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Cauchy Problem for the Vlasov–Poisson–Boltzmann System

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Abstract

The dynamics of dilute electrons can be modelled by the fundamental Vlasov–Poisson–Boltzmann system which describes mutual interactions of the electrons through collisions in the self-consistent electric field. In this paper, it is shown that any smooth perturbation of a given global Maxwellian leads to a unique global-in-time classical solution when either the mean free path is small or the background charge density is large. Moreover, the solution converges to the global Maxwellian when time tends to infinity. The analysis combines the techniques used in the study of conservation laws with the decomposition of the Boltzmann equation introduced in [15, 17] by obtaining new entropy estimates for this physical model.

1. Introduction

The flow of dilute charged aggregate particles (e.g. electrons) in the absence of the magnetic field is governed by the fundamental Vlasov–Poisson–Boltzmann system:

$$\begin{cases} f_t+\xi\cdot\nabla_x f+\nabla_x\Phi\cdot\nabla_\xi f=\frac{1}{\kappa}Q(f,f), \\ \qquad\qquad \Delta_x\Phi=\rho-\rho_0=\int_{\mathbf{R}^3} f\,d\xi-\rho_0, \quad |\Phi|\to 0,\ \text{as } |x|\to+\infty \end{cases} \tag{1.1}$$

with initial data given by

$$f(0,x,\xi)=\overline{f}_0(x,\xi),$$

where $f(t,x,\xi)$ is the distribution function for the particles at time $t \geqq 0$ located at $x=(x_1,x_2,x_3)\in\mathbf{R}^3$ with velocity $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbf{R}^3$, and the constant $\kappa>0$ is the Knudsen number proportional to the mean free path. The self-consistent electric potential $\Phi(t,x)$ is coupled with the distribution function $f(t,x,\xi)$

through the Poisson equation. The constant background charge density is denoted by $\rho_0 > 0$. The short-range interaction between particles is accounted by the standard Boltzmann collision operator $Q(f, g)$ for the hard-sphere model:

$$Q(f,g)(\xi) \equiv \frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}^2_+}\Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi)\Big)$$
$$\times |(\xi - \xi_*)\cdot\Omega|\ d\xi_* d\Omega.$$

Here $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 :\ (\xi - \xi_*)\cdot\Omega \geqq 0\}$, and

$$\begin{cases} \xi' = \xi - [\,(\xi - \xi_*)\cdot\Omega]\ \Omega, \\ \xi'_* = \xi_* + [\,(\xi - \xi_*)\cdot\Omega]\ \Omega \end{cases}$$

is the relation between velocities ξ', ξ'_* after and the velocities ξ, ξ_* before the collision, which is induced by conservation of momentum and energy.

Previous work has been carried out on the Vlasov–Poisson–Boltzmann system. Global-in-time renormalized solutions with arbitrary amplitude were constructed in [14] and the result has been generalized to the case with boundary in [18]. The large-time behavior of weak solutions under extra regularity assumptions was studied in [4]. As to classical solutions, there has been some progress only recently. That is, for any smooth periodic perturbation of a global Maxwellian that preserves mass, momentum, and total energy, the first global existence result on smooth periodic solutions was obtained in [10]. As regards the whole space, to our knowledge the only result so far is [12] where global smooth small-amplitude solutions near vacuum were constructed for a class of "soft" collision kernels. Therefore, our global existence result is new for the Vlasov–Poisson–Boltzmann system near a given global Maxwellian in the whole space.

To state the main result, we first reformulate the problem by the scaling

$$\begin{cases} f(t,x,\xi) \to \dfrac{\overline{\rho}}{\rho_0} f\left(\dfrac{\overline{\rho}}{\kappa\rho_0}t, \dfrac{\overline{\rho}}{\kappa\rho_0}x, \xi\right), \\ \Phi(t,x) \to \Phi\left(\dfrac{\overline{\rho}}{\kappa\rho_0}t, \dfrac{\overline{\rho}}{\kappa\rho_0}x\right), \end{cases}$$

where $\overline{\rho}$ is any fixed constant. It is easy to see that after scaling the solution satisfies

$$\begin{cases} f_t + \xi\cdot\nabla_x f + \nabla_x\Phi\cdot\nabla_\xi f = Q(f,f), \\ \lambda\Delta_x\Phi = \rho - \overline{\rho} = \displaystyle\int_{\mathbf{R}^3} f d\xi - \overline{\rho}, \quad |\Phi| \to 0, \text{ as } |x| \to +\infty, \\ f(0,x,\xi) = f_0(x,\xi) = \frac{\overline{\rho}}{\rho_0}\overline{f}_0\left(\frac{\overline{\rho}}{\kappa\rho_0}x, \xi\right), \end{cases} \tag{1.2}$$

with $\lambda = \frac{\rho_0}{\kappa^2\overline{\rho}}$. In this paper, we will assume $\lambda > 0$ is suitably large which means that either the Knudsen number κ (i.e. the mean free path) is sufficiently small, or that the constant background charge density ρ_0 is sufficiently large.

A micro–macro decomposition of the Boltzmann equation and its solution was introduced in [15, 17]. The Boltzmann equation can be rewritten as a system akin to the equations of fluid dynamics, which henceforth will be dubbed the *macroscopic system*, coupled with an equation for a quantity describing microscopic properties of the gas, which henceforth will be referred to as the *microscopic components*, cf. [15]. Similarly, the quantity describing macroscopic properties of the gas will be referred to as the *macroscopic components*. As an illustration of this method, the global existence of classical solutions around a global Maxwellian was proved in [15] by using a simple energy method through the construction of entropy–entropy flux pairs. This method is used here for the study of the Boltzmann equation with self-induced electric field. As in fluid dynamics, the entropy–entropy flux pair plays an important role in the lower order energy estimate. Moreover, the dissipation induced by the electric field which is governed by the Poisson equation is crucial for closing the basic *a priori* estimate. The basic estimate in turn implies the uniform space–time integrability of the square of the perturbation of the density function. Note that the corresponding integral diverges for the Boltzmann equation or even for the Navier–Stokes equations without forcing.

To be precise, we decompose the solution of the Vlasov–Poisson–Boltzmann system $f(t,x,\xi)$ into the macroscopic component, i.e. the local Maxwellian $\mathbf{M} = \mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$, and the microscopic component, i.e. $\mathbf{G} = \mathbf{G}(t,x,\xi)$ as follows:

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi).$$

The local Maxwellian $\mathbf{M}$ is defined by the five conserved quantities, that is, the density $\rho(t,x)$, the momentum $m(t,x) = \rho(t,x)u(t,x)$, and the energy density $\mathbf{E}(t,x) + \frac{1}{2}|u(t,x)|^2$ given by:

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ m^i(t,x) \equiv \int_{\mathbf{R}^3} \psi_i(\xi) f(t,x,\xi)d\xi \text{ for } i = 1,2,3, \\ \left[\rho\left(\mathbf{E} + \frac{1}{2}|u|^2\right)\right](t,x) \equiv \int_{\mathbf{R}^3} \psi_4(\xi) f(t,x,\xi)d\xi, \end{cases} \tag{1.3}$$

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\left(-\frac{|\xi - u(t,x)|^2}{2R\theta(t,x)}\right). \tag{1.4}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy $\mathbf{E}$ by $\mathbf{E} = 3/2R\theta$ where R is the gas constant, and $u(t,x)$ is the fluid velocity; $\psi_\alpha(\xi)$, $\alpha = 0,1,\cdots,4$, are the five collision invariants, cf. [1, 2]:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i, \text{ for } i = 1,2,3, \quad \text{or } \psi = \xi, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases} \tag{1.5}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_\alpha(\xi) Q(h, g) d\xi = 0, \quad \text{for } \alpha = 0, 1, 2, 3, 4.$$

In what follows, we define an inner product in $\xi \in \mathbf{R}^3$ with respect to a given Maxwellian $\tilde{\mathbf{M}}$ as:

$$\langle h, g \rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} h(\xi) g(\xi) d\xi,$$

for functions h, g of ξ so that the above integral is well defined. With respect to the inner product $\langle h, g \rangle_{\mathbf{M}}$, the following functions which span the space of the macroscopic components of the solution, are pairwise orthogonal:

$$\begin{cases} \chi_0(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{\rho}} \mathbf{M}, \\ \chi_i(\xi; \rho, u, \theta) \equiv \frac{\xi_i - u_i}{\sqrt{R\rho\theta}} \mathbf{M} \quad \text{for } i = 1, 2, 3, \\ \chi_4(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{6\rho}} \left(\frac{|\xi - u|^2}{R\theta} - 3 \right) \mathbf{M}, \\ \quad \left\langle \chi_i, \chi_j \right\rangle_{\mathbf{M}} = \delta_{ij}, \quad \text{for } i, j = 0, 1, 2, 3, 4. \end{cases} \tag{1.6}$$

By using these five functions, the macroscopic projection $\mathbf{P}_0$ and microscopic projection $\mathbf{P}_1$ are:

$$\begin{cases} \mathbf{P}_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j \rangle_{\mathbf{M}} \chi_j, \\ \mathbf{P}_1 h \equiv h - \mathbf{P}_0 h. \end{cases} \tag{1.7}$$

Note that the operators $\mathbf{P}_0$ (and therefore $\mathbf{P}_1$) are orthogonal self-adjoint projections with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbf{M}}$.

Note that $h(\xi)$ is called a microscopic component if it is orthogonal to the subspace spanned by the collision invariants, that is,

$$\int_{\mathbf{R}^3} h(\xi) \psi_\alpha(\xi) d\xi = 0, \quad \text{for } \alpha = 0, 1, 2, 3, 4. \tag{1.8}$$

It is clear that such a function is in the range of the microscopic projection $\mathbf{P}_1$. Under the above decomposition, the solution $f(t, x, \xi)$ of the Vlasov–Poisson–Boltzmann system satisfies,

$$\mathbf{P}_0 f = \mathbf{M}, \quad \mathbf{P}_1 f = \mathbf{G}.$$

Then, by using $f(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi)$, the Vlasov–Poisson–Boltzmann system $(1.2)_1$ becomes:

$$\begin{aligned} &(\mathbf{M} + \mathbf{G})_t + \xi \cdot \nabla_x (\mathbf{M} + \mathbf{G}) + \nabla_x \Phi \cdot \nabla_\xi (\mathbf{M} + \mathbf{G}) \\ &\quad = (2Q(\mathbf{G}, \mathbf{M}) + Q(\mathbf{G}, \mathbf{G})). \end{aligned} \tag{1.9}$$

By applying $\mathbf{P}_0$ to (1.9), we have

$$\mathbf{M}_t + \mathbf{P}_0 (\xi \cdot \nabla_x \mathbf{M}) + \mathbf{P}_0 (\xi \cdot \nabla_x \mathbf{G}) + \nabla_x \Phi \cdot \nabla_\xi \mathbf{M} = 0.$$

As usual, the system of five conservation laws can be obtained by taking the inner product of the Vlasov–Poisson–Boltzmann system $(1.2)_1$ with the collision invariants $\psi_\alpha(\xi)$:

$$\begin{cases} \rho_t + \mathrm{div}_x\, m = 0, \\ m_{it} + \sum_{j=1}^{3} \left(u_i m_j\right)_{x_j} + p_{x_i} - \rho \Phi_{x_i} = -\int_{\mathbf{R}^3} \psi_i \,(\xi \cdot \nabla_x \mathbf{G})\, d\xi, \quad i = 1, 2, 3, \\ \left[\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right)\right]_t + \sum_{j=1}^{3} \left(u_j \left(\rho \left(\frac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_{x_j} - m \cdot \nabla_x \Phi \\ \qquad = -\int_{\mathbf{R}^3} \psi_4 \,(\xi \cdot \nabla_x \mathbf{G})\, d\xi. \end{cases} \tag{1.10}$$

Here p is the pressure for monatomic gases:

$$p = \frac{2}{3}\rho \mathbf{E} = R\rho\theta.$$

Moreover, the equation for the microscopic components of $\mathbf{G}$ is obtained by applying the microscopic projection $\mathbf{P}_1$ to (1.9):

$$\begin{aligned} &\mathbf{G}_t + \mathbf{P}_1 (\xi \cdot \nabla_x \mathbf{G} + \xi \cdot \nabla_x \mathbf{M}) + \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} \\ &\quad = L_{\mathbf{M}} \mathbf{G} + Q(\mathbf{G}, \mathbf{G}), \end{aligned} \tag{1.11}$$

i.e.

$$\begin{aligned} \mathbf{G} \;=\; & L_{\mathbf{M}}^{-1} \left(\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\right) \\ & + L_{\mathbf{M}}^{-1} \left(\mathbf{G}_t + \mathbf{P}_1(\xi \cdot \nabla_x \mathbf{G}) + \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} - Q(\mathbf{G}, \mathbf{G})\right) \\ := \; & L_{\mathbf{M}}^{-1} \left(\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\right) + \Theta, \end{aligned} \tag{1.12}$$

where

$$L_{\mathbf{M}} g = L_{[\rho,u,\theta]} g \equiv Q\left(\mathbf{M} + g, \mathbf{M} + g\right) - Q(g, g),$$

is the usual linearized collision operator.

By plugging (1.12) into (1.10), we now have another representation of the Vlasov–Poisson–Boltzmann system which contains a macroscopic system

$$\begin{cases} \rho_t + \mathrm{div}_x m = 0, \\ m_{it} + \sum_{j=1}^{3} \left(u_i m_j\right)_{x_j} + p_{x_i} - \rho \Phi_{x_i} \\ \quad = -\int_{\mathbf{R}^3} \psi_i \left(\xi \cdot \nabla_x L_{\mathbf{M}}^{-1} \left(\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\right)\right) d\xi \\ \qquad -\int_{\mathbf{R}^3} \psi_i \left(\xi \cdot \nabla_x \Theta\right) d\xi, \quad i = 1, 2, 3, \\ \left[\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right)\right]_t + \sum_{j=1}^{3} \left(u_j \left(\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_{x_j} - m \cdot \nabla_x \Phi \\ \quad = -\int_{\mathbf{R}^3} \psi_4 \left(\xi \cdot \nabla_x L_{\mathbf{M}}^{-1} \left(\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\right)\right) d\xi \\ \qquad -\int_{\mathbf{R}^3} \psi_4 \left(\xi \cdot \nabla_x \Theta\right) d\xi, \end{cases} \tag{1.13}$$

the equation (1.11) for the microscopic component $\mathbf{G}$, and the Poisson equation $(1.2)_2$ for the electric potential. Note that if all the terms containing Θ are dropped, then the above system reduces to the compressible Navier–Stokes–Poisson equations. Later in this paper, we will work on this reformulated system by applying the techniques used in the study of conservation laws together with the dissipative effect in the Boltzmann equation.

