Propagation of Singularities in the Solutions to the Boltzmann Equation near Equilibrium

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Abstract

This paper is about the propagation of the singularities in the solutions to the Cauchy problem of the spatially inhomogeneous Boltzmann equation with angular cutoff assumption. It is motivated by the work of Boudin-Desvillettes on the propagation of singularities in solutions near vacuum. It shows that for the solution near a global Maxwellian, singularities in the initial data propagate like the free transportation. Precisely, the solution is the sum of two parts in which one keeps the singularities of the initial data and the other one is regular with locally bounded derivatives of fractional order in some Sobolev space. In addition, the dependence of the regularity on the cross-section is also given.

1 Introduction

Consider the Cauchy problem of the Boltzmann equation

$$\partial_t f + \xi \cdot \nabla_x f = Q(f, f), \tag{1.1}$$

with initial data

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.2}$$

Here $f = f(t, x, \xi)$ is a non-negative function standing for the number density of gas particles with position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t > 0$. Q is the bilinear collision operator defined by

$$Q(f, g) = Q^+(f, g) - fLg,$$

$$Q^+(f, g) = \iint\limits_{\mathbb{R}^3\times S^2} f(t, x, \xi')g(t, x, \xi'_*)B\left(\xi - \xi_*, \frac{\xi - \xi_*}{|\xi - \xi_*|} \cdot \omega\right) d\xi_* d\omega,$$

$$Lg = \iint\limits_{\mathbb{R}^3\times S^2} g(t, x, \xi_*)B\left(\xi - \xi_*, \frac{\xi - \xi_*}{|\xi - \xi_*|} \cdot \omega\right) d\xi_* d\omega,$$

where the relation between the post-collision velocity pair (ξ', ξ'_*) of two particles with the pre-collision velocity pair (ξ, ξ_*) is given by

$$\xi' = \xi - (\xi - \xi_*) \cdot \omega\omega, \quad \xi'_* = \xi_* + (\xi - \xi_*) \cdot \omega\omega, \quad \omega \in S^2.$$

Here, $B(\cdot, \cdot)$ depending only on $|\xi - \xi_*|$ and $(\xi - \xi_*) \cdot \omega / |\xi - \xi_*|$ is called the cross section characterizing the collision of gas particles for various interaction potentials. As usual, set

$$A(z) = \int_{S^2} B\left(z, \frac{z}{|z|} \cdot \omega\right) d\omega, \quad z \in \mathbb{R}^3.$$

Then Lg can be written as

$$Lg = A *_\xi g = \int_{\mathbb{R}^3} A(\xi - \xi_*) g(t, x, \xi_*) d\xi_*.$$

In what follows, we assume that

A1. $B(\cdot, \cdot)$ is a non-negative measurable function in the form of

$$B(z, \cos\theta) = |z|^\gamma b(\cos\theta), \quad \cos\theta = \frac{z}{|z|} \cdot \omega.$$

Here $0 \leq \gamma \leq 1$, and $b(\cdot)$ satisfies the Grad's angular cutoff assumption [15], with

$$\int_{S^2} b(\cos\theta) d\omega = b_0, \quad 0 \leq b(\cos\theta) \leq b_1,$$

where b_0, b_1 are positive constants.

Recently, Ukai-Yang [26] proved that under the assumption A1, the Cauchy problem (1.1)-(1.2) for the Boltzmann equation is well-posed globally in time near a global Maxwellian in function spaces without any regularity condition on the derivatives. To be precise, without loss of generality, let the global Maxwellian $\mathbf{M}(\cdot)$ be

$$\mathbf{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|\xi|^2}{2}\right).$$

The function space X_β is defined by

$$X_\beta = L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^\infty(\mathbb{R}^3_x; L^\infty_\beta(\mathbb{R}^3_\xi)),$$

with norm

$$\|g\|_{X_\beta} = \|g\|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)} + \|\langle\xi\rangle^\beta g\|_{L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)},$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and $g = g(x, \xi)$. Then the following existence result was proved in [26].

Proposition 1.1. *Let $\beta > 3/2$ and the condition A1 hold. There are positive constants δ_0 and C_0 such that if the initial data satisfies*

$$f_0(x, \xi) = \mathbf{M}(\xi) + \sqrt{\mathbf{M}(\xi)} u_0(x, \xi) \geq 0, \quad x \in \mathbb{R}^3, \xi \in \mathbb{R}^3,$$
$$u_0 \in X_\beta, \quad \|u_0\|_{X_\beta} \leq \delta_0,$$

then the Cauchy problem (1.1)-(1.2) *has a unique solution*

$$f(t, x, \xi) = \mathbf{M}(\xi) + \sqrt{\mathbf{M}(\xi)} u(t, x, \xi) \geq 0, \quad t \geq 0, x \in \mathbb{R}^3, \xi \in \mathbb{R}^3,$$
$$u \in L^\infty(\mathbb{R}_+; X_\beta), \quad \sup_{t \geq 0} \|u(t)\|_{X_\beta} \leq C_0 \|u_0\|_{X_\beta}.$$

The purpose of this paper is to study the regularity of the solution f, or equivalently, the perturbation u obtained in the above proposition. As in the case of small perturbation near vacuum studied by Boudin-Desvillettes [8], we prove that the solution f in Proposition 1.1 can also be written into a sum of two parts in which one corresponds to the free transportation of the initial data with a coefficient decaying exponentially in time and having fractional derivatives in some Sobolev space, while the other one is just regular with locally bounded fractional derivatives. This shows that for the solutions near a global equilibrium, the singularities of the initial data also propagate along the free transportation. To state this theorem, we need one more assumption given below.

