

STABILITY OF A RELAXATION MODEL WITH A NONCONVEX FLUX*

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Abstract. In this paper, we study the nonlinear stability of travelling wave solutions with shock profile for a relaxation model with a nonconvex flux, which is proposed by Jin and Xin [*Comm. Pure Appl. Math.*, 48 (1995), pp. 555–563] to approximate an original hyperbolic system numerically under the subcharacteristic condition introduced by T. P. Liu [*Comm. Math. Phys.*, 108 (1987), pp. 153–175]. The travelling wave solutions with strong shock profile are shown to be asymptotically stable under small disturbances with integral zero using an elementary but technical energy method. Proofs involve detailed study of the error equation for disturbances using the same weight function introduced in [*Comm. Math. Phys.*, 165 (1994), pp. 83–96].

Key words. relaxation model, stability, travelling wave

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1. Introduction. Relaxation occurs when the underlying material is in nonequilibrium and usually takes the form of hyperbolic conservation laws with source terms. Relaxation is often stiff when the relaxation rate is much shorter than the scales of other physical quantities.

The relaxation limit for nonlinear systems of the following form was first studied by Liu [4]:

$$(1.0) \quad \begin{cases} \partial_t u + \partial_x f(u, v) &= 0, \\ \partial_t v + \partial_x g(u, v) &= \frac{v_*(u) - v}{\tau(u)}, \end{cases}$$

provided that the travelling waves are weak and $f(u, v_*(u))$ is a convex function. And the subcharacteristic condition for stability was formulated in [4]. The dissipative entropy condition was formulated for general nonlinear relaxation systems later by Chen, Levermore, and Liu [1].

Recently, a class of relaxation models were proposed by Jin and Xin [10] to approximate the original conservation laws numerically. The special structure of these relaxation systems enables one to solve them numerically with underresolved stable discretization without using either Riemann solvers spatially or nonlinear systems of algebraic equations solvers temporally.

In this paper, we study the following relaxation model introduced in [10]:

$$(1.1) \quad \begin{cases} u_t + v_x = 0, & x \in R^1, \\ v_t + au_x = -\frac{1}{\varepsilon}(v - f(u)), \end{cases}$$

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with the initial data

$$(1.2) \quad (u, v)(x, 0) = (u_0, v_0)(x) \rightarrow (u_{\pm}, v_{\pm}) \text{ as } x \rightarrow \pm\infty, \quad v_{\pm} = f(u_{\pm}),$$

where a is a positive constant satisfying

$$(1.3) \quad -\sqrt{a} < f'(u) < \sqrt{a}$$

for all u under consideration. (1.3) is the *subcharacteristic condition* introduced by Liu [4]. We will show that the travelling wave solutions are stable as $\epsilon \rightarrow 0$.

In the relaxation limit, $\epsilon \rightarrow 0^+$, the leading order of the relaxation system (1.1) is

$$(1.4) \quad \begin{aligned} v &= f(u), \\ u_t + f(u)_x &= 0. \end{aligned}$$

In fact, (1.1) was the prototype of the relaxation model introduced in [10] to solve (1.4) using a local relaxation approximation.

Using Chapman–Enskog expansion [4], the first-order approximation to (1.1) is

$$(1.5) \quad \begin{aligned} v &= f(u) - \epsilon(a - f'(u)^2)u_x, \\ u_t + f(u)_x &= \epsilon((a - f'(u)^2)u_x)_x. \end{aligned}$$

Since (1.5) is dissipative provided that condition (1.3) is satisfied, then similar to the diffusion, the relaxation term has smoothing and dissipative effects for the hyperbolic conservation laws. The stability of the viscous travelling waves with nonconvex flux was investigated by many authors, cf. [2], [5], [7], [8], etc. Using a weight function introduced in [7], we study the stability of strong travelling waves for the relaxation model (1.1) with a nonconvex flux. The behavior of solutions as $\epsilon \rightarrow 0$ when subcharacteristic condition is violated was investigated by R. Leveque and J. Wang [3] under the assumption that the relaxation term is linear.

Under the scaling $(x, t) \rightarrow (\epsilon x, \epsilon t)$, equation (1.1) becomes

$$(1.6) \quad \begin{cases} u_t + v_x = 0, & x \in R^1, \\ v_t + au_x = f(u) - v. \end{cases}$$

The behavior of the solution (u, v) of (1.1) and (1.2) at any fixed time t as $\epsilon \rightarrow 0^+$ is equivalent to the long time behavior of (u, v) of (1.6) as $t \rightarrow \infty$.