For preparation, we now recall some properties of the linearized collision operator $L_{\mathbf{M}}$. By definition, $L_{\mathbf{M}}$ is self-adjoint with respect to the inner product $\langle h, g\rangle_{\mathbf{M}}$, i.e.

$$\langle h, L_{\mathbf{M}} g\rangle_{\mathbf{M}} = \langle L_{\mathbf{M}} h, g\rangle_{\mathbf{M}},$$

and the null space $\mathcal{N}$ of $L_{\mathbf{M}}$ is spanned by the macroscopic variables:

$$\chi_j, \quad j = 0, \cdots, 4.$$

For the hard-sphere model, $L_{\mathbf{M}}$ takes the form, cf. [3, 5, 9],

$$\left(L_{\mathbf{M}} h\right)(\xi) = -\nu(\xi; \rho, u, \theta) h(\xi) + \sqrt{\mathbf{M}(\xi)} K_{\mathbf{M}} \left(\left(\frac{h}{\sqrt{\mathbf{M}}}\right)(\xi)\right). \tag{1.14}$$

Here $K_{\mathbf{M}}(\cdot) = -K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2 operator, and the collision frequency $\nu(\xi; \rho, u, \theta)$ and $K_{i\mathbf{M}}(\cdot)$ have the following expressions

$$\begin{cases} \nu(\xi;\rho,u,\theta) = \dfrac{2\rho}{\sqrt{2\pi R\theta}}\left\{\left(\dfrac{R\theta}{|\xi-u|}+|\xi-u|\right)\displaystyle\int_0^{|\xi-u|}\exp\left(-\dfrac{y^2}{2R\theta}\right)dy \right.\\ \qquad\qquad\qquad \left. +R\theta\exp\left(-\dfrac{|\xi-u|^2}{2R\theta}\right)\right\}, \\ k_{1\mathbf{M}}(\xi,\xi_*) = \dfrac{\pi\rho}{\sqrt{(2\pi R\theta)^3}}|\xi-\xi_*|\exp\left(-\dfrac{|\xi-u|^2}{4R\theta}-\dfrac{|\xi_*-u|^2}{4R\theta}\right), \\ k_{2\mathbf{M}}(\xi,\xi_*) = \dfrac{2\rho}{\sqrt{2\pi R\theta}}|\xi-\xi_*|^{-1}\exp\left(-\dfrac{|\xi-\xi_*|^2}{8R\theta}-\dfrac{(|\xi|^2-|\xi_*|^2)^2}{8R\theta|\xi-\xi_*|^2}\right), \end{cases}$$

where $k_{i\mathbf{M}}(\xi,\xi_*)(i=1,2)$ is the kernel of the operator $K_{i\mathbf{M}}(i=1,2)$ respectively, and $\nu(\xi;\rho,u,\theta)\sim(1+|\xi|)$ as $|\xi|\to+\infty$. Furthermore, there exists $\sigma_0(\rho,u,\theta)>0$ such that for any function $h(\xi)\in\mathcal{N}^\perp$

$$\langle h, L_{\mathbf{M}}h\rangle_{\mathbf{M}} \leqq -\sigma_0(\rho,u,\theta)\langle h,h\rangle_{\mathbf{M}},$$

which implies, cf. [9]

$$\langle h, L_{\mathbf{M}}h\rangle_{\mathbf{M}} \leqq -\sigma(\rho,u,\theta)\langle \nu(\xi)h,h\rangle_{\mathbf{M}}, \tag{1.15}$$

with some constant $\sigma(\rho,u,\theta)>0$.

For later use, notice that the projections $\mathbf{P}_0$ and $\mathbf{P}_1$ have the following properties:

$$\begin{cases} \mathbf{P}_0(\psi_j\mathbf{M}) = \psi_j\mathbf{M},\ \mathbf{P}_1(\psi_j\mathbf{M})=0,\ j=0,1,2,3,4, \\ L_{\mathbf{M}}\mathbf{P}_1 = \mathbf{P}_1L_{\mathbf{M}} = L_{\mathbf{M}},\ \mathbf{P}_1(Q(h,h)) = Q(h,h), \\ L_{\mathbf{M}}\mathbf{P}_0 = \mathbf{P}_0L_{\mathbf{M}} = 0,\ \mathbf{P}_0(Q(h,h))=0, \\ \langle\psi_j\mathbf{M},h\rangle_{\mathbf{M}} = \langle\psi_j\mathbf{M},\mathbf{P}_0h\rangle_{\mathbf{M}},\ j=0,1,2,3,4, \\ \langle h, L_{\mathbf{M}}g\rangle_{\mathbf{M}} = \langle\mathbf{P}_1h, L_{\mathbf{M}}(\mathbf{P}_1g)\rangle_{\mathbf{M}}, \\ \left\langle h, L^{-1}_{\mathbf{M}}(\mathbf{P}_1g)\right\rangle_{\mathbf{M}} = \left\langle L^{-1}_{\mathbf{M}}(\mathbf{P}_1h),\mathbf{P}_1g\right\rangle_{\mathbf{M}} = \left\langle \mathbf{P}_1h, L^{-1}_{\mathbf{M}}(\mathbf{P}_1g)\right\rangle_{\mathbf{M}}. \end{cases}$$

For a fixed temperature $\overline{\theta}>0$, we will study the existence of classical solutions for (1.2) near the global Maxwellian

$$\overline{\mathbf{M}} = \mathbf{M}_{[\overline{\rho},0,\overline{\theta}]} = \frac{\overline{\rho}}{\sqrt{\left(2\pi R\overline{\theta}\right)^3}}\exp\left(-\frac{|\xi|^2}{2R\overline{\theta}}\right).$$

As in [16], two sets of energy estimates are used, i.e. the energy estimates with respect to the local Maxwellian $\mathbf{M}_{[\rho,u,\theta]}(t,x,\xi)$ and a suitably chosen global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}(\xi)$. For this, a variation of the microscopic H-theorem is needed in order to relate the dissipation estimates with different weights as in Lemma 4.2 of [16]. That is, there exists a positive constant $\eta_0=\eta_0(\rho,u,\theta;\tilde{\rho},\tilde{u}.\tilde{\theta})>0$, which is not necessary small, such that if $\frac{\theta}{2}<\tilde{\theta}<\theta$ and $|u-\tilde{u}|+|\theta-\tilde{\theta}|<\eta_0$, the following microscopic H-theorem

$$-\int_{\mathbf{R}^3} \frac{\mathbf{G}L_{\mathbf{M}}\mathbf{G}}{\tilde{\mathbf{M}}} d\xi \geqq \overline{\sigma} \int_{\mathbf{R}^3} \frac{\nu(\xi)\mathbf{G}^2}{\tilde{\mathbf{M}}} d\xi, \tag{1.16}$$

holds for some positive constant $\overline{\sigma} = \overline{\sigma}(\rho, u, \theta; \tilde{\rho}, \tilde{u}, \tilde{\theta}) > 0$ with $\tilde{\mathbf{M}} = \mathbf{M}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}$.

Throughout this paper, we choose positive constants ρ_- and θ_- such that

$$\begin{cases} \rho_- = \overline{\rho}, \\ \frac{\overline{\theta}}{2} < \theta_- < \overline{\theta}, \\ \left|\theta_- - \overline{\theta}\right| < \eta_0. \end{cases} \tag{1.17}$$

It is easy to see that if $\mathbf{M}(t, x, \xi)$ is a small perturbation of $\overline{\mathbf{M}}(\xi)$, (1.16) holds for such chosen ρ_- and θ_- when $\tilde{\mathbf{M}} \equiv \mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}$.

The following are the spaces for the solution considered in this paper.

$$\begin{cases} \mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right) \\ \quad = \left\{ g(t,x,\xi) \;\middle|\; \begin{array}{c} \frac{\partial_t^{\gamma_0}\partial_x^{\alpha}\partial_\xi^{\beta} g(t,x\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in C\left([0,\infty), L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right) \\ |\gamma_0| + |\alpha| + |\beta| \leqq N \end{array} \right\}, \\ \overline{\mathbf{H}}^N = \left\{ g(t,x,\xi) \;\middle|\; \begin{array}{c} \frac{\partial_t^{\gamma_0}\partial_x^{\alpha}\partial_\xi^{\beta} g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in C\left([0,\infty), L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right), \\ |\gamma_0| \leqq 1, \quad |\gamma_0| + |\alpha| + |\beta| \leqq N \end{array} \right\}. \end{cases}$$

Here $g(t, x, \xi) = f(t, x, \xi) - \overline{\mathbf{M}}(\xi)$.

The main result in this paper can be stated as follows.

Theorem 1. *Assume that* $f_0(x, \xi) \geqq 0$ *and* $N \geqq 4$. *A sufficiently small constant* $\varepsilon > 0$ *and a sufficiently large constant* λ_0 *exist such that if*

$$\begin{cases} \lambda > \lambda_0, \qquad \lambda_0\varepsilon < 1, \\ \mathcal{E}(f_0) = \left\|\nabla_x \Delta_x^{-1}\left(\rho_0(x) - \overline{\rho}\right)\right\|_{L^2_x(\mathbf{R}^3)} \\ \qquad + \sum_{|\alpha|+|\beta| \leqq N} \left\| \frac{\partial_x^{\alpha}\partial_\xi^{\beta}(f_0(x,\xi) - \overline{\mathbf{M}}(\xi))}{\sqrt{\mathbf{M}_-(\xi)}} \right\|_{L^2_{x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)} \\ \quad \leqq \varepsilon, \end{cases} \tag{1.18}$$

then there exists a unique global classical solution $f(t, x, \xi)$ *to the Vlasov–Poisson–Boltzmann system* (1.2) *which satisfies* $f(t, x, \xi) \geqq 0$ *and which is uniformly bounded in* $\overline{\mathbf{H}}^N$. *Furthermore,*

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}^3} \sum_{|\alpha| \leqq N-4} \int_{\mathbf{R}^3} \frac{\left|\partial_x^{\alpha}\left(f(t,x,\xi) - \overline{\mathbf{M}}(\xi)\right)\right|^2}{\mathbf{M}_-(\xi)} d\xi = 0. \tag{1.19}$$

Remark 1. Note that in the space $\overline{\mathbf{H}}^N$, the order of the differentiation on $f(t,x,\xi)$ with respect to time is at most one. In general, the solutions may not be uniformly bounded in the space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)$. However, for any fixed $T>0$, there exists a positive constant $C(T)>0$ such that

$$\sup_{0\leqq t\leqq T}\sum_{\gamma_0+|\alpha|+|\beta|\leqq N}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(f(t,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right|^2}{\mathbf{M}_-(\xi)}d\xi dx$$
$$\leqq C(T). \tag{1.20}$$

Here, we relate and contrast our work with earlier contributions by other authors. For the Vlasov–Poisson–Boltzmann system, the behavior of the solutions to the linearized system around a global Maxwellian $\overline{\mathbf{M}}$ was studied in [6, 7]. When the initial data is a small perturbation of vacuum, a global existence result in the whole space was given in [12] for a class of "soft" potentials. The analysis there relies on the time decay estimate of the electric potential $\Phi(t,x)$ which follows from the $L^1_x\left(\mathbf{R}^3\right)$ estimate on $\rho(t,x)$ and the assumptions on the collision potentials. As pointed out in [10], this argument cannot be used for establishing existence of a global classical solution to the Vlasov–Poisson–Boltzmann system near a Maxwellian $\overline{\mathbf{M}}$, because it is very difficult, if not impossible, to obtain the desired $L^1_x\left(\mathbf{R}^3\right)$ estimate on $\rho(t,x)-\overline{\rho}$.

To construct global-in-time classical solutions around a global Maxwellian, a nonlinear energy method based on the decomposition with respect to the global Maxwellian was used in [10] for periodic data. For perturbations that preserve mass, momentum, and total energy, the existence of global smooth periodic solutions to the Vlasov–Poisson–Boltzmann system was obtained in the above reference. The analysis is based on a new estimate in the form

$$-\sum_{|\alpha|\leqq N}\int_0^t\int_{\mathbf{T}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\left(f(\tau,x,\xi)-\overline{\mathbf{M}}(\xi)\right)L_{\overline{\mathbf{M}}}\left[\partial_x^\alpha\left(f(\tau,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right]}{\overline{\mathbf{M}}(\xi)}d\xi dx d\tau$$
$$\geqq C\sum_{|\alpha|\leqq N}\int_0^t\int_{\mathbf{T}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial_x^\alpha\left(f(\tau,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right|^2}{\overline{\mathbf{M}}(\xi)}d\xi dx d\tau. \tag{1.21}$$

Here $\mathbf{T}^3=[-\pi,\pi]^3$. In fact, for periodic solution, the zero-th order estimate of the solution in (1.21) follows directly from the Poisson equation and the Poincaré inequality which give

$$\|\nabla_x\Phi(t,x)\|_{L^2_x(\mathbf{T}^3)}\leqq O(1)\|\rho(t,x)-\overline{\rho}\|_{L^2_x(\mathbf{T}^3)}\leqq O(1)\|\nabla_x\rho(t,x)\|_{L^2_x(\mathbf{T}^3)}.$$

This together with the conservation laws imply that both $|\nabla_x\Phi(t,x)|^2$ and $(\rho(x,t)-\overline{\rho})^2$ are uniformly integrable over space–time. Notice that for the problem on the whole space, the above argument based on the Poincaré inequality is not valid.

In this paper, we apply the decomposition with respect to the local Maxwellian as in the case for the Boltzmann equation without forcing, which was studied in [15].

To capture the dissipation on both the macroscopic and microscopic components of the solution, we use the macroscopic system (1.13) applying techniques from the theory of conservation laws. In this way, the behavior of the local Maxwellian is clear and the dissipative effects from the viscosity, heat conductivity and the one from the linearized collision operator on the microscopic component are clearly analyzed. Furthermore, this method would be helpful for studying the problem for the fluid dynamic limit, i.e. the behavior of the solutions when the Knudsen number tends to zero.

Besides the decomposition, note also that the *a priori* energy estimate is closed in the space $\overline{\mathbf{H}}^N$, not the usual space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)$ as for the Boltzmann equation without forcing. The derivatives on the macroscopic components with respect to time and space are equivalent for the Boltzmann equation without forcing because the time derivatives can be recovered from the space derivatives through the conservation laws. However, this may not be the case for the Boltzmann equation with forcing. In the momentum equation, the force may not be integrable with respect to space and time. In the system considered in this paper, there is no $L^2_{t,x}\left(\mathbf{R}^+ \times \mathbf{R}^3\right)$ estimate on $\nabla_x \Phi(t,x)$ on the whole space. Therefore, even though the estimates on the space derivatives of the solution can be obtained through the dissipation induced by viscosity and heat conductivity, a similar estimate on the time derivatives may not be obtained. Fortunately, the estimate in the space $\overline{\mathbf{H}}^N$ can be closed because we obtain a new space and time integrability estimate on $|\rho(t,x) - \overline{\rho}|^2$ induced by the dissipative effect of the Poisson equation $(1.2)_2$.

Finally, the energy estimates are worked out both with respect to the local Maxwellian $\mathbf{M}(t,x,\xi)$ and the global Maxwellian $\mathbf{M}_-(\xi)$ as in [16]. The energy estimate with respect to a global Maxwellian is used because some polynomials in ξ arise from the derivatives of the local Maxwellian, while the collision frequency $\nu(\xi)$ in (1.14) is only of order $(1 + |\xi|)$ for the hard-sphere model. However, to obtain the higher order energy estimates on the macroscopic component $\mathbf{M}$, there is no need to use the energy estimate with respect to $\mathbf{M}_-$.

The rest of the paper is organized as follows. The microscopic and macroscopic versions of the H-theorems will be stated in Section 2. The main energy estimates are given for the case when $N = 4$ in Section 3. The case when $N > 4$ can be discussed in a similar fashion. The proof of Theorem 1 will be given in Section 4, and the proofs of some technical lemmas stated in Section 3 are given in the Appendix.

Notation. Throughout the paper, $O(1)$ and C denote generic positive constants independent of λ, and ε, $C(\cdot,\cdot)$ denotes a positive constant depending on the quantities in the parenthesis, and μ is a sufficiently small positive constant. Note that constants may vary from line to line.

For $\gamma = (\gamma_0, \alpha, \beta)$, we use ∂^γ to denote the differential operator $\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta$. Here γ_0 is a non-negative integer, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices with length $|\alpha|$ and $|\beta|$, respectively. C_b^a means $\begin{pmatrix} a \\ b \end{pmatrix}$.

