Commun. Math. Phys. 268, 569–605 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-0103-4

Communications in
**Mathematical
Physics**

Global Existence of Classical Solutions to the Vlasov-Poisson-Boltzmann System

Tong Yang[1], **Huijiang Zhao**[2]

[1] Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong. E-mail: matyang@cityu.edu.hk
[2] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

Received: 7 November 2005 / Accepted: 4 April 2006
Published online: 7 September 2006 –

Abstract: The time evolution of the distribution function for the charged particles in a dilute gas is governed by the Vlasov–Poisson–Boltzmann system when the force is self-induced and its potential function satisfies the Poisson equation. In this paper, we give a satisfactory global existence theory of classical solutions to this system when the initial data is a small perturbation of a global Maxwellian. Moreover, the convergence rate in time to the global Maxwellian is also obtained through the energy method. The proof is based on the theory of compressible Navier–Stokes equations with forcing and the decomposition of the solutions to the Boltzmann equation with respect to the local Maxwellian introduced in [23] and elaborated in [31].

Contents

1. Introduction

Consider the Vlasov–Poisson–Boltzmann system:

$$\begin{cases} f_t + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f, f), \\ \Delta_x \Phi = \rho - \bar{\rho} = \int_{\mathbf{R}^3} f d\xi - \bar{\rho}, \quad |\Phi| \to 0, \text{ as } |x| \to +\infty, \end{cases} \tag{1.1}$$

with initial data

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.2}$$

Here $f(t, x, \xi)$ is the distribution function for the particles located at $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ with velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ at time $t \geq 0$. The self-consistent electric potential $\Phi(t, x)$ is coupled with the distribution function $f(t, x, \xi)$ through the Poisson equation in (1.1). The constant background charge density is denoted by $\bar{\rho} > 0$. The short-range binary interaction between particles is given by the standard Boltzmann collision operator $Q(f, g)$ for the hard-sphere model:

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}_+^2} \Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) \\ - f(\xi_*)g(\xi) \Big) \, |(\xi - \xi_*) \cdot \Omega| \; d\xi_* d\Omega.$$

Here $\mathbf{S}_+^2 = \{\Omega \in \mathbf{S}^2 : \ (\xi - \xi_*) \cdot \Omega \geq 0\}$, and

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \, \Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \, \Omega \end{cases}$$

is the relation between velocities ξ', ξ'_* after and the velocities ξ, ξ_* before the collision, which is induced by the conservation of momentum and energy.

In this paper, we will give a satisfactory global existence theory of classical solutions to the Cauchy problem (1.1) and (1.2) near a given global Maxwellian with density $\bar{\rho}$ which itself is a trivial solution to the system (1.1):

$$\bar{\mathbf{M}} = \mathbf{M}_{[\bar{\rho}, 0, \bar{\theta}]} = \frac{\bar{\rho}}{(2\pi R\bar{\theta})^{\frac{2}{3}}} \exp\left(-\frac{|\xi|^2}{2R\bar{\theta}} \right). \tag{1.3}$$

Here $\bar{\theta} > 0$ is a constant.

The Vlasov–Poisson–Boltzmann system is a classical physical model for the time evolution of charged particles, like electrons with the external force generated by the self-induced electronic field. It can also be viewed as a limiting model when the light speed tends to infinity of the Vlasov-Maxwell-Boltzmann system for the time evolution of ions and electrons under the influence of the self-induced electric and magnetic fields. A lot of work has been done on these systems. In what follows, we only mention those related to this paper. The global existence of classical solutions was obtained in [14, 15] when the solution is spatially periodic. In fact, for the spatially periodic solution, the Poincaré inequality is used to control the solution by the celebrated H-theorem about the dissipation of the collision operator on the microscopic components. To be precise, the viscosity and heat conductivity in the compressible Navier–Stokes level coming from the leading order of the microscopic component give the dissipation on the spatial derivatives of the velocity and temperature, but usually not on the perturbation of these two macroscopic components themselves. This dissipation, nevertheless, yields the L_x^2 estimate on the macroscopic components for the spatially periodic solutions because the Poincaré inequality holds.

The global existence of classical solutions to the Cauchy problem (1.1) and (1.2) was first obtained in [30] under the condition that either the mean free path is sufficiently small or the background charge density $\bar{\rho}$ is sufficiently large. Notice that the uniform boundedness on the derivatives of the solution was given only for those with temporal differentiation no more than one in [30]. The main purpose of this paper is to remove these restrictions and then give the global existence of classical solutions in the general setting. One of the main techniques used here is the elaborated energy method for the Boltzmann equation, [23, 25, 31]. Before [31], the a priori energy estimate is closed by deducing energy estimates on the microscopic component $\mathbf{G}$ and the macroscopic component $\mathbf{M}$ respectively. This was done by applying an energy method with respect to two weighted functions, that is, the local Maxwellian and an appropriately chosen global Maxwellian. Based on this method, the nonlinear stability of basic wave patterns for the Boltzmann equation with slab symmetry was obtained in [19, 24, 25, 32]. The observation in [31] is that the microscopic projection of the local Maxwellian with respect to the unperturbed global Maxwellian is of quadratic order of the perturbation so that two sets of energy estimates are not necessary. Hence, it not only simplifies the analysis but also gives better understanding of the solution behavior.

Besides the classical solutions, the global existence of renormalized solutions with large initial data to the Vlasov–Poisson–Boltzmann system was proved in [22] and this result was later generalized to the case with boundary in [27]. The time asymptotic behavior of the renormalized solutions with extra regularity assumptions was studied in [6]. The decay property of the solutions to the linearized Vlasov–Poisson–Boltzmann system around $\bar{\mathbf{M}}$ was studied in [9, 10]. Finally, for the perturbation around vacuum, the results in [7, 16] give the global existence of smooth small-amplitude solutions for the cut-off potentials and the hard sphere model.

In what follows, the function space of solutions is the standard Sobolev space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)$:

$$\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right) = \left\{ f(t,x,\xi) \;\middle|\; \begin{array}{l} \dfrac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \left(f(t,x,\xi) - \bar{\mathbf{M}}(\xi)\right)}{\sqrt{\bar{\mathbf{M}}}} \in C\left([0,\infty),\right. \\ \left. L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right), \ |\alpha| + \beta + |\gamma| \le N \end{array} \right\}. \tag{1.4}$$

The result in this paper can be stated as follows.

Theorem 1.1. *Assume that* $f_0(x,\xi) \ge 0$ *and* $N \ge 4$. *A sufficiently small constant* $\varepsilon > 0$ *exists such that if*

$$\mathcal{E}(f_0) = \left\| \nabla_x \Delta_x^{-1}\left(\rho_0(x) - \bar{\rho}\right) \right\|_{L^2_x(\mathbf{R}^3)} + \sum_{|\alpha|+\beta \le N} \left\| \frac{\partial_x^\alpha \partial_\xi^\beta (f_0(x,\xi) - \bar{\mathbf{M}}(\xi))}{\bar{\mathbf{M}}(\xi)} \right\|_{L^2_{x,\xi}(\mathbf{R}^3 \times \mathbf{R}^3)} \le \varepsilon, \tag{1.5}$$

then there exists a unique global classical solution $f(t,x,\xi)$ *to the Vlasov–Poisson–Boltzmann system (1.1) and (1.2) which satisfies* $f(t,x,\xi) \ge 0$ *and*

$$\sum_{|\alpha|+\beta \le N} \left\| \partial_x^\alpha \partial_t^\beta (\rho - \bar{\rho}, u, \theta - \bar{\theta})(t,x) \right\|^2 + \|\nabla_x \Phi(t,x)\|^2$$
$$+ \sum_{|\alpha|+\beta+|\gamma| \le N} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\left| \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f(t,x,\xi) \right|^2}{\bar{\mathbf{M}}} d\xi dx$$

$$
\begin{aligned}
&+\sum_{|\alpha|+\beta+|\gamma|\le N}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f(\tau,x,\xi)\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&+\int_0^t\left(\|\rho-\bar\rho\|^2+\sum_{|\alpha|+\beta\le N-1}\left\|\nabla_x\partial_x^\alpha\partial_t^\beta u\right\|^2+\sum_{1\le|\alpha|+\beta\le N}\left\|\partial_x^\alpha\partial_t^\beta(\rho,\theta)\right\|^2\right)d\tau\\
\le\ & O(1)\mathcal{E}(f_0)^2,
\end{aligned}
\tag{1.6}
$$

where $\rho_0(x)$ is the initial density function, and $\nu(\xi)$ is the collision frequency defined in (2.11).

Furthermore, we have

$$
\begin{aligned}
&\sup_{x\in\mathbf{R}^3}\left\{\sum_{|\alpha|+\beta+|\gamma|\le N-3}\left(\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\left(f(t,x,\xi)-\bar{\mathbf{M}}(\xi)\right)\right|^2}{\bar{\mathbf{M}}(\xi)}d\xi\right)^{\frac12}+|\nabla_x\Phi(t,x)|\right\}\\
&\le O(1)(1+t)^{-\frac12}.
\end{aligned}
\tag{1.7}
$$

Here and in what follows, $O(1)$ is used to denote a generic positive constant.

Remark 1.1. The convergence rate given here is less than the convergence rate for the Boltzmann without forcing which is $(1+t)^{-\frac34}$. However, it is better than the one for the stationary potential force obtained in [29] because of the dependence of the force on the solution. To obtain the optimal convergence rate is an interesting problem, but we will not pursue this in this paper. Another possible improvement of the above result is to study the case when the background charge density $\bar\rho$ is a function of x rather than a constant considered in this paper. In this case, the stationary solution is no longer a global Maxwellian but a local Maxwellian as a function of x and ξ. For this, the existence of stationary solutions was given in [11]. Based on the existence of stationary solutions and their decay properties, the analysis in this paper could be elaborated for the study of the stability of stationary solutions.

Before concluding this section, it is worth pointing out that the energy estimate is also closed by using the decomposition with respect to the global Maxwellian $\bar{\mathbf{M}}$ in [17] for the Boltzmann equation without forcing. To be precise, based on the macroscopic projection defined by (2.6), i.e.,

$$
\mathbf{P}_0^{\bar{\mathbf{M}}}f=\left(a(t,x)+\sum_{j=1}^3 b^j(t,x)\xi^j+c(t,x)|\xi|^2\right)\bar{\mathbf{M}},
$$

one can derive a set of equations for the functions $a(t,x)$, $b^j(t,x)(j=1,2,3)$ and $c(t,x)$ so that estimates on these functions can be obtained. However, the time evolution of the conserved quantities $\left(\rho,m,\rho\left(\frac12|u|^2+\mathcal{E}\right)\right)$, which are defined by (2.1) and governed by the conservation laws (2.8), is not clear by just analyzing these functions.

The main observation in this paper is to close the energy estimates with respect to the global Maxwellian $\bar{\mathbf{M}}$ by estimating the conserved quantities $\left(\rho,\rho u,\rho\left(\frac12|u|^2+\mathcal{E}\right)\right)$ governed by the conservation laws in the form of the compressible Navier–Stokes equations with the non-fluid component $\mathbf{P}_1^{\mathbf{M}}f$ defined in (2.6) appearing in the source terms.

Hence, the analytic techniques for the system of compressible Navier–Stokes equations can be used. Therefore, it not only simplifies the analysis in the previous works, but also gives better stability analysis for the Vlasov–Poisson–Boltzmann system.

The rest of the paper is organized as follows. Some preliminaries such as the reformulation of the problem through the decomposition and the standard estimate on the microscopic projection will be given in Sect. 2. Section 3 concerns the energy estimates for the global existence in the space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)$. And the convergence rate in time of the solution to the global Maxwellian is proved in Sect. 4. Finally, the proofs of some technical lemmas are given in the Appendix.

2. Preliminaries

As in [30], we first reformulate the Vlasov–Poisson–Boltzmann system through the decomposition around the local Maxwellian introduced in [23] for the Boltzmann equation. For the convenience of the readers, we briefly give the derivation as follows.

Let $f(t,x,\xi)$ be the solution to the system (1.1). By the five conserved quantities, i.e., the mass density $\rho(t,x)$, momentum density $m(t,x)=\rho(t,x)u(t,x)$ and energy density $\mathcal{E}(t,x)+|u(t,x)|^2/2$ given by

$$\begin{cases} \rho(t,x)\equiv\int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ m_i(t,x)\equiv\int_{\mathbf{R}^3}\psi_i(\xi)f(t,x,\xi)d\xi,\quad i=1,2,3, \\ \left[\rho\left(\mathcal{E}+\frac{1}{2}|u|^2\right)\right](t,x)\equiv\int_{\mathbf{R}^3}\psi_4(\xi)f(t,x,\xi)d\xi, \end{cases}\tag{2.1}$$

the local Maxwellian is

$$\mathbf{M}\equiv\mathbf{M}_{[\rho,u,\theta]}(\xi)\equiv\frac{\rho}{\sqrt{(2\pi R\theta)^3}}\exp\left(-\frac{|\xi-u|^2}{2R\theta}\right).\tag{2.2}$$

Here $\theta(t,x)$ is the temperature related to the internal energy $\mathcal{E}(t,x)$ by $\mathcal{E}=\frac{3}{2}R\theta$, $u(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))$ is the fluid velocity, and R is the gas constant. As usual, $\psi_\alpha(\xi)$, $\alpha=0,1,\ldots,4$, are the five collision invariants:

$$\begin{cases} \psi_0(\xi)\equiv 1, \\ \psi_i(\xi)\equiv\xi_i,\quad i=1,2,3,\ \text{or}\ \psi(\xi)=\xi, \\ \psi_4(\xi)\equiv\frac{1}{2}|\xi|^2. \end{cases}\tag{2.3}$$

In both the Hilbert and Chapman–Enskog expansions, the leading order term in the expansion of the solution $f(t,x,\xi)$ is a local Maxwellian. In particular, the solution is expanded as the summation of the local Maxwellian and an expansion of the microscopic component with respect to the Knudsen number in the Chapman–Enskog expansion. The main idea in [23] is not to expand the microscopic component into a series with respect to the Knudsen number. Instead, one can simply write the solution as the sum of the local Maxwellian and microscopic component $\mathbf{G}=\mathbf{G}(x,t,\xi)$:

$$f(t,x,\xi)=\mathbf{M}(t,x,\xi)+\mathbf{G}(t,x,\xi).\tag{2.4}$$

As shown later, the Vlasov–Poisson–Boltzmann system can then be reformulated into a coupled system containing the conservation laws for the macroscopic components, the equation for microscopic component and the Poisson equation for the potential of the force.

