Available online at www.sciencedirect.com
ScienceDirect

J. Differential Equations 229 (2006) 698–742

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of contact discontinuity for the Boltzmann equation

Feimin Huang [a,*,1], Tong Yang [b,2]

[a] *Courant Institute of Mathematical Sciences, New York University, Academy of Mathematics and System Sciences, Academia Sinica, China*

[b] *Lie Bie Ju Centre of Mathematics, City University of Hong Kong, Hong Kong*

Received 24 November 2005

Available online 19 January 2006

Abstract

The Boltzmann equation which describes the time evolution of a large number of particles through the binary collision in statistics physics has close relation to the systems of fluid dynamics, that is, Euler equations and Navier–Stokes equations. As for a basic wave pattern to Euler equations, we consider the nonlinear stability of contact discontinuities to the Boltzmann equation. Even though the stability of the other two nonlinear waves, i.e., shocks and rarefaction waves has been extensively studied, there are few stability results on the contact discontinuity because unlike shock waves and rarefaction waves, its derivative has no definite sign, and decays slower than a rarefaction wave. Moreover, it behaves like a linear wave in a nonlinear setting so that its coupling with other nonlinear waves reveals a complicated interaction mechanism. Based on the new definition of contact waves to the Boltzmann equation corresponding to the contact discontinuities for the Euler equations, we succeed in obtaining the time asymptotic stability of this wave pattern with a convergence rate. In our analysis, an intrinsic dissipative mechanism associated with this profile is found and used for closing the energy estimates.

Keywords: Contact discontinuity; Boltzmann equations

* Corresponding author.
E-mail addresses: fhuang@amt.ac.cn (F.M. Huang), matyang@math.cityu.edu.hk (T. Yang).

1 Research supported by the National Natural Science Foundation of China under contract 10471138.
2 Research supported by the RGC Competitive Earmarked Research Grant, CityU 103004.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2005.12.004

Contents

1. Introduction

Consider the Boltzmann equation with "slab symmetry"

$$f_t + \xi_1 f_x = Q(f, f), \quad (f, x, t, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^3, \tag{1.1}$$

where $f(x,t,\xi)$ represents the distributional density of particles at space–time (x,t) with velocity ξ. For monatomic gas, the rotational invariance of the molecule leads to the collision operator $Q(f,f)$ as a bilinear collision operator in the form of, cf. [6]:

$$\begin{aligned} &Q(f,g)(\xi) \\ &\quad \equiv \frac{1}{2} \int\limits_{\mathbf{R}^3} \int\limits_{\mathbf{S}^2_+} \big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi)\big) B\big(|\xi - \xi_*|, \theta\big)\, d\xi_*\, d\Omega, \end{aligned}$$

with θ being the angle between the relative velocity and the unit vector Ω. Here $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2\colon (\xi - \xi_*) \cdot \Omega \geqslant 0\}$. The conservation of momentum and energy give the following relation between velocities before and after collision:

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\Omega. \end{cases}$$

In this paper, we will consider the Boltzmann equation for the two basic models, i.e., the hard sphere model and the hard potential with angular cut-off. In these two cases, the collision kernel $B(|\xi - \xi_*|, \theta)$ takes the forms

$$B\big(|\xi - \xi_*|, \theta\big) = \big|(\xi - \xi_*, \Omega)\big|$$

and

$$B\big(|\xi - \xi_*|, \theta\big) = |\xi - \xi_*|^{\frac{n-5}{n-1}} b(\theta), \quad b(\theta) \in L^1\big([0, \pi]\big),\ n > 5,$$

respectively. Here, n is the index in the inverse power potentials proportional to r^{1-n} with r being the distance between two particles. Notice that the following analysis also applies to the case for Maxwellian molecule, $n = 5$, and other abstract models with some restriction on the collision kernel and frequency. However, we will not discuss them here.

The Boltzmann equation has close relations to the systems of fluid dynamics, that is, the Euler equations and the Navier–Stokes equations. As typical solution profiles for hyperbolic conservation laws, the solutions to the Euler equations contain three basic wave patterns, i.e., shock, rarefaction wave and contact discontinuity. The cases on the nonlinear waves, shocks and rarefaction waves have been extensively studied, cf. [17,23,26,27,32,33,35,39] and reference therein. However, there are very few results on contact discontinuities even for the Navier–Stokes equations, cf. [20,22,28]. One of the reasons is that the contact discontinuity is associated with the linear degenerate field in the nonlinear system. Therefore, the coupling and interaction with the nonlinear fields require new techniques in the stability analysis.

As a continuation of our work on the nonlinear stability of wave patterns to the Boltzmann equation, we will consider the stability of the contact discontinuity in this paper. The stability of such a linear wave in a nonlinear setting requires some subtle analysis as we will present later through the intrinsic dissipation of the solution around the solution profile.

For a nontrivial solution profile connecting two different global Maxwellians at $x = \pm\infty$, it is reasonable and better to decompose the Boltzmann equation and its solution with respect to the local Maxwellian. This kind of decomposition was introduced in [29,31] by rewriting the Boltzmann equation into a fluid-type dynamics system with the nonfluid component appearing in the source terms, coupled with an equation for the time evolution of the nonfluid component. In fact, set, cf. [29,34],

$$f(x,t,\xi) = M(x,t,\xi) + G(x,t,\xi),$$

where the local Maxwellian M and G represent the fluid and nonfluid components in the solution, respectively. Here, the local Maxwellian M is defined by the five conserved quantities, that is, the mass density $\rho(x,t)$, momentum $m(x,t) = \rho(x,t)u(x,t)$, and energy density $(E(x,t) + 1/2|u(x,t)|^2)$:

$$\begin{cases} \rho(x,t) \equiv \int_{\mathbf{R}^3} f(x,t,\xi)\,d\xi, \\ m^i(x,t) \equiv \int_{\mathbf{R}^3} \psi_i(\xi) f(x,t,\xi)\,d\xi \quad \text{for } i = 1,2,3, \\ \big[\rho\big(E + \frac{1}{2}|u|^2\big)\big](x,t) \equiv \int_{\mathbf{R}^3} \psi_4(\xi) f(x,t,\xi)\,d\xi, \end{cases} \tag{1.2}$$

as

$$M \equiv M_{[\rho,u,\theta]}(x,t,\xi) \equiv \frac{\rho(x,t)}{\sqrt{(2\pi R\theta(x,t))^3}} \exp\left(-\frac{|\xi - u(x,t)|^2}{2R\theta(x,t)}\right). \tag{1.3}$$

Here $\theta(x,t)$ is the temperature which is related to the internal energy E by $E = (3/2)R\theta$ with R being the gas constant, and $u(x,t)$ is the fluid velocity. It is well known that the collision invariants $\psi_\alpha(\xi)$ are given by (cf. [6])

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i \quad \text{for } i = 1,2,3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_j(\xi) Q(h, g)\, d\xi = 0, \quad \text{for } j = 0, 1, 2, 3, 4.$$

In the sequel, the inner product of h, g in $L^2_\xi(\mathbf{R}^3)$ with respect to a given Maxwellian $\widetilde{M}$ is defined by

$$\langle h, g \rangle_{\widetilde{M}} \equiv \int_{\mathbf{R}^3} \frac{1}{\widetilde{M}} h(\xi) g(\xi)\, d\xi,$$

when the integral is well defined. If $\widetilde{M}$ is the local Maxwellian M, with respect to the corresponding inner product, the macroscopic space is spanned by the following five pairwise orthogonal functions

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}} M, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}} M \quad \text{for } i = 1, 2, 3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}} \left(\frac{|\xi - u|^2}{R\theta} - 3 \right) M, \\ \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4. \end{cases}$$

Using these five basic functions, we define the macroscopic projection P_0 and microscopic projection P_1 as follows:

$$\begin{cases} P_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j \rangle \chi_j, \\ P_1 h \equiv h - P_0 h. \end{cases}$$

The projections P_0 and P_1 are orthogonal and satisfy

$$P_0 P_0 = P_0, \qquad P_1 P_1 = P_1, \qquad P_0 P_1 = P_1 P_0 = 0.$$

A function $h(\xi)$ is called microscopic or nonfluid if

$$\int h(\xi) \psi_j(\xi)\, d\xi = 0, \quad j = 0, 1, 2, 3, 4.$$

Under this decomposition, the solution $f(x, t, \xi)$ of the Boltzmann equations satisfies

$$P_0 f = M, \qquad P_1 f = G,$$

and the Boltzmann equation becomes

$$(M + G)_t + \xi_1 (M + G)_x = 2Q(M, G) + Q(G, G),$$

which is equivalent to the following fluid-type system for the fluid components (see [29–31] for details):

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\int \xi_1^2 G_x \, d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = -\int \xi_1 \xi_i G_x \, d\xi, \quad i = 2, 3, \\ \big(\rho\big(e + \frac{|u|^2}{2}\big)\big)_t + \big(\rho u_1\big(e + \frac{|u|^2}{2}\big) + p u_1\big)_x = -\int \frac{1}{2}\xi_1 |\xi|^2 G_x \, d\xi, \end{cases} \tag{1.4}$$

or more precisely,

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4}{3}(\mu(\theta) u_{1x})_x - \int \xi_1^2 \Theta_x \, d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Theta_x \, d\xi, \quad i = 2, 3, \\ \big(\rho\big(e + \frac{|u|^2}{2}\big)\big)_t + \big(\rho u_1\big(e + \frac{|u|^2}{2}\big) + p u_1\big)_x \\ \quad = (\lambda(\theta)\theta_x)_x + \frac{4}{3}(\mu(\theta) u_1 u_{1x})_x + \sum_{i=2}^3 (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_x \, d\xi, \end{cases} \tag{1.5}$$

together with the equation for the nonfluid component G:

$$G_t + P_1(\xi_1 M_x) + P_1(\xi_1 G_x) = L_M G + Q(G, G). \tag{1.6}$$

(1.6) implies that

$$G = L_M^{-1}\big(P_1(\xi_1 M_x)\big) + \Theta$$

with

$$\Theta = L_M^{-1}\big(G_t + P_1(\xi_1 G_x) - Q(G, G)\big).$$

Here L_M is the linearized operator of the collision operator with respect to the local Maxwellian M:

$$L_M h = Q(M, h) + Q(h, M),$$

and the null space N of L_M is spanned by the macroscopic variables:

$$\chi_j, \quad j = 0, 1, 2, 3, 4.$$

Furthermore, there exists a positive constant $\sigma_0(\rho, u, \theta) > 0$ such that for any function $h(\xi) \in N^\perp$, see [18],

$$\langle h, L_M h \rangle \leqslant -\sigma_0 \big\langle \nu\big(|\xi|\big) h, h \big\rangle,$$

where $\nu(|\xi|)$ is the collision frequency. For the hard sphere and the hard potential with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property

$$0 < \nu_0 < \nu\big(|\xi|\big) < c\big(1 + |\xi|\big)^\beta,$$

for some positive constants ν_0, c and $0 < \beta \leqslant 1$. For later use, we list some basic properties of the projections P_0, P_1 and the linearized collision operator L_M as follows:

$$\begin{cases} P_0(\psi_j M) = \psi_j M, \qquad P_1(\psi_j M) = 0, \quad j = 0,1,2,3,4, \\ L_M P_1 = P_1 L_M = L_M, \quad P_1(Q(h,h)) = Q(h,h), \\ L_M(p_0) = P_0 L_M = 0, \quad P_0(Q(h,h)) = 0, \\ \langle \psi_j M, h\rangle = \langle \psi_j M, P_0 h\rangle, \quad j = 0,1,2,3,4, \\ \langle h, L_M g\rangle = \langle P_1 h, L_M(P_1 g)\rangle, \\ \langle h, L_M^{-1}(P_1 g)\rangle = \langle L_M^{-1}(P_1 h), P_1 g\rangle = \langle P_1 h, L_M^{-1}(P_1 g)\rangle. \end{cases}$$

In the above presentation, we normalize the gas constant R to be $2/3$ for simplicity so that $e = \theta$ and $p = (2/3)\rho\theta$. Notice also that the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\lambda(\theta) > 0$ are smooth functions of the temperature θ. And the following relation holds between these two functions [18]:

$$\lambda(\theta) = \frac{15}{4} R\mu(\theta) = \frac{5}{2}\mu(\theta), \tag{1.7}$$

after taking $R = 2/3$. (1.7) is used in the following energy estimation. In fact, in our analysis, it is required that $\inf_\theta \lambda(\theta) > 5/12 \sup_\theta \mu(\theta)$ for all θ under consideration. By (1.7), we know that this holds when the variation of the temperature is small.

Since our problem is in one-dimensional space $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to rewrite the system and the equation by using the *Lagrangian* coordinates as in the study of conservation laws. That is, consider the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(y,t)\,dy, \qquad t \Rightarrow t.$$

We will still denote the *Lagrangian* coordinates by (x,t) for simplicity of notation. The system (1.1) and (1.4) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = Q(f,f) \tag{1.8}$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = -\int \xi_1^2 G_x\,d\xi, \\ u_{it} = -\int \xi_1 \xi_i G_x\,d\xi, \quad i = 2,3, \\ \left(e + \frac{|u|^2}{2}\right)_t + (pu_1)_x = -\int \frac{1}{2}\xi_1 |\xi|^2 G_x\,d\xi. \end{cases} \tag{1.9}$$

Moreover, (1.4) and (1.5) take the form

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4}{3}\left(\frac{\mu(\theta)}{v}u_{1x}\right)_x - \int \xi_1^2 \Theta_{1x}\, d\xi, \\ u_{it} = \left(\frac{\mu(\theta)}{v}u_{ix}\right)_x - \int \xi_1\xi_i \Theta_{1x}\, d\xi, \quad i = 2, 3, \\ \left(e + \frac{|u|^2}{2}\right)_t + (pu_1)_x = \left(\frac{\lambda(\theta)}{v}\theta_x\right)_x + \frac{4}{3}\left(\frac{\mu(\theta)}{v}u_1u_{1x}\right)_x \\ \quad + \sum_{i=2}^3 \left(\frac{\mu(\theta)}{v}u_iu_{ix}\right)_x - \int \frac{1}{2}\xi_1|\xi|^2\Theta_{1x}\, d\xi \end{cases} \tag{1.10}$$

and

$$G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 M_x) + \frac{1}{v}P_1(\xi_1 G_x) = L_M G + Q(G, G), \tag{1.11}$$

with

$$G = L_M^{-1}\left(\frac{1}{v}P_1(\xi_1 M_x)\right) + \Theta_1$$

and

$$\Theta_1 = L_M^{-1}\left(G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 G_x) - Q(G, G)\right). \tag{1.12}$$

Since we will investigate the stability of the contact discontinuity for the Boltzmann equation here, it is worthy to recalling of the contact discontinuity for the hyperbolic conservation laws. For the Euler equations

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ u_{it} = 0, \\ \left(e + \frac{|u|^2}{2}\right)_t + (pu_1)_x = 0, \end{cases}$$

with a given Riemann data

$$\begin{cases} (v, u, \theta)(x, 0) = (v_-, 0, \theta_-), & \text{if } x < 0, \\ (v, u, \theta)(x, 0) = (v_+, 0, \theta_+), & \text{if } x > 0, \end{cases}$$

where $u = (u_1, u_2, u_3)$, and $v_\pm > 0$ and $\theta_\pm > 0$ are given constants, the solution is a contact discontinuity $(v^c, u^c, \theta^c)(x, t)$ located at $x = 0$ given by

$$\left(v^c, u^c, \theta^c\right)(x, t) = \begin{cases} (v_-, 0, \theta_-), & x < 0, \\ (v_+, 0, \theta_+), & x > 0, \end{cases} \tag{1.13}$$

provided that

$$p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}. \tag{1.14}$$

Since the Boltzmann equation contains the viscosity and heat conductivity in the macroscopic level as for the Navier–Stokes equations, the above contact discontinuity spreads out and becomes smooth with these two dissipative effects. Indeed, as shown in Section 2, it behaves like a nonlinear diffusion wave. Furthermore, coming from the microscopic effect in the Boltzmann equation and the slow decay of the nonlinear diffusion wave, the definition of the contact wave for the Boltzmann equation is more complicated than the one for the Navier–Stokes equations which includes some higher order terms from the nonfluid component besides the viscosity and heat conductivity. Without giving the detailed construction of this profile in the introduction, we now simply denote the profile by $(\bar\rho, \bar u, \bar\theta)(x,t)$. Detailed definition will be given in the next section.

To state our main theorem, we need to introduce the following notations. First, denote the perturbation around the above profile by

$$\phi(x,t) = v - \bar v, \qquad \psi(x,t) = u - \bar u, \qquad \zeta(x,t) = \theta - \bar\theta. \tag{1.15}$$

Then set

$$\begin{aligned} \Phi(x,t) &= \int_{-\infty}^{x} \phi(y,t)\,dy, \qquad \Psi(x,t) = \int_{-\infty}^{x} \psi(y,t)\,dy, \\ \overline{W}(x,t) &= \int_{-\infty}^{x} \left(e + \frac{|u|^2}{2} - \bar e - \frac{|\bar u|^2}{2}\right)(y,t)\,dy. \end{aligned} \tag{1.16}$$

Since the Boltzmann equation (1.10) and the system for the contact wave defined later are in the conservation forms, the quantities $(\Phi, \Psi, \overline{W})$ can be defined in some Sobolev space if the initial perturbation has zero mass, i.e., $\Phi(\infty,0) = \Psi(\infty,0) = \overline{W}(\infty,0) = 0$. The stability with general initial perturbation for the contact discontinuity is left for future investigation. Notice that even for shock wave to the Boltzmann equation, the stability with nonzero mass perturbation is also unsolved.

The main theorem can be stated as follows.

Theorem 1.1. *Let $(\bar v, \bar u, \bar\theta)(x,t)$ be the contact wave with strength $\delta = |\theta_+ - \theta_-| \leqslant \delta_0$ for some small positive constant δ_0. Then there exist a global Maxwellian $M_* = M_{[\rho_*, u_*, \theta_*]}$ and a small positive constant ϵ, such that if the initial data satisfies*

$$\left\{ \big\|(\Phi, \Psi, \overline{W})\big\|_{H_x^2} + \sum_{|\alpha|=2} \big\|\partial^\alpha f\big\|_{L_x^2(L_\xi^2(\frac{1}{\sqrt{M_*}}))} + \sum_{0 \leqslant |\alpha| \leqslant 1} \big\|\partial^\alpha G\big\|_{L_x^2(L_\xi^2(\frac{1}{\sqrt{M_*}}))} \right\}\Bigg|_{t=0} \leqslant \epsilon, \tag{1.17}$$

then the Cauchy problem (1.8) *admits a unique global solution $f(x,t,\xi)$ satisfying*

$$\begin{cases} \|f(x,t,\xi) - M_{[\bar v, \bar u, \bar\theta]}\|(t)_{L_x^\infty(L_\xi^2(\frac{1}{\sqrt{M_*}}))} \leqslant C(\epsilon + \delta_0)(1+t)^{-1/4}, \\ \|(\Phi, \Psi, \overline{W})\|_{L^\infty} \leqslant C(\epsilon + \delta_0)(1+t)^{-1/8 + \overline{C}_0\sqrt{\delta}}, \end{cases} \tag{1.18}$$

where C and $\overline{C}_0$ are positive constants. Here $f(\xi) \in L_\xi^2(\frac{1}{\sqrt{M_*}})$ means that $\frac{f(\xi)}{\sqrt{M_*}} \in L_\xi^2(\mathbf{R}^3)$.