In the following energy estimates, we will use the following sets of indices for different cases.

$$\begin{cases} \Lambda_1 = \{\gamma = (\gamma_0, \alpha, 0) : \quad \gamma_0 \leqq 1,\ \gamma_0 + |\alpha| \leqq 4\}, \\ \Lambda_2^j = \{\gamma = (\gamma_0, \alpha, \beta) : \quad \gamma_0 \leqq j,\ \gamma_0 + |\alpha| + |\beta| \leqq 4\}, \quad j = 1, 2, 3, 4, \\ \Lambda_3 = \{\gamma = (\gamma_0, \alpha, \beta) : \quad \gamma \in \Lambda_2^1,\ |\beta| \geqq 1\}, \\ \Lambda_4 = \{\gamma = (\gamma_0, \alpha, 0) : \quad \gamma_0 \leqq 1,\ \gamma_0 + |\alpha| \leqq 3\}, \\ \Lambda_5 = \{\gamma = (0, \alpha, \beta) : \quad |\alpha| \geqq 1,\ |\beta| \geqq 1,\ |\alpha| + |\beta| \leqq 4\}, \\ \Lambda_6 = \{\gamma = (1, \alpha, \beta) : \quad |\beta| \geqq 1,\ |\alpha| + |\beta| \leqq 3\}, \\ \Lambda_7 = \{\gamma = (\gamma_0, \alpha, 0) : \quad \gamma_0 \leqq 1,\ 2 \leqq \gamma_0 + |\alpha| \leqq 4\}. \end{cases}$$

2. *H*-theorem

The celebrated H-theorem of the Boltzmann equation is based on the bilinear structure of $Q(f, f)$, that is,

$$\int_{\mathbf{R}^3} Q(f, f) \ln f d\xi \leqq 0,$$

where the equality holds only when $f(t, x, \xi)$ is a Maxwellian.

Corresponding to the macroscopic and microscopic components, the H-theorem can be seen from two viewpoints. Firstly, it is apparent from (1.15) and (1.16) that dissipation occurs due to the effect of the linearized collision operator $L_{\mathbf{M}}$ acting on the microscopic components. Secondly, as apparent from the conservation laws (1.13), dissipation occurs due to the viscosity and heat conductivity at the macroscopic level.

In the following, we derive some inequalities regarding the nonlinear and linearized collision operators $Q(f, f)$ and $L_{\mathbf{M}}$. The first lemma is from [8].

Lemma 1. *There exists a positive constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} Q(f, g)^2}{\tilde{\mathbf{M}}} d\xi \leqq C \left\{ \int_{\mathbf{R}^3} \frac{\nu(\xi) f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int_{\mathbf{R}^3} \frac{g^2}{\tilde{\mathbf{M}}} d\xi + \int_{\mathbf{R}^3} \frac{f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int_{\mathbf{R}^3} \frac{\nu(\xi) g^2}{\tilde{\mathbf{M}}} d\xi \right\}, \tag{2.1}$$

where $\tilde{\mathbf{M}}$ *is any Maxwellian such that the above integrals are well defined.*

Based on Lemma 1, the following result was proved in [16].

Lemma 2. *If* $\frac{\theta}{2} < \tilde{\theta} < \theta$, *then there exist two positive constants* $\overline{\sigma} = \overline{\sigma}(\rho, u, \theta; \tilde{\rho}, \tilde{u}, \tilde{\theta})$ *and* $\eta_0 = \eta_0(\rho, u, \theta; \tilde{\rho}, \tilde{u}, \tilde{\theta})$ *such that if* $|u - \tilde{u}| + |\theta - \tilde{\theta}| < \eta_0$ *and* $h(\xi) \in \mathcal{N}^{\perp}$, *we have*

$$-\int_{\mathbf{R}^3} \frac{hL_{\mathbf{M}}h}{\tilde{\mathbf{M}}} d\xi \geqq \overline{\sigma} \int_{\mathbf{R}^3} \frac{\nu(\xi)h^2}{\tilde{\mathbf{M}}} d\xi.$$

Here $\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t, x, \xi)$ *and* $\tilde{\mathbf{M}}(t, x, \xi) = \tilde{\mathbf{M}}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}(t, x, \xi)$.

As a direct consequence of Lemma 2 and the Cauchy inequality, we have the following corollary, cf. [16].

Corollary 1. *Under the assumptions in* Lemma 2, *for* $h(\xi) \in \mathcal{N}^{\perp}$, *we have*

$$\begin{cases} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leqq \sigma^{-2} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} h^2(\xi)}{\mathbf{M}} d\xi, \\ \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}_-} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leqq \overline{\sigma}^{-2} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} h^2(\xi)}{\mathbf{M}_-} d\xi. \end{cases} \tag{2.2}$$

To construct the entropy–entropy flux pairs to the Vlasov–Poisson–Boltzmann system, we first derive the macroscopic version of the H-theorem similar to that for the Boltzmann equation without any external force [15]. We first set

$$-\frac{3}{2}\rho S \equiv \int_{\mathbf{R}^3} \mathbf{M} \ln \mathbf{M} d\xi. \tag{2.3}$$

As a result of direct calculation, we obtain

$$\begin{aligned} &-\frac{3}{2}(\rho S)_t - \frac{3}{2}\mathrm{div}_x(\rho u S) + \nabla_x \left(\int_{\mathbf{R}^3} (\xi \ln \mathbf{M}) \mathbf{G} d\xi \right) \\ &= \int_{\mathbf{R}^3} \frac{\mathbf{G} \mathbf{P}_1 \left(\xi \cdot \nabla_x \mathbf{M} \right)}{\mathbf{M}} d\xi, \end{aligned} \tag{2.4}$$

and

$$\begin{cases} S = -\frac{2}{3} \ln \rho + \ln(2\pi R\theta) + 1, \\ p = \frac{2}{3}\rho\theta = k\rho^{\frac{5}{3}} \exp(S), \\ \mathbf{E} = \theta, \quad R = \frac{2}{3}. \end{cases} \tag{2.5}$$

Remark 2. When the macroscopic entropy S is defined as (2.3), the gas constant R is normalized to be $\frac{2}{3}$ so that $\mathbf{E} = \theta$.

A convex entropy–entropy flux pair (η, q) around the global Maxwellian $\overline{\mathbf{M}} = \mathbf{M}_{[\overline{\rho},0,\overline{\theta}]}$ can be given as follows, cf. [15]. We begin by writing the conservation laws (1.10) in the following form:

$$\mathbf{m}_t + \mathrm{div}_x \mathbf{n} = - \begin{pmatrix} 0 \\ \int_{\mathbf{R}^3} \psi_1 (\xi \cdot \nabla_x \mathbf{G}) d\xi \\ \int_{\mathbf{R}^3} \psi_2 (\xi \cdot \nabla_x \mathbf{G}) d\xi \\ \int_{\mathbf{R}^3} \psi_3 (\xi \cdot \nabla_x \mathbf{G}) d\xi \\ \int_{\mathbf{R}^3} \psi_4 (\xi \cdot \nabla_x \mathbf{G}) d\xi \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \Phi_{x_1} \\ \rho \Phi_{x_2} \\ \rho \Phi_{x_3} \\ m \cdot \nabla_x \Phi \end{pmatrix}.$$

Here,

$$\begin{cases} \mathbf{m} = (m_0, m_1, m_2, m_3, m_4)^t = \left(\rho, \rho u_1, \rho u_2, \rho u_3, \rho \left(\frac{1}{2}|u|^2 + \theta\right)\right)^t, \\ \mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3), \\ \mathbf{n}_j = (n_0^j, n_1^j, n_2^j, n_3^j, n_4^j)^t \\ \quad = \left(\rho u_j, u_1 m_j + \frac{2}{3}\rho\theta, u_2 m_j + \frac{2}{3}\rho\theta, u_3 m_j + \frac{2}{3}\rho\theta, \rho u_j \left(\frac{1}{2}|u|^2 + \frac{5}{3}\theta\right)\right)^t, \\ \qquad j = 1, 2, 3. \end{cases}$$

The entropy–entropy flux pair (η, q) can then be defined by

$$\begin{cases} \eta = \overline{\theta} \left\{ -\frac{3}{2}\rho S + \frac{3}{2}\overline{\rho}\overline{S} + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{m} - \overline{\mathbf{m}}) \right\}, \\ q_j = \overline{\theta} \left\{ -\frac{3}{2}\rho u_j S + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{n}_j - \overline{\mathbf{n}}_j) \right\}, \ j = 1, 2, 3. \end{cases} \tag{2.6}$$

Since

$$\begin{cases} (\rho S)_{m_0} = S + \frac{|u|^2}{2\theta} - \frac{5}{3}, \\ (\rho S)_{m_i} = -\frac{u_i}{\theta}, \ i = 1, 2, 3, \\ (\rho S)_{m_4} = \frac{1}{\theta}, \end{cases}$$

we have

$$\begin{cases} \eta = \frac{3}{2} \left\{ \rho\theta - \overline{\theta}\rho S + \rho \left[\left(\overline{S} - \frac{5}{3} \right) \overline{\theta} + \frac{|u|^2}{2} \right] + \frac{2}{3}\overline{\rho}\overline{\theta} \right\}, \\ q_j = u_j \eta + u_j \left(\rho\theta - \overline{\rho}\overline{\theta} \right), \quad j = 1, 2, 3. \end{cases} \tag{2.7}$$

Note that for $\mathbf{m}$ in any closed bounded region $\mathcal{D} \subset \Sigma = \{\mathbf{m} : \rho > 0, \theta > 0\}$, there exists a positive constant C that depends on $\mathcal{D}$, such that the entropy–entropy flux pair in (2.7) satisfies, cf. [15, 16],

$$C^{-1} |\mathbf{m} - \overline{\mathbf{m}}|^2 \leqq \eta \leqq C |\mathbf{m} - \overline{\mathbf{m}}|^2, \tag{2.8}$$

and (η, q_1, q_2, q_3) solves the following equation

$$\eta_t + \mathrm{div}_x q = -\nabla_x \left(\int_{\mathbf{R}^3} \left(\xi \mathbf{G} \ln \mathbf{M} + \frac{3}{2} \psi_4 \xi \mathbf{G} \right) d\xi \right) + \frac{3}{2} m \cdot \nabla_x \Phi + \int_{\mathbf{R}^3} \frac{\mathbf{P}_1 \left(\xi \cdot \nabla_x \mathbf{M} \right) \mathbf{G}}{\mathbf{M}} d\xi. \tag{2.9}$$

Integrating (2.9) with respect to x over $\mathbf{R}^3$ gives

$$\frac{d}{dt} \int_{\mathbf{R}^3} \eta(t) dx = \frac{3}{2} \int_{\mathbf{R}^3} m \cdot \nabla_x \Phi dx + \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1 \left(\xi \cdot \nabla_x \mathbf{M} \right) \mathbf{G}}{\mathbf{M}} d\xi dx. \tag{2.10}$$

Since

$$\int_{\mathbf{R}^3} m \cdot \nabla_x \Phi dx = -\int_{\mathbf{R}^3} \mathrm{div}_x m \Phi dx = \int_{\mathbf{R}^3} (\rho - \overline{\rho})_t \Phi dx = \lambda \int_{\mathbf{R}^3} \Phi \Delta \Phi_t dx = -\frac{\lambda}{2} \frac{d}{dt} \int_{\mathbf{R}^3} |\nabla_x \Phi|^2 dx, \tag{2.11}$$

we obtain the entropy estimate

$$\frac{d}{dt} \left\{ \int_{\mathbf{R}^3} \left(\eta + \frac{3\lambda}{4} |\nabla_x \Phi|^2 \right) dx \right\} = \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1 \left(\xi \cdot \nabla_x \mathbf{M} \right) \mathbf{G}}{\mathbf{M}} d\xi dx, \tag{2.12}$$

which will be crucial in the later analysis of the macroscopic components of the solutions.

3. Energy estimates

In this section, we will give the entropy estimates used in the proof of the global existence of solutions. For this, we first assume the following *a priori* estimate,

$$\begin{aligned} N(t)^2 = \sup_{0 \leqq \tau \leqq t} \Bigg\{ & \sum_{\gamma \in \Lambda_1} \int_{\mathbf{R}^3} \left(|\partial^\gamma \left(\rho - \overline{\rho}, u, \theta - \overline{\theta} \right) (\tau, x)|^2 + \lambda |\nabla_x \partial^\gamma \Phi(\tau, x)|^2 \right) dx \\ & + \sum_{\gamma \in \Lambda_2^1} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}(\tau, x, \xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx \Bigg\} \\ & + \sum_{\gamma \in \Lambda_2^1} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) |\partial^\gamma \mathbf{G}(\tau, x, \xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx d\tau \\ \leqq & \ \delta_0^2. \end{aligned} \tag{3.1}$$

Here $\delta_0 > 0$ is a sufficiently small constant such that $\lambda\delta_0 < 1$.

First, from the Poisson equation $(1.2)_2$ and the conservation laws (1.10), we have

$$N(0) \leqq O(1)\mathcal{E}(f_0), \tag{3.2}$$

and

$$\begin{cases} \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha \Phi_t(t,x)\right|^2 dx \leqq \frac{4}{\lambda^2}\int_{\mathbf{R}^3} \left|\partial_x^\alpha m(t,x)\right|^2 dx, \quad |\alpha| \leqq 4, \\ \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha \Phi(t,x)\right|^2 dx \leqq \frac{4}{\lambda^2} \sum_{|\alpha'|=|\alpha|-1} \int_{\mathbf{R}^3} \left|\partial_x^{\alpha'}(\rho(t,x)-\overline{\rho})\right|^2 dx, \\ \qquad 1 \leqq |\alpha| \leqq 5. \end{cases} \tag{3.3}$$

Sobolev's inequality, (3.1) and (3.3) imply that

$$\begin{cases} \left|(\rho(t,x)-\overline{\rho}, u(t,x), \theta(t,x)-\overline{\theta})\right| \\ \quad + \sum_{|\alpha|\leqq 1} \left(|\partial_x^\alpha \partial_t(\rho,u,\theta)| + |\nabla_x \partial_x^\alpha(\rho,u,\theta)|\right)(t,x) \leqq O(1)\delta_0, \\ \sum_{|\alpha|\leqq 3} \left|\nabla_x \partial_x^\alpha \Phi(t,x)\right| + \sum_{|\alpha|\leqq 2} \left|\nabla_x \partial_x^\alpha \Phi_t(t,x)\right| \leqq O(1)\frac{\delta_0}{\lambda} \leqq O(1)\delta_0, \\ \int_{\mathbf{R}^3} \left\{ \frac{1}{\mathbf{M}_-} \left(|\mathbf{G}|^2 + \sum_{|\alpha|\leqq 1} \left(|\nabla_x \partial_x^\alpha \mathbf{G}|^2 + |\partial_x^\alpha \mathbf{G}_t|^2 \right) \right) \right\}(t,x,\xi) d\xi \leqq O(1)\delta_0^2. \end{cases} \tag{3.4}$$

The L^2 estimates applied to the collision operators $Q\left(\partial^\gamma \mathbf{G}, \partial^{\gamma'}\mathbf{G}\right)$ and $Q(\partial^\gamma \mathbf{M}, \partial^{\gamma'}\mathbf{G}\big)$ with respect to $\mathbf{M}$ and $\mathbf{M}_-$ are given in the following lemma.