For later use, we need to define the projection with respect to a given Maxwellian, either global or local for any function in the corresponding L^2_ξ space. To be precise, for any given Maxwellian $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}_{[\tilde\rho,\tilde u,\tilde\theta]}$, we define an inner product in $\xi \in \mathbf{R}^3$ by

$$\langle h, g\rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{h(\xi)g(\xi)}{\tilde{\mathbf{M}}} d\xi,$$

for functions h and g of ξ such that the integral is well defined. By using this inner product, the subspace spanned by the collision invariants has the following set of orthogonal basis:

$$\begin{cases} \chi_0^{\tilde{\mathbf{M}}} = \chi_0(\xi;\tilde\rho,\tilde u,\tilde\theta) \equiv \frac{1}{\sqrt{\tilde\rho}}\tilde{\mathbf{M}}, \\ \chi_i^{\tilde{\mathbf{M}}} = \chi_i(\xi;\tilde\rho,\tilde u,\tilde\theta) \equiv \frac{\xi_i-\tilde u_i}{\sqrt{R\tilde\rho\tilde\theta}}\tilde{\mathbf{M}}, \quad i=1,2,3, \\ \chi_4^{\tilde{\mathbf{M}}} = \chi_4(\xi;\tilde\rho,\tilde u,\tilde\theta) \equiv \frac{1}{\sqrt{6\tilde\rho}}\left(\frac{|\xi-\tilde u|^2}{R\tilde\theta}-3\right)\tilde{\mathbf{M}}, \\ \left\langle \chi_\alpha^{\tilde{\mathbf{M}}}, \chi_\beta^{\tilde{\mathbf{M}}}\right\rangle_{\tilde{\mathbf{M}}} = \delta_{\alpha\beta}, \quad \text{for } \alpha,\ \beta = 0,1,2,3,4. \end{cases} \tag{2.5}$$

The macroscopic projection $\mathbf{P}_0^{\tilde{\mathbf{M}}}$ and microscopic projection $\mathbf{P}_1^{\tilde{\mathbf{M}}}$ can then be defined by:

$$\begin{cases} \mathbf{P}_0^{\tilde{\mathbf{M}}} h \equiv \sum_{\alpha=0}^{4} \left\langle h, \chi_\alpha^{\tilde{\mathbf{M}}}\right\rangle_{\tilde{\mathbf{M}}} \chi_\alpha^{\tilde{\mathbf{M}}}, \\ \mathbf{P}_1^{\tilde{\mathbf{M}}} h \equiv h - \mathbf{P}_0^{\tilde{\mathbf{M}}} h. \end{cases} \tag{2.6}$$

Notice that the operators $\mathbf{P}_0^{\tilde{\mathbf{M}}}$ and $\mathbf{P}_1^{\tilde{\mathbf{M}}}$ are projections, that is,

$$\mathbf{P}_0^{\tilde{\mathbf{M}}}\mathbf{P}_0^{\tilde{\mathbf{M}}} = \mathbf{P}_0^{\tilde{\mathbf{M}}}, \quad \mathbf{P}_1^{\tilde{\mathbf{M}}}\mathbf{P}_1^{\tilde{\mathbf{M}}} = \mathbf{P}_1^{\tilde{\mathbf{M}}}, \quad \mathbf{P}_0^{\tilde{\mathbf{M}}}\mathbf{P}_1^{\tilde{\mathbf{M}}} = \mathbf{P}_1^{\tilde{\mathbf{M}}}\mathbf{P}_0^{\tilde{\mathbf{M}}} = 0.$$

And the system of conservation laws

$$\int_{\mathbf{R}^3} \psi_\alpha\Big(f_t + \xi\cdot\nabla_x f + \nabla_x\Phi\cdot\nabla_\xi f\Big)d\xi = 0, \quad \alpha = 0,1,\ldots,4, \tag{2.7}$$

becomes

$$\begin{cases} \rho_t + \mathrm{div}_x\, m = 0, \\ m_{it} + \left(\sum_{j=1}^{3} u_j m_i\right)_{x_j} + p_{x_i} - \rho\Phi_{x_i} + \int_{\mathbf{R}^3} \psi_i(\xi)\Big(\xi\cdot\nabla_x \mathbf{G}\Big)d\xi = 0, \quad i=1,2,3, \\ \left[\rho\left(\frac{|u|^2}{2}+\mathcal{E}\right)\right]_t + \sum_{j=1}^{3}\left\{u_j\left[\rho\left(\frac{|u|^2}{2}+\mathcal{E}\right)+p\right]\right\}_{x_j} - m\cdot\nabla_x\Phi \\ + \int_{\mathbf{R}^3}\psi_4(\xi)\Big(\xi\cdot\nabla_x\mathbf{G}\Big)d\xi = 0. \end{cases} \tag{2.8}$$

Here, the equation of state for the monatomic gas is given by,

$$p = \frac{2}{3}\rho\mathcal{E}, \quad \mathcal{E} = \theta,$$

where the gas constant R is chosen to be $\frac{2}{3}$ without loss of generality. Notice also that the macroscopic entropy S is given by

$$S = -\frac{2}{3}\ln\rho + \ln\left(\frac{4}{3}\pi\theta\right) + 1.$$

By noticing that $\mathbf{G}_t$, $\nabla_\xi \mathbf{G}$, $L_{\mathbf{M}}\mathbf{G}$, and $Q(\mathbf{G}, \mathbf{G})$ are microscopic components, the microscopic equation is obtained by applying the microscopic projection $\mathbf{P}_1^{\mathbf{M}}$ to the Vlasov–Poisson–Boltzmann system (1.1):

$$\mathbf{G}_t + \mathbf{P}_1^{\mathbf{M}}\Big(\xi\cdot\nabla_x\mathbf{G} + \xi\cdot\nabla_x\mathbf{M}\Big) + \nabla_x\Phi\cdot\nabla_\xi\mathbf{G} = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G}, \mathbf{G}), \tag{2.9}$$

where $L_{\mathbf{M}}$ is the linearized collision operator defined by

$$L_{\mathbf{M}}g = L_{\mathbf{M}_{[\rho,u,\theta]}}g \equiv 2Q(\mathbf{M}_{[\rho,u,\theta]}, g). \tag{2.10}$$

The null space of $L_{\mathbf{M}}$, denoted by $\mathcal{N}$, is spanned by the collision invariants $\chi_j^{\mathbf{M}},\ j = 0, 1, 2, 3, 4$.

For the hard sphere model, $L_{\mathbf{M}}$ takes the form, cf. [18], and [1, 3, 4, 8, 13] for the discussions on other collision kernels,

$$(L_{\mathbf{M}}h)(\xi) = -\nu(\xi;\rho,u,\theta)h(\xi) + \sqrt{\mathbf{M}(\xi)}K_{\mathbf{M}}\left(\left(\frac{h}{\sqrt{\mathbf{M}}}\right)(\xi)\right).$$

Here $K_{\mathbf{M}}(\cdot) = -K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2_ξ-operator. The collision frequency $\nu(\xi;\rho,u,\theta)$ and $k_{i\mathbf{M}}(\xi,\xi_*)$ which is the kernel of the operator $K_{i\mathbf{M}}$ $(i = 1, 2)$ have the following expressions:

$$\begin{cases} \nu(\xi;\rho,u,\theta) = \frac{2\rho}{\sqrt{2\pi R\theta}}\left\{\left(\frac{R\theta}{|\xi-u|} + |\xi-u|\right)\int_0^{|\xi-u|}\exp\left(-\frac{y^2}{2R\theta}\right)dy \right. \\ \qquad\qquad\qquad \left. + R\theta\exp\left(-\frac{|\xi-u|^2}{2R\theta}\right)\right\}, \\ k_{1\mathbf{M}}(\xi,\xi_*) = \frac{\pi\rho}{\sqrt{(2\pi R\theta)^3}}|\xi-\xi_*|\exp\left(-\frac{|\xi-u|^2}{4R\theta} - \frac{|\xi_*-u|^2}{4R\theta}\right), \\ k_{2\mathbf{M}}(\xi,\xi_*) = \frac{2\rho}{\sqrt{2\pi R\theta}}|\xi-\xi_*|^{-1}\exp\left(-\frac{|\xi-\xi_*|^2}{8R\theta} - \frac{(|\xi|^2-|\xi_*|^2)^2}{8R\theta|\xi-\xi_*|^2}\right). \end{cases} \tag{2.11}$$

Moreover, for the hard sphere model,

$$0 < \nu_0 \le \frac{\nu(\xi;\rho,u,\theta)}{1+|\xi|} < \infty$$

holds uniformly in ξ for some constant ν_0.

Since $L_{\mathbf{M}}$ is a bounded and one to one operator on $\mathcal{N}^\perp$, from (2.9), we have

$$\begin{aligned} \mathbf{G} &= L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1^{\mathbf{M}}(\xi\cdot\nabla_x\mathbf{M})\right) + L_{\mathbf{M}}^{-1}\left(\partial_t\mathbf{G} + \mathbf{P}_1^{\mathbf{M}}(\xi\cdot\nabla_x\mathbf{G}) + \nabla_x\Phi\cdot\nabla_\xi\mathbf{G} - Q(\mathbf{G},\mathbf{G})\right) \\ &= L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1^{\mathbf{M}}(\xi\cdot\nabla_x\mathbf{M})\right) + \Theta. \end{aligned} \tag{2.12}$$

Here

$$\mathcal{N}^{\perp} = \left\{ f(\xi) : \int_{\mathbf{R}^3} \frac{\chi_j^{\mathbf{M}} f(\xi)}{\mathbf{M}} d\xi = 0, \ j = 0, 1, 2, 3, 4. \right\}.$$

Substituting (2.12) into (2.8) yields the following conservative system for the macroscopic components:

$$\begin{cases} \rho_t + \mathrm{div}_x\, m = 0, \\ m_{it} + \left(\sum_{j=1}^{3} u_j m_i \right)_{x_j} + p_{x_i} - \rho \Phi_{x_i} + \int_{\mathbf{R}^3} \psi_i(\xi) \left(\xi \cdot \nabla_x L_{\mathbf{M}}^{-1} \left(\mathbf{P}_1^{\mathbf{M}} (\xi \cdot \nabla_x \mathbf{M}) \right) \right) d\xi \\ \quad + \int_{\mathbf{R}^3} \psi_i(\xi) \left(\xi \cdot \nabla_x \Theta \right) d\xi = 0, \quad i = 1, 2, 3, \\ \left[\rho \left(\frac{|u|^2}{2} + \mathcal{E} \right) \right]_t + \sum_{j=1}^{3} \left\{ u_j \left[\rho \left(\frac{|u|^2}{2} + \mathcal{E} \right) + p \right] \right\}_{x_j} - m \cdot \nabla_x \Phi \\ \quad + \int_{\mathbf{R}^3} \psi_4(\xi) \left(\xi \cdot \nabla_x L_{\mathbf{M}}^{-1} \left(\mathbf{P}_1^{\mathbf{M}} (\xi \cdot \nabla_x \mathbf{M}) \right) \right) d\xi + \int_{\mathbf{R}^3} \psi_4(\xi) \left(\xi \cdot \nabla_x \Theta \right) d\xi = 0. \end{cases} \tag{2.13}$$

Notice that in the above system, the viscosity and heat conductivity terms are the same as those in the compressible Navier–Stokes equations which are independent of the density gradient $\nabla_x \rho$. In fact, by using the Burnett functions A and B, the viscosity coefficient $\mu(\theta)$ and heat conductivity coefficient $\kappa(\theta)$ are represented by:

$$\begin{cases} A_j(\xi) = \frac{|\xi|^2 - 5}{2} \xi_j, \quad j = 1, 2, 3, \\ B_{ij}(\xi) = \xi_i \xi_j - \frac{1}{3} \delta_{ij} |\xi|^2, \quad i, j = 1, 2, 3, \\ \mu(\theta) = -R\theta \int_{\mathbf{R}^3} B_{ij} \left(\frac{\xi}{\sqrt{R\theta}} \right) L_{\mathbf{M}_{[1,0,\theta]}}^{-1} \left(B_{ij} \left(\frac{\xi}{\sqrt{R\theta}} \right) \mathbf{M}_{[1,0,\theta]} \right) d\xi > 0, \quad i \neq j, \\ \kappa(\theta) = -R^2\theta \int_{\mathbf{R}^3} A_l \left(\frac{\xi}{\sqrt{R\theta}} \right) L_{\mathbf{M}_{[1,0,\theta]}}^{-1} \left(A_l \left(\frac{\xi}{\sqrt{R\theta}} \right) \mathbf{M}_{[1,0,\theta]} \right) d\xi > 0, \end{cases} \tag{2.14}$$

and the system (2.13) can be rewritten as

$$\begin{cases} \rho_t + \mathrm{div}_x m = 0, \\ m_{it} + \sum_{j=1}^{3} \left(u_j m_i \right)_{x_j} + p_{x_i} - \rho \Phi_{x_i} = \sum_{j=1}^{3} \left[\mu(\theta) \left(u_{ix_j} + u_{jx_i} - \frac{2}{3} \delta_{ij} \mathrm{div}_x u \right) \right]_{x_j} \\ \qquad - \int_{\mathbf{R}^3} \psi_i(\xi) \left(\xi \cdot \nabla_x \Theta \right) d\xi, \quad i = 1, 2, 3, \\ \left[\rho (\frac{1}{2}|u|^2 + \mathcal{E}) \right]_t + \sum_{j=1}^{3} \left(u_j \left(\rho \left(\frac{1}{2}|u|^2 + \mathcal{E} \right) + p \right) \right)_{x_j} - m \cdot \nabla_x \Phi \\ \quad = \sum_{i,j=1}^{3} \left\{ \mu(\theta) u_i \left(u_{ix_j} + u_{jx_i} - \frac{2}{3} \delta_{ij} \mathrm{div}_x u \right) \right\}_{x_j} \\ \quad + \sum_{j=1}^{3} \left(\kappa(\theta) \theta_{x_j} \right)_{x_j} - \int_{\mathbf{R}^3} \psi_4(\xi) \left(\xi \cdot \nabla_x \Theta \right) d\xi. \end{cases} \tag{2.15}$$

In (2.15), the structures of the compressible Euler–Poisson and the compressible Navier–Stokes–Poisson equations are clear. For instance, when the microscopic component $\mathbf{G}$ is set to be zero, the system (2.15) becomes part of the compressible Euler–Poisson equations. On the other hand, when Θ is set to be zero in (2.15), it becomes part of the compressible Navier–Stokes–Poisson equations. Both Euler–Poisson and Navier–Stokes–Poisson systems are approximations to the Vlasov–Poisson–Boltzmann system if they are derived through the Hilbert and Chapman-Enskog expansions. Nevertheless, the above reformulation is exact to the Vlasov–Poisson–Boltzmann system and is consistent in spirit with those expansions where the first order approximation is a local Maxwellian.

In what follows, we will use the system (2.15) to analyze the macroscopic components and the Vlasov–Poisson–Boltzmann system itself for the microscopic component. This approach follows from the elaborated analysis on the Boltzmann equation in [31] which improves the previous energy method from [23] so that the analytic techniques from the theory of conservation laws and the H-theorem for the Boltzmann equation are fully and suitably used.

Finally in this section, we state some known estimates for later use. The first lemma is about some Sobolev inequalities.

Lemma 2.1. *For $g(x) \in H^1(\mathbf{R}^3)$, we have*

$$\|g(x)\|_{L^6(\mathbf{R}^3)} \leq C_0\|\nabla_x g(x)\|, \tag{2.16}$$

where C_0 is a uniform positive constant. Consequently, for $g(x) \in H^2(\mathbf{R}^3)$, there exists a uniform positive constant C_1 such that

$$\begin{cases} \|g(x)\|_{L^\infty(\mathbf{R}^3)} \leq C_1\|\nabla_x g(x)\|_1, \\ \|g(x)\|_{L^4(\mathbf{R}^3)} \leq C_1\|\nabla_x g(x)\|^{\frac{3}{4}}\|g(x)\|^{\frac{1}{4}}. \end{cases} \tag{2.17}$$

Here and in what follows, $\|\cdot\|$ and $\|\cdot\|_s$ denote the standard $L^2(\mathbf{R}^3)$ and $H^s(\mathbf{R}^3)$ norms respectively.

For the nonlinear and linearized collision operators $Q(f, f)$ and $L_{\mathbf{M}}\mathbf{G}$, we have the following estimates from [12].

Lemma 2.2. *There exists a uniform constant $C_2 > 0$ such that*

$$\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}Q(f,g)^2}{\tilde{\mathbf{M}}}d\xi \leq C_2\left\{\int_{\mathbf{R}^3}\frac{\nu(\xi)f^2}{\tilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{g^2}{\tilde{\mathbf{M}}}d\xi+\int_{\mathbf{R}^3}\frac{f^2}{\tilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{\nu(\xi)g^2}{\tilde{\mathbf{M}}}d\xi\right\}, \tag{2.18}$$

where $\tilde{\mathbf{M}}$ is any Maxwellian such that the above integrals are well defined.

