Arch. Rational Mech. Anal. 181 (2006) 333–371
Digital Object Identifier (DOI) 10.1007/s00205-005-0414-1

Nonlinear Stability of Rarefaction Waves for the Boltzmann Equation

TAI-PING LIU, TONG YANG, SHIH-HSIEN YU & HUI-JIANG ZHAO

Communicated by the Editors

Abstract

It is well known that the Boltzmann equation is related to the Euler and Navier-Stokes equations in the field of gas dynamics. The relation is either for small Knudsen number, or, for dissipative waves in the time-asymptotic sense. In this paper, we show that rarefaction waves for the Boltzmann equation are time-asymptotic stable and tend to the rarefaction waves for the Euler and Navier-Stokes equations. Our main tool is the combination of techniques for viscous conservation laws and the energy method based on micro-macro decomposition of the Boltzmann equation. The expansion nature of the rarefaction waves and the suitable microscopic version of the H-theorem are essential elements of our analysis.

1. Introduction

Consider the one space dimensional Boltzmann equation

$$f_t + \xi_1 f_x = Q(f, f), \;\; (f, t, x, \xi) \in \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^3, \tag{1.1}$$

where $f(t, x, \xi)$ represents the distributional density of particles at time-space (t, x) with velocity ξ, and $Q(f, f)$ is a bilinear collision operator, cf. [5]. We consider the hard sphere model, for which $Q(f, g)$ is:

$$Q(f, g)(\xi) \equiv \frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}^2_+} \Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi) \Big)$$

$$\times \; |(\xi - \xi_*) \cdot \Omega| \;\; d\xi_* d\Omega.$$

Here $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 : \; (\xi - \xi_*) \cdot \Omega \geqq 0\}$, and

$$\xi' = \xi - [\,(\xi - \xi_*) \cdot \Omega]\;\; \Omega,$$

$$\xi'_* = \xi_* + [\,(\xi - \xi_*) \cdot \Omega]\;\; \Omega.$$

It is well known that the Boltzmann equation is related to the systems of fluid dynamics, i.e., Euler equations and Navier-Stokes equations (cf. [4, 6, 7, 14, 21, 22, 29, 34, 35, 38, 41] and references therein). The relation is usually studied in the limit of zero mean free path, when the Boltzmann solutions become locally thermally equilibrated away from shocks, cf. [7, 29, 34, 41]. In this paper we consider the rarefaction waves of the fluid equations. Owing to its expansive nature, a rarefaction wave tends to constant states locally in space as the time variable goes to infinity. Thus, it would expect that there is a corresponding Boltzmann wave that is time-asymptotically in local thermal equilibrium and tends to the fluid rarefaction wave. The purpose of the present paper is to confirm this correspondense rigorously. Our analysis is based on the micro-macro decomposition of the Boltzmann equation introduced in [29] and further elaborated in [28]. The Boltzmann equation is decomposed into a conservation law mainly for the fluid components, together with an equation mainly for the non-fluid component. Similarly, to the studies of the Boltzmann shocks in [29] & [41], the decomposition allows for the application of the theory of the nonlinear stability of rarefaction wave for the fluid dynamics. Unlike the study of shock waves, for which locating the waves through the conservation laws is a primary concern ([26, 41]), the construction of accurate approximate rarefaction Boltzmann waves is essential for our stability analysis. Even though time-asymptotically, the Euler, Navier-Stokes and Boltzmann equations are equivalent on the level of the rarefaction waves, there are basic differences between these equations. For Euler equations, the rarefaction waves can be constructed exactly. For Navier-Stokes and Boltzmann equations, we can approximately construct rarefaction waves which are only accurate time-asymptotically. For the Boltzmann equation, accurate approximate rarefaction waves are in thermo-equilibrium only time-asymptotically. The key element for the fluid theory of the stability of rarefaction waves is the expansiveness of these waves, see [27, 32, 30, 36] and references therein.

For a given solution $f(t,x,\xi)$ of the Boltzmann equation, there are five conserved macroscopic quantities: the mass density $\rho(t,x)$, momentum $m(t,x)=\rho(t,x)u(t,x)$, and energy density $\mathbf{E}(t,x)+\frac{1}{2}|u(t,x)|^2$:

$$\begin{aligned}\rho(t,x) &\equiv \int_{\mathbf{R}^3} f(t,x,\xi)d\xi,\\ m_i(t,x) &\equiv \int_{\mathbf{R}^3} \xi_i f(t,x,\xi)d\xi, \text{ for } i=1,2,3,\\ \left[\rho\left(\mathbf{E}+\frac{1}{2}|u|^2\right)\right](t,x) &\equiv \int_{\mathbf{R}^3}\frac{1}{2}|\xi|^2 f(t,x,\xi)d\xi.\end{aligned}\tag{1.2}$$

The local Maxwellian $\mathbf{M}$ associated to the Boltzmann solution $f(t,x,\xi)$ is defined in terms of the conserved fluid variables:

$$\mathbf{M}\equiv\mathbf{M}_{[\rho,u,\theta]}(t,x,\xi)\equiv\frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}}\exp\left(-\frac{|\xi-u(t,x)|^2}{2R\theta(t,x)}\right).\tag{1.3}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy $\mathbf{E}$ by $\mathbf{E}=\frac{3}{2}R\theta=\theta$ with the gas constant R taken to be $\frac{2}{3}$ in this paper for convenience, and $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))^t$ is the fluid velocity. It is well known that the Boltzmann equation is reduced to the compressible Euler equations when the gas is in local thermo-equilibrium i.e., $f = \mathbf{M}$:

$$\begin{aligned} \rho_t + (\rho u_1)_x &= 0, \\ (\rho u_1)_t + \left(\rho u_1^2 + p\right)_x &= 0, \\ (\rho u_2)_t + (\rho u_1 u_2)_x &= 0, \\ (\rho u_3)_t + (\rho u_1 u_3)_x &= 0, \\ \left[\rho(\frac{1}{2}|u|^2 + \mathbf{E})\right]_t + \left(u_1\left(\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_x &= 0, \end{aligned} \tag{1.4}$$

The equation of the state is that for the monatomic gases (with the above choice of the gas constant $R = \frac{2}{3}$):

$$p = \frac{2}{3}\rho\mathbf{E}.$$

The entropy S is constant across the Euler rarefaction waves, [13],

$$S = -\frac{2}{3}\ln\rho + \ln\left(\frac{4}{3}\pi\theta\right) + 1.$$

The Euler waves propagate with Euler characteristics

$$\begin{aligned} \lambda_1 &= u_1 - \frac{\sqrt{15k}}{3}\rho^{\frac{1}{3}}\exp\left(\frac{S}{2}\right), \\ \lambda_2 &= u_1, \\ \lambda_3 &= u_1 + \frac{\sqrt{15k}}{3}\rho^{\frac{1}{3}}\exp\left(\frac{S}{2}\right) \end{aligned}$$

where $k = \frac{1}{2\pi e}$.

The first and third characteristic fields are genuinely nonlinear and can give rise to rarefaction waves, [23]. Time-asymptotically, all the rarefaction waves for the Euler equations are equivalent to centered rarefaction waves with jump initial data, [25]:

$$(\rho, u, \theta)(t,x)|_{t=0} = \left(\rho_0^r, u_0^r, \theta_0^r\right)(x) = \begin{cases} (\rho_l, u_l, \theta_l), & x < 0, \\ (\rho_r, u_r, \theta_r), & x > 0. \end{cases} \tag{1.5}$$

Notice that since we are concerned with plane waves $x \in \mathbf{R}$, we will assume $u_l = (u_{1l}, 0, 0)$, $u_r = (u_{1r}, 0, 0)$ in the following.

We are interested in the situation where the solution of the Riemann problem (1.4), (1.5) consists of a 1-rarefaction wave $\left(\rho^{R_1}, u^{R_1}, \theta^{R_1}\right)(x/t)$ connecting (ρ_l, u_l, θ_l) and (ρ_m, u_m, θ_m), and a 3-rarefaction wave $\left(\rho^{R_3}, u^{R_3}, \theta^{R_3}\right)(x/t)$ connecting (ρ_m, u_m, θ_m) and (ρ_r, u_r, θ_r). The wave $\left(\rho^{R_1}, u^{R_1}, \theta^{R_1}\right)(x/t)$ (or $\left(\rho^{R_3}, u^{R_3}, \theta^{R_3}\right)(x/t)$) of first (or third) type travels with speed $x/t = \lambda_1$ (or

$x/t = \lambda_3$) and takes values along the Riemann invariant curves $(\rho_m, u_m, \theta_m) \in R_1(\rho_l, u_l, \theta_l)$ (or $(\rho_m, u_m, \theta_m) \in R_3(\rho_r, u_r, \theta_r)$), [13], [37]:

$$\begin{aligned} R_1(\rho_l, u_l, \theta_l) \equiv \Big\{ (\rho, u, \theta) \,\Big|\, & S = S_l,\ u_1 + \sqrt{15k}\rho^{\frac{1}{3}} \exp\left(\tfrac{S}{2}\right) \\ & = u_{1l} + \sqrt{15k}\rho_l^{\frac{1}{3}} \exp\left(\tfrac{S_l}{2}\right), u_2 = u_3 = 0, \\ & u_1 > u_{1l},\ \rho < \rho_l \Big\}, \\ R_3(\rho_r, u_r, \theta_r) = \Big\{ (\rho, u, \theta) \,\Big|\, & S = S_r,\ u_1 - \sqrt{15k}\rho^{\frac{1}{3}} \exp\left(\tfrac{S}{2}\right) \\ & = u_{1r} - \sqrt{15k}\rho_r^{\frac{1}{3}} \exp\left(\tfrac{S_r}{2}\right), u_2 = u_3 = 0, \\ & u_1 < u_{1r},\ \rho < \rho_r \Big\}. \end{aligned} \tag{1.6}$$

Set

$$\begin{aligned} \left(\rho^R, u^R, \theta^R\right)\left(\frac{x}{t}\right) = \left(\rho^{R_1}, u^{R_1}, \theta^{R_1}\right)\left(\frac{x}{t}\right) + \left(\rho^{R_3}, u^{R_3}, \theta^{R_3}\right)\left(\frac{x}{t}\right) \\ - (\rho_m, u_m, \theta_m). \end{aligned}$$

In fact, more general Euler i-th rarefaction waves can be constructed along any given R_i curve when the i-th characteristic satisfies the inviscid Burgers equation, [24, 26],

$$\lambda_{it} + \lambda_i \lambda_{ix} = 0,$$

with increasing initial data. Here we adopt the construction introduced in [32] with an initial value with a gradient that is proportional to the parameter $\varepsilon > 0$:

$$\begin{aligned} & \lambda_{it} + \lambda_i \lambda_{ix} = 0, \\ & \lambda_i(0, x) = \frac{1}{2}(\lambda_{i+} + \lambda_{i-}) + \frac{1}{2}(\lambda_{i+} - \lambda_{i-}) \tanh(\varepsilon x), \quad i = 1, 3, \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \lambda_{1-} = \lambda_1(\rho_l, u_l, \theta_l),\ \lambda_{1+} = \lambda_1(\rho_m, u_m, \theta_m), \\ \lambda_{3-} = \lambda_3(\rho_m, u_m, \theta_m),\ \lambda_{3+} = \lambda_3(\rho_r, u_r, \theta_r). \end{aligned}$$

This gives rise to two smooth rarefaction waves, $(\rho^{A_1}, u^{A_1}, \theta^{A_1})(t, x)$ and $(\rho^{A_3}, u^{A_3}, \theta^{A_3})(t, x)$, which are defined as follows:

$$\begin{aligned} & u_1^{A_i}(t, x) + (-1)^{\frac{1+i}{2}} \frac{\sqrt{15k}}{3} \left(\rho^{A_i}(t, x)\right)^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) = \lambda_i(t, x),\ i = 1, 3, \\ & u_1^{A_1}(t, x) + \sqrt{15k} \left(\rho^{A_1}(t, x)\right)^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) = u_{1l} + \sqrt{15k}\rho_l^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right), \\ & u_1^{A_3}(t, x) - \sqrt{15k} \left(\rho^{A_3}(t, x)\right)^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) = u_{1r} - \sqrt{15k}\rho_r^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right), \\ & \theta^{A_i}(t, x) = \frac{3}{2}k \left(\rho^{A_i}(t, x)\right)^{\frac{2}{3}} \exp(\overline{S}),\ u_2^{A_i} = u_3^{A_i} = 0,\ i = 1, 3. \end{aligned} \tag{1.8}$$

We set the approximate rarefaction waves as the linear superposition of the above two rarefaction waves. Since we are interested in the time-asymptotic behavior, and also for the consideration of the accuracy of the approximation, we start with a large time $t_0 = \frac{1}{d_1 \varepsilon^2}$, $d_1 = \frac{\sqrt{15k}}{3} \rho_m^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) > 0$:

$$\begin{aligned}\left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t, x) \\ = \left(\rho^{A_1} + \rho^{A_3} - \rho_m, u^{A_1} + u^{A_3} - u_m, \theta^{A_1} + \theta^{A_3} - \theta_m\right)(t + t_0, x),\end{aligned} \tag{1.9}$$

An approximate Boltzmann solution is defined to be in local thermo-equilibrium based on the Euler approximate rarefaction waves:

$$\mathbf{M}_{[\overline{\rho}, \overline{u}, \overline{\theta}]}(t, x, \xi) = \frac{\overline{\rho}(t, x)}{\sqrt{(2\pi R \overline{\theta}(t, x))^3}} \exp\left(-\frac{|\xi - \overline{u}(t, x)|^2}{2R\overline{\theta}(t, x)}\right). \tag{1.10}$$

Another way of constructing approximate rarefaction wave profiles is to start with the Navier-Stokes equations. In this case, the characteristic values are well approximated by the Burgers equation, [24, 26],

$$\lambda_{it} + \lambda_i \lambda_{ix} = \mu \lambda_{ixx}.$$

As mentioned previously, all these approximate rarefaction wave profiles are time-asymptotically equivalent and tend to the centered rarefaction waves. We choose to use the Euler equations for simplicity. In fact, for our stability analysis, we will later need to construct more accurate approximate Boltzmann rarefaction waves which are not in local thermo-equilibrium. The function space for the difference

$$g(t, x, \xi) = f(t, x, \xi) - \mathbf{M}_{[\overline{\rho}, \overline{u}, \overline{\theta}]}(t, x, \xi),$$

is the following:

$$\mathbf{H}^s_{t,x,\xi}(\mathbf{R}^+) = \left\{ g(t, x, \xi) \left| \begin{array}{c} \frac{\partial_t^{\alpha_0} \partial_x^{\alpha_1} g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left(\mathbf{R}^+, L^2_{x,\xi}\left(\mathbf{R} \times \mathbf{R}^3\right)\right) \\ \frac{\sqrt{1+|\xi|}\partial_t^{\alpha_0} \partial_x^{\alpha_1} g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in L^2_{t,x,\xi}\left(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^3\right) \\ \alpha_0 + \alpha_1 \leqq s \end{array} \right. \right\}.$$

We also use the notation $f(\xi) \in L^2_\xi\left(\frac{1}{\sqrt{\mathbf{M}_-}}\right)$ to mean that $\frac{f(\xi)}{\sqrt{\mathbf{M}_-}} \in L^2_\xi$. Here $\mathbf{M}_- = \mathbf{M}_{[\rho_-, u_-, \theta_-]}$ is a global Maxwellian satisfying

$$\begin{aligned} \tfrac{1}{2}\theta(t, x) < \theta_- < \theta(t, x), \\ |\rho(t, x) - \rho_-| + |u(t, x) - u_-| + |\theta(t, x) - \theta_-| < \eta_0 \end{aligned} \tag{1.11}$$

for all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$.

Moreover, we use $\mathcal{R}(\varepsilon, \eta_0; \rho_l, u_l, \theta_l)$ to denote the class of approximate rarefaction waves which are of the form (1.7)–(1.9) with the following amplitude condition

$$\begin{aligned} \delta = |\rho_l - \rho_r| + |u_l - u_r| + |\theta_l - \theta_r| < \eta_0, \\ \frac{1}{2} \sup_{(t,x)\in\mathbf{R}_+\times\mathbf{R}} \overline{\theta}(t, x) < \inf_{(t,x)\in\mathbf{R}_+\times\mathbf{R}} \overline{\theta}(t, x). \end{aligned} \tag{1.12}$$

Notice that ε appears in the initial data for (1.7), while η_0 enters the amplitude condition (1.12).