A2. The angular part $b(\cdot)$ in the cross section $B(\cdot,\cdot)$ satisfies

$$\sup_{|y|\le 1}\left|\frac{\partial b}{\partial y}(y)\right| \le b_2,$$

where $b_2>0$ is a constant.

The main result in this paper can be stated as follows.

Theorem 1.1. *Let the conditions A1 and A2 hold. Suppose that there is a solution f to the Cauchy problem* (1.1)-(1.2) *such that*

$$f(t,x,\xi)=\mathbf{M}(\xi)+\sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\ge 0,\quad t\ge 0, x\in\mathbb{R}^3,\xi\in\mathbb{R}^3, \tag{1.3}$$

$$u\in L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)\cap L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)). \tag{1.4}$$

Then f can be written as

$$f(t,x,\xi)=f_0(x-\xi t,\xi)\Gamma_1(t,x,\xi)+\Gamma_2(t,x,\xi), \tag{1.5}$$

for all $t\ge 0$, $x\in\mathbb{R}^3$, $\xi\in\mathbb{R}^3$, where there exits $\alpha_0=\alpha_0(\gamma)>0$ defined by

$$\alpha_0(\gamma)=\begin{cases}\frac{1}{25}, & \text{if } \gamma=0,\\ \frac{\gamma}{5(3+2\gamma)}, & \text{if } 0<\gamma\le 1,\end{cases} \tag{1.6}$$

such that

$$\Gamma_1,\Gamma_2\in H^\alpha_{loc}(\mathbb{R}_+\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi) \tag{1.7}$$

for all $\alpha\in(0,\alpha_0)$. Equivalently, for the perturbation u, we have

$$u(t,x,\xi)=u_0(x-\xi t,\xi)\widetilde{\Gamma}_1(t,x,\xi)+\widetilde{\Gamma}_2(t,x,\xi), \tag{1.8}$$

$$\widetilde{\Gamma}_1,\widetilde{\Gamma}_2\in H^\alpha_{loc}(\mathbb{R}_+\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi),\quad 0<\alpha<\alpha_0. \tag{1.9}$$

Remark 1.1. *By Proposition 1.1, there exist solutions satisfying* (1.3)-(1.4) *required in Theorem 1.1. However, for the study of propagation of singularities, so far we don't need the decay of the scaled perturbation u in the velocity variable even though it is needed in the existence theorem, that is, the index β in the space X_β is greater than $\frac{3}{2}$ in the existence theorem while β is zero in Theorem 1.1. It would be interesting to find out whether the Cauchy problem* (1.1)-(1.2) *is well-posed in X_β when $\beta\le\frac{3}{2}$.*

Remark 1.2. *Notice that the index $\alpha_0(\gamma)$ obtained in Theorem 1.1 has a jump when $\gamma=0$. This index in the Sobolev space shows the effect of the kinetic part $|z|^\gamma$ in the cross section on the regularity in the solutions. The index given here is by no means to be optimal even though it gives some interesting relation between two indices.*

Now let us review some related research to the problem considered here. In the last two decades, there is an enormous literature on the study of regularity properties of solutions to the Boltzmann equation with or without the Grad's angular cutoff assumption. In the non-cutoff case, the solution is more regular than the initial data because the non-cutoff collision operator behaves like a fraction of Laplacian which has regularizing effect on the solution, cf. [2] and references therein. In fact, there are a lot of works and there is a satisfactory theory for the spatially homogeneous Boltzmann equation, cf. [11] and references therein. For example, the C^∞ regularization property of weak solutions for the Maxwellian molecule and the regularized hard potentials was proved in [3, 4] by using the Littlewood-Paley decomposition, and the Gevrey regularity for the Maxwellian molecule was obtained in [10, 22]. However, the progress on the spatially inhomogeneous Boltzmann equation without cutoff is much less, cf. [5] and references therein.

On the other hand, in the angular cutoff case, regularity properties of solutions are completely different from those for the non-cutoff potentials. In the cutoff case, the regularities as well as singularities of the initial data propagate in time according to the hyperbolicity of the equation. In the content of the spatially inhomogeneous Boltzmann equation, this kind of propagation for solutions near a vacuum was first studied in [8]. It was shown that the solution at time $t > 0$ has the same regularity as the initial data. An extension of this result to the Vlasov-Poisson-Boltzmann system was recently given in [6]. We would also like to mention that the case of the space homogeneous equation has been extensively studied in [23].

