regularization operator $S_N(D_v)$ w.r.t. v variable has the property that, for any p

$$\|2^{-N}S_N(D_v)g(t, x, \cdot)\|_{H_p^s(\mathbb{R}_v^3)} \leq \|2^{-N}S_N(D_v)g(t, x, \cdot)\|_{H_p^1(\mathbb{R}_v^3)} \leq C\|g(t, x, \cdot)\|_{L_p^2(\mathbb{R}_v^3)},$$

where C is a constant independent on N . It follows that

$$(3.7) \quad \left| \left(S_N(D_x)Q(f, S_N(D_v)g) - Q(f, S_N(D_x)S_N(D_v)g), h \right)_{L^2(\mathbb{R}^7)} \right| \\ \leq C\|\nabla_x f\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|g\|_{L^2(\mathbb{R}_{t,x}^4, L_{\ell-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}_{t,x}^4, H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3))}.$$

Completion of proof of Proposition 3.1.

As regards the commutator terms in (3.4), we have

$$\begin{aligned} & \left(P_{N,\ell}Q(f, F) - Q(f, P_{N,\ell}F), P_{N,\ell}F \right)_{L^2(\mathbb{R}^7)} \\ &= \left(S_N(D_v)Q(f, F) - Q(f, S_N(D_v)F), S_N^*(D_x)\psi_2(x)W_\ell P_{N,\ell}F \right)_{L^2(\mathbb{R}^7)} \\ &+ \left(S_N(D_x)Q(f, S_N(D_v)F) - Q(f, S_N(D_x)S_N(D_v)F), \psi_2(x)W_\ell P_{N,\ell}F \right)_{L^2(\mathbb{R}^7)} \\ &+ \left(W_\ell Q(f, S_N(D_x)S_N(D_v)F) - Q(f, P_{N,\ell}F), \psi_2(x)P_{N,\ell}F \right)_{L^2(\mathbb{R}^7)}. \\ &= (1) + (2) + (3). \end{aligned}$$

Note that $[W_\ell, S_N(D_v)]$ is also a uniformly bounded operator from L^2 to $L_{\ell-1}^2$ with respect to the parameter N .

Using Corollary 2.17 with $\lambda = 0$, we have, for $0 < s' < s, \gamma + 2s' > -3/2$,

$$\begin{aligned} |(1)| &\leq C\|f_2\|_{L^\infty(\mathbb{R}_{t,x}^4, H_{3/2+\gamma'+(2s-1)^+}^{s'}(\mathbb{R}_v^3))} \|F\|_{L^2(\mathbb{R}_{t,x}^4, L_{\gamma'+2s-1}^2(\mathbb{R}_v^3))} \|W_\ell P_{N,\ell}F\|_{L^2(\mathbb{R}_{t,x}^4, H^s(\mathbb{R}_v^3))} \\ &\leq \varepsilon\|\Lambda_v^s W_{\gamma/2} P_{N,\ell}g\|_{L^2(\mathbb{R}^7)}^2 + C_\varepsilon\|f_3\|_{H_{2\ell}^s(\mathbb{R}^7)}^4. \end{aligned}$$

We can use (3.7) to show that

$$\begin{aligned} |(2)| &\leq C\|\nabla_x f_2\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|F\|_{L^2(\mathbb{R}_{t,x}^4, L_{\ell-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|W_\ell P_{N,\ell}F\|_{L^2(\mathbb{R}_{t,x}^4, H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3))} \\ &\leq \varepsilon\|\Lambda_v^s W_{\gamma/2} P_{N,\ell}g\|_{L^2(\mathbb{R}_{t,x,v}^7)}^2 + C_\varepsilon\|f_3\|_{H_{2\ell}^s(\mathbb{R}^7)}^4. \end{aligned}$$

Finally, (2.9) implies that

$$\begin{aligned} |(3)| &\leq C\|f_2\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2+(2s-1)^++\gamma'}^2(\mathbb{R}_v^3))} \|S_N(D_x)S_N(D_v)F\|_{L^2(\mathbb{R}_{t,x}^4, H_{\ell+(2s-1)^++\gamma'}^{2s-1+s}(\mathbb{R}_v^3))} \|P_{N,\ell}F\|_{L^2(\mathbb{R}^7)} \\ &\leq \varepsilon\|\Lambda_v^s W_{\gamma/2} P_{N,\ell}F\|_{L^2(\mathbb{R}_{t,x,v}^7)}^2 + C_\varepsilon\|f_3\|_{H_{2\ell}^s(\mathbb{R}^7)}^4. \end{aligned}$$

In summary, we have obtained the following estimate for the second term on the right hand side of (3.4)

$$\begin{aligned} & \left| \left(P_{N,\ell}Q(f_2, F) - Q(f_2, P_{N,\ell}F), P_{N,\ell}g \right)_{L^2(\mathbb{R}^7)} \right| \\ & \leq \varepsilon\|\Lambda_v^s W_{\gamma/2} P_{N,\ell}F\|_{L^2(\mathbb{R}_{t,x,v}^7)}^2 + C_\varepsilon\|f_3\|_{H_{2\ell}^s(\mathbb{R}^7)}^4. \end{aligned}$$

Finally, it holds that from (3.4) and (2.11) that

$$\|\Lambda_v^s W_{\gamma/2} P_{N,\ell}F\|_{L^2(\mathbb{R}^7)}^2 \leq C\left(1 + \|f_3\|_{H_{2\ell}^s(\mathbb{R}^7)}^4\right),$$

where the constant C is independent of N . Therefore, Proposition 3.1 is proved by taking the limit $N \rightarrow \infty$.

3.2. Gain of regularity in (t, x) . In [5], by using a generalized uncertainty principle, we proved a hypo-elliptic estimate, as regards a transport equation in the form of

$$(3.8) \quad f_t + v \cdot \nabla_x f = g \in D'(\mathbb{R}^{2n+1}),$$

where $(t, x, v) \in \mathbb{R}^{1+n+n} = \mathbb{R}^{2n+1}$.

Lemma 3.4. *Assume that $g \in H^{-s'}(\mathbb{R}^{2n+1})$, for some $0 \leq s' < 1$. Let $f \in L^2(\mathbb{R}^{2n+1})$ be a weak solution of the transport equation (3.8), such that $\Lambda_v^s f \in L^2(\mathbb{R}^{2n+1})$ for some $0 < s \leq 1$. Then it follows that*

$$\Lambda_x^{s(1-s')/(s+1)} f \in L_{-\frac{s'}{s+1}}^2(\mathbb{R}^{2n+1}), \quad \Lambda_t^{s(1-s')/(s+1)} f \in L_{-\frac{s}{s+1}}^2(\mathbb{R}^{2n+1}),$$

where $\Lambda_\bullet = (1 + |\mathbf{D}_\bullet|^2)^{1/2}$.

As mentioned earlier, this hypo-elliptic estimate together with Proposition 3.1 are used to obtain the partial regularity in the variable (t, x) . With this partial regularity in (t, x) , by applying the Leibniz type estimate on the fractional differentiation on the solution, we will show some improved regularity in all variables, v and (x, t) . Then the hypo-elliptic estimate can be used again to get higher regularity in the variable (x, t) . This procedure can be continued to obtain at least one order higher differentiation regularity in (t, x) variable.

To proceed, recall (see for example [6]) a Leibniz type formula for fractional derivatives with respect to variable (t, x) . Let $0 < \lambda < 1$. Then there exists a positive constant $C_\lambda \neq 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^n)$, one has

$$(3.9) \quad |D_y|^\lambda f(y) = \mathcal{F}^{-1}(|\xi|^\lambda \hat{f}(\xi)) = C_\lambda \int_{\mathbb{R}^n} \frac{f(y) - f(y+h)}{|h|^{n+\lambda}} dh.$$

First of all, we have the following proposition on the gain of regularity in the variable (t, x) through uncertainty principle as in [6].

Proposition 3.5. *Under the hypothesis of Theorem 1.1, one has*

$$(3.10) \quad \Lambda_{t,x}^{s_0} f_1 \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$ and $0 < s_0 = \frac{s(1-s)}{(s+1)}$.

Therefore, under the hypothesis $f \in L^\infty([0, T]; H_\ell^5(\mathbb{R}^6))$ for all $\ell \in \mathbb{N}$, it follows that for any $\ell \in \mathbb{N}$ we have

$$(3.11) \quad \Lambda_v^s(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)), \quad \Lambda_{t,x}^{s_0}(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)).$$

This partial regularity in (t, x) variable will now be improved.

Proposition 3.6. *Let $0 < \lambda < 1$. Suppose that $f \in L^\infty([0, T]; H_\ell^5(\Omega \times \mathbb{R}^3))$ for all $\ell \in \mathbb{N}$ is a solution of the equation (1.1), and for any cutoff functions φ, ψ , we have*

$$(3.12) \quad \Lambda_v^s(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)), \quad \Lambda_{t,x}^\lambda(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)).$$

Then, one has

$$\Lambda_v^s \Lambda_{t,x}^\lambda(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$ and any cutoff functions φ, ψ .

Set

$$F_{N,\ell} = P_{N,\ell} F = \psi_2(x) S_N(D_x) W_\ell S_N(D_v) \partial^\alpha(\varphi(t)\psi(x)f),$$

where $\alpha \in \mathbb{N}^6$, $|\alpha| \leq 6$ and $\ell \in \mathbb{N}$. Then (3.12) yields

$$\|\Lambda_v^s F_{N,\ell}\|_{L^2(\mathbb{R}^7)} \leq C \|\Lambda_v^s \partial^\alpha(\varphi(t)\psi(x)f)\|_{L_\ell^2(\mathbb{R}^7)},$$

and

$$\|\Lambda_{t,x}^\lambda F_{N,\ell}\|_{L^2(\mathbb{R}^7)} \leq C \|\Lambda_{t,x}^\lambda \partial^\alpha(\varphi(t)\psi(x)f)\|_{L_\ell^2(\mathbb{R}^7)},$$

where the constant C is independent on N .

It follows that $F_{N,\ell}$ satisfies the following equation

$$(3.13) \quad \partial_t(F_{N,\ell}) + v \cdot \partial_x(F_{N,\ell}) = Q(f, F_{N,\ell}) + G_{N,\ell},$$

where $G_{N,\ell}$ is given by

$$G_{N,\ell} = W_\ell \left[S_N(D_v), v \right] \cdot \nabla_x S_N(D_x) F + \left(P_{N,\ell} Q(f_2, F) - Q(f_2, P_{N,\ell} F) \right) + P_{N,\ell} G,$$

with G defined in (3.2).

We now choose $|D_{t,x}|^\lambda |D_{t,x}|^\lambda F_{N,\ell}$ as a test function for equation (3.13). It follows that

$$\begin{aligned} & \|\Lambda_v^s \Lambda_{t,x}^\lambda P_{N,\ell} F\|_{L^2(\mathbb{R}^7)} \\ & \leq C \left| \left(|D_{t,x}|^\lambda Q(f_2, F_{N,\ell}) - Q(f_2, |D_{t,x}|^\lambda F_{N,\ell}), |D_{t,x}|^\lambda F_{N,\ell} \right)_{L^2(\mathbb{R}^7)} \right| \\ & \quad + \left| \left(|D_{t,x}|^\lambda G_{N,\ell}, |D_{t,x}|^\lambda F_{N,\ell} \right)_{L^2(\mathbb{R}^7)} \right|. \end{aligned}$$

Using the formula (3.9), the proof of the Proposition 3.6 is similar to the corresponding result in [6], here we omit the cut-off function, it is easy to treat as before.

We can then get the following regularity result on the solution with respect to the (t, x) variable.

Proposition 3.7. *Under the hypothesis of Theorem 1.1, one has*

$$(3.14) \quad \Lambda_{t,x}^{1+\varepsilon}(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$ and some $\varepsilon > 0$.

Proof. By fixing $s_0 = \frac{s(1-s)}{(s+1)}$, then (3.11) and Proposition 3.6 with $\lambda = s_0$ imply

$$\Lambda_v^s \Lambda_{t,x}^{s_0} F \in L_\ell^2(\mathbb{R}^7).$$

It follows that,

$$(\Lambda_{t,x}^{s_0} F)_t + v \cdot \partial_x (\Lambda_{t,x}^{s_0} F) + L_1(\Lambda_{t,x}^{s_0} F) = \Lambda_{t,x}^{s_0} Q(f_2, F) + \Lambda_{t,x}^{s_0} G \in H_\ell^{-s}(\mathbb{R}^7).$$

By applying Lemma 3.4 with $s' = s$, we can deduce that

$$\Lambda_{t,x}^{s_0+s_0}(F) \in L_\ell^2(\mathbb{R}^7),$$

for any $\ell \in \mathbb{N}$. If $2s_0 < 1$, by using again Proposition 3.6 with $\lambda = 2s_0$ and Lemma 3.4 with $s' = s$, we have

$$\Lambda_v^s(\varphi(t)\psi(x)f), \Lambda_{t,x}^{2s_0}(F) \in L_\ell^2(\mathbb{R}^7) \Rightarrow \Lambda_{t,x}^{3s_0}(F) \in L_\ell^2(\mathbb{R}^7).$$

Choose $k_0 \in \mathbb{N}$ such that

$$k_0 s_0 < 1, \quad (k_0 + 1)s_0 = 1 + \varepsilon > 1.$$

Finally, (3.14) follows from (3.10) and Proposition 3.6 with $\lambda = k_0 s_0$ by induction. And this completes the proof of the proposition 3.7. \square

3.3. Full regularity of solution. The above preparations will be used for the proof of the full regularity of solution in Theorem 1.1, by using an induction argument.

From Proposition 3.6 and Proposition 3.7, it follows that for any $\alpha \in \mathbb{N}$, $|\alpha| \leq 5$ and any $\ell \in \mathbb{N}$,

$$\Lambda_v^s \partial^\alpha(\varphi(t)\psi(x)f), \Lambda_{t,x} \partial^\alpha(\varphi(t)\psi(x)f) \in L_\ell^2(\mathbb{R}^7).$$

These will be used to get the high order regularity with respect to the variable v .

Proposition 3.8. *Let $s \leq \lambda < 1$. Suppose that, for any cutoff functions $\varphi \in C_0^\infty([0, T[)$, $\psi \in C_0^\infty(\mathbb{R}^3)$, any $\alpha \in \mathbb{N}$, $|\alpha| \leq 5$ and all $\ell \in \mathbb{N}$,*

$$(3.15) \quad \Lambda_v^\lambda \partial^\alpha(\varphi(t)\psi(x)f), \Lambda_{t,x} \partial^\alpha(\varphi(t)\psi(x)f) \in L_\ell^2(\mathbb{R}^7).$$

Then, for all cutoff function and all $\alpha \in \mathbb{N}$, $|\alpha| \leq 5$, $\ell \in \mathbb{N}$,

$$(3.16) \quad \Lambda_v^{\lambda+s} \partial^\alpha(\varphi(t)\psi(x)f) \in L_\ell^2(\mathbb{R}^7).$$

Proof. Recall that, for $|\alpha| \leq 5$, $F = \partial^\alpha(\varphi(t)\psi(x)f)$ is the weak solution of the equation :

$$\frac{\partial F}{\partial t} + v \cdot \partial_x F - Q(f, F) = G, \quad (t, x, v) \in \mathbb{R}^7,$$

where G is given in (3.2). Set

$$P_{N,\ell,\lambda} = \psi_2(x) W_\ell S_N(D_x) S_N(D_v) \Lambda_v^\lambda,$$

we take now

$$P_{N,\ell,\lambda}^* P_{N,\ell,\lambda} F = P_{N,\ell,\lambda}^* F_{N,\ell,\lambda} \in C_0^1([T_1, T_2]; H_p^{+\infty}(\mathbb{R}^6))$$

as test function. Then, one has

$$\begin{aligned} & \left([P_{N,\ell,\lambda}, v] \cdot \partial_x F, P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} - \left(Q(f, F_{N,\ell,\lambda}), F_{N,\ell,\lambda} \right)_{L^2(\mathbb{R}^7)} \\ &= \left(P_{N,\ell,\lambda} Q(f, F) - Q(f, P_{N,\ell,\lambda} F), P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \\ & \quad + \left(P_{N,\ell,\lambda} G, P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)}. \end{aligned}$$

Since

$$[\Lambda_v^\lambda, v] \cdot \partial_x = \lambda \Lambda_v^{\lambda-2} \partial_v \cdot \partial_x,$$

and $\Lambda_v^{\lambda-2} \partial_v$ are bounded operators in L^2 , for any $0 < \lambda < 1$, we have, by using the hypothesis (3.15) that

$$\left| \left([P_{N,\ell,\lambda}, v] \cdot \partial_x F, P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \right| \leq C \|\Lambda_v^\lambda F\|_{L_t^2(\mathbb{R}^7)} \|\Lambda_x F\|_{L_t^2(\mathbb{R}^7)}.$$

