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Nonlinear Stability of Boundary Layers of the Boltzmann Equation, I. The case $\mathscr{M}^\infty < -1$

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Abstract: This is a continuation of the paper [15] on nonlinear boundary layers of the Boltzmann equation where the existence is established and shown to be strongly dependent on the Mach number $\mathscr{M}^\infty$ of the Maxwellian state at far field. In this paper, when $\mathscr{M}^\infty < -1$, we will show that the linearized operator has the exponential decay in time property and therefore a bootstrapping argument yields nonlinear stability of the boundary layers.

1. Introduction and Main Result

The nonlinear Milné problem can be stated as follows. Consider the 3-dimensional half-space $D = \{(x, y, z) \in \mathbb{R}^3 | x > 0\}$, in which the mass density F of gas particles is assumed constant on each plane parallel to the boundary $\partial D = \{x = 0\}$ although the particle motion is 3-dimensional. That is, F is assumed to be a function of position x (but not of y, z) and particle velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Here, ξ_1 stands for the velocity component along the x-axis. Then, F is governed by the stationary Boltzmann equation

$$\begin{cases} \xi_1 F_x = Q(F, F), & x > 0,\ \xi \in \mathbb{R}^3, \\ F|_{x=0} = F_b(\xi), & \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ F \to M_\infty(\xi) \quad (x \to \infty), & \xi \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where

$$M_\infty(\xi) = M[\rho_\infty, u_\infty, T_\infty](\xi) = \frac{\rho_\infty}{(4\pi T_\infty)^{3/2}} \exp\left(-\frac{|\xi - u_\infty|^2}{2T_\infty}\right), \tag{1.2}$$

is a Maxwellian with constants $\rho_\infty > 0$, $u_\infty = (u_{\infty,1}, u_{\infty,2}, u_{\infty,3}) \in \mathbb{R}^3$, and $T_\infty > 0$ which are the macroscopic components in the particle distribution F. By a shift of the variable ξ in the direction orthogonal to the x-axis, we can assume without loss of generality that $u_{\infty,2} = u_{\infty,3} = 0$, and then, the sound speed and Mach number of this equilibrium state are given by

$$c_\infty = \sqrt{\frac{5}{3}T_\infty}, \qquad \mathscr{M}^\infty = \frac{u_{\infty,1}}{c_\infty}, \tag{1.3}$$

respectively, see [4]. Here, Q, the *collision operator*, is a bilinear integral operator

$$Q(F,G) = \frac{1}{2}\int_{\mathbb{R}^3\times S^2}\Big(F(\xi')G(\xi'_*) + F(\xi'_*)G(\xi') - F(\xi)G(\xi_*) - F(\xi_*)G(\xi)\Big) \times q(\xi-\xi_*,\omega)\, d\xi_* d\omega, \tag{1.4}$$

with

$$\xi' = \xi - [(\xi-\xi_*)\cdot\omega]\,\omega, \qquad \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\omega]\,\omega, \tag{1.5}$$

where "$\cdot$" is the inner product of $\mathbb{R}^3$. We restrict ourselves to the hard sphere gas for which the *collision kernel* q is given by

$$q(\zeta,\omega) = \sigma_0|\zeta\cdot\omega|,$$

where σ_0 is the surface area of the hard sphere.

The existence of stationary solutions, called boundary layer solutions, to the problem (1.1) is studied recently in [15]. The result there shows that the existence of boundary layer solutions depends on the Mach number $\mathscr{M}^\infty$ at $x = \infty$. When $\mathscr{M}^\infty \neq 0, \pm 1$, a solvability condition is given implicitly so that the co-dimensions of the manifold for boundary data $F_b(\xi)$ is obtained. In the simplest case, i.e., $\mathscr{M}^\infty < -1$, there is no extra solvability condition because all the information at infinity goes into the layer, which means that as long as the boundary data F_b is close to the Maxwellian at $x = \infty$ under some suitable norm, the boundary layer solution always exists. As the first step, to study the stability of the boundary layer solutions obtained in [15], we will study the case when $\mathscr{M}^\infty < -1$. The main reason why this case is easiest is that the linearized problem has exponential decay phenomena. And this decay estimate is easier to be handled in the bootstrapping argument for nonlinear stability. For the other case, the decay rate should be algebraic as for the Cauchy problem so that it is more difficult and will be pursued by authors in the future.