In section 2, we will show that there exist travelling wave solutions with shock profile for (1.6), i.e.,

$$(u, v)(x, t) = (U, V)(x - st) \equiv (U, V)(z), \quad (U, V)(z) \rightarrow (u_{\pm}, v_{\pm}) \text{ as } z \rightarrow \pm\infty,$$

if the shock speed s lies between $-\sqrt{a}$ and \sqrt{a} and (u_-, u_+) is an admissible shock of (1.4), that is, the constants u_{\pm} and s (shock speed) satisfy the Rankine–Hugoniot condition

$$(1.7) \quad -s(u_+ - u_-) + f(u_+) - f(u_-) = 0$$

and the entropy condition

$$(1.8) \quad Q(u) \equiv f(u) - f(u_{\pm}) - s(u - u_{\pm}) \begin{cases} < 0 & \text{for } u_+ < u < u_-, \\ > 0 & \text{for } u_- < u < u_+. \end{cases}$$

Note that the U component of a travelling wave solution of (1.6) is a travelling wave solution of the viscous conservation law

$$(1.9) \quad u_t + f(u)_x = \mu u_{xx}$$

with $\mu = a - s^2$. This also gives another justification of the dynamic subcharacteristic condition $s^2 < a$ [4].

The purpose of this paper is to show the stability of the strong travelling wave satisfying $s^2 < a$ for any nonconvex flux f which satisfies the entropy condition (1.7) and (1.8); our result also gives a justification of relaxation schemes introduced in [4] for the case of scalar nonconvex conservation laws.

Notation. Hereafter, C denotes a generic positive constant. L^2 denotes the space of square integrable functions on R with the norm

$$\|f\| = \left(\int_R |f|^2 dx \right)^{1/2}.$$

Without any ambiguity, the integral region R will be omitted. H^j ($j > 0$) denotes the usual j th-order Sobolev space with the norm

$$\|f\|_{H^j} = \|f\|_j = \left(\sum_{k=0}^j \|\partial_x^k f\|^2 \right)^{1/2}.$$

For a weight function $w > 0$, L_w^2 denotes the space of measurable functions f satisfying $\sqrt{w}f \in L^2$ with the norm

$$\|f\|_w = \left(\int w(x) |f(x)|^2 dx \right)^{1/2}.$$

2. Preliminaries and theorem. We first state the existence of the travelling wave solution with shock profile for the system (1.6). Substituting

$$(u, v)(x, t) = (U, V)(z), \quad z = x - st,$$

into (1.6), we have

$$(2.1) \quad \begin{cases} -sU_z + V_z = 0, \\ -sV_z + aU_z = f(U) - V, \end{cases}$$

hence

$$(2.2) \quad (a - s^2)U_z = f(U) - V.$$

Integrating the first equation of (2.1) over $(\pm\infty, z)$ and using $(U, V)(\pm\infty) = (u_{\pm}, v_{\pm})$ and $v_{\pm} = f(u_{\pm})$ yields

$$(2.3) \quad -sU + V = -su_{\pm} + v_{\pm} = -su_{\pm} + f(u_{\pm}).$$

Combining (2.2) and (2.3), we obtain

$$(2.4) \quad U_z = \frac{Q(U)}{a - s^2},$$

where $Q(U) \equiv f(U) - f(u_{\pm}) - s(U - u_{\pm})$ and

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Since (2.4) is a scalar ordinary differential equation of U , the trajectories satisfying boundary conditions $U(\pm\infty) = u_{\pm}$ necessarily connect adjacent equilibria u_- and u_+ . It is easy to check that there is a trajectory from u_- to u_+ if and only if condition $(u_+ - u_-)\frac{Q(U)}{a-s^2} > 0$ that holds for u lies strictly between u_+ and u_- . By virtue of $s^2 < a$, this implies

$$Q(u)(u_+ - u_-) > 0$$

for u that lies strictly between u_+ and u_- , i.e., if and only if

$$u = \begin{cases} u_-, x - st < 0, \\ u_+, x - st > 0 \end{cases}$$

is an admissible shock for (1.4).

Without loss of generality, we study only the following case:

$$(2.5) \quad u_+ < u_- \quad \text{and} \quad U_z < 0.$$

Then the ordinary differential equation (2.4) with boundary condition $U(\pm\infty) = u_{\pm}$ has a unique smooth solution. Moreover, if $f'(u_+) < s < f'(u_-)$ or $Q'(u_{\pm}) \neq 0$, then $Q(U) \sim -|U - u_{\pm}|$ as $U \rightarrow u_{\pm}$. Hence $|(U - u_{\pm}, V - v_{\pm})(z)| \sim \exp(-c_{\pm}|z|)$ as $z \rightarrow \pm\infty$ for some constants $c_{\pm} > 0$. While if $s = f'(u_+)$ or $Q'(u_+) = 0$, $|(U - u_+, V - v_+)(z)| \sim z^{-\frac{1}{k_+}}$ as $z \rightarrow +\infty$ provided $Q(U) \sim -|U - u_+|^{1+k_+}$ for $k_+ > 0$. Note $k_+ = n$ if $Q'(u_+) = \dots = Q^{(n)}(u_+) = 0$ and $Q^{(n+1)}(u_+) \neq 0$.

Thus we have the existence of travelling wave solutions.