Lemma 3. *Under assumption* (3.1)*, for each* $\gamma \in \Lambda_1$, $\gamma' \in \Lambda_1$, $|\gamma| + |\gamma'| \leqq 4$*, we have the following estimates with respect to the weight* $\mathbf{M}$*,*

$$\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial^\gamma \mathbf{G}, \partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}} d\xi dx d\tau \leqq O(1)\delta_0^2 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)\left(|\partial^\gamma \mathbf{G}|^2 + |\partial^{\gamma'}\mathbf{G}|^2\right)}{\mathbf{M}} d\xi dx d\tau. \tag{3.5}$$

In the case $|\gamma| > 0$*, in which a stronger estimate can be obtained*

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial^\gamma \mathbf{M}, \partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}} d\xi dx d\tau \\ &\quad \leqq O(1)\delta_0^2 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ &\qquad + O(1)\delta_0^2 \sum_{|\alpha|\leqq|\gamma|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha(\rho,u,\theta)\right|^2 dx d\tau. \end{aligned} \tag{3.6}$$

The corresponding estimates with respect to the weight $\mathbf{M}_-$ *are different when* $|\gamma| \geqq 3$ *and are given by*

$$\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial^\gamma \mathbf{G}, \partial^{\gamma'} \mathbf{G}\right)\right|^2}{\mathbf{M}_-} d\xi dx d\tau$$

$$\leqq \begin{cases} O(1)\delta_0^2 \displaystyle\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)\left(|\partial^\gamma \mathbf{G}|^2 + |\partial^{\gamma'} \mathbf{G}|^2\right)}{\mathbf{M}_-} d\xi dx d\tau, \\ \quad \textit{if } \max\{|\gamma|, |\gamma'|\} \leqq 2, \\ O(1)\delta_0^2 \displaystyle\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}_-} \left(\sum_{|\alpha| \leqq 3} |\partial_x^\alpha \partial^{\gamma'} \mathbf{G}|^2 + |\partial^\gamma \mathbf{G}|^2 \right) d\xi dx d\tau, \\ \quad \textit{if } \max\{|\gamma|, |\gamma'|\} = |\gamma| \geqq 3, \end{cases} \tag{3.7}$$

and

$$\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial^\gamma \mathbf{M}, \partial^{\gamma'} \mathbf{G}\right)\right|^2}{\mathbf{M}_-} d\xi dx d\tau$$

$$\leqq \begin{cases} O(1)\delta_0^2 \displaystyle\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau, \\ \quad \textit{if } 0 < |\gamma| \leqq 2, \\ O(1)\delta_0^2 \displaystyle\sum_{|\alpha| \leqq 3} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_x^\alpha \partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau, \\ \quad + O(1)\delta_0^2 \displaystyle\sum_{|\alpha| \leqq |\gamma| - 1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha (\rho, u, \theta)\right|^2 dx d\tau, \\ \quad \textit{if } |\gamma| \geqq 3. \end{cases} \tag{3.8}$$

Proof. We only prove (3.5) and (3.8) because the proof for the other estimates is similar. Firstly, (3.4) together with the fact that $\frac{\theta}{2} < \theta_- < \theta$ imply

$$\sum_{\gamma \in \Lambda_2^1, |\gamma| \leqq 2} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi \leqq \sum_{\gamma \in \Lambda_2^1, |\gamma| \leqq 2} \int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi \leqq O(1)\delta_0^2.$$

From this, Lemma 1, and the fact that $|\gamma| + |\gamma'| \leqq 4$, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial^\gamma \mathbf{G}, \partial^{\gamma'} \mathbf{G}\right)\right|^2}{\mathbf{M}} d\xi dx d\tau \\
&\leqq O(1) \int_0^t \int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}} d\xi \right.\\
&\quad \left. + \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi \int_{\mathbf{R}^3} \frac{|\partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}} d\xi \right) dx d\tau \\
&\leqq O(1)\delta_0^2 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)\left(|\partial^\gamma \mathbf{G}|^2 + |\partial^{\gamma'} \mathbf{G}|^2\right)}{\mathbf{M}} d\xi dx d\tau,
\end{aligned}$$

which yields (3.5).

For (3.8) and Lemma 1, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial^\gamma \mathbf{M}, \partial^{\gamma'} \mathbf{G}\right)\right|^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\leqq O(1) \int_0^t \int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{M}|^2}{\mathbf{M}_-} d\xi \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right.\\
&\quad \left. + \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{M}|^2}{\mathbf{M}_-} d\xi \int_{\mathbf{R}^3} \frac{|\partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) dx d\tau \\
&= I_1^{\gamma,\gamma'} + I_2^{\gamma,\gamma'}. \qquad (3.9)
\end{aligned}$$

When $1 \leqq |\gamma| \leqq 2$, we have $\gamma, \gamma' \in \Lambda_1$ so that (3.4) gives

$$I_1^{\gamma,\gamma'} + I_2^{\gamma,\gamma'} \leqq O(1)\delta_0^2 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'} \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau.$$

On the other hand, if $|\gamma| \geqq 3$, then $|\gamma'| \leqq 1$ because $|\gamma| + |\gamma'| \leqq 4$. Hence, by (3.3), (3.4) and the conservation laws (1.10), $I_1^{\gamma,\gamma'}$ satisfies

$$\begin{aligned}
I_2^{\gamma,\gamma'} &\leqq O(1)\delta_0^2 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{M}|^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\leqq O(1)\delta_0^2 \sum_{|\alpha| \leqq |\gamma|-1} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha (\rho, u, \theta)|^2 dx d\tau.
\end{aligned}$$

Here we have used the fact that $|\gamma| \geqq 3$.

For $I_1^{\gamma,\gamma'}$, by using the identity

$$\begin{aligned}
f^2(x_1, x_2, x_3) &= 2 \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} \left(f f_{x_1}\right)_{x_2 x_3} dx_1 dx_2 dx_3 \\
&\leqq \sum_{|\alpha| \leqq 3} \int_{\mathbf{R}^3} |\partial_x^\alpha f|^2 dx,
\end{aligned}$$

from the *a priori* assumption (3.1), we have

$$\int_0^t \left(\sup_{x\in\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) d\tau \leqq O(1) \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'}\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau.$$

Thus,

$$\begin{aligned} I_1^{\gamma,\gamma'} &\leqq O(1) \sup_{0\leqq\tau\leqq t, x\in\mathbf{R}^3} \left\{ \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{M}|^2}{\mathbf{M}_-} d\xi dx \right\} \\ &\quad \times \int_0^t \left(\sup_{x\in\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) d\tau \\ &\leqq O(1)\delta_0^2 \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_x^\alpha \partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau. \end{aligned}$$

The above estimates on $I_1^{\gamma,\gamma'}$ and $I_2^{\gamma,\gamma'}$ give (3.8) by using (3.9). This completes the proof of the lemma. □

Remark 3. As a direct consequence of Lemma 3, we have the following estimates because of the bilinear forms of the operator $Q(f, g)$ and $\mathbf{L}_\mathbf{M}(h)$.

$$\begin{aligned} &\sum_{\gamma\in\Lambda_1} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1}|\partial^\gamma Q(\mathbf{G},\mathbf{G})|^2}{\tilde{\mathbf{M}}} d\xi dx d\tau \\ &\quad \leqq O(1)\delta_0^2 \sum_{\gamma\in\Lambda_1} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\tilde{\mathbf{M}}} d\xi dx d\tau, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\sum_{\gamma\in\Lambda_1} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left|\partial^\gamma (L_\mathbf{M}\mathbf{G}) - L_\mathbf{M}\left(\partial^\gamma \mathbf{G}\right)\right|^2}{\tilde{\mathbf{M}}} d\xi dx d\tau \\ &\quad \leqq O(1)\delta_0^2 \sum_{\gamma\in\Lambda_1} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\tilde{\mathbf{M}}} d\xi dx d\tau \\ &\quad\quad + O(1)\delta_0^2 \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha (\rho, u, \theta)|^2 dx d\tau. \end{aligned} \tag{3.11}$$

Here $\tilde{\mathbf{M}}$ can be taken as $\mathbf{M}$ or $\mathbf{M}_-$.

Using the above results, we now give energy estimates for the solution in the following three subsections. The first subsection concerns estimates on the entropy $\eta(\rho, u, \theta)$ and the microscopic component $\mathbf{G}$. The other two subsections concern the derivatives of the solution with respect to the weights of the local Maxwellian $\mathbf{M}$ and the global Maxwellian $\mathbf{M}_-$ respectively.

3.1. Lower order estimates

In this subsection, we will give the energy estimates on the entropy $\eta(\rho, u, \theta)$ and the microscopic component $\mathbf{G}(t, x, \xi)$.

First, integrating (2.12) with respect to t over $[0, t]$ yields

$$\int_{\mathbf{R}^3} \left(\eta + \frac{3\lambda}{4}|\nabla_x \Phi|^2\right) dx \Big|_0^t = \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\mathbf{G}}{\mathbf{M}} d\xi dx d\tau. \tag{3.12}$$

From (1.12), Lemma 1 and Corollary 1, and the fact that the following estimate

$$\begin{aligned}
&-\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M}) L_{\mathbf{M}}^{-1} (\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M}))}{\mathbf{M}} d\xi dx d\tau \\
&\geqq C \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})|^2}{\mathbf{M}} d\xi dx d\tau \\
&\geqq C \int_0^t \int_{\mathbf{R}^3} |\nabla_x (u, \theta)|^2 dx d\tau,
\end{aligned}$$

holds for some positive constant C, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\mathbf{G}}{\mathbf{M}} d\xi dx d\tau \\
&= \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M}) L_{\mathbf{M}}^{-1} (\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M}))}{\mathbf{M}} d\xi dx d\tau \\
&\quad + \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M}) L_{\mathbf{M}}^{-1}(\Theta)}{\mathbf{M}} d\xi dx d\tau \\
&\leqq -C \int_0^t \int_{\mathbf{R}^3} |\nabla_x (u, \theta)|^2 dx d\tau \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(|\mathbf{G}_t|^2 + |\nabla_x \mathbf{G}|^2 + \delta_0^2 \left(|\nabla_\xi \mathbf{G}|^2 + |\mathbf{G}|^2\right)\right) d\xi dx d\tau.
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12) yields

$$\begin{aligned}
&\int_{\mathbf{R}^3} \left(\eta + \lambda |\nabla_x \Phi|^2\right)(t) dx + \int_0^t \int_{\mathbf{R}^3} |\nabla_x (u, \theta)|^2 dx d\tau \leqq O(1) N(0)^2 \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(|\mathbf{G}_t|^2 + |\nabla_x \mathbf{G}|^2 + \delta_0^2 \left(|\nabla_\xi \mathbf{G}|^2 + |\mathbf{G}|^2\right)\right) d\xi dx d\tau.
\end{aligned} \tag{3.14}$$

For the microscopic component $\mathbf{G}$, multiplying (1.11) by $\frac{\mathbf{G}}{\mathbf{M}}$ and integrating the result with respect to t, x, and ξ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$, and using (1.15) and Lemma 3, we obtain

$$\begin{aligned}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\ \leqq O(1)N(0)^2+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\ +O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau.\end{aligned}\tag{3.15}$$

Similarly, if we replace the weight $\mathbf{M}$ by the global Maxwellian $\mathbf{M}_-$, we have

$$\begin{aligned}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\ \leqq O(1)N(0)^2+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dxd\tau.\end{aligned}\tag{3.16}$$

The equations (3.14)–(3.16) give the complete lower order energy estimates.

3.2. Higher order energy estimates with respect to $\mathbf{M}$

In this subsection, we will consider higher order energy estimates, i.e. $\partial^\gamma\mathbf{M}$, $\partial^\gamma\mathbf{G}$, and $\partial^\gamma f$ for $|\gamma|\geqq 1$ with respect to the local Maxwellian $\mathbf{M}$. Since the proofs are tedious and technical, we just state the results in this subsection and present the proofs in the appendix.

Firstly, for $1\leqq|\alpha|\leqq 3$, we have the following estimate that applies to $\partial_x^\alpha\mathbf{M}$.

Lemma 4. *Under the assumptions in* Lemma 3 *and for* $j=1,2,3$*, we have*

$$\begin{aligned}\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\left(\lambda\left|\nabla_x\partial_x^\alpha\Phi\right|^2+\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi\right)dx\\ +\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2dxd\tau\\ \leqq O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|\leqq j}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2dxd\tau\\ +O(1)\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\partial_x^\alpha\mathbf{G}_t|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)}{\mathbf{M}}d\xi dxd\tau\end{aligned}$$

$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leqq j}\left|\nabla_\xi\partial_x^\alpha\mathbf{G}\right|^2+\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha\mathbf{G}\right|^2\right)d\xi dx d\tau$$
$$+O(1)\delta_{1,j}\sum_{|\alpha|=1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau, \qquad (3.17)$$

where we have used the assumption that $\delta_0\lambda<1$. *Hereinafter,* $\delta_{i,j}$ *is the Kronecker symbol, i.e.*

$$\delta_{i,j}=\begin{cases}0, & i\neq j,\\ 1, & i=j.\end{cases}$$

Secondly, for $\gamma\in\Lambda_1$, $|\gamma|\leqq 3$, we have the following estimate that applies to $\partial^\gamma\mathbf{G}$.

Lemma 5. *Under the assumptions in* Lemma 3 *and for* $j=0,1,2$, *we have*

$$\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau$$
$$\leqq O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau$$
$$+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=j+1}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\delta_0\sum_{|\alpha|\leqq j}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dxd\tau$$
$$+O(1)\delta_0\sum_{|\alpha|\leqq j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_x^\alpha\mathbf{G}|^2+|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau, \qquad (3.18)$$

and

$$\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau$$
$$\leqq O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau$$
$$+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=j+1}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\delta_0\sum_{|\alpha|\leqq j}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dxd\tau$$
$$+O(1)\sum_{|\alpha|=j+1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_t\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau$$
$$+O(1)\delta_0\sum_{|\alpha|\leqq j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_t\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau$$
$$+O(1)\delta_{1,j+1}\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau$$

$$+O(1)\left(1-\delta_{1,j+1}\right)\lambda^{-2}\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\rho|^2dxd\tau$$

$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leqq j}|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2+\sum_{|\alpha|<j}|\nabla_\xi\partial_t\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau. \tag{3.19}$$

A suitable linear combination of (3.17), (3.18), and (3.19) yields the following estimates.

$$\int_{\mathbf{R}^3}\left(\lambda\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2+\int_{\mathbf{R}^3}\frac{|\nabla_x\mathbf{M}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2}{\mathbf{M}}d\xi\right)dx$$
$$+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=1}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi\right)dxd\tau$$
$$\leqq O(1)N(0)^2+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dxd\tau$$
$$+O(1)\sum_{|\alpha|=1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_t\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau$$
$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 1}|\nabla_x\partial_x^\alpha\rho|^2+|\nabla_x(u,\theta)|^2\right)dxd\tau$$
$$+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau$$
$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\beta|\leqq 1}|\nabla_x\partial_\xi^\beta\mathbf{G}|^2+\sum_{|\alpha|\leqq 4}|\partial_x^\alpha\mathbf{G}|^2\right.$$
$$\left.+|\nabla_\xi\mathbf{G}|^2+|\nabla_\xi\mathbf{G}_t|^2\right)d\xi dxd\tau, \tag{3.20}$$

$$\int_{\mathbf{R}^3}\left(\lambda\sum_{|\alpha|=3}|\partial_x^\alpha\Phi|^2+\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{M}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}}d\xi\right)dx$$
$$+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=2}|\nabla_x\partial_x^\alpha(u,\theta)|^2\right.$$
$$\left.+\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi\right)dxd\tau$$

$$\leqq O(1)N(0)^2 + O(1)\delta_0 \sum_{|\alpha|=1} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_x \partial_x^\alpha \mathbf{G}|^2 + |\partial_x^\alpha \mathbf{G}_t|^2}{\mathbf{M}_-} d\xi dx d\tau$$

$$+O(1) \sum_{|\alpha|=2} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(|\partial_t \partial_x^\alpha \mathbf{G}|^2 + |\nabla_x \partial_x^\alpha \mathbf{G}|^2\right) d\xi dx d\tau$$

$$+O(1)\left(\delta_0 + \lambda^{-2}\right) \int_0^t \int_{\mathbf{R}^3} \left(\sum_{|\alpha|\leqq 2} |\nabla_x \partial_x^\alpha \rho|^2 + \sum_{|\alpha|\leqq 1} |\nabla_x \partial_x^\alpha (u,\theta)|^2 \right) dx d\tau$$

$$+O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(\sum_{|\alpha|\leqq 2} |\nabla_\xi \partial_x^\alpha \mathbf{G}|^2 + \sum_{|\alpha|\leqq 4} |\partial_x^\alpha \mathbf{G}|^2 \right.$$

$$\left. + \sum_{|\beta|\leqq 1} |\partial_\xi^\beta \mathbf{G}_t|^2 \right) d\xi dx d\tau, \tag{3.21}$$

and

$$\int_{\mathbf{R}^3} \left(\lambda \sum_{|\alpha|=4} |\partial_x^\alpha \Phi|^2 + \sum_{|\alpha|=2} \int_{\mathbf{R}^3} \frac{|\nabla_x \partial_x^\alpha \mathbf{M}|^2 + |\nabla_x \partial_x^\alpha \mathbf{G}|^2 + |\partial_x^\alpha \mathbf{G}_t|^2}{\mathbf{M}} d\xi \right) dx$$

$$+ \int_0^t \int_{\mathbf{R}^3} \left(\sum_{|\alpha|=3} |\nabla_x \partial_x^\alpha (u,\theta)|^2 \right.$$

$$\left. + \sum_{|\alpha|=2} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(|\nabla_x \partial_x^\alpha \mathbf{G}|^2 + |\partial_x^\alpha \mathbf{G}_t|^2\right) d\xi \right) dx d\tau$$

$$\leqq O(1)N(0)^2 + O(1)\delta_0 \sum_{|\alpha|=2} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_x \partial_x^\alpha \mathbf{G}|^2 + |\partial_x^\alpha \mathbf{G}_t|^2}{\mathbf{M}_-} d\xi dx d\tau$$

$$+O(1) \sum_{|\alpha|=3} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(|\partial_t \partial_x^\alpha \mathbf{G}|^2 + |\nabla_x \partial_x^\alpha \mathbf{G}|^2\right) d\xi dx d\tau$$

$$+O(1)\lambda^{-2} \sum_{|\alpha|=2} \int_0^t \int_{\mathbf{R}^3} |\partial_x^\alpha \rho|^2 dx d\tau$$

$$+O(1)\delta_0 \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha (\rho, u, \theta)|^2 dx d\tau$$

$$+O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(\sum_{|\alpha|\leqq 3} \left(|\nabla_\xi \partial_x^\alpha \mathbf{G}|^2 + |\nabla_x \partial_x^\alpha \mathbf{G}|^2\right) \right.$$

$$\left. + \sum_{|\alpha|\leqq 1} \left(|\partial_x^\alpha \mathbf{G}_t|^2 + |\nabla_\xi \partial_x^\alpha \mathbf{G}_t|^2\right) \right) d\xi dx d\tau. \tag{3.22}$$

Combining (3.14), (3.15), (3.20), (3.21), and (3.22) gives the following corollary.