Using the coercivity (1.3), we get as (3.4),

$$(3.17) \quad \begin{aligned} \eta_0 \|\Lambda_v^s W_{\gamma/2} F_{N,\ell,\lambda}\|_{L^2(\mathbb{R}^7)}^2 &\leq C \|\Lambda_v^s F\|_{L^2(\mathbb{R}^7)} \|\Lambda_x F\|_{L^2(\mathbb{R}^7)} + \left| (P_{N,\ell,\lambda} G, P_{N,\ell,\lambda} F)_{L^2(\mathbb{R}^7)} \right| \\ &+ \left| (P_{N,\ell,\lambda} Q(f_2, F) - Q(f_2, P_{N,\ell,\lambda} F), P_{N,\ell,\lambda} F)_{L^2(\mathbb{R}^7)} \right|. \end{aligned}$$

We conclude the proof of Proposition 3.8 by using the following Lemma. \square

Lemma 3.9. *Let $f \in L^\infty([0, T]; H_\ell^5(\Omega \times \mathbb{R}^3))$, $\ell \geq \ell_0$ (large). Then, we have, for any $\varepsilon > 0$,*

$$\begin{aligned} \left| (P_{N,\ell,\lambda} G, P_{N,\ell,\lambda} F)_{L^2(\mathbb{R}^7)} \right| &\leq \varepsilon \|\Lambda_v^s W_{\gamma/2} F_{N,\ell,\lambda}\|_{L^2(\mathbb{R}^7)}^2 \\ &+ C_\varepsilon \left(\|\Lambda_v^s f_3\|_{L^2([0,T]; H_{2\ell}^5(\mathbb{R}^6))}^2 \|\Lambda_t^1 f_3\|_{L^2([0,T]; H_{2\ell}^5(\mathbb{R}^6))}^2 + \|\Lambda_x^1 f_3\|_{L^2([0,T]; H_{2\ell}^5(\mathbb{R}^6))}^2 \right). \end{aligned}$$

Proof. By using the decomposition in (3.2), it is obvious that for the linear terms

$$\left| (P_{N,\ell,\lambda}((B) + (C)), P_{N,\ell,\lambda} F)_{L^2(\mathbb{R}^7)} \right| \leq C \|\Lambda_v^s f_2\|_{L^2([0,T]; H_{\ell+1}^5(\mathbb{R}^6))}^2.$$

For the term (A), recall that $\alpha_1 + \alpha_2 = \alpha$, $|\alpha| \leq 5$ and $|\alpha_2| < 5$. Then, by separating the cases $|\alpha_1| \leq 3$ and $|\alpha_1| > 3$, we get, with $\Lambda_N^s(D_v) = \Lambda_v^s S_N(D_v)$,

$$\begin{aligned} &\left| (P_{N,\ell,\lambda} Q(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1), P_{N,\ell,\lambda} F)_{L^2(\mathbb{R}^7)} \right| \\ &= \left| (\Lambda_N^s(D_v)(Q(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1)), W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda})_{L^2(\mathbb{R}^7)} \right| \\ &\leq \left| (\Lambda_N^s(D_v)(Q(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) \right. \\ &\quad \left. - (Q(\partial^{\alpha_1} f_2, \Lambda_N^s(D_v) \partial^{\alpha_2} f_1)), W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda})_{L^2(\mathbb{R}^7)} \right| \\ &\quad + \left| ((Q(\partial^{\alpha_1} f_2, \Lambda_N^s(D_v) \partial^{\alpha_2} f_1)), W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda})_{L^2(\mathbb{R}^7)} \right|. \end{aligned}$$

Using Corollary 2.17, we have

$$\begin{aligned} &\left| (\Lambda_N^s(D_v)(Q(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1) - (Q(\partial^{\alpha_1} f_2, \Lambda_N^s(D_v) \partial^{\alpha_2} f_1)), W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda})_{L^2(\mathbb{R}^7)} \right| \\ &\leq C \int \|\partial^{\alpha_1} f_2\|_{H_{2+\gamma+(2s-1)}^s(\mathbb{R}_v^3)} \|\partial^{\alpha_2} f_1\|_{H_{\gamma+(2s-1)}^s(\mathbb{R}_v^3)} \|W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda}\|_{H^s(\mathbb{R}_v^3)} dx dt \\ &\leq C \begin{cases} \|\partial^{\alpha_1} f_2\|_{L^\infty(\mathbb{R}_{t,x}^4; H_{2+\gamma+(2s-1)}^s(\mathbb{R}_v^3))} \|\Lambda_{x,v}^s \Lambda_v^s f_1\|_{L_{\gamma+(2s-1)}^2(\mathbb{R}^7)} \|\Lambda_v^s F_{N,\ell,\lambda}\|_{L_t^2(\mathbb{R}^7)}, & |\alpha_1| \leq 3 \\ \|\partial^{\alpha_1} f_2\|_{L^2(\mathbb{R}_{t,x}^4; H_{2+\gamma+(2s-1)}^s(\mathbb{R}_v^3))} \|\Lambda_{x,v}^{1+3/2+\delta} \Lambda_v^s \Lambda_t^{1/2+\delta} f_1\|_{L_{\gamma+(2s-1)}^2(\mathbb{R}^7)} \|\Lambda_v^s F_{N,\ell,\lambda}\|_{L_t^2(\mathbb{R}^7)}, & |\alpha_1| > 3 \end{cases} \\ &\leq C \|\Lambda_{x,v}^5 \Lambda_t f_2\|_{L_{2+\gamma+(2s-1)}^2(\mathbb{R}^7)} \|\Lambda_{x,v}^5 \Lambda_v^s f_2\|_{L_{\gamma+(2s-1)}^2(\mathbb{R}^7)} \|\Lambda_v^s F_{N,\ell,\lambda}\|_{L_t^2(\mathbb{R}^7)} \\ &\leq \varepsilon \|\Lambda_v^s W_{\gamma/2} F_{N,\ell,\lambda}\|_{L^2(\mathbb{R}^7)}^2 + C_\varepsilon \|\Lambda_{x,v}^5 \Lambda_t f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2 \|\Lambda_{x,v}^5 \Lambda_v^s f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2, \end{aligned}$$

Proposition 2.9 with $m = 0$ and Sobolev embedding for $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$ give

$$\begin{aligned} &\left| ((Q(\partial^{\alpha_1} f_2, \Lambda_N^s(D_v) \partial^{\alpha_2} f_1)), W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda})_{L^2(\mathbb{R}^7)} \right| \\ &\leq C \int \|\partial^{\alpha_1} f_2\|_{L_{2+(\gamma+2s)+\ell-\gamma/2}^2(\mathbb{R}_v^3)} \|\partial^{\alpha_2} f_1\|_{H_{\ell-\gamma/2+(\gamma+2s)}^{\ell+s}(\mathbb{R}_v^3)} \|W_\ell S_N^*(D_x) \psi_2 F_{N,\ell,\lambda}\|_{H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3)} dx dt \\ &\leq \varepsilon \|\Lambda_v^s W_{\gamma/2} F_{N,\ell,\lambda}\|_{L^2(\mathbb{R}^7)}^2 + C_\varepsilon \|\Lambda_{x,v}^5 \Lambda_t f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2 \|\Lambda_{x,v}^5 \Lambda_v^s f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2. \end{aligned}$$

This ends the proof of Lemma 3.9. \square

Recall $\Lambda_N^\lambda(D_v) = \Lambda_v^\lambda S_N(D_v)$, we have for the last term of (3.17), (the term involving μ is omitted since is easier than f_2),

$$\begin{aligned} & \left(P_{N,\ell,\lambda} Q(f_2, F) - Q(f_2, P_{N,\ell,\lambda} F), P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \\ &= \left(\Lambda_N^\lambda(D_v) Q(f_2, F) - Q(f_2, \Lambda_N^\lambda(D_v) F), W_\ell S_N^*(D_x) P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \\ &+ \left(S_N(D_x) Q(f_2, \Lambda_N^\lambda(D_v) F) - Q(f_2, S_N(D_x) \Lambda_N^\lambda(D_v) F), W_\ell P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \\ &+ \left(W_\ell Q(f_2, S_N(D_x) \Lambda_N^\lambda(D_v) F) - Q(f_2, P_{N,\ell,\lambda} F), P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \\ &= (I) + (II) + (III). \end{aligned}$$

Using again Corollary 2.17, we have by interpolation,

$$\begin{aligned} |(I)| &\leq C \int \|f_2\|_{H_{2+(2s-1)^++\gamma^+}^s} \|F\|_{H_{(2s-1)^++\gamma^+}^1} \|W_\ell S_N^*(D_x) P_{N,\ell,\lambda} F\|_{H^s} dx dt \\ &\leq C \|\Lambda_t \Lambda_x^2 \Lambda_v^s f_2\|_{L_{2+(2s-1)^++\gamma^+}^2(\mathbb{R}^7)} \|\Lambda_v^\lambda F\|_{L_{(2s-1)^++\gamma^+}^2} \|\Lambda_v^s W_\ell S_N^*(D_x) P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)} \\ &\leq C \|\Lambda_t \Lambda_{x,v}^4 f_2\|_{L_{2+(2s-1)^++\gamma^+}^2(\mathbb{R}^7)} \|\Lambda_v^\lambda F\|_{L_{(2s-1)^++\gamma^+}^2} \left(\epsilon \|\Lambda_v^s W_{\gamma/2} P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)} + C_\epsilon \|\Lambda_v^\lambda F\|_{L_{3\ell}^2(\mathbb{R}^7)} \right) \\ &\leq \epsilon \|\Lambda_v^s W_{\gamma/2} P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)}^2 + C_\epsilon \|\Lambda_t \Lambda_{x,v}^4 f_2\|_{L_{2+(2s-1)^++\gamma^+}^2(\mathbb{R}^7)} \|\Lambda_v^\lambda \Lambda_{x,v}^5 f_1\|_{L_{2\ell}^2(\mathbb{R}^7)}. \end{aligned}$$

Using now (3.6), similarly as for (3.7), we have

$$\begin{aligned} |(II)| &\leq C 2^{-N} \|\nabla_x f_2\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}_v^3))} \|M_\lambda(D_v) F\|_{L^2(\mathbb{R}_{t,x}^4, H_{\ell-\gamma/2+(2s+\gamma)^+}^s(\mathbb{R}_v^3))} \|W_\ell P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}_{t,x}^4, H_{-\ell+\gamma/2}^s(\mathbb{R}_v^3))} \\ &\leq C \|\Lambda_t \Lambda_x^3 f_2\|_{L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}^7)} \|\Lambda_v^\lambda F\|_{L_{\ell-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}^7)} \|\Lambda_v^s W_{\gamma/2} P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)}. \end{aligned}$$

For the term (III), we use (2.9)

$$\begin{aligned} |(III)| &= \left| \left(W_\ell Q(f_2, S_N(D_x) \Lambda_N^\lambda(D_v) F) - Q(f_2, P_{N,\ell,\lambda} F), P_{N,\ell,\lambda} F \right)_{L^2(\mathbb{R}^7)} \right| \\ &\leq C \|f_2\|_{L^\infty(\mathbb{R}_{t,x}^4, L_{\ell+2+(2s-1)^++\gamma^+}^2(\mathbb{R}_v^3))} \|S_N(D_x) \Lambda_N^\lambda(D_v) F\|_{L^2(\mathbb{R}_{t,x}^4, H_{\ell+(2s-1)^++\gamma^+}^{2s-1+\epsilon}(\mathbb{R}_v^3))} \|P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)} \\ &\leq \epsilon \|\Lambda_v^s W_{\gamma/2} P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)}^2 + C_\epsilon \left(\|\Lambda_v^\lambda F\|_{L_{2\ell}^2(\mathbb{R}^7)}^2 + \|\Lambda_t \Lambda_x^2 f_2\|_{L_{\ell+2-\gamma/2+(2s+\gamma)^+}^2(\mathbb{R}^7)}^2 \|\Lambda_v^\lambda F\|_{L_\ell^2(\mathbb{R}^7)}^2 \right). \end{aligned}$$

Finally, from (3.17), choose $\epsilon > 0$ small enough, we get for big ℓ ,

$$\begin{aligned} \frac{\eta_0}{2} \|\Lambda_v^s W_{\gamma/2} P_{N,\ell,\lambda} F\|_{L^2(\mathbb{R}^7)}^2 &\leq \|\Lambda_v^\lambda \Lambda_{x,v}^5 f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2 + \|\Lambda_x \Lambda_{x,v}^5 f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2 \\ &\quad + \|\Lambda_t \Lambda_{x,v}^5 f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2 \|\Lambda_v^\lambda \Lambda_{x,v}^5 f_2\|_{L_{2\ell}^2(\mathbb{R}^7)}^2. \end{aligned}$$

Taking the limit $N \rightarrow \infty$, we have proved (3.16), and ended the proof of Proposition 3.8.

We can now conclude that the following regularity result with respect to the variable v holds true.

Proposition 3.10. *Under the hypothesis of Theorem 1.1, one has*

$$(3.18) \quad \Lambda_v^{1+\varepsilon}(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)),$$

for any $\ell \in \mathbb{N}$ and some $\varepsilon > 0$.

Again, this result is indeed obtained by noticing that there exists $k_0 \in \mathbb{N}$ such that

$$k_0 s < 1, \quad (k_0 + 1)s = 1 + \varepsilon > 1.$$

Then we get (3.18) from (3.3), Proposition 3.8 with $\lambda = k_0 s$ by induction.

High order regularity by iterations

From Proposition 3.7 and Proposition 3.10, we can now deduce that, for any $\ell \in \mathbb{N}$, and any cutoff functions $\varphi(t)$ and $\psi(x)$,

$$\Lambda_v(\varphi(t)\psi(x)f), \Lambda_{t,x}(\varphi(t)\psi(x)f) \in L^2([0, T]; H_\ell^5(\mathbb{R}^6)),$$

which is

$$\varphi(t)\psi(x)f \in L^2([0, T]; H_\ell^6(\mathbb{R}^6)) \cap H^1([0, T]; H_\ell^5(\mathbb{R}^6))$$

The proof of full regularity is then completed by induction for (x, v) variable

$$\varphi(t)\psi(x)f \in L^2([0, T]; H_\ell^7(\mathbb{R}^6)) \cap H^1([0, T]; H_\ell^6(\mathbb{R}^6))$$

and using the equation to prove the regularity for t variable.

4. UNIQUENESS OF SOLUTIONS

In this section, we prove precise versions for uniqueness results which will cover more general cases than those presented in Theorem 1.2.

We need the coercive estimate in a global version: For suitable function f , we say that f satisfies the global coercive estimate, if there exist constants $c_0 > 0$ and $C > 0$ independent of $t \in]0, T[$ such that

$$(4.1) \quad -(Q(f(t), h), h)_{L^2(\mathbb{R}^6)} \geq c_0 \|h\|_{L^2(\mathbb{R}_x^3; H_{\gamma/2}^s(\mathbb{R}_v^3))}^2 - C \|h\|_{L^2(\mathbb{R}_x^3; L^2_{(\gamma/2+s)^+}(\mathbb{R}_v^3))}^2$$

for any $h \in L^2(\mathbb{R}_x^3; \mathcal{S}(\mathbb{R}_v^3))$. Using the notations introduced in Section 1, we prove the following precise version of Theorem 1.2, where we do not assume that solution is a perturbation around a normalized Maxwellian.

Theorem 4.1. *Assume that $0 < s < 1$ and $\max\{-3, -3/2 - 2s\} < \gamma < 2 - 2s$. Let $f_0 \in \tilde{\mathcal{E}}_0^0(\mathbb{R}^6)$, $0 < T < +\infty$ and suppose that $f \in \tilde{\mathcal{E}}^m([0, T] \times \mathbb{R}_{x,v}^6)$, $m \geq s$ is a weak solution to the Cauchy problem (1.1). If f is non-negative, then solution f is unique in the function space $\tilde{\mathcal{E}}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$.*

Moreover, if f is non-negative and satisfies the global coercive estimate (4.1), then the solution f is unique in the function space $\tilde{\mathcal{E}}^s([0, T] \times \mathbb{R}_{x,v}^6)$. The same conclusion holds without the non-negativity of f if the term $\|h\|_{L^2(\mathbb{R}_x^3; H_{\gamma/2}^s(\mathbb{R}_v^3))}^2$ in the condition (4.1) is replaced by $\int \|h\|_{\Phi_\gamma}^2 dx$.