Based on the existence result stated in Proposition 1.1, the study in this paper is motivated by the work of [8] for the perturbation of vacuum. What plays the key role in the whole analysis is still the combination of the velocity regularization properties of the positive part in the Boltzmann operator [7, 20, 21, 28] and the spatial regularization properties from the averaging lemma [9, 14]. However, some differences in the proof for Theorem 1.1 from the one for the result near vacuum can be explained as follows. Under the current consideration, the perturbation does not decay exponentially in the space variable which is the case for the perturbation of vacuum studied in [8]. In fact, the solution studied in [8] satisfies

$$0 \leq f(t,x,\xi) \leq C_T \exp\left(-\frac{1}{2}|x-\xi t|^2 + |\xi|^2\right), \tag{1.10}$$

for any $0 \leq t \leq T$, $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ even though it can be generalized to algebraic decay in the space variable. Now the perturbation of the global Maxwellian still decays exponentially in ξ, but it is only bounded in $L^\infty(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ in x variable. Hence, instead of using the direct pointwise bound like (1.10) for the vacuum perturbation, we need to use the uniform bound on the perturbation in the function space (1.4). In addition, only the bounded cross section satisfying

$$B(z,\cos\theta) \in L^\infty([-1,1]; W^{1,\infty}(\mathbb{R}^3)),$$

was considered in [6, 8]. In this paper, both the hard potentials with angular cutoff and the hard sphere model are considered so that the analysis involves more subtle estimates. Notice that the analysis in this paper can be applied to the study on the perturbation of vacuum provided that the pointwise bound (1.10) is replaced by a weaker assumption, that is,

$$0 \leq \exp\left(\frac{|\xi|^2}{4}\right) f(t,x,\xi) \in L^\infty([0,T]; L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)),$$

where the exponential weight can be changed to some algebraic weight $\langle\xi\rangle^k$ for some k large enough.

Finally, we would like to mention some other related works on the well-posedness theory of the Cauchy problem for the Boltzmann equation, that is, the global renormalized solution in [12], global solutions in $\mathbb{R}^3$ near Maxwellians in [13, 18, 24, 25, 26], global solutions near a vacuum in [16, 17, 19]. Interested reader can find the review paper [27] for more detailed references.

The rest of this paper is arranged as follows. In Section 2, we give the line of proof for Theorem 1.1 and list some basic lemmas. The regularities of the loss term and the gain term are obtained in Section 3 and Section 4, respectively. The proof of Theorem 1.1 is given at the end of Section 4.

Notations. Let $N \geq 1$, $\ell \geq 0$ be integers and $\Omega \subset \mathbb{R}^N$ be an open set with smooth boundary. $L^p_k(\Omega)$, with $1 \leq p \leq \infty$, $k \in \mathbb{R}$ denotes the weighted Lebesgue spaces with norms

$$\|f\|_{L^p_k(\Omega)} = \left(\int_\Omega \langle\xi\rangle^{kp}|f(\xi)|^p\right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_{L^\infty_k(\Omega)} = \sup_{\xi\in\Omega}\langle\xi\rangle^k|f(\xi)|.$$

In addition, $W^{\ell,p}(\Omega)$, $\dot{W}^{\ell,p}(\Omega)$, $1 \leq p \leq \infty$ denote the usual Sobolev spaces and homogeneous Sobolev spaces respectively, with the convection $H^\ell = W^{\ell,2}$, $\dot{H}^\ell = \dot{W}^{\ell,2}$. Further, we will use $W^{s,p}(\Omega)$, with $0 < s < 1, 1 \leq p < \infty$ to denote the fractional Sobolev spaces with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \left(\|f\|^p_{L^p(\Omega)} + \iint\limits_{\Omega\times\Omega} \frac{|f(\xi) - f(\eta)|^p}{|\xi - \eta|^{N+ps}} d\xi d\eta\right)^{1/p}.$$

Throughout this paper, C denotes a generic constant which may vary from line to line. If the dependence of the constant on some parameter, for example a, needs to be specified, then the notation C_a will be used. Finally, B_R denotes a ball with center at origin and radius R.

2 Mild Form and Basic Lemmas

Write the solution f to the Cauchy problem (1.1)-(1.2) in the mild form

$$\begin{aligned} f(t,x,\xi) &= f_0(x-\xi t,\xi)\exp\left(-\int_0^t Lf(\theta,x-\xi\theta,\xi)d\theta\right) \\ &\quad + \int_0^t Q^+(f,f)(s,x-\xi s,\xi)\exp\left(-\int_s^t Lf(\theta,x-\xi\theta,\xi)d\theta\right)ds, \end{aligned}$$

and set

$$\begin{aligned} \Gamma_1 &= \exp\left(-\int_0^t Lf(\theta,x-\xi\theta,\xi)d\theta\right), \\ \Gamma_2 &= \int_0^t Q^+(f,f)(s,x-\xi s,\xi)\exp\left(-\int_s^t Lf(\theta,x-\xi\theta,\xi)d\theta\right)ds. \end{aligned}$$