For any approximate rarefaction wave $\left(\overline{\rho}(t,x), \overline{u}(t,x), \overline{\theta}(t,x)\right) \in \mathcal{R}(\varepsilon, \eta_0; \rho_l, u_l, \theta_l)$, we define $\mathcal{I}\left(\varepsilon_0, \varepsilon, \eta_0; \overline{\rho}, \overline{u}, \overline{\theta}\right)$ to be the class of initial data $f_0(x,\xi)$ satisfying

$$\left\| f_0(x,\xi) - \mathbf{M}_{[\overline{\rho}(0,x), \overline{u}(0,x), \overline{\theta}(0,x)]} \right\|_{H_x^s\left(L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_-}}\right)\right)} \leqq \varepsilon_0 \tag{1.13}$$

for a global Maxwellian $\mathbf{M}_-$ which satisfies (1.11).

Then, the main result of this paper can be stated as follows:

Theorem 1.1. *Given* (ρ_l, u_l, θ_l) *with* $\rho_l > 0, \theta_l > 0$, *let* $\left(\overline{\rho}(t,x), \overline{u}(t,x), \overline{\theta}(t,x)\right) \in \mathcal{R}(\varepsilon, \eta_0; \rho_l, u_l, \theta_l)$, *and pick* ε *and* ε_0 *small enough. Then, for each* $f_0(x,\xi) \in \mathcal{I}\left(\varepsilon_0, \varepsilon, \eta_0; \overline{\rho}, \overline{u}, \overline{\theta}\right)$, *the Cauchy problem for the Boltzmann equation* (1.1) *with initial data* $f_0(x,\xi)$ *yields a unique global solution* $f(t,x,\xi) \in \mathbf{H}^s_{t,x,\xi}\ (\mathbf{R}^+)$ *satisfying, for some positive constant* $\delta_0 = O(1)(\varepsilon_0 + \varepsilon)$,

$$\left\| f(t,x,\xi) - \mathbf{M}_{[\overline{\rho}, \overline{u}, \overline{\theta}]} \right\|_{H_x^s\left(L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_-}}\right)\right)} \leqq \delta_0, \tag{1.14}$$

and tends to the local thermo-equilibrium rarefaction waves time asymptotically:

$$\lim_{t\to\infty} \left\| f(t,x,\xi) - \mathbf{M}_{[\rho^R, u^R, \theta^R]} \right\|_{L_x^\infty\left(L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_-}}\right)\right)} = 0. \tag{1.15}$$

Remark 1.2. The constant $s \geqq 4$ in Theorem 1.1 is any given integer, the constant ε comes from the definition of the approximate rarefaction wave profile in (1.7). The existence of the global Maxwellian $\mathbf{M}_-$ is a consequence of the energy estimate. The constant $\eta_0 > 0$ is mainly for validity of the microscopic H-theorem, (1.17), and will be specified later in Lemma 4.2.

For the energy method, the Boltzmann equation and its solutions are decomposed into fluid, and non-fluid, parts (see Section 2). The fluid part of the Boltzmann equation contains Navier-Stokes type dissipations. The complete understanding of the dissipation requires the decomposition of the $\mathbf{H}$-theorems, see Section 4. A basic element here, beyond [28, 29], comes from the fact that the strength of a nonlinear wave pattern may not be small. In its simplest form, the micorscopic $\mathbf{H}$-theorem states that the linearized collision operator $L_{\mathbf{M}_0}$ around a fixed Mawellian state $\mathbf{M}_0$ is negative definite when applied to an non-fluid element $\mathbf{G}$, [8],

$$-\int_{\mathbf{R}^3} \frac{\mathbf{G} L_{\mathbf{M}_0} \mathbf{G}}{\mathbf{M}_0} d\xi \geqq \sigma \int_{\mathbf{R}^3} \frac{(1+|\xi|)\mathbf{G}^2}{\mathbf{M}_0} d\xi \tag{1.16}$$

for a positive constant σ.

Here, with the varying Maxwellian $\mathbf{M}$, we show that

$$-\int_{\mathbf{R}^3} \frac{\mathbf{G} L_{\mathbf{M}} \mathbf{G}}{\mathbf{M}_-} d\xi \geqq \overline{\sigma} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\mathbf{G}^2}{\mathbf{M}_-} d\xi, \tag{1.17}$$

for a global Maxwellian $\mathbf{M}_-$ with $\theta/2 < \theta_- < \theta$. The restriction $\delta < \eta_0$ on the strength δ of the rarefaction wave is mainly to ensure that this microscopic version of $\mathbf{H}$-theorem holds for some $\overline{\sigma} = \overline{\sigma}(\rho, u, \theta; \rho_-, u_-, \theta_-) > 0$. Clearly, (1.17) would hold for $\mathbf{M}$ varying over a small neighborhood of a fixed Maxwellian. We will show in Section 4 that it holds for rarefaction waves that are not necessarily small in strength.

The accurate approximate Boltzmann rarefaction waves are constructed based on a Chapman-Enskog type expansion, see Section 3. These waves are not in thermo-equilibrium. As in [29], the fluid component is first estimated by the entropy method, as for the Navier-Stokes equations (see Section 5), and then we make use of the coupling property of the Euler equations to complete the estimate of the density function, omitted by the entropy estimate because of the degeneracy of the Navier-Stokes dissipations, see Section 5. The microscopic $\mathbf{H}$-theorem (1.17) is used in Sections 6 and 7 to estimate the non-fluid component.

For the entropy estimate with respect to local Maxwellian (see Section 5) the key monotonic property of the characteristic fields across rarefaction waves is used. To treat the nonlinear term by Sobolev analysis, we need to carry out the energy estimates up to fourth-order differentiations in Section 6. However, a complication arises because the microscopic H-theorem, (2.13) and (1.17), has dissipation on the microscopic component of the order of $\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau$ where the order of growth in ξ is only 1. However, the energy estimate by using the weight of local Maxwellian has error terms with a polynomial of ξ with order greater than 1 because of the derivatives on the local Maxwellian. Hence, another set of energy estimates based on a global suitably chosen Maxwellian $\mathbf{M}_-$ is needed to complete the analysis, see Section 7. When we perform the energy estimates with respect to the local Maxwellian $\mathbf{M}$, there is a typical error term $\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}^2} \mathbf{M}_t d\xi dx d\tau$ which appears and satisfies

$$\left| \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}^2} \mathbf{M}_t d\xi dx d\tau \right|$$
$$\leqq C(\theta - \theta_-, \rho, u, \rho_-, u_-)(\varepsilon + \delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau, \quad (1.18)$$

where $\mathbf{M}_t \equiv \partial_t \mathbf{M}$, $\partial^\alpha = \partial_t^{\alpha_0} \partial_x^{\alpha_1}$ for $\alpha = (\alpha_0, \alpha_1) \geqq 0$. Notice that the error term in the above inequality is now an integral with the weight $\mathbf{M}_-$ and a small factor of the order of $\varepsilon + \delta_0$. We thank the anonymous referee who pointed out [6], where arguments similar to this last estimate have been used. Although an additional term in the form of integrals of the fluid components and their derivatives appears because the orthogonality property of $\mathbf{M}$ and $\mathbf{G}$ fails with respect to weight $\mathbf{M}_-$, the small factor $\varepsilon + \delta_0$ in (1.18) helps to yield the desired estimates. For the higher-order energy estimates on the macroscopic component $\mathbf{M}$, there is no need for the use of a global Maxwellian $\mathbf{M}_-$ because all polynomials of ξ (if any) can be absorbed by the local Maxwellian $\mathbf{M}$.

Before the energy method based on the decomposition (2.5) is used, an elegant analysis using the spectral properties of the linearized collision operator $L_{\mathbf{M}}$ is used to obtain the existence and large-time behavior of solutions to the Boltzmann equation, see [21, 35, 39] and references therein.

2. Micro-macro decomposition and fluid equations

The collision operator has five collision invariants $\psi_\alpha(\xi)$, cf. [5]:

$$\begin{aligned}&\psi_0(\xi) \equiv 1,\\&\psi_i(\xi) \equiv \xi_i \ \text{ for } \ i = 1, 2, 3, \ \text{ or } \psi(\xi) = \xi,\\&\psi_4(\xi) \equiv \frac{1}{2}|\xi|^2,\end{aligned} \tag{2.1}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_j(\xi) Q(h, g) d\xi = 0 \text{ for } j = 0, 1, 2, 3, 4.$$

In the following, we define an inner product in $\xi \in \mathbf{R}^3$ with respect to the local Maxwellian $\mathbf{M}$ as:

$$\langle h, g\rangle \equiv \int_{\mathbf{R}^3} \frac{1}{\mathbf{M}} h(\xi) g(\xi) d\xi$$

for functions $h, \ g$ of ξ such that the above integral is well defined. With respect to this inner product, the following functions spanning the space of macroscopic, i.e. fluid components of the solution, are pairwise orthogonal:

$$\begin{aligned}\chi_0(\xi; \rho, u, \theta) &\equiv \frac{1}{\sqrt{\rho}}\mathbf{M},\\ \chi_i(\xi; \rho, u, \theta) &\equiv \frac{\xi^i - u^i}{\sqrt{R\rho\theta}}\mathbf{M} \ \text{ for } \ i = 1, 2, 3,\\ \chi_4(\xi; \rho, u, \theta) &\equiv \frac{1}{\sqrt{6\rho}}\left(\frac{|\xi - u|^2}{R\theta} - 3\right)\mathbf{M},\\ \left\langle\chi_i, \chi_j\right\rangle &= \delta_{ij}, \text{ for } i, j = 0, 1, 2, 3, 4.\end{aligned} \tag{2.2}$$

The macroscopic projection $\mathbf{P}_0$ and microscopic projection $\mathbf{P}_1$ can be defined as:

$$\begin{cases}\mathbf{P}_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j\rangle \chi_j,\\ \mathbf{P}_1 h \equiv h - \mathbf{P}_0 h.\end{cases} \tag{2.3}$$

Notice that the operators $\mathbf{P}_0$ (and therefore $\mathbf{P}_1$) are orthogonal (and thus self-adjoint) projections for the inner product $\langle\cdot, \cdot\rangle$.

A function $h(\xi)$ is called microscopic, or non-fluid, if it has no fluid components, i,e.

$$\int_{\mathbf{R}^3} h(\xi)\psi_j(\xi) d\xi = 0, \ \text{ for } j = 0, 1, 2, 3, 4. \tag{2.4}$$

It is clear that such a function is in the range of the microscopic projection $\mathbf{P}_1$.

The solution of the Boltzmann equation $f(t,x,\xi)$ is decomposed into the macroscopic (fluid) component, i.e. the local Mawellian $\mathbf{M} = \mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}$ and the microscopic (non-fluid) component, i.e. $\mathbf{G} = \mathbf{G}(t,x,\xi)$:

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi), \ \mathbf{P}_0 f = \mathbf{M}, \quad \mathbf{P}_1 f = \mathbf{G}. \tag{2.5}$$

The Boltzmann equation hence becomes:

$$(\mathbf{M}+\mathbf{G})_t + \xi_1(\mathbf{M}+\mathbf{G})_x = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G},\mathbf{G}), \tag{2.6}$$

where $L_{\mathbf{M}}$ is the linearized collision operator around the local Maxwellian $\mathbf{M}$:

$$L_{\mathbf{M}} g = L_{[\rho,u,\theta]} g = 2Q(g,\mathbf{M}).$$

With the micro-macro decomposition, the Boltzmann equation (2.6) can be decomposed as follows: the conserved variables are governed by the conservation laws, which are obtained by taking the inner product of the Boltzmann equation with the collision invariants $\psi_\alpha(\xi)$:

$$\begin{aligned}
\rho_t + (\rho u_1)_x &= 0, \\
(\rho u_1)_t + \left(\rho u_1^2 + p\right)_x &= -\left(\int_{\mathbf{R}^3} \xi_1^2 \mathbf{G} d\xi\right)_x, \\
(\rho u_2)_t + (\rho u_1 u_2)_x &= -\left(\int_{\mathbf{R}^3} \xi_1\xi_2 \mathbf{G} d\xi\right)_x, \\
(\rho u_3)_t + (\rho u_1 u_3)_x &= -\left(\int_{\mathbf{R}^3} \xi_1\xi_3 \mathbf{G} d\xi\right)_x, \\
\left[\rho(\tfrac{1}{2}|u|^2 + \mathbf{E})\right]_t + \left(u_1\left(\rho\left(\tfrac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_x &= -\frac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1|\xi|^2 \mathbf{G} d\xi\right)_x.
\end{aligned} \tag{2.7}$$

Another component of the Boltzmann equation, the microscopic equation for $\mathbf{G}$, is obtained by applying the microscopic projection $\mathbf{P}_1$ to (2.6):

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1\mathbf{G}_x + \xi_1\mathbf{M}_x) = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G},\mathbf{G}), \tag{2.8}$$

whence

$$\begin{aligned}
\mathbf{G} =& \ L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi_1\mathbf{M}_x)\Big) + L_{\mathbf{M}}^{-1}\Big(\mathbf{G}_t + \mathbf{P}_1(\xi_1\mathbf{G}_x) - Q(\mathbf{G},\mathbf{G})\Big) \\
:=& \ L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi_1\mathbf{M}_x)\Big) + \Theta.
\end{aligned} \tag{2.9}$$

Substitute (2.9) into (2.7), the conservation laws now become:

$$
\begin{aligned}
\rho_t + (\rho u_1)_x &= 0,\\
(\rho u_1)_t + \Big(\rho u_1^2 + p\Big)_x &= -\left(\int_{\mathbf{R}^3} \xi_1^2 L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi_1 \mathbf{M}_x)\Big) d\xi\right)_x\\
&\quad - \left(\int_{\mathbf{R}^3} \xi_1^2 \Theta d\xi\right)_x,\\
(\rho u_2)_t + (\rho u_1 u_2)_x &= -\left(\int_{\mathbf{R}^3} \xi_1 \xi_2 L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi_1 \mathbf{M}_x)\Big) d\xi\right)_x\\
&\quad - \left(\int_{\mathbf{R}^3} \xi_1 \xi_2 \Theta d\xi\right)_x,\\
(\rho u_3)_t + (\rho u_1 u_3)_x &= -\left(\int_{\mathbf{R}^3} \xi_1 \xi_3 L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi_1 \mathbf{M}_x)\Big) d\xi\right)_x\\
&\quad - \left(\int_{\mathbf{R}^3} \xi_1 \xi_3 \Theta d\xi\right)_x,\\
\Big[\rho(\tfrac{1}{2}|u|^2 + \mathbf{E})\Big]_t &+ \left(u_1\Big(\rho\left(\tfrac{1}{2}|u|^2 + \mathbf{E}\right) + p\Big)\right)_x\\
&= -\tfrac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi_1 \mathbf{M}_x)\Big) d\xi\right)_x\\
&\quad - \tfrac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 \Theta d\xi\right)_x.
\end{aligned}
\tag{2.10}
$$

The fluid equations can now be viewed as being a part of the Boltzmann equation. In (2.7), if the gas is assumed to be in thermo-equilibrium, that is, setting $\mathbf{G}$ to be zero, then the conservation laws become the Euler equations for gas dynamics, (1.4). If we neglect all the terms containing Θ, then (2.10) becomes the Navier-Stokes equations for gas dynamics. By rewriting the Boltzmann equation in this form, we can later construct the time-asymptotic rarefaction waves for the Boltzmann equation based on the fluid equations. It also allows us to perform the energy analysis, based in part, on the energy estimate for the fluid equations.