Remark 1.2. The estimate for the higher derivative can be obtained similarly, provided that the initial data has the same order regularity.

The stability for the contact discontinuity is a long lasting open problem because of its special linear degenerate property. Under some restrictive conditions, the stability was discussed in [28]. For the system of Navier–Stokes equations, the stability with a free boundary as particle path was studied in [20,22], and the Cauchy problem with zero perturbation mass in [21]. There are two main analytical difficulties for stability of the contact discontinuity. One is that the space derivative of the nonlinear diffusion wave has no definite sign unlike the shock profile and rarefaction wave. The other is that the time decay of the background profile behaves like a solution to a heat equation, which is slower than the one for the rarefaction wave.

There are two main steps to overcome these difficulties. First, unlike the nonlinear diffusion waves defined for the Navier–Stokes equations [20], our Boltzmann profile includes more terms in the definition of the diffusion wave taken from the nonfluid component G and its derivatives. Such a nonlinear diffusion equation will be introduced in Section 2 with the properties of the solutions given in various norm spaces. Then, in Section 4, we will show that there is an intrinsic dissipation associated with the profile besides those from the viscosity and heat conductivity. Let us try to explain this intuitively as follows. The laws of Boyle and Gay-Lussac for ideal gas give $p = R\rho\theta$ and for a contact discontinuity we also have $u_1 = 0$. To view this in the perturbation equations from the quantities defined in (1.16), these two quantities correspond to the two linear combination of the antiderivatives of the perturbation, that is:

$$\bar{b}_1 = \Phi - \frac{2}{3p_+}\overline{W}, \qquad \bar{b}_3 = \Psi_1. \tag{1.19}$$

Since the solution approaches to the contact wave time asymptotically when the nonlinear waves spread out, the above two quantities $\bar{b}_1$ and $\bar{b}_3$ should approach to zero as time tends to infinity. This implies that there is some dissipative mechanism on these two quantities. In fact, this kind of dissipation takes the form of

$$\iint |\bar{\theta}_x|\big(\bar{b}_1^2 + \bar{b}_3^2\big)\,dx\,dt, \tag{1.20}$$

where $\bar{\theta}$ is the temperature function in the definition of the contact wave. It is shown in (4.25) that the intrinsic dissipation takes another form which is exactly equivalent to (1.20). Here and in the sequel, $\int dx$ means the integral $\int_R dx$, $\int d\xi$ means $\int_{R^3} d\xi$ and $\int dt$ means $\int_0^\infty dt$ for simplicity of notations. With this dissipation and those classical ones from the viscosity and heat conductivity, we can close the energy estimates in some Sobolev space and thus obtain the time asymptotic stability with a convergence rate.

Moreover, even for the Navier–Stokes equations, so far there is no convergence rates obtained for the shock wave and rarefaction wave. The convergence rate of the solution to the Boltzmann equation for the contact wave given in Theorem 1.1 is quite particular to this kind of degenerate waves. Hence, even though the convergence rate given here may not be optimal, it is quite reasonable in the setting of the contact waves in the fully nonlinear system. As we can see in the following analysis, the lowest order estimate may grow in time. The stability and decay rate in time are obtained by the compensation of the time decay in higher order derivatives estimates. Furthermore, the growth rate of the lowest order to the contact wave depends on the strength of the contact wave. Therefore, the smallness assumption on the wave strength is essential in this paper.

At the end of the introduction, we should mention that the Boltzmann equation has been extensively studied and important contribution has been made in many aspects, such as the renormalized solution, fluid dynamic limits, global existence around a global Maxwellian, regularity of the solutions, cf. [1–5,7–15,24,25,36,37,40] and references therein. Since they are not directly related to our problem considered here, we will not discuss them in details. On the other hand, the energy method making uses of the spectrum properties of the linearized operator which was from Grad to Ukai gives a good description of the perturbation of a global Maxwellian, cf. [18, 38,41,42]. Recently, the energy method based on the decomposition around a global Maxwellian is also introduced in [19] for the problems on space periodic solutions with or without forces.

The rest of the paper will be arranged as follows. In Section 2, the Boltzmann contact wave is constructed. In Section 3, the Boltzmann equation is reformulated to an integrated system. And Section 4 is devoted to the lower order estimate, while Section 5 is for the derivative estimate. The stability and convergence rate of the contact discontinuity for the Boltzmann equation will be given in Section 6.

2. Contact wave for the Boltzmann equation

We now construct the contact wave $(\bar{v}, \bar{u}, \bar{\theta})(x,t)$ for the Boltzmann equation. First let us recall the contact wave $(v^{ns}, u_1^{ns}, \theta^{ns})$ for the one-dimensional Navier–Stokes equations introduced in [20]. Corresponding to the fluid-type system from the Boltzmann equation, the system of the one-dimensional Navier–Stokes equations is

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4}{3}\big(\frac{\mu(\theta)}{v} u_{1x}\big)_x, \\ \big(e + \frac{u_1^2}{2}\big)_t + (pu_1)_x = \big(\frac{\lambda(\theta)}{v}\theta_x\big)_x + \frac{4}{3}\big(\frac{\mu(\theta)}{v} u_1 u_{1x}\big)_x. \end{cases} \tag{2.1}$$

Notice that (2.1) is exactly (1.10) if $\Theta_1 = 0$. In this situation, the temperature function θ^{ns} of the contact wave can be defined as a self-similar solution $\theta^{ns}(x/\sqrt{1+t})$ to the following nonlinear diffusion equation

$$\theta_t^{ns} = \big(a\big(\theta^{ns}\big)\theta_x^{ns}\big)_x, \quad \theta^{ns}(-\infty,t) = \theta_-, \ \theta^{ns}(+\infty,t) = \theta_+, \tag{2.2}$$

where the function $a(\theta) = (9p_+\lambda(\theta))/(10\theta) > 0$. Here, we have used the assumption of polytropic gas with $\gamma = 5/3$ for monatomic gas. In fact, this nonlinear diffusion equation is derived from the first and the third equation of (2.1) by letting $R\theta^{ns}/v^{ns} = p^{ns} = p_+$, and dropping the faster time-decay terms $u_1^{ns}u_{1t}^{ns}$ and $\frac{4}{3}(\frac{\mu(\theta^{ns})}{v^{ns}}u_1^{ns}u_{1x}^{ns})_x$, see [20] for details. Let $\delta = |\theta_+ - \theta_-|$, it is easy to check that

$$\big|\theta_x^{ns}\big| = O(\delta)(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \quad \text{as } x \to \pm\infty. \tag{2.3}$$

The velocity u_1^{ns} is then defined as $\frac{R}{p_+}a(\theta^{ns})\theta_x^{ns}$ with the decay rate $1/\sqrt{1+t}$. And $v^{ns} = \frac{2}{3p_+\theta^{ns}}$.

For the Boltzmann equation, if we still use the Navier–Stokes profile $(v^{ns}, u_1^{ns}, \theta^{ns})$, some non-$t$-integrable error terms, coming from the nonfluid component, exist for the integrated equation for $(\Phi, \Psi, \overline{W})$. To see this, we first notice that the principle part of the nonfluid component in the solution G and part of it Θ_1, defined in (1.12), are given by

$$w = \frac{1}{v} L_M^{-1}\big(P_1(\xi_1 M_x)\big) = \frac{1}{Rv\theta} L_M^{-1}\left\{P_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\theta_x + \xi\cdot u_x\right)M\right]\right\} \tag{2.4}$$

and

$$\widehat{\Theta}_1 = L_M^{-1}\left(\frac{1}{v}P_1(\xi_1 w_x) - Q(w,w)\right), \tag{2.5}$$

respectively. If we substitute $(v^{ns}, u_1^{ns}, \theta^{ns})$ into (2.5), then $\widehat{\Theta}_1$ decays with the rate $\frac{1}{1+t}$ which is nonintegrable in time and not easy to be estimated. To eliminate these non-t-integrable terms, we instead use the Boltzmann contact wave, which will be constructed below. First let us distinguish the leading part coming from the nonfluid component. We rewrite the Boltzmann equation (1.10) as

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4}{3}\big(\frac{\mu(\theta)}{v}u_{1x}\big)_x - \sum_{j=1}^2 \int \xi_1^2 \Theta_{1x}^j \, d\xi, \\ u_{it} = \big(\frac{\mu(\theta)}{v}u_{ix}\big)_x - \sum_{j=1}^2 \int \xi_1\xi_i \Theta_{1x}^j \, d\xi, \quad i = 2,3, \\ \big(e + \frac{|u|^2}{2}\big)_t + (pu_1)_x = \big(\frac{\lambda(\theta)}{v}\theta_x\big)_x - \sum_{j=1}^2 \int \frac{1}{2}\xi_1|\xi|^2 \Theta_{1x}^j \, d\xi + H_{1x}, \end{cases} \tag{2.6}$$

with

$$\begin{aligned} \widetilde{G}_t - L_M\widetilde{G} = {} & -\frac{1}{Rv\theta}P_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}(\theta-\bar\theta)_x + \xi\cdot(u-\bar u)_x\right)M\right] \\ & + \frac{u_1}{v}G_x - \frac{1}{v}P_1(\xi_1 G_x) + Q(G,G) - \overline{G}_t, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} &\Theta_1 = L_M^{-1}\left(G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 G_x) - Q(G,G)\right) = \sum_{j=1}^2 \Theta_1^j, \\ &\Theta_1^1 = L_M^{-1}\left(\frac{1}{v}P_1(\xi_1\overline{G}_x) - Q(\overline{G},\overline{G})\right), \\ &\Theta_1^2 = L_M^{-1}\left(G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1\widetilde{G}_x) - 2Q(\widetilde{G},\overline{G}) - Q(\widetilde{G},\widetilde{G})\right), \\ &\overline{G} = \frac{1}{Rv\theta}L_M^{-1}\left\{P_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\bar\theta_x + \xi\cdot\bar u_x\right)M\right]\right\}, \\ &\widetilde{G} = G - \overline{G}, \qquad H_1 = \frac{4}{3}\left(\frac{\mu(\theta)}{v}u_1u_{1x}\right) + \sum_{i=2}^3\left[\frac{\mu(\theta)}{v}u_iu_{ix}\right], \end{aligned} \tag{2.8}$$

where the function $(\bar v, \bar u, \bar\theta)(x,t)$ is the contact wave which will be constructed later.

Since the velocity u decays faster than (v,θ) in time, the leading part of the energy equation in (2.6) is

$$\theta_t + pv_t = \left(\frac{\lambda(\theta)}{v}\theta_x\right)_x - \int \frac{1}{2}\xi_1|\xi|^2\Theta^1_{1x}\,d\xi. \tag{2.9}$$

By the definition of Θ^1_1, we have

$$\begin{aligned} &-\int \frac{1}{2}\xi_1|\xi|^2\Theta^1_1\,d\xi = N_1 + F_1, \quad N_1 = f_{11}\theta_x\bar\theta_x + f_{12}v_x\bar\theta_x + f_{13}\bar\theta_x^2 + f_{14}\bar\theta_{xx}, \\ &|F_1| = O(1)\big(\big(|v_x| + |\theta_x| + |\bar\theta_x| + |u_x| + |\bar u_x|\big)|\bar u_x| + |u_x||\bar\theta_x| + |\bar u_{xx}|\big), \end{aligned} \tag{2.10}$$

where the coefficients f_{1j}, $j = 1,2,3,4$, are smooth functions of (v,u,θ). Motivated by the work on the Navier–Stokes equations, we expect that the contact wave $(\bar v,\bar u,\bar\theta)$ for the Boltzmann equation satisfies $\bar p = 2/3\bar\theta/\bar v \approx p_+$ as t is large. Thus, by choosing only the leading term in (2.9), we have

$$\theta_t = \big(a(\theta)\theta_x\big) + \frac{3}{5}N_{1x}, \tag{2.11}$$

where $a(\theta)$ is defined in (2.2). To include more microscopic effect, let the contact wave $\bar\theta \approx \theta^{ns}(x/\sqrt{1+t}) + \theta^{nf}(x,t)$, where $\theta^{nf}(x,t)$ represents the part of the nonlinear diffusion wave coming from the nonfluid component not appearing in the Navier–Stokes level. Moreover, the term $\theta^{nf}(x,t)$ in the form $\frac{1}{\sqrt{1+t}}D_1(\frac{x}{\sqrt{1+t}})$ is from N_1 in (2.10). Note that $\theta^{nf}(x,t)$ decays faster than $\theta^{ns}(x,t)$ so that it can be viewed as the perturbation around the Navier–Stokes profile $\theta^{ns}(x,t)$. To construct $\theta^{nf}(x,t)$, we linearize Eq. (2.11) around $\theta^{ns}(x,t)$ and drop all the higher order terms. This leads to a linear equation for $\theta^{nf}(x,t)$ from (2.2)

$$\theta^{nf}_t = \big(a\big(\theta^{ns}\big)\theta^{nf}_x\big)_x + \big(a'\big(\theta^{ns}\big)\theta^{ns}_x\theta^{nf}\big)_x + \frac{3}{5}\widetilde N_{1x}, \tag{2.12}$$

where $\widetilde N_1 = (\tilde f_{11} + \frac{2}{3p_+}\tilde f_{12} + \tilde f_{13})(\theta^{ns}_x)^2 + \tilde f_{14}\theta^{ns}_{xx}$ with $\tilde f_{1j} = f_{1j}(v^{ns},0,\theta^{ns})$, $j = 1,2,3,4$. Integrating (2.12) with respect to x yields

$$g_{1t} = a\big(\theta^{ns}\big)g_{1xx} + a'\big(\theta^{ns}\big)\theta^{ns}_x g_{1x} + \frac{3}{5}\widetilde N_1, \tag{2.13}$$

where

$$g_1(x,t) = \int_{-\infty}^{x} \theta^{nf}(x,t)\,dx.$$

Note that $\widetilde N_1$ takes the form $\frac{1}{1+t}D_2(\frac{x}{\sqrt{1+t}})$ and satisfies the property (2.3). It is straightforward to check that there exists a self-similar solution $g_1(\eta)$, $\eta = x/\sqrt{1+t}$, for (2.13) with the boundary condition $g_1(-\infty,t) = d_1$, $g_1(+\infty,t) = d_1 + \delta_1$. Here d_1 can be any given constant and δ_1 satisfies $0 < \delta_1 < \delta$. It is worthy to pointing out that even though the function $g_1(x,t)$ depends on the constants d_1 and δ_1, $\theta^{nf}(x,t)=g_{1x}(x,t)\to 0$ as $t\to\infty$ which gives the same time asymptotic state. That is, the choice of the constants d_1 and δ_1 has no influence on our main result as long as $0<\delta_1<\delta$. From now on, we fix d_1 and δ_1 so that the function g_1 is uniformly determined. And the derivative $g_{1x}=\theta^{nf}$ satisfies the property (2.3).

Now we follow the same procedure to construct the second and third components of the velocity of the contact wave denoted by $\bar{u}_i$, $i=2,3$. Similarly, the leading part of the equation for u_i coming from (2.6) is

$$u_{it}=\left(\frac{p_+\mu(\theta)}{R\theta}u_{ix}\right)_x-\int \xi_1\xi_i\Theta_{1x}^1\,d\xi. \tag{2.14}$$

For $i=2,3$, we have

$$\begin{aligned}&-\int \xi_1\xi_i\Theta_1^1\,d\xi=N_i+F_i,\qquad N_i=f_{i1}\theta_x\bar{\theta}_x+f_{i2}v_x\bar{\theta}_x+f_{i3}\bar{\theta}_x^2+f_{i4}\bar{\theta}_{xx},\\ &|F_i|=O(1)\big(\big(|v_x|+|\theta_x|+|\bar{\theta}_x|+|u_x|+|\bar{u}_x|\big)|\bar{u}_x|+|u_x||\bar{\theta}_x|+|\bar{u}_{xx}|\big),\end{aligned} \tag{2.15}$$

with smooth functions f_{ij}, $i=2,3$, $j=1,2,3,4$. Notice that the symbols N_i and F_i, $i=2,3$, used here are for the convenience of notation. And N_1 and F_1 defined in (2.10) are not the case when $i=1$.

From (2.14) and (2.15), we expect that the contact wave $\bar{u}_i(x,t)$ takes the form $(1/\sqrt{1+t})\times h_i(x/\sqrt{1+t})$ and satisfies the following linear equation:

$$\bar{u}_{it}=\left(\frac{p_+\mu(\theta^{ns})}{R\theta^{ns}}\bar{u}_{ix}\right)_x+\widetilde{N}_{ix},\quad i=2,3, \tag{2.16}$$

where $\widetilde{N}_i=(\tilde{f}_{i1}+\frac{2}{3p_+}\tilde{f}_{i2}+\tilde{f}_{i3})(\theta_x^{ns})^2+\tilde{f}_{i4}\theta_{xx}^{ns}$, $\tilde{f}_{ij}=f_{ij}(v^{ns},0,\theta^{ns})$, $i=2,3$, $j=1,2,3,4$. Integrating (2.16) with respect to x, we have

$$g_{it}=\frac{p_+\mu(\theta^{ns})}{R\theta^{ns}}g_{ixx}+\widetilde{N}_i, \tag{2.17}$$

where $g_i(x,t)=\int_{-\infty}^x\bar{u}_i(x,t)\,dx$. For given θ^{ns}, it is easy to check that there exists a unique solution $g_i(\eta)$ with $\eta=x/\sqrt{1+t}$ satisfying $g_i(-\infty,t)=d_i$, $g_i(+\infty,t)=d_i+\delta_i$, where d_i can be any fixed constants and δ_i satisfies $0<\delta_i<\delta$. As we explained before, the choice of d_i and δ_i is not important to our result. From (2.3), it is easy to check that the solution $\bar{u}_i=g_{ix}$, $i=2,3$, has the following property

$$|\bar{u}_i|=|g_{ix}|=O(\delta)(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4b(\theta_\pm)(1+t)}},\quad \text{as } x\to\pm\infty, \tag{2.18}$$

where $b(\theta_\pm)=\max\{a(\theta_\pm),\frac{3p_+}{2\theta_\pm}\mu(\theta_\pm)\}$.