LEMMA 2.1. *Assume that $Q(U) < 0$ for $U \in (u_+, u_-)$, $s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$, $v_{\pm} = f(u_{\pm})$, and $|Q(U)| \sim |U - u_+|^{1+k_+}$ as $U \rightarrow u_+$ with $k_+ \geq 0$. Then there exists a travelling wave solution $(U, V)(x - st)$ of (1.1) with $(U, V)(\pm\infty) = (u_{\pm}, v_{\pm})$, which is unique up to a shift and the speed satisfies*

$$(2.6) \quad s^2 < a.$$

Moreover, it holds as $z \rightarrow \pm\infty$

$$\begin{aligned} |(U - u_{\pm}, V - v_{\pm})(z)| &\sim \exp(-c_{\pm}|z|) \quad \text{if } f'(u_+) < s < f'(u_-); \\ |(U - u_+, V - v_+)(z)| &\sim z^{-\frac{1}{k_+}} \quad \text{if } s = f'(u_+). \end{aligned}$$

For the initial disturbance, without loss of generality, we assume

$$(2.7) \quad \int_{-\infty}^{+\infty} (u_0 - U)(x) dx = 0.$$

For a pair of travelling wave solutions given by Lemma 2.1, we let

$$(2.8) \quad (\phi_0, \psi_0)(x) = \left(\int_{-\infty}^x (u_0 - U)(y) dy, (v_0 - V)(x) \right).$$

Our goal is to show that the solution $(u, v)(x, t)$ of (1.6), (1.2) will approach the travelling wave solution $(U, V)(x - st)$ as $t \rightarrow \infty$; the main theorem is as follows.

THEOREM 2.2 (stability). *Suppose that (1.7)–(1.8) hold and $f'(u)^2 < a$, where $a > 0$ is a suitably large constant and $f(u)$ is a smooth function. Let $(U, V)(x - st)$ be a travelling wave solution determined by (2.7) with speed $s^2 < a$, and assume that*

$u_0 - U$ is integrable on R and $\phi_0 \in H^3, \psi_0 \in H^2$. Then there exists a constant $\varepsilon_0 > 0$ independent of (u_\pm, v_\pm) such that if

$$N(0) \equiv \|u_0 - U, v_0 - V\|_2 + \|\phi_0, \psi_0\| < \varepsilon_0,$$

the initial value problem (1.6), (1.2) has a unique global solution $(u, v)(x, t)$ satisfying

$$(u - U, v - V) \in C^0(0, \infty; H^2) \cap L^2(0, \infty; H^2).$$

Furthermore, the solution satisfies

$$(2.9) \quad \sup_{x \in R} |(u, v)(x, t) - (U, V)(x - st)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

3. Reformulation of the problem. The proof of Theorem 2.2 is based on L^2 energy estimates. We first rewrite the problem (1.6), (1.2) using the moving coordinate $z = x - st$. Under the assumption of (2.7), we will look for a solution of the following form:

$$(3.1) \quad (u, v)(x, t) = (U, V)(z) + (\phi_z, \psi)(z, t),$$

where (ϕ, ψ) is in some space of integrable functions which will be defined later.

We substitute (3.1) into (1.6), by virtue of (2.1), and integrate the first equation once with respect to z ; the perturbation (ϕ, ψ) satisfies

$$(3.2) \quad \begin{cases} \phi_t - s\phi_z + \psi = 0, \\ \psi_t - s\psi_z + a\phi_{zz} = f(U + \phi_z) - f(U) - \psi. \end{cases}$$

The first equation of (3.2) gives

$$(3.3) \quad \psi = -(\phi_t - s\phi_z).$$

Substituting (3.3) into the second equation of (3.2), we get a closed equation for ϕ :

$$(3.4) \quad L(\phi) \equiv (\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - a\phi_{zz} + \phi_t + \lambda\phi_z = -F(U, \phi_z),$$

where $F(U, \phi_z) = f(U + \phi_z) - f(U) - f'(U)\phi_z = O(1)(\phi_z^2)$ is a higher order term and $\lambda = Q'(U) = f'(U) - s$.

The corresponding initial data for (3.4) becomes

$$(3.5) \quad \phi(z, 0) = \phi_0(z), \quad \phi_t(z, 0) = s\phi_0'(z) - \psi_0 = \phi_1(z).$$

The asymptotic stability of the profile (U, V) means that the perturbation (ϕ_z, ψ) decays to zero as $t \rightarrow \infty$. The left-hand side of (3.4) contains a first-order term with speed λ which plays the essential role of governing the large-time behavior of the solution.

Now, we introduce the solution space of the problem (3.4), (3.5) as follows:

$$X(0, T) = \{\phi(z, t) : \phi \in C^0([0, T]; H^3) \cap C^1(0, T; H^2), (\phi_z, \phi_t) \in L^2(0, T; H^2)\},$$

with $0 < T \leq +\infty$. By virtue of (3.3), we have

$$\psi \in C^0([0, T]; H^2) \cap L^2(0, T; H^2).$$

By the Sobolev embedding theorem, if we let

$$N(t) = \sup_{0 \leq \tau \leq t} \{\|\phi(\tau)\|_3 + \|\phi_t(\tau)\|_2\},$$

then

$$(3.6) \quad \sup_{z \in \mathbb{R}} \{|\phi|, |\phi_z|, |\phi_{zz}|, |\phi_t|, |\phi_{tz}|\} \leq CN(t).$$

Thus Theorem 2.2 is a consequence of the following theorem.