For the boundary layer problem, there are a lot of results on the linear existence, stability and the numerical computation, cf. [1, 2, 5–8, 12–14]. Since we will discuss the stability problem in this paper, we will not present their works in details.

The main result in this paper can be stated as follows. Let $\bar{F} = \bar{F}(x,\xi)$ be the stationary solution to the problem (1.1). Consider the initial boundary value problem,

$$\begin{cases} F_t + \xi_1 F_x = Q(F,F), & t>0,\ x>0,\ \xi\in\mathbb{R}^3, \\ F|_{t=0} = F_0(x,\xi), & x>0,\ \xi\in\mathbb{R}^3, \\ F|_{x=0} = F_b(\xi), & t>0,\ \xi_1>0,\ (\xi_2,\xi_3)\in\mathbb{R}^2, \\ F \to M_\infty(\xi) \quad (x\to\infty), & t>0,\ \xi\in\mathbb{R}^3. \end{cases} \tag{1.6}$$

Theorem 1.1. *When $\mathscr{M}^\infty < -1$, under the assumption that*

$$|F_b(\xi) - M_\infty(\xi)| \le \epsilon_0 W_\beta(\xi), \qquad \xi \in R^3_+, \quad \beta > 5/2,$$

with the weight function $W_\beta(\xi)$ defined in (2.1) *and ϵ_0 being a sufficiently small positive constant, there exists a boundary layer solution $\bar{F}(x, \xi)$ to* (1.1) *proved in [15]. For* (1.6)*, when $[[F_0(x, \xi) - \bar{F}(x, \xi)]] < \epsilon_1$ with $\beta > 5/2$, where $\epsilon_1 > 0$ is a sufficiently small constant and the norm $[[\cdot]]$ is defined in* (2.28)*, there exists a unique solution $F(t, x, \xi)$ to the problem* (1.6) *which decays exponentially in time to the stationary solution $\bar{F}(x, \xi)$. In other words, the boundary layer solution in this case is nonlinearly stable.*

Remark 1.2. We prove the global existence in the setting of the contraction mapping principle associated to the reduced problem (2.7) related to the quantity $F - \bar{F}$, in the space endowed with the norm (2.30). Hence, the asymptotic stability is a straightforward consequence of it. As for the existence, the method in [11] may work for (1.1). □

The proof of our theorem is given in the following section. We will first consider two semigroups associated with two linearized problems of (1.6) and show that they both have exponential decay property. Then by applying the bootstrapping argument and the smallness of the strength of the boundary layer, we will have the nonlinear stability result stated in Theorem 1.1. In the following, c is used to denote a generic positive constant.

2. Stability Analysis

The stability problem to (1.6) can be discussed in two steps. The first step is to consider the corresponding linearized problem by the energy method for $L^2_{x,\xi}$ and then the bootstrapping argument for $L^\infty_{x,\xi}$. The exponential decay in time estimate obtained in the first step can be used in the second step for nonlinear stability by using Grad's estimate on the nonlinear Boltzmann collision term to obtain an a priori estimate on the solution for the application of the fixed point theorem.

In the following, we will use the following weighted function:

$$W_\beta(\xi) = (1 + |\xi|)^{-\beta}\Big(M[1, u_\infty, T_\infty](\xi)\Big)^{1/2}, \tag{2.1}$$

with $\beta \in \mathbb{R}$.

First, we shall look for the solution of (1.6) in the form

$$F(t, x, \xi) = M_\infty(\xi) + W_0(\xi) f(t, x, \xi), \tag{2.2}$$

where W_0 is the weight of (2.1) with $\beta = 0$. Then, the problem (1.6) reduces to

$$\begin{cases} f_t + \xi_1 f_x - Lf = \Gamma(f), & t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ f|_{t=0} = f_0(x, \xi), & x > 0,\ \xi \in \mathbb{R}^3, \\ f|_{x=0} = a_0(\xi), & t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ f \to 0\ (x \to \infty),\ t > 0,\ \xi \in \mathbb{R}^3, \end{cases} \tag{2.3}$$