For $\mathbf{P}_1^{\mathbf{M}_0}f$ which is the microscopic projection of the solution $f(t,x,\xi)$ with respect to a given Maxwellian $\mathbf{M}_0$, the microscopic version of the H-theorem states that the linearized collision operator $L_{\mathbf{M}_0}$ is negative definite on $\mathbf{P}_1^{\mathbf{M}_0}f$, cf. [2], i.e.,

$$-\int_{\mathbf{R}^3}\frac{\mathbf{P}_1^{\mathbf{M}_0}fL_{\mathbf{M}_0}\left(\mathbf{P}_1^{\mathbf{M}_0}f\right)}{\mathbf{M}_0}d\xi \geq \sigma\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\mathbf{P}_1^{\mathbf{M}_0}f\right|^2}{\mathbf{M}_0}d\xi,$$

for some positive constant σ. In fact, the Maxwellian around which the linearized collision operator is defined can be different from the Maxwellian used as the weight function in L^2_ξ. That is, we also have the following estimate by Lemma 2.2, cf. [24].

Lemma 2.3. *When* $\frac{\theta}{2} < \tilde{\theta}$*, there exist two positive constants* $\overline{\sigma} = \overline{\sigma}(u, \theta; \tilde{u}, \tilde{\theta})$ *and* $\eta_0 = \eta_0(u, \theta; \tilde{u}, \tilde{\theta})$ *such that if* $|u - \tilde{u}| + |\theta - \tilde{\theta}| < \eta_0$ *and* $h(\xi) \in \mathcal{N}^{\perp}$*, we have*

$$-\int_{\mathbf{R}^3} \frac{hL_{\mathbf{M}}h}{\tilde{\mathbf{M}}} d\xi \geq \overline{\sigma} \int_{\mathbf{R}^3} \frac{\nu(\xi)h^2}{\tilde{\mathbf{M}}} d\xi. \tag{2.19}$$

Here, $\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(\xi)$ *and* $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}(\xi)$.

Remark 2.1. η_0 in the above lemma is some positive constant depending on the first non-zero eigenvalue of the linearized collision operator $L_{\mathbf{M}}$ which does not need to be small, cf. [24].

A direct consequence of Lemma 2.3 and the Cauchy-Schwarz inequality is the following corollary, cf. [24].

Corollary 2.1. *Under the assumptions in Lemma 2.3, for* $h(\xi) \in \mathcal{N}^{\perp}$*, we have*

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)}{\tilde{\mathbf{M}}} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leq \overline{\sigma}^{-2} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} h^2(\xi)}{\tilde{\mathbf{M}}} d\xi. \tag{2.20}$$

The following estimates are based on the following a priori estimate which we will prove by energy method in Sect. 3. Since the analysis for the case when $N > 4$ is similar to the one when $N = 4$. In what follows, we only give the estimates when $N = 4$.

Set

$$\begin{aligned} N(t)^2 = \sup_{0 \leq \tau \leq t} \Bigg\{ & \sum_{|\alpha|+\beta \leq 4} \left\| \partial_x^\alpha \partial_t^\beta (\rho - \bar{\rho}, u, \theta - \bar{\theta})(\tau, x) \right\|^2 + \|\nabla_x \Phi(\tau, x)\|^2 \\ & + \sum_{|\alpha|+\beta+|\gamma| \leq 4} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\left| \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f(\tau, x, \xi) \right|^2}{\bar{\mathbf{M}}} d\xi dx \Bigg\} < \varepsilon^2. \end{aligned} \tag{2.21}$$

By the Sobolev inequality, for $x \in \mathbf{R}^3$ and $0 \leq \tau \leq t$, the a priori estimate (2.21) implies that

$$\begin{cases} \sum_{|\alpha|+\beta \leq 2} \left(\left| \partial_x^\alpha \partial_t^\beta (\rho - \bar{\rho}, u, \theta - \bar{\theta})(\tau, x) \right| + \left| \nabla_x \partial_x^\alpha \partial_t^\beta \Phi(\tau, x) \right| \right) \leq O(1)\varepsilon, \\ \sum_{|\alpha|+\beta+|\gamma| \leq 2} \int_{\mathbf{R}^3} \frac{\left| \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f(\tau, x, \xi) \right|^2}{\bar{\mathbf{M}}} d\xi \leq O(1)\varepsilon^2. \end{cases} \tag{2.22}$$

To perform the energy estimates, we first give the estimate on the difference between the microscopic projections of $f(t, x, \xi)$ with respect to the local Maxwellian $\mathbf{M}$ and the global Maxwellian $\bar{\mathbf{M}}$. Note that

$$\mathbf{P}_1^{\bar{\mathbf{M}}} f = \mathbf{G} + \mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M} = \mathbf{P}_1^{\mathbf{M}} f + \mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M}.$$

Since $\mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M}$ is a smooth function of ρ, u, θ and ξ satisfying

$$\begin{cases} \mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M} \Big|_{(u,\theta)=(0,\bar{\theta})} = 0, \\ \nabla_{(u,\theta)} \mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M} \Big|_{(u,\theta)=(0,\bar{\theta})} = \mathbf{P}_1^{\bar{\mathbf{M}}} \left(\nabla_{(u,\theta)} \mathbf{M} \right) \Big|_{(u,\theta)=(0,\bar{\theta})} = 0, \end{cases}$$

$\mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M}$ is of the quadratic order of u and $\theta - \bar{\theta}$ which are the perturbations in the velocity and temperature variables. Hence, Lemma 2.1 and the a priori estimate (2.21) give the following lemma, cf. [31].

Lemma 2.4. *For $|\alpha| + \beta + |\gamma| \le 4$, we have*

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{G}\right|^2}{\bar{\mathbf{M}}} d\xi dx \le O(1) \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx$$

$$+ \ O(1)\varepsilon^2 \sum_{|\alpha'|+\beta' \le 3} \left\| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (u, \theta) \right\|^2. \qquad (2.23)$$

The next lemma concerns the estimate on the nonlinear collision operator $Q(f, f)$.

Lemma 2.5. *Under the a priori estimate (2.21), we have*

$$\sum_{|\alpha|+\beta+|\gamma| \le 4} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma Q\left(\mathbf{P}_1^{\bar{\mathbf{M}}} f, \mathbf{P}_1^{\bar{\mathbf{M}}} f\right)\right|^2}{\bar{\mathbf{M}}} d\xi dx$$

$$\le O(1)\varepsilon^2 \sum_{|\alpha'|+\beta'+|\gamma'| \le 4, \beta' \le \beta} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^{\alpha'} \partial_t^{\beta'} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx, \qquad (2.24)$$

$$\sum_{|\alpha|+\beta+|\gamma| \le 4} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma Q\left(\mathbf{P}_1^{\bar{\mathbf{M}}} f, \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M} - \bar{\mathbf{M}})\right)\right|^2}{\bar{\mathbf{M}}} d\xi dx$$

$$\le O(1)\varepsilon^2 \sum_{|\alpha'|+\beta'+|\gamma'| \le 4, \beta' \le \beta} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^{\alpha'} \partial_t^{\beta'} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx, \qquad (2.25)$$

and

$$\sum_{|\alpha|+\beta+|\gamma| \le 4} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma Q\left(\mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M} - \bar{\mathbf{M}}), \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M} - \bar{\mathbf{M}})\right)\right|^2}{\bar{\mathbf{M}}} d\xi dx$$

$$\le O(1)\varepsilon^2 \sum_{|\alpha'|+\beta' \le 3} \left\| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta) \right\|^2. \qquad (2.26)$$

Proof. We only prove (2.24) because the proof of (2.25) and (2.26) is similar. Since

$$\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma Q\left(\mathbf{P}_1^{\bar{\mathbf{M}}} f, \mathbf{P}_1^{\bar{\mathbf{M}}} f\right) = \sum_{(\alpha'',\beta'',\gamma'') \le (\alpha,\beta,\gamma)} C_{\alpha,\beta,\gamma}^{\alpha'',\beta'',\gamma''} Q\left(\partial_x^{\alpha''} \partial_t^{\beta''} \partial_\xi^{\gamma''} \mathbf{P}_1^{\bar{\mathbf{M}}} f,\right.$$

$$\left. \partial_x^{\alpha-\alpha''} \partial_t^{\beta-\beta''} \partial_\xi^{\gamma-\gamma''} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right),$$

from Lemma 2.2, we have

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|\partial_x^{\alpha}\partial_t^{\beta}\partial_\xi^{\gamma}Q\left(\mathbf{P}_1^{\bar{\mathbf{M}}}f,\mathbf{P}_1^{\bar{\mathbf{M}}}f\right)\right|^2}{\bar{\mathbf{M}}}d\xi dx\\
&\leq O(1)\sum_{(\alpha'',\beta'',\gamma'')\leq(\alpha,\beta,\gamma)}\int_{\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha''}\partial_t^{\beta''}\partial_\xi^{\gamma''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi\int_{\mathbf{R}^3}\frac{\left|\partial_x^{\alpha-\alpha''}\partial_t^{\beta-\beta''}\partial_\xi^{\gamma-\gamma''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi\right\}dx\\
&:=O(1)\sum_{(\alpha'',\beta'',\gamma'')\leq(\alpha,\beta,\gamma)}J_{\alpha,\beta,\gamma}^{\alpha'',\beta'',\gamma''}. \qquad (2.27)
\end{aligned}
$$

If $|\alpha''|+\beta''+|\gamma''|\leq 2$, then from (2.22) we have

$$
J_{\alpha,\beta,\gamma}^{\alpha'',\beta'',\gamma''}\leq O(1)\varepsilon^2\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha-\alpha''}\partial_t^{\beta-\beta''}\partial_\xi^{\gamma-\gamma''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx.
$$

On the other hand, when $|\alpha''|+\beta''+|\gamma''|>2$ which implies that $|\alpha-\alpha''|+\beta-\beta''+|\gamma-\gamma''|\leq 1$, we have

$$
\begin{aligned}
J_{\alpha,\beta,\gamma}^{\alpha'',\beta'',\gamma''}&\leq O(1)\left\|\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha-\alpha''}\partial_t^{\beta-\beta''}\partial_\xi^{\gamma-\gamma''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi\right\|_{L_x^\infty}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^{\alpha''}\partial_t^{\beta''}\partial_\xi^{\gamma''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\right)\\
&\leq O(1)\varepsilon^2\left\|\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha-\alpha''}\partial_t^{\beta-\beta''}\partial_\xi^{\gamma-\gamma''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi\right\|_{L_x^\infty}\\
&\leq O(1)\varepsilon^2\sum_{|\alpha'|+\beta'+|\gamma'|\leq 4,\beta'\leq\beta}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx.
\end{aligned}
$$

Thus, for any $(\alpha'',\beta'',\gamma'')\leq(\alpha,\beta,\gamma)$, we have

$$
J_{\alpha,\beta,\gamma}^{\alpha'',\beta'',\gamma''}\leq O(1)\varepsilon^2\sum_{|\alpha'|+\beta'+|\gamma'|\leq 4,\beta'\leq\beta}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx. \qquad (2.28)
$$

This gives (2.24) and then completes the proof of the lemma. □

The last lemma in this section concerns the estimate on $\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}|\xi|^k\left|\partial_x^\alpha\Theta\right|^2d\xi dx$ for any $k\in\mathbf{Z}^+$ and $|\alpha|\leq 3$.

Lemma 2.6. *Under the a priori estimate (2.21), for any $k \in \mathbf{Z}^+$ and $|\alpha| \le 3$, we have*

$$\begin{aligned}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}|\xi|^k\left|\partial_x^\alpha\Theta\right|^2 d\xi dx \le C(k)&\sum_{|\alpha'|+\beta'\le 4,\beta'\le 1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\\ &+C(k)\varepsilon^2\sum_{|\alpha'|+|\gamma'|\le 4,\gamma'\le 1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\\ &+C(k)\varepsilon^2\sum_{|\alpha'|+\beta'\le 3,\beta'\le 1}\left\|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2,\end{aligned} \tag{2.29}$$

where $C(k)$ is a positive constant depending on k.

Proof. Choose ε in (2.21) sufficiently small such that

$$\varepsilon < \min\left\{\frac{\eta_0}{2},\ \overline{\theta}\right\}. \tag{2.30}$$

Then for any θ_- satisfying

$$\overline{\theta} < \theta_- < \overline{\theta}+\varepsilon, \tag{2.31}$$

we have

$$\begin{cases}\theta \le \overline{\theta}+|\theta-\overline{\theta}| < \overline{\theta}+\varepsilon < 2\overline{\theta} < 2\theta_-,\\ |u|+|\theta-\theta_-| \le \left(|u|+|\theta-\overline{\theta}|\right)+\theta_- -\overline{\theta} < 2\varepsilon < \eta_0.\end{cases} \tag{2.32}$$

Consequently, by using $\mathbf{M}_- = \mathbf{M}_{[\bar{\rho},0,\theta_-]}$, Lemma 2.2, 2.4, 2.5, Corollary 2.1, and (2.21), we have

$$\begin{aligned}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}|\xi|^k|\Theta|^2 d\xi dx &\le O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|L_{\mathbf{M}}^{-1}\left(\mathbf{G}_t+\mathbf{P}_1^{\mathbf{M}}(\xi\cdot\nabla_x\mathbf{G})+\nabla_x\Phi\cdot\nabla_\xi\mathbf{G}-Q(\mathbf{G},\mathbf{G})\right)\right|^2}{\mathbf{M}_-}d\xi dx\\ &\le O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left(|\mathbf{G}_t|^2+|\xi|^2|\nabla_x\mathbf{G}|^2+|\nabla_x\Phi|^2|\nabla_\xi\mathbf{G}|^2+|Q(\mathbf{G},\mathbf{G})|^2\right)}{\mathbf{M}_-}d\xi dx\\ &\le O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}_t|^2+|\nabla_x\mathbf{G}|^2+\varepsilon^2\left(\mathbf{G}^2+|\nabla_\xi\mathbf{G}|^2\right)}{\bar{\mathbf{M}}}d\xi dx\\ &\le O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_t\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2+\left|\nabla_x\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2+\varepsilon^2\left(\left|\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2+\left|\nabla_\xi\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2\right)}{\bar{\mathbf{M}}}d\xi dx\\ &\quad+O(1)\varepsilon^2\sum_{|\alpha'|+\beta'\le 3,\beta'\le 1}\left\|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2.\end{aligned} \tag{2.33}$$

This is exactly (2.29) when $\alpha = 0$.

The case when $1 \le |\alpha| \le 3$ can be proved similarly by using

$$\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}h\right\} = L_{\mathbf{M}}^{-1}(\partial_x^\alpha h)-\sum_{j=0}^{|\alpha|-1}\sum_{|\alpha_j|=j}C_{\alpha_j}L_{\mathbf{M}}^{-1}\left(Q\left(\partial_x^{\alpha_j}\left(L_{\mathbf{M}}^{-1}h\right),\partial_x^{\alpha-\alpha_j}\mathbf{M}\right)\right).$$

This completes the proof of the lemma. □

3. Energy Estimates

In this section, we will prove the energy estimates for the global existence of classical solutions. Notice that the local existence in the space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^+ \times \mathbf{R}^3 \times \mathbf{R}^3\right)$ follows directly from the corresponding result on the Boltzmann equation with external force, [28]. Hence, we only need to close the a priori estimate (2.21).