Corollary 2. *Under the assumptions listed in* Lemma 3*, we have*

$$\begin{aligned}
&\int_{\mathbf{R}^3}\Bigg(\eta+\lambda\sum_{1\leqq|\alpha|\leqq 4}|\partial_x^\alpha\Phi|^2 \\
&\quad+\int_{\mathbf{R}^3}\frac{1}{\mathbf{M}}\Bigg(\sum_{|\alpha|\leqq 2}|\nabla_x\partial_x^\alpha\mathbf{M}|^2+\sum_{\gamma\in\Lambda_4}|\partial^\gamma\mathbf{G}|^2\Bigg)d\xi\Bigg)dx \\
&\quad+\int_0^t\int_{\mathbf{R}^3}\Bigg(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{\gamma\in\Lambda_4}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\Bigg)dxd\tau \\
&\leqq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau \\
&\quad+O(1)\sum_{|\alpha|=3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi dxd\tau \\
&\quad+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau \\
&\quad+O(1)\left(\delta_0+\lambda^{-2}\right)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau \\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\Bigg(\sum_{|\alpha|\leqq 3}|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2 \\
&\quad+\sum_{|\alpha|\leqq 1}|\nabla_\xi\partial_x^\alpha\mathbf{G}_t|^2+\sum_{|\beta|\leqq 1}|\partial_\xi^\beta\mathbf{G}_t|^2\Bigg)d\xi dxd\tau. \qquad (3.23)
\end{aligned}$$

Now we turn to the estimates including the derivatives of the solutions $f(t,x,\xi)$ to the Vlasov–Poisson–Boltzmann system (1.2) with respect to the velocity ξ. It is clear that we only need to obtain the corresponding estimates on the microscopic component. The idea in the following estimate is to transfer the derivatives with respect to ξ to the derivatives with respect to the space and time variables. The following lemma provides an estimate of $\partial^\gamma\mathbf{G}$ with $\gamma\in\Lambda_3$.

Lemma 6. *Under the assumptions in* Lemma 3*, we have*

$$\sum_{\gamma\in\Lambda_3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau$$

$$
\begin{aligned}
&\leqq O(1)N(0)^2 + O(1)\delta_0 \sum_{\gamma\in\Lambda_3} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\quad + O(1) \sum_{|\alpha|=2} \int_0^t \int_{\mathbf{R}^3} |\partial_x^\alpha \Phi|^2 dx d\tau \\
&\quad + O(1) \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha (\rho, u, \theta)|^2 dx d\tau \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left(|\mathbf{G}|^2 + \sum_{|\alpha|\leqq 3} \left(|\nabla_x \partial_x^\alpha \mathbf{G}|^2 + |\partial_x^\alpha \mathbf{G}_t|^2 \right) \right) d\xi dx d\tau.
\end{aligned}
\tag{3.24}
$$

Notice that the proof of the above lemma is based on the following three estimates for the derivatives of $\mathbf{G}$ with respect to ξ whose proofs can be found in the Appendix.

$$
\begin{aligned}
&\sum_{|\beta|\leqq 4} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial_\xi^\beta \mathbf{G}|^2}{\mathbf{M}} d\xi dx \\
&\quad + \sum_{|\beta|\leqq 4} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_\xi^\beta \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\leqq O(1)N(0)^2 + O(1)\delta_0 \sum_{|\beta|\leqq 4} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial_\xi^\beta \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}^3} |\nabla_x (u, \theta)|^2 dx d\tau \\
&\quad + O(1) \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\sum_{\gamma\in\Lambda_5} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx \\
&\quad + \sum_{\gamma\in\Lambda_5} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\leqq O(1)N(0)^2 + O(1)\delta_0 \sum_{\gamma\in\Lambda_5} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}^3} \left(\sum_{1\leqq|\alpha|\leqq 3} |\nabla_x \partial_x^\alpha (u, \theta)|^2 \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\delta_0\left(|\nabla_x(u,\theta)|^2+\sum_{|\alpha|\leqq 2}|\nabla_x\partial_x^\alpha\rho|^2\right)\Bigg)dx\,d\tau\\
&+O(1)\sum_{1\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi\,dx\,d\tau\\
&+O(1)\delta_0\sum_{1\leqq|\beta|\leqq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\,dx\,d\tau,
\end{aligned}\tag{3.26}
$$

and

$$
\begin{aligned}
&\sum_{\gamma\in\Lambda_6}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\,dx\\
&\quad+\sum_{\gamma\in\Lambda_6}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\,dx\,d\tau\\
&\leqq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_6}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\,dx\,d\tau\\
&\quad+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dx\,d\tau\\
&\quad+O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\Bigg(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\\
&\quad+\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi\Bigg)dx\,d\tau\\
&\quad+O(1)\delta_0\sum_{|\beta|\geqq 1,|\alpha|+|\beta|\leqq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\,dx\,d\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}}d\xi\,dx\,d\tau.
\end{aligned}\tag{3.27}
$$

The following corollary is a direct consequence of (3.23) and (3.24).

Corollary 3. *Under the assumptions in* Lemma 3*, we have*

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\Bigg(\eta+\lambda\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha\Phi|^2\\
&\quad+\int_{\mathbf{R}^3}\frac{1}{\mathbf{M}}\left(\sum_{|\alpha|\leqq 2}|\nabla_x\partial_x^\alpha\mathbf{M}|^2+\sum_{\gamma\in\Lambda_3\cup\Lambda_4}|\partial^\gamma\mathbf{G}|^2\right)d\xi\Bigg)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{\gamma\in\Lambda_3\cup\Lambda_4}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\
&\leqq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_3\cup\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau\\
&\quad+O(1)\left(\delta_0+\lambda^{-2}\right)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau. \qquad (3.28)
\end{aligned}
$$

To obtain the fourth-order derivatives of $\mathbf{G}$ with respect to the spatial variables, we need to use the original Vlasov–Poisson–Boltzmann equation to avoid the appearance of the fifth-order derivatives. This can be summarized in the following lemma.

Lemma 7. *Under the assumptions in* Lemma 3*, we have*

$$
\begin{aligned}
&\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leqq O(1)\left(\delta_0+\lambda^{-1}\right)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau\\
&\quad+O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau. \qquad (3.29)
\end{aligned}
$$

Remark 4. To obtain (3.29), we use the property that the macroscopic components and the microscopic components are orthogonal with respect to the local Maxwellian $\mathbf{M}$, in particular, the following identity

$$
\begin{aligned}
&\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi\\
&\quad=\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi\\
&\quad=0. \qquad (3.30)
\end{aligned}
$$

We are now ready to complete the energy estimates with respect to $\mathbf{M}$. First, since

$$\int_{\mathbf{R}^3} \frac{|\partial^\gamma f|^2}{\mathbf{M}} d\xi = \int_{\mathbf{R}^3} \frac{|\mathbf{P}_0(\partial^\gamma \mathbf{M})|^2 + |\mathbf{P}_1(\partial^\gamma \mathbf{M}) + \partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi \geqq \int_{\mathbf{R}^3} \frac{|\mathbf{P}_0(\partial^\gamma \mathbf{M})|^2}{\mathbf{M}} d\xi, \tag{3.31}$$

we have by induction that

$$\begin{aligned}\sum_{|\alpha|\leqq 3} \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha (\rho, u, \theta)|^2 dx &\leqq O(1) \sum_{\gamma\in\Lambda_7} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\mathbf{P}_0(\partial^\gamma \mathbf{M})|^2}{\mathbf{M}} d\xi dx + O(1) \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\nabla_x \mathbf{M}|^2}{\mathbf{M}} d\xi dx \\ &\quad + O(1) \sum_{\gamma\in\Lambda_2^1} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx. \end{aligned}\tag{3.32}$$

For $\lambda_1 > 0$ sufficiently large, we multiply (3.28) and (3.29) by λ_1 and λ_1^2 respectively and take the sum. Then using (3.32) we have

$$\begin{aligned}&\int_{\mathbf{R}^3} \left(\eta + \lambda \sum_{|\alpha|\leqq 3} |\nabla_x \partial_x^\alpha \Phi|^2 + \sum_{|\alpha|\leqq 3} |\nabla_x \partial_x^\alpha (\rho, u, \theta)|^2 \right) dx \\ &\quad + \sum_{\gamma\in\Lambda_2^1} \left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx + \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \right) \\ &\quad + \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha (u, \theta)|^2 dx d\tau \\ &\leqq O(1) N(0)^2 + O(1)\delta_0 \sum_{\gamma\in\Lambda_2^1} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ &\quad + O(1) \sum_{|\alpha|=2} \int_0^t \int_{\mathbf{R}^3} |\partial_x^\alpha \Phi|^2 dx d\tau \\ &\quad + O(1)(\delta_0 + \lambda^{-1}) \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^\alpha \rho|^2 dx d\tau. \end{aligned}\tag{3.33}$$

To recover the estimates on $\nabla_x \partial_x^\alpha \rho$ in (3.33), we need to use the conservation laws (1.10) as in [13]. In fact, since

$$\frac{2\theta}{3\rho} \nabla_x \rho - \nabla_x \Phi = -u_t - u \cdot \nabla_x u - \frac{2}{3} \nabla_x \theta - \int_{\mathbf{R}^3} \frac{\psi(\xi \cdot \nabla_x \mathbf{G})}{\rho} d\xi, \tag{3.34}$$

for $|\alpha| \leqq 3$, we have

$$\begin{aligned}
&\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\frac{2\theta}{3\rho}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau\\
&\quad=\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\nabla_x\partial_x^\alpha\Phi dxd\tau\\
&\qquad-\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\partial_x^\alpha u_tdxd\tau\\
&\qquad-\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\partial_x^\alpha(u\cdot\nabla_xu)dxd\tau\\
&\qquad-\frac{2}{3}\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\nabla_x\partial_x^\alpha\theta dxd\tau\\
&\qquad-\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\psi\xi\cdot\partial_x^\alpha\left(\frac{\nabla_x\mathbf{G}}{\rho}\right)dxd\tau\\
&\qquad-\sum_{|\alpha|\leqq 3}\left(1-\delta_{1,|\alpha|+1}\right)\sum_{0<\alpha'\leqq\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\nabla_x\partial_x^{\alpha-\alpha'}\rho\partial_x^{\alpha'}\left(\frac{2\theta}{3\rho}\right)dxd\tau\\
&\quad=\sum_{j=1}^6 I_j,
\end{aligned}\tag{3.35}$$

where I_j, $j=1,\cdots,6$, are the corresponding terms in the above equation.

Now we estimate $I_j (1\leqq j\leqq 6)$ term by term. Firstly, for I_1, from the Poisson equation $(1.2)_2$, we have

$$\begin{aligned}
I_1&=\lambda\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha(\Delta_x\Phi)\cdot\nabla_x\partial_x^\alpha\Phi dxd\tau\\
&=-\frac{\lambda}{2}\sum_{1\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau.
\end{aligned}\tag{3.36}$$

For I_2, from the conservation laws (1.10), we have

$$\begin{aligned}
I_2=&\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\partial_x^\alpha\left(\rho-\overline{\rho}\right)\mathrm{div}_x\left(\partial_x^\alpha u_t\right)dxd\tau\\
=&\sum_{|\alpha|\leqq 3}\int_{\mathbf{R}^3}\partial_x^\alpha\left(\rho-\overline{\rho}\right)\mathrm{div}_x\left(\partial_x^\alpha u\right)dx\Big|_0^t\\
&+\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\partial_x^\alpha\left(\mathrm{div}_x(\rho u)\right)\mathrm{div}_x\left(\partial_x^\alpha u\right)dxd\tau
\end{aligned}$$

$$\leqq O(1)N(0)^2 + O(1)\sum_{|\alpha|\leqq 3}\int_{\mathbf{R}^3}\left|\left(\nabla_x\partial_x^\alpha u,\, \partial_x^\alpha(\rho-\overline{\rho})\right)\right|^2 dx$$

$$+O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\left(\delta_0\left|\nabla_x\partial_x^\alpha\rho\right|^2 + \left| nabla_x\partial_x^\alpha u\right|^2\right) dx d\tau. \quad (3.37)$$

A similar argument leads to

$$I_3 \leqq O(1)\delta_0\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u)|^2 dx d\tau, \quad (3.38)$$

$$I_4 \leqq \mu\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2 dx d\tau$$

$$+ O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\theta|^2 dx d\tau, \quad (3.39)$$

$$I_5 \leqq \mu\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2 dx d\tau$$

$$+ O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau, \quad (3.40)$$

and

$$I_6 \leqq O(1)\delta_0\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,\theta)|^2 dx d\tau. \quad (3.41)$$

By choosing $\mu > 0$ sufficiently small and substituting (3.36)–(3.41) into (3.35), we have

$$\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2 dx d\tau + \lambda\sum_{1\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2 dx d\tau$$

$$\leqq O(1)N(0)^2 + O(1)\sum_{|\alpha|\leqq 3}\int_{\mathbf{R}^3}\left|\left(\nabla_x\partial_x^\alpha u,\, \partial_x^\alpha(\rho-\overline{\rho})\right)\right|^2 dx$$

$$+O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(u,\theta)|^2 dx d\tau$$

$$+O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau. \quad (3.42)$$

When $\lambda > 0$ is chosen to be sufficiently large, the combination of (3.33) and (3.42) gives

$$\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dx$$
$$+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\right)$$
$$+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\lambda\sum_{1\leqq|\alpha|\leqq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau$$
$$\leqq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_2^1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau. \tag{3.43}$$

To complete the energy estimates with respect to $\mathbf{M}$, it remains to obtain estimates of the derivatives of the macroscopic components with respect to the time variable t. From

$$\mathbf{M}_t+\mathbf{P}_0\left(\xi\cdot\nabla_x\mathbf{M}\right)+\mathbf{P}_0\left(\xi\cdot\nabla_x\mathbf{G}\right)+\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}=0,$$

we have

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}_t|^2}{\mathbf{M}}d\xi dx\leqq O(1)\int_{\mathbf{R}^3}\left(|\nabla_x\Phi|^2+\int_{\mathbf{R}^3}\frac{|\nabla_x\mathbf{M}|^2+|\nabla_x\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx. \tag{3.44}$$

Secondly, from the conservation laws (1.10), we have

$$\sum_{|\alpha|\leqq 3}\int_{\mathbf{R}^3}|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2dx$$
$$\leqq O(1)\sum_{|\alpha|\leqq 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx$$
$$+O(1)\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 1}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dx, \tag{3.45}$$

and

$$\sum_{|\alpha|\leqq 2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau$$
$$\leqq O(1)\sum_{|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau$$
$$+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dxd\tau. \tag{3.46}$$

Combining (3.43)–(3.46) yields the following corollary.