THEOREM 3.1. *Under the conditions of Theorem 2.2, there exists a positive constant δ_1 such that if*

$$(3.7) \quad N(0) = \|\phi_0\|_3 + \|\phi_1\|_2 \leq \delta_1,$$

then the problem (3.4), (3.5) has a unique global solution $\phi \in X(0, +\infty)$ satisfying

$$(3.8) \quad \|\phi(t)\|_3^2 + \|\phi_t\|_2^2 + \int_0^t \|(\phi_t, \phi_z)(\tau)\|_2^2 d\tau \leq CN(0)^2$$

for $t \in [0, +\infty)$. Furthermore,

$$(3.9) \quad \sup_{z \in \mathbb{R}} |(\phi_z, \phi_t)(z, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

For the solution ϕ in the above theorem, we define (ϕ, ψ) by (3.3). Then it becomes a global solution of the problem (3.2) with $(\phi, \psi)(z, 0) = (\phi_0, \psi_0)(z)$, and consequently we have the desired solution of the problem (1.6), (1.2) through the relation (3.1). On the other hand the solution of (1.6) is unique in the space $C^0(0, T; H^2)$. Therefore Theorem 2.2 follows from Theorem 3.1. Global existence for ϕ will be derived from the following local existence theorem for ϕ combined with an a priori estimate. (3.8) gives

$$(3.10) \quad \|(\phi_t, \phi_z)\|_1^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

from which we have

$$\begin{aligned} \phi_t^2 + \phi_z^2 &= \int_{-\infty}^z (2\phi_t\phi_{tz} + 2\phi_z\phi_{zz})(y, t) dy \\ &\leq \left(\int_{-\infty}^{+\infty} (\phi_t^2 + \phi_z^2) dy \right)^{1/2} \left(\int_{-\infty}^{+\infty} (\phi_{tz}^2 + \phi_{zz}^2) dy \right)^{1/2} \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

PROPOSITION 3.2 (local existence). *For any $\delta_0 > 0$, there exists a positive constant T_0 depending on δ_0 such that if $\phi_0 \in H^3$ and $\phi_1 \in H^2$, with $N(0) < \delta_0/2$, then the problem (3.4), (3.5) has a unique solution $\phi \in X(0, T_0)$ satisfying*

$$(3.11) \quad N(t) < 2N(0)$$

for any $0 \leq t \leq T_0$.

PROPOSITION 3.3 (a priori estimate). *Let $\phi \in X(0, T)$ be a solution for a positive constant T ; then there exists a positive constant δ_2 independent of T such that if*

$$N(t) < \delta_2, \quad t \in [0, T],$$

then ϕ satisfies (3.8) for any $0 \leq t \leq T$.

Proposition 3.2 can be proved in the standard way, so we omit the proof; cf. [9]. To prove Proposition 3.3 is our main task in the following section.

Here we prove Theorem 3.1 by the continuation arguments based on Proposition 3.2 and Proposition 3.3.

Proof of Theorem 3.1. By the definition of $N(t)$, we have

$$(3.12) \quad N(t)^2 \leq 2 \sup_{0 \leq \tau \leq t} [\|\phi(\tau)\|_3^2 + \|\phi_t(\tau)\|_2^2].$$

Then the inequality (3.8) implies

$$(3.13) \quad N(t) < \sqrt{2C}N(0).$$

Choose δ_1 such that $\delta_1 = \min\{\frac{\delta_2}{2}, \frac{\delta_2}{2\sqrt{2C}}\}$; then the local solution of (3.4) can be continued globally in time, provided the smallness condition $N(0) \leq \delta_1$ is satisfied. In fact we have $N(0) < \delta_1 \leq \delta_2/2$. Therefore, by Proposition 3.2, there is a positive constant $T_0 = T_0(\delta_2)$ such that a solution exists on $[0, T_0]$ and satisfies $N(t) < 2N(0) \leq \delta_2$ for $t \in [0, T_0]$.

Hence we can apply Proposition 3.3 with $T = T_0$ and get the estimate (3.8), that is, $N(t) \leq \sqrt{2C}N(0) \leq \frac{\delta_2}{2}$ for $t \in [0, T_0]$. Then we apply Proposition 3.2 by taking $t = T_0$ as the new initial time. We have a solution on $[T_0, 2T_0]$ with the estimate $N(t) \leq 2N(T_0) \leq \delta_2$ for $t \in [T_0, 2T_0]$. Therefore $N(t) \leq \delta_2$ holds on $[0, 2T_0]$. Hence this again gives the estimate (3.8) for $t \in [0, 2T_0]$. In the same way we can extend the solution to the interval $[0, nT_0]$ successively, $n = 1, 2, \dots$, and get a global solution ϕ . This completes the proof of Theorem 3.1. \square

4. Energy estimates. In this section, we will complete the proof of our stability theorem. We establish the basic L^2 estimate as follows.