Remark 4.2. *In the case where $\gamma > -3/2$ and $f \in \tilde{\mathcal{E}}^s([0, T] \times \mathbb{R}_{x,v}^6)$ is non-negative, it follows that f coincides with any another solution $f_2 \in \tilde{\mathcal{E}}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$ without the coercivity condition (4.1).*

The next result proves the uniqueness of perturbative solutions around a normalized Maxwellian obtained in [9, 10] where we do not assume the non-negativity of solutions.

Theorem 4.3. *Assume that $0 < s < 1$, $\max\{-3, -3/2 - 2s\} < \gamma$. Let $\ell_1 > 3/2 + \max\{(\gamma + 2s)^+, |\gamma|/2\}$. Then there exists an $\varepsilon_0 > 0$ such that if $f_1(t), f_2(t) \in \tilde{\mathcal{E}}^s([0, T] \times \mathbb{R}_{x,v}^6)$ are two solutions of the Cauchy problem (1.1) with the properties*

$$\mu^{-1/2}(f_j(t) - \mu) \in L^\infty([0, T] \times \mathbb{R}_x^3; H_{\ell_1}^s), \quad j = 1, 2,$$

and the smallness condition for f_1

$$(4.2) \quad \|\mu^{-1/2}(f_1(t) - \mu)\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \leq \varepsilon_0,$$

then $f_1(t) \equiv f_2(t)$ for all $t \in [0, T]$.

To study the uniqueness of solutions constructed in Theorem 1.4 of [9], we define another function space with exponential decay in the velocity variable as follows: For $m \in \mathbb{R}$ and for $T > 0$, set

$$\begin{aligned} \tilde{\mathcal{B}}^m([0, T] \times \mathbb{R}_{x,v}^6) &= \{f \in C^0([0, T]; \mathcal{D}'(\mathbb{R}_{x,v}^6)); \exists \rho > 0 \\ &\text{s.t. } e^{\rho(v)^2} f \in L^\infty([0, T] \times \mathbb{R}_x^3; L^2(\mathbb{R}_v^3)) \cap L^2([0, T]; L^\infty(\mathbb{R}_x^3; H^m(\mathbb{R}_v^3)))\}. \end{aligned}$$

We get the following refinement of the last part of Theorem 4.1, in the case $\gamma + 2s \leq 0$.

Theorem 4.4. *Assume that $0 < s < 1$ and $\max\{-3, -3/2 - 2s\} < \gamma \leq -2s$. Let $0 < T < +\infty$ and suppose that $f_1(t) \in \tilde{\mathcal{B}}^s([0, T] \times \mathbb{R}_{x,v}^6)$ is a solution to the Cauchy problem (1.1) satisfying the global coercivity estimate (4.1) with the term $\|h\|_{L^2(\mathbb{R}_x^3; H_{\gamma/2}^s(\mathbb{R}_v^3))}^2$ replaced by $\int \|h\|_{\Phi_\gamma}^2 dx$. Then $f_1(t)$ coincides with any another solution $f_2(t) \in \tilde{\mathcal{B}}^s([0, T] \times \mathbb{R}_{x,v}^6)$.*

If the Cauchy problem (1.1) admits two solutions $f_1(t), f_2(t) \in \tilde{\mathcal{E}}^s([0, T] \times \mathbb{R}_{x,v}^6)$, then there exist $\rho_0, \rho_1, \rho_2 > 0$ such that

$$e^{\rho_0(v)^2} f_0 \in L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3)), \quad e^{\rho_1(v)^2} f_1, \quad e^{\rho_2(v)^2} f_2 \in L^\infty([0, T] \times \mathbb{R}_x^3; H^s(\mathbb{R}_v^3)).$$

Take $0 < \rho < \min\{\rho_0, \rho_1, \rho_2\}$ and $\kappa > 0$ sufficiently small such that $\frac{\rho}{2\kappa} > T$. Then we have

$$g_0 = e^{\rho\langle v \rangle^2} f_0 \in L^\infty(\mathbb{R}_x^3; L_l^2(\mathbb{R}_v^3)), \quad g_1 = e^{(\rho-\kappa t)\langle v \rangle^2} f_1, \quad g_2 = e^{(\rho-\kappa t)\langle v \rangle^2} f_2 \in L^\infty([0, T] \times \mathbb{R}_x^3; H_l^s(\mathbb{R}_v^3))$$

for any $l \in \mathbb{N}$, and g_1, g_2 are two solutions of the following Cauchy problem

$$(4.3) \quad \begin{cases} \tilde{g}_t + v \cdot \nabla_x \tilde{g} + \kappa(1 + |v|^2) \tilde{g} = \Gamma^t(\tilde{g}, \tilde{g}), \\ \tilde{g}|_{t=0} = g_0, \end{cases}$$

where

$$\Gamma^t(g, h) = \mu_\kappa(t)^{-1} \mathcal{Q}(\mu_\kappa(t)g, \mu_\kappa(t)h), \quad \mu_\kappa(t) = \mu(t, v) = e^{-(\rho-\kappa t)(1+|v|^2)}.$$

Set $g = g_1 - g_2$. Then we have

$$(4.4) \quad \begin{cases} g_t + v \cdot \nabla_x g + \kappa(1 + |v|^2)g = \Gamma^t(g_1, g) + \Gamma^t(g, g_2), \\ g|_{t=0} = 0. \end{cases}$$

4.1. Estimates for modified collisional operator. We now prepare several lemmas concerning the estimates for $(\Gamma^t(f, g), h)_{L^2}$, where $L^2 = L^2(\mathbb{R}_v^3)$. In this subsection, variables t and x are regarded as parameters. For the brevity we often write Γ and μ instead of Γ^t and $\mu_\kappa(t, v)$, respectively. All constants in estimates are uniform with respect to $t \in [0, T]$ and moreover they hold with $\mu_\kappa(t)$ replaced by $\mu^{1/2}$.

Lemma 4.5. *Let $0 < s < 1$ and $\gamma > \max\{-3, -2s - 3/2\}$. Then for any $\beta \in \mathbb{R}$ we have*

$$(4.5) \quad \begin{aligned} & \left| (\Gamma^t(f, g), h)_{L^2} - (\mathcal{Q}(\mu_\kappa f, g), h)_{L^2} \right| \\ & \lesssim (\mathcal{D}(\mu_\kappa |f|, \langle v \rangle^\beta g))^{1/2} \|f\|_{L^2}^{1/2} \|h\|_{L^2_{s+\gamma/2-\beta}} + \|f\|_{L^2} \|g\|_{L^2_{s+\gamma/2+\beta}} \|h\|_{L^2_{s+\gamma/2-\beta}}. \end{aligned}$$

Proof. We write

$$\begin{aligned} (\Gamma(f, g), h)_{L^2} - (\mathcal{Q}(\mu f, g), h)_{L^2} &= \iiint B(\mu'_* - \mu_*) (f_*) g h' d\sigma dv_* dv \\ &= 2 \iiint B(\mu'_*{}^{1/2} - \mu_*^{1/2}) (\mu_*^{1/2} f_*) g' h' d\sigma dv_* dv \\ &\quad + \iiint B(\mu'_*{}^{1/2} - \mu_*^{1/2})^2 f_* g h' d\sigma dv_* dv \\ &\quad + 2 \iiint B(\mu'_*{}^{1/2} - \mu_*^{1/2}) h' f_* \mu_*^{1/2} (g - g') d\sigma dv_* dv \\ &= D_1 + D_2 + D_3. \end{aligned}$$

By the Cauchy-Schwarz inequality we have for any $\beta \in \mathbb{R}$

$$\begin{aligned} |D_3| &\lesssim \left(\iiint B(\mu'_*{}^{1/2} - \mu_*^{1/2})^2 |f_*| (\langle v \rangle^{-\beta} h)^2 d\sigma dv_* dv \right)^{1/2} \\ &\quad \times \left(\iiint B \mu_* |f_*| \langle v \rangle^{2\beta} (g - g')^2 d\sigma dv_* dv \right)^{1/2} \\ &= (\bar{D}_3(f, \langle v \rangle^{-\beta} h))^{1/2} (\mathcal{D}_\beta(\mu f, g))^{1/2}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{D}_\beta(\mu f, g) &\leq 2(\mathcal{D}(\mu |f|, \langle v \rangle^\beta g) + \iiint B \mu_* |f_*| (\langle v \rangle^\beta - \langle v \rangle^\beta)^2 g^2 dv dv_* d\sigma) \\ &\lesssim \mathcal{D}(\mu |f|, \langle v \rangle^\beta g) + \|f\|_{L^2} \|g\|_{L^2_{\beta+\gamma/2}}^2, \end{aligned}$$

because it follows from the same arguments in the proof of Lemma 2.12 that

$$(4.6) \quad \left| \langle v \rangle^\beta - \langle v \rangle^\beta \right| \leq \sin \frac{\theta}{2} (\langle v \rangle^\beta \langle v_* \rangle^{2\beta+1} \mathbf{1}_{|v-v_*|>1} + \langle v \rangle^{\beta-1} |v - v_*| \mathbf{1}_{|v-v_*|\leq 1}).$$

Note that

$$\begin{aligned} \left((\mu'_*)^{1/2} - \mu_{\kappa_*}^{1/2} \right)^2 &\lesssim (\mu'_*)^{1/4} - \mu_*^{1/4} \left((\mu'_*)^{1/2} + \mu_*^{1/2} \right) \\ &\lesssim \min(|v' - v_*|, 1) \min(|v' - v'_*|, 1) (\mu'_*)^{1/2} + \left(\min(|v' - v_*|, 1) \right)^2 \mu_*^{1/2}. \end{aligned}$$

By this decomposition we estimate

$$\widetilde{D}_3(f, \langle v \rangle^{-\beta} h) \lesssim \widetilde{D}_3^{(1)} + \widetilde{D}_3^{(2)}.$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \left| \widetilde{D}_3^{(1)} \right|^2 &\lesssim \left(\iiint B |v' - v'_*|^{\gamma+2s} \left(\min(|v' - v'_*|, 1) \right)^2 \mu'_* (W_{-\beta} h)^2 \langle v' \rangle^{-(\gamma+2s)} d\sigma dv_* dv dx \right) \\ &\quad \times \left(\iiint B |v' - v_*|^{-(\gamma+2s)} \left(\min(|v' - v_*|, 1) \right)^2 f_*^2 (W_{-\beta} h)^2 \langle v' \rangle^{\gamma+2s} d\sigma dv_* dv \right) \\ &\lesssim \|h\|_{L^2_{s+\gamma/2-\beta}}^2 \|f\|_{L^2}^2 \|h\|_{L^2_{s+\gamma/2-\beta}}^2. \end{aligned}$$

Here, we have taken the change of variables $(v, v_*) \rightarrow (v', v'_*)$ and $v \rightarrow v'$ in the first and second factors, respectively, and moreover, in view of $2(\gamma + 2s) > -3$, we have used the fact that

$$\int \left(\int B |v - v_*|^{\gamma+2s} \min(|v - v_*|^2, 1) d\sigma \right) \mu_* dv_* \lesssim \int |v - v_*|^{2(\gamma+2s)} \mu_* dv_* \lesssim \langle v \rangle^{2(\gamma+2s)}.$$

Since the estimation of $\widetilde{D}_3^{(2)}$ is quite similar as $\widetilde{D}_3^{(1)}$ we obtain

$$(4.7) \quad \widetilde{D}_3(f, \langle v \rangle^{-\beta} h) \lesssim \|f\|_{L^2} \|h\|_{L^2_{s+\gamma/2-\beta}}^2$$

and hence

$$|D_3| \lesssim \left(\mathcal{D}(\mu|f|, \langle v \rangle^\beta g) \right)^{1/2} \|f\|_{L^2}^{1/2} \|h\|_{L^2_{s+\gamma/2-\beta}}^2 + \|f\|_{L^2} \|g\|_{L^2_{\beta+\gamma/2}} \|h\|_{L^2_{s+\gamma/2-\beta}}$$

The Cauchy-Schwarz inequality shows

$$|D_2|^2 \leq \widetilde{D}_3(f, \langle v \rangle^{-\beta} h) \left(\iiint B \left((\mu'_*)^{1/2} - \mu_*^{1/2} \right)^2 |f_*| \left(\langle v' \rangle^\beta g \right)^2 d\sigma dv_* dv \right),$$

so that it is easy to see, in view of (4.6),

$$|D_2| \lesssim \|f\|_{L^2} \|g\|_{L^2_{s+\gamma/2+\beta}} \|h\|_{L^2_{s+\gamma/2-\beta}}.$$

Take the change of variables $(v', v'_*) \rightarrow (v, v_*)$ and $(v, v_*) \rightarrow (v_*, v)$ for D_1 . Then we consider

$$\begin{aligned} D_1 &= 2 \iiint B \left(\mu^{1/2} - (\mu')^{1/2} \right) (\mu^{1/2} f)' (gh)_* d\sigma dv_* dv \\ &= 2 \iiint B (\nabla \mu^{1/2})(v') \cdot (v - v') (\mu^{1/2} f)' (gh)_* d\sigma dv_* dv \\ &\quad + \int_0^1 \iiint B (\nabla^2 \mu^{1/2})(v' + \tau(v - v')) (v - v')^2 (\mu^{1/2} f)' (gh)_* d\sigma dv_* dv d\tau \\ &= D_{1,1} + D_{1,2}, \end{aligned}$$

by using the Taylor formula

$$\mu^{1/2} - \mu'^{1/2} = (\nabla \mu^{1/2})(v') \cdot (v - v') + \frac{1}{2} \int_0^1 (\nabla^2 \mu^{1/2})(v' + \tau(v - v')) (v - v')^2 d\tau.$$

Note

$$|(\nabla^2 \mu^{1/2})(v' + \tau(v - v')) (v - v')^2| \lesssim |v' - v_*|^2 (1 - \cos \theta).$$

and deduce

$$D_{1,1} = 2 \left(\iiint_{\{|v' - v_*|^2 (1 - \cos \theta) \leq 1\}} + \iiint_{\{|v' - v_*|^2 (1 - \cos \theta) > 1\}} \right).$$

Then it follows from the spherical symmetry that the first term of the decomposition $D_{1,1}$ vanishes, so that we can estimate by the change of variables $v \rightarrow v'$

$$\begin{aligned} |D_1| &\lesssim \iint |v' - v_*|^\gamma (\mu^{1/2} |f|)' \left(\int_{S^2} b(\cos \theta) \min(\theta^2 |v' - v_*|^2, 1) d\sigma \right) (gh)_* dv' dv_* \\ &\lesssim \left(\iint |v' - v_*|^{2(\gamma+2s)} \mu' g_*^2 \langle v_* \rangle^{-(\gamma+2s)+\beta} dv' dv_* \right)^{1/2} \\ &\quad \times \left(\iint f'^2 h_*^2 \langle v_* \rangle^{\gamma+2s-\beta} dv' dv_* \right)^{1/2} \\ &\lesssim \|f\|_{L^2} \|g\|_{L^2_{s+\gamma/2+\beta}} \|h\|_{L^2_{s+\gamma/2-\beta}} \end{aligned}$$

Summing up above estimates we obtain the desired estimate. \square

Since Lemma 2.11 holds with $\sqrt{\mu}$ replaced by $\mu_\kappa(t, v)$, the combination of Lemma 2.11 and Lemma 4.5 with $\beta = 0$ implies

Lemma 4.6. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$. If $f \geq 0$ then we have*

$$(4.8) \quad \begin{aligned} (\Gamma^t(f, g), g)_{L^2} &\leq -\frac{1}{4} \mathcal{D}(\mu_\kappa f, g) \\ &\quad + C \min \left\{ \|f\|_{L^2} (\|g\|_{H_{\gamma/2}^{s'}} + \|g\|_{L^2_{s+\gamma/2}}), \|f\|_{H^{2s'}} \|g\|_{L^2_{s+\gamma/2}} \right\} \end{aligned}$$

for any $s' \in]0, s[$ satisfying $\gamma + 2s' > -3/2$ and $s' < 3/4$.

Furthermore, if $\gamma > -3/2$ then the second term on the right hand side can be replaced by $C \|f\|_{L^2} \|g\|_{L^2_{s+\gamma/2}}^2$.