There are two steps in closing the a priori estimate for the global existence. One is the estimation on the conserved quantities $\left(\rho, \rho u, \rho\left(\mathcal{E} + \frac{1}{2}|u|^2\right)\right)$ by analyzing the system (2.15) using the techniques from the theory of conservation laws, [20, 26]. Compared to the classical compressible Navier–Stokes equations, the extra term in the system (2.15) defined by the microscopic component $\mathbf{G}$ and viewed as a source term is of higher order either in the nonlinearity or differentiation, so that it can be viewed as an error term for small and smooth perturbations.

On the other hand, the source term in (2.15) defined by the microscopic component $\mathbf{G}$ comes from the projection with respect to the local Maxwellian. However, the differentiation on the local Maxwellian with respect to the x variable yields some extra factor in ξ which could cause complexity in the analysis. The direct estimation on $\mathbf{G}$ itself was used in the previous work before [31] where two sets of energy estimates are needed. However, by Lemma 2.4, the estimation on the microscopic component $\mathbf{G}$ is equivalent to the estimation on $\mathbf{P}_1^{\bar{\mathbf{M}}} f$ which can be obtained directly from the Boltzmann equation with the global Maxwellian $\bar{\mathbf{M}}$ being the weight function.

The energy estimates will be performed as follows. Firstly, we will present some estimates on the potential function $\Phi(t, x)$ by using the Poisson equation and the conservation laws in the form of (2.8). Secondly, we will give the estimation on the macroscopic components by using again the conservation laws now in the form of (2.15). Finally, the estimation on the microscopic component will be obtained by directly using the Vlasov–Poisson–Boltzmann equation and the H-theorem.

Under the a priori assumption (2.21), by the Poisson in (1.1), the conservation laws (2.8) and Lemma 2.4, we have the following estimates on the potential $\Phi(t, x)$:

Lemma 3.1. $\Phi(t, x)$ *satisfies the following estimates:*

$$\begin{cases} \left\|\nabla_x \partial_x^\alpha \Phi\right\|^2 \leq O(1) \displaystyle\sum_{|\alpha'|=|\alpha|-1} \left\|\partial_x^{\alpha'}(\rho - \bar{\rho})\right\|^2, \quad 1 \leq |\alpha| \leq 5, \\ \left\|\nabla_x \partial_x^\alpha \Phi_t\right\|^2 \leq O(1) \left\|\partial_x^\alpha m\right\|^2, \quad |\alpha| \leq 4, \\ \left\|\nabla_x \partial_x^\alpha \Phi_{tt}\right\|^2 \leq O(1) \displaystyle\sum_{|\alpha'|\leq|\alpha|} \left\|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right\|^2 + O(1) \left\|\nabla_x \partial_x^\alpha \Phi\right\|^2 \\ \qquad + O(1) \displaystyle\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\nabla_x \partial_x^\alpha \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx, \quad |\alpha| \leq 3, \\ \left\|\nabla_x \partial_x^\alpha \Phi_{ttt}\right\|^2 \leq O(1) \displaystyle\sum_{|\alpha'|\leq|\alpha|, \beta'\leq 1} \left\|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'}(\rho, u, \theta)\right\|^2 + O(1) \left\|\nabla_x \partial_x^\alpha \Phi_t\right\|^2 \\ \qquad + O(1) \displaystyle\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\nabla_x \partial_x^\alpha \partial_t \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx, \quad |\alpha| \leq 2, \\ \left\|\nabla_x \partial_x^\alpha \Phi_{tttt}\right\|^2 \leq O(1) \displaystyle\sum_{|\alpha'|\leq|\alpha|, \beta'\leq 2} \left\|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'}(\rho, u, \theta)\right\|^2 + O(1) \left\|\nabla_x \partial_x^\alpha \Phi_{tt}\right\|^2 \\ \qquad + O(1) \displaystyle\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\nabla_x \partial_x^\alpha \partial_t^2 \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx, \quad |\alpha| \leq 1. \end{cases} \tag{3.1}$$

From (3.1) and the energy estimates given below, we will see that $\nabla_x \partial_x^\alpha \partial_t^\beta \Phi(t, x) \in L^2_{t,x}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)$ only when $|\alpha| \geq 1$.

Now we turn to the estimates on the conserved quantities $\left(\rho, \rho u, \rho\left(\mathcal{E} + \frac{1}{2}|u|^2\right)\right)$. As usual, the lowest order estimate can be obtained by using the macroscopic entropy-entropy flux pair (η, q) around $(\bar\rho, 0, \bar\theta)$, cf. [30]:

$$\begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \bar\theta\rho S + \rho\left[\left(\bar S - \frac{5}{3}\right)\bar\theta + \frac{|u|^2}{2}\right] + \frac{2}{3}\bar\rho\bar\theta\right\}, \\ q_j = u_j\eta + u_j(\rho\theta - \bar\rho\bar\theta), \quad j = 1, 2, 3. \end{cases} \tag{3.2}$$

From (2.15), (η, q) satisfies

$$\begin{aligned} \eta_t + \mathrm{div}_x q = & \sum_{i,j=1}^3 \frac{3\bar\theta u_i}{2\theta}\left[\mu(\theta)\left(u_{ix_j} + u_{jx_i} - \frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right]_{x_j} + \sum_{j=1}^3 \frac{3(\theta - \bar\theta)}{2\theta}\left(\kappa(\theta)\theta_{x_j}\right)_{x_j} \\ & + \frac{3}{2} m \cdot \nabla_x \Phi + \sum_{i,j=1}^3 \frac{3(\theta-\bar\theta)}{2\theta}\left\{\mu(\theta)u_i\left[u_{ix_j} + u_{jx_i} - \frac{2}{3}\delta_{ij}\mathrm{div}_x u\right]\right\}_{x_j} \\ & - \int_{\mathbf{R}^3}\left(\frac{3\bar\theta}{2\theta}u \cdot \psi + \frac{3(\theta-\bar\theta)}{2\theta}\psi_4\right)(\xi \cdot \nabla_x \Theta) d\xi. \end{aligned} \tag{3.3}$$

By using (2.21), (2.22), (3.2), (3.3), Lemma 2.4 and Lemma 2.6, we have the following estimate on the thermodynamic variables (ρ, u, θ) and their derivatives with respect to the spatial variable x. Since we will use the system (2.15) to deduce the estimates, to avoid the appearance of the fifth order derivatives of (ρ, u, θ), we first consider these variables and their derivatives up to the third order.

Lemma 3.2. *(ρ, u, θ) satisfies*

$$\begin{aligned} & \sum_{|\alpha|\leq 3}\left(\left\|\partial_x^\alpha(\rho - \bar\rho, u, \theta - \bar\theta)\right\|^2 + \left\|\nabla_x\partial_x^\alpha\Phi\right\|^2 + \int_0^t \left\|\nabla_x \partial_x^\alpha(u, \theta)\right\|^2 d\tau\right) \\ & \quad \leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \sum_{|\alpha|+|\gamma|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ & + O(1)\varepsilon\sum_{|\alpha|\leq 4}\int_0^t \left\|\partial_x^\alpha(\rho - \bar\rho)\right\|^2 d\tau \\ & + O(1)\sum_{1\leq|\alpha|+\beta\leq 4, \beta\leq 1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau. \end{aligned} \tag{3.4}$$

Note that even though the last term does not have the small factor ε, its order of differentiation is at least one.

Proof. By integrating the entropy identity (3.3) with respect to t and x over $[0, t] \times \mathbf{R}^3$ and using Lemma 2.4 and Lemma 2.6, we have (3.4) when $\alpha = 0$.

For $1 \le |\alpha| \le 3$, we first rewrite the conservation laws (2.15) as

$$\begin{cases} \rho_t = -(\rho-\bar\rho)\mathrm{div}_x u - \nabla_x\rho\cdot u - \bar\rho\,\mathrm{div}_x u, \\ u_{it} + \sum_{j=1}^3 u_j u_{ix_j} + \frac{2}{3\rho}(\rho\theta)_{x_i} - \Phi_{x_i} = -\int_{\mathbf{R}^3}\frac{\psi_i(\xi\cdot\nabla_x\Theta)}{\rho}d\xi \\ \qquad +\frac{1}{\rho}\sum_{j=1}^3\left\{\mu(\theta)\left(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right\}_{x_j}, \quad i=1,2,3, \\ \theta_t + \sum_{j=1}^3\left(u_j\theta_{x_j}+\frac{2}{3}\theta u_{jx_j}\right) = -\int_{\mathbf{R}^3}\frac{\psi_4-\xi\cdot u}{\rho}(\xi\cdot\nabla_x\Theta)d\xi \\ \qquad +\frac{1}{\rho}\left\{\sum_{j=1}^3\left(\kappa(\theta)\theta_{x_j}\right)_{x_j} + \frac{1}{2}\mu(\theta)\sum_{i,j=1}^3\left(u_{ix_j}+u_{jx_i}\right)^2 - \frac{2}{3}\mu(\theta)(\mathrm{div}_x u)^2\right\}. \end{cases} \tag{3.5}$$

Equation (3.4) can be proved by using the argument similar to the one in [26] for the compressible Navier–Stokes equations with external forces because the system (3.5) has the main structure of the compressible Navier–Stokes equations. The only difference is to estimate the terms containing Θ and the term containing the potential function in the form of

$$I = -\int_0^t\int_{\mathbf{R}^3}\rho\partial_x^\alpha u\cdot\nabla_x\partial_x^\alpha\Phi dxd\tau.$$

Since the terms containing Θ can be estimated by using Lemma 2.6 in a standard way, for brevity, we only estimate I by using the Poisson equation $(1.1)_2$, the conservation laws (2.8) and Lemma 3.1 as follows.

$$\begin{aligned} I &= \int_0^t\int_{\mathbf{R}^3}\mathrm{div}_x\left(\rho\partial_x^\alpha u\right)\partial_x^\alpha\Phi dxd\tau \\ &= \int_0^t\int_{\mathbf{R}^3}\partial_x^\alpha\Big(\mathrm{div}_x(\rho u)\Big)\partial_x^\alpha\Phi dxd\tau \\ &\quad - \sum_{0<\alpha'\le\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\mathrm{div}_x\left(\partial_x^{\alpha'}\rho\partial_x^{\alpha-\alpha'}u\right)\partial_x^\alpha\Phi dxd\tau \\ &= -\int_0^t\int_{\mathbf{R}^3}\partial_x^\alpha\Delta_x\Phi\partial_x^\alpha\Phi dxd\tau \\ &\quad - \sum_{0<\alpha'\le\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\mathrm{div}_x\left(\partial_x^{\alpha'}\rho\partial_x^{\alpha-\alpha'}u\right)\partial_x^\alpha\Phi dxd\tau \\ &\ge \frac{1}{2}\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dx\Big|_0^t - O(1)\varepsilon\sum_{\alpha'\le\alpha}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^3dxd\tau. \end{aligned} \tag{3.6}$$

And this completes the proof of the lemma. □

By using the conservation laws (2.8), the estimates on the temporal derivatives of (ρ,u,θ) can be transferred to the spatial derivatives, cf. [28]. Thus, a direct consequence of (3.4) is the following corollary.

Corollary 3.1. *(ρ, u, θ) satisfies*

$$\sum_{|\alpha|+\beta\leq 3}\left(\left\|\partial_x^\alpha\partial_t^\beta(\rho-\bar{\rho},u,\theta-\bar{\theta})\right\|^2+\left\|\nabla_x\partial_x^\alpha\partial_t^\beta\Phi\right\|^2+\int_0^t\left\|\nabla_x\partial_x^\alpha\partial_t^\beta(u,\theta)\right\|^2d\tau\right)$$
$$\leq O(1)\mathcal{E}(f_0)^2+O(1)\sum_{1\leq|\alpha|+\beta\leq 4,\beta\leq 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx$$
$$+O(1)\varepsilon\sum_{|\alpha|+|\gamma|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau$$
$$+O(1)\sum_{|\alpha|\leq 4}\int_0^t\left\|\partial_x^\alpha(\rho-\bar{\rho})\right\|^2d\tau$$
$$+O(1)\sum_{1\leq|\alpha|+\beta\leq 4,\beta\leq 2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau. \tag{3.7}$$

Since the highest order derivatives on the macroscopic components can be obtained by the system (1.1) directly as shown later, we now turn to the estimates on the microscopic component $\mathbf{P}_1^{\bar{\mathbf{M}}}f$ satisfying

$$\left(\mathbf{P}_1^{\bar{\mathbf{M}}}f\right)_t+\mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi\cdot\nabla_x\mathbf{P}_0^{\bar{\mathbf{M}}}f\right)+\mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi\cdot\nabla_x\mathbf{P}_1^{\bar{\mathbf{M}}}f\right)$$
$$+\mathbf{P}_1^{\bar{\mathbf{M}}}\left(\nabla_x\Phi\cdot\nabla_\xi\mathbf{P}_0^{\bar{\mathbf{M}}}f\right)+\nabla_x\Phi\cdot\nabla_\xi\mathbf{P}_1^{\bar{\mathbf{M}}}f$$
$$=L_{\bar{\mathbf{M}}}\left(\mathbf{P}_1^{\bar{\mathbf{M}}}f\right)+Q\left(\mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}})+\mathbf{P}_1^{\bar{\mathbf{M}}}f,\mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}})+\mathbf{P}_1^{\bar{\mathbf{M}}}f\right). \tag{3.8}$$

The following lemma gives the lowest order estimate on the microscopic component $\mathbf{P}_1^{\bar{\mathbf{M}}}f$. And we will present its proof in the Appendix.

Lemma 3.3. *For $\mathbf{P}_1^{\bar{\mathbf{M}}}f$, we have*

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau$$
$$\leq O(1)\mathcal{E}(f_0)^2+O(1)\int_0^t\left(\|\nabla_x(u,\theta)\|^2+\varepsilon\|\nabla_x\rho\|^2\right)d\tau. \tag{3.9}$$

Since the derivative of the solution with respect to the velocity variable ξ appears in the system (1.1), we need to obtain the corresponding estimates on the microscopic component $\mathbf{P}_1^{\bar{\mathbf{M}}}f$. One way to achieve this is to transfer the estimates on the derivatives with respect to ξ into those on the derivatives with respect to the spatial and temporal variables as follows:

Lemma 3.4. *Denote $\Lambda=\{(\alpha,\beta,\gamma):\ |\alpha|+\beta+|\gamma|\leq 4,|\gamma|\geq 1\}$. We have*

$$\sum_{(\alpha,\beta,\gamma)\in\Lambda}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx+\sum_{(\alpha,\beta,\gamma)\in\Lambda}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau$$
$$\leq O(1)\mathcal{E}(f_0)^2+O(1)\sum_{|\alpha|+\beta\leq 3}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right|^2dx d\tau$$
$$+O(1)\sum_{|\alpha|+\beta\leq 4,\beta\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau. \tag{3.10}$$

Since the proof of Lemma 3.4 contains some tedious calculations, we also present it in the Appendix. Here, we only point out that the following estimates are used in the proof. For $|\alpha| + \beta \leq 4$, we have

$$\left\|\partial_x^\alpha \partial_t^\beta \rho\right\|^2 \leq O(1) \sum_{|\alpha'|+\beta' \leq 4, \beta' \leq \beta-1} \left\|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u)\right\|^2, \tag{3.11}$$

and

$$\begin{aligned}\left\|\partial_x^\alpha \partial_t^\beta \theta\right\|^2 \leq O(1) &\sum_{|\alpha'|+\beta' \leq 4, \beta' \leq \beta-1} \left\|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right\|^2 \\ &+ O(1) \sum_{|\alpha'|+\beta' \leq 4, \beta' \leq \beta-1} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\left|\partial_x^{\alpha'} \partial_t^{\beta'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx.\end{aligned} \tag{3.12}$$

To close the a priori estimate, the following estimate on the derivatives with respect to the spatial and temporal variables is crucial because all error terms on its right-hand side have the small factor ε. It can then be combined with the previous estimates to yield the desired result.