LEMMA 4.1. *There are positive constants C such that if*

$$-\sqrt{a} < f'(u) < \sqrt{a}, \quad u \in (u_+, u_-),$$

and a is sufficiently large, then

$$(4.1) \quad \begin{aligned} & \|\phi(t)\|_1^2 + \|\phi_t(t)\|^2 + \int_0^t \|(\phi_t, \phi_z)(\tau)\|^2 d\tau + \int_0^t \int_R |U_z| \phi^2 dz d\tau \\ & \leq C\{\|\phi_0\|_1^2 + \|\phi_1\|^2 + \int_0^t \int_R |F|(|\phi| + |(\phi_t, \phi_z)|) dz d\tau\} \end{aligned}$$

holds for $t \in [0, T]$.

Proof. When f is a nonconvex function, the standard energy method used in [6] does not work for our problem (3.4), (3.5). To overcome this difficulty, we use a weight function $w(U)$ introduced in [7] depending on the shock profile U .

First, by multiplying (3.4) by $2w(U)\phi$, we obtain

$$(4.2) \quad 2w(U)\phi \cdot L(\phi) = -2Fw(U)\phi.$$

The left-hand side of (4.2) can be reduced to

$$(4.3) \quad \begin{aligned} & 2[(\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - a\phi_{zz}]w\phi + 2(\phi_t + \lambda\phi_z)w\phi \\ & = [2w\phi(\phi_t - s\phi_z)]_t - 2w\phi_t(\phi_t - s\phi_z) - 2s[w\phi(\phi_t - s\phi_z)]_z \\ & \quad + 2sw_z\phi(\phi_t - s\phi_z) + 2sw\phi_z(\phi_t - s\phi_z) - 2a(w\phi\phi_z)_z + 2aw\phi_z^2 \\ & \quad + (aw_z\phi^2)_z - aw_{zz}\phi^2 + (w\phi^2)_t + (\lambda w\phi^2)_z - \phi^2(\lambda w)_z \\ & = [w\phi^2 + 2w\phi(\phi_t - s\phi_z)]_t - 2w(\phi_t - s\phi_z)^2 + 2aw\phi_z^2 - aw_{zz}\phi^2 \\ & \quad - (\lambda w)_z\phi^2 + sw_z(\phi^2)_t - s^2\{w_z(\phi^2)\}_z + s^2w_{zz}\phi^2 \\ & \quad + \{-2sw\phi(\phi_t - s\phi_z) - 2aw\phi\phi_z + aw_z\phi^2 + \lambda w\phi^2\}_z \\ & = [w\phi^2 + 2w\phi(\phi_t - s\phi_z) + sw_z\phi^2]_t - 2w(\phi_t - s\phi_z)^2 + 2aw\phi_z^2 + A\phi^2 + \{\dots\}_z; \end{aligned}$$

here $A = (s^2 - a)w_{zz} - (\lambda w)_z$, $\{\dots\}_z$ denotes the terms which will disappear after integration with respect to $z \in R$.

Secondly, we calculate

$$(4.4) \quad 2(\phi_t - s\phi_z)w \cdot L(\phi) = -2F(\phi_t - s\phi_z)w.$$

The left-hand side of (4.4) is

$$(4.5) \quad \begin{aligned} & 2[(\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - a\phi_{zz}]w(\phi_t - s\phi_z) \\ & \quad + 2w(\phi_t - s\phi_z)(\phi_t - s\phi_z + f'(U)\phi_z) \\ & = [w(\phi_t - s\phi_z)^2]_t - s[w(\phi_t - s\phi_z)^2]_z + sw_z(\phi_t - s\phi_z)^2 \\ & \quad - 2a[w\phi_z(\phi_t - s\phi_z)]_z + 2aw_z\phi_z(\phi_t - s\phi_z) + 2aw\phi_z(\phi_t - s\phi_z)_z \\ & \quad + 2w(\phi_t - s\phi_z)^2 + 2wf'(U)\phi_z(\phi_t - s\phi_z) \\ & = [w(\phi_t - s\phi_z)^2]_t + (2w + sw_z)(\phi_t - s\phi_z)^2 + 2aw_z\phi_z(\phi_t - s\phi_z) \\ & \quad + 2wf'(U)\phi_z(\phi_t - s\phi_z) + [aw\phi_z^2]_t - [asw\phi_z^2]_z + asw_z\phi_z^2 \\ & \quad - [sw(\phi_t - s\phi_z)^2 + 2aw\phi_z(\phi_t - s\phi_z)]_z \\ & = [aw\phi_z^2 + w(\phi_t - s\phi_z)^2]_t + (2w + sw_z)(\phi_t - s\phi_z)^2 \\ & \quad + saw_z\phi_z^2 + 2f'(U)w\phi_z(\phi_t - s\phi_z) + 2aw_z\phi_z(\phi_t - s\phi_z) \\ & \quad - [sw(\phi_t - s\phi_z)^2 + 2aw\phi_z(\phi_t - s\phi_z) + asw\phi_z^2]_z. \end{aligned}$$