where

$$a_0 = W_0^{-1}\Big(F_b - M_\infty\Big),$$

and

$$Lf = W_0^{-1}\Big\{Q(M_\infty, W_0 f) + Q(W_0 f, M_\infty)\Big\}, \qquad \Gamma(f) = \Gamma(f,\ f), \tag{2.4}$$

with

$$\Gamma(f, g) = W_0^{-1} Q(W_0 f,\ W_0 g).$$

The operator L is linear while the remainder Γ is quadratic, both acting only on the variable ξ. The following properties (and nothing else) from them will be used in the sequel. Set $L^p_\xi = L^p(\mathbb{R}^3_\xi)$ and $L^\infty_{\xi,\beta} = L^\infty(\mathbb{R}^3_\xi,\ W_\beta(\xi)d\xi)$.

Proposition 2.1. *For the hard sphere model, the following holds with some positive constants ν_0, ν_1, k_0, k_1, k_2 depending only on $\rho_\infty, u_\infty, T_\infty$.*

(i) *L has the decomposition*

$$L = -\nu(\xi) \times +K,$$

where $\nu(\xi)$ is a positive function satisfying

$$\nu_0 \le \nu(\xi) \le \nu_0^{-1}(1 + |\xi|), \quad \xi \in \mathbb{R}^3,$$

whereas K is an integral operator

$$Kh = \int_{\mathbb{R}^3} K(\xi, \xi')h(\xi')d\xi'$$

with the kernel enjoying the estimate

$$|K(\xi, \xi')| \le k_0(|\xi - \xi'| + |\xi - \xi'|^{-1})e^{-k_1|\xi-\xi'|^2}.$$

(ii) *L is non-positive self-adjoint on L^2_ξ, with the estimate*

$$(Lh, h)_{L^2_\xi} \le -\nu_1 ||(1 + |\xi|)^{1/2} P^\perp h||^2_{L^2_\xi}, \tag{2.5}$$

where $P^\perp = I - P$, P being the orthogonal projection onto the null space N of L.

(iii) *K has the regularizing property that it is bounded as an operator*

$$K : L^\infty_{\xi,\beta} \to L^\infty_{\xi,\beta+1} \qquad \textit{and} \qquad K : L^2_\xi \to L^\infty_\xi$$

for all $\beta \ge 0$.

(iv) *The bilinear operator $\Gamma(f, g)$ enjoys the estimate*

$$||\nu^{-1}\Gamma(f, g)||_{L^\infty_{\xi,\beta}} \le k_3||f||_{L^\infty_{\xi,\beta}}||g||_{L^\infty_{\xi,\beta}}$$

for all β.

Proof. For $\rho_\infty = 1$, $u_\infty = 0$, and $T_\infty = 1$, that is, for the case of the standard Maxwellian $M^0(\xi) = M[1, 0, 1](\xi)$, all the statements in the above are found in, e.g. [4], pp. 197-198, except for (2.5) which is stated in [6]. Let $\nu^0(\xi)$ and $K^0(\xi, \xi')$ be ones corresponding to the standard Maxwellian M^0. Their explicit formulas go back to [10, 3] (see also [4], pp. 196–197). Since

$$M[\rho_\infty, u_\infty, T_\infty](\xi) = \alpha M^0(\gamma(\xi - u_\infty)),$$

for $\alpha = \rho_\infty / T_\infty^{3/2}$ and $\gamma = 1/T_\infty^{1/2}$, it follows from (2.4) that

$$\nu(\xi) = c_0 \nu^0(\gamma(\xi - u_\infty)), \qquad K(\xi, \xi') = c_0 K^0(\gamma(\xi - u_\infty)), \gamma(\xi' - u_\infty)),$$

with $c_0 = \alpha/\gamma = \rho_\infty / T_\infty$, whence the proposition follows for the general Maxwellian. □

This proposition is also valid for Grad's cut-off hard potential [9] with due modification, particularly with $(|\xi| + 1)^\delta$ ($\delta \in [0, 1]$) in place of $(|\xi| + 1)$ in (2.5). Since the model we consider is the hard sphere ($\delta = 1$), we can let $f = e^{-\sigma x} g$ in (2.3) and control by (2.5) (and by P) the term $\sigma \xi_1$ appearing in the deduced problem

$$\begin{cases} g_t + \xi_1 g_x - \sigma \xi_1 g - Lg = e^{-\sigma x} \Gamma(g), & t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ g|_{t=0} = g_0(x, \xi), & x > 0,\ \xi \in \mathbb{R}^3, \\ g|_{x=0} = a_0(\xi), & t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ g \to 0\ (x \to \infty),\ t > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.6}$$