In summary, we can define the contact wave $(\bar{v},\bar{u},\bar{\theta})$ for the Boltzmann equation as follows. To satisfy the conservation of mass, we expect

$$\bar{v}_t-\bar{u}_{1x}=0. \tag{2.19}$$

By letting $\bar v=\frac{2}{3p_+}(\theta^{ns}+\theta^{nf})$, we have

$$\bar u_1=\frac{2}{3p_+}\big[a\big(\theta^{ns}\big)\theta^{ns}_x+a\big(\theta^{ns}\big)\theta^{nf}_x+a'\big(\theta^{ns}\big)\theta^{ns}_x\theta^{nf}\big]+\frac{2}{5p_+}\widetilde N_1. \tag{2.20}$$

However, by plugging (2.20) into the momentum equation of (2.6), we have a nonconservative term containing $\widetilde N_{1t}$. To avoid this, we define

$$\bar u_1=\frac{2}{3p_+}\big[a\big(\theta^{ns}\big)\theta^{ns}_x+a\big(\theta^{ns}\big)\theta^{nf}_x+a'\big(\theta^{ns}\big)\theta^{ns}_x\theta^{nf}\big]. \tag{2.21}$$

Similarly, to avoid the nonconservative term $(|\bar u|^2)_t$ in the energy equation, we set

$$\bar\theta=\theta^{ns}+\theta^{nf}-\frac12|\bar u|^2. \tag{2.22}$$

Therefore, the Boltzmann contact wave $(\bar v,\bar u,\bar\theta)$ is finally defined as:

$$\begin{aligned}
&\bar v=\frac{2}{3p_+}\big(\theta^{ns}+\theta^{nf}\big),\\
&\bar u_1=\frac{2}{3p_+}\big[a\big(\theta^{ns}\big)\theta^{ns}_x+a\big(\theta^{ns}\big)\theta^{nf}_x+a'\big(\theta^{ns}\big)\theta^{ns}_x\theta^{nf}\big],\\
&\bar u_i=g_{ix},\quad i=2,3,\\
&\bar\theta=\theta^{ns}+\theta^{nf}-\frac12|\bar u|^2,
\end{aligned} \tag{2.23}$$

where θ^{ns} is given by (2.2), θ^{nf} by (2.12) and g_i, $i=2,3$, by (2.17).

Notice that even though the Boltzmann profile $(\bar v,\bar u,\bar\theta)$ includes some other nonfluid terms, this profile and the $(v^{ns},u^{ns},\theta^{ns})$ for the Navier–Stokes are equivalent as t tends to infinity because all the extra nonfluid terms decay with the rate $1/\sqrt{1+t}$. In another word, the contact wave $(\bar v,\bar u,\bar\theta)$ is also an appropriate approximation to the original contact discontinuity (v^c,u^c,θ^c) for the Euler equations. In addition, a direct but tedious computation shows that

$$\begin{cases}
\bar v_t-\bar u_{1x}=\frac{2}{5p_+}\widetilde N_{1x},\\
\bar u_{1t}+\bar p_x=\frac43\big(\frac{\mu(\bar\theta)}{\bar v}\bar u_{1x}\big)_x+R_{1x},\\
\bar u_{it}=\big(\frac{\mu(\bar\theta)}{\bar v}\bar u_{ix}\big)_x+\overline N_{ix}+R_{ix},\quad i=2,3,\\
\big(\bar e+\frac{|\bar u|^2}{2}\big)_t+(\bar p\bar u_1)_x=\big(\frac{\lambda(\bar\theta)}{\bar v}\bar\theta_x\big)_x-\frac25\widetilde N_{1x}+\overline N_{1x}+\overline H_{1x}+R_{4x},
\end{cases} \tag{2.24}$$

where

$$\begin{aligned}
R_1&=\frac{2}{3p_+}\big[a\big(\theta^{ns}\big)\theta_{nst}+\big(a\big(\theta^{ns}\big)\theta^{nf}\big)_t\big]+\bar p-p_+-\frac43\bigg(\frac{\mu(\bar\theta)}{\bar v}\bar u_{1x}\bigg)\\
&=O(\delta)(1+t)^{-1}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}},\quad \text{as } |x|\to\infty,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}R_i&=\left[\frac{3p_+\mu(\theta^{ns})}{2\theta^{ns}}-\frac{\mu(\bar\theta)}{\bar v}\right]\bar u_{ix}+\widetilde N_i-\overline N_i\\&=O(\delta)(1+t)^{-\frac32}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}},\quad \text{as } |x|\to\infty,\ i=2,3,\end{aligned}\tag{2.26}$$

$$\begin{aligned}R_4&=\left[\frac53\big(a(\theta^{ns})\theta^{ns}_x+a(\theta^{ns})\theta^{nf}_x+a'(\theta^{ns})\theta^{ns}_x\theta^{nf}\big)-\frac{\lambda(\bar\theta)}{\bar v}\bar\theta_x\right]\\&\quad+(\bar p-p_+)\bar u_1+\widetilde N_1-\overline N_1-\overline H_1\\&=O(\delta)(1+t)^{-\frac32}e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}},\quad \text{as } |x|\to\infty,\end{aligned}\tag{2.27}$$

$$\widetilde N_i=O(\delta)(1+t)^{-1}e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}},\quad \text{as } |x|\to\infty,\ i=1,2,3,\tag{2.28}$$

with $c(\theta_\pm)=\max\{a(\theta_\pm),\frac12 b(\theta_\pm)\}$, $\overline N_i$, $i=1,2,3$, and $\overline H_1$ are the corresponding functions defined in (2.8), (2.10) and (2.15) by substituting the variable (v,u,θ) by the contact wave profile $(\bar v,\bar u,\bar\theta)$. It is worthy to pointing out that the time-decay rate of R_i, $i=2,3,4$, is of order $(1+t)^{-3/2}$ which is much better than $(1+t)^{-1}$. This is the main improvement when we use the Boltzmann contact wave instead of the Navier–Stokes profile. Furthermore, even though the time-decay rate of R_1 is still $(1+t)^{-1}$, it is sufficient to give the desired a priori estimates through a subtle analysis coming from the intrinsic dissipation mechanism mentioned in the introduction because it appears in the first equation of the momentum equations.

3. Reformulated system

To prove the main theorem, we now derive the system for the perturbation (ϕ,ψ,ζ) around the contact wave $(\bar v,\bar u,\bar\theta)$. Set

$$\phi=v-\bar v,\qquad \psi=u-\bar u,\qquad \zeta=\theta-\bar\theta,\tag{3.1}$$

and

$$\begin{aligned}&\Phi=\int_{-\infty}^x\phi(y,t)\,dy,\qquad \Psi=\int_{-\infty}^x\psi(y,t)\,dy,\\&\overline W=\int_{-\infty}^x\left(e+\frac{|u|^2}{2}-\bar e-\frac{|\bar u|^2}{2}\right)(y,t)\,dy.\end{aligned}\tag{3.2}$$

As mentioned before, we impose $\Phi(\infty,0)=\Psi(\infty,0)=\overline W(\infty,0)=0$ so that the quantities Φ,Ψ and $\overline W$ can be defined in some Sobolev space. The initial data satisfying this condition is called zero mass perturbation condition. Naturally, we have $(\phi,\psi)=(\Phi,\Psi)_x$ and $\zeta+\frac12|\Psi_x|^2+\sum_{i=1}^3\bar u_i\Psi_{ix}=\overline W_x$.

Subtracting (2.24) from Eq. (2.6) and integrating the resulting system yield

$$\begin{cases} \Phi_t - \Psi_{1x} = -\frac{2}{5p_+}\widetilde{N}_1, \\ \Psi_{1t} + p - \bar{p} = \frac{4}{3}\frac{\mu(\theta)}{v}u_{1x} - \frac{4}{3}\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1x} - \sum_{j=1}^{2}\int \xi_1^2 \Theta_1^j\,d\xi - R_1, \\ \Psi_{it} = \frac{\mu(\theta)}{v}u_{ix} - \frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{ix} + N_i - \overline{N}_i + F_i - \int \xi_1\xi_i\Theta_1^2\,d\xi - R_i, \quad i=2,3, \\ \overline{W}_t + pu_1 - \bar{p}\bar{u}_1 = \frac{\lambda(\theta)}{v}\theta_x - \frac{\lambda(\bar{\theta})}{\bar{v}}\bar{\theta}_x + \frac{2}{5}\widetilde{N}_1 + N_1 - \overline{N}_1 + F_1 + H_1 - \overline{H}_1 \\ \qquad - \int \frac{1}{2}\xi_1|\xi|^2\Theta_1^2\,d\xi - R_4. \end{cases} \tag{3.3}$$

Since the variable $\overline{W}$ is the antiderivative of the total energy, not the temperature, it is more convenient to introduce another variable

$$W = \overline{W} - \bar{u}_1\Psi_1. \tag{3.4}$$

It follows that

$$\zeta = W_x - Y, \quad \text{with } Y = \frac{1}{2}|\Psi_x|^2 - \bar{u}_{1x}\Psi_1 + \bar{u}_2\Psi_{2x} + \bar{u}_3\Psi_{3x}. \tag{3.5}$$

Using the new variable W and linearizing the left-hand side of the system (3.3) by using the formula (2.8) for H_1, we have

$$\begin{cases} \Phi_t - \Psi_{1x} = -\frac{2}{5p_+}\widetilde{N}_1, \\ \Psi_{1t} - \frac{p_+}{\bar{v}}\Phi_x + \frac{2}{3\bar{v}}W_x = \frac{4}{3}\frac{\mu(\bar{\theta})}{\bar{v}}\Psi_{1xx} + \frac{4}{3}\big(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}}\big)u_{1x} \\ \quad - \sum_{j=1}^{2}\int \xi_1^2\Theta_1^j\,d\xi + J_1 + \frac{2}{3\bar{v}}Y - R_1 \doteq \frac{4}{3}\frac{\mu(\bar{\theta})}{\bar{v}}\Psi_{1xx} + Q_1, \\ \Psi_{it} = \frac{\mu(\bar{\theta})}{\bar{v}}\Psi_{ixx} + \big(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}}\big)u_{ix} + N_i - \overline{N}_i + F_i \\ \quad - \int \xi_1\xi_i\Theta_1^2\,d\xi - R_i \doteq \frac{\mu(\bar{\theta})}{\bar{v}}\Psi_{ixx} + Q_i, \quad i=2,3, \\ W_t + p_+\Psi_{1x} = \frac{\lambda(\bar{\theta})}{\bar{v}}W_{xx} + \big(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar{\theta})}{\bar{v}}\big)\theta_x + \frac{2}{5}\widetilde{N}_1 + N_1 - \overline{N}_1 \\ \quad + F_1 + \frac{4u_{1x}}{3}\frac{\mu(\theta)}{v}\Psi_{1x} + \sum_{i=2}^{3}\big[\frac{\mu(\theta)}{v}u_iu_{ix} - \frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_i\bar{u}_{ix}\big] - R_4 - \bar{u}_{1t}\Psi_1 + J_2 \\ \quad - \int \frac{1}{2}\xi_1|\xi|^2\Theta_1^2\,d\xi + \bar{u}_1R_1 + \bar{u}_1\sum_{j=1}^{2}\int \xi_1^2\Theta_1^j\,d\xi - \frac{\lambda(\bar{\theta})}{\bar{v}}Y_x \\ \doteq \frac{\lambda(\bar{\theta})}{\bar{v}}W_{xx} + \frac{2}{5}\widetilde{N}_1 + Q_4, \end{cases} \tag{3.6}$$

where

$$J_1 = \frac{\bar{p} - p_+}{\bar{v}}\Phi_x - \left[p - \bar{p} + \frac{\bar{p}}{\bar{v}}\Phi_x - \frac{2}{3\bar{v}}(\theta - \bar{\theta})\right] = O(1)\big(\Phi_x^2 + (\theta - \bar{\theta})^2 + |\bar{u}|^4\big), \tag{3.7}$$

$$J_2 = (p_+ - p)\Psi_{1x} = O(1)\big(\Phi_x^2 + \Psi_{1x}^2 + (\theta - \bar{\theta})^2 + |\bar{u}|^4\big), \tag{3.8}$$

$$Q_1 = \frac{4}{3}\left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}}\right)u_{1x} - \sum_{j=1}^{2}\int \xi_1^2\Theta_1^j\,d\xi + J_1 + \frac{2}{3\bar{v}}Y - R_1, \tag{3.9}$$

$$Q_i = \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar\theta)}{\bar v}\right)u_{ix} + N_i - \overline{N}_i + F_i - \int \xi_1\xi_i\Theta_1^2\,d\xi - R_i, \quad i = 2,3, \tag{3.10}$$

$$\begin{aligned} Q_4 = {} & \left(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar\theta)}{\bar v}\right)\theta_x + N_1 - \overline{N}_1 F_1 + \frac{4u_{1x}}{3}\frac{\mu(\theta)}{v}\Psi_{1x} - R_4 - \bar u_{1t}\Psi_1 + \bar u_1 R_1 \\ & + \sum_{i=2}^{3}\left[\frac{\mu(\theta)}{v}u_i u_{ix} - \frac{\mu(\bar\theta)}{\bar v}\bar u_i \bar u_{ix}\right] - \int \frac{1}{2}\xi_1|\xi|^2\Theta_1^2\,d\xi + J_2 \\ & + \bar u_1 \sum_{j=1}^{2}\int \xi_1^2\Theta_1^j\,d\xi - \frac{\lambda(\bar\theta)}{\bar v}Y_x. \end{aligned} \tag{3.11}$$

In the next section, we will work on the reformulated system (3.6). Since the local existence follows similarly from the discussions in [19,41], we will omit it here for brevity. To prove the global existence, we only need to close the following a priori estimate:

$$\begin{aligned} N(T) = \sup_{0\leqslant t\leqslant T}\Bigg\{ & \big\|(\Phi,\Psi,W)\big\|_{L^\infty}^2 + \big\|(\phi,\psi,\zeta)\big\|_{H^1}^2 \\ & + \int_R\int_{R^3}\left(\frac{\widetilde{G}^2}{M_*} + \sum_{|\alpha|=1}\frac{(\partial^\alpha G)^2}{M_*} + \sum_{|\alpha|=2}\frac{(\partial^\alpha f)^2}{M_*}\,d\xi\,dx\right)\Bigg\} \leqslant \varepsilon_0^2, \end{aligned} \tag{3.12}$$

where ε_0 is positive small constant depending on the initial data and M_* is a global Maxwellian chosen later for any $T > 0$. Here, it is worthy to pointing out that (3.12) also gives the a priori assumptions on $\|(\phi_t,\psi_t,\zeta_t)\|$, $\|\partial^\alpha(\phi,\psi,\zeta)\|$ and $\iint |\partial^\alpha G|^2/M_*\,d\xi\,dx$ ($|\alpha| = 2$). In fact, from (1.9) and (3.12), we have

$$\begin{aligned} \big\|(\phi_t,\psi_t,\zeta_t)\big\|^2 & \leqslant C\left(\big\|(\phi_x,\psi_x,\zeta_x)\big\|^2 + \iint \frac{G_x^2}{M_*}\,d\xi\,dx + \delta^2(1+t)^{-\frac12}\right) \\ & \leqslant C(\varepsilon_0+\delta)^2, \end{aligned} \tag{3.13}$$

where we have used

$$\left(\int \xi_1^2 G_x\,d\xi\right)^2 \leqslant C\int \frac{G_x^2}{M_*}\,d\xi \tag{3.14}$$

and

$$\begin{aligned} \big\|(\phi_t,\psi_t,\zeta_t)\big\|^2 & \leqslant C\big\|(v_t,u_t,\theta_t)\big\|^2 + C\delta^2(1+t)^{-\frac32}, \\ \big\|(\phi_x,\psi_x,\zeta_x)\big\|^2 & \leqslant C\big\|(v_x,u_x,\theta_x)\big\|^2 + C\delta^2(1+t)^{-\frac12}. \end{aligned} \tag{3.15}$$

To derive the a priori assumption on $\|\partial^\alpha(\phi,\psi,\zeta)\|$ ($|\alpha| = 2$), we use the definition of ρ, $m = \rho u$ and $\rho(\theta + (1/2)|u|^2)$. Let $|\alpha| = 2$, by (1.2), we obtain

$$\left\|\partial^\alpha\left(\rho, m, \rho\left(\theta + \frac12|u|^2\right)\right)\right\|^2 \leqslant C\iint \frac{|\partial^\alpha f|^2}{M_*}\,d\xi\,dx \leqslant C\varepsilon_0^2. \tag{3.16}$$

This yields that

$$\left\|\partial^\alpha(\phi,\psi,\zeta)\right\|^2 \leqslant C\varepsilon_0 + C\delta^2(1+t)^{-3/2} \leqslant C(\varepsilon_0+\delta)^2, \quad |\alpha|=2. \tag{3.17}$$

Finally, we have

$$\begin{aligned}\iint \frac{|\partial^\alpha G|^2}{M_*}\,d\xi\,dx &\leqslant \iint \frac{|\partial^\alpha f|^2}{M_*}\,d\xi\,dx + \iint \frac{|\partial^\alpha M|^2}{M_*}\,d\xi\,dx \\ &\leqslant C(\varepsilon_0+\delta)^2, \quad |\alpha|=2.\end{aligned} \tag{3.18}$$

4. Lower-order estimate

Before proving the a priori estimate (3.12), we list the following basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [16].

Lemma 4.1. *There exists a positive constant $C>0$ such that*

$$\int_{R^3} \frac{\nu(|\xi|)^{-1}Q(f,g)^2}{M}\,d\xi \leqslant C\left\{\int_{R^3}\frac{\nu(|\xi|)f^2}{M}\,d\xi\int_{R^3}\frac{g^2}{M} + \int_{R^3}\frac{f^2}{M}\,d\xi\int_{R^3}\frac{\nu(|\xi|)g^2}{M}\right\},$$

where M can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 4.1, the following three lemmas are proved in [30]. The proofs are straightforward by using the Cauchy inequality.

Lemma 4.2. *If $\theta/2<\theta_*<\theta$, then there exist two positive constants $\bar\sigma=\bar\sigma(\rho,u,\theta;\rho_*,u_*,\theta_*)>0$ and $\eta_0=\eta_0(\rho,u,\theta;\rho_*,u_*,\theta_*)>0$ such that if $|\rho-\rho_*|+|u-u_*|+|\theta-\theta_*|<\eta_0$, we have for $h(\xi)\in N^\perp$,*

$$-\int_{R^3}\frac{hL_Mh}{M_*}\,d\xi \geqslant \bar\sigma\int_{R^3}\frac{\nu(|\xi|)h^2}{M_*}\,d\xi, \tag{4.1}$$

where $M_*=M_{[\rho_*,u_*,\theta_*]}$ and the definition of $M_{[\rho,u,\theta]}$ can be found in (1.3).

Lemma 4.3. *Under the assumptions in Lemma* 4.2, *we have*

$$\begin{cases}\int_{R^3}\frac{\nu(|\xi|)}{M}|L_M^{-1}h|^2\,d\xi \leqslant \bar\sigma^{-2}\int_{R^3}\frac{\nu(|\xi|)^{-1}h^2}{M}\,d\xi,\\ \int_{R^3}\frac{\nu(|\xi|)}{M_*}|L_M^{-1}h|^2\,d\xi \leqslant \bar\sigma^{-2}\int_{R^3}\frac{\nu(|\xi|)^{-1}h^2}{M_*}\,d\xi\end{cases} \tag{4.2}$$

for each $h(\xi)\in N^\perp$.