Lemma 4.7. *Let $0 < s < 1$, $\gamma > \max\{-3, -2s - 3/2\}$. For any $\ell \in \mathbb{R}$ and $m \in [0, s]$ we have*

$$(4.9) \quad \left| (\Gamma^t(f, g), h)_{L^2} \right| \lesssim \|f\|_{L^2} \|g\|_{H_{\ell+(\gamma+2s)^+}^{s+m}} \|h\|_{H_{-\ell}^{s-m}}.$$

Furthermore

$$(4.10) \quad \left| (\Gamma^t(f, g), g)_{L^2} \right| \lesssim \|f\|_{L^2} \|g\|_{\Phi_\gamma}^2.$$

Proof. Since Lemma 2.12 holds with $\sqrt{\mu}$ replaced by $\mu_\kappa(t, v)$, in view of (2.11) we have

$$\left| \mathcal{D}(\mu_\kappa f, g) \right| \lesssim \|f\|_{L^2} \|g\|_{\Phi_\gamma}^2 \lesssim \|f\|_{L^2} \|g\|_{H_{(s+\gamma/2)^+}^s}^2.$$

Applying this to the right hand side of (4.5), by Proposition 2.9 we obtain the desired estimate (4.9). The second estimate can be obtained by using Proposition 2.13 instead of Proposition 2.9. \square

For $\alpha > 3/2$, set $\varphi(v, x) = (1 + |v|^2 + |x|^2)^{\alpha/2}$ and

$$W_{\varphi, \ell} = \frac{\langle v \rangle^\ell}{\varphi(v, x)} = \frac{(1 + |v|^2)^{\ell/2}}{(1 + |v|^2 + |x|^2)^{\alpha/2}}.$$

Lemma 4.8. *If $\varphi(v, x) = (1 + |v|^2 + |x|^2)^{\alpha/2}$ for $\alpha > 3/2$ and*

$$W_{\varphi, \ell} = \frac{\langle v \rangle^\ell}{\varphi(v, x)} = \frac{(1 + |v|^2)^{\ell/2}}{(1 + |v|^2 + |x|^2)^{\alpha/2}}.$$

for $\ell \in \mathbb{R}$ then we have

$$(4.11) \quad |\partial_x^{\beta_1} \partial_v^{\beta_2} W_{\varphi, \ell}(v)| \lesssim \langle v \rangle^{-|\beta_1| - |\beta_2|} W_{\varphi, \ell}(v),$$

$$(4.12) \quad \left| W_{\varphi, \ell}(v') - W_{\varphi, \ell}(v) \right| \lesssim \sin\left(\frac{\theta}{2}\right) \frac{|v - v'_*| \langle v'_* \rangle^{\alpha+|\ell|-1}}{\langle v \rangle} W_{\varphi, \ell}(v),$$

$$(4.13) \quad \left| \nabla^2 W_{\varphi, \ell}(v + \tau(v' - v)) \right| \lesssim \frac{\langle v_* \rangle^{\alpha+|\ell|-2}}{\langle v \rangle^2} W_{\varphi, \ell}(v), \quad \tau \in [0, 1].$$

Proof. The first inequality follows from the direct calculation. Since

$$W_{\varphi,\ell}(v') - W_{\varphi,\ell}(v) = \int_0^1 \nabla_v W_{\varphi,\ell}(v + \tau(v' - v)) d\tau \cdot (v' - v)$$

and since $|v - v'| = \sin(\theta/2)|v - v_*|$, $|v - v_*| \sim |v - v'_*|$, for the proof of (4.12) it suffices to show

$$(4.14) \quad \left| \nabla W_{\varphi,\ell}(v_\tau) \right| \lesssim \frac{W_{\varphi,\ell}(v_\tau)}{\langle v_\tau \rangle} \lesssim \frac{\langle v'_* \rangle^{\alpha+(\ell-1)^+}}{\langle v \rangle} W_{\varphi,\ell}(v) \quad , \quad v_\tau = v + \tau(v' - v).$$

For $a \in \mathbb{R}$ we have

$$\begin{aligned} 1 + a^2 + |v|^2 &\lesssim 1 + a^2 + |v - v'_*|^2 + |v'_*|^2 \lesssim 1 + a^2 + |v_\tau - v'_*|^2 + |v'_*|^2 \\ &\lesssim 1 + a^2 + |v_\tau|^2 + |v'_*|^2 \lesssim (1 + a^2 + |v_\tau|^2) \langle v'_* \rangle^2, \end{aligned}$$

from which we get $\varphi(v_\tau, x)^{-1} \lesssim \varphi(v, x)^{-1} \langle v'_* \rangle^\alpha$ by setting $a = |x|$. Putting $a = 0$ in the above inequality we have $\langle v \rangle \lesssim \langle v_\tau \rangle \langle v'_* \rangle$. Since $\langle v_\tau \rangle \lesssim \langle v \rangle \langle v'_* \rangle$ holds similarly we have $\langle v_\tau \rangle^{\ell-1} \lesssim \langle v \rangle^{\ell-1} \langle v'_* \rangle^{\ell-1}$, which concludes the second inequality of (4.14). (4.13) also follows from the similar observation. \square

Lemma 4.9. *Let $0 < s < 1$ and $\gamma > \max\{-3, -2s - 3/2\}$. Then for any $\ell \geq 0$ we have*

$$\begin{aligned} &\left| (W_{\varphi,\ell} \Gamma^t(f, g) - \Gamma^t(f, W_{\varphi,\ell} g), h)_{L^2} \right| \\ &\lesssim (\mathcal{D}(\mu_\kappa |f|, h))^{1/2} \|f\|_{L^2}^{1/2} \|W_{\varphi,\ell} g\|_{L^2_{\gamma/2}} + \|f\|_{H^{(2s-1)^+}} \|W_{\varphi,\ell} g\|_{L^2_{s'+\gamma/2}} \|h\|_{L^2_{s+\gamma/2}}, \end{aligned}$$

for any $s' \in (2s - 1)^+$, s' satisfying $\gamma + 2s' > -3/2$.

Proof. Note that

$$\begin{aligned} &(W_{\varphi,\ell} \Gamma(f, g) - \Gamma(f, W_{\varphi,\ell} g), h)_{L^2} = \iiint B f_* (\mu_* h)' g(W'_{\varphi,l} - W_{\varphi,l}) dv dv_* d\sigma \\ &= \iiint B f_* \{(\mu_*^{1/2} h)' - (\mu_*^{1/2} h)\} \mu_*^{1/2} g(W'_{\varphi,l} - W_{\varphi,l}) dv dv_* d\sigma \\ &\quad + \iiint B f_* (\mu_*^{1/2} h) (\mu_*^{1/2} - \mu_*^{1/2}) g(W'_{\varphi,l} - W_{\varphi,l}) dv dv_* d\sigma \\ &\quad + \iiint B f_* (\mu_*^{1/2} h) \mu_*^{1/2} g(W'_{\varphi,l} - W_{\varphi,l}) dv dv_* d\sigma \\ &= \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \mathcal{A}_1^2 &\leq \iiint B |f_*| \{(\mu_*^{1/2} h)' - (\mu_*^{1/2} h)\}^2 dv dv_* d\sigma \\ &\quad \times \iiint B |f_*| (\mu_*^{1/2} g(W'_{\varphi,l} - W_{\varphi,l}))^2 dv dv_* d\sigma \\ &= \mathcal{A}_{1,1} \times \mathcal{A}_{1,2}. \end{aligned}$$

Writing

$$(\mu_*^{1/2} h)' - (\mu_*^{1/2} h) = \mu_*^{1/2} (h' - h) + h' (\mu_*^{1/2} - \mu_*^{1/2})$$

we obtain easily

$$\begin{aligned} \mathcal{A}_{1,1} &\leq 2 \left\{ \iiint B (\mu |f|)_* (h' - h)^2 dv dv_* d\sigma \right. \\ &\quad \left. + \iiint B |f_*| (\mu_*^{1/2} - \mu_*^{1/2})^2 h'^2 dv dv_* d\sigma \right\} \\ &= 2(\mathcal{D}(\mu |f|, h) + \widetilde{\mathcal{D}}_3(f, h)) \\ &\lesssim \mathcal{D}(\mu |f|, h) + \|f\|_{L^2} \|h\|_{L^2_{s+\gamma/2}}^2, \end{aligned}$$

where we have used (4.7). By (4.12) and the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\mathcal{A}_{1,2}|^2 &\lesssim \left(\iiint b \sin^2(\theta/2) \frac{|v-v_*|^{\gamma+2}}{\langle v \rangle^2} \mu_*'^{1/2} |f_*| (W_{\varphi,l}g)^2 dv dv_* d\sigma \right)^2 \\ &\lesssim \left(\iiint b \sin^2(\theta/2) \frac{|v-v_*|^{2(\gamma+2)}}{\langle v \rangle^{\gamma+4}} \mu_*' (W_{\varphi,l}g)^2 d\sigma dv_* dv \right) \\ &\times \left(\iiint b \sin^2(\theta/2) f_*^2 \langle v \rangle^\gamma (W_{\varphi,l}g)^2 d\sigma dv_* dv \right) \\ &\lesssim \|W_{\varphi,l}g\|_{L_{\gamma/2}^2}^2 \left(\|f\|_{L^2}^2 \|W_{\varphi,l}g\|_{L_{\gamma/2}^2}^2 \right), \end{aligned}$$

where we have used the change of variables $v_* \rightarrow v'_*$. Hence we have

$$|\mathcal{A}_1| \lesssim \left(\mathcal{D}(\mu|f, h) + \|f\|_{L^2}^{1/2} \|h\|_{L_{s+\gamma/2}^2} \right) \|f\|_{L^2}^{1/2} \|W_{\varphi,l}g\|_{L_{\gamma/2}^2}.$$

By using the similar formula as (4.12) with v'_* replaced by v_* we have

$$\left| \mu_*^{1/4} (\mu_*'^{1/2} - \mu_*^{1/2}) (W'_{\varphi,l} - W_{\varphi,l}) \right| \lesssim \min(|v-v_*|^2 \theta^2, |v-v_*| \theta) W_{\varphi,l} \langle v \rangle^{-1},$$

so that for any $\delta > 0$

$$\begin{aligned} |\mathcal{A}_2| &\lesssim \iiint |v-v_*|^\gamma \left(|v-v_*|^{2s} + \mathbf{1}_{|v-v_*| \geq 1} |v-v_*|^{(2s-1+\delta)^+ + 1} \right) \mu_*^{1/4} \left| f_* \frac{(W_{\varphi,l}g)h}{\langle v \rangle} \right| dv dv_* \\ &\lesssim \|f\|_{L^2} \|W_{\varphi,l}g\|_{L_{(2s-1+\delta)^+/2+\gamma/2}^2} \|h\|_{L_{(2s-1+\delta)^+/2+\gamma/2}^2}. \end{aligned}$$

In order to estimate \mathcal{A}_3 we use the Taylor expansion for $W_{\varphi,l} - W'_{\varphi,l}$ of second order. Then we have

$$\begin{aligned} \mathcal{A}_3 &= \iiint B f_* \mu_* g h (\nabla_v W_{\varphi,l})(x, v) \cdot (v' - v) dv dv_* d\sigma \\ &\quad + \frac{1}{2} \int_0^1 d\tau \iiint B f_* \mu_* g h (\nabla_v^2 W_{\varphi,l})(x, v + \tau(v' - v)) (v' - v)^2 dv dv_* d\sigma \\ &= \mathcal{A}_{3,1} + \mathcal{A}_{3,2}. \end{aligned}$$

Setting $\mathbf{k} = \frac{v-v_*}{|v-v_*|}$ and writing

$$v' - v = \frac{1}{2} |v-v_*| (\sigma - (\sigma \cdot \mathbf{k}) \mathbf{k}) + \frac{1}{2} ((\sigma \cdot \mathbf{k}) - 1) (v - v_*),$$

we have

$$\mathcal{A}_{3,1} = \frac{1}{2} \iiint B f_* \mu_* g h (\nabla_v W_{\varphi,l})(x, v) \cdot (v - v_*) (\cos \theta - 1) dv dv_* d\sigma$$

because it follows from the symmetry that $\int_{S^2} b(\sigma \cdot \mathbf{k}) (\sigma - (\sigma \cdot \mathbf{k}) \mathbf{k}) d\sigma = 0$. Therefore, for any $0 \leq s' < s$ satisfying $\gamma + 2s' < -3/2$ we have, in view of (4.11),

$$\begin{aligned} |\mathcal{A}_{3,1}| &\lesssim \int \left(\int |v-v_*|^{2(\gamma+2s')} \mu_*^2 dv_* \right)^{1/2} \left(\int \frac{f_*^2}{|v-v_*|^{2(2s'-1)^+}} dv_* \right)^{1/2} |W_{\varphi,l-1}g| |h| dv \\ &\lesssim \|f\|_{H^{(2s'-1)^+}} \|W_{\varphi,l}g\|_{L_{s'+\gamma/2}^2} \|h\|_{L_{s'+\gamma/2}^2}, \end{aligned}$$

by means of $|\nabla_v (W_{\varphi,l})(x, v)| \lesssim W_{\varphi,l-1}$. The better bound holds for $|\mathcal{A}_{3,2}|$ since it follows from (4.13) that

$$|(\nabla_v^2 W_{\varphi,l})(v + \tau(v' - v)) (v' - v)^2| \lesssim \langle v_* \rangle^{\alpha+|\ell-2|} W_{\varphi,l-2} \theta^2 |v - v_*|^2.$$

Therefore we have

$$|\mathcal{A}_3| \lesssim \|f\|_{H^{(2s'-1)^+}} \|W_{\varphi,l}g\|_{L_{s'+\gamma/2}^2} \|h\|_{L_{s'+\gamma/2}^2}.$$

Summing up above estimates we obtain the conclusion. \square

4.2. Proofs of the uniqueness Theorems. Using the notations introduced in subsection ??, the proof of Theorem 4.1 is reduced to

Proposition 4.10. *Assume that $0 < s < 1$ and $\max(-3, -3/2-2s) < \gamma < 2-2s$. Let $\ell_0 > 3/2 + \max\{1, (\gamma+2s)^+\}$ and $g_0 \in L^\infty(\mathbb{R}_x^3; L_{\ell_0}^2(\mathbb{R}_v^3))$. Suppose that the Cauchy problem (4.3) admits two solutions*

$$g_1, g_2 \in L^\infty([0, T] \times \mathbb{R}_x^3; H_{\ell_0}^m(\mathbb{R}_v^3)).$$

Then $g_1 \equiv g_2$ in $[0, T]$,

- 1) if $m = 2s$ and $g_1 \geq 0$. When $\gamma > -3/2$, we can suppose $g_1 \in L^\infty([0, T] \times \mathbb{R}_x^3; H_{\ell_0}^s(\mathbb{R}_v^3))$.
- 2) if $m = s$ and the coercivity inequality (4.1) is satisfied for $f_1 = \mu_\kappa(t)g_1 \geq 0$.
- 3) if $m = s$ and $f = \mu_\kappa(t)g_1$ satisfies the following strong coercivity estimate

$$(4.15) \quad -(Q(f(t), h), h)_{L^2(\mathbb{R}^6)} \geq c_0 \int \|h\|_{\Phi_\gamma}^2 dx - C \|h\|_{L^2(\mathbb{R}_x^3; L_{(\gamma/2+s)^+}^2(\mathbb{R}_v^3))}^2.$$

Proof. Let $S(\tau) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq S \leq 1$ and

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$

Set $S_N(D_x) = S(2^{-2N}|D_x|^2)$ and multiply $W_{\varphi, l} S_N(D_x)^2 W_{\varphi, l} g$ to (4.4), where we choose ℓ, α such that

$$\ell_0 - \max\{1, (\gamma + 2s)^+\} > \ell > \alpha > 3/2.$$

Integrating and letting $N \rightarrow \infty$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{\varphi, l} g(t)\|_{L^2(\mathbb{R}^6)}^2 + \kappa \|W_{\varphi, l+1} g(t)\|_{L^2(\mathbb{R}^6)}^2 &= (W_{\varphi, l} \Gamma^t(g_1, g) + W_{\varphi, l} \Gamma^t(g, g_2), W_{\varphi, l} g)_{L^2(\mathbb{R}^6)} \\ &\quad - (v \cdot \nabla_x (\varphi^{-1}) W_l g, W_{\varphi, l} g)_{L^2(\mathbb{R}^6)}, \end{aligned}$$

because $(v \cdot \nabla_x S_N(D_x) W_{\varphi, l} g, S_N(D_x) W_{\varphi, l} g)_{L^2(\mathbb{R}^6)} = 0$. The second term on the right hand side is estimated by $\|W_{\varphi, l} g\|_{L^2(\mathbb{R}^6)}^2$ because of (4.11). Write the first term on the right hand side as

$$\begin{aligned} &(\Gamma^t(g_1, W_{\varphi, l} g), W_{\varphi, l} g)_{L^2(\mathbb{R}^6)} + (W_{\varphi, l} \Gamma^t(g_1, g) - \Gamma^t(g_1, W_{\varphi, l} g), W_{\varphi, l} g)_{L^2(\mathbb{R}^6)} \\ &+ (\Gamma^t(g, g_2), W_{\varphi, l} W_{\varphi, l} g)_{L^2(\mathbb{R}^6)} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3. \end{aligned}$$