The approximate rarefaction waves for the Boltzmann equation, as defined by Euler equations in Section 1 and studied later in Section 3, are locally Maxwellian. However, to carry out the energy method, we will need to include the non-equilibrium Navier-Stokes effects of the viscosity and heat conductivity through the microscopic component $\mathbf{G}$ in (2.10). To this end, we subtract from $\mathbf{G}(t,x,\xi)$ the term $\overline{\mathbf{G}}(t,x,\xi)$:

$$
\begin{aligned}
&\overline{\mathbf{G}}(t,x,\xi)\\
&\quad = \frac{L_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1\left[\xi_1\left(\frac{|\xi - u(t,x)|^2}{2\theta(t,x)}\overline{\theta}_x(t,x) + \xi_1 \cdot \overline{u}_{1x}\right)\mathbf{M}(t,x)\right]\right\}}{\theta(t,x)},
\end{aligned}
\tag{2.11}
$$

which is the first term in the Chapman-Enskog expansion, cf. (2.9) and (2.10). The reason for this subtraction is that the approximate rarefaction waves defined by the

Euler equations through the inviscid Burgers equation are not sufficiently accurate for the energy method. In fact, $\left\|\left(\overline{u}_x, \overline{\theta}_x\right)(t)\right\|_{L^2}^2$ is not integrable with respect to t. Thus we include the Navier-Stokes term from $\overline{\mathbf{G}}(t, x, \xi)$ here.

For later use, we now collect some properties of the linearized collision operator $L_{\mathbf{M}}$ in the following lemma (cf. [17, 19]).

Lemma 2.1. *$L_{\mathbf{M}}$ has the following properties:*

(i) *$L_{\mathbf{M}}$ is self-adjoint, that is, to say:*

$$\langle h, L_{\mathbf{M}} g\rangle = \langle L_{\mathbf{M}} h, g\rangle.$$

(ii) *The null space $\mathcal{N}$ of $L_{\mathbf{M}}$ contains only the macroscopic fluid variables χ_j, $j = 0, \cdots, 4$.*

(iii) *For the hard sphere model, $L_{\mathbf{M}}$ takes the form, cf.* [19, 17],

$$(L_{\mathbf{M}} h)(\xi) = -\nu(\xi; \rho, u, \theta) h(\xi) + \sqrt{\mathbf{M}(\xi)} K_{\mathbf{M}} \left(\left(\frac{h}{\sqrt{\mathbf{M}}}\right)(\xi)\right). \tag{2.12}$$

Here $K_{\mathbf{M}}(\cdot) = -K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2 operator, and $\nu(\xi; \rho, u, \theta)$ and $K_{i\mathbf{M}}(\cdot)$ have the following expressions

$$\begin{aligned}
\nu(\xi; \rho, u, \theta) &= \frac{2\rho}{\sqrt{2\pi R\theta}} \left\{ \left(\frac{R\theta}{|\xi - u|} + |\xi - u| \right) \int_0^{|\xi - u|} \exp\left(-\frac{y^2}{2R\theta}\right) dy \right. \\
&\qquad \left. + R\theta \exp\left(-\frac{|\xi - u|^2}{2R\theta}\right) \right\}, \\
k_{1\mathbf{M}}(\xi, \xi_*) &= \frac{\pi\rho}{\sqrt{(2\pi R\theta)^3}} |\xi - \xi_*| \exp\left(-\frac{|\xi - u|^2}{4R\theta} - \frac{|\xi_* - u|^2}{4R\theta}\right), \\
k_{2\mathbf{M}}(\xi, \xi_*) &= \frac{2\rho}{\sqrt{2\pi R\theta}} |\xi - \xi_*|^{-1} \exp\left(-\frac{|\xi - \xi_*|^2}{8R\theta} - \frac{(|\xi|^2 - |\xi_*|^2)^2}{8R\theta|\xi - \xi_*|^2}\right),
\end{aligned}$$

where $k_{i\mathbf{M}}(\xi, \xi_*)(i = 1, 2)$ is the kernel of the operator $K_{i\mathbf{M}}(i = 1, 2)$, respectively.

(iv) *There exists $\sigma_0(\rho, u, \theta) > 0$ such that for any microscopic, non-fluid function $h(\xi) \in \mathcal{N}^\perp$*

$$\langle h, L_{\mathbf{M}} h\rangle \leqq -\sigma_0(\rho, u, \theta)\langle h, h\rangle,$$

which implies, cf. [17],

$$\langle h, L_{\mathbf{M}} h\rangle \leqq -\sigma(\rho, u, \theta)\langle (1 + |\xi|) h, h\rangle, \tag{2.13}$$

with some constant $\sigma(\rho, u, \theta) > 0$.

Before concluding this section, notice also that the projections $\mathbf{P}_0$ and $\mathbf{P}_1$ have the following basic properties:

$$\begin{aligned}
\mathbf{P}_0(\psi_j\mathbf{M}) &= \psi_j\mathbf{M},\ \mathbf{P}_1(\psi_j\mathbf{M}) = 0,\ j = 0, 1, 2, 3, 4,\\
L_{\mathbf{M}}\mathbf{P}_1 &= \mathbf{P}_1 L_{\mathbf{M}} = L_{\mathbf{M}},\ \mathbf{P}_1(Q(h,h)) = Q(h,h),\\
L_{\mathbf{M}}\mathbf{P}_0 &= \mathbf{P}_0 L_{\mathbf{M}} = 0,\ \mathbf{P}_0(Q(h,h)) = 0,\\
\langle\psi_j\mathbf{M}, h\rangle &= \langle\psi_j\mathbf{M}, \mathbf{P}_0 h\rangle,\ j = 0, 1, 2, 3, 4,\\
\langle h, L_{\mathbf{M}} g\rangle &= \langle\mathbf{P}_1 h, L_{\mathbf{M}}(\mathbf{P}_1 g)\rangle,\\
\left\langle h, L_{\mathbf{M}}^{-1}(\mathbf{P}_1 g)\right\rangle &= \left\langle L_{\mathbf{M}}^{-1}(\mathbf{P}_1 h), \mathbf{P}_1 g\right\rangle = \left\langle\mathbf{P}_1 h, L_{\mathbf{M}}^{-1}(\mathbf{P}_1 g)\right\rangle.
\end{aligned}$$

3. Approximate rarefaction waves

The solutions of the Riemann problem for the Euler equations are self-similar and governed by the inviscid Burgers equation for $i = 1, 3$,

$$\begin{aligned}
&\lambda_{it}^{R_i} + \lambda_i^{R_i}\lambda_{ix}^{R_i} = 0,\\
&\lambda_i^{R_i}(0,x) = \lambda_{i0}^{R_i}(x) = \begin{cases} \lambda_{i-}, & x < 0\\ \lambda_{i+}, & x > 0 \end{cases}
\end{aligned} \tag{3.1}$$

with $\lambda_{i-} \leqq \lambda_{i+}$, which have continuous solutions of the form $\lambda_i^{R_i}\left(\frac{x}{t}\right)$ given by

$$\lambda_i^{R_i}(z) = \begin{cases} \lambda_{i-}, & z \leqq \lambda_{i-},\\ z, & \lambda_{i-} \leqq z \leqq \lambda_{i+},\\ \lambda_{i+}, & z \geqq \lambda_{i+}. \end{cases} \tag{3.2}$$

In [32], it is shown that $\lambda_i^{R_i}\left(\frac{x}{t}\right)$ is approximated by the solution to the Cauchy problem (1.7) with decay rates when t is sufficiently large. This approximation is summarized in the following lemmas (the interested reader is referred to [32] for the proof).

Lemma 3.1. *Let $\delta_i = \lambda_{i+} - \lambda_{i-}$ be the wave strength of the i-th rarefaction wave, and have that the Cauchy problem (1.7) has a unique global smooth solution $\lambda_i(t,x)$ which satisfies the conditions:*

(i) $\lambda_{i-} < \lambda_i(t,x) < \lambda_{i+},\ \lambda_{ix}(t,x) > 0,\ \forall (t,x) \in \mathbf{R}_+ \times \mathbf{R}$.
(ii) *For any $p(1 \leqq p \leqq \infty)$, there exists a constant $C(p)$, depending only on p, such that*

$$\begin{aligned}
\|\lambda_{ix}(t,x)\|_{L^p} &\leqq C(p)\min\left\{\delta_i\varepsilon^{1-\frac{1}{p}},\ \delta_i^{\frac{1}{p}} t^{-1+\frac{1}{p}}\right\},\\
\left\|\frac{\partial^j}{\partial x^j}\lambda_i(t,x)\right\|_{L^p} &\leqq C(p)\min\left\{\delta_i\varepsilon^{j-\frac{1}{p}},\ \varepsilon^{j-1-\frac{1}{p}} t^{-1}\right\},\ j \geqq 2,
\end{aligned}$$

and in the region between two waves, we have

$$|(\lambda_1(t,x)-\lambda_{1+})\,\lambda_{3x}(t,x)| \leqq O(1)\delta_1\delta_3\varepsilon\exp(-2d_1\varepsilon t)\,,$$

$$|(\lambda_3(t,x)-\lambda_{3-})\,\lambda_{1x}(t,x)| \leqq O(1)\delta_1\delta_3\varepsilon\exp(-2d_1\varepsilon t)\,.$$

Specifically, there exists a constant $C(p)>0$ *such that for* $p>1$

$$\|(\lambda_1(t,x)-\lambda_{1+})\,\lambda_{3x}(t,x)\|_{L^p} \leqq C(p)\,(\delta_1\delta_3\varepsilon)^{1-\frac{1}{p}}\exp\left(-2d_1\left(1-\frac{1}{p}\right)\varepsilon t\right),$$

$$\|(\lambda_3(t,x)-\lambda_{3-})\,\lambda_{1x}(t,x)\|_{L^p} \leqq C(p)\,(\delta_1\delta_3\varepsilon)^{1-\frac{1}{p}}\exp\left(-2d_1\left(1-\frac{1}{p}\right)\varepsilon t\right).$$

(iii) $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbf{R}}\left|\lambda_i(t,x)-\lambda_i^{R_i}(\frac{x}{t})\right|=0.$

From Lemma 3.1 and (1.8), we can easily deduce that $\left(\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)\right)$ is globally (with respect to t and x) defined and smooth. As we mentioned before, from (1.7) and (1.8), $\left(\rho^{A_i},u^{A_i},\ \theta^{A_i}\right)(t,x)(i=1,3)$ is an exact solution of the Euler equations, [26],

$$\begin{aligned}
\rho_t^{A_i}+\left(\rho^{A_i}u_1^{A_i}\right)_x&=0,\\
u_{1t}^{A_i}+u_1^{A_i}u_{1x}^{A_i}+\frac{2}{3}\theta_x^{A_i}+\frac{2\theta^{A_i}}{3\rho^{A_i}}\rho_x^{A_i}&=0,\\
\theta_t^{A_i}+u_1^{A_i}\theta_x^{A_i}+\frac{2}{3}\theta^{A_i}u_{1x}^{A_i}&=0.
\end{aligned}\tag{3.3}$$

Moreover, from (3.3), (1.9) and (1.8), we have the following expressions of the approximate rarefaction waves which we list here for later use:

$$\begin{aligned}
\overline{\rho}_x(t,x)&=-\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\exp\left(-\frac{\overline{S}}{2}\right)u_{1x}^{A_1}(t+t_0,x)\\
&\quad+\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\exp\left(-\frac{\overline{S}}{2}\right)u_{1x}^{A_3}(t+t_0,x)+E_1(t,x),\\
\overline{\rho}_t(t,x)&=\left\{\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\frac{\overline{S}}{2}\right)-\overline{\rho}(t,x)\right\}u_{1x}^{A_1}(t+t_0,x)\\
&\quad-\left\{\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\frac{\overline{S}}{2}\right)+\overline{\rho}(t,x)\right\}u_{1x}^{A_3}(t+t_0,x)\\
&\quad+E_2(t,x),\\
\overline{u}_{1t}(t,x)&=\left\{\frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{1}{3}}(t,x)\exp\left(\frac{\overline{S}}{2}\right)-\overline{u}_1(t,x)\right\}u_{1x}^{A_1}(t+t_0,x)\\
&\quad-\left\{\frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{1}{3}}(t,x)\exp\left(\frac{\overline{S}}{2}\right)+\overline{u}_1(t,x)\right\}u_{1x}^{A_3}(t+t_0,x)\\
&\quad+E_3(t,x),
\end{aligned}\tag{3.4}$$

where

$$
\begin{aligned}
E_1(t,x) =& -\left\{\frac{3}{\sqrt{15k}}\left(\rho^{A_1}(t+t_0,x)\right)^{\frac{2}{3}} - \frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\right\}\exp\left(-\tfrac{\overline{S}}{2}\right)u_{1x}^{A_1}(t+t_0,x)\\
&+\frac{3\exp\left(-\frac{\overline{S}}{2}\right)}{\sqrt{15k}}\left\{\left(\rho^{A_3}(t+t_0,x)\right)^{\frac{2}{3}} - \overline{\rho}^{\frac{2}{3}}(t,x)\right\}u_{1x}^{A_3}(t+t_0,x),\\
E_2(t,x) =& \left\{\left[\frac{3}{\sqrt{15k}}\left(\rho^{A_1}(t+t_0,x)\right)^{\frac{2}{3}}u_1^{A_1}(t+t_0,x)\exp\left(-\tfrac{\overline{S}}{2}\right) - \rho^{A_1}(t+t_0,x)\right]\right.\\
&\left.-\left[\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\tfrac{\overline{S}}{2}\right) - \overline{\rho}(t,x)\right]\right\}u_{1x}^{A_1}(t+t_0,x)\\
&-\left\{\left[\frac{3}{\sqrt{15k}}\left(\rho^{A_3}(t+t_0,x)\right)^{\frac{2}{3}}u_1^{A_3}(t+t_0,x)\exp\left(-\tfrac{\overline{S}}{2}\right) + \rho^{A_3}(t+t_0,x)\right]\right.\\
&\left.-\left[\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(\tfrac{\overline{S}}{2}\right) + \overline{\rho}(t,x)\right]\right\}u_{1x}^{A_3}(t+t_0,x),\\
E_3(t,x) =& -\left\{\left[u_1^{A_1}(t+t_0,x) - \frac{\sqrt{15k}}{3}\left(\rho^{A_1}(t+t_0,x)\right)^{\frac{1}{3}}\exp\left(\tfrac{\overline{S}}{2}\right)\right]\right.\\
&\left.-\left[\overline{u}_1(t,x) - \frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{1}{3}}(t,x)\exp\left(\tfrac{\overline{S}}{2}\right)\right]\right\}u_{1x}^{A_1}(t+t_0,x)\\
&-\left\{\left[u_1^{A_3}(t+t_0,x) + \frac{\sqrt{15k}}{3}\left(\rho^{A_3}(t+t_0,x)\right)^{\frac{1}{3}}\exp\left(\tfrac{\overline{S}}{2}\right)\right]\right.\\
&\left.-\left[\overline{u}_1(t,x) + \frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{2}{3}}(t,x)\exp\left(\tfrac{\overline{S}}{2}\right)\right]\right\}u_{1x}^{A_3}(t+t_0,x). \qquad (3.5)
\end{aligned}
$$

From these expressions and Lemma 3.1 we find that $\left(\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)\right)$ has the following decay properties in the L^p norms. Note also that all the space and time derivatives of $\left(\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)\right)$ are dominated by $\bar{u}_{1x}$ which is positive as stated in the following lemma from [32].

Lemma 3.2. *The approximate rarefaction wave* $(\overline{\rho},\overline{u},\overline{\theta})(t,x)$ *constructed in* (1.9) *has the following properties:*

(i) $u_{1x}^{A_i}(t,x) > 0,\ \forall (t,x)\in \mathbf{R}_+\times\mathbf{R},\ i=1,3.$
(ii) *For any* $p(1\leqq p\leqq\infty)$*, there exists a constant* $C(p)>0$*, depending only on* p*, such that*

$$
\left\|\left(\overline{\rho},\overline{u},\overline{\theta}\right)_x(t,x)\right\|_{L^p} \leqq C(p)\min\left\{\varepsilon^{1-\frac{1}{p}},\ (t+t_0)^{-1+\frac{1}{p}}\right\},
$$

$$
\left\|\frac{\partial^j}{\partial x^j}\left(\overline{\rho},\overline{u},\overline{\theta}\right)(t,x)\right\|_{L^p} \leqq C(p)\min\left\{\varepsilon^{j-\frac{1}{p}},\ \varepsilon^{j-1-\frac{1}{p}}t^{-1}\right\},\ j\geqq 2,
$$

and

$$\begin{aligned} &\left|\left(\overline{\rho}_x, \overline{\theta}_x\right)(t,x)\right| \leqq O(1)\left(\overline{u}_{1x}(t,x) + |E_1|\right), \\ &\|(E_1, E_2, E_3)(t,x)\|_{L^p} \leqq O(1)\varepsilon^{1-\frac{1}{p}} \exp\left(-2\left(\frac{1}{\varepsilon} + d_1\varepsilon t\right)\left(1-\frac{1}{p}\right)\right). \end{aligned} \tag{3.6}$$

(iii) $\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left|\left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t,x) - \left(\rho^R, u^R, \theta^R\right)\left(\frac{x}{t}\right)\right| = 0.$

Remark 3.3. Notice that the quantities $E_i (i = 1, 2, 3)$ measure the interactions of rarefaction waves from different families and $(3.6)_2$ means that if we introduce a positive parameter $t_0 = \frac{1}{d_1\varepsilon^2}$ in the approximation of the rarefaction waves profile as in (1.9), such quantities can be suitably controlled. This choice of t_0 simply implies that if two rarefaction waves are seperated enough initially, then the interactions between them are sufficiently weak.