As in [8], in order to prove Theorem 1.1 for the regularity of Γ_1, Γ_2 with fractional Sobolev derivatives, it is equivalent to show

$$Lf, Q^+(f,f) \in L^2_{loc}(\mathbb{R}_+; H^\alpha_{loc}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)), \quad \forall\, \alpha \in (0, \alpha_0),$$

for some $\alpha_0 > 0$. In fact, one has

$$\begin{aligned} Lf &= A *_\xi \mathbf{M} + A *_\xi (\sqrt{\mathbf{M}}u), && (2.1)\\ Q^+(f,f) &= \mathbf{M}A *_\xi \mathbf{M} + Q^+(\mathbf{M}, \sqrt{\mathbf{M}}u) + Q^+(\sqrt{\mathbf{M}}u, \mathbf{M}) \\ &\quad + Q^+(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u). && (2.2) \end{aligned}$$

Notice that $A *_\xi \mathbf{M}$, $\mathbf{M}A *_\xi \mathbf{M}$ depending only on ξ are C^1-smooth functions. Thus, it suffices to consider the regularity of other terms. Firstly, for the convolution term in Lf, the velocity regularity follows naturally from that of $A(\cdot)$, whereas the spatial regularity will be obtained by the averaging lemma proved in [9] stated as follows.

Lemma 2.1 ([9]). *Let $T > 0$ and $f \in C([0,T]; L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$. Suppose*

$$g(t,x,\xi) := \partial_t f(t,x,\xi) + \xi \cdot \nabla_x f(t,x,\xi) \in L^2([0,T] \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi).$$

Then for any $\psi \in C^\infty_c(\mathbb{R}^3_\xi)$, the average quantity

$$\rho_\psi(t,x) := \int\limits_{\mathbb{R}^3} f(t,x,\xi)\psi(\xi)d\xi$$

satisfies

$$\rho_\psi \in L^2([0,T]; H^{1/2}(\mathbb{R}^3_x)).$$

In fact, for any $s > 1$, we have

$$\|\rho_\psi\|_{L^2([0,T];H^{1/2}(\mathbb{R}^3_x))} \le C_s \left(\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} |f(0,x,\xi)|^2 |\psi(\xi)|^2 \langle\xi\rangle^{2s} dx d\xi \right.$$
$$\left. + \iiint\limits_{[0,T]\times\mathbb{R}^3\times\mathbb{R}^3} |g(t,x,\xi)|^2 |\psi(\xi)|^2 \langle\xi\rangle^{2s} dx d\xi dt \right),$$

where C_s is a constant depending only on s.

Actually, we can apply the above lemma to the following equation for $\sqrt{\mathbf{M}}u$:

$$\begin{aligned} &\partial_t(\sqrt{\mathbf{M}}u) + \xi \cdot \nabla_x(\sqrt{\mathbf{M}}u) \\ &= Q^+(\mathbf{M}, \sqrt{\mathbf{M}}u) + Q^+(\sqrt{\mathbf{M}}u, \mathbf{M}) + Q^+(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u) \\ &\quad - \mathbf{M}A *_\xi (\sqrt{\mathbf{M}}u) - \sqrt{\mathbf{M}}uA *_\xi \mathbf{M} - \sqrt{\mathbf{M}}uA *_\xi (\sqrt{\mathbf{M}}u). \qquad (2.3) \end{aligned}$$

On the other hand, the terms in $Q^+(f,f)$ are regular in the velocity variable by the following lemma proved in [7], which in turn will lead to the regularity in the spatial variable with the help of the velocity mollifier and the aforementioned velocity averaging lemma.

Lemma 2.2 ([7]). *Under conditions A1 and A2, for any $\epsilon > 0$, there is a constant C_ϵ depending only on ϵ such that for any $f, g \in L^1_1(\mathbb{R}^3_\xi) \cap L^2_{(3+\epsilon)/2}(\mathbb{R}^3_\xi)$, one has*

$$Q^+(f,g) \in \dot{H}^1(\mathbb{R}^3_\xi),$$

and

$$\|Q^+(f,g)\|_{\dot{H}^1(\mathbb{R}^3_\xi)} \leq C_\epsilon(b_1+b_2)\|f\|_{L^2_{(3+\epsilon)/2}(\mathbb{R}^3_\xi)}\|g\|_{L^2_{(3+\epsilon)/2}(\mathbb{R}^3_\xi)}.$$

Finally, for later use, we list some basic estimates whose proofs can be found in [1, 25].