Lemma 4.4. *Under the conditions in Lemma* 4.2*, there exists a constant* $C > 0$ *such that for positive constants* k *and* λ*, we have*

$$\left|\int_{R^3} \frac{g_1 P_1(|\xi|^k g_2)}{M_*}\,d\xi - \int_{R^3} \frac{g_1|\xi|^k g_2}{M_*}\,d\xi\right| \leqslant C\int_{R^3} \frac{\lambda|g_1|^2+\lambda^{-1}|g_2|^2}{M_*}\,d\xi.$$

We are now ready to derive the lower order estimates. Multiplying $(3.6)_1$ by $p_+\Phi$, $(3.6)_2$ by $\bar{v}\Psi_1$, $(3.6)_3$ by Ψ_i, $(3.6)_4$ by $\frac{2}{3p_+}W$ respectively and adding all the resulting equations, we have

$$\begin{aligned}
&\left(\frac{p_+}{2}\Phi^2+\frac{1}{3p_+}W^2+\frac{\bar{v}}{2}\Psi_1^2+\frac{1}{2}\sum_{i=2}^3\Psi_i^2\right)_t \\
&\quad= -\frac{4\mu(\bar{\theta})}{3}\Psi_{1x}^2-\sum_{i=2}^3\frac{\mu(\bar{\theta})}{\bar{v}}\Psi_{ix}^2-\frac{2\lambda(\bar{\theta})}{3p_+\bar{v}}W_x^2+\frac{2}{5}\widetilde{N}_1\left(-\Phi+\frac{2}{3p_+}W\right)+\frac{1}{2}\bar{v}_t\Psi_1^2+\bar{v}Q_1\Psi_1 \\
&\qquad-\left(\frac{4\mu(\bar{\theta})}{3}\right)_x\Psi_1\Psi_{1x}-\sum_{i=2}^3\left(\frac{\mu(\bar{\theta})}{\bar{v}}\right)_x\Psi_i\Psi_{ix}-\left(\frac{2\lambda(\bar{\theta})}{3p_+\bar{v}}\right)_x WW_x+\sum_{i=2}^3 Q_i\Psi_i \\
&\qquad+\frac{2}{3p_+}WQ_4+(\cdots)_x.
\end{aligned}\tag{4.3}$$

Here and in the sequel the notation $(\cdots)_x$ represents the term in the conservative form so that it vanishes after integration. Since it has no effect on the energy estimates, we do not write them out in details for clear presentation.

Note that the term $Q_1\Psi_1$ contains $(1+t)^{-1}\Psi_1$ which cannot be controlled by the dissipation from the viscosity and heat conductivity. So is the term $\widetilde{N}_1(-\Phi+2W/(3p_+))$. As we will see later, an intrinsic dissipation associated with the contact discontinuity is derived by the diagonal method and weighted energy estimate to control the above two terms. Let us consider the equations for the conservation of the mass, the first component of velocity and energy by defining

$$m=(\Phi,\Psi_1,W)^{\mathrm{t}},\tag{4.4}$$

where $(\cdot,\cdot,\cdot)^{\mathrm{t}}$ means the transpose of the vector $(\cdot,\cdot,\cdot)$, then from (3.6), we have

$$m_t+A_1m_x=A_2m_{xx}+A_3,\tag{4.5}$$

where

$$A_1=\begin{pmatrix}0&-1&0\\-\frac{p_+}{\bar{v}}&0&\frac{2}{3\bar{v}}\\0&p_+&0\end{pmatrix},\qquad A_2=\begin{pmatrix}0&0&0\\0&\frac{4\mu(\bar{\theta})}{3\bar{v}}&0\\0&0&\frac{\lambda(\bar{\theta})}{\bar{v}}\end{pmatrix},\tag{4.6}$$

$$A_3=\left(-\frac{2}{5p_+}\widetilde{N}_1,\,Q_1,\,Q_4+\frac{2}{5}\widetilde{N}_1\right)^{\mathrm{t}}.\tag{4.7}$$

Direct computation shows that the eigenvalues of the matrix A_1 are $\lambda_1, 0, \lambda_3$. Here $\lambda_3 = -\lambda_1 = \sqrt{5p_+/(3\bar{v})}$. The corresponding normalized left and right eigenvectors can be chosen as

$$l_1 = \sqrt{3/10}\left(-1, -\frac{5}{3\lambda_3}, \frac{2}{3p_+}\right), \qquad l_2 = \sqrt{2/5}\left(1, 0, \frac{1}{p_+}\right),$$
$$l_3 = \sqrt{3/10}\left(-1, \frac{5}{3\lambda_3}, \frac{2}{3p_+}\right), \tag{4.8}$$

$$r_1 = \sqrt{3/10}(-1, -\lambda_3, p_+)^{\mathrm{t}}, \qquad r_2 = \sqrt{2/5}\left(1, 0, \frac{3}{2}p_+\right)^{\mathrm{t}},$$
$$r_3 = \sqrt{3/10}(-1, \lambda_3, p_+)^{\mathrm{t}}. \tag{4.9}$$

Hence,

$$l_i r_j = \delta_{ij}, \quad i, j = 1, 2, 3, \qquad LA_1R = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{4.10}$$

where

$$L = (l_1, l_2, l_3)^{\mathrm{t}}, \qquad R = (r_1, r_2, r_3).$$

Let

$$B = Lm = (b_1, b_2, b_3), \tag{4.11}$$

then multiplying Eqs. (4.5) by the matrix L yields that

$$B_t + \Lambda B_x = LA_2RB_{xx} + 2LA_2R_xB_x + \left[(L_t + \Lambda L_x)R + LA_2R_{xx}\right]B + LA_3. \tag{4.12}$$

A direct computation shows that

$$LA_2R = A_4 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{12} \\ b_{13} & b_{12} & b_{11} \end{pmatrix}, \tag{4.13}$$

with

$$\bar{v}b_{11} = \frac{2\mu(\bar{\theta})}{3} + \frac{1}{5}\lambda(\bar{\theta}), \qquad \bar{v}b_{12} = \frac{\sqrt{3}}{5}\lambda(\bar{\theta}), \tag{4.14}$$

$$\bar{v}b_{13} = -\frac{2\mu(\bar{\theta})}{3} + \frac{1}{5}\lambda(\bar{\theta}), \qquad \bar{v}b_{22} = \frac{3}{5}\lambda(\bar{\theta}). \tag{4.15}$$

It is easy to check that the symmetric matrix A_4 is nonnegative and its eigenvalues are $0, \frac{4\mu(\bar{\theta})}{3\bar{v}}, \frac{\lambda(\bar{\theta})}{\bar{v}}$. From (4.12), we shall use a weighted energy method to derive the intrinsic dissipation. For definiteness, we assume that $\theta_x^{ns} > 0$. The case when $\theta_x^{ns} < 0$ can be discussed similarly. Let $v_1 = \theta^{ns}/\theta_+$, then $|v_1 - 1| \leqslant C\delta$. Multiplying (4.12) by $\overline{B} = (v_1^n b_1, b_2, v_1^{-n} b_3)$ with a large positive integer n which will be chosen later, we have

$$\begin{aligned}
&\left(\frac{v_1^n}{2}b_1^2 + \frac{1}{2}b_2^2 + \frac{v_1^{-n}}{2}b_3^2\right)_t - \left(\frac{v_1^n}{2}\right)_t b_1^2 - \left(\frac{v_1^{-n}}{2}\right)_t b_3^2 + \overline{B}_x A_4 B_x + \overline{B} A_{4x} B_x \\
&\quad - \frac{v_1^{n-1}}{2}(n\lambda_1 v_{1x} + v_1\lambda_{1x})b_1^2 + \frac{v_1^{-n-1}}{2}(n\lambda_3 v_{1x} - v_1\lambda_{3x})b_3^2 + (\cdots)_x \\
&= 2\overline{B}LA_2R_xB_x + \overline{B}[L_tR + LA_2R_{xx}]B + \overline{B}\Lambda L_xRB + \overline{B}LA_3. \qquad (4.16)
\end{aligned}$$

Let

$$\begin{aligned}
E_1 &= \int \left(\frac{p_+}{2}\Phi^2 + \frac{1}{3p_+}W^2 + \frac{\bar{v}}{2}\Psi_1^2 + \frac{1}{2}\sum_{i=2}^{3}\Psi_i^2\right)dx \\
&\quad + \int \left(\frac{v_1^n}{2}b_1^2 + \frac{1}{2}b_2^2 + \frac{v_1^{-n}}{2}b_3^2\right)dx, \qquad (4.17)
\end{aligned}$$

$$K_1 = \int \left(\frac{4\mu(\bar{\theta})}{3}\Psi_{1x}^2 + \sum_{i=2}^{3}\frac{\mu(\bar{\theta})}{\bar{v}}\Psi_{ix}^2 + \frac{2\lambda(\bar{\theta})}{3p_+\bar{v}}W_x^2 + B_xA_4B_x\right)dx. \qquad (4.18)$$

Note that

$$\begin{aligned}
\left|\int (\overline{B} - B)_x A_4 B_x\,dx\right| &\leqslant C\delta \int |B_x|^2\,dx + C\delta \int |B|^2\left|\theta_x^{ns}\right|^2 dx \\
&\leqslant C\delta(1+t)^{-1}E_1 + C\delta K_1 + C\delta \int |\Phi_x|^2\,dx. \qquad (4.19)
\end{aligned}$$

Similarly, the terms in the last second line of (4.3), $\overline{B}A_{4x}B_x$, $\overline{B}LA_2R_xB_x$ and $\overline{B}[L_tR + LA_2R_{xx}]B$ satisfy the same estimate. For $\overline{B}\Lambda L_xRB$ and $\overline{B}LA_3$, we need to use the explicit presentation. By the choice of the characteristic matrix L and R, we have

$$\Lambda L_xR = \frac{1}{2}\lambda_{3x}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad (4.20)$$

$$LA_3 = \begin{pmatrix} \sqrt{\frac{2}{15}}\frac{1}{p_+}\left(\widetilde{N}_1 + Q_4\right) - \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3} \\ \sqrt{\frac{2}{5}}\frac{Q_4}{p_+} \\ \sqrt{\frac{2}{15}}\frac{1}{p_+}\left(\widetilde{N}_1 + Q_4\right) + \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3} \end{pmatrix}. \qquad (4.21)$$

Thus

$$\overline{B}\Lambda L_xRB = \frac{1}{2}\lambda_{3x}\left(v_1^n b_1^2 + v_1^{-n}b_1b_3 - v_1^n b_1b_3 - v_1^{-n}b_3^2\right), \qquad (4.22)$$

$$\overline{B}LA_3 = \sqrt{\frac{2}{15}}\frac{1}{p_+}\widetilde{N}_1\left(v_1^n b_1 + v_1^{-n}b_3\right) + q_1v_1^n b_1 + q_2b_2 + q_3v_1^{-n}b_3, \qquad (4.23)$$

where

$$q_1 = \sqrt{\frac{2}{15}}\frac{1}{p_+}Q_4 - \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3}, \qquad q_2 = \sqrt{\frac{2}{5}}\frac{Q_4}{p_+}, \qquad q_3 = \sqrt{\frac{2}{15}}\frac{1}{p_+}Q_4 + \sqrt{\frac{5}{6}}\frac{Q_1}{\lambda_3}. \quad (4.24)$$

Combine (4.3), (4.16), (4.19), (4.22), (4.23) and use the Cauchy inequality, we have by choosing n sufficiently large,

$$E_{1t} + \frac{1}{2}K_1 + \int |\theta_x^{ns}|\big(b_1^2 + b_3^2\big)\,dx \leqslant C\delta(1+t)^{-1}(E_1+1) + C\delta K_1 + C\delta\int \Phi_x^2\,dx + I_{nf}, \quad (4.25)$$

where

$$I_{nf} = \int \bar{v}Q_1\Psi_1\,dx + \int\sum_{i=2}^{3}Q_i\Psi_i\,dx + \int\frac{2}{3p_+}WQ_4\,dx + \int\big(q_1v_1^nb_1 + q_2b_2 + q_3v_1^{-n}b_3\big)\,dx. \quad (4.26)$$

Here we have used the fact that

$$-\Phi + \frac{2}{3p_+}W = \sqrt{5/6}(b_1+b_3), \quad (4.27)$$

and

$$\int |\widetilde{N}_1|\big(|b_1|+|b_3|\big)\,dx \leqslant C\delta\int|\theta_x^{ns}|\big(b_1^2+b_3^2\big)\,dx + C\delta(1+t)^{-1}, \quad (4.28)$$

and for n large enough,

$$-\frac{1}{2}v_1^{n-1}(n\lambda_1v_{1x} + 2v_1\lambda_{1x}) + \frac{1}{2}v_1^{-n-1}(n\lambda_3v_{1x} - 2v_1\lambda_{3x}) - \overline{B}\Lambda L_xRB > 2\theta_x^{ns}\big(b_1^2+b_3^2\big). \quad (4.29)$$

Even though Q_1 contains the term R_1 with the decay rate $\frac{1}{1+t}$, the terms in (4.26) involving Q_1 have factor b_1 or b_3 because

$$\Psi_1 = \sqrt{3/10}\,\lambda_3(b_3 - b_1). \quad (4.30)$$

Thus the terms $\bar{v}Q_1\Psi_1$, $q_1v_1^nb_1$ and $q_3v_1^{-n}b_3$ can be controlled by the intrinsic dissipation on b_1 and b_3 as shown later. The estimates on the other terms involving Q_i, $i = 2,3,4$, are straightforward because from (2.25)–(2.28) and (3.10), (3.11), the lowest decay terms of Q_i decay as $(1+t)^{-3/2}$. For brevity, we only estimate $\int\bar{v}Q_1\Psi_1\,dx$ and $\int q_2b_2\,dx$ as follows for illustration.

Estimation on $\int \bar{v} Q_1 \Psi_1 \, dx$

From (4.30), we have

$$\int \bar{v} Q_1 \Psi_1 \, dx = \sqrt{\frac{3}{10}} \int \bar{v} Q_1 \lambda_3 (b_3 - b_1) \, dx. \tag{4.31}$$

Here we only consider the integral

$$I_1 = \int \bar{v} Q_1 \lambda_3 b_1 \, dx, \tag{4.32}$$

and the other term in (4.31) can be estimated similarly. By the definition of Q_1 in (3.9), we have

$$\begin{aligned} I_1 &= \int \bar{v} \lambda_3 b_1 \left[\frac{4}{3} \left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1x} + J_1 + \frac{2}{3\bar{v}} Y \right] dx \\ &\quad - \int \bar{v} \lambda_3 b_1 R_1 \, dx - \int \bar{v} \lambda_3 b_1 \int \xi_1^2 \Theta_1 \, d\xi \, dx \\ &=: I_1^1 + I_1^2 + I_1^3. \end{aligned} \tag{4.33}$$

Since

$$\int |b_1 Y| \, dx \leqslant C(\delta + \varepsilon_0) \|\Psi_x\|^2 + C\delta(1+t)^{-1} E_1$$

and

$$\int |b_1 J_1| \, dx \leqslant C\varepsilon_0 \big(\|\Phi_x\|^2 + K_1 \big) + C\delta(1+t)^{-\frac{3}{2}},$$

from (3.7) and the Cauchy inequality, it is straightforward to show that

$$\left| I_1^1 \right| \leqslant C(\delta + \varepsilon_0) \big(K_1 + \|\Phi_x\|^2 \big) + C\delta(1+t)^{-1} E_1 + C\delta(1+t)^{-\frac{3}{2}} + C\varepsilon_0 \|\psi_{1x}\|^2. \tag{4.34}$$

On the other hand, from (2.25), we have

$$R_1 = O(\delta)(1+t)^{-1} e^{-\frac{x^2}{4c(\theta_\pm)(1+t)}}, \quad \text{as } |x| \to \infty.$$

From (2.3), θ_x^{ns} satisfies

$$\left| \theta_x^{ns} \right| = O(\delta)(1+t)^{-1} e^{-\frac{x^2}{4a(\theta_\pm)(1+t)}}, \quad \text{as } |x| \to \infty.$$

Thus, by (1.7) and the assumption on the weak contact wave, we have

$$\lambda(\theta_\pm) > \frac{5}{12} \mu(\theta_\pm). \tag{4.35}$$

(4.35) implies that $a(\theta_\pm) > (1/2)c(\theta_\pm)$ which leads to

$$\left|I_1^2\right| \leqslant C\delta \int \left|\theta_x^{ns}\right| b_1^2\, dx + C\delta(1+t)^{-1}. \tag{4.36}$$

We now estimate the integral I_1^3. Let M_* be a global Maxwellian with the state (ρ_*, u_*, θ_*) satisfying $1/2\theta < \theta_* < \theta$ and $|\rho - \rho_*| + |u - u_*| + |\theta - \theta_*| \leqslant \eta_0$ so that Lemma 4.2 holds. From (2.8),

$$I_1^3 = -\int \bar{v}\lambda_3 b_1 \int \xi_1^2 \Theta_1^1\, d\xi\, dx - \int \bar{v}\lambda_3 b_1 \int \xi_1^2 \Theta_1^2\, d\xi\, dx =: I_1^{31} + I_1^{32}. \tag{4.37}$$

The estimation on I_1^{31} is straightforward by using the intrinsic dissipation on b_1 and (2.8) as follows:

$$\begin{aligned}
\left|I_1^{31}\right| &= \left|\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\left[\frac{1}{v} P_1(\xi_1 \overline{G}_x) - Q(\overline{G}, \overline{G})\right] d\xi\, dx\right| \\
&\leqslant C \int |b_1|\left(\left|(\bar{u}_x, \bar{\theta}_x)\right|^2 + \left|(\bar{u}_{xx}, \bar{\theta}_{xx})\right| + \left|(\bar{u}_x, \bar{\theta}_x)\right|\left|(v_x, u_x, \theta_x)\right|\right) dx \\
&\leqslant C\delta \int \left|\theta_x^{ns}\right| b_1^2\, dx + C\delta(1+t)^{-1} + C\delta\left\|(\phi_x, \psi_x, \zeta_x)\right\|^2.
\end{aligned} \tag{4.38}$$

The estimation on I_1^{32} is more complicated and it can be done by dividing it into five parts as follows. From (2.8), we have

$$\begin{aligned}
I_1^{32} = &-\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}(G_t)\, d\xi\, dx + \int \bar{v}\lambda_3 b_1 \int \xi_1^2 \frac{u_1}{v} L_M^{-1}(G_x)\, d\xi\, dx \\
&- \int \bar{v}\lambda_3 b_1 \frac{1}{v} \int \xi_1^2 L_M^{-1}\left[P_1\left(\xi_1 \widetilde{G}_x\right)\right] d\xi\, dx + 2\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\left[Q(\widetilde{G}, \overline{G})\right] d\xi\, dx \\
&+ \int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\left[Q(\widetilde{G}, \widetilde{G})\right] d\xi\, dx =: \sum_{i=1}^{5} I_1^{32i}.
\end{aligned} \tag{4.39}$$

For the integral I_1^{321}, we have

$$\begin{aligned}
I_1^{321} &= -\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\left(\widetilde{G}_t\right) d\xi\, dx - \int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}(\overline{G}_t)\, d\xi\, dx \\
&=: I_1^{3211} + I_1^{3212}.
\end{aligned} \tag{4.40}$$

Note that the linearized operator L_M^{-1} satisfies, for any $h \in N^\perp$,

$$\begin{aligned}
\left(L_M^{-1} h\right)_t &= L_M^{-1}(h_t) - 2L_M^{-1}\left\{Q\left(L_M^{-1} h, M_t\right)\right\}, \\
\left(L_M^{-1} h\right)_x &= L_M^{-1}(h_x) - 2L_M^{-1}\left\{Q\left(L_M^{-1} h, M_x\right)\right\}.
\end{aligned} \tag{4.41}$$

Then we have

$$\begin{aligned} I_1^{3211} &= -\int \bar{v}\lambda_3 b_1 \int \xi_1^2 \big(L_M^{-1}\widetilde{G}\big)_t \, d\xi \, dx - 2\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\big\{Q\big(L_M^{-1}\widetilde{G}, M_t\big)\big\} \, d\xi \, dx \\ &= -\bigg(\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\widetilde{G} \, d\xi \, dx\bigg)_t + \int (\bar{v}\lambda_3 b_1)_t \int \xi_1^2 L_M^{-1}\widetilde{G} \, d\xi \, dx \\ &\quad - 2\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\big\{Q\big(L_M^{-1}\widetilde{G}, M_t\big)\big\} \, d\xi \, dx. \end{aligned} \tag{4.42}$$