Precisely, Lemma 3.2 implies that for $\varepsilon > 0$ sufficiently small

$$\begin{aligned} &\int_0^t \left(\|(\overline{u}_{1x}, E_1)\|_{L^2}^{\frac{5}{2}} + \|\overline{u}_{1x}\|_{L^3}^3 + \|E_1\|_{L^2}^2 + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})\right\|_{L^1}^{\frac{5}{4}}\right)(\tau)d\tau \\ &\qquad \leqq O(1)\varepsilon^{\frac{1}{8}}, \\ &\int_0^t \left(\|(\overline{u}_{1x}, E_1)\|_{L^2}^3 + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})\right\|_{L^1}^{\frac{3}{2}}\right)(\tau)\left\|(\tilde{\rho}, \tilde{u}, \tilde{\theta})\right\|_{L^2}^2 d\tau \\ &\qquad \leqq O(1)\int_0^t \left((1+\tau)^{-\frac{3}{2}} + \varepsilon^{\frac{3}{2}}\exp(-3d_1\varepsilon\tau)\right)\left\|(\tilde{\rho}, \tilde{u}, \tilde{\theta})\right\|_{L^2}^2 d\tau, \end{aligned} \tag{3.7}$$

where
$\left(\tilde{\rho}(t,x), \tilde{u}(t,x), \tilde{\theta}(t,x)\right) = \left(\rho(t,x) - \overline{\rho}(t,x), u(t,x) - \overline{u}(t,x), \theta(t,x) - \overline{\theta}(t,x)\right).$

4. *H*-Theorem

The H-theorem of the Boltzmann equation is based on the special property of the bilinear structure of $Q(f, f)$ which satisfies

$$\int_{\mathbf{R}^3} Q(f, f) \ln f d\xi \leqq 0,$$

and the equality only holds when the solution $f(t, x, \xi)$ is a Maxwellian, i.e. $f(t, x, \xi) = \mathbf{M}_{[\rho,u,\theta]}$. According to the dissipative effects on the macroscopic and microscopic components, the H-theorem can be viewed as two aspects. The first mainly considers the linearized collision operator $L_{\mathbf{M}}$ acting on the microscopic components stated in (2.13) and (1.17) which is called the microscopic H-theorem. The second originates from the nonlinear collision operator, where by the viscosity and heat conductivity can be expressed.

In the following, we first give a proof of (1.17). To do this, we need Lemma B.1 in [29], cf. also [15].

Lemma 4.1. *There exists a positive constant $C_2 > 0$ such that*

$$\int_{\mathbf{R}^3} \frac{(1+|\xi|)^{-1}Q(f,g)^2}{\mathbf{M}} d\xi \leqq \frac{C_2}{2}\left\{\int_{\mathbf{R}^3} \frac{(1+|\xi|)f^2}{\mathbf{M}} d\xi \cdot \int_{\mathbf{R}^3} \frac{g^2}{\mathbf{M}} d\xi + \int_{\mathbf{R}^3} \frac{f^2}{\mathbf{M}} d\xi \cdot \int_{\mathbf{R}^3} \frac{(1+|\xi|)g^2}{\mathbf{M}} d\xi\right\}, \tag{4.1}$$

where $\mathbf{M}$ *can be any Maxwellian so that the above integrals are well defined.*

Lemma 4.2. *If $\frac{\theta}{2} < \theta_- < \theta$, then there exist two positive constants $\overline{\sigma} = \overline{\sigma}(\rho, u, \theta; \rho_-, u_-, \theta_-)$ and $\eta_0 = \eta_0(\rho, u, \theta; \rho_-, u_-, \theta_-)$ such that if $|\rho - \rho_-| + |u - u_-| + |\theta - \theta_-| < \eta_0$, we have for $h(\xi) \in \mathcal{N}^\perp$,*

$$-\int_{\mathbf{R}^3} \frac{hL_{\mathbf{M}}h}{\mathbf{M}_-} d\xi \geqq \overline{\sigma} \int_{\mathbf{R}^3} \frac{(1+|\xi|)h^2}{\mathbf{M}_-} d\xi,$$

Where, here, $\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}$, $\mathbf{M}_- \equiv \mathbf{M}_{[\rho_-,u_-,\theta_-]}$.

Proof. By Lemma 4.1, we have

$$\begin{aligned}
-\int_{\mathbf{R}^3} \frac{hL_{\mathbf{M}}h}{\mathbf{M}_-} d\xi &= -\int_{\mathbf{R}^3} \frac{hL_{\mathbf{M}_-}h}{\mathbf{M}_-} d\xi + 2\int_{\mathbf{R}^3} \frac{hQ(\mathbf{M}_- - \mathbf{M}, h)}{\mathbf{M}_-} d\xi \\
&\geqq \sigma(\rho_-, u_-, \theta_-)\int_{\mathbf{R}^3} \frac{(1+|\xi|)h^2}{\mathbf{M}_-} d\xi + 2\int_{\mathbf{R}^3} \frac{hQ(\mathbf{M}_- - \mathbf{M}, h)}{\mathbf{M}_-} d\xi \\
&\geqq \frac{3}{4}\sigma(\rho_-, u_-, \theta_-)\int_{\mathbf{R}^3} \frac{(1+|\xi|)h^2}{\mathbf{M}_-} d\xi \\
&\quad - \frac{4}{\sigma(\rho_-, u_-, \theta_-)}\int_{\mathbf{R}^3} \frac{(1+|\xi|)^{-1}Q(\mathbf{M}_- - \mathbf{M}, h)^2}{\mathbf{M}_-} d\xi \\
&\geqq \frac{3}{4}\sigma(\rho_-, u_-, \theta_-)\int_{\mathbf{R}^3} \frac{(1+|\xi|)h^2}{\mathbf{M}_-} d\xi \\
&\quad - \frac{4C_2}{\sigma(\rho_-, u_-, \theta_-)}\int_{\mathbf{R}^3} \frac{(1+|\xi|)(\mathbf{M}_- - \mathbf{M})^2}{\mathbf{M}_-} d\xi \cdot \int_{\mathbf{R}^3} \frac{(1+|\xi|)h^2}{\mathbf{M}_-} d\xi.
\end{aligned} \tag{4.2}$$

Since $\frac{\theta}{2} < \theta_- < \theta$, we can choose a positive constant $C_3 = C_3(\rho, u, \theta; \rho_-, u_-, \theta_-)$ such that

$$\int_{|\xi| \geqq C_3} \frac{(1+|\xi|)(\mathbf{M}_- - \mathbf{M})^2}{\mathbf{M}_-} d\xi \leqq \frac{\sigma(\rho_-, u_-, \theta_-)^2}{16C_2}. \tag{4.3}$$

On the other hand,

$$\int_{|\xi| \leqq C_3} \frac{(1+|\xi|)(\mathbf{M}_- - \mathbf{M})^2}{\mathbf{M}_-} d\xi \leqq \frac{C_4}{4C_2}\Big(|\rho - \rho_-| + |u - u_-| + |\theta - \theta_-|\Big)^2 \tag{4.4}$$

holds for some positive constant $C_4 = C_4(\rho, u, \theta; \rho_-, u_-, \theta_-)$.

By substituting (4.3) and (4.4) into (4.2), and choosing

$$\eta_0 = \frac{\sigma(\rho_-, u_-, \theta_-)}{\sqrt{2C_4(\rho, u, \theta; \rho_-, u_-, \theta_-)}},$$

(4.2) gives the statement of the lemma.

The following is a direct corollary of Lemma 4.2 and the Cauchy inequality.

Corollary 4.3. *Under the assumptions listed in Lemma 4.2, we have*

$$\begin{aligned}\int_{\mathbf{R}^3} \frac{1+|\xi|}{\mathbf{M}} \left|L_{\mathbf{M}}^{-1} h\right|^2 d\xi \leqq \sigma^{-2} \int_{\mathbf{R}^3} \frac{(1+|\xi|)^{-1} h^2(\xi)}{\mathbf{M}} d\xi,\\ \int_{\mathbf{R}^3} \frac{1+|\xi|}{\mathbf{M}_-} \left|L_{\mathbf{M}}^{-1} h\right|^2 d\xi \leqq \overline{\sigma}^{-2} \int_{\mathbf{R}^3} \frac{(1+|\xi|)^{-1} h^2(\xi)}{\mathbf{M}_-} d\xi,\end{aligned} \tag{4.5}$$

which holds for each $h(\xi) \in \mathcal{N}^{\perp}$.

Finally, we include the following lemma for later use.

Lemma 4.4. *Under the conditions in Lemma 4.2, there exists a constant* $C_5 > 0$*, such that for each* $k \in \mathbf{Z}_+$ *and constant* $\lambda > 0$ *we have*

$$\left|\int_{\mathbf{R}^3} \frac{g_1 \mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_-} d\xi - \int_{\mathbf{R}^3} \frac{g_1 |\xi|^k g_2}{\mathbf{M}_-} d\xi\right| \leqq C_5 \int_{\mathbf{R}^3} \frac{\lambda |g_1|^2 + \lambda^{-1} |g_2|^2}{\mathbf{M}_-} d\xi. \tag{4.6}$$

Proof. Since

$$\mathbf{P}_1\left(|\xi|^k g_2\right) - |\xi|^k g_2 = -\mathbf{P}_0\left(|\xi|^k g_2\right) = -\sum_{j=0}^{4} \left\langle |\xi|^k g_2, \chi_j \right\rangle \chi_j,$$

we have from the assumption that $\frac{1}{2}\theta(t,x) < \theta_- < \theta(t,x)$ and Hölder's inequality that for $j = 0, 1, 2, 3, 4$

$$\begin{aligned}\left\langle |\xi|^k g_2, \chi_j \right\rangle &= \int_{\mathbf{R}^3} \frac{|\xi|^k g_2 \chi_j}{\mathbf{M}} d\xi\\ &\leqq \left(\int_{\mathbf{R}^3} \frac{g_2^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} \frac{|\xi|^{2k} \chi_j^2 \mathbf{M}_-}{\mathbf{M}^2} d\xi\right)^{\frac{1}{2}}\\ &\leqq O(1) \left(\int_{\mathbf{R}^3} \frac{g_2^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}^3} \frac{g_1 \chi_j}{\mathbf{M}_-} d\xi &\leqq \left(\int_{\mathbf{R}^3} \frac{g_1^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} \frac{\chi_j^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}}\\ &\leqq O(1) \left(\int_{\mathbf{R}^3} \frac{g_1^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

From the above two inequalities, we conclude for each $\lambda > 0$ that

$$\left|\int_{\mathbf{R}^3} \frac{g_1 \mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_-} d\xi - \int_{\mathbf{R}^3} \frac{g_1 |\xi|^k g_2}{\mathbf{M}_-} d\xi \right| \leqq O(1) \left(\int_{\mathbf{R}^3} \frac{g_1^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} \frac{g_2^2}{\mathbf{M}_-} d\xi\right)^{\frac{1}{2}}$$
$$\leqq C_5 \int_{\mathbf{R}^3} \frac{\lambda |g_1|^2 + \lambda^{-1} |g_2|^2}{\mathbf{M}_-} d\xi.$$

This completes the proof of Lemma 4.4.

In order to study the nonlinear wave behavior of the solutions, we also need to use the macroscopic version of the H-theorem studied in [28], see also [3]. To be self-contained, we include the macroscopic version as follows. Set the macroscopic entropy S by

$$-\frac{3}{2}\rho S \equiv \int_{\mathbf{R}^3} \mathbf{M} \ln \mathbf{M} d\xi. \tag{4.7}$$

Here, we have normalized the gas constant R to be $\frac{2}{3}$ so that $\mathbf{E} = \theta$ and $p = \frac{2}{3}\rho\theta$.

Direct calculation yields

$$-\tfrac{3}{2}(\rho S)_t - \tfrac{3}{2}(\rho u_1 S)_x + \left(\int_{\mathbf{R}^3} (\xi_1 \ln \mathbf{M}) \mathbf{G} d\xi\right)_x = \int_{\mathbf{R}^3} \frac{\mathbf{G} \mathbf{P}_1\left(\xi_1 \partial_x \mathbf{M}\right)}{\mathbf{M}} d\xi \tag{4.8}$$

and

$$\begin{aligned} S &= -\frac{2}{3} \ln \rho + \ln(2\pi R\theta) + 1, \\ p &= \frac{2}{3}\rho\theta = k\rho^{\frac{5}{3}} \exp(S), \quad k = \frac{1}{2\pi e}, \\ \mathbf{E} &= \frac{3}{2} R\theta. \end{aligned} \tag{4.9}$$

A convex entropy-entropy flux pair (η, q) around a Maxwellian $\overline{\mathbf{M}} = \mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]}$ ($\overline{u}_i = 0, i = 2, 3$) can be given as follows. Denote the conservation laws (2.7) by

$$\mathbf{m}_t + \mathbf{n}_x = -\begin{pmatrix} 0 \\ \int_{\mathbf{R}^3} \xi_1^2 \mathbf{G} d\xi \\ \int_{\mathbf{R}^3} \xi_1 \xi_2 \mathbf{G} d\xi \\ \int_{\mathbf{R}^3} \xi_1 \xi_3 \mathbf{G} d\xi \\ \frac{1}{2}\int_{\mathbf{R}^3} \xi_1 |\xi|^2 \mathbf{G} d\xi \end{pmatrix}_x. \tag{4.10}$$

Here

$$\begin{aligned}\mathbf{m} &= (m_0, m_1, m_2, m_3, m_4)^t \\ &= \left(\rho, \rho u_1, \rho u_2, \rho u_3, \rho\left(\frac{1}{2}|u|^2+\theta\right)\right)^t, \\ \mathbf{n} &= (n_0, n_1, n_2, n_3, n_4)^t \\ &= \left(\rho u_1, \rho u_1^2+\frac{2}{3}\rho\theta, \rho u_1 u_2, \rho u_1 u_3, \rho u_1\left(\frac{1}{2}|u|^2+\frac{5}{3}\theta\right)\right)^t.\end{aligned}$$

We now define an entropy-entropy flux pair (η, q) as

$$\begin{aligned}\eta &= \overline{\theta}\left\{-\tfrac{3}{2}\rho S + \tfrac{3}{2}\overline{\rho}\overline{S} + \tfrac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{m}-\overline{\mathbf{m}})\right\}, \\ q &= \overline{\theta}\left\{-\tfrac{3}{2}\rho u_1 S + \tfrac{3}{2}\overline{\rho}\overline{u}_1\overline{S} + \tfrac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{n}-\overline{\mathbf{n}})\right\}.\end{aligned} \tag{4.11}$$

Since

$$\begin{aligned}(\rho S)_{m_0} &= S + \tfrac{|u|^2}{2\theta} - \tfrac{5}{3}, \\ (\rho S)_{m_i} &= -\frac{u_i}{\theta},\ i = 1, 2, 3, \\ (\rho S)_{m_4} &= \tfrac{1}{\theta},\end{aligned}$$

we have

$$\begin{aligned}\eta &= \tfrac{3}{2}\left\{\rho\theta - \overline{\theta}\rho S + \rho\left[\left(\overline{S}-\tfrac{5}{3}\right)\overline{\theta} + \tfrac{|u-\overline{u}|^2}{2}\right] + \tfrac{2}{3}\overline{\rho}\overline{\theta}\right\}, \\ q &= u_1\eta + (u_1-\overline{u}_1)\left(\rho\theta - \overline{\rho}\overline{\theta}\right).\end{aligned} \tag{4.12}$$