Lemma 2.3. *Let $s > -3$, $\lambda > 0$. There is a constant $C_{s,\lambda}$ depending only on s, λ, such that for all $\xi \in \mathbb{R}^3$, it holds*

$$\int_{\mathbb{R}^3} |\xi - \xi_*|^s \exp(-\lambda|\xi_*|^2)d\xi_* \leq C_{s,\lambda}\langle\xi\rangle^s.$$

Lemma 2.4. *Let $f = f(x) \in W^{1,\infty}_{loc}(\mathbb{R}^3)$. Then $f \in H^{1/2}_{loc}(\mathbb{R}^3)$, and there is a constant C such that for any $R > 0$, it holds that*

$$\|f\|_{H^{1/2}(B_R)} \leq CR^{\frac{3}{2}}\|f\|_{W^{1,\infty}(B_R)}.$$

Lemma 2.5. *Let $h \in \mathbb{R}^3$. Denote the shift operator τ_h by $\tau_h f(x) = f(x-h)$. Then for any $f \in H^{1/2}(\mathbb{R}^3)$, it holds*

$$\|\tau_h f - f\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}|h|^{\frac{1}{2}}.$$

Lemma 2.6. *Let $\psi(\xi)$ be the standard mollifier, and $\psi_\delta(\cdot) = \frac{1}{\delta^3}\psi(\frac{\cdot}{\delta})$, for $\delta > 0$. Then for any $f = f(\xi) \in H^1(\mathbb{R}^3)$, it holds*

$$\|f - \psi_\delta *_\xi f\|_{L^2(\mathbb{R}^3)} \leq C_\psi\|f\|_{\dot{H}^1(\mathbb{R}^3)}\delta,$$

where C_ψ is a constant depending only on ψ.

3 Regularity of Lf

Recall the representation (2.1) of Lf. Firstly, we consider its regularity in the velocity variable.

Lemma 3.1. *Under the assumptions of Theorem 1.1, it holds*

$$A *_\xi (\sqrt{\mathbf{M}}u) \in L^2([0,T]\times\mathbb{R}^3_x; H^{1/2}(B_{R_\xi})),$$

with

$$\|A *_\xi (\sqrt{\mathbf{M}}u)\|_{L^2([0,T]\times\mathbb{R}^3_x;H^{1/2}(B_{R_\xi}))} \leq CR^{\frac{3}{2}}\langle R\rangle^\gamma\sqrt{T}\|u\|_{L^\infty([0,T];L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))}.$$

Proof. By the Hölder inequality and Lemma 2.3, we have

$$\begin{aligned}|A *_\xi (\sqrt{\mathbf{M}}u)|^2 &\leq \int_{\mathbb{R}^3} A(\xi-\xi_*)^2\mathbf{M}(\xi_*)d\xi_* \int_{\mathbb{R}^3}|u(t,x,\xi_*)|^2 d\xi_* \\ &\leq Cb_0^2\langle\xi\rangle^{2\gamma}\int_{\mathbb{R}^3}|u(t,x,\xi_*)|^2 d\xi_*.\end{aligned}$$

Similarly, it holds that

$$
\begin{aligned}
&|\nabla_\xi (A *_\xi (\sqrt{\mathbf{M}}u))|^2 = |\nabla_\xi A *_\xi (\sqrt{\mathbf{M}}u)|^2 \\
\le\ & Cb_0^2 \langle \xi \rangle^{2\gamma-2} \int\limits_{\mathbb{R}^3} |u(t,x,\xi_*)|^2 d\xi_* \le Cb_0^2 \int\limits_{\mathbb{R}^3} |u(t,x,\xi_*)|^2 d\xi_*.
\end{aligned}
$$

Thus for any $t \ge 0, x \in \mathbb{R}^3$, we have

$$
\|A *_\xi (\sqrt{\mathbf{M}}u)\|_{W^{1,\infty}(B_{R_\xi})} \le Cb_0 \langle R \rangle^\gamma \|u(t,x,\cdot)\|_{L^2(\mathbb{R}^3_\xi)}.
$$

This together with the imbedding $W^{1,\infty}_{loc}(\mathbb{R}^3_\xi) \hookrightarrow H^{1/2}_{loc}(\mathbb{R}^3_\xi)$ stated in Lemma 2.4 completes the proof of the lemma.

□

As in [8], the regularity in the space variable follows from the velocity averaging lemma.

Lemma 3.2. *Under the assumptions of Theorem 1.1, it holds*

$$
A *_\xi (\sqrt{\mathbf{M}}u) \in L^2([0,T] \times B_{R_\xi}; H^{1/2}(\mathbb{R}^3_x)),
$$

with

$$
\|A *_\xi (\sqrt{\mathbf{M}}u)\|_{L^2([0,T]\times B_{R_\xi}; H^{1/2}(\mathbb{R}^3_x))} \le C R^{\frac{3}{2}} \langle R \rangle^\gamma N_T(u),
$$

where

$$
N_T(u) = \|u_0\|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)} + \sqrt{T} \|u\|_{L^\infty([0,T]; L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))} \left(1 + \|u\|_{L^\infty[0,T]\times \mathbb{R}^3_x \times \mathbb{R}^3_\xi}\right). \tag{3.1}
$$