Corollary 4. *There exists a sufficiently large positive constant* $\lambda_0 > 1$ *such that for* $\lambda > \lambda_0$*, we have*

$$\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leqq 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx$$
$$+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\right)$$
$$+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\leqq 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right.$$
$$\left.+\lambda\sum_{1\leqq|\alpha|\leqq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau$$
$$\leqq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_2^1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau. \tag{3.47}$$

3.3. Higher order energy estimates with respect to $\mathbf{M}_-$

In this subsection, we will consider certain higher order energy estimates with respect to the global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}$ in order to close the *a priori* estimate (3.1). Compared to those estimates with respect to the local Maxwellian $\mathbf{M}$, the only difference is that the macroscopic and microscopic components with respect to $\mathbf{M}$ are no longer orthogonal with respect to the global Maxwellian $\mathbf{M}_-$. More precisely, in the proof of Lemma 7, we have used the following identities

$$\begin{cases}\displaystyle\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi=0,\\ \displaystyle\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi=0.\end{cases}$$

However, if the weight $\mathbf{M}$ is replaced by $\mathbf{M}_-$, the above identities do not hold. As a result, there is an extra error term that takes the form

$$\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right.$$
$$\left.+\sum_{|\alpha|\leqq 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{1\leqq|\alpha|\leqq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau. \tag{3.48}$$

By noting this difference and using the same argument for (3.47), we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leqq 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx\\
&\quad+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\right)\\
&\leqq O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\leqq 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right.\\
&\quad\left.+\sum_{1\leqq|\alpha|\leqq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dx d\tau+O(1)N(0)^2.
\end{aligned}
\tag{3.49}
$$

Combining (3.49) with (3.47), we finally obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leqq 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx\\
&\quad+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\right)\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\leqq 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right.\\
&\quad\left.+\lambda\sum_{1\leqq|\alpha|\leqq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dx d\tau\\
&\leqq O(1)N(0)^2\leqq O(1)\mathcal{E}(f_0)^2,
\end{aligned}
\tag{3.50}
$$

which closes the *a priori* estimate (3.1) provided that $\varepsilon > 0$ is sufficiently small that

$$
\begin{cases}
\mathcal{E}(f_0)<\varepsilon,\\
O(1)\varepsilon^2<\delta_0^2.
\end{cases}
\tag{3.51}
$$

Remark 5. The equation (3.50) and the Poisson equation $(1.2)_2$ imply that,

$$
\int_0^\infty\int_{\mathbf{R}^3}|\rho(t,x)-\overline{\rho}|^2\,dx d\tau\leqq O(1)\varepsilon^2,
$$

which is not true for the force-free Boltzmann equation.

4. The proof of Theorem 1

We are now ready to prove the main result Theorem 1. The idea is to use the continuity argument to extend the local solution to be valid for all time by using the closed *a priori* estimate. To do so, we first need to prove the local existence of solutions to the Vlasov–Poisson–Boltzmann system (1.2) in the space

$$\mathbf{H}^4_{x,\xi}([0,T)) = \left\{ g(t,x,\xi) \;\middle|\; \begin{array}{c} \frac{\partial_x^\alpha \partial_\xi^\beta g(t,x\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in C\left([0,T), L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right) \\ |\alpha| + |\beta| \leqq 4 \end{array} \right\}. \tag{4.1}$$

Here $T > 0$ is some positive constant and $g(t,x,\xi) = f(t,x,\xi) - \overline{\mathbf{M}}(\xi)$.

For periodic initial data, the corresponding local existence result was given in [10]. By a straightforward modification of the argument there, we have the following local existence result for the Vlasov–Poisson–Boltzmann system (1.2) that holds on the whole space. Thus, for brevity, we omit the proof.

Lemma 8 (Local existence)**.** *For any sufficiently small constant $M > 0$, there exists a positive constant $T^*(M) > 0$ such that if*

$$\begin{aligned} \mathcal{E}(f_0) &= \left\| \nabla_x \Delta_x^{-1}\left(\rho_0(x) - \overline{\rho}\right) \right\|_{L^2_x(\mathbf{R}^3)} \\ &\quad + \sum_{|\alpha|+|\beta|\leqq 4} \left\| \frac{\partial_x^\alpha \partial_\xi^\beta (f_0(x,\xi) - \overline{\mathbf{M}}(\xi))}{\sqrt{\mathbf{M}_-(\xi)}} \right\|_{L^2_{x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)} \\ &\leqq \frac{M}{2}, \end{aligned}$$

then there is a unique classical solution $f(t,x,\xi) \in \mathbf{H}^4_{x,\xi}([0,T^*(M)))$ to the Vlasov–Poisson–Boltzmann system (1.2) *on* $[0,T^*(M)) \times \mathbf{R}^3 \times \mathbf{R}^3$ *such that $f(t,x,\xi) \geqq 0$ and*

$$\sup_{0\leqq t\leqq T^*(M)} \sum_{|\alpha|+|\beta|\leqq 4} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial_x^\alpha \partial_\xi^\beta (f(t,x,\xi) - \overline{\mathbf{M}}(\xi))|^2}{\mathbf{M}_-(\xi)} d\xi dx \leqq M.$$

By using this local existence result and the energy estimates obtained in Section 3, we can conclude that the Vlasov–Poisson–Boltzmann system (1.2) has a unique global classical solution $f(t,x,\xi) \in \overline{\mathbf{H}}^4$ satisfying $f(t,x,\xi) \geqq 0$.

To complete the proof of Theorem 1, it remains to show that (1.19) holds. In fact, from (3.50), we have

$$\begin{cases} \sum_{|\alpha|\leqq 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau \leqq O(1), \\ \sum_{|\alpha|\leqq 3}\int_0^\infty\left|\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-}d\xi dx\right| d\tau \\ \leqq O(1)\sum_{|\alpha|\leqq 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha \mathbf{G}|^2+|\partial_x^\alpha \mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx d\tau \\ \leqq O(1), \\ \sum_{|\alpha|\leqq 2}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha \mathbf{M}|^2}{\mathbf{M}_-}d\xi dx d\tau \leqq O(1), \\ \sum_{|\alpha|\leqq 2}\int_0^\infty\left|\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha \mathbf{M}|^2}{\mathbf{M}_-}d\xi dx\right| d\tau \\ \leqq O(1)\sum_{|\alpha|\leqq 2}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha \mathbf{M}|^2+|\nabla_x\partial_x^\alpha \mathbf{M}_t|^2}{\mathbf{M}_-}d\xi dx d\tau \\ \leqq O(1). \end{cases} \tag{4.2}$$

Consequently,

$$\lim_{t\to\infty}\sum_{|\alpha|\leqq 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha(\mathbf{M}-\overline{\mathbf{M}})|^2+|\nabla_x\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-}d\xi dx = 0. \tag{4.3}$$

Since

$$\begin{aligned} &\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}d\xi \\ &\leqq O(1)\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\sum_{|\alpha|=3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha \mathbf{M}|^2+|\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12} \\ &\quad +O(1)\left(\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha \mathbf{M}|^2+|\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\sum_{|\alpha|=2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha \mathbf{M}|^2+|\partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}, \end{aligned}$$

from (4.3) we have

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\left(\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi = 0. \tag{4.4}$$

Thus,

$$\begin{aligned} &\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|f(t,x,\xi)-\overline{\mathbf{M}}(\xi)|^2}{\mathbf{M}_-(\xi)}d\xi \\ &\leqq O(1)\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}(t,x,\xi)-\overline{\mathbf{M}}(\xi)|^2+|\mathbf{G}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi, \\ &= 0, \end{aligned}$$

which is (1.19), and therefore this completes the proof of Theorem 1.

Finally, we give an estimate in the $\mathbf{H}^4_{t,x,\xi}(\mathbf{R}^3 \times \mathbf{R}^3)$ norm of the solution thus obtained. That is, we prove the estimate (1.20) given in Remark 1.

By (3.3) and (3.50), from the conservation laws (1.10), we have

$$\int_{\mathbf{R}^3} \left(\sum_{|\alpha| \leqq 3} |\partial_t \partial_x^\alpha (\rho, u, \theta)|^2 + \sum_{|\alpha| \leqq 2} |\partial_t^2 \partial_x^\alpha (\rho, u, \theta)|^2 \right) dx \leqq O(1). \tag{4.5}$$

Thus, for any fixed $T > 0$, the global solution $f(t, x, \xi)$ to the Vlasov–Poisson–Boltzmann system (1.2) satisfies

$$\int_0^T \int_{\mathbf{R}^3} \left(\sum_{|\alpha| \leqq 3} |\partial_t \partial_x^\alpha (\rho, u, \theta)|^2 + \sum_{|\alpha| \leqq 2} |\partial_t^2 \partial_x^\alpha (\rho, u, \theta)|^2 + \sum_{\gamma \in \Lambda_2^1} \int_{\mathbf{R}^3} \frac{\nu(\xi) |\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) dx d\tau \\ \leqq C(T). \tag{4.6}$$

Based on (4.6) and using a proof similar to that for (3.29), we have

$$\sum_{\gamma \in \Lambda_2^2} \left(\int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma (f - \overline{\mathbf{M}})|^2}{\mathbf{M}_-} d\xi dx + \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma (f - \overline{\mathbf{M}})|^2}{\mathbf{M}_-} d\xi dx d\tau \right) \\ \leqq C(T). \tag{4.7}$$

Equation (4.7) together with (3.50) imply

$$\sum_{\gamma \in \Lambda_2^2} \int_{\mathbf{R}^3} \left(|\partial^\gamma (\rho - \overline{\rho}, u, \theta - \overline{\theta})|^2 + \int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) dx \\ + \sum_{\gamma \in \Lambda_2^2} \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) |\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ \leqq C(T). \tag{4.8}$$

This and the conservation laws (1.10) yields

$$\int_{\mathbf{R}^3} \left(\sum_{\gamma \in \Lambda_2^3} |\partial^\gamma (\rho - \overline{\rho}, u, \theta - \overline{\theta})|^2 + \sum_{\gamma \in \Lambda_2^2} \int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) dx \\ + \sum_{\gamma \in \Lambda_2^2} \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) |\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ \leqq C(T). \tag{4.9}$$

Therefore,

$$\int_0^T \int_{\mathbf{R}^3} \left(\sum_{\gamma \in \Lambda_2^3} |\partial^\gamma (\rho - \overline{\rho}, u, \theta - \overline{\theta})|^2 + \sum_{\gamma \in \Lambda_2^2} \int_{\mathbf{R}^3} \frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) dx + \sum_{\gamma \in \Lambda_2^2} \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi) |\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \leqq C(T). \tag{4.10}$$

With (4.10), a similar argument that gave (3.29) leads to

$$\sum_{\gamma \in \Lambda_2^3} \left(\int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma (f - \overline{\mathbf{M}})|^2}{\mathbf{M}_-} d\xi dx + \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma (f - \overline{\mathbf{M}})|^2}{\mathbf{M}_-} d\xi dx d\tau \right) \leqq C(T). \tag{4.11}$$

Furthermore, the same argument gives

$$\sum_{\gamma \in \Lambda_2^4} \left(\int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma (f - \overline{\mathbf{M}})|^2}{\mathbf{M}_-} d\xi dx + \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma (f - \overline{\mathbf{M}})|^2}{\mathbf{M}_-} d\xi dx d\tau \right) \leqq C(T), \tag{4.12}$$

which is exactly (1.20).

5. Appendix

In the last section, we will give the proofs of the Lemma 4–7. Since the proof of Lemma 6 is essentially the same as the one for Lemma 5, we will only prove Lemmas 4, 5, and 7 in the following subsections.

5.1. The proof of Lemma 4

The local Maxwellian $\mathbf{M}$ solves

$$\mathbf{M}_t + \mathbf{P}_0 (\xi \cdot \nabla_x \mathbf{M}) + \nabla_x \Phi \cdot \nabla_\xi \mathbf{M} = -\mathbf{P}_0 \left(\xi \cdot \nabla_x \left(L_{\mathbf{M}}^{-1} (\mathbf{P}_1 (\xi \cdot \nabla_x \mathbf{M})) \right) \right) - \mathbf{P}_0 (\xi \cdot \nabla_x \Theta),$$

and by applying $\partial_x^\alpha (1 \leqq |\alpha| \leqq 3)$ to this equation and integrating its product with $\frac{\partial^\alpha \mathbf{M}}{\mathbf{M}}$ over $[0, t] \times \mathbf{R}^3 \times \mathbf{R}^3$, we have

$$\frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial_x^\alpha \mathbf{M}|^2}{\mathbf{M}} d\xi dx \Big|_{\tau=0}^{\tau=t} = -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \left(\frac{|\partial_x^\alpha \mathbf{M}|^2}{\mathbf{M}^2} \mathbf{M}_t + \frac{\partial_x^\alpha \mathbf{M}}{\mathbf{M}} \partial_x^\alpha [\mathbf{P}_0 (\xi \cdot \nabla_x \mathbf{M})] \right) d\xi dx d\tau$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\mathbf{M}\partial_x^\alpha\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dxd\tau\\
&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\mathbf{M}\partial_x^\alpha\left(\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}\right)}{\mathbf{M}}d\xi dxd\tau\\
&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\mathbf{M}\partial_x^\alpha\left\{\mathbf{P}_0\left(\xi\cdot\nabla_x\Theta\right)\right\}}{\mathbf{M}}d\xi dxd\tau\\
:=&\sum_{j=7}^{10}I_j,
\end{aligned}
\tag{5.1}
$$

where I_j, $j=7,\cdots,10$, are the corresponding terms in the above equation.

In the following, we estimate $I_j(j=7,8,9,10)$ term by term. Firstly, the *a priori* estimate (3.1) and the properties of the operators $\mathbf{P}_0$ and $\mathbf{P}_1$ give

$$
|I_7|\leqq O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau. \tag{5.2}
$$

Note that

$$
\begin{aligned}
I_8=&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dxd\tau\\
&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dxd\tau\\
:=&I_8^1+I_8^2,
\end{aligned}
$$

and

$$
\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}h\right\}=L_{\mathbf{M}}^{-1}(\partial_x^\alpha h)-\sum_{j=0}^{|\alpha|-1}\sum_{|\alpha_j|=j}C_{\alpha_j}L_{\mathbf{M}}^{-1}\left(Q\left(\partial_x^{\alpha_j}\left(L_{\mathbf{M}}^{-1}h\right),\partial_x^{\alpha-\alpha_j}\mathbf{M}\right)\right),
$$

where C_{α_j} are some positive constants. Then,

$$
\begin{aligned}
I_8^1=&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}}{\mathbf{M}}d\xi dxd\tau\\
=&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1[\xi\cdot\nabla_x(\mathbf{P}_0(\partial_x^\alpha\mathbf{M}))]\,\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})]\right\}}{\mathbf{M}}d\xi dxd\tau
\end{aligned}
$$

$$+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})]\right\}}{\mathbf{M}^2}\xi\cdot\nabla_x\mathbf{M}d\xi dxd\tau$$

$$\leqq -d\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dxd\tau$$

$$+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau,$$

and

$$I_8^2=\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^{\alpha_1'}(\mathbf{P}_1(\partial_x^\alpha\mathbf{M}))\partial_x^{\alpha-\alpha_1'}\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dxd\tau$$

$$-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial_x^\alpha\mathbf{M})\partial_x^{\alpha-\alpha_1'}\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}^2}\partial_x^{\alpha_1'}\mathbf{M}d\xi dxd\tau$$

$$\leqq\frac{d}{3}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dxd\tau$$

$$+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau.$$

Here d is a positive constant that arises from the microscopic H-theorem (1.15) with $\alpha_1'=(1,0,0)$, $(0,1,0)$ or $(0,0,1)$.