Lemma 3.5. *Under the a priori assumption (2.21), we have*

$$\begin{aligned}&\sum_{1 \leq |\alpha|+\beta \leq 4} \left(\int_{\mathbf{R}^3} \left(\left|\nabla_x \partial_x^\alpha \partial_t^\beta \Phi\right|^2 + \int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta f\right|^2}{\bar{\mathbf{M}}} d\xi \right) dx \right. \\ &\left. + \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \right) \\ &\leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \int_0^t \left(\|\rho - \bar{\rho}\|^2 + \sum_{|\alpha|+\beta \leq 3} \left\|\nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta)\right\|^2 \right) d\tau \\ &+ O(1)\varepsilon \sum_{|\alpha|+\beta+|\gamma| \leq 4, \beta \leq 3} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau.\end{aligned} \tag{3.13}$$

Again, we will prove Lemma 3.5 in the Appendix. Combining (3.1), (3.4), (3.7), (3.9)–(3.13) yields the following corollary.

Corollary 3.2. *Under the a priori assumption (2.21), we have*

$$\sum_{|\alpha|+\beta\le 4}\left\|\nabla_x\partial_x^\alpha\partial_t^\beta\Phi\right\|^2+\sum_{|\alpha|+\beta\le 3}\left(\left\|\partial_x^\alpha\partial_t^\beta(\rho-\bar{\rho},u,\theta-\bar{\theta})\right\|^2\right.$$
$$\left.+\int_0^t\left\|\nabla_x\partial_x^\alpha\partial_t^\beta(u,\theta)\right\|^2d\tau\right)$$
$$+\sum_{|\alpha|+\beta+|\gamma|\le 4}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma f\right|^2}{\bar{\mathbf{M}}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dxd\tau\right)$$
$$\le O(1)\mathcal{E}(f_0)^2+O(1)\varepsilon\sum_{|\alpha|\le 4}\int_0^t\left\|\partial_x^\alpha(\rho-\bar{\rho})\right\|^2d\tau. \tag{3.14}$$

To obtain the estimates on $\partial_x^\alpha(\rho-\bar{\rho})$, in the next lemma, it is standard to use the conservation laws as for the compressible Navier–Stokes equations considered in [20] so that we omit its proof for brevity.

Lemma 3.6. *Under the a priori assumption (2.21), we have*

$$\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau+\sum_{1\le|\alpha|\le 4}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau$$
$$\le O(1)N(0)^2+O(1)\sum_{|\alpha|\le 3}\int_{\mathbf{R}^3}\left|\left(\nabla_x\partial_x^\alpha u,\,\partial_x^\alpha(\rho-\bar{\rho})\right)\right|^2dx$$
$$+O(1)\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(u,\theta)|^2dxd\tau$$
$$+O(1)\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\nabla_x\partial_x^\alpha\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\mathbf{M}}d\xi dxd\tau. \tag{3.15}$$

By combining (3.7)–(3.12), (3.14) and (3.15), we obtain the following estimate:

$$\sum_{|\alpha|+\beta\le 4}\left(\left\|\partial_x^\alpha\partial_t^\beta(\rho-\bar{\rho},u,\theta-\bar{\theta})(t,x)\right\|^2+\left\|\nabla_x\partial_x^\alpha\partial_t^\beta\Phi(t,x)\right\|^2\right)$$
$$+\sum_{|\alpha|+\beta+|\gamma|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f(t,x,\xi)\right|^2}{\bar{\mathbf{M}}}d\xi dx$$
$$+\sum_{|\alpha|+\beta+|\gamma|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f(\tau,x,\xi)\right|^2}{\bar{\mathbf{M}}}d\xi dxd\tau$$
$$+\int_0^t\left(\|\rho-\bar{\rho}\|^2+\sum_{|\alpha|+\beta\le 3}\left\|\nabla_x\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2\right)d\tau$$
$$\le O(1)\mathcal{E}(f_0)^2, \tag{3.16}$$

which closes the a priori assumption (2.21) if we choose δ_0 to satisfy

$$\begin{cases}\mathcal{E}(f_0)\le\delta_0,\\ O(1)\delta_0^2<\varepsilon^2.\end{cases}$$

Therefore, this together with the local existence complete the proof of global existence of classical solutions to (1.1) and (1.2). Before concluding this section, we give a remark on the above energy estimates.

Remark 3.1. Compared to the estimates on the Boltzmann equation without forcing in [17, 23, 31], there is no uniform estimate on $\int_0^t \int_{\mathbf{R}^3} |\partial_t^\beta u|^2 dx d\tau$, $1 \le \beta \le 4$, for the Vlasov–Poisson–Boltzmann system because there is no uniform bound on $\int_0^t \int_{\mathbf{R}^3} |\nabla_x \Phi|^2 dx d\tau$. On the other hand, (3.16) gives the uniform space-time integrability of $|\rho - \bar{\rho}|^2$ which does not hold for the Boltzmann equation or even compressible Navier–Stokes equations without forcing.

4. Convergence Rate

In this section, we will prove the time decay estimate (1.7) by the method similar to the study on the Boltzmann equation with external force, cf. [29]. We will only consider the case when $N = 4$ because other cases when $N > 4$ can be proved similarly. Note that in the following discussion, we will use the properties of the solution given by (1.6) when $N = 4$ in Theorem 1.1.

For preparation, we state an inequality from [5] and some Sobolev inequalities in the next two lemmas respectively.

Lemma 4.1. *Let $f(t) \in C^1([t_0, \infty))$ satisfying $f(t) \ge 0$, $A = \int_{t_0}^\infty f(t)dt < +\infty$, and $f'(t) \le a(t)f(t)$, for all $t \ge t_0$. If $a(t) \ge 0$, and $B = \int_{t_0}^\infty a(t)dt < +\infty$, then*

$$f(t) \le \frac{(t_0 f(t_0) + 1)\exp(A + B) - 1}{t}, \quad \forall t \ge t_0.$$

Lemma 4.2. *Let $f(x) \in H^2(\mathbf{R}^3)$, $g(x) \in H^1(\mathbf{R}^3)$, $h(x) \in L^2(\mathbf{R}^3)$. Then for any $\delta > 0$, we have*

$$\begin{cases} \int_{\mathbf{R}^3} f(x) \cdot g(x) \cdot h(x) dx \le \delta \|\nabla_x f(x)\|^2 + C_\delta \|g(x)\|_1^2 \|h(x)\|^2, \\ \int_{\mathbf{R}^3} f(x) \cdot g(x) \cdot h(x) dx \le \delta \|g(x)\|^2 + C_\delta \|\nabla_x f(x)\|_1^2 \|h(x)\|^2. \end{cases} \tag{4.1}$$

Here C_δ is some positive constant depending only on δ.

By using Lemma 4.1, the convergence rate can be obtained by constructing a functional $H(t)$ which is equivalent to

$$\sum_{|\alpha|+\beta+|\gamma|\le 2} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma f(t,x,\xi)\right|^2}{\bar{\mathbf{M}}} d\xi dx + \sum_{|\alpha|+\beta\le 2} \int_{\mathbf{R}^3} \left|D_x^2 \partial_x^\alpha \partial_t^\beta \Phi(t,x)\right|^2 dx,$$

and satisfies

$$\frac{dH(t)}{dt} \le O(1)\phi(t)H(t), \tag{4.2}$$

for some non-negative function $\phi(t) \in L^1(\mathbf{R}^+)$. The existence of the functional $H(t)$ follows from the following lemmas. Since the proofs of these lemmas are similar to those of the corresponding lemmas either in the above section or in [29], we will only present the main estimates for brevity. Compared to the analysis in [29], the main difference in the functional $H(t)$ is that it consists of the differentiations with respect to x at least once here while it consists of the differentiations with respect to t at least once in [29]. And the reason is that the potential force considered in [29] is stationary so that the differentiation with respect to t is zero, and the potential force here is coupled with the solution through the Poisson equation which gives sufficient integrability in space and time for spatial differentiations.

Firstly, we consider the estimate on the macroscopic component $\nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta)$ for $|\alpha| + \beta \le 1$.

Lemma 4.3. *Set*

$$\begin{cases} H_0(t) = \sum_{|\alpha|+\beta\le 1} \int_{\mathbf{R}^3} \left[\left(\frac{\bar\theta}{3\bar\rho} \left|\nabla_x \partial_x^\alpha \partial_t^\beta \rho\right|^2 + \frac{\rho}{2} \left|\nabla_x \partial_x^\alpha \partial_t^\beta u\right|^2 + \frac{\rho}{2\bar\theta} \left|\nabla_x \partial_x^\alpha \partial_t^\beta \theta\right|^2 \right) \right. \\ \qquad \left. + \frac{1}{2} \left| D_x^2 \partial_x^\alpha \partial_t^\beta \Phi \right|^2 \right] (t,x) dx, \\ \bar H_0(t) = \sum_{|\alpha|+\beta\le 1} \left\| D_x^2 \partial_x^\alpha \partial_t^\beta (u,\theta)(t) \right\|^2, \\ \phi(t) = \sum_{|\alpha|+\beta\le 3} \left\| \nabla_x \partial_x^\alpha \partial_t^\beta u(t) \right\|^2 + \sum_{1\le|\alpha|+\beta\le 4} \left\| \partial_x^\alpha \partial_t^\beta (\rho,\theta)(t) \right\|^2 + \|(\rho-\bar\rho)(t)\|^2 \\ \qquad + \sum_{|\alpha|+\beta+|\gamma|\le 4} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f(t,x,\xi) \right|^2}{\bar{\mathbf{M}}} d\xi dx. \end{cases} \tag{4.3}$$

Then, there exists a positive constant d_0 such that

$$\begin{aligned} \frac{d}{dt} H_0(t) + d_0 \bar H_0(t) \le\ & O(1)\phi(t) \sum_{|\alpha|+\beta\le 2} \left(\left\| \nabla_x \partial_x^\alpha \partial_t^\beta (\rho,u,\theta) \right\|^2 + \left\| D_x^2 \partial_x^\alpha \partial_t^\beta \Phi \right\|^2 \right) \\ & + O(1) \sum_{|\alpha|+\beta\le 2} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx \\ & + O(1)\varepsilon \sum_{|\alpha|+\beta+|\gamma|\le 2, |\gamma|\ge 1} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx \\ & + \varepsilon \sum_{|\alpha|+\beta\le 1} \left(\left\| D_x^2 \partial_x^\alpha \partial_t^\beta \rho \right\|^2 + \left\| D_x^3 \partial_x^\alpha \partial_t^\beta \Phi \right\|^2 \right). \end{aligned} \tag{4.4}$$

Here and in what follows, for $k \in \mathbf{Z}_+$, $\left|D_x^k g(x)\right|^2 := \sum_{|\alpha|=k} \left|\partial_x^\alpha g(x)\right|^2$, $\left\|D_x^k g\right\|^2 := \sum_{|\alpha|=k} \left\|\partial_x^\alpha g\right\|^2$, etc.

Proof. The proof of (4.4) is similar to that of Lemma 3.1 in [29]. The only difference is the estimation on the term containing Φ which is

$$J = -\sum_{l=1}^{3} \int_{\mathbf{R}^3} \operatorname{div}_x \left(\rho \partial_x^\alpha \partial_t^\beta \partial_{x_l} u \right) \partial_x^\alpha \partial_t^\beta \partial_{x_l} \Phi dx.$$

By Lemma 2.1 and Lemma 4.2, for $|\alpha| + \beta \leq 1$, we have

$$
\begin{aligned}
J \leq & -\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3} \left|D_x^2 \partial_x^\alpha \partial_t^\beta \Phi\right|^2 dx + \varepsilon \sum_{|\alpha'|+\beta'\leq 1} \left(\left\|D_x^2 \partial_x^{\alpha'} \partial_t^{\beta'} \rho\right\|^2 + \left\|D_x^3 \partial_x^{\alpha'} \partial_t^{\beta'} \Phi\right\|^2\right) \\
& + O(1)\phi(t) \sum_{|\alpha'|+\beta'\leq 2} \left(\left\|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right\|^2 + \left\|D_x^2 \partial_x^{\alpha'} \partial_t^{\beta'} \Phi\right\|^2\right). \qquad (4.5)
\end{aligned}
$$

Note that we have used the uniform space-time integrability of $|\rho - \bar{\rho}|^2$, and this is why we should include $\|\rho - \bar{\rho}\|^2$ in the definition of $\phi(t)$ in (4.3).

With (4.5), the argument used in the proof of Lemma 3.1 in [29] shows that there exists a positive constant d_0 such that

$$
\begin{aligned}
\frac{d}{dt} H_0(t) + d_0 \bar{H}_0(t) \leq & \ \varepsilon \sum_{|\alpha|+\beta\leq 1} \left(\left\|D_x^2 \partial_x^\alpha \partial_t^\beta \rho\right\|^2 + \left\|D_x^3 \partial_x^\alpha \partial_t^\beta \Phi\right\|^2\right) \\
& + O(1)\phi(t) \sum_{|\alpha|+\beta\leq 2} \left(\left\|\nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta)\right\|^2 + \left\|D_x^2 \partial_x^\alpha \partial_t^\beta \Phi\right\|^2\right) \\
& + O(1) \sum_{|\alpha|+\beta\leq 1} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}_-} \left(\left|\partial_x^\alpha \partial_t^\beta \left(\frac{\psi(\xi\cdot\nabla_x\Theta)}{\rho}\right)\right|^2\right. \\
& \left. + \left|\partial_x^\alpha \partial_t^\beta \left(\frac{(\psi_4 - \xi\cdot u)(\xi\cdot\nabla_x\Theta)}{\rho}\right)\right|^2\right) dx. \qquad (4.6)
\end{aligned}
$$

Following the argument for Lemma 2.6, the last term on the right-hand side of (4.6) can be estimated by using Lemma 2.1, the a priori estimate (2.21), (2.32) and Corollary 2.1 as

$$
\begin{aligned}
& \sum_{|\alpha|+\beta\leq 1} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}_-} \left(\left|\partial_x^\alpha \partial_t^\beta \left(\frac{\psi(\xi\cdot\nabla_x\Theta)}{\rho}\right)\right|^2 + \left|\partial_x^\alpha \partial_t^\beta \left(\frac{(\psi_4 - \xi\cdot u)(\xi\cdot\nabla_x\Theta)}{\rho}\right)\right|^2\right) dx \\
& \leq O(1)\phi(t) \sum_{|\alpha|+\beta\leq 2} \left(\left\|\nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta)\right\|^2 + \left\|D_x^2 \partial_x^\alpha \partial_t^\beta \Phi\right\|^2\right) \\
& + O(1) \sum_{|\alpha|+\beta\leq 2} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx \\
& + O(1)\varepsilon \sum_{|\alpha|+\beta+|\gamma|\leq 2, |\gamma|\geq 1} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx. \qquad (4.7)
\end{aligned}
$$

Equations (4.6) and (4.7) imply (4.4) and this completes the proof of the lemma. □

Notice that $\phi(t) \in L^1(\mathbf{R}^+)$ by using (1.6) when $N = 4$. In what follows, we will construct some functionals to control the terms on the right-hand side of (4.4) except the one with factor $\phi(t)$. To be precise, the second term on the right-hand side in (4.4) will be estimated by $H_2(t)$, the third term by $H_1(t)$ and the last term in Lemma 4.6 as follows.

The next lemma concerns the estimates on the differentiations on microscopic component governed by Eq. (3.8). It shows that the differentiation with respect to ξ can be reduced to the differentiations with respect to x and t.