Proof. Firstly, by applying Lemma 2.1, we can obtain

$$
\begin{aligned}
&\|A *_\xi (\sqrt{\mathbf{M}}u)\|^2_{L^2([0,T]\times B_{R_\xi}; H^{1/2}(\mathbb{R}^3_x))} \le \int\limits_{B_{R_\xi}} \|A *_\xi (\sqrt{\mathbf{M}}u)\|^2_{L^2([0,T]; H^{1/2}(\mathbb{R}^3_x))} d\xi \\
&= \int\limits_{B_{R_\xi}} \left\| \rho_{A(\xi - \cdot)e_\lambda} \left(\frac{\sqrt{\mathbf{M}}u}{e_\lambda} \right) \right\|^2_{L^2([0,T]; H^{1/2}(\mathbb{R}^3_x))} d\xi \le C_s \int\limits_{B_{R_\xi}} [I_1(\xi) + I_2(\xi)] d\xi,
\end{aligned} \tag{3.2}
$$

where

$$
I_1(\xi) = \iint\limits_{x,\xi_*} \left| \frac{\sqrt{\mathbf{M}(\xi_*)}u(0,x,\xi_*)}{e_\lambda(\xi_*)} \right|^2 |A(\xi - \xi_*)|^2 e_\lambda(\xi_*)^2 \langle \xi_* \rangle^{2s}, \tag{3.3}
$$

$$
I_2(\xi) = \iiint\limits_{t,x,\xi_*} \left| (\partial_t + \xi_* \cdot \nabla_x) \frac{\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*)}{e_\lambda(\xi_*)} \right|^2 |A(\xi - \xi_*)|^2 e_\lambda(\xi_*)^2 \langle \xi_* \rangle^{2s}. \tag{3.4}
$$

Here and hereafter, e_λ denotes

$$
e_\lambda = e_\lambda(\xi) = \exp(-\lambda |\xi|^2),
$$

for $\lambda > 0$ to be chosen later. For simplicity, the following multiple integral notions are used

$$\iint_{x,\xi_*}(\cdots) = \iint_{\mathbb{R}^3\times\mathbb{R}^3}(\cdots)dxd\xi_*, \quad \iiint_{t,x,\xi_*}(\cdots) = \iint_{[0,T]\times\mathbb{R}^3\times\mathbb{R}^3}(\cdots)dtdxd\xi_*.$$

We estimate $I_1(\xi), I_2(\xi)$ as follows. For $I_1(\xi)$, notice that

$$A(\xi-\xi_*) \le b_1|\xi-\xi_*|^\gamma \le b_1(|\xi|+|\xi_*|)^\gamma \le b_1\langle\xi\rangle^\gamma\langle\xi_*\rangle^\gamma.$$

Define the function $M_{\lambda,\gamma}(\xi)$ by

$$M_{\lambda,\gamma}(\xi) = \langle\xi\rangle^\gamma \exp(-\lambda|\xi|^2),$$

and the corresponding constant $M_{\lambda,\gamma,\infty}$ by

$$M_{\lambda,\gamma,\infty} = \sup_{\xi\in\mathbb{R}^3} M_{\lambda,\gamma}(\xi).$$

Then we have

$$\begin{aligned} I_1(\xi) &\le Cb_1^2M_{\lambda,s+\gamma,\infty}^2\langle\xi\rangle^{2\gamma}\iint_{x,\xi_*}\exp\left(-\frac{|\xi_*|^2}{2}+2\lambda|\xi_*|^2\right)|u_0(x,\xi_*)|^2 \\ &\le Cb_1^2M_{\lambda,s+\gamma,\infty}^2\langle\xi\rangle^{2\gamma}\|u_0\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}, \end{aligned} \tag{3.5}$$

when $\lambda \le 1/4$. Similarly, for $I_2(\xi)$, it holds that

$$I_2(\xi) \le b_1^2M_{\lambda,s+\gamma,\infty}^2\langle\xi\rangle^{2\gamma}\iiint_{t,x,\xi_*}\left|(\partial_t+\xi_*\cdot\nabla_x)\frac{\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*)}{e_\lambda(\xi_*)}\right|^2. \tag{3.6}$$

Recall the equation (2.3) which is satisfied by $\sqrt{\mathbf{M}}u$. We need to estimate the $L^2([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$ norm for all the terms on the right hand of (2.3). Firstly, for $Q^+(\mathbf{M},\sqrt{\mathbf{M}}u)$, we have

$$\begin{aligned} &\frac{|Q^+(\mathbf{M},\sqrt{\mathbf{M}}u)|}{e_\lambda(\xi)} \le \frac{Q^+(\sqrt{\mathbf{M}},\sqrt{\mathbf{M}}|u|)}{e_\lambda(\xi)} \le \frac{\mathbf{M}(\xi)^{\frac16}}{e_\lambda(\xi)}Q^+(\mathbf{M}^{\frac13},\mathbf{M}^{\frac13}|u|) \\ &= \frac{\mathbf{M}(\xi)^{\frac16}}{e_\lambda(\xi)}\iint_{\xi_*,\omega}B(\xi-\xi_*,\cos\theta)\mathbf{M}(\xi')^{\frac13}\mathbf{M}(\xi'_*)^{\frac13}|u(t,x,\xi'_*)| \\ &\le Ce_{1/12-\lambda}(\xi)\left[\iint_{\xi_*,\omega}B(\xi-\xi_*,\cos\theta)^2\mathbf{M}(\xi)^{\frac13}\mathbf{M}(\xi_*)^{\frac13}\right]^{\frac12}\left[\iint_{\xi_*,\omega}\mathbf{M}(\xi')^{\frac13}\mathbf{M}(\xi'_*)^{\frac13}|u(t,x,\xi'_*)|^2\right]^{\frac12} \\ &\le Cb_1e_{1/12-\lambda}(\xi)\left[\iint_{\xi_*,\omega}\mathbf{M}(\xi')^{\frac13}\mathbf{M}(\xi'_*)^{\frac13}|u(t,x,\xi'_*)|^2\right]^{\frac12}, \end{aligned}$$