The Hölder inequality and Lemma 4.3 yield

$$\bigg|\int \xi_1^2 L_M^{-1}\widetilde{G} \, d\xi\bigg|^2 \leqslant C\int \xi_1^4 \nu\big(|\xi|\big)^{-1} M_* \, d\xi \int \frac{\nu(|\xi|)}{M_*}\big|L_M^{-1}\widetilde{G}\big|^2 \, d\xi \leqslant C\int \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2 \, d\xi. \tag{4.43}$$

Moreover, from Lemmas 4.1–4.3, we have

$$\begin{aligned} &\bigg|\int \xi_1^2 L_M^{-1}\big\{Q\big(L_M^{-1}\widetilde{G}, M_t\big)\big\} \, d\xi\bigg|^2 \\ &\quad \leqslant C\int \frac{\nu(|\xi|)}{M_*}\big|L_M^{-1}\big\{Q\big(L_M^{-1}\widetilde{G}, M_t\big)\big\}\big|^2 \, d\xi \\ &\quad \leqslant C\int \frac{\nu(|\xi|)^{-1}}{M_*}\big|Q\big(L_M^{-1}\widetilde{G}, M_t\big)\big|^2 \, d\xi \leqslant C\int \frac{\nu(|\xi|)}{M_*}\big|L_M^{-1}\widetilde{G}\big|^2 \, d\xi \int \frac{\nu(|\xi|)}{M_*}|M_t|^2 \, d\xi \\ &\quad \leqslant C\big(v_t^2 + u_t^2 + \theta_t^2\big)\int \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2 \, d\xi. \end{aligned} \tag{4.44}$$

Combining (4.22)–(4.44) gives

$$\begin{aligned} I_1^{3211} &\leqslant -\bigg(\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\widetilde{G} \, d\xi \, dx\bigg)_t + C\delta(1+t)^{-\frac{3}{2}} + C\varepsilon_1\int |b_{1t}|^2 \, dx \\ &\quad + C_{\varepsilon_1}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2 \, d\xi \, dx + C\varepsilon_0\big\|(\phi_t, \psi_t, \zeta_t)\big\|^2, \end{aligned} \tag{4.45}$$

where ε_1 is small positive constant chosen later. By the definition of $\overline{G}$ in (2.8), similar to the estimate in (4.38), we have

$$\begin{aligned} \big|I_1^{3212}\big| &= \bigg|\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}(\overline{G}_t) \, d\xi \, dx\bigg| \leqslant C\int |b_1|\big(\big|(\bar{u}_{xt}, \bar{\theta}_{xt})\big| + \big|(\bar{u}_x, \bar{\theta}_x)\big|\big|(v_t, u_t, \theta_t)\big|\big) \, dx \\ &\leqslant C\delta(1+t)^{-1}E_1 + C\delta(1+t)^{-\frac{3}{2}} + C\delta\big\|(\phi_t, \psi_t, \zeta_t)\big\|^2. \end{aligned} \tag{4.46}$$

Therefore, (4.45) and (4.46) imply

$$I_1^{321} \leqslant -\left(\int \bar{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\widetilde{G}\,d\xi\,dx\right)_t + C\delta(1+t)^{-1}E_1 + C\varepsilon_1 \int |b_{1t}|^2\,dx$$
$$+ C_{\varepsilon_1}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2\,d\xi\,dx + C\delta(1+t)^{-\frac{3}{2}} + C(\delta+\varepsilon_0)\big\|(\phi_t,\psi_t,\zeta_t)\big\|^2. \quad (4.47)$$

The estimates for I_1^{32i}, $i=2,4,5$, are relatively easy by using the Cauchy inequality and Lemmas 4.1–4.3 which are given as follows. First, we have

$$\big|I_1^{322}\big| \leqslant C\iint \frac{\nu(|\xi|)}{M_*}|G_x|^2\,d\xi\,dx + C\int b_1^2 u_1^2\,dx$$
$$\leqslant C\delta(1+t)^{-1}E_1 + C\varepsilon_0 K_1 + C\iint \frac{\nu(|\xi|)}{M_*}|G_x|^2\,d\xi\,dx. \quad (4.48)$$

Since

$$\left|\int \xi_1^2 L_M^{-1}\big\{Q\big(\widetilde{G},\overline{G}\big)\big\}\,d\xi\right|^2$$
$$\leqslant C\int \frac{\nu(|\xi|)}{M_*}\big|L_M^{-1}\big\{Q(\widetilde{G},\overline{G})\big\}\big|^2\,d\xi \leqslant C\int \frac{\nu(|\xi|)^{-1}}{M_*}\big|Q(\widetilde{G},\overline{G})\big|^2\,d\xi$$
$$\leqslant C\int \frac{\nu(|\xi|)}{M_*}\big|L_M^{-1}\widetilde{G}\big|^2\,d\xi \int \frac{\nu(|\xi|)}{M_*}|\overline{G}|^2\,d\xi \leqslant C\big|(\bar{u}_x,\bar{\theta}_x)\big|^2\int \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2\,d\xi, \quad (4.49)$$

and

$$\left|\int \xi_1^2 L_M^{-1}\big\{Q(\widetilde{G},\widetilde{G})\big\}\,d\xi\right| \leqslant C\left(\int \frac{\nu(|\xi|)}{M_*}\big|L_M^{-1}\big\{Q(\widetilde{G},\widetilde{G})\big\}\big|^2\,d\xi\right)^{1/2}$$
$$\leqslant C\left(\int \frac{\nu(|\xi|)^{-1}}{M_*}\big|Q(\widetilde{G},\widetilde{G})\big|^2\,d\xi\right)^{1/2} \leqslant C\int \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2\,d\xi, \quad (4.50)$$

it is straightforward to show that

$$\big|I_1^{324}\big| + \big|I_1^{325}\big| \leqslant C(\delta+\varepsilon_0)\iint \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2\,d\xi\,dx + C\delta(1+t)^{-1}E_1. \quad (4.51)$$

The estimate on I_1^{323} is similar to the one for I_1^{321}. First,

$$P_1(\xi_1\widetilde{G}_x) = \big\{P_1(\xi_1\widetilde{G})\big\}_x + \sum_{j=0}^{4}\langle\xi_1\widetilde{G},\chi_j\rangle P_1(\chi_{jx}). \quad (4.52)$$

From (4.41) and Lemmas 4.1–4.4, we have

$$
\begin{aligned}
I_1^{323} &= \int \left(\frac{\bar v}{v}\lambda_3 b_1\right)_x \int \xi_1^2 L_M^{-1}\big[P_1(\xi_1\widetilde G)\big]\,d\xi\,dx \\
&\quad - \int \frac{\bar v}{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\left[\sum_{j=0}^{4}\langle \xi_1\widetilde G,\chi_j\rangle P_1(\chi_{jx})\right]d\xi\,dx \\
&\quad - 2\int \frac{\bar v}{v}\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\big\{Q\big(L_M^{-1}\big[P_1(\xi_1\widetilde G)\big], M_x\big)\big\}\,d\xi\,dx \\
&\leqslant C_{\varepsilon_1}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde G|^2\,d\xi\,dx + C\delta(1+t)^{-1}E_1 + C(\varepsilon_0+\varepsilon_1)\big(K_1+\|\Phi_x\|^2\big) \\
&\quad + C\varepsilon_0\big\|(\phi_x,\psi_x,\zeta_x)\big\|^2, \qquad (4.53)
\end{aligned}
$$

where we have used the fact that

$$
\big|\langle \xi_1\widetilde G,\chi_j\rangle\big|^2 \leqslant C\int \frac{\nu(|\xi|)\widetilde G^2}{M_*}\,d\xi.
$$

By (4.39), (4.47), (4.48), (4.51) and (4.53), we have

$$
\begin{aligned}
I_1^{32} &\leqslant -\left(\int \bar v\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\widetilde G\,d\xi\,dx\right)_t + C\delta(1+t)^{-1}E_1 + C\varepsilon_1\int |b_{1t}|^2\,dx \\
&\quad + C(\varepsilon_0+\varepsilon_1)\big(K_1+\|\Phi_x\|^2\big) + C_{\varepsilon_1}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde G|^2\,d\xi\,dx + C\iint \frac{\nu(|\xi|)}{M_*}|G_x|^2\,d\xi\,dx \\
&\quad + C(\delta+\varepsilon_0)\sum_{|\alpha|=1}\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2 + C\delta(1+t)^{-\frac{3}{2}}, \qquad (4.54)
\end{aligned}
$$

which implies by (4.37) and (4.38) that

$$
\begin{aligned}
I_1^{3} &\leqslant -\left(\int \bar v\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\widetilde G\,d\xi\,dx\right)_t + C\delta(1+t)^{-1}(E_1+1) \\
&\quad + C\varepsilon_1\int |b_{1t}|^2\,dx + C_{\varepsilon_1}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde G|^2\,d\xi\,dx + C\delta\int \big|\theta_x^{ns}\big| b_1^2\,dx \\
&\quad + C\iint \frac{\nu(|\xi|)}{M_*}|G_x|^2\,d\xi\,dx + C(\varepsilon_0+\varepsilon_1)\big(K_1+\|\Phi_x\|^2\big) \\
&\quad + C(\delta+\varepsilon_0)\sum_{|\alpha|=1}\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2. \qquad (4.55)
\end{aligned}
$$

And finally, (4.33), (4.34), (4.36) and (4.55) yield the estimate on I_1,

$$
\begin{aligned}
I_1 &\leqslant -\left(\int \bar v\lambda_3 b_1 \int \xi_1^2 L_M^{-1}\widetilde G\,d\xi\,dx\right)_t + C\delta(1+t)^{-1}(E_1+1) \\
&\quad + C(\delta+\varepsilon_0+\varepsilon_1)\big(K_1+\|\Phi_x\|^2\big) + C\varepsilon_1\int |b_{1t}|^2\,dx + C_{\varepsilon_1}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde G|^2\,d\xi\,dx
\end{aligned}
$$

$$+ C\delta \int \left|\theta_x^{ns}\right| b_1^2\, dx + C \iint \frac{\nu(|\xi|)}{M_*} |G_x|^2\, d\xi\, dx + C(\delta + \varepsilon_0) \sum_{|\alpha|=1} \left\| \partial^\alpha(\phi, \psi, \zeta)\right\|^2. \quad (4.56)$$

And this completes the discussion on the term $\int \bar{v} Q_1 \Psi_1\, dx$.

Estimation on $\int q_2 b_2\, dx$

Notice that the profile has no intrinsic dissipation on b_2. Fortunately, $q_2 = \sqrt{2/5}\, Q_4/p_+$ and the lowest decay terms of Q_4 decay as $(1+t)^{-3/2}$. Thus the estimation on $\int q_2 b_2\, dx$ can be directly obtained even though there is no intrinsic dissipation on b_2. For example,

$$\left| \int \bar{u}_1 R_1 b_2\, dx \right| \leqslant C\delta(1+t)^{-1} E_1 + C\delta(1+t)^{-\frac{3}{2}},$$

$$\left| \iint \bar{u}_1 b_2 \xi_1^2 \Theta_1\, d\xi\, dx \right| \leqslant C\delta(1+t)^{-1} E_1 + C\delta(1+t)^{-\frac{3}{2}} + C(\delta + \varepsilon_0) \iint \frac{\nu(|\xi|)\widetilde{G}^2}{M_*}\, d\xi\, dx$$
$$+ C \iint \frac{\nu(|\xi|)}{M_*} \left(G_t^2 + G_x^2\right) d\xi\, dx. \quad (4.57)$$

And the term $\iint \xi_1 |\xi|^2 \Theta_1^2 b_2\, d\xi\, dx$ can be done by the same estimate for I_1^{32}, in which the intrinsic dissipation on b_1 is not used. Notice also that all the other terms in q_2 are of higher order terms. Therefore, we have

$$I_2 = \int q_2 b_2\, d\xi\, dx$$
$$\leqslant \left(\iint \hat{A}(\xi, B) L_M^{-1} \widetilde{G}\, d\xi\, dx \right)_t + C\delta(1+t)^{-1} E_1 + C\delta(1+t)^{-\frac{3}{2}}$$
$$+ C(\delta + \varepsilon_0 + \varepsilon_1)\left(K_1 + \|\Phi_x\|^2\right) + C\varepsilon_1 \int |b_{2t}|^2\, dx + C_{\varepsilon_1} \iint \frac{\nu(|\xi|)}{M_*} |\widetilde{G}|^2\, d\xi\, dx$$
$$+ C \iint \frac{\nu(|\xi|)}{M_*} \left(|G_x|^2 + |G_t|^2\right) d\xi\, dx + C(\delta + \varepsilon_0) \sum_{|\alpha|=1} \left\| \partial^\alpha(\phi, \psi, \zeta)\right\|^2, \quad (4.58)$$

where $\hat{A}$ is a linear function of b_2 and a polynomial of ξ. Using (4.25), (4.56) and (4.58), we get

$$E_{1t} + \left(\iint \hat{A}_1(\xi, B) L_M^{-1} \widetilde{G}\, d\xi\, dx \right)_t + \frac{1}{4} K_1 + \int \left|\theta_x^{ns}\right| \left(b_1^2 + b_3^2\right) dx$$
$$\leqslant C_1\delta(1+t)^{-1}(E_1 + 1) + C_1(\delta + \varepsilon_0 + \varepsilon_1)\left(\left\|(\Phi_t, \Psi_t, W_t)\right\|^2 + \|\Phi_x\|^2 \right)$$
$$+ C_{\varepsilon_1} \iint \frac{\nu(|\xi|)}{M_*} |\widetilde{G}|^2\, d\xi\, dx + C_1 \iint \frac{\nu(|\xi|)}{M_*} \left(|G_x|^2 + |G_t|^2\right) d\xi\, dx$$
$$+ C_1(\delta + \varepsilon_0) \sum_{|\alpha|=1} \left\| \partial^\alpha(\phi, \psi, \zeta)\right\|^2, \quad (4.59)$$

where we have used the smallness of δ and ε_0. Here $\hat{A}_1$ is a linear function of B and a polynomial of ξ.

Note that K_1 does not contain the norm $\|\Phi_x\|$. To complete the lower order inequality, we have to estimate Φ_x. From $(3.6)_2$, we have

$$\frac{4\mu(\bar{\theta})}{3\bar{v}}\Phi_{xt} - \Psi_{1t} + \frac{p_+}{\bar{v}}\Phi_x = \frac{2}{3\bar{v}}W_x - \frac{8\mu(\bar{\theta})}{15p_+\bar{v}}\widetilde{N}_{1x} - Q_1. \tag{4.60}$$

Multiplying (4.60) by Φ_x yields

$$\left(\frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_x^2\right)_t - \left(\frac{2\mu(\bar{\theta})}{3\bar{v}}\right)_t\Phi_x^2 - \Phi_x\Psi_{1t} + \frac{p_+}{\bar{v}}\Phi_x^2 = \left(\frac{2}{3\bar{v}}W_x - \frac{8\mu(\bar{\theta})}{15p_+\bar{v}}\widetilde{N}_{1x} - Q_1\right)\Phi_x. \tag{4.61}$$

Since

$$\Phi_x\Psi_{1t} = (\Phi_x\Psi_1)_t - (\Phi_t\Psi_1)_x + \Psi_{1x}^2 - \frac{2}{5p_+}\widetilde{N}_1\Psi_{1x}, \tag{4.62}$$

we obtain

$$\begin{aligned}
&\left(\int \frac{2\mu(\bar{\theta})}{3\bar{v}}\Phi_x^2 - \Phi_x\Psi_1\,dx\right)_t + \int \frac{p_+}{2\bar{v}}\Phi_x^2\,dx \\
&\quad\leqslant C\int \left(\Psi_{1x}^2 + W_x^2\right)dx + C\delta(1+t)^{-\frac{3}{2}} + \int Q_1^2\,dx.
\end{aligned} \tag{4.63}$$

The Q_1 formula (3.9) and the Cauchy inequality directly yield

$$\begin{aligned}
\int Q_1^2\,dx &\leqslant C\varepsilon_0\big(K_1 + \|\Phi_x\|^2\big) + C\delta(1+t)^{-\frac{3}{2}} + C\varepsilon_0\sum_{|\alpha|=1}\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2 \\
&\quad + C\int\left|\int \xi_1^2\Theta_1\,d\xi\right|^2 d\xi\,dx.
\end{aligned} \tag{4.64}$$

And using Lemmas 4.1–4.3 implies

$$\begin{aligned}
&\int\left|\int \xi_1^2\Theta_1\,d\xi\right|^2 d\xi\,dx \\
&\quad\leqslant C\iint \frac{\nu(|\xi|)}{M_*}\big(|G_x|^2 + |G_t|^2\big)\,d\xi\,dx + C\int |\bar{\theta}_x|^4\,dx + C(\delta+\varepsilon_0)\iint \frac{\nu(|\xi|)|\widetilde{G}|^2}{M_*}\,d\xi\,dx \\
&\quad\leqslant C(\delta+\varepsilon_0)\iint \frac{\nu(|\xi|)|\widetilde{G}|^2}{M_*}\,d\xi\,dx + C\iint \frac{\nu(|\xi|)}{M_*}\big(|G_x|^2 + |G_t|^2\big)\,d\xi\,dx + C\delta(1+t)^{-\frac{3}{2}}.
\end{aligned} \tag{4.65}$$

Plugging (4.64) and (4.65) into (4.63) yields

$$\left(\int \frac{2\mu(\bar\theta)}{3\bar v}\Phi_x^2-\Phi_x\Psi_1\,dx\right)_t+\int \frac{p_+}{4\bar v}\Phi_x^2\,dx$$
$$\leqslant C_2K_1+C_2\delta(1+t)^{-\frac32}+C_2\iint\frac{\nu(|\xi|)}{M_*}\big(|G_x|^2+|G_t|^2\big)\,d\xi\,dx$$
$$+C_2(\delta+\varepsilon_0)\iint\frac{\nu(|\xi|)|\widetilde G|^2}{M_*}+C_2(\delta+\varepsilon_0)\sum_{|\alpha|=1}\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2. \tag{4.66}$$

The microscopic component $\widetilde G$ can be estimated through Eq. (2.7). Multiplying (2.7) by $\widetilde G/M_*$, we get

$$\left(\frac{\widetilde G^2}{2M_*}\right)_t-\frac{\widetilde G}{M_*}L_M\widetilde G=\left\{-\frac{1}{Rv\theta}P_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}(\theta-\bar\theta)_x+\xi\cdot(u-\bar u)_x\right)M\right]\right.$$
$$\left.+\frac{u_1}{v}G_x-\frac1v P_1(\xi_1G_x)+Q(G,G)-\overline G_t\right\}\cdot\frac{\widetilde G}{M_*}. \tag{4.67}$$