Hence, the combination (4.2) $\times \mu$ + (4.4) with a positive constant μ yields

$$(4.6) \quad \begin{aligned} & \{E_1(\phi, (\phi_t - s\phi_z)) + E_3(\phi_z)\}_t + E_2(\phi_z, (\phi_t - s\phi_z)) + E_4(\phi) + \{\dots\}_z \\ & = -2Fw\{\mu\phi + (\phi_t - s\phi_z)\}, \end{aligned}$$

where

$$(4.7) \quad \begin{aligned} E_1(\phi, (\phi_t - s\phi_z)) & = w(\phi_t - s\phi_z)^2 + 2\mu w\phi(\phi_t - s\phi_z) + \mu(w + sw_z)\phi^2, \\ E_3(\phi_z) & = aw\phi_z^2, \\ E_2(\phi_z, (\phi_t - s\phi_z)) & = (2w + sw_z - 2\mu w)(\phi_t - s\phi_z)^2 \\ & \quad + 2(f'(U)w + aw_z)\phi_z(\phi_t - s\phi_z) + a(2\mu w + sw_z)\phi_z^2, \\ E_4(\phi) & = \mu A\phi^2. \end{aligned}$$

Due to $(a - s^2)U_z = Q(U)$ and $w = w(U)$, we have

$$(4.8) \quad \begin{aligned} A & = -\{(a - s^2)w'(U)U_z + \lambda w\}_z \\ & = -\{w'(U)Q(U) + Q'(U)w\}_z \\ & = -\{wQ\}''U_z. \end{aligned}$$

The monotonicity of the shock profile U implies $U_z < 0$; thus we need to choose $w \in C^2[u_+, u_-]$ such that

$$(4.9) \quad (wQ)'' \geq \nu > 0.$$

On the other hand, we need to choose a constant $\mu > 0$ and w such that the discriminants of E_i ($i = 1, 2$) are negative; that is, the inequalities

$$(4.10) \quad \sup_j D_j < 0, \quad j = 1, 2,$$

hold uniformly in (u_{\pm}, v_{\pm}) , where D_j is the discriminant of the functions $E_j (j = 1, 2)$, respectively.

$$D_1 = 4\mu w[(\mu - 1)w - sw_z],$$

$$D_2 = 4\{(f'w + aw_z)^2 - a(2\mu w + sw_z)(2w + sw_z - 2\mu w)\},$$

and $2\mu w + sw_z > 0$. For this choice of μ and w , there exist positive constants c and C such that

$$(4.11) \quad \begin{cases} c\{\phi^2 + (\phi_t - s\phi_z)^2\} \leq E_1 \leq C\{\phi^2 + (\phi_t - s\phi_z)^2\}, \\ c\{\phi_z^2 + (\phi_t - s\phi_z)^2\} \leq E_2. \end{cases}$$

On the other hand, (4.8) and $a > 0$ gives

$$(4.12) \quad \begin{cases} 0 \leq E_3 = aw\phi_z^2, \\ E_4 \geq \mu\nu|U_z|\phi^2 \geq 0. \end{cases}$$

Thus the equality (4.6) together with the estimates (4.11)–(4.12) give the desired estimate (4.1) after integration with respect to t and z .

It remains to check conditions (4.8)–(4.10). First we choose the weight function $w(U)$ introduced in [7] for the scalar viscous conservation laws with nonconvex flux

$$(4.13) \quad w(U) = \frac{(U - u_+)(U - u_-)}{Q(U)}.$$

Then $w \in C^2[u_+, u_-]$ and (4.8) holds, i.e., $(wQ)'' = \nu = 2$. Furthermore, choosing $\mu = \frac{1}{2}$, the two inequalities in (4.10) are equivalent to

$$(4.14) \quad 1 + 2s\frac{w_z}{w} > 0,$$

$$(4.15) \quad \left(f' + a\frac{w_z}{w}\right)^2 < a\left(1 + s\frac{w_z}{w}\right)^2,$$

since

$$\frac{w_z}{w} = \frac{w'}{w} \frac{Q}{a - s^2} = \frac{O(1)}{a - s^2},$$

which is small provided a is suitably large. This fact, together with $f'^2 < a$, gives us (4.14) and (4.15); thus conditions (4.8) and (4.10) are satisfied. This completes the proof of Lemma 4.1. \square

Next we estimate the higher derivatives of ϕ , multiplying the derivative of (3.4) with respect to z by ϕ_z and $(\phi_t - s\phi_z)_z$, respectively; we have

$$2\partial_z L(\phi) \cdot \phi_z = -2F_z \phi_z,$$

$$2\partial_z L(\phi) \cdot (\phi_t - s\phi_z)_z = -2F_z (\phi_t - s\phi_z)_z.$$

Letting $\phi_z = \Phi$, then

$$(4.16) \quad \begin{aligned} \partial_z L(\phi) &= (\phi_{zt} - s\phi_{zz})_t - s(\phi_{zt} - s\phi_{zz})_z - a\phi_{zzz} + \phi_{zt} + \lambda\phi_{zz} + \lambda_z\phi_z \\ &= L(\phi_z) + \lambda_z\phi_z = L(\Phi) + \lambda_z\Phi. \end{aligned}$$