where we have used

$$\begin{aligned} \iint_{\xi_*,\omega}B(\xi-\xi_*,\cos\theta)^2\mathbf{M}(\xi)^{\frac13}\mathbf{M}(\xi_*)^{\frac13} &\le Cb_1^2\int_{\xi_*}\langle\xi\rangle^{2\gamma}\langle\xi_*\rangle^{2\gamma}\mathbf{M}(\xi)^{\frac13}\mathbf{M}(\xi_*)^{\frac13} \\ &\le Cb_1^2M_{1/6,2\gamma,\infty}. \end{aligned}$$

Similarly, for $Q^+(\sqrt{\mathbf{M}}u, \mathbf{M})$ and $Q^+(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u)$, it holds that

$$\frac{|Q^+(\sqrt{\mathbf{M}}u, \mathbf{M})|}{e_\lambda(\xi)} \le Cb_1 e_{1/12-\lambda}(\xi) \left[\iint\limits_{\xi_*,\omega} \mathbf{M}(\xi')^{\frac{1}{3}} \mathbf{M}(\xi'_*)^{\frac{1}{3}} |u(t,x,\xi')|^2 \right]^{\frac{1}{2}},$$

and

$$\frac{|Q^+(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u)|}{e_\lambda(\xi)} \le \frac{Q^+(\sqrt{\mathbf{M}}|u|, \sqrt{\mathbf{M}})}{e_\lambda(\xi)} \|u\|_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}$$
$$\le Cb_1 e_{1/12-\lambda}(\xi) \|u\|_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \left[\iint\limits_{\xi_*,\omega} \mathbf{M}(\xi')^{\frac{1}{3}} \mathbf{M}(\xi'_*)^{\frac{1}{3}} |u(t,x,\xi')|^2 \right]^{\frac{1}{2}}.$$

For the loss terms, straightforward calculation shows that

$$\frac{|\mathbf{M}A *_\xi (\sqrt{\mathbf{M}}u)|}{e_\lambda(\xi)} \le \frac{\sqrt{\mathbf{M}}A *_\xi (\sqrt{\mathbf{M}}|u|)}{e_\lambda(\xi)} \le Cb_1 e_{1/12-\lambda}(\xi) \left[\int\limits_{\xi_*} |u(t,x,\xi_*)|^2 \right]^{\frac{1}{2}},$$
$$\frac{|\sqrt{\mathbf{M}}uA *_\xi \mathbf{M}|}{e_\lambda(\xi)} \le Cb_1 e_{1/12-\lambda}(\xi) |u(t,x,\xi)|,$$

and

$$\frac{|\sqrt{\mathbf{M}}uA *_\xi (\sqrt{\mathbf{M}}u)|}{e_\lambda(\xi)} \le Cb_1 e_{1/12-\lambda}(\xi) \|u\|_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} |u(t,x,\xi)|.$$

Therefore, by combining all the above estimates, we have

$$\iiint\limits_{t,x,\xi} \left| (\partial_t + \xi\cdot\nabla_x) \frac{\sqrt{\mathbf{M}(\xi)}u(t,x,\xi)}{e_\lambda(\xi)} \right|^2$$
$$\le Cb_1^2 \left(1 + \|u\|^2_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\right) \times \int\limits_{t,x,\xi,\xi_*,\omega} \cdots \int e_{1/6-2\lambda}(\xi) \mathbf{M}(\xi')^{\frac{1}{3}} \mathbf{M}(\xi'_*)^{\frac{1}{3}} \left(|u(t,x,\xi'_*)|^2 + |u(t,x,\xi')|^2\right)$$
$$+ Cb_1^2 \left(1 + \|u\|^2_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\right) \iiint\limits_{t,x,\xi} e_{1/6-2\lambda}(\xi)|u(t,x,\xi)|^2 + Cb_1^2 \iiiint\limits_{t,x,\xi,\xi_*} e_{1/6-2\lambda}(\xi)|u(t,x,\xi_*)|^2.$$

Thus by choosing $\lambda = 1/24$ and further making the change of variable $(\xi,\xi_*) \to (\xi',\xi'_*)$, the above inequality together with (3.6) yield

$$I_2(\xi) \le Cb_1^2 T M^2_{\lambda,s+\gamma,\infty} \langle\xi\rangle^{2\gamma} \|u\|^2_{L^\infty([0,T];L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))} \left(1 + \|u\|^2_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\right). \tag{3.7}$$

Putting (3.5) and (3.7) into (3.2) gives the desired estimate. This completes the proof of the lemma.