Integrating (4.67) with respect to ξ and x and using the Cauchy inequality and Lemmas 4.1–4.4, we have

$$\left(\iint\frac{\widetilde G^2}{2M_*}\,d\xi\,dx\right)_t+\frac{\bar\sigma}{2}\iint\frac{\nu(|\xi|)\widetilde G^2}{M_*}\,d\xi\,dx$$
$$\leqslant C_3\delta(1+t)^{-\frac32}+C_3\big(\|\psi_x\|^2+\|\zeta_x\|^2\big)+C_3\iint\frac{\nu(|\xi|)G_x^2}{M_*}\,d\xi\,dx. \tag{4.68}$$

On the other hand, since $(\Phi,\Psi,W)_t$ can be represented by $(\Phi,\Psi,W)_x$ and $(\Phi,\Psi,W)_{xx}$ from Eq. (3.6), we can get an estimate for (Φ_t,Ψ_t,W_t) as follows:

$$\int\big|(\Phi,\Psi,W)_t\big|^2\,dx$$
$$\leqslant C_4K_1+C_4\int|\Phi_x|^2\,dx+C_4\sum_{|\alpha|=1}\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2+C_4\delta(1+t)^{-\frac32}$$
$$+C_4\iint\frac{\nu(|\xi|)|\widetilde G|^2}{M_*}\,d\xi\,dx+C_4\iint\frac{\nu(|\xi|)}{M_*}\big(|G_x|^2+|G_t|^2\big)\,d\xi\,dx. \tag{4.69}$$

We now complete the lower order estimate. Since $\hat A_1$ is a linear function of the vector B and a polynomial of ξ, we get

$$\left|\iint\hat A_1(\xi,B)L_M^{-1}\widetilde G\,d\xi\,dx\right|\leqslant\frac14E_1+C\iint\frac{\widetilde G^2}{M_*}\,d\xi\,dx. \tag{4.70}$$

We choose large constants $\overline C_1>1$, $\overline C_2>1$, $\overline C_3>1$ and small constant ε_1 so that

$$\overline{C}_1E_1+\overline{C}_1\iint \hat{A}_1L_M^{-1}\widetilde{G}\,d\xi\,dx+\overline{C}_2\int \frac{2\mu(\bar\theta)}{3\bar v}\Phi_x^2-\Phi_x\Psi_1\,dx+\overline{C}_3\iint \frac{\widetilde{G}^2}{2M_*}\,d\xi\,dx$$
$$\geqslant \frac{1}{2}\overline{C}_1E_1+\overline{C}_2\int \frac{\mu(\bar\theta)}{3\bar v}\Phi_x^2\,dx+\frac{\overline{C}_3}{4}\iint \frac{\widetilde{G}^2}{M_*}\,d\xi\,dx, \tag{4.71}$$
$$\left(\frac{\overline{C}_1}{4}-C_2\overline{C}_2-\overline{C}_1C_1\varepsilon_1C_4\right)K_1+\int\left(\overline{C}_2\frac{p_+}{4\bar v}-\overline{C}_1C_1\varepsilon_1(1+C_4)\right)\Phi_x^2\,dx$$
$$>\frac{\overline{C}_1}{8}K_1+\overline{C}_2\int \frac{p_+}{8\bar v}\Phi_x^2\,dx \tag{4.72}$$

and

$$\frac{\bar\sigma}{2}\overline{C}_3-\overline{C}_1C_1\varepsilon_1C_4-C_{\varepsilon_1}\overline{C}_1>\frac{\bar\sigma}{4}\overline{C}_3. \tag{4.73}$$

Hence, by multiplying (4.59) by $\overline{C}_1$, (4.66) by $\overline{C}_2$, (4.68) by $\overline{C}_3$, (4.69) by $C_1(\delta+\varepsilon_0+\varepsilon_1)\overline{C}_1$ and adding all these inequalities together, we have

$$E_{2t}+K_2+\int\left|\theta_x^{ns}\right|\left(b_1^2+b_3^2\right)dx$$
$$\leqslant C_5\delta(1+t)^{-1}(E_2+1)+C_5\iint \frac{\nu(|\xi|)}{M_*}\left(|G_x|^2+|G_t|^2\right)d\xi\,dx$$
$$+C_5\sum_{|\alpha|=1}\left\|\partial^\alpha(\phi,\psi,\zeta)\right\|^2, \tag{4.74}$$

where

$$E_2=\overline{C}_1E_1+\overline{C}_1\iint \hat{A}_1L_M^{-1}\widetilde{G}\,d\xi\,dx+\overline{C}_2\int \frac{2\mu(\bar\theta)}{3\bar v}\Phi_x^2-\Phi_x\Psi_1\,dx$$
$$+\overline{C}_3\iint \frac{\widetilde{G}^2}{2M_*}\,d\xi\,dx, \tag{4.75}$$
$$K_2=\frac{\overline{C}_1}{8}K_1+\overline{C}_2\int \frac{p_+}{8\bar v}\Phi_x^2\,dx+\left\|(\Phi,\Psi,W)_t\right\|^2+\frac{\bar\sigma}{4}\overline{C}_3\iint \frac{\widetilde{G}^2}{M_*}\,d\xi\,dx. \tag{4.76}$$

5. Derivative estimate

To obtain the estimate for the first order derivative of (Φ_x,Ψ_x,W_x). We shall use the convex entropy based on the macroscopic version of the Boltzmann equations. From (1.10) and (2.24), we have

$$\begin{cases}\phi_t-\psi_{1x}=-\frac{2}{5p_+}\widetilde{N}_{1x},\\ \psi_{1t}+(p-\bar p)_x=\frac{4}{3}\left(\frac{\mu(\theta)}{v}u_{1x}-\frac{\mu(\bar\theta)}{\bar v}\bar u_{1x}\right)_x+Q_5,\\ \psi_{it}=\left(\frac{\mu(\theta)}{v}u_{ix}-\frac{\mu(\bar\theta)}{\bar v}\bar u_{ix}\right)_x+Q_{i+4},\quad i=2,3,\\ \zeta_t+pu_{1x}-\bar p\bar u_{1x}=\left(\frac{\lambda(\theta)}{v}\theta_x-\frac{\lambda(\bar\theta)}{\bar v}\bar\theta_x\right)_x+\frac{2}{5}\widetilde{N}_{1x}+Q_8,\end{cases} \tag{5.1}$$

where

$$Q_5 = -\int \xi_1^2 \Theta_{1x}\, d\xi - R_{1x}, \tag{5.2}$$

$$Q_{i+4} = -\int \xi_1\xi_i \Theta_{1x}\, d\xi - \overline{N}_{ix} - \overline{R}_{ix}, \quad i = 2, 3, \tag{5.3}$$

$$\begin{aligned} Q_8 = {} & \frac{4}{3}\frac{\mu(\theta)}{v}u_{1x}^2 + \sum_{i=2}^{3}\frac{\mu(\theta)}{v}u_{ix}^2 + \sum_{i=1}^{3} u_i \int \xi_1\xi_i\Theta_{1x}\, d\xi - \frac{1}{2}\int \xi_1|\xi|^2\Theta_{1x}\, d\xi \\ & - \overline{N}_{1x} - \overline{H}_{1x} - R_{4x} + \frac{1}{2}\big(|\bar{u}|^2\big)_t + \bar{p}_x\bar{u}_1. \end{aligned} \tag{5.4}$$

Multiplying $(5.1)_2$ by ψ_1, $(5.1)_3$ by ψ_i, we have

$$\begin{aligned} & \left(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2\right)_t - (p - \bar{p})\psi_{1x} + \frac{4}{3}\left(\frac{\mu(\theta)}{v}u_{1x} - \frac{\mu(\bar\theta)}{\bar v}\bar{u}_{1x}\right)\psi_{1x} + \left(\frac{\mu(\theta)}{v}u_{ix} - \frac{\mu(\bar\theta)}{\bar v}\bar{u}_{ix}\right)\psi_{ix} \\ & \quad = \sum_{i=1}^{3} Q_{i+4}\psi_i + (\cdots)_x. \end{aligned} \tag{5.5}$$

Since $p - \bar{p} = R\bar\theta(\frac{1}{v} - \frac{1}{\bar v}) + \frac{R\zeta}{v}$, we get

$$\begin{aligned} & \left(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2\right)_t - R\bar\theta\left(\frac{1}{v} - \frac{1}{\bar v}\right)\phi_t - \frac{R}{v}\zeta\psi_{1x} + \frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1x}^2 \\ & \qquad + \sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{ix}^2 + \frac{4}{3}\left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar\theta)}{\bar v}\right)\bar{u}_{1x}\psi_{1x} + \sum_{i=2}^{3}\left(\frac{\mu(\theta)}{v} - \frac{\mu(\bar\theta)}{\bar v}\right)\bar{u}_{ix}\psi_{ix} \\ & \quad = \sum_{i=1}^{3} Q_{i+4}\psi_i + \frac{2R}{5p_+}\bar\theta\left(\frac{1}{v} - \frac{1}{\bar v}\right)\widetilde{N}_{1x} + (\cdots)_x. \end{aligned} \tag{5.6}$$

Let

$$\widehat{\Phi}(s) = s - 1 - \ln s. \tag{5.7}$$

It is easy to check that $\widehat{\Phi}'(1) = 0$ and $\widehat{\Phi}(s)$ is strictly convex around $s = 1$. Moreover,

$$\begin{aligned} \left\{R\bar\theta\widehat{\Phi}\left(\frac{v}{\bar v}\right)\right\}_t &= R\bar\theta_t\widehat{\Phi}\left(\frac{v}{\bar v}\right) + R\bar\theta\left(-\frac{1}{v} + \frac{1}{\bar v}\right)\phi_t + R\bar\theta\left(-\frac{v}{\bar v^2} + \frac{1}{\bar v}\right)\bar{v}_t + R\bar\theta\left(-\frac{1}{v} + \frac{1}{\bar v}\right)\bar{v}_t \\ &= R\bar\theta\left(-\frac{1}{v} + \frac{1}{\bar v}\right)\phi_t - \bar{p}\widehat{\Psi}\left(\frac{v}{\bar v}\right)\bar{v}_t + \bar{v}\bar{p}_t\widehat{\Phi}\left(\frac{v}{\bar v}\right), \end{aligned} \tag{5.8}$$

where

$$\widehat{\Psi}(s) = s^{-1} - 1 + \ln s. \tag{5.9}$$

Substituting (5.8) into (5.6) yields

$$\begin{aligned}
&\left(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2+R\bar{\theta}\widehat{\Phi}\left(\frac{v}{\bar{v}}\right)\right)_t+\bar{p}\widehat{\Psi}\left(\frac{v}{\bar{v}}\right)\bar{v}_t-\frac{R}{v}\zeta\psi_{1x}+\frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1x}^2\\
&\qquad+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{ix}^2+\frac{4}{3}\left(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}}\right)\bar{u}_{1x}\psi_{1x}+\sum_{i=2}^{3}\left(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}}\right)\bar{u}_{ix}\psi_{ix}\\
&\quad=\sum_{i=1}^{3}Q_{i+4}\psi_i+\frac{2R}{5p_+}\bar{\theta}\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)\widetilde{N}_{1x}+\bar{v}\bar{p}_t\widehat{\Phi}\left(\frac{v}{\bar{v}}\right)+(\cdots)_x.
\end{aligned}\tag{5.10}$$

On the other hand, we calculate

$$\left[\bar{\theta}\widehat{\Phi}\left(\frac{\theta}{\bar{\theta}}\right)\right]_t=\left(1-\frac{\bar{\theta}}{\theta}\right)\zeta_t-\widehat{\Psi}\left(\frac{\theta}{\bar{\theta}}\right)\bar{\theta}_t\tag{5.11}$$

and

$$\begin{aligned}
&\left(1-\frac{\bar{\theta}}{\theta}\right)\zeta_t\\
&\quad=\left(1-\frac{\bar{\theta}}{\theta}\right)\left\{-pu_{1x}+\bar{p}\bar{u}_{1x}+\left(\frac{\lambda(\theta)\theta_x}{v}-\frac{\lambda(\bar{\theta})\bar{\theta}_x}{\bar{v}}\right)_x+\frac{2}{5}\widetilde{N}_{1x}+Q_8\right\}\\
&\quad=-\frac{R}{v}\zeta\psi_{1x}+\frac{\zeta}{\theta}(\bar{p}-p)\bar{u}_{1x}-\left(\frac{\zeta}{\theta}\right)_x\left(\frac{\lambda(\theta)\theta_x}{v}-\frac{\lambda(\bar{\theta})\bar{\theta}_x}{\bar{v}}\right)+\frac{2}{5}\frac{\zeta}{\theta}\widetilde{N}_{1x}+\frac{\zeta}{\theta}Q_8+(\cdots)_x\\
&\quad=-\frac{R}{v}\zeta\psi_{1x}+\frac{\zeta}{\theta}(\bar{p}-p)\bar{u}_{1x}-\frac{\lambda(\theta)}{v\theta}\zeta_x^2-\frac{\zeta_x}{\theta}\left(\frac{\lambda(\theta)}{v}-\frac{\lambda(\bar{\theta})}{\bar{v}}\right)\bar{\theta}_x\\
&\qquad+\frac{\zeta\theta_x}{\theta^2}\left(\frac{\lambda(\theta)\theta_x}{v}-\frac{\lambda(\bar{\theta})\bar{\theta}_x}{\bar{v}}\right)+\frac{2}{5}\frac{\zeta}{\theta}\widetilde{N}_{1x}+\frac{\zeta}{\theta}Q_8+(\cdots)_x.
\end{aligned}\tag{5.12}$$

Substituting (5.11) and (5.12) into (5.10) gives

$$\begin{aligned}
&\left(\frac{1}{2}\sum_{i=1}^{3}\psi_i^2+R\bar{\theta}\widehat{\Phi}\left(\frac{v}{\bar{v}}\right)+\bar{\theta}\widehat{\Phi}\left(\frac{\theta}{\bar{\theta}}\right)\right)_t+\frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1x}^2+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{ix}^2+\frac{\lambda(\theta)}{v\theta}\zeta_x^2\\
&\quad=-\bar{p}\widehat{\Psi}\left(\frac{v}{\bar{v}}\right)\bar{v}_t+\bar{v}\bar{p}_t\widehat{\Phi}\left(\frac{v}{\bar{v}}\right)-\frac{4}{3}\left(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}}\right)\bar{u}_{1x}\psi_{1x}-\sum_{i=2}^{3}\left(\frac{\mu(\theta)}{v}-\frac{\mu(\bar{\theta})}{\bar{v}}\right)\bar{u}_{ix}\psi_{ix}\\
&\qquad+\sum_{i=1}^{3}Q_{i+4}\psi_i+\frac{2R}{5p_+}\bar{\theta}\left(\frac{1}{v}-\frac{1}{\bar{v}}\right)\widetilde{N}_{1x}+\frac{\zeta}{\theta}(\bar{p}-p)\bar{u}_{1x}-\frac{\zeta_x}{\theta}\left(\frac{\lambda(\theta)}{v}-\frac{\lambda(\bar{\theta})}{\bar{v}}\right)\bar{\theta}_x\\
&\qquad+\frac{\zeta\theta_x}{\theta^2}\left(\frac{\lambda(\theta)\theta_x}{v}-\frac{\lambda(\bar{\theta})\bar{\theta}_x}{\bar{v}}\right)+\frac{2}{5}\frac{\zeta}{\theta}\widetilde{N}_{1x}+\frac{\zeta}{\theta}Q_8-\widehat{\Psi}\left(\frac{\theta}{\bar{\theta}}\right)\bar{\theta}_t+(\cdots)_x.
\end{aligned}\tag{5.13}$$

Let

$$E_3 = \int \frac{1}{2}\sum_{i=1}^{3}\psi_i^2 + R\bar{\theta}\widehat{\Phi}\left(\frac{v}{\bar{v}}\right) + \bar{\theta}\widehat{\Phi}\left(\frac{\theta}{\bar{\theta}}\right)dx \tag{5.14}$$

and

$$K_3 = \int \frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1x}^2 + \sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{ix}^2 + \frac{\lambda(\theta)}{v\theta}\zeta_x^2\,dx. \tag{5.15}$$

Note that $\widehat{\Phi}(s)$ is strictly convex around $s = 1$ so that there exist positive constants c_1 and c_2,

$$c_1\phi^2 \leqslant \widehat{\Phi}\left(\frac{v}{\bar{v}}\right) \leqslant c_2\phi^2, \qquad c_1\zeta^2 \leqslant \widehat{\Phi}\left(\frac{\theta}{\bar{\theta}}\right) \leqslant c_2\zeta^2. \tag{5.16}$$

$\widehat{\Psi}(s)$ is also convex around $s = 1$ and this leads to

$$\int\left|\widehat{\Psi}\left(\frac{v}{\bar{v}}\right)\bar{v}_t\right|dx + \int\left|\widehat{\Psi}\left(\frac{\theta}{\bar{\theta}}\right)\bar{\theta}_t\right|dx \leqslant C\delta(1+t)^{-1}K_2 + C\delta(1+t)^{-\frac{5}{2}}, \tag{5.17}$$

where we have used $(\phi, \psi) = (\Phi_x, \Psi_x)$, and $\zeta = W_x - Y$. On the other hand, we have

$$\int\left(|\xi| + \left|\left(\frac{1}{v} - \frac{1}{\bar{v}}\right)\right|\right)|\widetilde{N}_x|\,dx \leqslant C\delta(1+t)^{-1}K_2 + C\delta(1+t)^{-\frac{3}{2}}.$$

Integrating (5.13) with respect to x and using the Cauchy inequality, we get

$$E_{3t} + \frac{1}{2}K_3 \leqslant C\delta(1+t)^{-1}K_2 + C\delta(1+t)^{-\frac{3}{2}} + \int\sum_{i=1}^{3}Q_{i+4}\psi_i\,dx + \int\frac{\zeta}{\theta}Q_8\,dx. \tag{5.18}$$

Since

$$\int|R_{ix}\psi_i|\,dx \leqslant C\delta(1+t)^{-\frac{3}{2}} + C\delta(1+t)^{-1}K_2, \quad i = 1,2,3, \tag{5.19}$$

from (5.2) and (5.3), we only need to consider the slowest decay terms $\iint \xi_1\xi_i\Theta_{1x}\,d\xi\,\psi_i\,dx$, $i = 1,2,3$, in order to estimate $\int Q_{i+4}\psi_i\,dx$. From (2.8) and (4.65), we have

$$\begin{aligned}\left|\iint \xi_1\xi_i\Theta_{1x}\,d\xi\,\psi_i\,dx\right| &= \left|\iint \xi_1\xi_i\Theta_1\,d\xi\,\psi_{ix}\,dx\right| \\ &\leqslant \frac{1}{8}K_3 + C\delta(1+t)^{-\frac{3}{2}} + C\sum_{|\alpha|=1}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2 d\xi\,dx \\ &\quad + C(\delta+\varepsilon_0)\iint\frac{\nu(|\xi|)\widetilde{G}^2}{M_*}\,d\xi\,dx.\end{aligned} \tag{5.20}$$

Since $\int \frac{\zeta}{\theta} Q_8\,dx$ can be estimated similarly, we have

$$E_{3t}+\frac{1}{4}K_3\leqslant C_6(1+t)^{-1}K_2+C_6\delta(1+t)^{-\frac{3}{2}}+C_6\sum_{|\alpha|=1}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx$$
$$+C_6(\delta+\varepsilon_0)\iint\frac{\nu(|\xi|)\widetilde{G}^2}{M_*}\,d\xi\,dx. \tag{5.21}$$