If $g_1 \geq 0$ then it follows from Lemma 4.6 that

$$(4.16) \quad \begin{aligned} \mathcal{B}_1 &\leq -\frac{1}{4} \int \mathcal{D}(\mu_\kappa g_1, W_{\varphi, l} g) dx \\ &\quad + C \min\left(\|g_1\|_{L_{t,x}^\infty(L^2)} (\|W_{\varphi, l} g\|_{L_x^2(H_{\gamma/2}^{s'})})^2 + \|W_{\varphi, l} g\|_{L_x^2(L_{s+\gamma/2}^2)}^2\right), \quad \|g_1\|_{L_{t,x}^\infty(H^{2s})} \|W_{\varphi, l} g\|_{L_x^2(L_{s+\gamma/2}^2)}^2. \end{aligned}$$

We notice that the last term can be replaced by $\|g_1\|_{L_{t,x}^\infty(L^2)} \|W_{\varphi, l} g\|_{L_x^2(L_{s+\gamma/2}^2)}^2$ if $\gamma > -3/2$. By means of Lemma 4.9 we obtain for any $\delta > 0$

$$(4.17) \quad |\mathcal{B}_2| \leq \delta \int \mathcal{D}(\mu_\kappa |g_1|, W_{\varphi, l} g) dx + C_\delta \|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} \|W_{\varphi, l} g\|_{L_x^2(L_{s'+\gamma/2}^2)} \|W_{\varphi, l} g\|_{L_x^2(L_{s+\gamma/2}^2)}.$$

Lemma 4.7 with $\ell = \ell - \gamma/2$ implies that for $m = s, 0$

$$(4.18) \quad \begin{aligned} |\mathcal{B}_3| &\lesssim \int \|\langle x \rangle^{-\alpha} g\|_{L^2(\mathbb{R}^3)} \|g_2\|_{H_{\ell-\gamma/2+(2s+\gamma)^+}^{s+m}(\mathbb{R}^3)} \left\| \frac{\langle x \rangle^\alpha}{\langle v \rangle^\ell} W_{\varphi, l} W_{\varphi, l} g \right\|_{H_{\gamma/2}^{s-m}} dx \\ &\lesssim \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^{s+m})} \|W_{\varphi, \alpha} g\|_{L_x^2(L_{s+\gamma/2}^2)} \|W_{\varphi, \ell} g\|_{L_x^2(H_{\gamma/2}^{s-m})}, \end{aligned}$$

because $\langle x \rangle^{-\alpha} \leq W_{\varphi, \alpha}$ and $\langle x \rangle^\alpha \langle v \rangle^{-\ell} W_{\varphi, l}$ is a bounded operator on H^{s-m} . Note that for any $\delta > 0$

$$\|W_{\varphi, \ell+(s+\gamma/2)^+} g(t)\|_{L^2(\mathbb{R}^6)}^2 \leq \delta \|W_{\varphi, \ell+1} g(t)\|_{L^2(\mathbb{R}^6)}^2 + C_\delta \|W_{\varphi, \ell} g(t)\|_{L^2(\mathbb{R}^6)}^2.$$

If $g_1, g_2 \in \tilde{\mathcal{E}}^{2s}([0, T] \times \mathbb{R}^6)$ and $g_1 \geq 0$ then by summing up (4.16), (4.17) and (4.18) with $m = s$ we have

$$\frac{d}{dt} \|W_{\varphi, l} g(t)\|_{L^2(\mathbb{R}^6)}^2 \leq C_\kappa \left(\|g_1\|_{L_{t,x}^\infty(H^{2s})} + \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^{2s})} \right) \|W_{\varphi, l} g(t)\|_{L^2(\mathbb{R}^6)}^2,$$

where $\|g_1\|_{L_{t,x}^\infty(H^{2s})}$ can be replaced by $\|g_1\|_{L_{t,x}^\infty(L^2)}$ if $\gamma > -3/2$. Here it should be noted that the term $\mathcal{D}(\mu_\kappa g_1, W_{\varphi,\ell}g) \geq 0$ follows from the non-negativity of g_1 . Therefore, $\|W_{\varphi,\ell}g(0)\|_{L^2(\mathbb{R}^6)} = 0$ implies $\|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)} = 0$ for all $t \in [0, T]$. And this gives $g_1 = g_2$, and concludes the part 1) of Proposition 4.10.

For the part 2) of Proposition 4.10, using $\|h\|_{H_{\gamma/2}^s}^2 \leq \delta \|h\|_{H_{\gamma/2}^s}^2 + C_\delta \|h\|_{L_{\gamma/2}^2}$ and summing up (4.16), (4.17) and (4.18) with $m = 0$, we have

$$\begin{aligned} \frac{d}{dt} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2 &\leq -\frac{1}{16} \left(\int \mathcal{D}(f_1(t), W_{\varphi,\ell}g(t)) dx + \kappa \|W_{\varphi,\ell+1}g(t)\|_{L^2(\mathbb{R}^6)}^2 \right) \\ &+ \delta \left(\|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} + \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^s)} \right) \|W_{\varphi,\ell}g(t)\|_{L_x^2(H_{\gamma/2}^s)}^2 \\ &+ C_{\kappa,\delta} \left(\|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} + \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^s)} \right) \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2, \end{aligned}$$

then the coercivity condition (4.1)(with $(\gamma/2 + s)^+ < 1$) together with (2.12) leads us to

$$(4.19) \quad \frac{d}{dt} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2 \lesssim \left(\|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} + \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^s)} \right) \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2,$$

where it should be noted that (2.12) holds with $\mu^{1/2}$ replaced by $\mu_\kappa(t, \nu)$. Thus, the part 2) of Proposition 4.10 is proved.

When g_1 is not necessarily non-negative, by using Lemma 4.5 we obtain

$$\mathcal{B}_1 \leq \left(Q(\mu_\kappa g_1, W_{\varphi,\ell}g), W_{\varphi,\ell}g \right)_{L^2(\mathbb{R}^6)} + \delta \int \mathcal{D}(\mu_\kappa |g_1|, W_{\varphi,\ell}g) dx + C_\delta \|W_{\varphi,\ell}g\|_{L_x^2(L_{s+\gamma/2}^2)}^2$$

instead of (4.16). Since Lemma 2.12 holds with $\sqrt{\mu}$ replaced by μ_κ , by means of (4.15) we get

$$(4.20) \quad \mathcal{B}_1 \leq -(c_0 - \delta) \int \|W_{\varphi,\ell}g\|_{\Phi_\gamma}^2 dx + C_\delta \|W_{\varphi,\ell}g\|_{L_x^2(L_{(s+\gamma/2)^+}^2)}^2.$$

This estimate and (4.18), together with (4.17) applied by Lemma 2.12, imply (4.19). Hence the part 3) of Proposition 4.10 is also proved. \square

Proof of Theorem 4.3 :

If we set $g_j(t) = \mu^{-1/2} f_j(t)$ ($j = 1, 2$) and $g = g_1 - g_2$, then we have

$$\begin{cases} g_t + \nu \cdot \nabla_x g = \Gamma(g_1, g) + \Gamma(g, g_2), \\ g|_{t=0} = 0, \end{cases}$$

where $\Gamma(g, h) = \mu^{-1/2} Q(\mu^{1/2} g, \mu^{1/2} h)$. Take the inner product with $W_{\varphi,\ell} S_N(D_x)^2 W_{\varphi,\ell} g$ where we choose ℓ, α such that $\ell_1 - (\gamma + 2s) > \ell > \alpha > 3/2$. Then we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2 &= \left(W_{\varphi,\ell} \Gamma(g_1, g) + W_{\varphi,\ell} \Gamma(g, g_2), W_{\varphi,\ell}g \right)_{L^2(\mathbb{R}^6)} \\ &- (\nu \cdot \nabla_x (\varphi^{-1}) W_\ell g, W_{\varphi,\ell}g)_{L^2(\mathbb{R}^6)}, \end{aligned}$$

where the second term on the right hand side is estimated by $\|W_{\varphi,\ell}g\|_{L^2(\mathbb{R}^6)}^2$ because of (4.11). We write the first term on the right hand side as

$$\left(\Gamma(g_1, W_{\varphi,\ell}g), W_{\varphi,\ell}g \right)_{L^2(\mathbb{R}^6)} + \widetilde{\mathcal{B}}_2 + \widetilde{\mathcal{B}}_3,$$

where $\widetilde{\mathcal{B}}_2, \widetilde{\mathcal{B}}_3$ are defined by the same way as the above $\mathcal{B}_2, \mathcal{B}_3$ with Γ^t replaced by Γ and satisfy the similar estimates as (4.17) and (4.18), respectively, that is,

$$\begin{aligned} |\widetilde{\mathcal{B}}_2| &\leq \delta \int \mathcal{D}(\mu^{1/2} |g_1|, W_{\varphi,\ell}g) dx + C_\delta \|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} \|W_{\varphi,\ell}g\|_{L_x^2(L_{s+\gamma/2}^2)} \|W_{\varphi,\ell}g\|_{L_x^2(L_{s+\gamma/2}^2)}, \\ |\widetilde{\mathcal{B}}_3| &\lesssim \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^s)} \|W_{\varphi,\ell}g\|_{L_x^2(L_{\gamma/2}^2)} \|W_{\varphi,\ell}g\|_{L_x^2(H_{\gamma/2}^s)} \\ &\lesssim \|g_2\|_{L_{t,x}^\infty(H_{\ell-\gamma/2+(2s+\gamma)^+}^s)} \left(\delta \int \|W_{\varphi,\ell}g\|_{\Phi_\gamma}^2 dx + C_\delta \|W_{\varphi,\ell}g\|_{L^2(\mathbb{R}^6)}^2 \right), \end{aligned}$$

where the non-isotropic norm $\|\cdot\|_{\Phi_\gamma}$ is recalled in (2.10). By means of Lemma 2.12, we have

$$\left| \widetilde{\mathcal{B}}_2 \right| \leq \delta \|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} \int \|\|W_{\varphi,\ell}g\|\|_{\Phi_\gamma}^2 dx + C'_\delta \|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} \|W_{\varphi,\ell}g\|_{L^2(\mathbb{R}^6)}^2.$$

On the other hand, it follows from Proposition 2.1 of [9] and (4.10) that for suitable $C_1, C_2 > 0$ we have

$$(4.21) \quad \begin{aligned} & \left(\Gamma(g_1, W_{\varphi,\ell}g), W_{\varphi,\ell}g \right)_{L^2(\mathbb{R}^6)} = - \left(\mathcal{L}_1(W_{\varphi,\ell}g), W_{\varphi,\ell}g \right)_{L^2(\mathbb{R}^6)} + \left(\Gamma(\tilde{g}_1, W_{\varphi,\ell}g), W_{\varphi,\ell}g \right)_{L^2(\mathbb{R}^6)} \\ & \leq -C_1 \int \|\|W_{\varphi,\ell}g\|\|_{\Phi_\gamma}^2 dx \\ & \quad + C_2 \left(\sup_{[0,T] \times \mathbb{R}_x^3} \|\tilde{g}_1\|_{L^2(\mathbb{R}_v^3)} \int \|\|W_{\varphi,\ell}g\|\|_{\Phi_\gamma}^2 dx + \|W_{\varphi,\ell}g\|_{L^2(\mathbb{R}^6)}^2 \right), \end{aligned}$$

where $g_1 = \sqrt{\mu} + \tilde{g}_1$. Therefore, (2.11) and the smallness condition (4.2) imply

$$\begin{aligned} & \frac{d}{dt} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2 \\ & \leq - \left(C_1 - C_2 \varepsilon_0 - \delta (\|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} + \|g_2\|_{L_{t,x}^\infty(H_{\ell+2s+\gamma/2}^s)}) \right) \int \|\|W_{\varphi,\ell}g\|\|_{\Phi_\gamma}^2 dx \\ & \quad + C_\delta \left(\|g_1\|_{L_{t,x}^\infty(H^{(2s-1)^+})} + \|g_2\|_{L_{t,x}^\infty(H_{\ell+2s+\gamma/2}^s)} \right) \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

which shows $g(t) = 0$ for all $t \in [0, T]$ if $\varepsilon_0 < C_1/C_2$. Thus we have proved Theorem 4.3.

Proof of Theorem 4.4

Let now $f_j(t) \in \tilde{\mathcal{B}}^s([0, T] \times \mathbb{R}_{x,v}^6)$, ($j = 1, 2$) and set $g_j(t) = \mu_\kappa(t)^{-1} f_j(t)$ for a suitable $\mu_\kappa(t) = e^{-(\rho-\kappa t)(1+|v|^2)}$. Then we have for any $\ell \in \mathbb{N}$

$$g_j(t) \in L^\infty([0, T] \times \mathbb{R}_x^3; L_\ell^2(\mathbb{R}_v^3)) \cap L^2([0, T]; L^\infty(\mathbb{R}_x^3; H_\ell^m(\mathbb{R}_v^3))).$$

The proof of Theorem 4.4 is reduced to

Proposition 4.11. *Assume that $0 < s < 1$ and $\max\{-3, -3/2 - 2s\} < \gamma \leq -2s$. Let $0 < T < +\infty$ and $\ell_2 \geq 3$. Suppose that the Cauchy problem (4.3) admits two solutions*

$$g_1, g_2 \in L^\infty([0, T] \times \mathbb{R}_x^3; L_{\ell_2}^2(\mathbb{R}_v^3)) \cap L^2([0, T]; L^\infty(\mathbb{R}_x^3; H_{\ell_2}^s(\mathbb{R}_v^3))).$$

If (4.15) is satisfied for $f = \mu_\kappa(t)g_1$ then $g_1(t) \equiv g_2(t)$ for all $t \in [0, T]$.

Proof. Noting $\gamma + 2s \leq 0$, we estimate more carefully $\mathcal{B}_2, \mathcal{B}_3$ in the proof of Proposition 4.10. It follows from Lemma 4.9 and Lemma 2.12 that

$$\left| \mathcal{B}_2 \right| \leq \delta \int \|\|W_{\varphi,\ell}g\|\|_{\Phi_\gamma} dx + C_\delta \|g_1(t)\|_{L_x^\infty(H^s)} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2.$$

Lemma 4.7 with $\ell = \ell - \gamma/2$ and $m = 0$ yields

$$\begin{aligned} \left| \mathcal{B}_3 \right| & \leq \|g_2(t)\|_{L_x^\infty(H_{\ell+\gamma/2}^s)} \|W_{\varphi,\alpha}g(t)\|_{L^2(\mathbb{R}^6)} \|W_{\varphi,\ell}g(t)\|_{L_x^2(H_{\gamma/2}^s)} \\ & \leq \delta \|W_{\varphi,\ell}g(t)\|_{L_x^2(H_{\gamma/2}^s)}^2 + C_\delta \|g_2(t)\|_{L_x^\infty(H_{\ell+\gamma/2}^s)} \|W_{\varphi,\alpha}g(t)\|_{L^2(\mathbb{R}^6)}^2. \end{aligned}$$

Above estimates for \mathcal{B}_j ($j = 2, 3$) and (4.20) imply that

$$\max_{t \in [0, T_1]} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2 \leq \|W_{\varphi,\ell}g(0)\|_{L^2(\mathbb{R}^6)}^2 + \varepsilon(T_1) \max_{t \in [0, T_1]} \|W_{\varphi,\ell}g(t)\|_{L^2(\mathbb{R}^6)}^2,$$

where

$$\varepsilon(T_1) \lesssim T_1 + T_1 \|g_1\|_{L^2([0, T_1]; L_x^\infty(H^s))} + \|g_2\|_{L^2([0, T_1]; L_x^\infty(H_{\ell+\gamma/2}^s))}.$$

By assumption $\varepsilon(T_1) \rightarrow 0$ as $T_1 \rightarrow 0$. Therefore there exists a $T_* > 0$ such that $g(t) \equiv 0$ for $t \in [0, T_*]$. Replacing the initial time 0 by T_* if needed, we finally obtain $g(t) \equiv 0$ for $t \in [0, T]$. \square

4.3. Uniqueness of known solutions. Firstly we consider the uniqueness of global solutions given in [9, 10]. Theorem 4.3 is applicable to show the uniqueness of global solutions in Theorem 1.5 of [9], and also solutions in Theorem 1.1 of [10] because the global solutions given there are of the form $\mu + \sqrt{\mu}\tilde{g}$ with

$$\tilde{g}(t, x, v) \in L^\infty([0, \infty[; \mathcal{H}_\ell^m(\mathbb{R}^6))$$

for $m \geq 6$ and a suitable ℓ . It should be noted that the uniqueness holds under the smallness condition (4.2) of the perturbation \tilde{g} , without the non-negativity of solution $\mu + \mu^{1/2}\tilde{g}$.