Lemma 4.4. *For each (α, β, γ) satisfying $|\alpha| + \beta + |\gamma| \le 2$ and $|\gamma| \ge 1$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx+\frac{\sigma}{7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&\le O(1)\left\|D_x^2\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2+O(1)\phi(t)\sum_{|\alpha'|+\beta'\le 2}\left(\left\|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2\right.\\
&\left.+\left\|D_x^2\partial_x^{\alpha'}\partial_t^{\beta'}\Phi\right\|^2\right)\\
&+O(1)\varepsilon\sum_{|\alpha'|+\beta'+|\gamma'|\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\\
&+O(1)\sum_{\gamma'<\gamma,|\gamma'|=|\gamma|-1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|D_x^2\partial_x^\alpha\partial_t^\beta\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\\
&+O(1)\sum_{\gamma'<\gamma}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\\
&+O(1)\sum_{|\alpha'|+\beta'\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx. \qquad (4.8)
\end{aligned}$$

Notice that the terms on the left-hand side of (4.8) have differentiation with respect to x of at least order one. To obtain (4.8), we need to use

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx\le O(1)\mathcal{E}(f_0)^2, \qquad (4.9)$$

for $|\alpha|+\beta+|\gamma|\le 3$ which comes directly from the Vlasov–Poisson–Boltzmann equation and the estimate (1.6) when $N=4$ by using the Cauchy-Schwarz inequality.

With (4.9), the proof of Lemma 4.4 is similar to Lemma 3.4 which is based on Lemma 2.5 and the a priori estimate (2.21). Thus, we omit the details for brevity.

Set

$$\begin{cases}
H_1(t)=\displaystyle\sum_{|\alpha|+\beta+|\gamma|\le 2,|\gamma|\ge 1}C_{\alpha,\beta,\gamma}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx,\\
\bar{H}_1(t)=\displaystyle\sum_{|\alpha|+\beta+|\gamma|\le 2,|\gamma|\ge 1}C_{\alpha,\beta,\gamma}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx,
\end{cases} \qquad (4.10)$$

where $C_{\alpha,\beta,\gamma}>0$ are constants chosen later. Lemma 4.4 implies the following estimate on $H_1(t)$.

Corollary 4.1. *By suitably choosing the constants $C_{\alpha,\beta,\gamma}>0$, there exists a positive constant d_1 such that*

$$\begin{aligned}
\frac{d}{dt}H_1(t)+d_1\bar{H}_1(t)&\le O(1)\sum_{|\alpha|+\beta\le 1}\left\|D_x^2\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2\\
&+O(1)\phi(t)\sum_{|\alpha|+\beta\le 2}\left(\left\|\nabla_x\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2+\left\|D_x^2\partial_x^\alpha\partial_t^\beta\Phi\right\|^2\right)\\
&+O(1)\sum_{|\alpha|+\beta\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx. \qquad (4.11)
\end{aligned}$$

Now set

$$\begin{cases} H_2(t) = \sum_{|\alpha|+\beta\leq 2} \int_{\mathbf{R}^3} \Bigg[\frac{1}{2R\bar{\theta}} \left(\left| D_x^2 \partial_x^\alpha \partial_t^\beta \Phi(t,x) \right|^2 + (\rho(t,x) - \bar{\rho}) \left| \nabla_x \partial_t^{\beta-1} \Phi(t,x) \right|^2 \right) \\ \qquad + \frac{1}{2} \int_{\mathbf{R}^3} \frac{\left| \nabla_x \partial_x^\alpha \partial_t^\beta f(t,x,\xi) \right|^2}{\bar{\mathbf{M}}} d\xi \Bigg] dx, \\ \bar{H}_2(t) = \sum_{|\alpha|+\beta\leq 2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \nabla_x \partial_x^\alpha \partial_t^\beta f(t,x,\xi) \right|^2}{\bar{\mathbf{M}}} d\xi dx. \end{cases} \tag{4.12}$$

For $H_2(t)$, by mimicking the argument used in the proof of Lemma 3.5, we have

Lemma 4.5. *There exists a positive constant d_2 such that*

$$\begin{aligned} \frac{d}{dt} H_2(t) + d_2 \bar{H}_2(t) \leq & O(1)\phi(t) \sum_{|\alpha|+\beta\leq 2} \left(\left\| \nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta) \right\|^2 + \left\| D_x^2 \partial_x^\alpha \partial_t^\beta \Phi \right\|^2 \right) \\ & + O(1)\varepsilon \sum_{|\alpha|+\beta+|\gamma|\leq 2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx. \end{aligned} \tag{4.13}$$

Finally, we have the following estimate on the last term in (4.4) whose proof is similar to the one for Lemma 3.6.

Lemma 4.6. *There exists a positive constant $d_3 > 0$ such that*

$$\begin{aligned} & \sum_{|\alpha|+\beta\leq 1} \left(\left\| D_x^2 \partial_x^\alpha \partial_t^\beta \rho \right\|^2 + \left\| D_x^3 \partial_x^\alpha \partial_t^\beta \Phi \right\|^2 \right) \leq O(1)\phi(t) \sum_{|\alpha|+\beta\leq 2} \left\| \nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta) \right\|^2 \\ & + \frac{d}{dt} \left(d_3 \sum_{|\alpha|+\beta\leq 1} \int_{\mathbf{R}^3} \nabla_x \partial_x^\alpha \partial_t^\beta \rho \cdot \nabla_x \partial_x^\alpha \partial_t^\beta div_x u dx \right) + O(1) \sum_{|\alpha|+\beta\leq 1} \left\| D_x^2 \partial_x^\alpha \partial_t^\beta (u, \theta) \right\|^2 \\ & + O(1) \sum_{|\alpha|+\beta\leq 2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx. \end{aligned} \tag{4.14}$$

With the above estimates, firstly, by (4.13) and (4.4), there exists a positive constant d_4 such that

$$\begin{aligned} & \frac{d}{dt} \left(\lambda_1 H_2(t) + H_0(t) \right) + d_4 \left(\bar{H}_0(t) + \bar{H}_2(t) \right) \\ & \leq O(1)\varepsilon \sum_{|\alpha|+\beta\leq 1} \left(\left\| D_x^2 \partial_x^\alpha \partial_t^\beta \rho \right\|^2 + \left\| D_x^3 \partial_x^\alpha \partial_t^\beta \Phi \right\|^2 \right) \\ & + O(1)\phi(t) \sum_{|\alpha|+\beta\leq 2} \left(\left\| \nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta) \right\|^2 + \left\| D_x^2 \partial_x^\alpha \partial_t^\beta \Phi \right\|^2 \right) \\ & + O(1)\varepsilon \sum_{|\alpha|+\beta+|\gamma|\leq 2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx, \end{aligned} \tag{4.15}$$

where $\lambda_1 > 0$ is a sufficiently large constant.

Secondly, (4.15) and (4.14) imply that there exists a positive constant d_5 such that

$$
\begin{aligned}
&\frac{d}{dt}\left[\lambda_2\Big(\lambda_1 H_2(t)+H_0(t)\Big)-d_3\sum_{|\alpha|+\beta\le 1}\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\partial_t^\beta\rho\cdot\nabla_x\partial_x^\alpha\partial_t^\beta \mathrm{div}_x u\,dx\right]\\
&+d_5\left[\Big(\bar{H}_0(t)+\bar{H}_2(t)\Big)+\sum_{|\alpha|+\beta\le 1}\left(\left\|D_x^2\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2+\left\|D_x^3\partial_x^\alpha\partial_t^\beta\Phi\right\|^2\right)\right]\\
&\le O(1)\phi(t)\sum_{|\alpha|+\beta\le 2}\left(\left\|\nabla_x\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2+\left\|D_x^2\partial_x^\alpha\partial_t^\beta\Phi\right\|^2\right)\\
&+O(1)\varepsilon\sum_{|\alpha|+\beta+|\gamma|\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx,
\end{aligned}
\tag{4.16}
$$

where the constant $\lambda_2>0$ is also sufficiently large.

Finally, define

$$
\begin{cases}
H(t)=\lambda_3\Big[\lambda_2\Big(\lambda_1 H_2(t)+H_0(t)\Big)\\
\qquad\qquad -d_3\displaystyle\sum_{|\alpha|+\beta\le 1}\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\partial_t^\beta\rho\cdot\nabla_x\partial_x^\alpha\partial_t^\beta\mathrm{div}_x u\,dx\Big]+H_1(t),\\
\bar{H}(t)=\bar{H}_0(t)+\bar{H}_1(t)+\bar{H}_2(t)+\displaystyle\sum_{|\alpha|+\beta\le 1}\left(\left\|D_x^2\partial_x^\alpha\partial_t^\beta\rho\right\|^2+\left\|D_x^3\partial_x^\alpha\partial_t^\beta\Phi\right\|^2\right).
\end{cases}
\tag{4.17}
$$

Then by choosing the constant λ_3 sufficiently large, and using (4.11) and (4.16), we have

$$
\begin{aligned}
\frac{d}{dt}H(t)+d_6\bar{H}(t)&\le O(1)\phi(t)\sum_{|\alpha|+\beta\le 2}\left(\left\|\nabla_x\partial_x^\alpha\partial_t^\beta(\rho,u,\theta)\right\|^2+\left\|D_x^2\partial_x^\alpha\partial_t^\beta\Phi\right\|^2\right)\\
&\le O(1)\phi(t)H(t),
\end{aligned}
\tag{4.18}
$$

for some positive constant d_6.

Since when $\lambda_1>0,\ \lambda_2>0$ and $\lambda_3>0$ are sufficiently large,

$$
\begin{aligned}
H(t)\sim&\sum_{|\alpha|+\beta+|\gamma|\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma f(t,x,\xi)\right|^2}{\bar{\mathbf{M}}}d\xi dx\\
&+\sum_{|\alpha|+\beta\le 2}\int_{\mathbf{R}^3}\left|D_x^2\partial_x^\alpha\partial_t^\beta\Phi(t,x)\right|^2dx,
\end{aligned}
\tag{4.19}
$$

(4.18), (4.19) and Lemma 4.1 give

$$
\begin{aligned}
&\sum_{|\alpha|+\beta+|\gamma|\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma f(t,x,\xi)\right|^2}{\bar{\mathbf{M}}}d\xi dx+\sum_{|\alpha|+\beta\le 2}\int_{\mathbf{R}^3}\left|D_x^2\partial_x^\alpha\partial_t^\beta\Phi(t,x)\right|^2dx\\
&\le O(1)(1+t)^{-1}.
\end{aligned}
\tag{4.20}
$$

With (4.20), the time decay estimate in (1.7) follows from the Sobolev inequality.

5. Appendix

In the appendix, we present the proofs of Lemmas 3.3–3.5 in the following subsections respectively.

5.1. The proof of Lemma 3.3. Multiplying (3.8) by $\frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f}{\bar{\mathbf{M}}}$ and integrating the equation over $[0, t] \times \mathbf{R}^3 \times \mathbf{R}^3$ give

$$
\begin{aligned}
\frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi \bigg|_0^t = & -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f \ \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \nabla_x \mathbf{P}_0^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f \ \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_0^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f \ \nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_1^{\bar{\mathbf{M}}} f}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& +\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f \ L_{\bar{\mathbf{M}}}\left(\mathbf{P}_1^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& +\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f \ Q\left(\mathbf{P}_1^{\bar{\mathbf{M}}} f + \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}}), \mathbf{P}_1^{\bar{\mathbf{M}}} f + \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
:= & \sum_{j=1}^{5} I_j.
\end{aligned} \tag{5.1}
$$

I_j, $j = 1, 2, \ldots, 5$, can be estimated term by term as follows. Firstly, the a priori estimate (2.21), Lemma 2.3 and Lemma 2.5 give

$$
\begin{aligned}
I_3 &= -\frac{1}{2R\bar{\theta}} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\xi \cdot \nabla_x \Phi \left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&\le O(1)\varepsilon \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau,
\end{aligned} \tag{5.2}
$$

$$
I_4 \le -\sigma \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau, \tag{5.3}
$$

and

$$
I_5 \le \frac{2\sigma}{5} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau + O(1)\varepsilon \int_0^t \int_{\mathbf{R}^3} |\nabla_x(\rho, u, \theta)|^2 dx d\tau. \tag{5.4}
$$

Secondly, since

$$
\begin{aligned}
\mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \nabla_x \mathbf{P}_0^{\bar{\mathbf{M}}} f\right) &= \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \nabla_x \mathbf{P}_0^{\bar{\mathbf{M}}} \mathbf{M}\right) = \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \mathbf{P}_0^{\bar{\mathbf{M}}}\left(\nabla_x \mathbf{M}\right)\right) \\
&= \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \frac{\nabla_x \rho}{\rho} \mathbf{P}_0^{\bar{\mathbf{M}}} \mathbf{M}\right) + \rho \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \mathbf{P}_0^{\bar{\mathbf{M}}}\left(\nabla_x \mathbf{M}_{[1,u,\theta]}\right)\right) \\
&= O(1)|\nabla_x \rho|\, |(u, \theta - \bar{\theta})| + O(1)|\nabla_x(u, \theta)|,
\end{aligned}
$$

we have

$$
\begin{aligned}
I_1 \leq\ & \frac{\sigma}{5}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& + O(1)\int_0^t \int_{\mathbf{R}^3} \left(|\nabla_x \rho|^2 \ \ |(u, \theta-\bar{\theta})|^2 + |\nabla_x (u,\theta)|^2\right) dx d\tau \\
\leq\ & \frac{\sigma}{5}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& + O(1)\int_0^t \int_{\mathbf{R}^3} \left(\varepsilon\, |\nabla_x \rho|^2 + |\nabla_x (u,\theta)|^2\right) dx d\tau. \qquad (5.5)
\end{aligned}
$$

Finally, by Lemma 2.1 and (2.22), we have

$$
\begin{aligned}
I_2 =\ & -\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\mathbf{P}_1^{\bar{\mathbf{M}}} f\ \nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}})}{\bar{\mathbf{M}}} dx d\xi d\tau \\
\leq\ & \frac{\sigma}{5}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& + O(1)\int_0^t \int_{\mathbf{R}^3} |\nabla_x \Phi|^2 |(\rho-\bar{\rho}, u, \theta-\bar{\theta})|^2 dx d\tau \\
\leq\ & \frac{\sigma}{5}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& + O(1)\varepsilon \int_0^t \int_{\mathbf{R}^3} |\nabla_x (\rho, u, \theta)|^2\, dx d\tau. \qquad (5.6)
\end{aligned}
$$

Combining (5.2)–(5.6) yields (3.9) and completes the proof of the lemma.