Note that the norm $\|\phi_x\|$ in not included in K_3 (see (5.15)). To complete the first derivative estimate, we follow the same way to estimate Φ_x in the previous section. We rewrite Eq. $(5.1)_2$ as

$$\begin{aligned}&\frac{4}{3}\frac{\mu(\bar\theta)}{\bar v}\phi_{xt}-\psi_{1t}-(p-\bar p)_x\\&\quad=-\frac{4}{3}\left(\frac{\mu(\bar\theta)}{\bar v}\right)_x\psi_{1x}-\frac{8}{15p_+}\frac{\mu(\bar\theta)}{\bar v}\widetilde{N}_{1xx}-\frac{4}{3}\left[\left(\frac{\mu(\theta)}{v}-\frac{\mu(\bar\theta)}{\bar v}\right)u_{1x}\right]_x-Q_5,\end{aligned} \tag{5.22}$$

by using the equation of conservation of the mass (5.1). Multiplying (5.22) by ϕ_x, we get

$$\begin{aligned}&\frac{2}{3}\left(\frac{\mu(\bar\theta)}{\bar v}\phi_x^2\right)_t-\frac{2}{3}\left(\frac{\mu(\bar\theta)}{\bar v}\right)_t\phi_x^2-\psi_{1t}\phi_x-(p-\bar p)_x\phi_x\\&\quad=\left\{-\frac{4}{3}\left(\frac{\mu(\bar\theta)}{\bar v}\right)_x\psi_{1x}-\frac{8}{15p_+}\frac{\mu(\bar\theta)}{\bar v}\widetilde{N}_{1xx}-\frac{4}{3}\left[\left(\frac{\mu(\theta)}{v}-\frac{\mu(\bar\theta)}{\bar v}\right)u_{1x}\right]_x-Q_5\right\}\phi_x.\end{aligned} \tag{5.23}$$

Since

$$-(p-\bar p)_x=\frac{\bar p}{\bar v}\phi_x-\frac{R}{\bar v}\zeta_x+\left(\frac{p}{v}-\frac{\bar p}{\bar v}\right)v_x-\left(\frac{R}{v}-\frac{R}{\bar v}\right)\theta_x \tag{5.24}$$

and

$$\phi_x\psi_{1t}=(\phi_x\psi_1)_t-(\phi_t\psi_1)_x+\psi_{1x}^2-\frac{2}{5p_+}\widetilde{N}_{1x}\psi_{1x}, \tag{5.25}$$

integrating (5.23) with respect to x and using the Cauchy inequality yield

$$\begin{aligned}&\left(\int\frac{2\mu(\bar\theta)}{3\bar v}\phi_x^2-\phi_x\psi_1\,dx\right)_t+\int\frac{\bar p}{2\bar v}\phi_x^2\,dx\\&\quad\leqslant C_7K_3+C_7\delta(1+t)^{-1}K_2+C_7\delta(1+t)^{-\frac{5}{2}}+C_7\varepsilon_0\int\psi_{1xx}^2\,dx\\&\qquad+C_7\varepsilon_0\sum_{0\leqslant|\alpha|\leqslant1}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha\widetilde{G}\right|^2d\xi\,dx+C_7\sum_{|\alpha|=2}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx.\end{aligned} \tag{5.26}$$

Here we have used

$$\int\left|\left(\frac{p}{v}-\frac{\bar p}{\bar v}\right)v_x\phi_x\right|dx\leqslant C\varepsilon_0\|\phi_x\|^2+C\delta(1+t)^{-1}K_2$$

and

$$\int Q_5^2\,dx \leqslant C\delta(1+t)^{-\frac{5}{2}} + C(\delta+\varepsilon_0)\sum_{0\leqslant|\alpha|\leqslant 1}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha \widetilde{G}\big|^2\,d\xi\,dx + C\sum_{|\alpha|=2}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx, \tag{5.27}$$

due to (4.41) and Lemmas 4.1–4.4. To estimate $(\phi,\psi,\zeta)_t$, we use the original equation (1.9). For example, multiplying $(1.9)_2$ by ψ_{1t}, we have

$$\psi_{1t}^2 + \bar{u}_{1t}\psi_{1t} + (p-\bar{p})_x\psi_{1t} + \bar{p}_x\psi_{1t} = -\int \xi_1^2 G_x\,d\xi\,\psi_{1t}. \tag{5.28}$$

Integrating (5.28) with respect to x and using (5.24) give

$$\int \psi_{1t}^2\,dx \leqslant C_7\big(K_3+\|\phi_x\|^2\big) + C_7\delta(1+t)^{-1}K_2 + C_7\delta(1+t)^{-\frac{5}{2}} + C_7\sum_{|\alpha|=1}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx. \tag{5.29}$$

Similar estimates hold ϕ_t, ψ_{2t}, ψ_{3t} and ζ_t. Thus we choose large constants $\overline{C}_4$ and $\overline{C}_5$ so that

$$\overline{C}_4 E_3 + \overline{C}_5\int \frac{2\mu(\bar{\theta})}{3\bar{v}}\phi_x^2 - \phi_x\psi_1\,dx \geqslant \frac{\overline{C}_4}{2}E_3 + \overline{C}_5\int \frac{\mu(\bar{\theta})}{3\bar{v}}\phi_x^2\,dx \tag{5.30}$$

and

$$\frac{1}{4}\bar{C}_4 - (\overline{C}_5+5)C_7 \geqslant \frac{1}{8}\bar{C}_4, \qquad \overline{C}_5\int \frac{\bar{p}}{2\bar{v}}\phi_x^2\,dx - 5C_7\|\phi_x\|^2 \geqslant \frac{\overline{C}_5}{4}\int \frac{\bar{p}}{\bar{v}}\phi_x^2\,dx. \tag{5.31}$$

Let

$$E_4 = \overline{C}_4 E_3 + \overline{C}_5\int \frac{2\mu(\bar{\theta})}{3\bar{v}}\phi_x^2 - \phi_x\psi_1\,dx + \iint \frac{|\widetilde{G}|^2}{2M_*}\,d\xi\,dx, \tag{5.32}$$

$$K_4 = \frac{1}{8}\overline{C}_4 K_3 + \frac{\overline{C}_5}{4}\int \frac{\bar{p}}{\bar{v}}\phi_x^2\,dx + \int\big(\phi_t^2+|\psi_t|^2+\zeta_t^2\big)\,dx + \frac{\bar{\sigma}}{4}\iint \frac{\nu(|\xi|)}{M_*}|\widetilde{G}|^2\,dx. \tag{5.33}$$

Then we have estimate on the (ϕ,ψ,ζ), from (4.68), (5.21), (5.26) and (5.29)–(5.31),

$$E_{4t} + K_4 \leqslant C_8\delta(1+t)^{-\frac{3}{2}} + C_8\delta(1+t)^{-1}K_2 + C_8\varepsilon_0\int \psi_{1xx}^2\,dx + C_8\sum_{1\leqslant|\alpha|\leqslant 2}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx. \tag{5.34}$$

Next we derive the higher order derivative estimate. Applying ∂_x to (5.1) yields

$$\begin{cases} \phi_{xt} - \psi_{1xx} = -\frac{2}{5p_+}\widetilde{N}_{1xx}, \\ \psi_{1xt} - \frac{p}{v}\phi_{xx} + \frac{R}{v}\zeta_{xx} = \frac{4}{3}\big(\frac{\mu(\theta)}{v}\psi_{1xx}\big)_x + Q_9, \\ \psi_{ixt} = \big(\frac{\mu(\theta)}{v}\psi_{ixx}\big)_x + Q_{i+8}, \quad i = 2, 3, \\ \zeta_{xt} + p\psi_{1xx} = \big(\frac{\lambda(\theta)}{v}\zeta_{xx}\big)_x + Q_{12}, \end{cases} \tag{5.35}$$

where

$$Q_9 = \frac{p-\bar{p}}{v}\bar{v}_{xx} + \frac{\phi}{v}\bar{p}_{xx} + \frac{2v_x}{v}(p-\bar{p})_x + \frac{2\bar{p}_x}{v}\phi_x + \frac{4}{3}\bigg(\bigg(\frac{\mu(\theta)}{v}\bigg)_x\psi_{1x}\bigg)_x + \frac{4}{3}\bigg[\bigg(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}}\bigg)\bar{u}_{1x}\bigg]_{xx} + Q_{5x}, \tag{5.36}$$

$$Q_{i+8} = \bigg(\bigg(\frac{\mu(\theta)}{v}\bigg)_x\psi_{ix}\bigg)_x + \bigg[\bigg(\frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}}\bigg)\bar{u}_{ix}\bigg]_{xx} + (Q_{i+4})_x, \tag{5.37}$$

$$Q_{12} = -p_x u_{1x} - p\bar{u}_{1xx} + (\bar{p}\bar{u}_{1x})_x + \frac{2}{5}\widetilde{N}_{1xx} + Q_{8x} + \bigg(\bigg(\frac{\lambda(\theta)}{v}\bigg)_x\zeta_x\bigg)_x + \bigg[\bigg(\frac{\lambda(\theta)}{v} - \frac{\lambda(\bar{\theta})}{\bar{v}}\bigg)\bar{\zeta}_x\bigg]_{xx}. \tag{5.38}$$

Then multiplying $(5.35)_1$ by $p\phi_x$, $(5.35)_2$ by $v\psi_{1x}$, $(5.35)_3$ by ψ_{ix}, $(5.35)_4$ by $\frac{R}{p}\zeta_x$, we have

$$\begin{aligned} &\bigg(\frac{p}{2}\phi_x^2 + \frac{v}{2}\psi_{1x}^2 + \sum_{i=2}^3\psi_{ix}^2 + \frac{R}{2p}\zeta_x^2\bigg)_t - \frac{p_t}{2}\phi_x^2 - \frac{v_t}{2}\psi_{1x}^2 - \bigg(\frac{R}{2p}\bigg)_t\zeta_x^2 + p_x\psi_{1x}\phi_x \\ &\quad + \frac{4\mu(\theta)}{3}\psi_{1xx}^2 + \sum_{i=2}^3\frac{\mu(\theta)}{v}\psi_{ixx}^2 + \frac{R\lambda(\theta)}{vp}\zeta_{xx}^2 \\ &= -\frac{2p}{5p_+}\widetilde{N}_{1xx}\phi_x - \frac{4\mu(\theta)}{3v}\psi_{1xx}v_x\psi_{1x} - \frac{\lambda(\theta)}{v}\zeta_{xx}\bigg(\frac{R}{p}\bigg)_x\zeta_x + vQ_9\psi_{1x} \\ &\quad + \sum_{i=2}^3 Q_{i+8}\psi_{ix} + \frac{R}{p}Q_{12}\zeta_x + (\cdots)_x. \end{aligned} \tag{5.39}$$

Let

$$E_5 = \int \frac{p}{2}\phi_x^2 + \frac{v}{2}\psi_{1x}^2 + \sum_{i=2}^3\psi_{ix}^2 + \frac{R}{2p}\zeta_x^2\,dx, \tag{5.40}$$

$$K_5 = \int \frac{4\mu(\theta)}{3}\psi_{1xx}^2 + \sum_{i=2}^3\frac{\mu(\theta)}{v}\psi_{ixx}^2 + \frac{R\lambda(\theta)}{vp}\zeta_{xx}^2\,dx. \tag{5.41}$$

Integrating (5.39) with respect to x yields

$$E_{5t}+\frac{1}{2}K_5\leqslant C(\delta+\varepsilon_0)K_4+C\delta(1+t)^{-7/2}+\int\left(vQ_9\psi_{1x}+\sum_{i=2}^{3}Q_{i+8}\psi_{ix}+\frac{R}{p}Q_{12}\zeta_x\right)dx. \tag{5.42}$$

The estimate for $\int vQ_9\psi_{1x}\,dx$, $\int Q_{i+8}\psi_{ix}\,dx$, $\int\frac{R}{p}Q_{12}\zeta_x\,dx$ is easy. Here we only consider the integral $\int vQ_9\psi_{1x}\,dx$. The other integrals can be estimated similarly. By (5.24) and (5.36), we have

$$\int\left|\frac{2v_x}{v}(p-\bar{p})_x\psi_{1x}\right|dx\leqslant C(\delta+\varepsilon_0)K_4+C\delta(1+t)^{-2}K_2$$

and

$$\left|\int vQ_{5x}\psi_{1x}\,dx\right|\leqslant\left|\int v_xQ_5\psi_{1x}\,dx\right|+\left|\int vQ_5\psi_{1xx}\,dx\right|\leqslant C(\delta+\varepsilon_0)K_4+\frac{1}{16}K_5+\int Q_5^2\,dx,$$

which give

$$\left|\int vQ_9\psi_{1x}\,dx\right|\leqslant\frac{1}{8}K_5+C\delta(1+t)^{-\frac{5}{2}}+C(\delta+\varepsilon_0)K_4$$
$$+C\delta(1+t)^{-2}K_2+C\int Q_5^2\,dx. \tag{5.43}$$

From (5.27), (5.42) and (5.43), we get

$$E_{5t}+\frac{1}{4}K_5\leqslant C_9(\delta+\varepsilon_0)K_4+C_9\delta(1+t)^{-\frac{5}{2}}+C_9\sum_{|\alpha|=2}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx$$
$$+C_9(\delta+\varepsilon_0)\sum_{|\alpha|=1}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx+C_9\delta(1+t)^{-2}K_2. \tag{5.44}$$

To get the estimation for ϕ_{xx}, we use the first momentum equation of (1.9). Applying ∂_x on $(1.9)_2$, we have

$$\psi_{1xt}+\bar{u}_{1xt}+(p-\bar{p})_{xx}+\bar{p}_{xx}=-\int\xi_1^2G_{xx}\,d\xi. \tag{5.45}$$

Note that

$$(p-\bar{p})_{xx}=-\frac{p}{v}\phi_{xx}+\frac{R}{v}\zeta_{xx}-\frac{1}{v}(p-\bar{p})\bar{v}_{xx}-\frac{\phi}{v}\bar{p}_{xx}-\frac{2v_x}{v}(p-\bar{p})_x-\frac{2\bar{p}_x}{v}\phi_x.$$

Multiplying (5.45) by $-\phi_{xx}$ and using (5.24) imply

$$-(\psi_{1x}\phi_{xx})_t+\int\frac{p}{2v}\phi_{xx}^2\,dx\leqslant C_{10}K_5+C_{10}\delta(1+t)^{-\frac{7}{2}}+C_{10}(\delta+\varepsilon_0)K_4+C_{10}\delta(1+t)^{-2}K_2$$
$$+C_{10}\sum_{|\alpha|=2}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx. \tag{5.46}$$

To estimate $(\phi,\psi,\zeta)_{xt}$ and $(\phi,\psi,\zeta)_{tt}$, we use the original fluid-type equation (1.9). Here we only consider the case $\int\psi_{1xt}^2\,dx$ since the other terms can be estimated similarly. From $(1.9)_2$ and (5.45), we have

$$\psi_{1xt}=-(p-\bar p)_{xx}-\bar p_{xx}-\int\xi_1^2G_{xx}\,d\xi-\bar u_{1xt}. \tag{5.47}$$

Similarly, using the Cauchy inequality implies that

$$\int\psi_{1xt}^2\,dx\leqslant C_{11}\delta(1+t)^{-\frac{3}{2}}+C_{11}K_5+C_{11}\varepsilon_0K_4+C_{11}\delta(1+t)^{-2}K_2$$
$$+C_{11}\sum_{|\alpha|=2}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx. \tag{5.48}$$

Let $\overline{C}_6$ be a large constant, then we have

$$\overline{C}_6\left(E_5-\int\psi_{1x}\phi_{xx}\,dx\right)_t+\sum_{|\alpha|=2}\int\left|\partial^\alpha(\phi,\psi,\zeta)\right|^2dx$$
$$\leqslant C_{12}\delta(1+t)^{-\frac{5}{2}}+C_{12}(\delta+\varepsilon_0)K_4+C_{12}(\delta+\varepsilon_0)\sum_{|\alpha|=1}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx$$
$$+C_{12}\sum_{|\alpha|=2}\iint\frac{\nu(|\xi|)}{M_*}\left|\partial^\alpha G\right|^2d\xi\,dx+C_{12}\delta(1+t)^{-2}K_2. \tag{5.49}$$

To close our argument, we need to estimate the nonfluid component G. That is, we need the estimates for $\partial^\alpha G$, $1\leqslant|\alpha|\leqslant 2$. Applying ∂_x on (1.11), we have

$$G_{xt}-\left(\frac{u_1}{v}G_x\right)_x+\left\{\frac{1}{v}P_1(\xi_1M_x)\right\}_x+\left\{\frac{1}{v}P_1(\xi_1G_x)\right\}_x$$
$$=L_MG_x+2Q(M_x,G)+2Q(G_x,G). \tag{5.50}$$

Since

$$P_1(\xi_1M_x)=\frac{1}{Rv\theta}P_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\theta_x+\xi\cdot u_x\right)M\right],$$

we have

$$\left|\left\{\frac{1}{v}P_1(\xi_1 M_x)\right\}_x\right| \leqslant C\left(v_x^2+u_x^2+\theta_x^2+|\theta_{xx}|+|u_{xx}|\right)\left|\widehat{B}(\xi)\right|M,$$

where $\widehat{B}(\xi)$ is a polynomial of ξ. This yields that

$$\begin{aligned}&\iint\left|\left\{\frac{1}{v}P_1(\xi_1 M_x)\right\}_x\frac{G_x}{M_*}\right|d\xi\,dx\\&\quad\leqslant\frac{\bar\sigma}{8}\iint\frac{\nu(|\xi|)}{M_*}G_x^2\,d\xi\,dx+CK_5+C(\delta+\varepsilon_0)K_4+C\delta(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Also, we have

$$\begin{aligned}&\left|\iint Q(M_x,G)\frac{G_x}{M_*}\,d\xi\,dx\right|\\&\quad\leqslant\frac{\bar\sigma}{8}\iint\frac{\nu(|\xi|)}{M_*}G_x^2\,d\xi\,dx+C\int\left(\int\frac{\nu(|\xi|)}{M_*}M_x^2\,d\xi\iint\frac{\nu(|\xi|)}{M_*}G^2\,d\xi\right)dx\\&\quad\leqslant\frac{\bar\sigma}{8}\iint\frac{\nu(|\xi|)}{M_*}G_x^2\,d\xi\,dx+C(\delta+\varepsilon_0)K_4+C\delta(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Thus, multiplying (5.50) by $\frac{G_x}{M_*}$ and using the Cauchy inequality and Lemmas 4.1–4.4, we get

$$\begin{aligned}&\left(\iint\frac{G_x^2}{2M_*}\,d\xi\,dx\right)_t+\frac{\bar\sigma}{2}\iint\frac{\nu(|\xi|)}{M_*}G_x^2\,d\xi\,dx\\&\quad\leqslant C\delta(1+t)^{-\frac{3}{2}}+C(\delta+\varepsilon_0)K_4+C\iint\frac{\nu(|\xi|)}{M_*}G_{xx}^2\,d\xi\,dx+CK_5.\end{aligned}\tag{5.51}$$