Now, denote the stationary boundary layer solution to (2.6) by $\bar{g}$ and let the initial g_0 be a small perturbation of $\bar{g}$. Then the stability problem we consider can be formulated as follows:

$$\begin{cases} \tilde{g}_t + \xi_1 \tilde{g}_x - \sigma \xi_1 \tilde{g} - L\tilde{g} = e^{-\sigma x}\{\bar{L}\tilde{g} + \Gamma(\tilde{g})\}, & t > 0, x > 0,\ \xi \in \mathbb{R}^3, \\ \tilde{g}\big|_{t=0} = \tilde{g}_0(x, \xi), & x > 0,\ \xi \in \mathbb{R}^3, \\ \tilde{g}\big|_{x=0} = 0, & t > 0,\ \xi_1 > 0,\ (\xi_2, \xi_3) \in \mathbb{R}^2, \\ \tilde{g} \to 0\ (x \to \infty), & t > 0,\ \xi \in \mathbb{R}^3, \end{cases} \tag{2.7}$$

where $\tilde{g} = g - \bar{g}$, $\tilde{g}_0 = g_0 - \bar{g}$ and $\bar{L}\tilde{g} = 2\Gamma(\bar{g}, \tilde{g})$.

Let $S(t)$ be the solution operator (semi-group) of the linear problem

$$\begin{cases} h_t + \xi_1 h_x - \sigma \xi_1 h - Lh = 0, \quad t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ h\big|_{x=0} = 0\ (\xi_1 > 0), \quad h \to 0 (x \to \infty), \quad t > 0,\ \xi \in \mathbb{R}^3, \\ h\big|_{t=0} = h_0(x, \xi), \quad x > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.8}$$

Then we have $h = S(t) h_0$.

For the case $\mathscr{M}^\infty < -1$, the L^2 decay estimate for (2.8) is easy to establish. Recall that in this case, the operator $A = P\xi_1 P$ introduced in our previous paper [15] is negative definite on N, whereas L is also negative definite on $N^\perp$ with the estimate (2.5). Here, P and N are as in Proposition 2.1. Now for a small $\sigma > 0$, a straightforward energy estimate gives

$$\frac{1}{2}\frac{d}{dt}||h(t)||^2 + < |\xi_1| h^0, h^0 >_- + \nu_2 ||(1 + |\xi|)^{1/2} h(t)||^2 \le 0,$$

with a constant $\nu_2 > 0$ (say $\nu_2 = \nu_1/2$), where $||\cdot|| = ||\cdot||_{L^2_{x,\xi}}$, $< \cdot, \cdot >_- = (\cdot, \cdot)_{L^2(\xi_1>0)}$, and $h^0 = h|_{x=0}$. This implies that

$$\frac{d}{dt}\left(e^{\nu_2 t}||h(t)||^2\right) + e^{\nu_2 t}\left\{2 < |\xi_1|h^0(t), h^0(t) >_- + \nu_2||(1+|\xi|)^{1/2}h(t)||^2\right\} \leq 0.$$

Then it follows that

$$e^{\nu_2 t}||h(t)||^2 + \int_0^t e^{\nu_2 t}\left\{2 < |\xi_1|h^0(t), h^0(t) >_- + \nu_2||(1+|\xi|)^{1/2}h(t)||^2\right\}dt \leq ||h_0||^2, \tag{2.9}$$

and

$$||S(t)h_0|| \leq e^{-\kappa t}||h_0||, \qquad \kappa = \frac{\nu_2}{2}. \tag{2.10}$$

As for the existence analysis, we want to prove the following estimate which is sufficient for the application of the fixed point theorem to get the global existence of the solution to the nonlinear problem (2.5),

$$||S(t)h_0||_\beta \leq ce^{-\kappa t}\left\{||h_0||_\beta + ||h_0||_{L^2_{x,\xi}}\right\}, \tag{2.11}$$

for $\beta \geq 0$, where $||\cdot||_\beta$ is the norm of the space $L^\infty_{x,\xi}\left(W_\beta(\xi)dxd\xi\right) = L^\infty_\beta$.