By a similar argument to obtain (4.3) and (4.5) with $w = 1$, we have

$$(4.17) \quad \begin{aligned} & [\Phi^2 + 2\Phi(\Phi_t - s\Phi_z)]_t + 2a\Phi_z^2 - 2(\Phi_t - s\Phi_z)^2 - \lambda_z\Phi^2 + 2\lambda_z\Phi^2 + \{\dots\}_z \\ & = -2F_z\Phi, \end{aligned}$$

and

$$(4.18) \quad \begin{aligned} & [(\Phi_t - s\Phi_z)^2 + a\Phi_z^2]_t + 2(\Phi_t - s\Phi_z)^2 + 2f'(U)\Phi_z(\Phi_t - s\Phi_z) \\ & \quad + 2\lambda_z\Phi(\Phi_t - s\Phi_z) + \{\dots\}_z \\ & = -2F_z(\Phi_t - s\Phi_z). \end{aligned}$$

The combination (4.17) $\times \frac{1}{2}$ + (4.18) yields

$$(4.19) \quad \begin{aligned} & \{E_1(\Phi, (\Phi_t - s\Phi_z)) + E_2(\Phi_z)\}_t + E_3(\Phi_z, (\Phi_t - s\Phi_z)) + G + \{\dots\}_z \\ & = -F_z\{\Phi + 2(\Phi_t - s\Phi_z)\}, \end{aligned}$$

where

$$(4.20) \quad \begin{aligned} G &= \frac{\lambda_z}{2}\Phi^2 + 2\lambda_z\Phi(\Phi_t - s\Phi_z), \\ E_1(\Phi, (\Phi_t - s\Phi_z)) &= (\Phi_t - s\Phi_z)^2 + \Phi(\Phi_t - s\Phi_z) + \frac{1}{2}\Phi^2, \\ E_2(\Phi_z) &= a\Phi_z^2, \\ E_3(\Phi_z, (\Phi_t - s\Phi_z)) &= (\Phi_t - s\Phi_z)^2 + 2f'(U)\Phi_z(\Phi_t - s\Phi_z) + a\Phi_z^2. \end{aligned}$$

After integration with respect to t and z , (4.19) together with (4.20) gives the following estimate:

$$(4.21) \quad \begin{aligned} & \|\Phi(t)\|_1^2 + \|\Phi_t(t)\|^2 + \int_0^t \|(\Phi_t, \Phi_z)(\tau)\|^2 d\tau \\ & \leq C \left\{ \|\Phi_0\|_1^2 + \|\Phi_1\|^2 + \int_0^t \int |G| dz d\tau + \int_0^t \int_R |F_z| (|\Phi| + |(\Phi_t, \Phi_z)|) dz d\tau \right\}; \end{aligned}$$

here $\Phi_0 = \phi'_0$ and $\Phi_1 = \phi'_1$.

Using the estimate (4.1), we obtain

$$(4.22) \quad \begin{aligned} & \int_0^t \int |G| dz d\tau \leq \int_0^t \int \left[\frac{|\lambda_z|}{2}\Phi^2 + 2|\lambda_z|^2\Phi^2 + \frac{1}{2}\Phi_t^2 + 2s^2|\lambda_z|^2\Phi^2 + \frac{1}{2}\Phi_z^2 \right] dz d\tau \\ & \leq \frac{1}{2} \int_0^t \|(\Phi_t, \Phi_z)(\tau)\|^2 d\tau + C \int_0^t \int \Phi^2 dz d\tau \\ & \leq \frac{1}{2} \int_0^t \|(\Phi_t, \Phi_z)(\tau)\|^2 d\tau \\ & \quad + C \left\{ \|\phi_0\|_1^2 + \|\phi_1\|^2 + \int_0^t \int |F| (|\phi| + |(\phi_t, \phi_z)|) dz d\tau \right\}, \end{aligned}$$

where we have used Lemma 4.1 and the boundness of $|\lambda_z|$.

Substituting (4.22) into (4.21) and replacing Φ by $\partial_z\phi$, we have the following lemma.

LEMMA 4.2. *There are positive constants C such that if*

$$-\sqrt{a} < f'(u) < \sqrt{a} \text{ for } u \in (u_+, u_-),$$

then

$$\begin{aligned}
& \|\partial_z \phi(t)\|_1^2 + \|\partial_z \phi_t\|^2 + \frac{1}{2} \int_0^t \|(\partial_z \phi_t, \partial_z \phi_z)(\tau)\|^2 d\tau \\
& \leq C \left\{ \|\phi_0\|_2^2 + \|\phi_1\|_1^2 + \int_0^t \int |F_z| (|\partial_z \phi| + |(\partial_z \phi_t, \partial_z \phi_z)|) dz d\tau \right. \\
(4.23) \quad & \left. + \int_0^t \int |F| (|\phi| + |(\phi_t, \phi_z)|) dz d\tau \right\}
\end{aligned}$$

holds for $t \in [0, T]$.