Hence,

$$I_8\leqq-\frac{2d}{3}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dxd\tau$$

$$+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau. \tag{5.3}$$

Lemma 3 and the *a priori* estimate (3.1) imply that there exists a sufficiently small constant $\mu>0$ such that

$$|I_{10}|\leqq\mu\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dxd\tau$$

$$+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dxd\tau$$

$$+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau$$

$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{\alpha'\leqq\alpha}|\nabla_\xi\partial_x^{\alpha'}\ bfG|^2+\sum_{|\alpha'|\leqq3}|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2\right)d\xi dxd\tau. \tag{5.4}$$

To estimate I_9, we notice that

$$\begin{aligned} I_9 &= -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_0 \left(\partial_x^\alpha \mathbf{M}\right) \partial_x^\alpha \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{M}\right)}{\mathbf{M}} d\xi dx d\tau \\ &\quad - \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1 \left(\partial_x^\alpha \mathbf{M}\right) \partial_x^\alpha \left(\nabla_x \Phi \cdot \nabla_\xi \mathbf{M}\right)}{\mathbf{M}} d\xi dx d\tau \\ &= J_9^1 + J_9^2. \end{aligned} \tag{5.5}$$

It is straightforward to show that

$$J_9^2 \leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau. \tag{5.6}$$

We can estimate J_9^1 as follows. First,

$$\begin{aligned} J_9^1 &= -\sum_{\alpha' \leqq \alpha} C_\alpha^{\alpha'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_0 \left(\partial_x^\alpha \mathbf{M}\right)}{\mathbf{M}} \left(\nabla_x \partial_x^{\alpha'} \Phi \cdot \nabla_\xi \partial_x^{\alpha-\alpha'} \mathbf{M}\right) d\xi dx d\tau \\ &= \sum_{\alpha' \leqq \alpha} J_9^{1,\alpha'}. \end{aligned} \tag{5.7}$$

When $\alpha' < \alpha$, we have

$$J_9^{1,\alpha'} \leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau. \tag{5.8}$$

On the other hand, when $\alpha' = \alpha$, we have

$$\begin{aligned} J_9^{1,\alpha} &= -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_0 \left(\partial_x^\alpha \mathbf{M}\right)}{\mathbf{M}} \left(\nabla_x \partial_x^\alpha \Phi \cdot \nabla_\xi \mathbf{M}\right) d\xi dx d\tau \\ &\leqq -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \left(\nabla_x \partial_x^\alpha \Phi \cdot \nabla_\xi \mathbf{M}\right) \left[\left(\frac{\partial_x^\alpha \rho}{\rho} - \frac{3\partial_x^\alpha \theta}{2\theta}\right)\right. \\ &\qquad \left. + \frac{3}{2\theta}\left((\xi - u) \cdot \partial_x^\alpha u + \frac{|\xi - u|^2}{2\theta} \partial_x^\alpha \theta\right)\right] d\xi dx d\tau \\ &\qquad + O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau \qquad (5.9) \\ &= O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau \\ &\qquad + \frac{3}{2} \int_0^t \int_{\mathbf{R}^3} \frac{\rho \partial_x^\alpha u \cdot \nabla_x \partial_x^\alpha \Phi}{\theta} dx d\tau \\ &\leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau \\ &\qquad - \frac{3}{2} \int_0^t \int_{\mathbf{R}^3} \frac{\mathrm{div}_x (\rho \partial_x^\alpha u) \partial_x^\alpha \Phi}{\theta} dx d\tau. \end{aligned}$$

Since

$$\operatorname{div}_x\left(\rho\partial_x^\alpha u\right)=\operatorname{div}_x\left(\partial_x^\alpha(\rho u)\right)-\sum_{0<\alpha'<\alpha}C_\alpha^{\alpha'}\operatorname{div}_x\left(\partial_x^{\alpha'}\rho\partial_x^{\alpha-\alpha'}u\right)-\operatorname{div}_x\left(\partial^\alpha\rho u\right),$$

from the *a priori* assumption (3.1) and (3.3), we have

$$\begin{aligned}
&\left|\sum_{0<\alpha'<\alpha}\int_0^t\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\Phi\operatorname{div}_x\left(\partial_x^{\alpha'}\rho\partial_x^{\alpha-\alpha'}u\right)}{\theta}dxd\tau\right|\\
&\quad\leqq O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau,\\
&\frac{3}{2}\int_0^t\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\Phi\operatorname{div}_x\left(\partial_x^\alpha\rho u\right)}{\theta}dxd\tau\\
&\quad=-\frac{3}{2}\int_0^t\int_{\mathbf{R}^3}\frac{\left(\theta\nabla_x\partial_x^\alpha\Phi-\nabla_x\theta\partial_x^\alpha\Phi\right)\cdot\left(\partial_x^\alpha\rho u\right)}{\theta^2}dxd\tau\\
&\quad\leqq O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau\\
&\qquad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{3}{2}\int_0^t\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\Phi\partial_x^\alpha\operatorname{div}_x(\rho u)}{\theta}dxd\tau\\
&=\frac{3}{2}\int_0^t\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\Phi\partial_x^\alpha\rho_t}{\theta}dxd\tau\\
&=\frac{3}{2}\lambda\int_0^t\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\Phi\partial_x^\alpha\Delta_x\Phi_t}{\theta}dxd\tau\\
&=-\frac{3}{4}\lambda\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\bigg|_0^t-\frac{3}{4}\lambda\int_0^t\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta^2}\theta_tdxd\tau\\
&\quad+\frac{3}{2}\lambda\int_0^t\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\Phi\nabla_x\theta\cdot\nabla_x\partial_x^\alpha\Phi_t}{\theta^2}dxd\tau\\
&\leqq-\frac{3}{4}\lambda\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\bigg|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau\\
&\quad+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&-\frac{3}{2}\int_0^t\int_{\mathbf{R}^3}\frac{\operatorname{div}_x\left(\rho\partial_x^\alpha u\right)\partial_x^\alpha\Phi}{\theta}dxd\tau \\
&\leqq -\frac{3}{4}\lambda\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t \\
&+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau \\
&+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau. \qquad (5.10)
\end{aligned}
$$

Substituting (5.10) into (5.9) yields

$$
\begin{aligned}
J_9^{1,\alpha}\leqq{}& -\frac{3}{4}\lambda\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau \\
&+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau. \qquad (5.11)
\end{aligned}
$$

By combining (5.7), (5.8), and (5.11), we have

$$
\begin{aligned}
J_9^{1}\leqq{}& -\frac{3}{4}\lambda\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau \\
&+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau. \qquad (5.12)
\end{aligned}
$$

Therefore, (5.5), (5.6), and (5.12) give

$$
\begin{aligned}
I_9\leqq{}& -\frac{3}{4}\lambda\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau \\
&+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau. \qquad (5.13)
\end{aligned}
$$

Finally, combining (5.1), (5.2), (5.3), (5.4), and (5.13) gives

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\lambda\left|\nabla_x\partial_x^\alpha\Phi\right|^2+\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi\right)dx+\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2dxd\tau\\
&\leqq O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{\alpha'\leqq\alpha}\left|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}\right|^2+\sum_{|\alpha'|\leqq3}\left|\nabla_x\partial_x^{\alpha'}\mathbf{G}\right|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_{1,|\alpha|}\delta_0\lambda\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}\left|\partial_x^{\alpha'}\Phi\right|^2dxd\tau. \qquad (5.14)
\end{aligned}$$

Using (5.14) and noticing $\delta_0\lambda<1$, we can obtain (3.17) by summing over α with $|\alpha|=j$ for $j=1,2,3$, respectively. This completes the proof of Lemma 4.

5.2. The proof of Lemma 5

We only prove (3.19) because the proof for (3.18) is similar.

Applying $\partial_t\partial_x^\alpha(|\alpha|\leqq 2)$ to (1.11) and integrating its product with $\frac{\partial_t\partial_x^\alpha\mathbf{G}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$ yields

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_0^t\\
&=-\frac{1}{2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}^2}\mathbf{M}_t d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha\left(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha\left(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G})\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha\left(\nabla_x\Phi\cdot\nabla_\xi\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha\left(L_{\mathbf{M}}\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha\left(Q(\mathbf{G},\mathbf{G})\right)}{\mathbf{M}}d\xi dxd\tau\\
&:=\sum_{j=11}^{16}I_j, \qquad (5.15)
\end{aligned}$$

where I_j, $j=11,\cdots,16$, are the corresponding terms in the above equation.

As in the proof of Lemma 4, we have

$$|I_{11}| \leqq O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau, \tag{5.16}$$

$$\begin{aligned} |I_{13}| \leqq\ & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^\alpha \mathbf{G}_t|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)(|\partial_t \partial_x^{\alpha'} \mathbf{G}|^2 + |\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2)}{\mathbf{M}} d\xi dx d\tau \\ & + O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} (\rho, u, \theta) \right|^2 dx d\tau, \end{aligned} \tag{5.17}$$

$$\begin{aligned} |I_{15}| =\ & \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G} \cdot L_{\mathbf{M}}(\partial_t \partial_x^\alpha \mathbf{G})}{\mathbf{M}} d\xi dx d\tau \\ & + 2 \sum_{\alpha' < \alpha} C_\alpha^{\alpha'} \int_0^t \imath nt_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G} \cdot Q\left(\partial_x^{\alpha-\alpha'} \mathbf{M}, \partial_x^{\alpha'} \mathbf{G}_t\right)}{\mathbf{M}} d\xi dx d\tau \\ & + 2 \sum_{\alpha' \leqq \alpha} C_\alpha^{\alpha'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G} \cdot Q\left(\partial_x^{\alpha-\alpha'} \mathbf{M}_t, \partial_x^{\alpha'} \mathbf{G}\right)}{\mathbf{M}} d\xi dx d\tau \\ \leqq\ & -\frac{\sigma}{2} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} \left| \nabla_x \partial_x^{\alpha'} (\rho, u, \theta) \right|^2 dx d\tau \\ & + O(1)\delta_0 \sum_{|\alpha'| < |\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)(|\partial_t \partial_x^{\alpha'} \mathbf{G}|^2 + |\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2)}{\mathbf{M}} d\xi dx d\tau, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} |I_{16}| \leqq\ & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \sum_{\alpha' \leqq \alpha} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} \left| Q\left(\partial_x^{\alpha-\alpha'} \mathbf{G}, \partial_x^{\alpha'} \mathbf{G}_t\right) \right|^2}{\mathbf{M}} d\xi dx d\tau \\ \leqq\ & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)(|\partial_t \partial_x^{\alpha'} \mathbf{G}|^2 + |\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2)}{\mathbf{M}} d\xi dx d\tau. \end{aligned} \tag{5.19}$$

To estimate I_{14}, we first note that

$$
\begin{aligned}
I_{14} \;=\; & -\int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G}(\nabla_x \Phi \cdot \nabla_\xi (\partial_t \partial_x^\alpha \mathbf{G}))}{\mathbf{M}} d\xi dx d\tau \\
& - \sum_{\alpha' \leqq \alpha} C_\alpha^{\alpha'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G}(\nabla_x \partial_x^{\alpha'} \Phi_t \cdot \nabla_\xi \partial_x^{\alpha-\alpha'} \mathbf{G})}{\mathbf{M}} d\xi dx d\tau \\
& - \sum_{\alpha' < \alpha} C_\alpha^{\alpha'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G}(\nabla_x \partial_x^{\alpha-\alpha'} \Phi \cdot \nabla_\xi \partial_x^{\alpha'} \mathbf{G}_t)}{\mathbf{M}} d\xi dx d\tau \\
:= & \sum_{j=1}^3 J_{14}^j,
\end{aligned}
$$

where J_{14}^j, $j = 1, 2, 3$, are the corresponding terms in the above equation.

Since $|\alpha| \leqq 2$, from (3.1), (3.3), and (3.4), we have

$$
\begin{aligned}
J_{14}^1 &= -\frac{1}{2} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial_t \partial_x^\alpha \mathbf{G}|^2 \nabla_x \Phi \cdot \nabla_\xi \mathbf{M}}{\mathbf{M}} d\xi dx d\tau \\
&\leqq O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau, \\
J_{14}^2 &\leqq \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1) \sum_{\alpha' \leqq \alpha} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_x \partial_x^{\alpha'} \Phi_t|^2 |\nabla_\xi \partial_x^{\alpha-\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\leqq \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\delta_0 \sum_{\alpha' \leqq \alpha} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_\xi \partial_x^{\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau, \\
J_{14}^3 &\leqq \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\delta_0 \sum_{\alpha' < \alpha} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_\xi \partial^{\alpha'} \mathbf{G}_t|^2}{\mathbf{M}} d\xi dx d\tau.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I_{14} &\leqq (\mu + O(1)\delta_0) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\delta_0 \sum_{\alpha' < \alpha} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_\xi \partial_x^{\alpha'} \mathbf{G}_t|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\delta_0 \sum_{\alpha' \leqq \alpha} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\nabla_\xi \partial_x^{\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau. \qquad (5.20)
\end{aligned}
$$

Finally, we estimate I_{12} which represents the most difficult part of proving (3.19).

Since

$$\begin{aligned}
&\partial_t\partial_x^\alpha\left(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\right)\\
&\quad=\xi\cdot\nabla_x\partial_x^\alpha\Big(-\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}-\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{M})-\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{G})\Big)\\
&\qquad-\sum_{j=0}^{4}\partial_x^\alpha\left(\left\langle\xi\cdot\nabla_x\mathbf{M},\left(\frac{\chi_j}{\mathbf{M}}\right)_t\mathbf{M}\right\rangle_{\mathbf{M}}\chi_j+\langle\xi\cdot\nabla_x\mathbf{M},\chi_j\rangle_{\mathbf{M}}\chi_{jt}\right),
\end{aligned}$$

we have the expression

$$\begin{aligned}
I_{12}&=\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\xi\cdot\nabla_x\partial_x^\alpha(\nabla_x\Phi\cdot\nabla_\xi\mathbf{M})}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\xi\cdot\nabla_x\partial_x^\alpha\left(\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{M})\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\xi\cdot\nabla_x\partial_x^\alpha\left(\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{G})\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\sum_{j=0}^{4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\partial_x^\alpha\left(\left\langle\xi\cdot\nabla_x\mathbf{M},\left(\frac{\chi_j}{\mathbf{M}}\right)_t\mathbf{M}\right\rangle_{\mathbf{M}}\chi_j+\langle\xi\cdot\nabla_x\mathbf{M},\chi_j\rangle_{\mathbf{M}}\chi_{jt}\right)}{\mathbf{M}}d\xi dxd\tau\\
&:=\sum_{k=1}^{4}J_{12}^k,
\end{aligned}\tag{5.21}$$

where $J_{12}^k, k=1,\cdots,4$, are the corresponding terms in the above equation.