It follows from Corollary 2.15 that the smallness condition (4.2) implies (4.15) for the global solution given in Theorem 1.4 of [9], because \tilde{g} there satisfies $\|\tilde{g}\|_{L^\infty([0, \infty); H^3(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} < \varepsilon_0$ and for any $0 < T < +\infty$

$$\int_0^T \left(\sum_{|\alpha| \leq 3} \int \|\partial_x^\alpha \tilde{g}(t, x)\|_{\Phi_\gamma}^2 dx \right) dt < +\infty.$$

Therefore Theorem 4.4 shows the uniqueness of the solution given in Theorem 1.4 of [9] by means of the Sobolev embedding.

In [11], bounded solutions of the Boltzmann equation in the whole space have been constructed without specifying any limit behaviors at the spatial infinity and without assuming the smallness condition on initial data. More precisely, it has been shown that if the initial data is non-negative and belongs to a uniformly local Sobolev space with the Maxwellian decay property in the velocity variable, then the Cauchy problem of the Boltzmann equation possesses a non-negative local solution in the same function space, both for the cutoff and non-cutoff collision cross section with mild singularity. Since solutions there are non-negative and belong to $\mathcal{E}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$, Theorem 4.1 yields their uniqueness.

5. NON-NEGATIVITY OF SOLUTIONS

The purpose of this section is to show the non-negativity of solutions constructed in [9, 10], where the solution $f = \mu + \sqrt{\mu}g$ is a perturbation around a normalized Maxwellian distribution $\mu(v)$, that means g is solution of following Cauchy problem :

$$(5.1) \quad \begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}(g) = \Gamma(g, g), \\ g|_{t=0} = g_0, \end{cases}$$

where

$$\mathcal{L}(g) = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}) = \mathcal{L}_1(g) + \mathcal{L}_2(g).$$

It is the limit of a sequence constructed successively by the following linear Cauchy problem,

$$(5.2) \quad \begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = Q(f^n, f^{n+1}), \\ f^{n+1}|_{t=0} = f_0 = \mu + \mu^{1/2}g_0 \geq 0, \end{cases}$$

if one returns to the original Boltzmann equation. Hence the non-negativity of solution comes from the following induction argument: Let $f^0 = f_0 = \mu + \mu^{1/2}g_0 \geq 0$, suppose that

$$(5.3) \quad f^n = \mu + \mu^{1/2}\tilde{g}^n \geq 0,$$

for some $n \in \mathbb{N}$. Then (5.3) is true for $n + 1$.

Proposition 5.1. *Assume that $\max\{-3, -\frac{3}{2} - 2s\} < \gamma < 2 - 2s$. Let $\{f^n\}$ is a sequence of solutions of Cauchy problem (5.2) with*

$$\exists \rho > 0 ; e^{\rho(v)^2} f^n(t, x, v) \in L^\infty([0, T] \times \mathbb{R}_x^3; H^N(\mathbb{R}_v^3)) \text{ for } \forall n = 1, 2, 3, \dots,$$

for some $N \geq 4$. Then for any $n \in \mathbb{N}$, $f^n \geq 0$ on $[0, T]$ implies $f^{n+1} \geq 0$ on the same interval.

Proof. Taking a $\kappa > 0$ such that $\frac{\rho}{2\kappa} > T$, we set $g^n(t, x, v) = \mu_\kappa(v)^{-1} f^n(t, x, v)$ with $\mu_\kappa(t) = e^{-(\rho - \kappa t)(v)^2}$ then it follows from (5.2) that

$$(5.4) \quad \partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \kappa(v)^2 g^{n+1} = \Gamma'(g^n, g^{n+1}).$$

We notice that for any $\ell \in \mathbb{N}$

$$g^n \in L^\infty([0, T] \times \mathbb{R}_x^3; H_\ell^N(\mathbb{R}_v^3))$$

so that $\sup_{t,x} \|g^n\|_{\Phi_\gamma} < \infty$. If g satisfies $\|g\| < \infty$ and if $g_\pm = \pm \max(\pm g, 0)$, then we have

$$\|g_+\|_{\Phi_\gamma}^2 + \|g_-\|_{\Phi_\gamma}^2 \leq \|g\|_{\Phi_\gamma}^2,$$

because

$$\begin{aligned} \|g\|_{\Phi_\gamma}^2 &= \iiint B\mu_* ((g'_+ + g'_-) - (g_+ + g_-))^2 \\ &\quad + \iiint B(g_{*,+} + g_{*,-})^2 (\sqrt{\mu'} - \sqrt{\mu})^2 \\ &= \|g_+\|_{\Phi_\gamma}^2 + \|g_-\|_{\Phi_\gamma}^2 - 2 \iiint b(\cos \theta) \Phi(|v - v_*|) \mu_* (g'_+ g_- + g_+ g'_-) \end{aligned}$$

and the third term is non-negative. Therefore $g_-^n \in L_{t,x}^\infty(H_\ell^s(\mathbb{R}^3))$. Take the convex function $\beta(s) = \frac{1}{2}(s^-)^2 = \frac{1}{2}s(s^-)$ with $s^- = \min\{s, 0\}$. Let $\varphi(v, x) = (1 + |v|^2 + |x|^2)^{\alpha/2}$ with $\alpha > 3/2$, and notice that

$$\beta_s(g^{n+1})\varphi(v, x)^{-2} = \left(\frac{d}{ds}\beta\right)(g^{n+1})\varphi(v, x)^{-2} = g_-^{n+1}\varphi(v, x)^{-2} \in L^\infty([0, T]; L^1(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))),$$

because $\tilde{g}^{n+1} \in L_t^\infty(H^N(\mathbb{R}^6))$ with $N \geq 4$ implies $g^{n+1} \in L_{t,x}^\infty(L_v^2)$. Multiplying (5.4) by $\beta_s(g^{n+1})\varphi(v, x)^{-2} = g_-^{n+1}\varphi(v, x)^{-2}$ we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^6} \beta(g^{n+1})\varphi(v, x)^{-2} dx dv + \kappa \int_{\mathbb{R}^6} \langle v \rangle^2 \beta(g^{n+1})\varphi(v, x)^{-2} dx dv \\ &= \int_{\mathbb{R}^6} \Gamma^t(g^n, g^{n+1}) \beta_s(g^{n+1})\varphi(v, x)^{-2} dx dv \\ &\quad - \int_{\mathbb{R}^6} v \cdot \nabla_x (\beta(g^{n+1})\varphi(v, x)^{-2}) dx dv - \int_{\mathbb{R}^6} (\varphi(v, x)^2 v \cdot \nabla_x \varphi(v, x)^{-2}) \beta(g^{n+1})\varphi(v, x)^{-2} dx dv, \end{aligned}$$

where the first term on the right hand side is well defined because $g^{n+1}, g_-^{n+1} \in L_{t,x}^\infty(H_\ell^s)$. Since the second term vanishes and $|v \cdot \nabla_x \varphi(v, x)^{-2}| \leq C\varphi(v, x)^{-2}$, we obtain

$$(5.5) \quad \begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^6} \beta(g^{n+1})\varphi(v, x)^{-2} dx dv + \kappa \int_{\mathbb{R}^6} \langle v \rangle^2 \beta(g^{n+1})\varphi(v, x)^{-2} dx dv \\ &\leq \int_{\mathbb{R}^6} \Gamma^t(g^n, g^{n+1}) \beta_s(g^{n+1})\varphi(v, x)^{-2} dx dv + C \int_{\mathbb{R}^6} \beta(g^{n+1})\varphi(v, x)^{-2} dx dv. \end{aligned}$$

The first term on the right hand side is equal to

$$\begin{aligned} &\int_{\mathbb{R}^6} \Gamma^t(g^n, g_-^{n+1}) g_-^{n+1} \varphi(v, x)^{-2} dx dv + \iiint B\mu_{\kappa,*}(g_*^n)' (g_+^{n+1})' g_-^{n+1} \varphi(v, x)^{-2} dv dv_* d\sigma dx \\ &= A_1 + A_2. \end{aligned}$$

From the induction hypothesis, the second term A_2 is non-positive.

On the other hand, we have

$$\begin{aligned} A_1 &= \int (\Gamma^t(g^n, g_-^{n+1}), \varphi(v, x)^{-2} g_-^{n+1})_{L^2(\mathbb{R}_v^3)} dx \\ &= \int (\Gamma^t(g^n, \varphi(v, x)^{-1} g_-^{n+1}), \varphi(v, x)^{-1} g_-^{n+1})_{L^2(\mathbb{R}_v^3)} dx \\ &\quad + \int (\varphi(v, x)^{-1} \Gamma^t(g^n, g_-^{n+1}) - \Gamma^t(g^n, \varphi(v, x)^{-1} g_-^{n+1}), \varphi(v, x)^{-1} g_-^{n+1})_{L^2(\mathbb{R}_v^3)} dx. \\ &= A_{1,1} + A_{1,2}. \end{aligned}$$

It follows from Lemma 4.6 that

$$A_{1,1} \leq -\frac{1}{4} \int \mathcal{D}(\mu_\kappa g^n, \varphi(v, x)^{-1} g_-^{n+1}) dx + C \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; H^{2s})} \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L^2_{s+\gamma/2}}^2 dx.$$

By means of Lemma 4.9 we have

$$\begin{aligned} |A_{1,2}| &\leq \delta \int \left(\mathcal{D}(\mu_\kappa g^n, \varphi(v, x)^{-1} g_-^{n+1}) dx + \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; H^s)} \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L_{s+\gamma/2}^2}^2 dx \right) \\ &\quad + C_\delta \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; H^s)} \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L^2}^2 dx. \end{aligned}$$

Therefore

$$A_1 \lesssim \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; H^{2s})} \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L_{s+\gamma/2}^2}^2 dx,$$

where we have used the fact that $\mathcal{D}(\mu_\kappa g^n, \varphi(v, x)^{-1} g_-^{n+1}) \geq 0$ because of the induction hypothesis $\mu_\kappa g^n = f^n \geq 0$. If $\gamma + 2s < 2$ then we have

$$\int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L_{s+\gamma/2}^2}^2 dx \leq \delta \int_{\mathbb{R}^6} \langle v \rangle^2 \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv + C_\delta \int_{\mathbb{R}^6} \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv.$$

Therefore, from (5.5) we have

$$(5.6) \quad \frac{d}{dt} \int_{\mathbb{R}^6} \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv \lesssim \left(1 + \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; H^{2s})} \right) \int_{\mathbb{R}^6} \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv.$$

Since $\beta(g^{n+1})|_{t=0} = 0$ we obtain $\int_{\mathbb{R}^6} \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv = 0$ for all $t \in [0, T]$, which implies that $g^{n+1}(t, x, v) \geq 0$ for $(t, x, v) \in [0, T] \times \mathbb{R}^6$. This implies that $f^{n+1} \geq 0$ and then it completes the proof of the proposition. \square

Proposition 5.2. *Assume that $\gamma \geq 2 - 2s$. Let $\{f^n\}$ with $f^n = \mu + \mu^{\frac{1}{2}} \tilde{g}^n$ be sequence of solutions of Cauchy problem (5.2) with $\sup_{[0, T] \times \mathbb{R}_x^3} \|\tilde{g}^n\|_{\Phi_\gamma}$ being sufficiently small uniform in n . If*

$$e^{\frac{1}{2}\langle v \rangle^2} f^n(t, x, v) \in L^\infty([0, T] \times \mathbb{R}_x^3; H^N(\mathbb{R}_v^3)) \text{ for } \forall n = 1, 2, 3, \dots,$$

for some $N \geq 4$, then for any $n \in \mathbb{N}$, $f^n \geq 0$ on $[0, T]$ implies $f^{n+1} \geq 0$ on the same interval.

Proof. This case can be treated by the same way as the proof of Theorem 4.3. In fact, if we put $g^n = \mu^{-1/2} f^n$, then we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^6} \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv \\ &\leq \int_{\mathbb{R}^6} \Gamma(g^n, g^{n+1}) \beta_s(g^{n+1}) \varphi(v, x)^{-2} dx dv + C \int_{\mathbb{R}^6} \beta(g^{n+1}) \varphi(v, x)^{-2} dx dv \end{aligned}$$

instead of (5.5). We need to estimate $\tilde{A}_{1,1}$ and $\tilde{A}_{1,2}$ defined by replacing Γ' by Γ in above $A_{1,1}$ and $A_{1,2}$. By the same way as in (4.21) we have for suitable $C_1, C_2 > 0$

$$\begin{aligned} \tilde{A}_{1,1} &\leq -C_1 \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{\Phi_\gamma}^2 dx \\ &\quad + C_2 \left(\sup_{[0, T] \times \mathbb{R}_x^3} \|\tilde{g}^n\|_{\Phi_\gamma} \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{\Phi_\gamma}^2 dx + \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L^2}^2 dx \right). \end{aligned}$$

It follows from Lemma 4.9 and Lemma 2.12 that

$$\begin{aligned} |\tilde{A}_{1,2}| &\leq \delta \left(1 + \sup_{[0, T] \times \mathbb{R}_x^3} \|\tilde{g}^n\|_{L^2} \right) \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{\Phi_\gamma}^2 dx \\ &\quad + C_\delta \|g^n\|_{L^\infty([0, T] \times \mathbb{R}_x^3; H^s)} \int \|\varphi(v, x)^{-1} g_-^{n+1}\|_{L^2}^2 dx. \end{aligned}$$

If $\|\tilde{g}^n\|_{\Phi_\gamma}$ is sufficiently small, then both estimates lead us to (5.6). Hence we have $f^{n+1} \geq 0$ and then it completes the proof of the proposition. \square

Completion of the proof of Theorem 1.3.

We recall now the existence and convergence of the sequence $\{\tilde{g}^n\}$ constructed in [9, 10] for different cases of index :

The hard potential case $\gamma + 2s > 0$. (Theorem 1.1 of [10]) Let $g_0 \in H_{\ell_0}^k(\mathbb{R}^6)$ for some $k \geq 6$, $\ell_0 > 3/2 + 2s + \gamma$. There exists $\varepsilon_0 > 0$, such that if $\|g_0\|_{H_{\ell_0}^k(\mathbb{R}^6)} \leq \varepsilon_0$, then the sequence $\{\tilde{g}^n\}$ converges in $L^\infty([0, +\infty[; H_{\ell_0}^k(\mathbb{R}^6))$ to a global solution \tilde{g} with $\|\tilde{g}\|_{L^\infty([0, +\infty[; H_{\ell_0}^k(\mathbb{R}^6))} \leq C\varepsilon_0$.

The soft potential case $\gamma + 2s \leq 0$. (Theorem 1.5 of [9]) Assume $\gamma > \max\{-3, -\frac{3}{2} - 2s\}$. Let $g_0 \in \tilde{\mathcal{H}}_k^k(\mathbb{R}^6)$ for some $k \geq 6$. There exists $\varepsilon_0 > 0$ such that if $\|g_0\|_{\tilde{\mathcal{H}}_k^k(\mathbb{R}^6)} \leq \varepsilon_0$, then the sequence $\{\tilde{g}^n\}$ converges in $L^\infty([0, +\infty[; \tilde{\mathcal{H}}_k^k(\mathbb{R}^6))$ to a global solution \tilde{g} . Remark that the approximate sequence $\{\tilde{g}^n\}$ is convergent in $L^\infty([0, T]; H^k(\mathbb{R}^6))$.

So in both cases, the sequence $f^n = \mu + \mu^{1/2}\tilde{g}^n$ satisfies the conditions of Propositions 5.1 and 5.2 with $N = k - 2$, which implies that the limit $f = \mu + \mu^{1/2}\tilde{g} \geq 0$. We have proved Theorem 1.3.

6. CONVERGENCE TO THE EQUILIBRIUM STATE

In this section, the convergence rates of the solutions to the equilibrium will be discussed for both the soft and hard potentials. Precisely, for the hard potential, the optimal convergence rates in the Sobolev space can be obtained by combining the energy estimates proven previously and the $L^p - L^q$ estimate on the solution operator of the linearized equation. Such $L^p - L^q$ estimate can be obtained either by spectrum analysis [35] or by using the compensating functions introduced by Kawashima [27]. On the other hand, for soft potential, the convergence rate presented here is solely based on the energy estimate and is not optimal.

6.1. Hard potential. In this subsection, we will combine the compensating function and the energy estimate to obtain the optimal convergence rate for the hard potential case $\gamma + 2s > 0$, that is, the first part of Theorem 1.4. Note that the decay estimate in the theorem can be generalized to the case when the initial lies in $Z_q(\mathbb{R}^6)$ with $1 < q < 2$, where $Z_q(\mathbb{R}^6) = L^q(\mathbb{R}_x^3; L_v^2(\mathbb{R}^3))$.