In order to prove (2.11), we first consider another simpler linear solution operator. Let $\nu(\xi)$ be as in Proposition 2.1(i) and let $S_0(t)$ be the solution operator (semi-group) of

$$\begin{cases} h_t + \xi_1 h_x - \sigma\xi_1 h + \nu(\xi)h = 0, \quad t > 0,\ x > 0,\ \xi \in \mathbb{R}^3, \\ h\Big|_{x=0} = 0\ (\xi_1 > 0), \quad h \to 0 (x \to \infty), \quad t > 0,\ \xi \in \mathbb{R}^3, \\ h\Big|_{t=0} = h_0(x,\xi), \quad x > 0,\ \xi \in \mathbb{R}^3. \end{cases} \tag{2.12}$$

The solution to the above linear initial boundary value problem has the following explicit expression:

$$h = S_0(t)h_0 = e^{-(\nu(\xi)-\sigma\xi_1)t}\chi(x-\xi_1 t)h_0(x-\xi_1 t, \xi), \tag{2.13}$$

where $\chi(y)$ is the usual characteristic function for $y > 0$. Based on this expression and with the lower bound $\nu(\xi) \geq \nu_0 > 0$, a simple calculation yields the following estimate on S_0:

$$||S_0(t)h_0||_X \leq ce^{-(2\kappa-\varepsilon)t}||h_0||_X, \tag{2.14}$$

with κ chosen to be $\min(\nu_0, \nu_2)/2$, for some small constant $\epsilon > 0$. Here the space X can be either L^∞_β or $L^2_{x,\xi}$.

From (2.8) and (2.12), we have

$$\begin{cases} S(t)h_0 = S_0(t)h_0 + \int_0^t S_0(t-s)KS(s)h_0 ds \\ \qquad\quad = \sum_{j=0}^{m-1} I_j(t) + J_m(t) \\ I_0(t) \quad = S_0(t)h_0 \\ I_j(t) \quad = \int_0^t S_0(t-s)KI_{j-1}(s)ds = (S_0K) * I_{j-1} \\ J_m(t) \quad = \underbrace{(S_0K) * (S_0K) * \cdots * (S_0K)}_{m} * h, \end{cases} \tag{2.15}$$

with $h = S(t)h_0$. Here and hereafter, "$*$" stands for the convolution in t. By using the estimate (2.14) and the regularizing property of the compact operator K in Proposition 2.1(iii), we have for $\beta \geq j \geq 0$,

$$||I_j(t)||_\beta \leq c_j e^{(-2\kappa+\varepsilon)t}||h_0||_{\beta-j}. \tag{2.16}$$

The estimate on J_m is more complicated and can be stated in the following bootstrapping lemma.

Lemma 2.2. *For $\beta \geq 0$, we have*

$$||J_{\beta+3}(t)||_\beta \leq ce^{-\kappa t}||h_0||_{L^2_{x,\xi}}.$$

Proof. First, again by the regularizing property of K in Proposition 2.1(iii), we have

$$||J_{\beta+3}(t)||_\beta \leq \frac{C}{\beta!}\int_0^t (t-\tau)^\beta e^{-(2\kappa-\epsilon)(t-\tau)}||J_2||_{L^\infty_x(L^2_\xi)}(\tau)d\tau, \tag{2.17}$$

where

$$J_2(t) = (S_0K) * (S_0K) * h = S_0 * \bar{J}, \tag{2.18}$$

with

$$\bar{J} = KS_0K * h = \int_0^t KS_0(t-s)Kh(s)ds = \int_0^t \bar{J}_0(t-s,s)ds. \tag{2.19}$$

We now estimate $\bar{J}_0(t,s)$ as follows. Here, we need to use some integral property of the compact operator K. By definition, we have

$$\begin{aligned} \bar{J}_0(t,s) &= KS_0(t)Kh(s) \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} K(\xi,\xi')K(\xi',\xi'')e^{-(\nu(\xi')-\sigma\xi_1')t}\chi(y)h(s,y,\xi'')d\xi'd\xi'', \end{aligned} \tag{2.20}$$

where $y = x - \xi_1't$. Hence,

$$|\bar{J}_0(t,s)| \leq e^{-(\nu_0-\varepsilon)t}\int_{\mathbb{R}\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\chi(y)|h(s,y,\xi'')|d\xi_1'd\xi'', \tag{2.21}$$

where

$$K_0(\xi,\xi_1',\xi'') \equiv \int_{\mathbb{R}^2}|K(\xi,\xi')||K(\xi',\xi'')|d\xi_2'd\xi_3',$$

with $\xi' = (\xi_1',\xi_2',\xi_3')$.