The entropy-entropy flux thus constructed has the following properties: $\eta(\overline{\mathbf{m}}) = 0, \nabla_{\mathbf{m}}\eta(\overline{\mathbf{m}}) = 0$, and the Hessian $\nabla^2_{\mathbf{m}}\eta$ is positive definite for any $\mathbf{m}$ satisfying $\rho,\ \theta > 0$ (cf. [28]). Thus, for $\mathbf{m}$ in any closed bounded region in $\Sigma = \{\mathbf{m} : \rho > 0, \theta > 0\}$, there exists a positive constant C_6 such that

$$C_6^{-1}|\mathbf{m}-\overline{\mathbf{m}}|^2 \leqq \eta \leqq C_6|\mathbf{m}-\overline{\mathbf{m}}|^2. \tag{4.13}$$

From (4.12), it is straightforward to derive the following entropy equation

$$\begin{aligned}\eta_t + q_x &= \left\{\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_t + \nabla_{(\overline{\rho},\overline{u},\overline{S})}q\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_x\right\} \\ &\quad + \int_{\mathbf{R}^3}\left[\xi_1\partial_x(\overline{\theta}\ln\mathbf{M}) - \tfrac{3}{2}\overline{u}_{1x}\xi_1^2\right]\mathbf{G}d\xi \\ &\quad - \left\{\int_{\mathbf{R}^3}\left(\xi_1\overline{\theta}\ln\mathbf{M} - \tfrac{1}{2}\xi_1|\xi|^2 - \tfrac{3}{2}\xi_1^2\overline{u}_1\right)\mathbf{G}d\xi\right\}_x.\end{aligned} \tag{4.14}$$

Integrating the above identity over $[0, t) \times \mathbf{R} \times \mathbf{R}^3$ yields

$$\begin{aligned}\int_{\mathbf{R}} \eta(\tau)dx\Big|_{\tau=0}^{\tau=t} &= \int_0^t \int_{\mathbf{R}} \Big[\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta \cdot \big(\overline{\rho}, \overline{u}, \overline{S}\big)_t + \nabla_{(\overline{\rho},\overline{u},\overline{S})} q \cdot \big(\overline{\rho}, \overline{u}, \overline{S}\big)_x\Big] dx d\tau \\ &\quad + \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \Big[\xi_1 \partial_x (\overline{\theta} \ln \mathbf{M}) - \tfrac{3}{2}\overline{u}_{1x}\xi_1^2\Big] \mathbf{G} d\xi dx d\tau \\ &= I_1 + I_2. \end{aligned} \tag{4.15}$$

Here and in what follows, we denote the corresponding terms in the summation by I_i or I_i^j without any ambiguity.

We now use the microscopic H-theorem from Lemma 4.2 to get an estimate on I_2. The corresponding estimates on I_1 uses the particular expansion property of the rarefaction wave and will be given in Section 5.

Noticing that

$$\begin{aligned}&\mathbf{P}_1 \left[\left(\xi_1 \partial_x (\overline{\theta} \ln \mathbf{M}) - \frac{3}{2}\overline{u}_{1x}\xi_1^2\right)\mathbf{M}\right] \\ &\qquad = \frac{3}{2}\mathbf{P}_1 \left(\xi_1 \left(\xi \cdot \tilde{u}_x + \frac{\tilde{\theta}_x}{2\theta}|\xi - u|^2\right)\mathbf{M} + \xi_1 \tilde{\theta}\left(\frac{|\xi - u|^2}{2\theta}\right)_x \mathbf{M}\right), \\ &\mathbf{P}_1\Big(\xi_1 \partial_x \mathbf{M}\Big) = \frac{3}{2\theta}\mathbf{P}_1 \left(\xi_1 \left(\xi \cdot \tilde{u}_x + \frac{\tilde{\theta}_x}{2\theta}|\xi - u|^2\right)\mathbf{M} \right.\\ &\qquad\qquad \left. + \xi_1 \left(\frac{|\xi - u|^2}{2\theta}\overline{\theta}_x + \xi_1 \overline{u}_{1x}\right)_x \mathbf{M}\right),\end{aligned}$$

we have from (2.9), Lemmas 2.1, 3.1, 4.1, 4.2 and Corollary 4.3 that;

$$\begin{aligned} I_2 &\leqq \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{9}{4\theta} \frac{\mathbf{P}_1\left(\xi_1 \left(\xi \cdot \tilde{u}_x + \frac{\tilde{\theta}_x}{2\theta}|\xi - u|^2\right)\mathbf{M}\right)}{\mathbf{M}} \\ &\quad \times L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1 \left(\xi \cdot \tilde{u}_x + \frac{\tilde{\theta}_x}{2\theta}|\xi - u|^2\right)\mathbf{M}\right)\right) d\xi dx d\tau \\ &\quad + \lambda \int_0^t \int_{\mathbf{R}} \Big(|\tilde{u}_x|^2 + |\tilde{\theta}_x|^2 + \overline{u}_{1x}\left|\tilde{\theta}\right|^2\Big) dx d\tau \\ &\quad + O(1)\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\mathbf{G}_t^2 + (1 + |\xi|)\mathbf{G}_x^2 + (1 + |\xi|)^{-1}Q(\mathbf{G}, \mathbf{G})^2}{\mathbf{M}} d\xi dx d\tau \\ &\quad + O(1)\int_0^t \left(\|(\overline{u}_{1x}, E_1)(\tau)\|_{L^2}^3 + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{3}{2}}\right)\left\|\sqrt{\eta(\tau)}\right\|_{L^2}^2 d\tau \\ &\quad + O(1)\int_0^t \left(\|(\overline{u}_{1x}, E_1)(\tau)\|_{L^2}^{\frac{5}{2}} + \|\overline{u}_{1x}(\tau)\|_{L^3}^3 \right.\\ &\quad \left. + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{5}{4}} + \|E_1(\tau)\|_{L^2}^2\right) d\tau. \end{aligned} \tag{4.16}$$

Here we have used the fact that the derivatives of $\big(\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)\big)$ are dominated by $\overline{u}_{1x} > 0$ in Lemma 3.1 and $\lambda > 0$ denotes a sufficiently small constant.

In the above identity, the term

$$
\begin{aligned}
&\int_{\mathbf{R}^3} \frac{\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)}{\mathbf{M}}d\xi\\
&\quad=\left\langle \mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right),\right.\\
&\quad\left. L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)\right\rangle,
\end{aligned}
\tag{4.17}
$$

represents the entropy dissipation.

By Lemma 2.1, the operator $L_{\mathbf{M}}^{-1}:\mathcal{N}^{\perp}\to\mathcal{N}^{\perp}$ is one-to-one and bounded on any finite dimensional subspace of $\mathcal{N}^{\perp}$. Since the non-fluid function $\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)$ is of finite dimension in the ξ variables, there exists a positive constant $C>0$ such that

$$
\begin{aligned}
&C^{-1}\left\langle \mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right),\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right\rangle\\
&\quad\leqq\left\langle L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right),\right.\\
&\quad\left. L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)\right\rangle\\
&\quad\leqq C\left\langle \mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right),\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right\rangle.
\end{aligned}
$$

The above inequalities, together with the microscopic version of the H-theorem imply that there exist some positive constants σ_1 and σ_2 such that,

$$
\begin{aligned}
&\sigma_1\int_{\mathbf{R}^3}\frac{\left|\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right|^2}{\mathbf{M}}\\
&\leqq-\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)}{\mathbf{M}}d\xi\\
&\leqq\sigma_2\int_{\mathbf{R}^3}\frac{\left|\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right|^2}{\mathbf{M}}.
\end{aligned}
\tag{4.18}
$$

The first inequality in (4.18) follows from (iv) of Lemma 2.1:

$$
\begin{aligned}
&-\int_{\mathbf{R}^3} \frac{\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)}{\mathbf{M}}d\xi\\
&\quad= -\int_{\mathbf{R}^3} \frac{L_{\mathbf{M}}\left(L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)\right)L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\mathbf{M}\right)\right)}{\mathbf{M}}d\xi\\
&\quad\geqq \sigma_0\int_{\mathbf{R}^3} \frac{\left|L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right)\right|^2}{\mathbf{M}}d\xi\\
&\quad\geqq \sigma_1\int_{\mathbf{R}^3} \frac{\left|\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right|^2}{\mathbf{M}}d\xi.
\end{aligned}
$$

The second inequality of (4.18) follows from the boundedness of the operator $L_{\mathbf{M}}^{-1}$ on $\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)$.

Note that the dissipation order in (4.18) is

$$
\int_{\mathbf{R}^3} \frac{\left|\mathbf{P}_1\left(\xi_1\left(\xi\cdot\tilde{u}_x+\frac{\tilde{\theta}_x}{2\theta}|\xi-u|^2\right)\right)\right|^2}{\mathbf{M}}\,d\xi = O(1)\Big(|\tilde{u}_x|^2+|\tilde{\theta}_x|^2\Big), \tag{4.19}
$$

for some positive function $O(1)$. The left-hand side of (4.19) gives exactly the dissipative effect of the fluid components through viscosity and heat conductivity in the solution of the Boltzmann equation.

Putting (4.16), (4.18), and (4.19) together imply

$$
\begin{aligned}
I_2 \leqq\ & -d_2\int_0^t\int_{\mathbf{R}}\Big(|\tilde{u}_x|^2+|\tilde{\theta}_x|^2\Big)dx\,d\tau+\lambda\int_0^t\int_{\mathbf{R}}\overline{u}_{1x}\left|\tilde{\theta}\right|^2dx\,d\tau\\
&+O(1)\int_0^t\left(\|(\overline{u}_{1x},E_1)(\tau)\|_{L^2}^3+\left\|(\overline{u}_{1xx},\overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{3}{2}}\right)\left\|\sqrt{\eta(\tau)}\right\|_{L^2}^2d\tau\\
&+O(1)\int_0^t\left(\|(\overline{u}_{1x},E_1)(\tau)\|_{L^2}^{\frac{5}{2}}+\|\overline{u}_{1x}(\tau)\|_{L^3}^3\right.\\
&+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{G}_t^2+(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi\,dx\,d\tau\\
&\left.+\left\|(\overline{u}_{1xx},\overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{5}{4}}+\|E_1(\tau)\|_{L^2}^2\right)d\tau
\end{aligned} \tag{4.20}
$$

for a positive constant d_2 of the order of the viscosity and heat conductivity of the fluid. Here λ is a small constant.

5. Entropy estimates and lower-order estimates

The purpose of this section is to initiate the proof of Theorem 1.1 using the entropy pairs, (4.12), (4.15) and (4.19), for the first-order estimate of fluid variables. This, and the subsequent two sections, will complete the energy estimates. First we set up the overall framework for these estimates. Denote ∂^α the differential operator $\partial^\alpha = \partial^{(\alpha_0,\alpha_1)} = \partial_t^{\alpha_0}\partial_x^{\alpha_1}$, $|\alpha| = \alpha_0+\alpha_1$, where α_0 and α_1 are non-negative integers. Set

$$\begin{aligned}\tilde\rho(t,x) &= \rho(t,x) - \overline{\rho}(t,x),\\ \tilde u(t,x) &= u(t,x) - \overline{u}(t,x),\\ \tilde\theta(t,x) &= \theta(t,x) - \overline{\theta}(t,x),\\ \tilde{\mathbf{G}}(t,x,\xi) &= \mathbf{G}(t,x,\xi) - \overline{\mathbf{G}}(t,x,\xi)\end{aligned}$$

as usual, in order to extend the standard local existence theory of the Boltzmann equation (cf. [40, 18]) to the global existence theory, we need to close the following *a priori* assumption

$$\begin{aligned}N(t)^2 = &\sup_{0\leqq\tau\leqq t}\left\{\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\leqq|\alpha|\leqq3}\frac{(\partial^\alpha\mathbf{M})^2+(\partial^\alpha\mathbf{G})^2}{\mathbf{M}_-}+\sum_{|\alpha|=4}\frac{(\partial^\alpha f)^2}{\mathbf{M}_-}\right)(\tau)d\xi dx\right\}\\ &+\sup_{0\leqq\tau\leqq t}\left\{\int_{\mathbf{R}}\eta(\tau)dx\right\}\\ &< \delta_0^2.\end{aligned} \tag{5.1}$$

Here $\delta_0 > 0$ is a suitably chosen, sufficiently small, constant.

From (2.7), (5.1) yields the following $L^\infty_{(t,x)}$ estimates by the Sobolev imbedding theorem:

$$\begin{aligned}&\sup_{\tau\in[0,t],x\in\mathbf{R}}\left\{\sum_{1\leqq|\alpha|\leqq3}\left(\left|\partial^\alpha(\rho,u,\theta)(\tau,x)\right|+\left\|\frac{\partial^\alpha\mathbf{G}(\tau,x)}{\sqrt{\mathbf{M}_-(\tau,x)}}\right\|_{L^2_\xi}\right)\right\}\\ &\quad+\sup_{\tau\in[0,t],x\in\mathbf{R}}\left(\left|(\tilde\rho,\tilde u,\tilde\theta)(\tau,x)\right|\right)\\ &\quad< O(1)(\varepsilon+\delta_0),\end{aligned} \tag{5.2}$$

where ε is the constant in the definition of the approximate rarefaction waves.

Under the *a priori* assumption (5.1), by choosing δ_0 and ε to be sufficiently small, there exists a constant state (ρ_-, u_-, θ_-) with $\rho_- > 0$ and $\theta_- > 0$ such that for all $(\tau, x) \in [0, t] \times \mathbf{R}$

$$\begin{aligned}&\frac{1}{2}\theta(\tau,x) < \theta_- < \theta(\tau,x),\\ &|\theta(\tau,x)-\theta_-| + |u(\tau,x)-u_-| + |\rho(\tau,x)-\rho_-| < \eta_0.\end{aligned} \tag{5.3}$$

Thus, the microscopic H-theorem (1.17) holds for the global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,u_-,\theta_-]}$.

We now start with the entropy estimate. From (4.15) and (4.20), to obtain the entropy estimate around the approximate rarefaction wave profile, we only need to estimate I_1. It is worth pointing out that the expansion of the rarefaction waves, i.e. $\bar{u}_{1x} > 0$, plays an important role in the following estimation. In fact, notice that

$$\begin{aligned}
&\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_t+\nabla_{(\overline{\rho},\overline{u},\overline{S})}q\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_x\\
&\quad=\eta_{\overline{\rho}}E_2+\eta_{\overline{u}_1}E_3+q_{\overline{\rho}}E_1-H_1\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_1}\\
&\qquad-H_2\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_3}.
\end{aligned}\tag{5.4}$$

Here

$$\begin{aligned}
H_1\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)&=\tfrac{3}{2}\rho\left(u_1-\overline{u}_1\right)^2+\tfrac{3}{2}k\left(\rho^{\frac{5}{3}}\exp(S)-\overline{\rho}^{\frac{5}{3}}\exp\left(\overline{S}\right)\right)\\
&\quad-\tfrac{5}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\left(\rho-\overline{\rho}\right)-\tfrac{3}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\rho\left(S-\overline{S}\right)\\
&\quad-\frac{3\sqrt{15k}}{10}\overline{\rho}^{\frac{1}{3}}\exp\left(\tfrac{\overline{S}}{2}\right)\rho\left(u_1-\overline{u}_1\right)\left(S-\overline{S}\right),\\
H_2\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)&=\tfrac{3}{2}\rho\left(u_1-\overline{u}_1\right)^2+\tfrac{3}{2}k\left(\rho^{\frac{5}{3}}\exp(S)-\overline{\rho}^{\frac{5}{3}}\exp\left(\overline{S}\right)\right)\\
&\quad-\tfrac{5}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\left(\rho-\overline{\rho}\right)-\tfrac{3}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\rho\left(S-\overline{S}\right)\\
&\quad+\frac{3\sqrt{15k}}{10}\overline{\rho}^{\frac{1}{3}}\exp\left(\tfrac{\overline{S}}{2}\right)\rho\left(u_1-\overline{u}_1\right)\left(S-\overline{S}\right).
\end{aligned}\tag{5.5}$$