The expression J_{12}^k can be estimated as follows. For $|\alpha|\leqq 2$, using the Cauchy–Schwarz inequality, (3.1), (3.3), and (3.4), we have

$$\begin{aligned}
J_{12}^1&\leqq\mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^\alpha(\nabla_x\Phi\cdot\nabla_\xi\ bf M)|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leqq\mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_{1,|\alpha|+1}\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^{\alpha'}\Phi|^2dxd\tau\\
&\quad+O(1)\left(1-\delta_{1,|\alpha|+1}\right)\lambda^{-2}\sum_{|\alpha'|=|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\partial_x^{\alpha'}\rho|^2dxd\tau\\
&\quad+\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2dxd\tau,
\end{aligned}\tag{5.22}$$

$$
\begin{aligned}
J_{12}^{2} &= \int_{0}^{t}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^{\alpha}\mathbf{G}\mathbf{P}_1\left(\xi\cdot\nabla_x\partial_x^{\alpha}(\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{M}))\right)}{\mathbf{M}}d\xi dx d\tau\\
&\leqq \mu\int_{0}^{t}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^{\alpha}\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad + O(1)\sum_{|\alpha'|=1+|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(u,\theta)|^2 dx d\tau\\
&\quad + O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2 dx d\tau, \qquad (5.23)
\end{aligned}
$$

and

$$
\begin{aligned}
J_{12}^{3} &\leqq \mu\int_{0}^{t}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^{\alpha}\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad + O(1)\sum_{|\alpha'|=1+|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad + O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2 dx d\tau\\
&\quad + O(1)\delta_0\sum_{|\alpha'|\leqq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau. \qquad (5.24)
\end{aligned}
$$

Since,

$$
\begin{aligned}
&\partial_x^{\alpha}\left(\left\langle\xi\cdot\nabla_x\mathbf{M},\left(\frac{\chi_j}{\mathbf{M}}\right)_t\mathbf{M}\right\rangle_{\mathbf{M}}\chi_j+\langle\xi\cdot\nabla_x\mathbf{M},\chi_j\rangle_{\mathbf{M}}\chi_{jt}\right)\\
&\quad=\sum_{\alpha'+\alpha''\leqq\alpha}C_{\alpha}^{\alpha',\alpha''}\left\langle\xi\cdot\nabla_x\partial_x^{\alpha'}\mathbf{M},\partial_x^{\alpha''}\left(\frac{\chi_j}{\mathbf{M}}\right)_t\mathbf{M}\right\rangle_{\mathbf{M}}\partial_x^{\alpha-\alpha'-\alpha''}\chi_j\\
&\qquad+\sum_{\alpha'+\alpha''\leqq\alpha}C_{\alpha}^{\alpha',\alpha''}\left\langle\xi\cdot\nabla_x\partial_x^{\alpha'}\mathbf{M},\partial_x^{\alpha''}\left(\frac{\chi_j}{\mathbf{M}}\right)\mathbf{M}\right\rangle_{\mathbf{M}}\partial_x^{\alpha-\alpha'-\alpha''}\chi_{jt},
\end{aligned}
$$

we have

$$
\begin{aligned}
J_{12}^{4} &\leqq \mu\int_{0}^{t}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^{\alpha}\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad+O(1)\sum_{j=0}^{4}\sum_{\alpha'+\alpha''\leqq\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\left\langle\xi\cdot\nabla_x\partial_x^{\alpha'}\mathbf{M},\partial_x^{\alpha''}\left(\frac{\chi_j}{\mathbf{M}}\right)_t\mathbf{M}\right\rangle_{\mathbf{M}}\partial_x^{\alpha-\alpha'-\alpha''}\chi_j\right|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad+O(1)\sum_{j=0}^{4}\sum_{\alpha'+\alpha''\leqq\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\left\langle\xi\cdot\nabla_x\partial_x^{\alpha'}\mathbf{M},\partial_x^{\alpha''}\left(\frac{\chi_j}{\mathbf{M}}\right)\mathbf{M}\right\rangle_{\mathbf{M}}\partial_x^{\alpha-\alpha'-\alpha''}\chi_{jt}\right|^2}{\mathbf{M}}d\xi dx d\tau\\
&= K_1+K_2+K_3. \qquad (5.25)
\end{aligned}
$$

If $|\alpha'| \geqq 1$, then $|\alpha''| \leqq 1$, $|\alpha - \alpha' - \alpha''| \leqq 1$ because $|\alpha| \leqq 2$. Thus, from the conservation laws (1.10), (3.1), and (3.4), we have

$$K_2 \leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)|^2 dx d\tau.$$

On the other hand, if $\alpha' = 0$, we need to consider the following three cases:

(a) $\alpha'' = 0$,
(b) $|\alpha''| = 1$,
(c) $|\alpha''| = |\alpha| = 2$.

For the cases (a) and (b), the same argument for the case when $|\alpha'| \geqq 1$ leads to

$$K_2 \leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)|^2 dx d\tau.$$

As for the case (c), we have

$$\begin{aligned} |K_2| &\leqq O(1) \sum_{|\alpha'| \leqq 1} \int_0^t \int_{\mathbf{R}^3} |\nabla_x(\rho, u, \theta)|^2 \left|\partial_t \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau \\ &\quad + O(1) \sum_{|\alpha'| = 2} \int_0^t \int_{\mathbf{R}^3} |\nabla_x(\rho, u, \theta)|^2 \left|\partial_t \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau \\ &\leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 dx d\tau. \end{aligned}$$

Here we have used (3.1), (3.3), (3.4) and the conservation laws (1.10).

Thus, we have

$$K_2 \leqq O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)|^2 dx d\tau. \tag{5.26}$$

Similar estimate holds for K_3.

Consequently

$$\begin{aligned} J_{12}^4 &\leqq \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ &\quad + O(1)\delta_0 \sum_{|\alpha'| \leqq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)|^2 dx d\tau. \end{aligned} \tag{5.27}$$

By substituting (5.22), (5.23), (5.24) and (5.27) into (5.21), we have

$$
\begin{aligned}
I_{12} \leqq\; & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
& + O(1) \sum_{|\alpha'|=1+|\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
& + O(1)\delta_0 \sum_{|\alpha'|\leqq|\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)|^2 dx d\tau \\
& + O(1)\delta_0 \sum_{|\alpha'|\leqq|\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
& + O(1) \sum_{|\alpha'|=|\alpha|+1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(u, \theta)\right|^2 dx d\tau \\
& + O(1)\delta_{1,|\alpha|+1} \sum_{|\alpha'|=2} \int_0^t \int_{\mathbf{R}^3} |\partial_x^{\alpha'} \Phi|^2 dx d\tau \\
& + O(1)\left(1-\delta_{1,|\alpha|+1}\right) \lambda^{-2} \sum_{|\alpha'|=|\alpha|} \int_0^t \int_{\mathbf{R}^3} |\partial_x^{\alpha'} \rho|^2 dx d\tau.
\end{aligned} \tag{5.28}
$$

Combining (5.15)–(5.20) with (5.28), we obtain

$$
\begin{aligned}
& \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx + \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
& \quad \leqq O(1)N(0)^2 + O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\
& \qquad + O(1) \sum_{|\alpha'|=1+|\alpha|} \int_0^t \int_{\mathbf{R}^3} \left(\left|\nabla_x \partial_x^{\alpha'}(u, \theta)\right|^2 \right. \\
& \qquad \left. + \int_{\mathbf{R}^3} \frac{\nu(\xi)(\left|\nabla_x \partial_x^{\alpha'} \mathbf{G}\right|^2 + \left|\partial_x^{\alpha'} \mathbf{G}_t\right|^2)}{\mathbf{M}} d\xi \right) dx d\tau \\
& \qquad + O(1)\delta_0 \sum_{|\alpha'|\leqq|\alpha|} \int_0^t \int_{\mathbf{R}^3} \left(\left|\nabla_x \partial_x^{\alpha'}(\rho, u, \theta)\right|^2 \right. \\
& \qquad \left. + \int_{\mathbf{R}^3} \frac{\nu(\xi)\left(\left|\nabla_x \partial_x^{\alpha'} \mathbf{G}\right|^2 + \left|\partial_x^{\alpha'} \mathbf{G}_t\right|^2\right)}{\mathbf{M}} d\xi \right) dx d\tau
\end{aligned}
$$

$$+O(1)\delta_{1,1+|\alpha|}\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}\left|\partial_x^{\alpha'}\Phi\right|^2 dxd\tau$$
$$+O(1)\left(1-\delta_{1,1+|\alpha|}\right)\lambda^{-2}\sum_{|\alpha'|=|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\partial_x^{\alpha'}\rho\right|^2 dxd\tau$$
$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha'|\leqq|\alpha|}\left|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}\right|^2\right.$$
$$\left.+\sum_{|\alpha'|<|\alpha|}\left|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}_t\right|^2\right)d\xi dxd\tau. \tag{5.29}$$

(3.19) follows directly from (5.29). This completes the proof of the lemma.

5.3. The proof of Lemma 7

For Lemma 7, by applying $\partial^\gamma(\gamma\in\Lambda_7)$ to $(1.2)_1$, multiplying it by $\frac{\partial^\gamma f}{\mathbf{M}}$ and integrating the final equation with respect to t, x, and ξ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$, we have

$$\frac{1}{2}\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx\Big|_0^t$$
$$=-\frac{1}{2}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}^2}\left(\mathbf{M}_t+\xi\cdot\nabla_x\mathbf{M}\right)d\xi dxd\tau$$
$$-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\,\partial^\gamma(\nabla_x\Phi\cdot\nabla_\xi f)}{\mathbf{M}}d\xi dxd\tau$$
$$+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\,\partial^\gamma\left(L_{\mathbf{M}}\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau$$
$$+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\,\partial^\gamma\left(Q(\mathbf{G},\mathbf{G})\right)}{\mathbf{M}}d\xi dxd\tau$$
$$:=\sum_{j=17}^{20}I_j, \tag{5.30}$$

where I_j, $j=17,\cdots,20$ are the corresponding terms in the above equation.

Now we estimate I_j term by term as follows. Firstly, from Lemma 3, (3.1), (3.3) and (3.4), we have

$$
\begin{aligned}
I_{17} &\leqq O(1)\delta_0 \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}_-}d\xi dx d\tau \\
&\leqq O(1)\delta_0 \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau \\
&\quad + O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau, \qquad (5.31)
\end{aligned}
$$

$$
\begin{aligned}
I_{19} &= \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\mathbf{P}_1(\partial^\gamma\mathbf{M})+\mathbf{P}_1(\partial^\gamma\mathbf{G}))\,\partial^\gamma\,(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dx d\tau \\
&\leqq -\frac{\sigma}{2}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau \\
&\quad + O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi dx d\tau \\
&\quad + O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau, \qquad (5.32)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{20} &= \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\mathbf{P}_1(\partial^\gamma\mathbf{M})+\mathbf{P}_1(\partial^\gamma\mathbf{G}))\,\partial^\gamma\,(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dx d\tau \\
&\leqq \mu\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau \\
&\quad + O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi dx d\tau \\
&\quad + O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dxd\tau. \qquad (5.33)
\end{aligned}
$$

For I_{18}, we have the following expression

$$
\begin{aligned}
I_{18} =& -\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\nabla_x\Phi\cdot\nabla_\xi\partial^\gamma f}{\mathbf{M}}d\xi dx d\tau \\
&-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi f}{\mathbf{M}}d\xi dx d\tau \\
&-\sum_{\gamma\in\Lambda_7}\sum_{0<\gamma'<\gamma}C_\gamma^{\gamma'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\nabla_x\partial^{\gamma-\gamma'}\Phi\cdot\nabla_\xi\partial^{\gamma'} f}{\mathbf{M}}d\xi dx d\tau \\
:=& \sum_{i=1}^3 J_{18}^i, \qquad (5.34)
\end{aligned}
$$

where $J_{18}^i, i = 1, 2, 3$, are the corresponding terms in the above equation. From the conservation laws (1.10), (3.1), (3.3) and (3.4), we have

$$
\begin{aligned}
J_{18}^1 =& -\frac{1}{2}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}^2}\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}d\xi dx d\tau \\
\leqq& O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx d\tau \\
\leqq& O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau \\
&+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dx d\tau. \qquad (5.35)
\end{aligned}
$$

We firstly rewrite J_{18}^2 as

$$
\begin{aligned}
J_{18}^2 =& -\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0\left(\partial^\gamma\mathbf{M}\right)\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi\mathbf{M}}{\mathbf{M}}d\xi dx d\tau \\
&-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\left(\partial^\gamma\mathbf{M}\right)\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi\mathbf{G}}{\mathbf{M}}d\xi dx d\tau \\
&-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\mathbf{G}\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi\mathbf{G}}{\mathbf{M}}d\xi dx d\tau \\
:=& K_4+K_5+K_6. \qquad (5.36)
\end{aligned}
$$

Then, from (3.1), (3.3) and (3.4), we have

$$K_4 \leqq \frac{1}{\lambda} \sum_{\gamma\in\Lambda_7} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\mathbf{P}_0\left(\partial^\gamma \mathbf{M}\right)|^2}{\mathbf{M}} d\xi dx d\tau$$
$$+ O(1)\lambda \sum_{\gamma\in\Lambda_7} \int_0^t \int_{\mathbf{R}^3} \frac{|\nabla_x \partial^\gamma \Phi|^2}{\mathbf{M}} dx d\tau$$
$$\leqq O(1)\lambda^{-1} \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha(\rho, u, \theta)\right|^2 dx d\tau,$$
$$K_5 \leqq O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \left(\sum_{|\alpha|\leqq 3} \left|\nabla_x \partial_x^\alpha(\rho, u, \theta)\right|^2 + \sum_{|\alpha|=2} \left|\partial_x^\alpha \Phi\right|^2 + \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_\xi \mathbf{G}|^2}{\mathbf{M}} d\xi \right) dx d\tau,$$

and

$$K_6 \leqq O(1)\delta_0 \sum_{\gamma\in\Lambda_7} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau$$
$$+ O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_\xi \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau$$
$$+ O(1)\delta_0 \sum_{|\alpha|\leqq 3} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^\alpha(\rho, u, \theta)\right|^2 dx d\tau.$$

Thus,

$$J_{18}^2 \leqq O(1)\left(\delta_0 + \lambda^{-1}\right) \int_0^t \int_{\mathbf{R}^3} \left(\sum_{|\alpha|\leqq 3} \left|\nabla_x \partial_x^\alpha(\rho, u, \theta)\right|^2 + \sum_{|\alpha|=2} \left|\partial_x^\alpha \Phi\right|^2 \right) dx d\tau$$
$$+ O(1)\delta_0 \sum_{\gamma\in\Lambda_7} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau$$
$$+ O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_\xi \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau. \tag{5.37}$$

Finally, from the conservation laws (1.10), (3.1), (3.3) and (3.4) and using the fact that

$$J_{18}^3 = -\sum_{\gamma\in\Lambda_7} \sum_{0<\gamma'<\gamma} C_\gamma^{\gamma'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_0\left(\partial^\gamma \mathbf{M}\right) \nabla_x \partial^{\gamma-\gamma'} \Phi \cdot \nabla_\xi \left(\mathbf{P}_0\left(\partial^{\gamma'} \mathbf{M}\right)\right)}{\mathbf{M}} d\xi dx d\tau$$
$$-\sum_{\gamma\in\Lambda_7} \sum_{0<\gamma'<\gamma} C_\gamma^{\gamma'} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\mathbf{P}_0\left(\partial^\gamma \mathbf{M}\right) \nabla_x \partial^{\gamma-\gamma'} \Phi \cdot \nabla_\xi \left(\mathbf{P}_1\left(\partial^{\gamma'} \mathbf{M}\right) + \partial^{\gamma'} \mathbf{G}\right)}{\mathbf{M}} d\xi dx d\tau$$

$$
\begin{aligned}
&- \sum_{\gamma\in\Lambda_7}\sum_{0<\gamma'<\gamma} C_\gamma^{\gamma'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\mathbf{G}\nabla_x\partial^{\gamma-\gamma'}\Phi\cdot\nabla_\xi\left(\partial^{\gamma'}\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau\\
&:= K_7+K_8+K_9, \qquad (5.38)
\end{aligned}
$$

we have

$$
\begin{aligned}
K_7 &\leqq O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dxd\tau,\\
K_8 &\leqq O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dxd\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
K_9 &\leqq O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
J_{18}^2 &\leqq O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dxd\tau. \qquad (5.39)
\end{aligned}
$$

Combining (5.35), (5.37) and (5.39) yields

$$
\begin{aligned}
I_{18} &\leqq O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dxd\tau. \qquad (5.40)
\end{aligned}
$$

Substituting (5.31), (5.32), (5.33) and (5.40) into (5.30), we finally obtain

$$\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau$$
$$\leqq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau$$
$$+O(1)\left(\delta_0+\lambda^{-1}\right)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leqq 3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2+\sum_{|\alpha|=2}\left|\partial_x^\alpha\Phi\right|^2\right)dx d\tau$$
$$+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_\xi\mathbf{G}|^2+|\mathbf{G}_t|^2+\sum_{\gamma\Lambda_4}\left|\nabla_\xi\partial^\gamma\mathbf{G}\right|^2\right)d\xi dx d\tau, \tag{5.41}$$

which is (3.29). This completes the proof of the lemma.

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