5.2. The proof of Lemma 3.4. By applying $\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma$ ($|\alpha|+\beta+|\gamma| \leq 4,\ |\gamma| \geq 1$) to (3.8) and integrating its product with $\frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f}{\bar{\mathbf{M}}}$ over $[0,t] \times \mathbf{R}^3 \times \mathbf{R}^3$, we have

$$
\begin{aligned}
& \frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} dx d\xi \Bigg|_0^t \\
&= -\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\ \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \nabla_x \mathbf{P}_0^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&\quad -\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\ \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}}\left(\xi \cdot \nabla_x \mathbf{P}_1^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_0^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&- \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_1^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&+ \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \left(L_{\bar{\mathbf{M}}} \left(\mathbf{P}_1^{\bar{\mathbf{M}}} f\right)\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&+ \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma Q\left(\mathbf{P}_1^{\bar{\mathbf{M}}} f + \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}}), \mathbf{P}_1^{\bar{\mathbf{M}}} f + \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&:= \sum_{j=6}^{11} I_j. \qquad (5.7)
\end{aligned}
$$

Since $\bar{\mathbf{M}}$ depends only on ξ, we have

$$
\begin{cases}
\mathbf{P}_0^{\bar{\mathbf{M}}} f = \mathbf{P}_0^{\bar{\mathbf{M}}} \mathbf{M}. \quad \partial_x^\alpha \partial_t^\beta \mathbf{P}_i^{\bar{\mathbf{M}}} f = \mathbf{P}_i^{\bar{\mathbf{M}}} \left(\partial_x^\alpha \partial_t^\beta f\right), \quad i = 0, 1, \\
\mathbf{P}_1^{\bar{\mathbf{M}}} \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_0^{\bar{\mathbf{M}}} f\right) = \mathbf{P}_1^{\bar{\mathbf{M}}} \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_0^{\bar{\mathbf{M}}} (\mathbf{M} - \bar{\mathbf{M}})\right), \\
\partial_\xi^\gamma \left(\mathbf{P}_1^{\bar{\mathbf{M}}} \left(\xi \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right)\right) = \sum_{0<\gamma'\le\gamma} C_\gamma^{\gamma'} \partial_\xi^{\gamma'}(\xi) \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^{\gamma-\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f \\
\qquad - \sum_{j=0}^{4} \left\langle \xi \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f, \chi_j^{\bar{\mathbf{M}}} \right\rangle_{\bar{\mathbf{M}}} \partial_\xi^\gamma \chi_j^{\bar{\mathbf{M}}}.
\end{cases}
$$

By using these identities, for $|\gamma| \ge 1$, we have

$$
\begin{aligned}
I_6 &= -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} \left(\xi \cdot \mathbf{P}_0^{\bar{\mathbf{M}}} \left(\nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{M}\right)\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&\le \frac{\sigma}{7} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\
&\quad + O(1)\varepsilon \sum_{|\alpha'|+\beta'\le 3} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right|^2 dx d\tau \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha \partial_t^\beta (\rho, u, \theta)\right|^2 dx d\tau, \qquad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
I_7 &= -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} \left(\xi \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&= -\sum_{0<\gamma'\le\gamma} C_\gamma^{\gamma'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \partial_\xi^{\gamma'}(\xi) \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \partial_\xi^{\gamma-\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f}{\bar{\mathbf{M}}} d\xi dx d\tau \\
&\quad + \sum_{j=0}^{4} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f \ \left\langle \xi \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f, \chi_j^{\bar{\mathbf{M}}} \right\rangle_{\bar{\mathbf{M}}} \partial_\xi^\gamma \chi_j^{\bar{\mathbf{M}}}}{\bar{\mathbf{M}}} d\xi dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\sigma}{7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&\quad+O(1)\sum_{\gamma'<\gamma,|\gamma'|=|\gamma|-1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_x\partial_x^\alpha\partial_t^\beta\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
I_8 &= -\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\ \partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}\left(\nabla_x\Phi\cdot\nabla_\xi\mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M}-\bar{\mathbf{M}})\right)}{\bar{\mathbf{M}}}dx d\xi d\tau\\
&\leq \frac{\sigma}{7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&\quad+O(1)\sum_{(\alpha'+\alpha'',\beta'+\beta'')\leq(\alpha,\beta)}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}\Phi\right|^2\left|\partial_x^{\alpha''}\partial_t^{\beta''}(\rho-\bar{\rho},u,\theta-\bar{\theta})\right|^2dx d\tau\\
&\leq \frac{\sigma}{7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&\quad+O(1)\varepsilon\sum_{|\alpha'|+\beta'\leq 3}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right|^2dx d\tau.
\end{aligned}
\tag{5.10}
$$

Then, from Lemma 2.1, Lemma 3.1 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
I_9 &= -\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\ \nabla_x\Phi\cdot\nabla_\xi\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f}{\bar{\mathbf{M}}}dx d\xi d\tau\\
&\quad-\sum_{(\alpha',\beta')<(\alpha,\beta)}C_{\alpha,\beta}^{\alpha',\beta'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\ \nabla_x\partial_x^{\alpha-\alpha'}\partial_t^{\beta-\beta'}\Phi\cdot\nabla_\xi\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f}{\bar{\mathbf{M}}}dx d\xi d\tau\\
&\leq \left(\frac{\sigma}{7}+O(1)\varepsilon\right)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau\\
&\quad+O(1)\varepsilon\sum_{(\alpha',\beta')<(\alpha,\beta)}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\nabla_\xi\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dx d\tau.
\end{aligned}
\tag{5.11}
$$

Finally, since

$$
\begin{aligned}
\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma L_{\bar{\mathbf{M}}}\left(\mathbf{P}_1^{\bar{\mathbf{M}}}f\right) &= L_{\bar{\mathbf{M}}}\left(\partial_x^\alpha\partial_t^\beta\partial_\xi^\gamma\mathbf{P}_1^{\bar{\mathbf{M}}}f\right)\\
&\quad+2\sum_{0<\gamma'\leq\gamma}C_\gamma^{\gamma'}Q\left(\partial_\xi^{\gamma'}\bar{\mathbf{M}},\partial_x^\alpha\partial_t^\beta\partial_\xi^{\gamma-\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right),
\end{aligned}
$$

by using Lemma 2.2, Lemma 2.3 and Lemma 2.5, we have

$$I_{10} \le -\frac{6\sigma}{7}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau + O(1)\sum_{\gamma'<\gamma}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau, \tag{5.12}$$

and

$$\begin{aligned} I_{11} \le{}& \frac{\sigma}{7}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ &+ O(1)\varepsilon \sum_{|\alpha'|+\beta'+|\gamma'|<4,\,\beta'\le\beta}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha'} \partial_t^{\beta'} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ &+ O(1)\varepsilon \sum_{|\alpha'|+\beta'\le 3}\int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'}(\rho,u,\theta)\right|^2 dx d\tau. \end{aligned} \tag{5.13}$$

Substituting (5.8)–(5.13) into (5.7) gives

$$\begin{aligned} &\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx + \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^\alpha \partial_t^\beta \partial_\xi^\gamma \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ &\le O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \sum_{|\alpha'|+\beta'\le 3}\int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'}(\rho,u,\theta)\right|^2 dx d\tau \\ &+O(1)\int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha} \partial_t^{\beta}(\rho,u,\theta)\right|^2 dx d\tau \\ &+O(1)\varepsilon \sum_{|\alpha'|+\beta'+|\gamma'|\le 4,\,\beta'\le\beta}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha'} \partial_t^{\beta'} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ &+O(1)\sum_{\gamma'<\gamma,\,|\gamma'|=|\gamma|-1}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\nabla_x\partial_x^{\alpha} \partial_t^{\beta} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ &+O(1)\sum_{\gamma'<\gamma}\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha} \partial_t^{\beta} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\ &+O(1)\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\nabla_x\partial_x^{\alpha} \partial_t^{\beta} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau. \end{aligned} \tag{5.14}$$

Thus, (5.14) implies (3.10) and then this completes the proof of the lemma.

5.3. The proof of Lemma 3.5. Set $g(t,x,\xi) = f(t,x,\xi) - \bar{\mathbf{M}}(\xi)$ which satisfies

$$g_t + \xi\cdot\nabla_x g + \nabla_x\Phi\cdot\nabla_\xi g + \nabla_x\Phi\cdot\nabla_\xi \bar{\mathbf{M}} = L_{\bar{\mathbf{M}}} g + Q(g,g). \tag{5.15}$$

Applying $\partial_x^\alpha \partial_t^\beta$ $(1 \le |\alpha| + \beta \le 4)$ to (5.15) and integrating its product with $\frac{\partial_x^\alpha \partial_t^\beta g}{\bar{\mathbf{M}}}$ over $[0, t] \times \mathbf{R}^3 \times \mathbf{R}^3$ yield

$$
\begin{aligned}
\frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\left|\partial_x^\alpha \partial_t^\beta g\right|^2}{\bar{\mathbf{M}}} dx d\xi \Big|_0^t = & -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta g \ \partial_x^\alpha \partial_t^\beta (\nabla_x \Phi \cdot \nabla_\xi g)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta g \ \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \cdot \nabla_\xi \bar{\mathbf{M}}}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& +\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta g \ L_{\bar{\mathbf{M}}}\left(\partial_x^\alpha \partial_t^\beta g\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
& +\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta g \ \partial_x^\alpha \partial_t^\beta Q(g,g)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
:= & \sum_{j=12}^{15} I_j.
\end{aligned} \tag{5.16}
$$

For $|\alpha| + \beta \ge 1$, since

$$
\mathbf{P}_1^{\bar{\mathbf{M}}} \left(\partial_x^\alpha \partial_t^\beta g\right) = \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f,
$$

by Lemma 2.5, the conservation laws (2.8) and the Poisson equation $(1.1)_2$, we have

$$
\begin{aligned}
I_{13} &= \frac{1}{R\bar{\theta}} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \partial_x^\alpha \partial_t^\beta g \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \cdot \xi d\xi dx d\tau \\
&= \frac{1}{R\bar{\theta}} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \partial_x^\alpha \partial_t^\beta (\rho u) \cdot \nabla_x \partial_x^\alpha \partial_t^\beta \Phi dx d\tau \\
&= -\frac{1}{2R\bar{\theta}} \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha \partial_t^\beta \Phi\right|^2 dx \Big|_0^t,
\end{aligned} \tag{5.17}
$$

$$
\begin{aligned}
I_{14} &= \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f \ L_{\bar{\mathbf{M}}}\left(\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&\le -\sigma \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau,
\end{aligned} \tag{5.18}
$$

and

$$
\begin{aligned}
I_{15} \le & \frac{\sigma}{3} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\
& + O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 4, \beta' \le \beta} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left|\partial_x^{\alpha'} \partial_t^{\beta'} \mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\
& + O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 3} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta)\right|^2 dx d\tau.
\end{aligned} \tag{5.19}
$$

The following estimation on I_{12} is more subtle. First of all, since

$$
g = \mathbf{P}_0^{\bar{\mathbf{M}}}(\mathbf{M} - \bar{\mathbf{M}}) + \mathbf{P}_1^{\bar{\mathbf{M}}} f = \mathbf{M} - \bar{\mathbf{M}} + \mathbf{P}_1^{\bar{\mathbf{M}}}(f - \mathbf{M}),
$$

we have

$$
\begin{aligned}
I_{12} = &-\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \mathbf{M}\ \partial_x^\alpha \partial_t^\beta \left(\nabla_x \Phi \cdot \nabla_\xi (\mathbf{M}-\bar{\mathbf{M}})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&-\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \mathbf{M}\ \partial_x^\alpha \partial_t^\beta \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_1^{\bar{\mathbf{M}}}(f-\mathbf{M})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&-\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}}(f-\mathbf{M})\ \partial_x^\alpha \partial_t^\beta \left(\nabla_x \Phi \cdot \nabla_\xi (\mathbf{M}-\bar{\mathbf{M}})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&-\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}}(f-\mathbf{M})\ \partial_x^\alpha \partial_t^\beta \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{P}_1^{\bar{\mathbf{M}}}(f-\mathbf{M})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
:= &\sum_{j=1}^4 J_{12}^j. \qquad (5.20)
\end{aligned}
$$

By Lemma 2.4, J_{12}^2 satisfies

$$
\begin{aligned}
J_{12}^2 = &\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nabla_\xi \partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M} \cdot \partial_x^\alpha \partial_t^\beta \left(\nabla_x \Phi\ \mathbf{P}_1^{\bar{\mathbf{M}}}(f-\mathbf{M})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
&+ \frac{1}{R\bar{\theta}} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_x^\alpha \partial_t^\beta \mathbf{P}_1^{\bar{\mathbf{M}}} \mathbf{M}\ \xi \cdot \partial_x^\alpha \partial_t^\beta \left(\nabla_x \Phi\ \mathbf{P}_1^{\bar{\mathbf{M}}}(f-\mathbf{M})\right)}{\bar{\mathbf{M}}} dx d\xi d\tau \\
\le &\ O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 3} \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta) \right|^2 dx d\tau \\
&+ O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 4, \beta' \le \beta} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \partial_x^{\alpha'} \partial_t^{\beta'} \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau. \qquad (5.21)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J_{12}^3 \le &\ O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 3} \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta) \right|^2 dx d\tau \\
&+ O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 4, \beta' \le \beta} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \partial_x^{\alpha'} \partial_t^{\beta'} \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau, \qquad (5.22)
\end{aligned}
$$

and

$$
\begin{aligned}
J_{12}^4 \le &\ O(1)\varepsilon \sum_{|\alpha'|+\beta' \le 3} \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} (\rho, u, \theta) \right|^2 dx d\tau \\
&+ O(1)\varepsilon \sum_{|\alpha'|+\beta'+|\gamma'| \le 4, \beta' \le \beta} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) \left| \partial_x^{\alpha'} \partial_t^{\beta'} \partial_\xi^{\gamma'} \mathbf{P}_1^{\bar{\mathbf{M}}} f \right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau. \qquad (5.23)
\end{aligned}
$$

Similar to the estimates on J_{12}^k for $k = 2, 3, 4$, the only difference in estimating J_{12}^1 is to consider

$$
\begin{aligned}
K = &- \sum_{(\alpha', \beta') \le (\alpha, \beta)} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\left(\partial_x^\alpha \partial_t^\beta \mathbf{M} \right)_{HOT} \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} \Phi \cdot \nabla_\xi \left(\partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} (\mathbf{M} - \bar{\mathbf{M}}) \right)_{HOT}}{\bar{\mathbf{M}}} d\xi dx d\tau \\
= &- \sum_{(\alpha', \beta') \le (\alpha, \beta)} I_{\alpha, \beta}^{\alpha', \beta'}, \qquad (5.24)
\end{aligned}
$$

where

$$\left(\partial_x^\alpha \partial_t^\beta \mathbf{M}\right)_{HOT} = \left[\frac{\partial_x^\alpha \partial_t^\beta \rho}{\rho} + \frac{(\xi - u) \cdot \partial_x^\alpha \partial_t^\beta u + \frac{1}{2\theta}\left(\frac{|\xi-u|^2}{R\theta} - 3\right) \partial_x^\alpha \partial_t^\beta \theta}{R\theta} \right] \mathbf{M}.$$

When $(\alpha', \beta') = (\alpha, \beta)$, direct calculation gives

$$\begin{aligned} I_{\alpha,\beta}^{\alpha,\beta} =& \int_0^t \int_{\mathbf{R}^3} \tfrac{\rho}{R\bar{\theta}} \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \cdot \partial_x^\alpha \partial_t^\beta u \left(1 - \tfrac{\rho\bar{\theta}^5}{\bar{\rho}} \theta^{-\frac{5}{2}} (2\bar{\theta} - \theta)^{-\frac{5}{2}} \exp\left(\tfrac{|u|^2}{R(2\bar{\theta}-\theta)}\right)\right) dx d\tau \\ =& -\tfrac{1}{R\bar{\theta}} \int_0^t \int_{\mathbf{R}^3} (\rho - \bar{\rho}) \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \cdot \partial_x^\alpha \partial_t^\beta u dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} |(\rho - \bar{\rho}, u, \theta - \bar{\theta})|^2 \left| \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \cdot \partial_x^\alpha \partial_t^\beta u \right| dx d\tau. \end{aligned} \tag{5.25}$$

On the other hand, when $(\alpha', \beta') < (\alpha, \beta)$, we have

$$\begin{aligned} \left| I_{\alpha,\beta}^{\alpha',\beta'} \right| \leq& O(1) \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} \Phi \right| |u| \left| \partial_x^\alpha \partial_t^\beta u \right| \left| \partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} u \right| dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} \Phi \right| \left| \partial_x^\alpha \partial_t^\beta u \right| \left| \partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} (\rho, \theta) \right| dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} \Phi \right| \left| \partial_x^\alpha \partial_t^\beta (\rho, \theta) \right| \left| \partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} u \right| dx d\tau. \end{aligned} \tag{5.26}$$