Notice that the estimate of the kernel $K(\xi,\xi')$ stated in Proposition 2.1(i) gives

$$\int_{\mathbb{R}^3}|K(\xi,\xi')|\,d\xi' = \int_{\mathbb{R}^3}|K(\xi',\xi)|\,d\xi' \leq C_0,$$
$$\int_{\mathbb{R}^2}|K(\xi,\xi')|\,d\xi_2'\,d\xi_3' \leq C_1,$$

where C_0 and C_1 are some positive constants depending only on the parameters $\rho_\infty, u_\infty, T_\infty$. Thus, we have

$$\int_{\mathbb{R}\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')d\xi_1' d\xi'' = \int_{\mathbb{R}^3\times\mathbb{R}^3} |K(\xi,\xi')|\,|K(\xi',\xi'')|\,d\xi'\,d\xi'' \le C_0^2,$$
$$\int_{\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')d\xi \le C_0 \int_{\mathbb{R}^2} |K(\xi',\xi'')\,d\xi_2'\,d\xi_2' \le C_0C_1.$$

By (2.21) and the Schwartz inequality,

$$\begin{aligned}\left|\bar{J}_0(t,s)\right|^2 &\le e^{-2(2\kappa-\epsilon)t}\left[\int_{\mathbb{R}^2\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\,d\xi_1'\,d\xi''\right] \\ &\quad\times\left[\int_{\mathbb{R}^2\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\chi(y)\left|h(s,y,\xi'')\right|^2 d\xi_1'\,d\xi''\right] \\ &\le C_0^2 e^{-2(2\kappa-\epsilon)t}\int_{\mathbb{R}^2\times\mathbb{R}^3} K_0(\xi,\xi_1',\xi'')\chi(y)\left|h(s,y,\xi'')\right|^2 d\xi_1'\,d\xi''. \end{aligned} \tag{2.22}$$

Therefore, we have

$$\begin{aligned}\|\bar{J}_0(t,s)\|^2_{L^\infty_x(L^2_\xi)} &= \sup_{x>0}\int_{\mathbb{R}^3}\left|\bar{J}_0(t,s)\right|^2 d\xi \\ &\le C_0^2C_0C_1e^{-2(2\kappa-\epsilon)t}\int_{\mathbb{R}\times\mathbb{R}^3}\chi(y)\left|h(s,y,\xi'')\right|^2 d\xi_1'\,d\xi'' \\ &= \frac{c}{t}e^{-2(2k-\varepsilon)t}\int_0^\infty dy\int_{\mathbb{R}^3} d\xi''\left|h(s,y,\xi'')\right|^2 \\ &\le \frac{c}{t}e^{-2(2k-\varepsilon)t}e^{-2ks}\|h_0\|^2_{L^2_{x,\xi}}. \end{aligned} \tag{2.23}$$

Here, we have used the L^2 decay estimate (2.10). Hence (2.19) and (2.23) give

$$\begin{aligned}\|\bar{J}(t)\|_{L^\infty_x(L^2_\xi)} &\le \int_0^t \|\bar{J}_0(t-s,s)\|_{L^\infty_x(L^2_\xi)}\,ds \\ &\le c\int_0^t \frac{e^{-(2\kappa-\epsilon)(t-s)}}{\sqrt{t-s}}\,e^{-\kappa s}\|h_0\|_{L^2_{x,\xi}}\,ds \\ &\le c\,e^{-kt}\int_0^t \frac{e^{-(\kappa-\epsilon)(t-s)}}{\sqrt{t-s}}\,ds\|h_0\| \le ce^{-\kappa t}\|h_0\|. \end{aligned} \tag{2.24}$$