Similarly, we have

$$\begin{aligned}&\left(\iint\frac{G_t^2}{2M_*}\,d\xi\,dx\right)_t+\frac{\bar\sigma}{2}\iint\frac{\nu(|\xi|)}{M_*}G_t^2\,d\xi\,dx\\&\quad\leqslant C\delta(1+t)^{-\frac{3}{2}}+C(\delta+\varepsilon_0)K_4+C(\delta+\varepsilon_0)\iint\frac{\nu(|\xi|)}{M_*}G_x^2\,d\xi\,dx\\&\qquad+C\iint\frac{\nu(|\xi|)}{M_*}G_{xt}^2\,d\xi\,dx+C\int\psi_{xt}^2+\zeta_{xt}^2\,dx.\end{aligned}\tag{5.52}$$

Finally, we need the highest order estimate in our discussion to control $\int\psi_{1x}\phi_{xx}\,dx$ and $\iint\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2\,d\xi\,dx$, $|\alpha|=2$ in (5.49). To estimate $\int\psi_{1x}\phi_{xx}\,dx$, it is sufficient to study the a priori estimate for $\iint\frac{\nu(|\xi|)}{M_*}|\partial^\alpha f|^2\,d\xi\,dx$ ($|\alpha|=2$) due to (3.16), (3.17). To this end, apply ∂^α ($|\alpha|=2$) to (1.8), we have

$$\left(\partial^\alpha f\right)_t-\partial^\alpha\left(\frac{u_1-\xi_1}{v}f_x\right)=\partial^\alpha Q(f,f)=\partial^\alpha\left[L_M G+Q(G,G)\right].\tag{5.53}$$

Multiply (5.53) by $\frac{\partial^\alpha f}{M_*} = \frac{\partial^\alpha M}{M_*} + \frac{\partial^\alpha G}{M_*}$, we obtain

$$\begin{aligned}
&\left(\frac{|\partial^\alpha f|^2}{2M_*}\right)_t + \sum_{|\beta|=1} C(\alpha,\beta)\partial^{\alpha-\beta}\left(\frac{u_1-\xi_1}{v}\right)\partial^\beta f_x \frac{\partial^\alpha f}{M_*} - L_M\partial^\alpha G\cdot\frac{\partial^\alpha G}{M_*}\\
&\quad = \partial^\alpha\left(\frac{u_1-\xi_1}{v}\right) f_x \frac{\partial^\alpha f}{M_*} - \left(\frac{u_1-\xi_1}{2v}\right)_x \frac{|\partial^\alpha f|^2}{M_*} + L_M\partial^\alpha G\cdot\frac{\partial^\alpha M}{M_*}\\
&\qquad + \left(\sum_{|\beta|=1} 2Q\big(\partial^{\alpha-\beta}M, \partial^\alpha G\big) + 2Q\big(\partial^\alpha M, G\big)\right)\frac{\partial^\alpha f}{M_*} + \partial^\alpha Q(G,G)\frac{\partial^\alpha f}{M_*} + (\cdots)_x, \qquad (5.54)
\end{aligned}$$

where we have used

$$\partial^\alpha L_M G = L_M\partial^\alpha G + \sum_{|\beta|=1} 2Q\big(\partial^{\alpha-\beta}M, \partial^\beta G\big) + 2Q\big(\partial^\alpha M, G\big), \quad |\alpha| = 2.$$

Since $M_x, M_t \in N$, $P_1(\partial^\alpha M)$ does not contain $\partial^\alpha(v,u,\theta)$. Thus, we have

$$\begin{aligned}
\left|\iint \frac{L_M\partial^\alpha G\cdot\partial^\alpha M}{M}\,d\xi\,dx\right| &= \left|\iint \frac{L_M\partial^\alpha G\cdot P_1(\partial^\alpha M)}{M}\,d\xi\,dx\right|\\
&\leqslant C(\delta+\varepsilon_0)K_4 + C\delta(1+t)^{-\frac{3}{2}} + \frac{\bar\sigma}{8}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx \qquad (5.55)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\iint L_M\partial^\alpha G\cdot\partial^\alpha M\left(\frac{1}{M_*}-\frac{1}{M}\right)d\xi\,dx\right|\\
&\quad \leqslant \frac{\bar\sigma}{8}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx + C\eta_0\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2\\
&\qquad + C(\delta+\varepsilon_0)K_4 + C\delta(1+t)^{-\frac{3}{2}}, \qquad (5.56)
\end{aligned}$$

where the small constant η_0 is defined in Lemma 4.2. Thus integrating (5.54) and using $f = M + G$, Lemma 4.2 and Cauchy inequality give

$$\begin{aligned}
&\left(\sum_{|\alpha|=2}\iint \frac{|\partial^\alpha f|^2}{2M_*}\,d\xi\,dx\right)_t + \frac{1}{2}\bar\sigma\sum_{|\alpha|=2}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx\\
&\quad \leqslant C\delta(1+t)^{-\frac{3}{2}} + C(\delta+\varepsilon_0)\sum_{|\alpha|=1}\iint \frac{\nu(|\xi|)}{M_*}\big|\partial^\alpha G\big|^2\,d\xi\,dx + C(\delta+\varepsilon_0)K_4\\
&\qquad + C(\eta_0+\delta+\varepsilon_0)\sum_{|\alpha|=2}\big\|\partial^\alpha(\phi,\psi,\zeta)\big\|^2. \qquad (5.57)
\end{aligned}$$

By (3.16) and (3.17), we choose suitable constants $\widehat{C}_i > 1$, $i = 1, 2, 3, 4$, so that

$$\begin{aligned} E_6 &= \widehat{C}_1 E_4 + \widehat{C}_2 \overline{C}_6 \left(E_5 - \int \psi_{1x} \phi_{xx}\, dx \right) + \widehat{C}_3 \sum_{|\alpha|=1} \iint \frac{1}{M_*} \left| \partial^\alpha G \right|^2 d\xi\, dx \\ &\quad + \widehat{C}_4 \sum_{|\alpha|=2} \iint \frac{1}{M_*} \left| \partial^\alpha f \right|^2 d\xi\, dx \\ &\geqslant \left\| (\phi, \psi, \zeta) \right\|^2 + \left\| (\phi_x, \psi_x, \zeta_x) \right\|^2 - C\delta^2 (1+t)^{-\frac{3}{2}} + \iint \frac{1}{M_*} |\widetilde{G}|^2 d\xi\, dx \\ &\quad + \sum_{|\alpha|=1} \iint \frac{1}{M_*} \left| \partial^\alpha G \right|^2 d\xi\, dx + \sum_{|\alpha|=2} \iint \frac{1}{M_*} \left| \partial^\alpha f \right|^2 d\xi\, dx. \end{aligned} \tag{5.58}$$

Let

$$\begin{aligned} K_6 &= \iint \frac{\nu(|\xi|)}{M_*} |\widetilde{G}|^2 d\xi\, dx + \sum_{1 \leqslant |\alpha| \leqslant 2} \iint \frac{\nu(|\xi|)}{M_*} \left| \partial^\alpha G \right|^2 d\xi\, dx \\ &\quad + \sum_{1 \leqslant |\alpha| \leqslant 2} \left| \partial^\alpha (\phi, \psi, \zeta) \right|^2. \end{aligned} \tag{5.59}$$

Then using (5.34), (5.49), (5.51), (5.52) and (5.57), we obtain the final energy estimate

$$E_{6t} + K_6 \leqslant C\delta (1+t)^{-\frac{3}{2}} + C\delta (1+t)^{-1} K_2. \tag{5.60}$$

We note that the derivative estimate (5.60) is independent of the lower order one (4.74) except the term $(1+t)^{-1} K_2$. This kind of derivative estimate is crucial for the stability and convergence rate of the contact wave.

6. Stability and convergence rate

This section is devoted to the stability and convergence rate for the Boltzmann contact wave. By combining (4.74) and (5.60) and choosing a large constant $\widehat{C}_5$, we have

$$(E_2 + \widehat{C}_5 E_6)_t + K_2 + \widehat{C}_5 K_6 \leqslant C_0 \sqrt{\delta}\, (1+t)^{-1} (E_2 + \widehat{C}_5 E_6 + \sqrt{\delta}\,), \tag{6.1}$$

where we have used the smallness of δ. Let

$$E_7 = E_2 + \widehat{C}_5 E_6, \qquad K_7 = K_2 + \widehat{C}_5 K_6. \tag{6.2}$$

Then the Granwall's inequality yields

$$E_7 \leqslant C\big(E_7(0) + \sqrt{\delta}\big)(1+t)^{C_0 \sqrt{\delta}}, \qquad \int_0^t K_7\, dt \leqslant C\big(E_7(0) + \sqrt{\delta}\big)(1+t)^{C_0 \sqrt{\delta}}. \tag{6.3}$$

Hence we have

$$\|(\Phi,\Psi,W)\|^2 \leqslant C\big(E_7(0)+\delta\big)(1+t)^{C_0\sqrt{\delta}}. \tag{6.4}$$

Notice that (6.4) gives no uniform estimate in the lower order estimate. However, the growth rate of the L^2 norm of (Φ,Ψ,W) is slow with the power $C_0\sqrt{\delta}$ depending on the strength of the contact wave. Hence, if the L^2 norm of the derivative for the variable (Φ,Ψ,W) decreases with a decay rate independent of the small parameter δ. Then the Sobolev inequality gives the L^∞ norm decay of the perturbation. In fact, multiply (5.60) by $(1+t)$, we have

$$\big[(1+t)E_6\big]_t \leqslant C\delta(1+t)^{-\frac{1}{2}} + C\delta K_2 + E_6 \leqslant C\delta(1+t)^{-\frac{1}{2}} + CK_7, \tag{6.5}$$

where we have used the fact that $E_6 \leqslant CK_7 + C\delta^2(1+t)^{-\frac{3}{2}}$. Integrating (6.5) with respect to t and using (6.3) imply

$$E_6 \leqslant C\big(E_7(0)+\sqrt{\delta}\big)(1+t)^{-\frac{1}{2}}. \tag{6.6}$$

Notice that in the a priori assumption (3.12), all norms are included in E_6 except the norm $\|(\Phi,\Psi,W)\|_{L^\infty}$. To close the energy estimate, we have to show that the L^∞ norm of (Φ,Ψ,W) is uniformly bounded. Since we have already obtained the decay property (6.6) for the derivative variable (Φ_x,Ψ_x,W_x), the L^∞ estimate for (Φ,Ψ,W) is quite straightforward. In fact, since $(\phi,\psi)=(\Phi,\Psi)_x$ and $\zeta = W_x - Y$ and the norm $\|(\phi,\psi,\zeta)\|^2$ is included in E_6, we have

$$\begin{aligned}\|(\Phi,\Psi,W)\|_{L^\infty}^2 &\leqslant C\|(\Phi,\Psi,W)\|\,\|(\Phi_x,\Psi_x,W_x)\| \leqslant C\big(E_7(0)+\sqrt{\delta}\big)(1+t)^{-\frac{1}{4}+\frac{C_0}{2}\sqrt{\delta}}\\ &\leqslant C\big(E_7(0)+\sqrt{\delta}\big).\end{aligned} \tag{6.7}$$

The last step to prove our main Theorem 1.1 is to justify the first decay rate of (1.18). By (6.6), we have

$$\begin{aligned}\iint \frac{|f(x,t,\xi)-M_{[\bar\rho,\bar u,\bar\theta]}|^2}{M_*}\,d\xi\,dx &\leqslant \iint \frac{|M-M_{[\bar\rho,\bar u,\bar\theta]}|^2}{M_*}\,d\xi\,dx + \iint \frac{G^2}{M_*}\,d\xi\,dx\\ &\leqslant C\|(\phi,\psi,\zeta)\|^2 + C\iint \frac{G^2}{M_*}\,d\xi\,dx\\ &\leqslant C\big(E_7(0)+\sqrt{\delta}\big)(1+t)^{-\frac{1}{2}}\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}&\iint \frac{|f_x(x,t,\xi)-(M_{[\bar\rho,\bar u,\bar\theta]})_x|^2}{M_*}\,d\xi\,dx\\ &\quad\leqslant \iint \frac{|M_x-(M_{[\bar\rho,\bar u,\bar\theta]})_x|^2}{M_*}\,d\xi\,dx + \iint \frac{G_x^2}{M_*}\,d\xi\,dx\\ &\quad\leqslant C\delta^2(1+t)^{-1}\|(\phi,\psi,\zeta)\|^2 + C\|(\phi_x,\psi_x,\zeta_x)\|^2 + C\iint \frac{G^2}{M_*}\,d\xi\,dx\end{aligned}$$

$$\leqslant C\big(E_7(0)+\sqrt{\delta}\big)(1+t)^{-\frac{1}{2}}. \tag{6.9}$$

It is straightforward to imply (1.18) by (6.8) and (6.9) and the Sobolev inequality. Therefore the main Theorem 1.1 is proved.

References

[1] L. Arkeryd, A. Nouri, On the Milne problem and the hydrodynamic limit for a steady Boltzmann equation model, J. Statist. Phys. 99 (3/4) (2000) 993–1019.

[2] C. Bardos, R.E. Caflisch, B. Nicolaenko, The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas, Comm. Pure Appl. Math. 49 (1986) 323–452.

[3] C. Bardos, F. Golse, Different aspects de la notion d'entropie au niveau de l'equation de Boltzmann et de Navier–Stokes, C. R. Acad. Sci. Paris Sér. I Math. 299 (7) (1984) 225–228.

[4] C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations, I. Formal derivations, J. Statist. Phys. 63 (1991) 323–344;
C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations, II. Convergence proofs for the Boltzmann equation, Comm. Pure Appl. Math. 46 (1993) 667–753.

[5] C. Bardos, S. Ukai, The classical incompressible Navier–Stokes limit of the Boltzmann equation, Math. Models Methods Appl. Sci. 1 (2) (1991) 235–257.

[6] L. Boltzmann, Lectures on Gas Theory, Dover, New York, 1964, translated by S.G. Brush.

[7] C. Caflisch, The fluid dynamic limit of the nonlinear Boltzmann equation, Comm. Pure Appl. Math. 33 (5) (1980) 651–666.

[8] R.E. Caflish, B. Nicolaenko, Shock profile solutions of the Boltzmann equation, Comm. Math. Phys. 86 (1982) 161–194.

[9] T. Carleman, Sur la théorie de l'équation intégrodifférentielle de Boltzmann, Acta Math. 60 (1932) 91–142.

[10] C. Cercignani, The Boltzmann Equation and Its Applications, Springer-Verlag, New York, 1988.

[11] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, Springer-Verlag, Berlin, 1994.

[12] S. Chapman, T.G. Cowling, The Mathematical Theory of Non-Uniform Gases, third ed., Cambridge Univ. Press, Cambridge, 1990.

[13] F. Coron, F. Golse, C. Sulem, A classification of well-posed kinetic layer problems, Comm. Pure Appl. Math. 41 (1988) 409–435.

[14] R.J. Diperna, P.L. Lions, On the Cauchy problem for Boltzmann equation: Global existence and weak stability, Ann. of Math. 130 (1989) 321–366.

[15] F. Golse, S.-R. Laure, The Navier–Stokes limit for the Boltzmann equation, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001) 897–902.

[16] F. Golse, B. Perthame, C. Sulem, On a boundary layer problem for the nonlinear Boltzmann equation, Arch. Ration. Mech. Anal. 103 (1986) 81–96.

[17] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Ration. Mech. Anal. 95 (4) (1986) 325–344.

[18] H. Grad, Asymptotic theory of the Boltzmann equation II, in: J.A. Laurmann (Ed.), in: Rarefied Gas Dynamics, vol. 1, Academic Press, New York, 1963, pp. 26–59.

[19] Y. Guo, The Boltzmann equation in the whole space, Indiana Univ. Math. J. 53 (4) (2004) 1081–1094.

[20] F.M. Huang, A. Matsumura, X. Shi, On the stability of contact discontinuity for compressible Navier–Stokes equations with free boundary, Osaka J. Math. 41 (1) (2004) 193–210.

[21] F.M. Huang, A. Matsumura, Z.P. Xin, Stability of contact discontinuity for compressible Navier–Stokes equations, Arch. Ration. Mech. Anal. (2005), online.

[22] F.M. Huang, H.J. Zhao, On the global stability of contact discontinuity for compressible Navier–Stokes equations, Rend. Sem. Mat. Univ. Padova 109 (2003) 283–305.

[23] S. Kawashima, A. Matsumura, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, Comm. Math. Phys. 101 (1) (1985) 97–127.

[24] S. Kawashima, A. Matsumura, T. Nishida, On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier–Stokes equation, Comm. Math. Phys. 70 (2) (1979) 97–124.

[25] P.L. Lions, N. Masmoudi, Form the Boltzmann equations to the equations of incompressible fluid mechanics, I, Arch. Ration. Mech. Anal. 158 (2001) 173–193.

[26] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Amer. Math. Soc. 56 (329) (1985) 1–108.
[27] T.-P. Liu, Z.P. Xin, Nonlinear stability of rarefaction waves for compressible Navier–Stokes equations, Comm. Math. Phys. 118 (3) (1988) 451–465.
[28] T.-P. Liu, Z.P. Xin, Pointwise decay to contact discontinuities for systems of viscous conservation laws, Asian J. Math. 1 (1997) 34–84.
[29] T.-P. Liu, T. Yang, S.-H. Yu, Energy method for the Boltzmann equation, Phys. D 188 (3/4) (2004) 178–192.
[30] T.-P. Liu, T. Yang, S.-H. Yu, H.-J. Zhao, Nonlinear stability of rarefaction waves for the Boltzmann equation, Arch. Ration. Mech. Anal., submitted for publication.
[31] T.-P. Liu, S.-H. Yu, Boltzmann equation: Micro–macro decompositions and positivity of shock profiles, Comm. Math. Phys. 246 (1) (2004) 133–179.
[32] A. Matsumura, K. Nishihara, On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math. 2 (1) (1985) 17–25.
[33] A. Matsumura, K. Nishihara, Asymptotics toward the rarefaction waves of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math. 3 (1) (1985) 3–13.
[34] J.C. Maxwell, (a) On the dynamical theory of gases and (b) On stresses in rarefied gases arising from inequalities of temperature, in: The Scientific Papers of James Clerk Maxwell, vol. II, Cambridge Univ. Press, Cambridge, 1890, p. 26, p. 681.
[35] K. Nishihara, T. Yang, H.-J. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier–Stokes equations, SIAM J. Math. Anal. 35 (6) (2004) 1561–1597.
[36] B. Nicolaenko, Shock wave solutions of the Boltzmann equation as a nonlinear bifurcation problem from the essential spectrum, in: Theories Cinetiques Classiques et Relativistes, Paris, 1974, in: Colloq. Internat. Centre Nat. Recherche Sci., vol. 236, Centre Nat. Recherche Sci., Paris, 1975, pp. 127–150.
[37] T. Nishida, Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation, Comm. Math. Phys. 61 (2) (1978) 119–148.
[38] T. Nishida, K. Lmai, Global solutions to the initial value problem for the nonlinear Boltzmann equation, RIMS, Kyoto Univ. 12 (1976) 229–239.
[39] J. Smoller, Shock Waves and Reaction–Diffusion Equations, Springer, New York, 1994.
[40] Y. Sone, Kinetic Theory and Fluid Dynamics, Birkhäuser Boston, Boston, 2002.
[41] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, Proc. Japan Acad. 50 (1974) 179–184.
[42] S. Ukai, Les solutions globales de l'équation de Boltzmann dans l'espace tout entier et dans le demi-espace, C. R. Acad. Sci. Paris Sér. A 282 (1976) 317–320.