Hence, by Lemma 2.1, we obtain

$$\begin{aligned} & \int_0^t \int_{\mathbf{R}^3} |(\rho - \bar{\rho}, u, \theta - \bar{\theta})|^2 \left| \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \cdot \partial_x^\alpha \partial_t^\beta u \right| dx d\tau \\ & \leq O(1) \int_0^t \left\| (\rho - \bar{\rho}, u, \theta - \bar{\theta}) \right\|_{L^\infty}^2 \left\| \nabla_x \partial_x^\alpha \partial_t^\beta \Phi \right\| \left\| \partial_x^\alpha \partial_t^\beta u \right\| d\tau \\ & \leq O(1) \varepsilon^2 \sum_{|\alpha''| \leq 1} \int_0^t \left\| \nabla_x \partial_x^{\alpha''} (\rho, u, \theta) \right\|^2 d\tau. \end{aligned} \tag{5.27}$$

Moreover, since $|\alpha'| + \beta' < |\alpha| + \beta \leq 4$, from Lemma 2.1, Lemma 3.1, (3.7), (3.11), (3.12), and by using the fact that $|\alpha| + \beta \geq 1$, $|\alpha - \alpha'| + \beta - \beta' \geq 1$, we have

$$\begin{aligned} \left| I_{\alpha,\beta}^{\alpha',\beta'} \right| \leq& O(1) \varepsilon^{-1} \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} \partial_t^{\beta'} \Phi \right|^2 \left| \partial_x^\alpha \partial_t^\beta u \right|^2 dx d\tau \\ & + O(1) \varepsilon \int_0^t \int_{\mathbf{R}^3} |u|^2 \left| \partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} u \right|^2 dx d\tau \\ & + O(1) \varepsilon \int_0^t \int_{\mathbf{R}^3} \left(\left| \partial_x^{\alpha-\alpha'} \partial_t^{\beta-\beta'} (\rho, \theta) \right|^2 + \left| \partial_x^\alpha \partial_t^\beta (\rho, \theta) \right|^2 \right) dx d\tau \end{aligned}$$

$$\le O(1)\varepsilon\int_0^t\left(\|u\|_{L^\infty}^2+\left\|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}\Phi\right\|_{L^\infty}^2\right)d\tau$$

$$+O(1)\varepsilon\int_0^t\int_{\mathbf{R}^3}\left(\left|\partial_x^{\alpha-\alpha'}\partial_t^{\beta-\beta'}(\rho,\theta)\right|^2+\left|\partial_x^{\alpha}\partial_t^{\beta}(\rho,\theta)\right|^2\right)dxd\tau$$

$$\le O(1)\varepsilon\sum_{|\alpha''|+\beta''\le 3}\int_0^t\left\|\nabla_x\partial_x^{\alpha''}\partial_t^{\beta''}(\rho,u,\theta)\right\|^2d\tau$$

$$+O(1)\varepsilon\sum_{|\alpha''|+\beta''\le 4,\beta''\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_x^{\alpha''}\partial_t^{\beta''}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dxd\tau. \tag{5.28}$$

Thus, it remains to estimate the following term on the right-hand side of (5.25):

$$L=\int_0^t\int_{\mathbf{R}^3}(\rho-\bar\rho)\nabla_x\partial_x^{\alpha}\partial_t^{\beta}\Phi\cdot\partial_x^{\alpha}\partial_t^{\beta}udxd\tau. \tag{5.29}$$

For this, if $\beta\le 3$ and $1\le|\alpha|+\beta\le 4$, from Lemma 2.1, we have

$$L\le O(1)\varepsilon\int_0^t\left(\|\rho-\bar\rho\|^2+\sum_{|\alpha'|+\beta'\le 3}\left\|\nabla_x\partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2\right)d\tau$$

$$+O(1)\varepsilon\sum_{|\alpha'|+\beta'\le 4,\beta'\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\bar{\mathbf{M}}}f\right|^2}{\bar{\mathbf{M}}}d\xi dxd\tau. \tag{5.30}$$

When $(\alpha,\beta)=(0,4)$, L takes the form

$$L=\int_0^t\int_{\mathbf{R}^3}(\rho-\bar\rho)\nabla_x\partial_t^4\Phi\cdot\partial_t^4udxd\tau. \tag{5.31}$$

Since

$$\partial_t^4u=-\partial_t^3\Big((u\cdot\nabla_x)u\Big)-\frac{2}{3}\partial_t^3\left(\frac{\nabla_x(\rho\theta)}{\rho}\right)+\nabla_x\partial_t^3\Phi-\int_{\mathbf{R}^3}\partial_t^3\left(\frac{\psi(\xi\cdot\nabla_x\mathbf{G})}{\rho}\right)d\xi,$$

and

$$\int_0^t\int_{\mathbf{R}^3}(\rho-\bar\rho)\nabla_x\partial_t^4\Phi\cdot\nabla_x\partial_t^3\Phi dxd\tau$$

$$=\tfrac{1}{2}\int_{\mathbf{R}^3}(\rho-\bar\rho)\left|\nabla_x\partial_t^3\Phi\right|^2dx\Big|_0^t-\tfrac{1}{2}\int_0^t\int_{\mathbf{R}^3}\rho_t\left|\nabla_x\partial_t^3\Phi\right|^2dxd\tau$$

$$\le O(1)\mathcal{E}(f_0)^2+O(1)\varepsilon\left\|\nabla_x\partial_t^3\Phi\right\|^2$$

$$+O(1)\varepsilon\int_0^t\|\nabla_x(\rho,u)\|^2d\tau+O(1)\varepsilon^{-1}\int_0^t\left\|\nabla_x\partial_t^3\Phi\right\|_{L^4}^4d\tau$$

$$\le O(1)\mathcal{E}(f_0)^2+O(1)\varepsilon\left\|\nabla_x\partial_t^3\Phi\right\|^2$$

$$+O(1)\varepsilon\int_0^t\|\nabla_x(\rho,u)\|^2d\tau+O(1)\varepsilon\int_0^t\left\|\nabla_x\partial_t^3\Phi\right\|_{L^\infty}^2d\tau$$

$$
\begin{aligned}
&\leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \left\|\nabla_x \partial_t^3 \Phi\right\|^2 \\
&\quad + O(1)\varepsilon \int_0^t \|\nabla_x(\rho,u)\|^2 d\tau + O(1)\varepsilon \sum_{|\alpha''|\leq 1} \int_0^t \left\|D_x^2 \partial_x^{\alpha''}\partial_t^3 \Phi\right\|^2 d\tau \\
&\leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \left\|\nabla_x \partial_t^3 \Phi\right\|^2 \\
&\quad + O(1)\varepsilon \int_0^t \left(\|\rho-\bar\rho\|^2 + \sum_{|\alpha'|+\beta'\leq 3} \left\|\nabla_x \partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2\right) d\tau \\
&\quad + O(1)\varepsilon \sum_{|\alpha'|+\beta'\leq 4,\beta'\leq 3} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau, \qquad (5.32)
\end{aligned}
$$

we have

$$
\begin{aligned}
L &\leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \left\|\nabla_x \partial_t^3 \Phi\right\|^2 \\
&\quad + O(1)\varepsilon \int_0^t \left(\|\rho-\bar\rho\|^2 + \sum_{|\alpha'|+\beta'\leq 3} \left\|\nabla_x \partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2\right) d\tau \\
&\quad + O(1)\varepsilon \sum_{|\alpha'|+\beta'\leq 4,\beta'\leq 3} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau. \qquad (5.33)
\end{aligned}
$$

Here we have used Lemma 2.1, Lemma 3.1, (3.7), (3.11) and (3.12) in deducing (5.32) and (5.33).

Therefore, for $1 \leq |\alpha| + \beta \leq 4$, (5.21)-(5.23), (5.27), (5.28), (5.30) and (5.33) imply that I_{12} satisfies

$$
\begin{aligned}
I_{12} &\leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \left\|\nabla_x \partial_t^3 \Phi\right\|^2 \\
&\quad + O(1)\varepsilon \int_0^t \left(\|\rho-\bar\rho\|^2 + \sum_{|\alpha'|+\beta'\leq 3} \left\|\nabla_x \partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2\right) d\tau \\
&\quad + O(1)\varepsilon \sum_{|\alpha'|+\beta'+|\gamma'|\leq 4,\beta'\leq 3} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau. \qquad (5.34)
\end{aligned}
$$

Finally, by plugging (5.34), (5.17), (5.18) and (5.19) into (5.16), for $1 \leq |\alpha| + \beta \leq 4$, we have

$$
\begin{aligned}
&\int_{\mathbf{R}^3} \left(\left|\nabla_x \partial_x^{\alpha}\partial_t^{\beta}\Phi\right|^2 + \int_{\mathbf{R}^3} \frac{\left|\partial_x^{\alpha}\partial_t^{\beta} f\right|^2}{\bar{\mathbf{M}}} d\xi\right) dx + \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha}\partial_t^{\beta}\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau \\
&\leq O(1)\mathcal{E}(f_0)^2 + O(1)\varepsilon \left\|\nabla_x \partial_t^3 \Phi\right\|^2 \\
&\quad + O(1)\varepsilon \int_0^t \left(\|\rho-\bar\rho\|^2 + \sum_{|\alpha'|+\beta'\leq 3} \left\|\nabla_x \partial_x^{\alpha'}\partial_t^{\beta'}(\rho,u,\theta)\right\|^2\right) d\tau \\
&\quad + O(1)\varepsilon \sum_{|\alpha'|+\beta'+|\gamma'|\leq 4,\beta'\leq 3} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)\left|\partial_x^{\alpha'}\partial_t^{\beta'}\partial_\xi^{\gamma'}\mathbf{P}_1^{\bar{\mathbf{M}}} f\right|^2}{\bar{\mathbf{M}}} d\xi dx d\tau, \qquad (5.35)
\end{aligned}
$$

which gives (3.13). This completes the proof of the lemma.

Acknowledgement. The research of Tong Yang was supported by RGC Competitive Earmarked Research Grant CityU 102805. The research of Huijiang Zhao was supported by the National Natural Science Foundation of China under contract 10431060. The research was also supported in part by the National Natural Science Foundation of China under contract 10329101.

References

1. Boltzmann, L. (translated by Stephen G. Brush): *Lectures on Gas Theory*. New York: Dover Publications, Inc. 1964
2. Carleman, T.: Sur la théorie de l'équation intégrodifférentielle de Boltzmann. Acta Math. **60**, 91–142 (1932)
3. Cercignani, C.: The Boltzmann equation and its applications. Applied Mathematical Sciences **67**, New York: Springer-Verlag, 1988
4. Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases. Applied Mathematical Sciences **106**, New York: Springer-Verlag 1994
5. Deckelnick, K.: Decay estimates for the compressible Navier–Stokes equations in unbounded domains. Math. Z. **209**, 115–130 (1992)
6. Desvillettes, L., Dolbeault, J.: On long time asymptotics of the Vlasov–Poisson–Boltzmann equation. Comm. Partial Differ. Eqs. **16** (2–3), 451–489 (1991)
7. Duan, R., Yang, T., Zhu, C.-J.: Boltzmann equation with external force in infinite vacuum. J. Math. Phys. **46**(5), 053307 (2005)
8. Glassey, R.: The Cauchy Problem in Kinetic Theory. Philadelphia, PA: *Society for Industrial and Applied Mathematics (SIAM)*, 1996
9. Glassey, R., Strauss, W.A.: Decay of the linearized Boltzmann–Vlasov system. Transport Theory Statist. Phys. **28**(2), 135–156 (1999)
10. Glassey, R., Strauss, W.A.: Perturbation of essential spectra of evolution operators and the Vlasov–Poisson–Boltzmann system. Discrete Contin. Dynam. Systems **5**(3), 457–472 (1999)
11. Glassey, R., Schaeffer, J., Zheng Y.-X.: Steady states of the Vlasov–Poisson–Fokker–Planck system. J. Math. Anal. Appl. **202**(3), 1058–1075 (1996)
12. Golse, F., Perthame, B., Sulem, C.: On a boundary layer problem for the nonlinear Boltzmann equation. Arch. Rational Mech. Anal. **103**, 81–96 (1986)
13. Grad, H.: *Asymptotic Theory of the Boltzmann Equation II, Rarefied Gas Dynamics*. J. A. Laurmann, ed., **Vol. 1**, New York: Academic Press 1963, pp. 26–59
14. Guo, Y.: The Vlasov–Poisson–Boltzmann system near Maxwellians. Comm. Pure Appl. Math. **55**(9), 1104–1135 (2002)
15. Guo, Y.: The Vlasov–Maxwell–Boltzmann system near Maxwellians. Invent. Math. **153**(3), 593–630 (2003)
16. Guo, Y.: The Vlasov–Poisson–Boltzmann system near vacuum. Commun. Math. Phys. **218**(2), 293–313 (2001)
17. Guo, Y.: The Boltzmann equation in the whole space. Indiana Univ. Math. J. **53**(4), 1081–1094 (2004)
18. Hilbert, D.: *Grundzuge einer Allgemeinen Theorie der Linearen Integralgleichungen* (in German). New York, N.Y.: Chelsea Publishing Company, 1953
19. Huang, F.-M., Xin, Z.-P., Yang, T.: *Contact discontinuity with general perturbations for gas motions*. Preprint 2004, available at http://www.cityu.edu.hk/rcms/publication/preprint18.pdf
20. Kawashima, S., Matsumura, A.: Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. Commun. Math. Phys. **101**(1), 97–127 (1985)
21. Li, H.-L., Matsumura, A.: Asymptotic behavior of compressible Navier–Stokes–Poisson system. Preprint
22. Lions, P.-L.: On kinetic equations. In: *Proceedings of the International Congress of Mathematicians*, Kyoto: Math. Soc. Japan, 1991, pp.1173–1185
23. Liu, T.-P., Yang, T., Yu, S.-H.: Energy method for the Boltzmann equation. Physica D **188**(3-4), 178–192 (2004)
24. Liu, T.-P., Yang, T., Yu, S.-H., Zhao, H.-J.: Nonlinear stability of rarefaction waves for the Boltzmann equation. Arch. Rational Mech. Anal. **181**(2), 333–371 (2006)
25. Liu, T.-P., Yu, S.-H.: Boltzmann equation: Micro-macro decompositions and positivity of shock profiles. Commun. Math. Phys. **246**(1), 133–179 (2004)
26. Matsumura, A., Nishida, T.: Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. Commun. Math. Phys. **89**(4), 445–464 (1983)
27. Mischler, S.: On the initial boundary value problem for the Vlasov–Poisson–Boltzmann system. Commun. Math. Phys. **210**(2), 447–466 (2000)
28. Ukai, S., Yang, T., Zhao, H.-J.: Global solutions to the Boltzmann equation with external forces. Analysis and Applications **3**(2), 157–193 (2005)

29. Ukai, S., Yang, T., Zhao, H.-J.: Convergence rate to stationary solutions for Boltzmann equation with external force. To appear in Chin. Ann. Math. Ser. B. **27**(4), 363–378 (2006)
30. Yang, T., Yu, H.-J., Zhao, H.-J.: Cauchy problem for the Vlasov–Poisson–Boltzmann system. Arch. Rational Mech. Anal. **182**(3), (2006)
31. Yang, T., Zhao, H.-J.: A new energy method for the Boltzmann equation. J. Math. Phys. **47**(5), 053301 (2006)
32. Yang, T., Zhao, H.-J.: A half-space problem for the Boltzmann equation with specular reflection boundary condition. Commun. Math. Phys. **255**(3), 683–726 (2005)

Communicated by H.-T. Yau