This and (2.14), (2.18) give

$$\begin{aligned}\|J_2(t)\|_{L^\infty_x(L^2_\xi)} = \|S_0 * \bar{J}\| &\le \int_0^t e^{-(2\kappa-\varepsilon)(t-s)}\|\bar{J}(s)\|_{L^\infty_x(L^2_\xi)}\,ds \\ &\le c\int_0^t e^{-(2\kappa-\varepsilon)(t-s)}\,e^{-\kappa s}\,ds\,\|h_0\|_{L^2_{x,\xi}} \\ &\le ce^{-\kappa t}\,\|h_0\|_{L^2_{x,\xi}}. \end{aligned} \tag{2.25}$$

Plugging this into (2.17) yields

$$\begin{aligned}||J_{\beta+3}(t)||_\beta &\le e^{-\kappa t}\frac{c}{\beta!}\int_0^t (t-\tau)^\beta e^{-(\kappa-\epsilon)(t-\tau)}d\tau||h_0||_{L^2_{x,\xi}} \\ &\le ce^{-\kappa t}||h_0||_{L^2_{x,\xi}}. \end{aligned} \tag{2.26}$$

And this completes the proof of the lemma. □

This lemma and (2.16) complete the proof of the L^∞_β decay estimate (2.11).

In order to estimate the nonlinear term $\Gamma(\tilde{g})$ and the coupling term $\bar{L}\tilde{g}$ in (2.7) by Proposition 2.1(iv), we also need the following lemma.

Lemma 2.3. *When $\beta \geq 0$, for the two semigroups S_0 and S, we have*

$$||S_0 * \nu(\xi)h||_\beta(t) \leq ce^{-\kappa t} \sup_{0\leq\tau\leq t} \{e^{\kappa\tau}||h||_\beta(\tau)\},$$
$$||S * \nu(\xi)h||_\beta(t) \leq ce^{-\kappa/2t}\{ \sup_{0\leq\tau\leq t} (e^{\kappa/2\tau}||h||_\beta(\tau)) + \sup_{0\leq\tau\leq t} (e^{\kappa/2\tau}||\nu h||_{L^2_{x,\xi}}(\tau))\},$$

both for every function $h(t, x, \xi)$ with the relevant norm bounded.

Proof. First, by the special property of the semigroup S_0 and the linear growth rate of $\nu(\xi)$, we have

$$\begin{aligned}
||S_0 * \nu h||_\beta &\leq \sup_{x,\xi} \int_0^t (1 + |\xi|^\beta)e^{-(\nu(\xi)-\sigma\xi_1)(t-s)}\chi(x - \xi_1 s)\nu(\xi)|h(s, x - \xi_1 s, \xi)|ds \\
&\leq e^{-\kappa t} \sup_{0\leq\tau\leq t} \{e^{\kappa\tau}||h||_\beta(\tau)\} \sup_\xi \left\{ \int_0^t e^{-(\nu(\xi)-\kappa-\sigma\xi_1)(t-s)}\nu(\xi)ds \right\} \\
&\leq ce^{-\kappa t} \sup_{0\leq\tau\leq t} \{e^{\kappa\tau}||h||_\beta(\tau)\}.
\end{aligned}$$

To give the estimate for S, we use the relation between S and S_0,

$$S = S_0 + S_0 * KS.$$

First, write (2.11) as

$$||S(t)h_0||_\beta \leq ce^{-\kappa t}[[h_0]]_\beta, \tag{2.27}$$

with

$$[[\cdot]]_\beta = ||\cdot||_\beta + ||\cdot||_{L^2_{x,\xi}}. \tag{2.28}$$

We assume $\beta \geq 1$ but the proof is similar for other β. By the regularizing property of the operator K again, we have

$$\begin{aligned}
||S_0 * KS * \nu h||_\beta &\leq \int_0^t e^{-\kappa(t-s)}||KS * \nu h||_\beta(s)ds \\
&\leq c\int_0^t e^{-\kappa(t-s)}||S * \nu h||_{\beta-1}(s)ds \\
&\leq c\int_0^t e^{-\kappa(t-s)} \int_0^s e^{-\kappa(s-\tau)}[[\nu h]]_{\beta-1}(\tau)d\tau ds \quad \text{(by (2.27))} \\
&\leq c \sup_{0\leq\tau\leq t} \{e^{\kappa/2\tau}[[\nu h]]_{\beta-1}(\tau)\} \int_0^t e^{-\kappa(t-s)}e^{-\kappa/2s}sds \\
&\leq ce^{-\kappa/2t} \sup_{0\leq\tau\leq t} \{e^{\kappa/2\tau}[[\nu h]]_{\beta-1}(\tau)\}.
\end{aligned}$$