From (4.12) and (5.5), we obtain

$$\begin{aligned}
\eta_{\overline{\rho}}&=-\tfrac{3}{2}k\overline{\rho}^{-\frac{1}{3}}\exp\left(\overline{S}\right)\rho\left(S-\overline{S}\right)-\tfrac{5}{2}k\exp\left(\overline{S}\right)\overline{\rho}^{-\frac{1}{3}}\left(\rho-\overline{\rho}\right),\\
\eta_{\overline{u}_1}&=-\tfrac{3}{2}\rho\left(u_1-\overline{u}_1\right),\\
q_{\overline{\rho}}&=-\tfrac{3}{2}k\overline{\rho}^{-\frac{1}{3}}\exp\left(\overline{S}\right)\rho u_1\left(S-\overline{S}\right)-\tfrac{5}{2}k\exp\left(\overline{S}\right)\overline{\rho}^{-\frac{1}{3}}\left(\rho u_1-\overline{\rho}\,\overline{u}_1\right),
\end{aligned}\tag{5.6}$$

$$\begin{aligned}
&\nabla_{(\rho,u_1,S)}H_i\left(\overline{\rho},\overline{u},\overline{\theta};\overline{\rho},\overline{u},\overline{\theta}\right)=(0,0,0),\\
&\nabla^2_{(\rho,u_1,S)}H_i\left(\overline{\rho},\overline{u},\overline{\theta};\overline{\rho},\overline{u},\overline{\theta}\right)\\
&=\begin{pmatrix}
\frac{5}{3}k\overline{\rho}^{-\frac{1}{3}}\exp\left(\overline{S}\right) & 0 & k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\\
0 & 3\overline{\rho} & \frac{3(-1)^i\sqrt{15k}}{10}\overline{\rho}^{\frac{4}{3}}\exp\left(\frac{\overline{S}}{2}\right)\\
k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right) & \frac{3(-1)^i\sqrt{15k}}{10}\overline{\rho}^{\frac{4}{3}}\exp\left(\frac{\overline{S}}{2}\right) & \frac{3}{2}k\overline{\rho}^{\frac{5}{3}}\exp\left(\overline{S}\right)
\end{pmatrix}.
\end{aligned}\tag{5.7}$$

Since $\nabla^2_{(\rho,u_1,S)}H_i$, the Hessian of H_i are positive definite at $(\rho,u,\theta)=\left(\overline{\rho},\overline{u},\overline{\theta}\right)$. Thus around the approximate rarefaction waves profile $\left(\overline{\rho},\overline{u},\overline{\theta}\right)$, there exists a positive constant d_3 such that

$$\begin{aligned}
&-H_1\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_1}-H_2\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_3}\\
&\quad\leqq -d_3\overline{u}_{1x}\left(|\tilde{\rho}|^2+|\tilde{u}|^2+|\tilde{\theta}|^2\right).
\end{aligned}\tag{5.8}$$

Furthermore, (4.13), (5.6), and Lemma 3.2 imply

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}} \Big(\eta_{\overline{\rho}} E_2 + \eta_{\overline{u}_1} E_3 + q_{\overline{\rho}} E_1 \Big) dx d\tau \\
&\leqq O(1) \int_0^t \left\| \sqrt{\eta(\tau)} \right\|_{L^2} \|(E_1, E_2, E_3)(\tau)\|_{L^2} \, d\tau \\
&\leqq O(1) \left(\int_0^t \|(E_1, E_2, E_3)(\tau)\|_{L^2} \, d\tau \right. \\
&\quad \left. + \int_0^t \left\| \sqrt{\eta(\tau)} \right\|_{L^2}^2 \|(E_1, E_2, E_3)(\tau)\|_{L^2} \, d\tau \right) \\
&\leqq O(1) \varepsilon^{-\frac{1}{2}} \exp\left(-\frac{1}{\varepsilon} \right) \\
&\quad + O(1)\sqrt{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \right) \int_0^t \exp(-\varepsilon d_1 \tau) \left\| \sqrt{\eta(\tau)} \right\|_{L^2}^2 d\tau. \qquad (5.9)
\end{aligned}$$

Putting (3.7), (5.8), and (5.9) together, we can deduce that

$$\begin{aligned}
I_1 \leqq \quad & -d_3 \int_0^t \int_{\mathbf{R}^3} \overline{u}_{1x} \Big(|\tilde{\rho}|^2 + |\tilde{u}|^2 + |\tilde{\theta}|^2 \Big) dx d\tau \\
& + O(1) \varepsilon^{-\frac{1}{2}} \exp\left(-\tfrac{1}{\varepsilon}\right) \\
& + O(1)\sqrt{\varepsilon} \exp\left(-\tfrac{1}{\varepsilon}\right) \int_0^t \exp(-\varepsilon d_1 \tau) \left\| \sqrt{\eta(\tau)} \right\|_{L^2}^2 d\tau. \qquad (5.10)
\end{aligned}$$

Inserting (4.20) and (5.10) into (4.15), we have, from Gronwall's inequality, the following entropy estimate:

$$\begin{aligned}
&\int_{\mathbf{R}} \eta(t) dx + \int_0^t \int_{\mathbf{R}} \Big(|\tilde{u}_x|^2 + |\tilde{\theta}_x|^2 + \overline{u}_{1x} (|\tilde{\rho}|^2 + |\tilde{u}|^2 + |\tilde{\theta}|^2) \Big) dx d\tau \\
&\leqq O(1) \left(\int_{\mathbf{R}} \eta(0) dx + \varepsilon^{\frac{1}{8}} \right) \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\mathbf{G}_t^2 + (1+|\xi|)\mathbf{G}_x^2 + (1+|\xi|)^{-1} Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}} d\xi dx d\tau. \quad (5.11)
\end{aligned}$$

We now finish the lower-order estimate. As well as (5.11) on the macroscopic component, we still need to estimate the microscopic component $\mathbf{G}$ and to obtain the double integral for $|\tilde{\rho}_x|^2$. As mentioned in the introduction, $\|\mathbf{G}\|_{L^2_{x,\xi}}^2$ is not integrable with respect to time. Thus, we will first derive the lower-order estimates for the microscopic part $\tilde{\mathbf{G}}$ by using the microscopic H-theorem. We will then combine the microscopic part $\tilde{\mathbf{G}}$ with the entropy estimate on the fluid component in the last section and the coupling terms to obtain the lower-order estimate of the solution to the Boltzmann equation. For this estimate, $\tilde{\mathbf{G}}$ solves

$$\begin{aligned}
\tilde{\mathbf{G}}_t - L_{\mathbf{M}} \tilde{\mathbf{G}} = & -\frac{1}{R\theta} \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u|^2}{2\theta} \tilde{\theta}_x + \xi \cdot \tilde{u}_x \right) \mathbf{M} \right] \\
& - \mathbf{P}_1(\xi_1 \mathbf{G}_x) + Q(\mathbf{G},\mathbf{G}) - \overline{\mathbf{G}}_t. \qquad (5.12)
\end{aligned}$$

Multiplying (5.12) by $\frac{\tilde{\mathbf{G}}}{\mathbf{M}}$ and integrating the product over $[0,t]\times\mathbf{R}\times\mathbf{R}^3$, we obtain from (2.13)

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sigma\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau\\
&\leqq -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}^2}\mathbf{M}_t d\xi dx d\tau\\
&\quad -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}}{\mathbf{M}}\left\{\frac{\mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\tilde{\theta}_x+\xi\cdot\tilde{u}_x\right)\mathbf{M}\right]}{R\theta}\right.\\
&\quad\left.+\ \mathbf{P}_1(\xi_1\mathbf{G}_x)-Q(\mathbf{G},\mathbf{G})+\overline{\mathbf{G}}_t\right\}d\xi dx d\tau. \qquad (5.13)
\end{aligned}
$$

From (2.11), (5.1), (5.2), Lemma 4.1 and Corollary 4.3, direct calculations yield

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau\\
&\leqq O(1)\varepsilon^{\frac{1}{8}}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx d\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\left((\varepsilon+\delta_0)\,|\tilde{\rho}_x|^2+\left|(\tilde{u}_x,\tilde{\theta}_x)\right|^2\right)dx d\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi dx d\tau. \qquad (5.14)
\end{aligned}
$$

For later use in the higher-order estimates in Section 7, we also need the following estimate with respect to the weight $\mathbf{M}_-$ instead of $\mathbf{M}$ which can be obtained similarly:

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx d\tau\\
&\leqq O(1)\varepsilon^{\frac{1}{8}}+O(1)\int_0^t\int_{\mathbf{R}}\left((\varepsilon+\delta_0)\,|\tilde{\rho}_x|^2+\left|(\tilde{u}_x,\tilde{\theta}_x)\right|^2\right)dx d\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}_-}d\xi dx d\tau. \qquad (5.15)
\end{aligned}
$$

Since for $\mathbf{M}_i = \mathbf{M}_-$ or $\mathbf{M}$, we have from Lemma 4.1 and (5.2) that

$$
\begin{aligned}
&\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{((1+|\xi|)^{-1} Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}_i} d\xi dx d\tau \\
&\leqq O(1) \int_0^t \int_{\mathbf{R}} \left(\int_{\mathbf{R}^3} \frac{(1+|\xi|)\mathbf{G}^2}{\mathbf{M}_i} \right) \cdot \left(\int_{\mathbf{R}^3} \frac{\mathbf{G}^2}{\mathbf{M}_i} \right) dx d\tau \\
&\leqq O(1) \int_0^t \int_{\mathbf{R}} \left(\int_{\mathbf{R}^3} \frac{(1+|\xi|)(\tilde{\mathbf{G}}^2 + \overline{\mathbf{G}}^2)}{\mathbf{M}_i} d\xi \right) \cdot \left(\int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2 + \overline{\mathbf{G}}^2}{\mathbf{M}_i} d\xi \right) dx d\tau \\
&\leqq O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon + \delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\tilde{\mathbf{G}}_x^2}{\mathbf{M}_i} d\xi dx d\tau. \qquad (5.16)
\end{aligned}
$$

From (5.11), (5.14), and (5.15) the following two estimates on the entropy η and $\tilde{\mathbf{G}}$ hold:

$$
\begin{aligned}
&\int_{\mathbf{R}} \eta(t) dx + \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi dx \Big|_{t=0}^{\tau=t} + \int_0^t \int_{\mathbf{R}} \Big(|(\tilde{u}_x, \tilde{\theta}_x)|^2 + \overline{u}_{1x} |(\tilde{\rho}, \tilde{u}, \tilde{\theta})|^2 \Big) dx d\tau \\
&+ \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad \leqq O(1) \left(\varepsilon^{\frac{1}{8}} + \int_{\mathbf{R}} \eta(0) dx \right) + O(1)(\varepsilon + \delta_0) \int_0^t \int_{\mathbf{R}} |\tilde{\rho}_x|^2 \, dx d\tau \\
&+ O(1) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \left((\varepsilon + \delta_0) \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} + \frac{(1+|\xi|)\left(\mathbf{G}_t^2 + \mathbf{G}_x^2\right)}{\mathbf{M}} \right) d\xi dx d\tau,
\end{aligned}
\qquad (5.17)
$$

and

$$
\begin{aligned}
&\int_{\mathbf{R}} \eta(t) dx + \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} d\xi dx \Big|_{t=0}^{\tau=t} + \int_0^t \int_{\mathbf{R}} \Big(|(\tilde{u}_x, \tilde{\theta}_x)|^2 + \overline{u}_{1x} |(\tilde{\rho}, \tilde{u}, \tilde{\theta})|^2 \Big) dx d\tau \\
&+ \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\quad \leqq O(1) \left(\varepsilon^{\frac{1}{8}} + \int_{\mathbf{R}} \eta(0) dx \right) + O(1)(\varepsilon + \delta_0) \int_0^t \int_{\mathbf{R}} |\tilde{\rho}_x|^2 \, dx d\tau \\
&+ O(1) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\left(\mathbf{G}_t^2 + \mathbf{G}_x^2\right)}{\mathbf{M}_-} d\xi dx d\tau. \qquad (5.18)
\end{aligned}
$$

Finally the double integral of $\tilde{\rho}_x^2$ and $\tilde{\rho}_t^2$ is obtained from the coupling through the conservation laws as in [20]. From (2.7), (1.9), and (3.3), since $\left(\tilde{\rho}, \tilde{u}, \tilde{\theta}\right)(t, x)$

solves

$$\begin{aligned}
\tilde{\rho}_t + (\tilde{\rho}\tilde{u}_1)_x &= -E_4, \\
\tilde{u}_{1t} + \tilde{u}_1\tilde{u}_{1x} + \frac{2}{3}\tilde{\theta}_x + \frac{2\theta}{3\rho}\tilde{\rho}_x &= -\int_{\mathbf{R}^3} \frac{\xi_1^2 \mathbf{G}_x}{\rho} d\xi - E_5, \\
\tilde{u}_{2t} + \tilde{u}_1\tilde{u}_{2x} &= -\int_{\mathbf{R}^3} \frac{\xi_1\xi_2 \mathbf{G}_x}{\rho} d\xi - \overline{u}_1\tilde{u}_{2x}, \\
\tilde{u}_{3t} + \tilde{u}_1\tilde{u}_{3x} &= -\int_{\mathbf{R}^3} \frac{\xi_1\xi_3 \mathbf{G}_x}{\rho} d\xi - \overline{u}_1\tilde{u}_{3x}, \\
\tilde{\theta}_t + \frac{2}{3}\tilde{\theta}\tilde{u}_{1x} + \tilde{u}_1\tilde{\theta}_x &= \int_{\mathbf{R}^3} \frac{\xi_1\left(\xi\cdot u - \frac{1}{2}|\xi|^2\right)\mathbf{G}_x}{\rho} d\xi - E_6,
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
E_4 &= (\overline{\rho}\tilde{u}_1 + \overline{u}_1\tilde{\rho})_x + \rho_x^{A_1}\left(u_1^{A_3} - u_{1m}\right) + \rho_x^{A_3}\left(u_1^{A_1} - u_{1m}\right) \\
&\quad + u_{1x}^{A_1}\left(\rho^{A_3} - \rho_m\right) + u_{1x}^{A_3}\left(\rho^{A_1} - \rho_m\right), \\
E_5 &= \tilde{u}_1\overline{u}_{1x} + \overline{u}_1\tilde{u}_{1x} + u_{1x}^{A_1}\left(u_1^{A_3} - u_{1m}\right) + u_{1x}^{A_3}\left(u_1^{A_1} - u_{1m}\right) \\
&\quad + \frac{2}{3}\left\{\frac{\overline{\rho}\tilde{\theta} - \overline{\theta}\tilde{\rho}}{\rho\overline{\rho}}\overline{\rho}_x + \frac{\rho^{A_1}\left(\theta^{A_3} - \theta_m\right) - \theta^{A_1}\left(\rho^{A_3} - \rho_m\right)}{\overline{\rho}\rho^{A_1}}\rho_x^{A_1}\right. \\
&\quad \left. + \frac{\rho^{A_3}\left(\theta^{A_1} - \theta_m\right) - \theta^{A_3}\left(\rho^{A_1} - \rho_m\right)}{\overline{\rho}\rho^{A_3}}\rho_x^{A_3}\right\}, \\
E_6 &= \frac{2}{3}\left\{(\tilde{\theta}\overline{u}_{1x} + \overline{\theta}\tilde{u}_{1x}) + u_{1x}^{A_1}\left(\theta^{A_3} - \theta_m\right) + u_{1x}^{A_3}\left(\theta^{A_1} - \theta_m\right)\right\} \\
&\quad + \left\{(\tilde{u}_1\overline{\theta}_x + \overline{u}_1\tilde{\theta}_x) + \theta_x^{A_1}\left(u_1^{A_3} - u_{1m}\right) + \theta_x^{A_3}\left(u_1^{A_1} - u_{1m}\right)\right\}.
\end{aligned} \tag{5.20}$$

Thus we have by Lemma 3.2:

$$\begin{aligned}
\int_0^t \int_{\mathbf{R}} |\tilde{\rho}_x|^2 dx d\tau \leqq\ & O(1)\varepsilon^{\frac{1}{8}} + O(1)\int_{\mathbf{R}} |(\tilde{u}, \tilde{\rho}_x)|^2 (t) dx \\
& + O(1)\int_{\mathbf{R}} |(\tilde{u}, \tilde{\rho}_x)|^2 (0) dx \\
& + O(1)\int_0^t \int_{\mathbf{R}} \left(|\tilde{\theta}_x|^2 + |\tilde{u}_x|^2 + \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2}{\mathbf{M}} d\xi\right) dx d\tau \\
& + O(1)(\varepsilon + \delta_0)\int_0^t \int_{\mathbf{R}} \overline{u}_{1x}\left(|\tilde{u}|^2 + |\tilde{\theta}|^2 + |\tilde{\rho}|^2\right) dx d\tau,
\end{aligned} \tag{5.21}$$

and

$$\int_0^t \int_{\mathbf{R}} |\tilde{\rho}_t|^2 \, dx d\tau \leqq O(1)\varepsilon^{-\frac{1}{2}} \exp\left(-\frac{1}{\varepsilon}\right) + O(1)\int_0^t \int_{\mathbf{R}} \left(|(\tilde{\rho}_x, \tilde{u}_x)|^2 + \bar{u}_{1x}\,|(\tilde{\rho}, \tilde{u})|^2\right) dx d\tau. \tag{5.22}$$

Hence, (5.17)–(5.18) and (5.21)–(5.22) give the complete lower-order estimates:

$$\int_{\mathbf{R}} \eta(t) dx + \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2(t)}{\mathbf{M}_-} d\xi dx + \int_0^t \int_{\mathbf{R}} \left(|(\tilde{u}_x, \tilde{\theta}_x)|^2 + \bar{u}_{1x}|(\tilde{\rho}, \tilde{u}, \tilde{\theta})|^2\right) dx d\tau + \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-} d\xi dx d\tau \leqq O(1)\left(\varepsilon^{\frac{1}{8}} + \int_{\mathbf{R}} \eta(0) dx\right) \times \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2(0)}{\mathbf{M}_-} d\xi dx + O(1)\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)\left(\mathbf{G}_t^2 + \mathbf{G}_x^2\right)}{\mathbf{M}_-} d\xi dx d\tau. \tag{5.23}$$

6. Higher-order estimates with respect to M

To close the energy estimate, we need to consider the estimates on the derivatives of $\mathbf{M}$ and $\mathbf{G}$ with respect to x and t. As mentioned in the Introduction, both estimates with respect to the weight functions $\mathbf{M}$ and $\mathbf{M}_-$ are necessary. We will also make use of both the equations for the fluid and non-fluid components $\mathbf{M}$ and $\mathbf{G}$ together with the conservation laws.