The compensating function is useful in deriving $L^p - L^q$ estimates for linear dissipative kinetic equations in the form of

$$(6.1) \quad g_t + v \cdot \nabla_x g + \mathcal{L}g = h,$$

where h is a given function and \mathcal{L} is the linearized Boltzmann collision operator.

Let us now recall the definition of compensating function introduced by Kawashima [27].

Definition 6.1. $S(\omega)$ is called a compensating function if it has the following properties:

(i) $S(\cdot)$ is C^∞ on SS^2 (the unit sphere in \mathbb{R}^3) with values in the space of bounded linear operators on $L^2(\mathbb{R}^3)$, and $S(-\omega) = -S(\omega)$ for all $\omega \in SS^2$.

(ii) $iS(\omega)$ is self-adjoint on $L^2(\mathbb{R}^3)$ for all $\omega \in SS^2$.

(iii) There exist constants $\lambda > 0$ and $c_0 > 0$ such that for all $g \in L^2(\mathbb{R}^3)$ and $\omega \in SS^2$,

$$(6.2) \quad \operatorname{Re}(S(\omega)(v \cdot \omega)g, g)_{L^2(\mathbb{R}^3)} + (\mathcal{L}g, g)_{L^2(\mathbb{R}^3)} \geq c_0(\|\mathbf{P}g\|_{L^2(\mathbb{R}^3)}^2 + \|(\mathbf{I} - \mathbf{P})g\|^2).$$

The construction of $S(\omega)$ was given in [27], but for completeness and the convenience of the readers, we recall some basic derivation and estimates.

Let \mathcal{W} be the subspace spanned by the thirteen moments containing the null space \mathcal{N} of \mathcal{L} and the images of \mathcal{N} under the mappings $g(v) \mapsto v_j g(v)$ ($j = 1, 2, 3$) denoted by:

$$\mathcal{W} = \operatorname{span}\{e_j | j = 1, 2, \dots, 13\}.$$

Here, the orthonormal set of functions e_j is given by

$$e_1 = \mu^{\frac{1}{2}}, \quad e_{i+1} = v_i \mu^{\frac{1}{2}}, \quad i = 1, 2, 3, \quad e_5 = \frac{1}{\sqrt{6}}(|v|^2 - 3)\mu^{\frac{1}{2}},$$

and

$$\begin{aligned} e_{j+4} &= \sum_{i=1}^3 \frac{c_{ji}}{\sqrt{2}} (v_i^2 - 1)\mu^{\frac{1}{2}}, \quad j = 2, 3 \\ e_8 &= v_1 v_3 \mu^{\frac{1}{2}}, \quad e_9 = v_2 v_3 \mu^{\frac{1}{2}}, \quad e_{10} = v_3 v_1 \mu^{\frac{1}{2}}, \\ e_{i+10} &= \frac{1}{\sqrt{10}}(|v|^2 - 5)v_i \mu^{\frac{1}{2}}, \quad i = 1, 2, 3, \end{aligned}$$

where the constant vectors $c_i = (c_{i1}, c_{i2}, c_{i3})$, $i = 2, 3$ together with $c_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ form an orthonormal basis of \mathbb{R}^3 .

Let \mathbf{P}_0 be the orthogonal projection from $L^2(\mathbb{R}_v^3)$ onto \mathcal{W} , that is,

$$\mathbf{P}_0 g = \sum_{k=1}^{13} (g, e_k)_{L^2(\mathbb{R}_v^3)} e_k.$$

Set $W_k = \langle f, e_k \rangle$, $k = 1, 2, \dots, 13$, and $W = [W_1, \dots, W_{13}]^T$. For later use, set $W_l = [W_1, \dots, W_5]^T$, and $W_{II} = [W_6, \dots, W_{13}]^T$. Then we have

$$\partial_t W + \sum_j V^j \partial_{x_j} W + \bar{L} W = \bar{h} + R,$$

where V^j ($j = 1, 2, 3$) and \bar{L} are the symmetric matrices defined by

$$\bar{L} = \{(\mathcal{L}e_l, e_k)_{L^2(\mathbb{R}_v^3)}\}_{k,l=1}^{13}, \quad V^j = \{(v_j e_k, e_l)_{L^2(\mathbb{R}_v^3)}\}_{k,l=1}^{13},$$

and $\bar{h} = [(h, e_1)_{L^2(\mathbb{R}_v^3)}, \dots, (h, e_{13})_{L^2(\mathbb{R}_v^3)}]^T$. Here R denotes the remaining term which contains the factor $(\mathbf{I} - \mathbf{P}_0)g$. Straightforward calculation gives

$$V(\xi) = \sum_{j=1}^3 V^j \xi_j = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

with

$$V_{11}(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ \xi_1 & 0 & 0 & 0 & a_1 \xi_1 \\ \xi_2 & 0 & 0 & 0 & a_1 \xi_2 \\ \xi_3 & 0 & 0 & 0 & a_1 \xi_3 \\ 0 & a_1 \xi_1 & a_1 \xi_2 & a_1 \xi_3 & 0 \end{pmatrix},$$

and

$$V_{21}(\xi) = V_{12}(\xi)^T = \begin{pmatrix} 0 & a_{21} \xi_1 & a_{22} \xi_2 & a_{23} \xi_3 & 0 \\ 0 & a_{31} \xi_1 & a_{32} \xi_2 & a_{33} \xi_3 & 0 \\ 0 & \xi_2 & \xi_1 & 0 & 0 \\ 0 & 0 & \xi_3 & \xi_2 & 0 \\ 0 & \xi_3 & 0 & \xi_1 & 0 \\ 0 & 0 & 0 & 0 & a_4 \xi_1 \\ 0 & 0 & 0 & 0 & a_4 \xi_2 \\ 0 & 0 & 0 & 0 & a_4 \xi_3 \end{pmatrix},$$

where $a_1 = \sqrt{\frac{2}{3}}$, $a_{kj} = \sqrt{2}c_{kj}$, $k = 2, 3$, $j = 1, 2, 3$, and $a_4 = \sqrt{\frac{3}{5}}$. By setting

$$R(\xi) = \sum_{j=1}^3 R^j \xi_j = \begin{pmatrix} \alpha \tilde{R}_{11} & V_{12} \\ -V_{21} & 0 \end{pmatrix},$$

with

$$\tilde{R}_{11} = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ -\xi_1 & 0 & 0 & 0 & 0 \\ -\xi_2 & 0 & 0 & 0 & 0 \\ -\xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It was shown in [27] that there exist positive constants c_1 and c_2 such that

$$\operatorname{Re} \langle R(\omega) V(\omega) W, W \rangle \geq c_1 |W_I|^2 - c_2 \sum_{k=2}^4 |W_{II}|^2,$$

for any $\omega \in SS^2$ with the constant α suitably chosen. Here $\langle \cdot, \cdot \rangle$ represents the standard inner product in \mathbb{C}^{13} .

Hence, a compensating function $S(\omega)$ can be defined as follows. For any given $\omega \in SS^2$, set $R(\omega) \equiv \{r_{ij}(\omega)\}_{i,j=1}^4$ and let

$$S(\omega)g \equiv \sum_{k,\ell=1}^4 \lambda r_{k\ell}(\omega)(g, e_\ell)_{L^2(\mathbb{R}_+^3)} e_k, \quad f \in L^2(\mathbb{R}^3).$$

When the parameter $\lambda > 0$ is chosen small enough, it was shown in [27] that the estimate (6.2) holds because of the dissipation of \mathcal{L} on the space \mathcal{N}^\perp .

To obtain the $L^p - L^q$ estimate, by taking the Fourier transform in the variable x , the equation (6.1) yields

$$(6.3) \quad \hat{g}_t + i|\xi|(\nu \cdot \omega)\hat{g} + \mathcal{L}\hat{g} = \hat{h},$$

where $\omega = \frac{\xi}{|\xi|}$. Take the inner product of (6.3) with $((1 + |\xi|^2) - i\kappa S(\omega))\hat{g}$ and use the properties of the compensating function, to get

$$\begin{aligned} ((1 + |\xi|^2)\|\hat{g}\|_{L^2(\mathbb{R}_+^3)}^2) &- \kappa|\xi|(iS(\omega)\hat{g}, \hat{g})_{L^2(\mathbb{R}_+^3)} + \delta_0((1 + |\xi|^2)\|(\mathbf{I} - \mathbf{P})\hat{g}\|^2 + |\xi|^2\|\mathbf{P}\hat{g}\|_{L^2(\mathbb{R}_+^3)}^2) \\ &\leq C(1 + |\xi|^2)Re(\hat{g}, \hat{h})_{L^2(\mathbb{R}_+^3)}, \end{aligned}$$

which implies that

$$E(\hat{g})_t + \delta_0 \frac{|\xi|^2}{1 + |\xi|^2} E(\hat{g}) \leq C\|\hat{h}\|_{L^2(\mathbb{R}_+^3)}^2,$$

where

$$E(\hat{g}) = \|\hat{g}\|_{L^2(\mathbb{R}_+^3)}^2 - \kappa \frac{|\xi|}{1 + |\xi|^2} (iS(\omega)\hat{g}, \hat{g})_{L^2(\mathbb{R}_+^3)} \sim \|\hat{g}\|_{L^2(\mathbb{R}_+^3)}^2,$$

when κ is chosen to be small. And this estimate yields

$$(6.4) \quad \|\hat{g}\|_{L^2(\mathbb{R}_+^3)}^2 \leq C \exp\left\{-\frac{\delta_0|\xi|^2 t}{1 + |\xi|^2}\right\} \|\hat{g}_0\|_{L^2(\mathbb{R}_+^3)}^2 + C \int_0^t \exp\left\{-\frac{\delta_0|\xi|^2(t-s)}{1 + |\xi|^2}\right\} \|\hat{h}\|_{L^2(\mathbb{R}_+^3)}^2(s) ds.$$

Based on (6.4), we have the following $L^p - L^q$ estimate on the solution operator of (6.1) obtained in [27].

Lemma 6.2. *Let $k \geq k_1 \geq 0$ and $N \geq 4$. Assume that*

- (i) $g_0 \in H^N(\mathbb{R}^6) \cap Z_q$,
 - (ii) $h \in C^0([0, \infty[; H^N \cap Z_q)$,
 - (iii) $\mathbf{P}h(t, x, \nu) = 0$ for all $(t, x, \nu) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$.
 - (iv) $g(t, x, \nu) \in C^0([0, \infty[; H^N(\mathbb{R}^6)) \cap C^1([0, \infty[; H^{N-1}(\mathbb{R}^6))$ is a solution of (6.1).
- Then we have

$$\begin{aligned} \|\nabla_x^k g\|_{L^2(\mathbb{R}^6)}^2 &\leq C(1+t)^{-2\sigma_{q,m}} (\|\nabla_x^{k_1} g_0\|_{Z_q(\mathbb{R}^6)} + \|\nabla_x^k g_0\|_{L^2(\mathbb{R}^6)})^2 \\ &+ \int_0^t (1+t-s)^{-2\sigma_{q,m}} (\|\nabla_x^{k_1} h\|_{Z_q(\mathbb{R}^6)} + \|\nabla_x^k h\|_{L^2(\mathbb{R}^6)})^2 ds, \end{aligned} \quad (2.11)$$

for any integer $m = k - k_1 \geq 0$, where $q \in [1, 2]$ and

$$\sigma_{q,m} = \frac{3}{2} \left(\frac{1}{q} - \frac{1}{2} \right) + \frac{m}{2}.$$

We now recall the energy estimates obtained for the global existence of solutions for the hard potential in [10]. Firstly, we have when $N \geq 6$ and $l > 3/2 + 2s + \gamma$,

$$\frac{d}{dt} \mathcal{E} + D \leq 0,$$

where $\mathcal{E} = \|g\|_{H_l^N(\mathbb{R}^6)}^2$ and $D = \|\nabla_x \mathbf{P}g\|_{H^{N-1}(\mathbb{R}^6)}^2 + \|(\mathbf{I} - \mathbf{P})g\|_{\mathcal{B}_l^N(\mathbb{R}^6)}^2$. We claim that the following energy estimate also holds

$$(6.5) \quad \frac{d}{dt} \mathcal{E}_1 + D \leq C \|\nabla_x \mathbf{P}g\|_{L_{x,\nu}^2(\mathbb{R}^6)}^2,$$

where $\mathcal{E}_1 = \|\nabla_x \mathbf{P}g\|_{H^{N-1}(\mathbb{R}^6)}^2 + \|(\mathbf{I} - \mathbf{P})g\|_{H^1(\mathbb{R}^6)}^2$. In fact, for the energy estimate on the macroscopic component of the solution, by Lemma 4.4 of [10] and by taking the sum over $1 \leq |\alpha| \leq N-1$, we have

$$(6.6) \quad \begin{aligned} \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)}^2 &\lesssim -\frac{d}{dt} \sum_{1 \leq |\alpha| \leq N-1} \left\{ (\partial^\alpha r, \nabla_x \partial^\alpha (a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right\} \\ &\quad + \|g_2\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}^2 + \mathcal{E}^{\frac{1}{2}} D, \end{aligned}$$

where we use the same notations used in [10] except that we replace $E_{N,1} D_{N,0}$ by $\mathcal{E}^{\frac{1}{2}} D$ in an obvious way.

On the other hand, for the energy estimate on the microscopic component without weight, one can follow the proof of Lemma 4.5 in [10] except for $\alpha = 0$, we take the $L^2(\mathbb{R}_{x,v}^6)$ inner product of (6.1) with $(\mathbf{I} - \mathbf{P})g = g_2$ to have

$$\frac{d}{dt} \|g_2\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 \lesssim \|\nabla_x \mathbf{P}g\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \mathcal{E}^{\frac{1}{2}} D.$$

This together with the estimate given in Lemma 4.5 of [10] for $1 \leq |\alpha| \leq N$ gives

$$(6.7) \quad \frac{d}{dt} (\|g_2\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|\nabla_x g\|_{H^{N-1}(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}^2) + \|g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 + \sum_{1 \leq |\alpha| \leq N} \|\partial^\alpha g\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 \lesssim \|\nabla_x \mathbf{P}g\|^2 + \mathcal{E}^{\frac{1}{2}} D.$$

Moreover, for $\ell \geq 0$ and $\beta \neq 0$, (4.10) in [10] gives

$$(6.8) \quad \begin{aligned} &\frac{d}{dt} \|W_\ell \partial_\beta^\alpha g_2\|_{L^2(\mathbb{R}^6)}^2 + \|W_\ell \partial_\beta^\alpha g_2\|_{\mathcal{B}_0^0} \\ &\lesssim \|g_2\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}^2 + \mathcal{E}_{N,\ell}^{1/2} \mathcal{D}_{N,\ell} \\ &\quad + \delta_0 \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}^3)}^2 + \sum_{|\alpha'|=|\alpha|+1, |\beta'|=|\beta|-1} \|\partial_{\beta'}^{\alpha'} g_2\|_{\mathcal{B}_0^0}^2. \end{aligned}$$

Here $\delta_0 > 0$ is a small constant. By using induction on $|\beta|$ and $|\alpha| + |\beta|$, a suitable linear combination of (6.6), (6.7) and (6.8) gives (6.5) for sufficiently small \mathcal{E} .