Next we calculate the equality

$$\partial_z^2 \phi \cdot \partial_z^2 L(\phi) + 2\partial_z^2(\phi_t - s\phi_z) \cdot \partial_z^2 L(\phi) = -\partial_z^2 F \{ \partial_z^2 \phi + 2\partial_z^2(\phi_t - s\phi_z) \}$$

in the same way as for the proof of Lemma 4.2; it is easy to get the following equality for $\Psi = \partial_z^2 \phi$:

$$\begin{aligned}
(4.24) \quad & \left[(\Psi_t - s\Psi_z)^2 + a\Psi_z^2 + \Psi(\Psi_t - s\Psi_z) + \frac{1}{2}\Psi^2 \right]_t + (\Psi_t - s\Psi_z)^2 + 2f'(U)\Psi_z(\Psi_t - s\Psi_z) \\
& + a\Psi_z^2 + 4\lambda_z\Psi(\Psi_t - s\Psi_z) + \frac{3}{2}\lambda_z\Psi^2 + \lambda_{zz}\Psi\phi_z + 2\lambda_{zz}\phi_z(\Psi_t - s\Psi_z) + \{\dots\}_z \\
& = -F_{zz}[\Psi + 2(\Psi_t - s\Psi_z)].
\end{aligned}$$

Thus, noting $\Psi = \phi_{zz}$, we have from (4.24) that

$$\begin{aligned}
(4.25) \quad & \|\partial_z^2 \phi(t)\|_1^2 + \|\partial_z^2 \phi_t\|^2 + \frac{1}{3} \int_0^t \|(\partial_z^2 \phi_t, \partial_z^2 \phi_z)(\tau)\|^2 d\tau - C \int_0^t \{ \|\partial_z^2 \phi\|^2 + \|\phi_z\|^2 \} d\tau \\
& \leq C \left\{ \|\phi_0\|_3^2 + \|\phi_1\|_2^2 + \int_0^t \int |F_{zz}| (|\partial_z^2 \phi| + |(\partial_z^2 \phi_t, \partial_z^2 \phi_z)|) dz d\tau \right\},
\end{aligned}$$

where we have used the fact that λ_z, λ_{zz} are smooth bounded functions and the Young inequality for the terms $4\lambda_z\Psi(\Psi_t - s\Psi_z)$ and $2\lambda_{zz}\phi_z(\Psi_t - s\Psi_z)$. Combining successively the estimates (4.1), (4.23), and (4.25), we have

$$\begin{aligned}
(4.26) \quad & \|\phi(t)\|_3^2 + \|\phi_t(t)\|_2^2 + \int_0^t \int |\lambda_z|\phi^2 dz d\tau + \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau \\
& \leq C \left\{ \|\phi_0\|_3^2 + \|\phi_1\|_2^2 + \int_0^t \int \{ |F| (|\phi| + |(\phi_t, \phi_z)|) + |F_z| (|\partial_z \phi| + |(\partial_z \phi_t, \partial_z \phi_z)|) \right. \\
& \quad \left. + |F_{zz}| (|\partial_z^2 \phi| + |(\partial_z^2 \phi_t, \partial_z^2 \phi_z)|) \} dz d\tau \right\}.
\end{aligned}$$

Since $F = f(U + \phi_z) - f'(U)\phi_z - f(U)$, we have

$$|F| = O(1)(\phi_z^2), \quad |F_z| = O(1)(\phi_z^2 + \phi_{zz}^2),$$

$$|F_{zz}| = O(1)(\phi_z^2 + \phi_{zz}^2 + |\phi_z\phi_{zzz}|).$$

By virtue of (3.6), the integral on the right-hand side of (4.26) is majored by

$$CN(t) \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau;$$

then we have

$$N^2(t) + \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau + \int_0^t \int |\lambda_z| \phi^2 dz d\tau \leq N(0)^2 + CN(t) \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau.$$

Therefore, by assuming $N(T) \leq \frac{1}{2C}$, we obtain the desired estimate

$$N^2(t) + \int_0^t \|(\phi_t, \phi_z)\|_2^2 d\tau \leq CN(0)^2 \quad \text{for } t \in [0, T].$$

Thus the proof of Proposition 3.3 is completed. \square

Remark. When $s = f'(u_+)$ or $s = f'(u_-)$, we need a weight of the order $\langle x \rangle = \sqrt{1+x^2}$ as $x \rightarrow +\infty$ or $-\infty$ for a stability theorem. The stability analysis for ϕ in this case can be investigated similarly using the weighted function space

$$X_w(0, T) = \{\phi(z, t) : \phi \in C^0([0, T]; H^3 \cap L_w^2(U)) \cap C^1(0, T; H^2 \cap L_w^2(U)), \\ (\phi_z, \phi_t) \in L^2(0, T; H^2 \cap L_w^2(U))\},$$

where $w(U(z)) \sim \langle z \rangle$ as $z \rightarrow \pm\infty$ by virtue of Lemma 2.1 and the definition of $w(U)$ in (4.13).

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