Since the local Maxwellian depends on (t, x), the linearized operator $\mathbf{L}_{\mathbf{M}}$ is also a function of (t, x, ξ). Thus, the derivatives on the equation contain terms of derivatives on this operator. Moreover, the highest-order derivative, i.e. the fourth-order should be taken on the equation for $f(t, x, \xi)$ instead of the equations for the macroscopic or microscopic components, respectively. Otherwise, the fifth-order derivatives will appear and the estimates cannot be closed. However, the estimation on the lower-order estimates should be sharp on both the macroscopic and microscopic components, respectively, in order to obtain the dissipation on the fluid component in the form of viscosity and heat conductivity, and the dissipation on the non-fluid component by the linearized collision operator.

In this section, we will consider higher-order energy estimates of $\partial^\alpha \mathbf{M}$, $\partial^\alpha \mathbf{G}$ and $\partial^\beta f$ for $1 \leqq |\alpha| \leqq 3, 2 \leqq |\beta| \leqq 4$ with respect to the local Maxwellian

$\mathbf{M} = \mathbf{M}_{[\rho,u,\theta]}$. First, from (5.2) and (5.3), we have

$$\begin{aligned}
&\sup_{\tau\in[0,t],x\in\mathbf{R}} \left\{ \sum_{0\leqq|\alpha|\leqq 3} \left\| \frac{\sqrt{1+|\xi|}\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}}} \right\|_{L^2_\xi} \right\} \\
&\leqq O(1) \sup_{\tau\in[0,t],x\in\mathbf{R}} \left\{ \sum_{0\leqq|\alpha|\leqq 3} \left\| \frac{\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}} \right\|_{L^2_\xi} \right\} \\
&\leqq O(1)(\varepsilon+\delta_0).
\end{aligned} \tag{6.1}$$

To obtain the estimates on $\partial^\alpha \mathbf{M}$, we apply $\mathbf{P}_0$ to (2.6) to yield

$$\begin{aligned}
&\mathbf{M}_t + \mathbf{P}_0(\xi_1\partial_x \mathbf{M}) + \mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\} \\
&\quad = -\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{G}_t + \mathbf{P}_1(\xi_1\partial_x\mathbf{G}) - Q(\mathbf{G},\mathbf{G}))\right]\right\}.
\end{aligned} \tag{6.2}$$

Applying $\partial^\alpha(1\leqq|\alpha|\leqq 3)$ to (6.2) and integrating its product with $\frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}\times\mathbf{R}^3$ yields

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t} \\
&= -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}^2}\mathbf{M}_t + \frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}\partial^\alpha[\mathbf{P}_0(\xi_1\partial_x\mathbf{M})]\right)d\xi dx d\tau \\
&\quad -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}\partial^\alpha\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau \\
&\quad -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}\partial^\alpha\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{G}_t+\mathbf{P}_1(\xi_1\partial_x\mathbf{G})-Q(\mathbf{G},\mathbf{G}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau.
\end{aligned} \tag{6.3}$$

Notice that

$$\partial^\alpha\left\{L^{-1}_{\mathbf{M}}h\right\} = L^{-1}_{\mathbf{M}}(\partial^\alpha h) - \sum_{j=0}^{|\alpha|-1}\sum_{|\alpha_j|=j} C_{\alpha_j} L^{-1}_{\mathbf{M}}\left(Q\left(\partial^{\alpha_j}\left(L^{-1}_{\mathbf{M}}h\right), \partial^{\alpha-\alpha_j}\mathbf{M}\right)\right)$$

where C_{α_j} are some positive constants, and the dissipation is from

$$
\begin{aligned}
&-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\alpha\mathbf{M})\partial^\alpha\left\{\xi_1\partial_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}}{\mathbf{M}}d\xi dxd\tau\\
&\quad=-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1[\xi_1\partial_x(\mathbf{P}_0(\partial^\alpha\mathbf{M}))]\partial^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi_1\partial_x\mathbf{M})]\right\}}{\mathbf{M}}d\xi dxd\tau\\
&\qquad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1[\xi_1\mathbf{P}_0(\partial^\alpha\mathbf{M})]\partial^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi_1\partial_x\mathbf{M})]\right\}}{\mathbf{M}^2}\mathbf{M}_x d\xi dxd\tau\\
&\quad\leqq -d_4\int_0^t\int_{\mathbf{R}}|\partial^\alpha(u_x,\theta_x)|^2\,dxd\tau+O(1)\varepsilon^{\frac{1}{8}}\\
&\qquad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leqq|\alpha|}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau.
\end{aligned}
\tag{6.4}
$$

For $j=1,2,3$, straightforward calculations yield

$$
\begin{aligned}
&\sum_{|\alpha|=j}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}}|\partial^\alpha(u_x,\theta_x)|^2\,dxd\tau\\
&\leqq O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leqq j}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(|\partial^\alpha\mathbf{G}_x|^2+|\partial^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dxd\tau+O(1)\varepsilon^{\frac{1}{8}}\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{1\leqq|\beta|\leqq j}\frac{(1+|\xi|)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}\right)d\xi dxd\tau.
\end{aligned}
\tag{6.5}
$$

Similarly, we have the following estimates on $\partial^\alpha\mathbf{G}$ by using (2.8):

$$
\begin{aligned}
&\sum_{|\alpha|=j}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leqq O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leqq j}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(|\partial^\alpha\mathbf{G}_x|^2+|\partial^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{1\leqq|\beta|\leqq j}\frac{(1+|\xi|)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}}|\partial^\alpha(u_x,\theta_x)|^2\,dxd\tau+O(1)\varepsilon^{\frac{1}{8}}.
\end{aligned}
\tag{6.6}
$$

In summary, by appropriate combination of (6.5) and (6.6) for $j = 1, 2, 3$, we have

$$\begin{aligned}
&\sum_{1\leqq|\alpha|\leqq 3}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{G}|^2+|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}\\
&\quad+\sum_{1\leqq|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\sum_{1\leqq|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau\\
&\leqq O(1)\varepsilon^{\frac{1}{8}}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leqq 4}\left|\partial^\beta\rho\right|^2\right)dxd\tau\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\leqq|\beta|\leqq 3}\frac{(1+|\xi|)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=3}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(|\partial^\alpha\mathbf{G}_x|^2+|\partial^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dxd\tau. \qquad (6.7)
\end{aligned}$$

To have the dissipative estimate on the fourth-order derivatives of the non-fluid component $\mathbf{G}$, we need to consider the original equation for $f(t,x,\xi)$. By applying $\partial^\alpha(2\leqq|\alpha|\leqq 4)$ to (1.1) and integrating its product with $\frac{\partial^\alpha f}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}\times\mathbf{R}^3$ we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}=-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}^2}(\mathbf{M}_t+\xi_1\mathbf{M}_x)\,d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha f\partial^\alpha(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha f\partial^\alpha(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau. \qquad (6.8)
\end{aligned}$$

By calculations similar to those for $\mathbf{M}$ and $\mathbf{G}$, we have

$$\begin{aligned}
&\sum_{2\leqq|\alpha|\leqq 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{2\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leqq O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leqq 4}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{2\leqq|\beta|\leqq 4}\frac{(1+|\xi|)|\partial^\beta f|^2}{\mathbf{M}_-}\right)d\xi dxd\tau\\
&\quad+O(1)\varepsilon^{\frac{1}{8}}. \qquad (6.9)
\end{aligned}$$

By suitable linear combination of (6.5), (6.6) and (6.9), and choosing ε and δ_0 sufficiently small we obtain

$$\sum_{1\leqq|\alpha|\leqq 3}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha \mathbf{G}|^2+|\partial^\alpha \mathbf{M}|^2}{\mathbf{M}}d\xi dx\bigg|_{\tau=0}^{\tau=t}+\sum_{2\leqq|\alpha|\leqq 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\bigg|_{\tau=0}^{\tau=t}$$
$$+\sum_{1\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau+\sum_{1\leqq|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2 dx d\tau$$
$$\leqq O(1)\varepsilon^{\frac{1}{8}}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{1<|\beta|\leqq 4}\left|\partial^\beta\rho\right|^2\right)dx d\tau$$
$$+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\leqq|\beta|\leqq 4}\frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dx d\tau. \tag{6.10}$$

Similar to the discussions, (6.8)–(6.11), on the integral of $\tilde{\rho}_x^2$ and $\tilde{\rho}_t^2$ in Section 6, we can recover the estimate on $|\partial^\alpha\rho|^2$ with $|\alpha|=4$ by using the conservation laws (2.7). Notice also that

$$\begin{aligned}
&\int_{\mathbf{R}^3}\frac{|\partial^\beta f|^2}{\mathbf{M}}d\xi\\
&=\int_{\mathbf{R}^3}\frac{|\partial^\beta \mathbf{M}+\partial^\beta \mathbf{G}|^2}{\mathbf{M}}d\xi=\int_{\mathbf{R}^3}\frac{|\partial^\beta \mathbf{M}|^2+2\partial^\beta \mathbf{M}\partial^\beta \mathbf{G}+|\partial^\beta \mathbf{G}|^2}{\mathbf{M}}d\xi\\
&=\int_{\mathbf{R}^3}\frac{\left|\mathbf{P}_0\left(\partial^\beta \mathbf{M}\right)\right|^2+\left|\mathbf{P}_1\left(\partial^\beta \mathbf{M}\right)\right|^2+2\mathbf{P}_1\left(\partial^\beta \mathbf{M}\right)\partial^\beta \mathbf{G}+\left|\partial^\beta \mathbf{G}\right|^2}{\mathbf{M}}d\xi\\
&\geqq\int_{\mathbf{R}^3}\frac{\left|\mathbf{P}_0\left(\partial^\beta \mathbf{M}\right)\right|^2}{\mathbf{M}}d\xi.
\end{aligned}\tag{6.11}$$

Thus we have

$$\begin{aligned}
&\sum_{1\leqq|\alpha|\leqq 3}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha\rho_x\right|^2 dx d\tau\\
&\leqq O(1)\left(\varepsilon^{\frac{1}{8}}+N(0)^2\right)+O(1)\int_0^t\int_{\mathbf{R}}\Bigg((\varepsilon+\delta_0)\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2\\
&\quad+\sum_{1<|\beta|\leqq 4}\left(\left|\partial^\beta(u,\theta)\right|^2+\int_{\mathbf{R}^3}\frac{|\partial^\beta \mathbf{G}|^2}{\mathbf{M}}d\xi\right)\Bigg)dx d\tau\\
&\quad+O(1)\sum_{1\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx d\tau.
\end{aligned}\tag{6.12}$$

Finally, from (5.17), (5.21), (5.22), (6.10), and (6.12), we have

$$\begin{aligned}
&\int_{\mathbf{R}} \eta(t)dx + \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi dx \\
&+ \int_0^t \int_{\mathbf{R}} \left(|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)|^2 + \overline{u}_{1x}|(\tilde{\rho}, \tilde{u}, \tilde{\theta})|^2 + \sum_{1<|\beta|\leqq 4} \left|\partial^\beta(\rho, u, \theta)\right|^2 \right) dxd\tau \\
&+ \sum_{1\leqq|\alpha|\leqq 3} \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{M}|^2 + |\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx + \sum_{|\alpha|=4} \int_{\mathbf{R}}\int_{\mathbf{R}^3} \frac{|\partial^\alpha f|^2}{\mathbf{M}} d\xi dx \\
&+ \int_0^t \int_{\mathbf{R}}\int_{\mathbf{R}^3} \left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}} + \sum_{1\leqq|\alpha|\leqq 4} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} \right) d\xi dxd\tau \\
&\leqq O(1)\left(\varepsilon^{\frac{1}{8}} + N(0)^2\right) \\
&\quad + O(1)(\varepsilon+\delta_0)\int_0^t \int_{\mathbf{R}}\int_{\mathbf{R}^3} \left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} + \sum_{1\leqq|\alpha|\leqq 4} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_-} \right) d\xi dxd\tau. \qquad (6.13)
\end{aligned}$$

Here, we have used the fact that, because of $\theta_- < \theta$,

$$\int_{\mathbf{R}^3} \frac{f^2}{\mathbf{M}} d\xi \leqq O(1) \int_{\mathbf{R}^3} \frac{f^2}{\mathbf{M}_-} d\xi.$$

A direct consequence of (6.13) is

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}} \left(|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)|^2 + \sum_{1<|\beta|\leqq 4} \left|\partial^\beta(\rho, u, \theta)\right|^2 \right) dxd\tau \\
&\leqq O(1)\left(\varepsilon^{\frac{1}{8}} + N(0)^2\right) \\
&\quad + O(1)(\varepsilon+\delta_0)\int_0^t \int_{\mathbf{R}}\int_{\mathbf{R}^3} \left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} + \sum_{1\leqq|\alpha|\leqq 4} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_-} \right) d\xi dxd\tau. \qquad (6.14)
\end{aligned}$$

With (6.14), it remains to estimate the terms to non-fluid component $\mathbf{G}$ and its derivatives with respect to the weight function $\mathbf{M}_-$ which will be discussed in the next section.

7. Higher-order estimates with respect to $\mathbf{M}_-$

Finally, we will give the higher-order energy estimates with respect to the global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-, u_-, \theta_-]}$ to close the *a priori* estimate in (6.14). For this, we need to use the variation of H-theorem for non-fluid component $\mathbf{G}$ as stated in (1.17).

For brevity of presentation, we will only point out the main differences from those arguments in Section 6, and then give the corresponding estimates instead of giving all the detailed proofs.