In the following, we also need some L^p estimate on the nonlinear collision operator. Recall that Lemma 4.7 implies that

$$\|\Gamma(f, g)\|_{L^2(\mathbb{R}_x^3)} \lesssim \|f\|_{L^2(\mathbb{R}_x^3)} \|g\|_{H_{(\gamma+2s)^+}^{2s}(\mathbb{R}_v^3)}.$$

Hence, by using the fact that $N \geq 6$ and $\ell > 3/2 + 2s + \gamma$, Sobolev imbedding implies

$$\begin{aligned} \|\Gamma(g, g)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 + \|\nabla_x \Gamma(g, g)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 &\lesssim \mathcal{E}^2 \lesssim \mathcal{E}_1 \mathcal{E} + \|\mathbf{P}g\|_{L^2(\mathbb{R}_{x,v}^6)}^4, \\ \|\Gamma(g, g)\|_{Z_1}^2 &\lesssim \mathcal{E}_1 \mathcal{E} + \|\mathbf{P}g\|_{L^2(\mathbb{R}_{x,v}^6)}^4. \end{aligned}$$

Define

$$M(t) = \sup_{0 \leq s \leq t} \{(1+s)^{\frac{5}{2}} \mathcal{E}_1(s)\}, \quad M_0(t) = \sup_{0 \leq s \leq t} \{(1+s)^{\frac{3}{2}} \|g(s)\|_{L^2(\mathbb{R}_{x,v}^6)}^2\}.$$

Then by the $L^p - L^q$ estimate, we have

$$\begin{aligned} \|\nabla_x g(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 &\lesssim (1+t)^{-\frac{5}{2}} (\|g_0\|_{Z_1(\mathbb{R}^6)}^2 + \|\nabla_x g_0\|_{L^2(\mathbb{R}_{x,v}^6)}^2) \\ &\quad + \int_0^t (1+t-s)^{-\frac{5}{2}} (\|\Gamma(g, g)\|_{Z_1(\mathbb{R}^6)} + \|\nabla_x \Gamma(g, g)\|_{L^2(\mathbb{R}_{x,v}^6)})^2 ds \\ &\lesssim \eta (1+t)^{-\frac{5}{2}} + \int_0^t (1+t-s)^{-\frac{5}{2}} (\mathcal{E} \mathcal{E}_1 + \|\mathbf{P}g\|_{L^2(\mathbb{R}_{x,v}^6)}^4)(s) ds \\ &\lesssim \eta (1+t)^{-\frac{5}{2}} + \delta M(t) \int_0^t (1+t-s)^{-\frac{5}{2}} (1+s)^{-\frac{5}{2}} ds \\ &\quad + M_0^2(t) \int_0^t (1+t-s)^{-\frac{5}{2}} (1+s)^{-3} ds \\ &\lesssim \eta (1+t)^{-\frac{5}{2}} + \delta (1+t)^{-\frac{5}{2}} M(t) + (1+t)^{-\frac{5}{2}} M_0^2(t), \end{aligned}$$

where $\eta = \|g_0\|_{Z_1(\mathbb{R}^6)}^2 + \|g_0\|_{H^N(\mathbb{R}^6)}^2$. Here, we use $\delta > 0$ to denote the upper bound of \mathcal{E} for all time.

Thus, we have

$$\begin{aligned} \mathcal{E}_1(t) &\leq \mathcal{E}_1(0)e^{-t} + \int_0^t e^{-(t-s)} \|\nabla_x g\|_{L^2(\mathbb{R}_{x,v}^6)}^2(s) ds \\ &\lesssim \delta e^{-t} + \eta(1+t)^{-\frac{5}{2}} + \delta(1+t)^{-\frac{5}{2}} M(t) + (1+t)^{-\frac{5}{2}} M_0^2(t), \end{aligned}$$

that is,

$$M(t) \lesssim (\delta + \eta) + \delta M(t) + M_0^2(t).$$

By applying the $L^p - L^q$ estimate again, we have

$$\begin{aligned} \|g(t)\|_{L^2(\mathbb{R}_{x,v}^6)}^2 &\lesssim (1+t)^{-\frac{3}{2}} (\|g_0\|_{Z_1(\mathbb{R}^6)}^2 + \|g_0\|_{L^2(\mathbb{R}_{x,v}^6)}^2) \\ &\quad + \int_0^t (1+t-s)^{-\frac{3}{2}} (\|\Gamma(g, g)\|_{Z_1(\mathbb{R}^6)} + \|\Gamma(g, g)\|_{L^2(\mathbb{R}_{x,v}^6)})^2(s) ds \\ &\lesssim \eta(1+t)^{-\frac{3}{2}} + \int_0^t (1+t-s)^{-\frac{3}{2}} (\mathcal{E}\mathcal{E}_1 + \|\mathbf{P}g\|_{L^2(\mathbb{R}_{x,v}^6)}^4)(s) ds \\ &\lesssim \eta(1+t)^{-\frac{3}{2}} + \delta(1+t)^{-\frac{3}{2}} M(t) + (1+t)^{-\frac{3}{2}} M_0^2(t). \end{aligned}$$

Hence,

$$\begin{aligned} M_0(t) &\lesssim \eta + \delta M(t) + M_0^2(t) \\ &\lesssim (\eta + \delta) + M_0^2(t). \end{aligned}$$

By assumption, $\eta + \delta$ is small. The above estimate and the continuity argument give $M_0(t) \leq C_{\eta, \delta}$, and then $M(t) \leq \bar{C}_{\eta, \delta}$, where $C_{\eta, \delta}$ and $\bar{C}_{\eta, \delta}$ are two constants depending on η and δ only. This completes the proof of the first part in Theorem 1.4.

6.2. Soft Potential. Finally, in this subsection, we will prove the second part of Theorem 1.4 for the soft potential case, that is, when $2s + \gamma \leq 0$.

As for the case with angular cutoff, here we need to apply the following basic inequality from [15].

Lemma 6.3. *Let $f(t) \in C^1([t_0, \infty))$ such that $f(t) \geq 0$, $A = \int_{t_0}^{\infty} f(t) dt < +\infty$ and $f'(t) \leq a(t)f(t)$ for all $t \geq t_0$.*

Here $a(t) \geq 0$, $B = \int_{t_0}^{\infty} a(t) dt < +\infty$. Then

$$f(t) \leq \frac{(t_0 f(t_0) + 1) \exp(A + B) - 1}{t}, \quad \text{for all } t \geq t_0.$$

Now it remains to find the appropriate functionals $f(t)$ and $a(t)$ that satisfy the above differential inequality. First of all, the basic energy estimate derived in [9] for the global existence is

$$\frac{d}{dt} \mathcal{E}_{N, \ell} + c_0 \mathcal{D}_{N, \ell} \leq 0,$$

where $c_0 > 0$ is a constant. Here,

$$\begin{aligned} \mathcal{E}_{N, \ell} &\sim \|\mathcal{A}\|_{H^N(\mathbb{R}^3)}^2 + \|g_2\|_{\tilde{\mathcal{H}}_\ell^N(\mathbb{R}^6)}^2, \\ \mathcal{D}_{N, \ell} &= \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}^3)}^2 + \|g_2\|_{\tilde{\mathcal{B}}_\ell^N(\mathbb{R}^6)}^2, \end{aligned}$$

and

$$\tilde{\mathcal{B}}_\ell^N(\mathbb{R}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6); \|g\|_{\tilde{\mathcal{B}}_\ell^N(\mathbb{R}^6)}^2 = \sum_{|\alpha|+|\beta| \leq N} \int_{\mathbb{R}_x^3} \|\tilde{W}_{\ell-|\beta|} \partial_\beta^\alpha g(x, \cdot)\|_{\Phi_\gamma}^2 dx < +\infty \right\}.$$

Later we introduce another functional $\bar{\mathcal{E}}_{N-1, \ell-1}$ that has the following property

$$(6.9) \quad \bar{\mathcal{E}}_{N-1, \ell-1} \sim \|\nabla_x \mathcal{A}\|_{H^{N-2}(\mathbb{R}^3)}^2 + \|\nabla_x g_2\|_{\tilde{\mathcal{H}}_{\ell-1}^{N-2}(\mathbb{R}^6)}^2.$$

Clearly, by the property of the $\|\cdot\|$, $\bar{\mathcal{E}}_{N-1, \ell-1} \lesssim \mathcal{D}_{N, \ell}$ so that $\int_0^\infty \bar{\mathcal{E}}_{N-1, \ell-1}(t) dt < \infty$. Note that this functional contains spatial differentiation of at least one order, and the maximum order of differentiation is $N - 1$. The

reason for this is to have the time integrability of the functional coming from the dissipation effect. And the functional excludes the N -th order differentiation because we need to estimate the term like

$$\begin{aligned} \int_{\mathbb{R}^3} \|\mathcal{A}\|_{L^2(\mathbb{R}_x^3)} \|\partial_x^\alpha \mathcal{A}\|_{L^2(\mathbb{R}_x^3)} \|\partial_x^\alpha g_2\| dx &\lesssim \|\mathcal{A}\|_{L^6(\mathbb{R}_x^3)} \|\partial_x^\alpha \mathcal{A}\|_{L^3(\mathbb{R}_x^3)} \|\partial_x^\alpha g_2\|_{\mathcal{X}^0(\mathbb{R}^6)} \\ &\lesssim \|\nabla_x \mathcal{A}\|_{L^2(\mathbb{R}_x^3)} \|\partial_x^\alpha \mathcal{A}\|_{H^1(\mathbb{R}_x^3)} \|\partial_x^\alpha g_2\|_{\mathcal{X}^0(\mathbb{R}^6)}, \end{aligned}$$

where

$$\mathcal{X}^N(\mathbb{R}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6); \quad \|g\|_{\mathcal{X}^N(\mathbb{R}^6)}^2 = \sum_{|\alpha| \leq N} \int_{\mathbb{R}_x^3} \|\partial_x^\alpha g\|_{\Phi_\gamma}^2 dx < +\infty \right\}.$$

Hence, since the maximum order of differentiation is N , the above estimate requires that $|\alpha| \leq N - 1$.

We now construct $\bar{\mathcal{E}}_{N-1, \ell-1}$ following the argument used in Lemma 6.3, Lemma 6.4 and (6.15) in [9]. Firstly, by taking $1 \leq |\alpha| \leq N - 2$ in Lemma 6.3 of [9], we have

$$(6.10) \quad \begin{aligned} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}_x^3)}^2 &\leq -\frac{d}{dt} \left\{ (\partial^\alpha r, \nabla_x \partial^\alpha (a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right\} \\ &\quad + \|\nabla_x g_2\|_{\mathcal{X}^{N-2}(\mathbb{R}^6)}^2 + (\|\mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)} + \|g_2\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_x^3))}) \bar{\mathcal{D}}_{N-1} \\ &\quad + \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)}^2 (\|\nabla_x \mathcal{A}\|_{H^{N-2}(\mathbb{R}_x^3)}^2 + \|g_2\|_{L^2(\mathbb{R}^6)}^2), \end{aligned}$$

where

$$\bar{\mathcal{D}}_{N-1} = \sum_{1 \leq \alpha \leq N-2} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}_x^3)}^2 + \|\nabla_x g_2\|_{\mathcal{X}^{N-2}}^2.$$

Note that $\bar{\mathcal{D}}_{N-1}$ is different from \mathcal{D}_{N-1} defined in [9]. In particular, in $\bar{\mathcal{D}}_{N-1}$, the usual dissipation terms $\|\nabla_x \mathcal{A}\|_{L^2(\mathbb{R}_x^3)}^2$ and $\|g_2\|_{L^2(\mathbb{R}^6)}^2$ are not included. And this is also why there is the last term on the right hand side of (6.10).

Next, following the argument used for Lemma 6.4 in [9], we can derive

$$\frac{d}{dt} \bar{\mathcal{E}}_{N-1} + c_0 \|\nabla_x g_2\|_{\mathcal{X}^{N-2}}^2 \lesssim \mathcal{E}_N^{\frac{1}{2}} \bar{\mathcal{D}}_{N-1} + \sum_{0 \leq |\alpha| \leq N-2} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}_x^3)}^4,$$

where $\mathcal{E}_N = \sum_{0 \leq |\alpha| \leq N-1} \|\partial_x^\alpha g\|_{L^2(\mathbb{R}^6)}^2$ and $\bar{\mathcal{E}}_{N-1} = \sum_{1 \leq |\alpha| \leq N-2} \|\partial_x^\alpha g\|_{L^2(\mathbb{R}^6)}^2$.

Finally, corresponding to the weighted estimate (6.15) in [9], one can show that for $|\alpha| \geq 1$ and $|\beta| \geq 1$ with $|\alpha + \beta| \leq N - 1$, it holds

$$\begin{aligned} &\frac{d}{dt} \left(\|\partial_\beta^\alpha g_2\|_{L_{\ell-1-|\beta|}^2(\mathbb{R}^6)} + (\partial^\alpha r, \nabla_x \partial^\alpha (a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right) \\ &\quad + c_0 \|\|\tilde{W}_{\ell-1-|\beta|} \partial_\beta^\alpha g_2\|_{\Phi_\gamma}\|_{L^2(\mathbb{R}_x^3)}^2 \\ &\lesssim \|\partial^\alpha g_2\|_{L_{s+\gamma/2}^2(\mathbb{R}^6)}^2 + \mathcal{E}_{N, \ell-1}^{1/2} \bar{\mathcal{D}}_{N-1, \ell-1} \\ &\quad + \sum_{|\alpha'|=|\alpha|+1, |\beta'|=|\beta|-1} \|\|\tilde{W}^{\ell-1-(|\beta|-1)} \partial_x^{\alpha'} \partial_v^{\beta'} g_2\|_{\Phi_\gamma}\|_{L^2(\mathbb{R}_x^3)}^2 + \delta_0 \|\nabla_x g_2\|_{\mathcal{B}_\ell^{N-1}(\mathbb{R}^6)}^2 \\ &\quad + \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)}^2 (\|\nabla_x \mathcal{A}\|_{H^{N-2}(\mathbb{R}_x^3)}^2 + \|g_2\|_{L^2(\mathbb{R}^6)}^2), \end{aligned}$$

where $\delta_0 > 0$ is a small constant, and

$$\bar{\mathcal{D}}_{N-1, \ell-1} = \sum_{1 \leq |\alpha| \leq N-2} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}_x^3)}^2 + \|\nabla_x g_2\|_{\mathcal{B}_{\ell-1}^{N-2}(\mathbb{R}^6)}^2.$$

Here, we have used the assumption that $\ell - 1 \geq N$.

Now we can define the functional $\bar{\mathcal{E}}_{N-1, \ell-1}$ as follows:

$$\begin{aligned} \bar{\mathcal{E}}_{N-1, \ell-1} &= \bar{\mathcal{E}}_{N-1} + c_1 \sum_{1 \leq |\alpha| \leq N-2} \left\{ (\partial^\alpha r, \nabla_x \partial^\alpha (a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right\} \\ &\quad + \sum_{|\alpha|, |\beta| \geq 1, |\alpha+\beta| \leq N-1} c_{\alpha, \beta} \left(\|\partial_\beta^\alpha g_2\|_{L_{\ell-1-|\beta|}^2(\mathbb{R}^6)}^2 + (\partial^\alpha r, \nabla_x \partial^\alpha (a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)} \right), \end{aligned}$$

where $c_1 > 0$ and $c_{\alpha,\beta} > 0$ are small constants which can be chosen so that $\bar{\mathcal{E}}_{N,\ell-1}$ satisfies (6.9). It is straightforward to check that by induction on $|\beta|$, we have

$$\begin{aligned} \frac{d}{dt} \bar{\mathcal{E}}_{N-1,\ell-1} + \eta_0 \bar{\mathcal{D}}_{N,\ell-1} &\lesssim \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)}^2 (\|\nabla_x \mathcal{A}\|_{H^{N-2}(\mathbb{R}_x^3)}^2 + \|g_2\|_{L^2(\mathbb{R}^6)}^2) \\ &\lesssim \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)}^2 \bar{\mathcal{E}}_{N-1,\ell-1}, \end{aligned}$$

where $\eta_0 > 0$ is a constant. Here, we have used the fact that $\mathcal{E}_{N,\ell}(t)$ is sufficiently small for all time from the global existence. Since

$$\int_0^\infty (\bar{\mathcal{E}}_{N-1,\ell-1} + \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}_x^3)}^2) dt < \infty,$$

Lemma 6.3 implies that

$$\bar{\mathcal{E}}_{N-1,\ell-1} \lesssim (1+t)^{-1}.$$

By using the fact that $\ell-1 \geq N$, the Sobolev imbedding theorem implies the decay estimate given in the second part of Theorem 1.4.

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R. ALEXANDRE, DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY
SHANGHAI, 200240, P. R. CHINA

AND

IRENAV, ARTS ET METIERS PARIS TECH, ECOLE NAVALE,
LANVEOC POULMIC, BREST 29290 FRANCE

E-mail address: radjesvarane.alexandre@ecole-navale.fr

Y. MORIMOTO, GRADUATE SCHOOL OF HUMAN AND ENVIRONMENTAL STUDIES, KYOTO UNIVERSITY
KYOTO, 606-8501, JAPAN

E-mail address: morimoto@math.h.kyoto-u.ac.jp

S. UKAI, 17-26 IWASAKI-CHO, HODOGAYA-KU, YOKOHAMA 240-0015, JAPAN

E-mail address: ukai@kurims.kyoto-u.ac.jp

C.-J. XU, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA

AND

UNIVERSITÉ DE ROUEN, UMR 6085-CNRS, MATHÉMATIQUES
AVENUE DE L'UNIVERSITÉ, BP.12, 76801 SAINT ETIENNE DU ROUVRAY, FRANCE

E-mail address: Chao-Jiang.Xu@univ-rouen.fr

T. YANG, DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, HONG KONG, P. R. CHINA

E-mail address: matyang@cityu.edu.hk