Combining this with the estimate for S_0, we have

$$||S * \nu(\xi)h||_\beta(t) \le ce^{-\kappa/2t}\left\{\sup_{0\le\tau\le t}\left(e^{\kappa/2\tau}||h||_\beta(\tau)\right) + \sup_{0\le\tau\le t}\left(e^{\kappa/2\tau}[[\nu h]]_{\beta-1}(\tau)\right)\right\}.$$

Recalling the linear growth of $\nu(\xi)$ and the definition (2.28) completes the proof of the lemma. □

By using the estimates in the above lemmas and (2.27), we can now construct a global solution to the nonlinear problem (2.7). The definition of the semigroup implies that

$$\tilde{g} = S(t)\tilde{g}_0 + S * \{e^{-\sigma x}(\bar{L}\tilde{g} + \Gamma(\tilde{g}))\}. \tag{2.29}$$

Write the right-hand side by $\Phi[\tilde{g}]$. We have

$$\begin{aligned} ||\Phi[\tilde{g}]||_\beta &\le ||S(t)\tilde{g}_0||_\beta + ||S * \{\nu\nu^{-1}e^{-\sigma x}(\bar{L}\tilde{g} + \Gamma(\tilde{g}))\}||_\beta \\ &\le ce^{-\kappa/2t}\Big\{[[\tilde{g}_0]]_\beta + \sup_{\tau\ge 0}\left(e^{\kappa/2\tau}||e^{-\sigma x}\nu\nu^{-1}(\tilde{L}\bar{g} + \Gamma(\tilde{g}))||_\beta(\tau)\right) \\ &\qquad + \sup_{\tau\ge 0}\left(e^{\kappa/2\tau}||e^{-\sigma x}\nu\nu^{-1}(\tilde{L}\bar{g} + \Gamma(\tilde{g}))||_{L^2_{x,\xi}}(\tau)\right)\Big\} \\ &\le ce^{-\kappa/2t}\{[[\tilde{g}_0]]_\beta + ||\bar{g}||_\beta|||\tilde{g}||| + |||\tilde{g}|||^2\}, \end{aligned}$$

where

$$|||h||| = \sup_{t\ge 0}\{e^{\kappa/2t}||h||_\beta(t)\}. \tag{2.30}$$

In the above we have used the estimate in Proposition 2.1(iv) and the relation

$$\begin{aligned} ||e^{-\sigma x}\nu h||^2_{L^2_{x,\xi}} &\le \left(\int_0^\infty e^{-2\sigma x}dx\int_{\mathbb{R}^3}\nu^2(\xi)(1+|\xi|)^{-2\beta}d\xi\right)||h||^2_\beta \\ &= c||h||^2_\beta, \ \left(\beta > \frac{5}{2}\right). \end{aligned}$$

Consequently, we have

$$|||\Phi[\tilde{g}]|||_\beta \le c([[\tilde{g}_0]]_\beta + ||\bar{g}||_\beta|||\tilde{g}||| + |||\tilde{g}|||^2),$$

and similarly,

$$|||\Phi[\tilde{g}] - \Phi[\tilde{h}]|||_\beta \le c(||\bar{g}||_\beta|||\tilde{g} - \tilde{h}||| + |||\tilde{g} + \tilde{h}|||\,|||\tilde{g} - \tilde{h}|||),$$

with the same constant c.

The smallness assumption on $[[\bar{g}_0]]]_\beta$ and that on $|||\bar{g}|||$ which follows from the smallness assumption on the boundary data a_0 in (2.6) now assure that the nonlinear map Φ is a contraction map in a small ball of the Banach space defined with the norm (2.30) and therefore a unique fixed point exists. This implies, taking into account the choice of the norm (2.30), that (2.7) has a unique global in time solution converging exponentially to 0 as $t \to \infty$ in the norm (2.28). Thus Theorem 1.1 follows.

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