The first main difference is that the fluid part $\mathbf{P}_0(\partial^\alpha \mathbf{M})$ and the non-fluid part $\mathbf{G}$ are no longer orthogonal with respect to the global Maxwellian $\mathbf{M}_-$, i.e.

$$\int_{\mathbf{R}^3} \frac{\mathbf{P}_0(\partial^\alpha \mathbf{M}) \cdot \mathbf{G}}{\mathbf{M}_-} d\xi \neq 0.$$

As a result, there is an extra error term, in the form of

$$O(1)\int_0^t \int_{\mathbf{R}} \left(\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 + \sum_{2\leqq|\alpha|\leqq 4} \left|\partial^\alpha(\rho, u, \theta)\right|^2 \right) dx d\tau.$$

The second difference is that the corresponding *a priori* estimate similar to (6.1), i.e.

$$\sup_{\tau\in[0,t], x\in\mathbf{R}} \left\{ \sum_{0\leqq|\alpha|\leqq 3} \left\| \frac{\sqrt{1+|\xi|}\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}} \right\|_{L^2_\xi} \right\} \leqq O(1)(\varepsilon + \delta_0),$$

does not hold. The estimate on the derivatives on $Q(\mathbf{G}, \mathbf{G})$ is then different without the weight $\sqrt{1+|\xi|}$ in the *a priori* estimate. For illustration, we give the estimate on the following typical term when we differentiate $Q(\mathbf{G}, \mathbf{G})$, after applying Lemma 4.1:

$$\begin{aligned} I_{11} &= \sum_{j=0}^{|\alpha|} \sum_{|\alpha_j|=j} \int_0^t \int_{\mathbf{R}} \left(\int_{\mathbf{R}^3} \frac{(1+|\xi|)\,|\partial^{\alpha_j} \mathbf{G}|^2}{\mathbf{M}_-} d\xi \right) \cdot \left(\int_{\mathbf{R}^3} \frac{\left|\partial^{\alpha-\alpha_j} \mathbf{G}\right|^2}{\mathbf{M}_-} d\xi \right) dx d\tau \\ &:= \sum_{j=0}^{|\alpha|} \sum_{|\alpha_j|=j} I_{11}^{\alpha_j}, \quad 1 \leqq |\alpha| \leqq 4. \end{aligned} \tag{7.1}$$

For $|\alpha - \alpha_j| \leqq 2$, then we have from (5.2) that

$$\begin{aligned} I_{11}^{\alpha_j} &\leqq O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0)\int_0^t \int_{\mathbf{R}} |(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)|^2 dx d\tau \\ &\quad + O(1)(\varepsilon+\delta_0)\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-} + \sum_{1\leqq|\beta|\leqq|\alpha_j|} \frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}_-} \right) dx d\tau. \end{aligned}$$

For $|\alpha - \alpha_j| \geqq 3$, the above analysis does not apply, at least for $\alpha_j = 0$, because (5.2) holds only for $|\alpha| \leqq 3$. However, from (6.11), we have

$$\begin{aligned} \sum_{|\alpha|=4} \left\| \frac{\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}} \right\|_{L^2_x(L^2_\xi)} &\leqq O(1) \sum_{|\alpha|=4} \left(\left\| \frac{\partial^\alpha f}{\sqrt{\mathbf{M}_-}} \right\|_{L^2_x(L^2_\xi)} + \left\| \frac{\partial^\alpha \mathbf{M}}{\sqrt{\mathbf{M}_-}} \right\|_{L^2_x(L^2_\xi)} \right) \\ &\leqq O(1) \sum_{1\leqq|\alpha|\leqq 4} \left\| \frac{\partial^\alpha f}{\sqrt{\mathbf{M}_-}} \right\|_{L^2_x(L^2_\xi)} \\ &\leqq O(1)(\varepsilon+\delta_0). \end{aligned} \tag{7.2}$$

With (7.2), we now estimate $I_{11}^{\alpha_j}$ for the case $|\alpha-\alpha_j|\geqq 3$ as follows. Since

$$\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^{\alpha_j}\mathbf{G}|^2}{\mathbf{M}_-}d\xi \leqq O(1)\left(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^{\alpha_j}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx\right)^{\frac{1}{2}} \times\left(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^{\alpha_j}\mathbf{G}_x|^2}{\mathbf{M}_-}d\xi dx\right)^{\frac{1}{2}},$$

we have for $|\alpha_j|=1$

$$\begin{aligned} I_{11}^{\alpha_j} &\leqq O(1)\sup_{\tau\in[0,t]}\left\{\left\|\frac{\partial^{\alpha-\alpha_j}\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|^2_{L^2_x(L^2_\xi)}\right\}\int_0^t\prod_{k=0}^{1}\left\|\frac{\sqrt{1+|\xi|}\partial^{\alpha_j}\partial_x^k\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_x(L^2_\xi)}d\tau \\ &\leqq O(1)(\varepsilon+\delta_0)^2\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\big(|\partial^{\alpha_j}\mathbf{G}|^2+|\partial^{\alpha_j}\mathbf{G}_x|^2\big)}{\mathbf{M}_-}d\xi dx d\tau. \end{aligned}$$

Notice that $\|\overline{\mathbf{G}}\|^2_{L^2_x(L^2_\xi)}$ is not in $L^1_t(\mathbf{R}^+)$, when $\alpha_j=0$. We need to factor out the supremum of L^2_ξ norm of $\overline{\mathbf{G}}$ when it appears. Besides this, similar to the case when $\alpha_j=1$, we have

$$\begin{aligned} I_{11}^{\alpha_j} &\leqq O(1)\int_0^t\left\|\frac{\partial^{\alpha}\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|^2_{L^2_x(L^2_\xi)}\prod_{k=0}^{1}\left\|\frac{\sqrt{1+|\xi|}\partial_x^k\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_x(L^2_\xi)}d\tau \\ &\leqq O(1)\int_0^t\left\|\frac{\partial^{\alpha}\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|^2_{L^2_x(L^2_\xi)}\left\|\frac{\sqrt{1+|\xi|}\partial_x\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_x(L^2_\xi)} \\ &\quad\times\left(\left\|\frac{\sqrt{1+|\xi|}\tilde{\mathbf{G}}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_x(L^2_\xi)}+\left\|\frac{\sqrt{1+|\xi|}\overline{\mathbf{G}}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_x(L^2_\xi)}\right)d\tau \\ &\leqq O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(\tilde{\mathbf{G}}^2+\sum_{1\leqq|\beta|\leqq|\alpha|}\left|\partial^{\beta}\mathbf{G}\right|^2\right)}{\mathbf{M}_-}d\xi dx d\tau. \end{aligned}$$

In summary, we have the estimate on (7.1) which is similar to (6.9).

Keeping in mind the above two differences, we have, similar to previous estimates with the weight function $\mathbf{M}$, the following estimate on derivatives with the weight function $\mathbf{M}_-$:

$$\sum_{1\leqq|\alpha|\leqq 3}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^{\alpha}\mathbf{G}|^2+|\partial^{\alpha}\mathbf{M}|^2}{\mathbf{M}_-}d\xi dx\Big|_{\tau=0}^{\tau=t} + \sum_{2\leqq|\alpha|\leqq 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^{\alpha}f|^2}{\mathbf{M}_-}d\xi dx\Big|_{\tau=0}^{\tau=t} + \sum_{1\leqq|\alpha|\leqq 4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^{\alpha}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau$$

$$\leqq O(1)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{1\leqq|\alpha|\leqq 3}\left|\partial^\alpha(\rho_x,u_x,\theta_x)\right|^2\right)dxd\tau$$

$$+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau+O(1)\varepsilon^{\frac{1}{8}}. \tag{7.3}$$

We obtain the final energy estimate from (5.18), (6.14) and (7.3):

$$\int_{\mathbf{R}}\eta(t)dx+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx$$

$$+\int_0^t\int_{\mathbf{R}}\left(|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)|^2+\overline{u}_{1x}|(\tilde{\rho},\tilde{u},\tilde{\theta})|^2+\sum_{1<|\beta|\leqq 4}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau$$

$$+\sum_{1\leqq|\alpha|\leqq 3}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\sum_{|\alpha|=4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}_-}d\xi dx$$

$$+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\leqq|\alpha|\leqq 4}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau$$

$$\leqq O(1)\left(\varepsilon^{\frac{1}{8}}+N(0)^2\right). \tag{7.4}$$

This closes the *a priori* estimate (5.1) provided that we choose $\varepsilon_0>0$ and $\varepsilon>0$ sufficiently small such that

$$N(0)<\varepsilon_0,$$

$$O(1)\left(\varepsilon^{\frac{1}{8}}+\varepsilon_0^2\right)<\delta_0^2.$$

Combination of (7.4) and the convergence property on $(\overline{\rho},\overline{u},\overline{\theta})$ to (ρ^R,u^R,θ^R) in Section 3 give the global existence and the time asymptotic behavior of the solution as stated in Theorem 1.1.

Acknowledgements. The authors would like to thank the referee for helpful suggestions, which lead to a major revision of the paper.

The research of Tai-Ping Liu is supported by Institute of Mathematics, Academia Sinica and NSF Grant DMS-9803323. The research of Tong Yang is supported by the Strategic Research Grant of City University of Hong Kong # 7001439 and the National Natural Science Foundation of China #10329101. The research of Shih-Hsien Yu is supported by the CityU Strategic Research Grant 7001350. The research of Hui-Jiang Zhao is supported by the JSPS Research Fellowship for Foreign Researchers, the National Natural Science Foundation of China under contract 10431060, the National Key Program for Basic Research of China under contract 2002CCA03700, and the grant from the Chinese Academy of Sciences entitled "Yin Jin Guo Wai Jie Chu Ren Cai Ji Jin".

References

1. ARKERYD, L., NOURI, A.: On the Milne problem and the hydrodynamic limit for a steady Boltzmann equation model. *J. Statist. Phy.* **99**, 993–1019 (2000)
2. BARDOS, C., CAFLISCH, R.E., NICOLAENKO, B.: The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas. *Comm. Pure Appl. Math.* **49**, 323–452 (1986)
3. BARDOS, C., GOLSE, F.: Differents aspects de la notion d'entropie au niveau de l'equation de Boltzmann et de Navier-Stokes. *C. R. Acad. Sci. Paris Ser. I Math.* **299** (7), 225–228 (1984)
4. BARDOS, C., UKAI, S.: The classical incompressible Navier-Stokes limit of the Boltzmann equation. *Math. Models Methods Appl. Sci.* **1**, 235–257 (1991)
5. BOLTZMANN, L.: (translated by Stephen G. Brush), *Lectures on Gas Theory*. Dover Publications, Inc. New York, 1964
6. CAFLISCH, C.: The fluid dynamic limit of the nonlinear Boltzmann equation. *Comm. Pure Appl. Math.* **33**, 651–666 (1980)
7. CAFLISH, R.E., NICOLAENKO, B.: Shock profile solutions of the Boltzmann equation. *Comm. Math. Phys.* **86**, 161–194 (1982)
8. CARLEMAN, T.: Sur la théorie de l'équation intégrodifférentielle de Boltzmann. *Acta Mathematica* **60**, 91–142 (1932)
9. CERCIGNANI, C.: *The Boltzmann Equation and Its Applications*. Springer-Verlag, New York, 1988
10. CERCIGNANI, C., ILLNER, R., PULVIRENTI, M.: *The Mathematical Theory of Dilute Gases*. Springer-Verlag, Berlin, 1994
11. CHAPMAN, S., COWLING, T.G.: *The Mathematical Theory of Non-Uniform Gases*, 3rd edition. Cambridge University Press, 1990
12. CORON, F., GOLSE, F., SULEM, C.: A classification of well-posed kinetic layer problems. *Comm. Pure Appl. Math.* **41**, 409–435 (1988)
13. COURANT, R., FRIEDRICHS, K.O.: *Supersonic Flows and Shock Waves*. Wiley-Interscience: New York, 1948
14. ESPOSITO, R., PULVIRENTI, M.: From particles to fluid. In: S. FRIEDLANDER and D. SERRE, editors, *Handbooks of Mathematical Fluid Dynamics*, **Vol. III**, Elsevier, 2004
15. GOLSE, F., PERTHAME, B., SULEM, C.: On a boundary layer problem for the nonlinear Boltzmann equation. *Arch. Ration. Mech. Anal.* **103**, 81–96 (1986)
16. GOODMAN, J.: Nonlinear asymptotic stability of viscous shock profiles for conservation laws. *Arch. Rational Mech. Anal.* **95**, 325–344 (1986)
17. GRAD, H.: Asymptotic Theory of the Boltzmann Equation II. In: J.A. LAURMANN, editor, *Rarefied Gas Dynamics*, **Vol. 1**, Academic Press, New York, 26–59, 1963
18. GUO, Y.: The Boltzmann equation in the whole space. *Indiana University Mathematics Journal* **53**, 1081–1094 (2004)
19. HILBERT, D.: *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen*. Teubner, Leipzig, Chap. 22, 1953
20. KAWASHIMA, S., MATSUMURA, A.: Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. *Comm. Math. Phys.* **101**, 97–127 (1985)
21. KAWASHIMA, S., MATSUMURA, A., NISHIDA, T.: On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation. *Comm. Math. Phys.* **70**, 97–124 (1979)
22. LACHOWICZ, M.: On the initial layer and the existence theorem for the nonlinear Boltzmann equation. *Math. Methods Appl. Sci.* **9**, 342–366 (1987)
23. LAX, P.D.: Hyperbolic system of conservation laws, II. *Comm. Pure Appl. Math.* **10**, 537–566 (1957)
24. LIU, T.-P.: Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws. *Comm. Pure Appl. Math.* **30**, 586–611 (1977)
25. LIU, T.-P.: Linear and nonlinear large-time behaviors of solutions of hyperbolic conservation laws. *Comm. Pure Appl. Math.* **30**, 767–796 (1977)
26. LIU, T.-P.: Nonlinear Stability of Shock Waves for Viscous Conservation Laws. *Mem. Amer. Math. Soc.* **56**, 1–108 (1985)

27. LIU, T.-P., XIN, Z.-P.: Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations. *Comm. Math. Phys.* **118**, 451–465 (1988)
28. LIU, T.-P., YANG, T., YU, S.-H.: Energy method for the Boltzmann equation. *Physica D* **188**, 178–192 (2004)
29. LIU, T.-P., YU, S.-H.: Boltzmann equation: Micro-macro decompositions and positivity of shock profiles. *Commun. Math. Phys.* **246**, 133–179 (2004)
30. MATSUMURA, A.: Asymptotic toward rarefaction wave for solutions of the Broadwell model of a discrete velocity gas. *Japan J. Appl. Math.* **4**, 489–502 (1987)
31. MATSUMURA, A., NISHIHARA, K.: On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas. *Japan J. Appl. Math.* **2**, 17–25 (1985)
32. MATSUMURA, A., NISHIHARA, K.: Asymptotics toward the rarefaction waves of a one-dimensional model system for compressible viscous gas. *Japan J. Appl. Math.* **3**, 3–13 (1985)
33. MAXWELL, J.C.: *The Scientific Papers of James Clerk Maxwell*, Cambridge University Press, 1890: (a) *On the dynamical theory of gases*, **Vol. II**, p. 26. (b) *On stresses in rarefied gases arising from inequalities of temperature*, **Vol. II**, p. 681
34. NICOLAENKO, B.: Shock wave solutions of the Boltzmann equation as a nonlinear bifurcation problem from the essential spectrum. In: *Theories cinetiques classiques et relativistes* (Colloq. Internat. Centre Nat. Recherche Sci., No. **236**, Paris, 1974), pp. 127–150. Centre Nat. Recherche Sci., Paris, 1975
35. NISHIDA, T.: Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation. *Comm. Math. Phys.* **61**, 119–148 (1978)
36. NISHIHARA, K., YANG, T., ZHAO, H.-J.: Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations. *SIAM J. Math. Anal.* **35**, 1561–1597 (2004)
37. SMOLLER, J.: *Shock Waves and Reaction-diffusion Equations*. Springer, New York, 1994
38. SONE, Y.: *Kinetic Theory and Fluid Dynamics*. Birkhauser, Boston, 2002
39. UKAI, S.: On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad.* **50**, 179–184 (1974)
40. UKAI, S.: private communications
41. YU, S.-H.: Hydrodynamic limits with shock waves of the Boltzmann equations. *Commun. Pure Appl. Math.* **58**, 397–431 (2005)

Institute of Mathematics, Academia Sinica, Taipei

and

Department of Mathematics, Stanford University
e-mail: liu@math.stanford.edu

and

Department of Mathematics, City University of Hong Kong
e-mail: matyang@math.cityu.edu.hk, mashyu@math.cityu.edu.hk

and

School of Mathematics and Statistics,
Wuhan University, Wuhan 430072, China

and

School of Political Science and Economics
Waseda University, Tokyo 169-8050, Japan
e-mail: hhjjzhao@hotmail.com

(*Received July 9, 2005/ Accepted October 14, 2005*)
Published online: January 27, 2006 –