□

Corollary 3.1. *Under the assumptions of Theorem 1.1, it holds*

$$A *_\xi (\sqrt{\mathbf{M}}u) \in L^2([0,T]; H^{1/2}(\mathbb{R}^3_x \times B_{R_\xi})),$$
$$0 \le Lf \in L^2([0,T]; H^{1/2}_{loc}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)) \cap L^\infty([0,T]\times\mathbb{R}^3_x; L^\infty_{loc}(\mathbb{R}^3_\xi)).$$

4 Regularity of $Q^+(f,f)$

Recall the expansion (2.2) for $Q^+(f,f)$. For simplicity, denote

$$S_1^Q = Q^+(\mathbf{M}, \sqrt{\mathbf{M}}u), \quad S_2^Q = Q^+(\sqrt{\mathbf{M}}u, \mathbf{M}), \quad S_3^Q = Q^+(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u).$$

The regularity in the velocity variable for each term above follows essentially from smoothness of the operator $Q^+(\cdot,\cdot)$.

Lemma 4.1. *Under the conditions of Theorem 1.1, for each $i = 1,2,3$, it holds that*

$$S_i^Q \in L^2([0,T]\times\mathbb{R}_x^3; H^1(\mathbb{R}_\xi^3)) \cap L^\infty([0,T]\times\mathbb{R}_x^3; H^1(\mathbb{R}_\xi^3)),$$

and

$$\|S_i^Q\|_{L^2([0,T]\times\mathbb{R}_x^3;H^1(\mathbb{R}_\xi^3))} \le C(b_1+b_2)\sqrt{T}\|u\|_{L^\infty([0,T];L^2(\mathbb{R}_x^3\times\mathbb{R}_\xi^3))}\left(1+\|u\|_{L^\infty([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)}\right),$$
$$\|S_i^Q\|_{L^\infty([0,T]\times\mathbb{R}_x^3;H^1(\mathbb{R}_\xi^3))} \le C(b_1+b_2)\sqrt{T}\|u\|_{L^\infty([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)}\left(1+\|u\|_{L^\infty([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)}\right).$$

Proof. From the proof of Lemma 3.2, it is straightforward to show that

$$S_i^Q \in L^2([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3) \cap L^\infty([0,T]\times\mathbb{R}_x^3; L^2(\mathbb{R}_\xi^3)).$$

Then, it follows from Lemma 2.2 that

$$\|S_i^Q\|_{\dot H^1(\mathbb{R}_\xi^3)} \le C_\epsilon(b_1+b_2)\|\mathbf{M}\|_{L^2_{(3+\epsilon)/2}}\|\sqrt{\mathbf{M}}u\|_{L^2_{(3+\epsilon)/2}}, \quad i=1,2,$$
$$\|S_3^Q\|_{\dot H^1(\mathbb{R}_\xi^3)} \le C_\epsilon(b_1+b_2)\|\sqrt{\mathbf{M}}u\|^2_{L^2_{(3+\epsilon)/2}},$$

where $\epsilon > 0$ is fixed. Thus, taking $L^2([0,T]\times\mathbb{R}_x^3)$ or $L^\infty([0,T]\times\mathbb{R}_x^3)$ norms on both sides of the above inequalities gives the desired estimates. This completes the proof of the lemma.

□

Next, we consider the spatial regularity of S_i^Q. In order to use the velocity averaging lemma, as in [8], we use the mollifier to construct some velocity averaged quantities. For this, let $\phi \in C_c^\infty(\mathbb{R}_\xi^3)$ and consider the integral

$$\begin{aligned}\int_{\mathbb{R}^3} S_1^Q(t,x,\xi)\phi(\xi)d\xi &= \iiint_{\xi,\xi_*,\omega} \mathbf{M}(\xi')\sqrt{\mathbf{M}(\xi_*')}u(t,x,\xi_*')\phi(\xi)B(\xi-\xi_*,\cos\theta)\\ &= \iint_{\xi,\xi_*} \mathbf{M}(\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*)Z^\phi(\xi,\xi_*),\end{aligned} \tag{4.1}$$

where

$$Z^\phi(\xi,\xi_*) = \int_\omega B(\xi-\xi_*,\cos\theta)\phi(\xi-(\xi-\xi_*)\cdot\omega\omega)d\omega.$$

Similarly, corresponding to S_2^Q, S_3^Q, set

$$\int_{\mathbb{R}^3} S_2^Q(t,x,\xi)\phi(\xi)d\xi = \iint_{\xi,\xi_*} \sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\mathbf{M}(\xi_*)Z^\phi(\xi,\xi_*), \tag{4.2}$$

$$\int_{\mathbb{R}^3} S_3^Q(t,x,\xi)\phi(\xi)d\xi = \iint_{\xi,\xi_*} \sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*)Z^\phi(\xi,\xi_*). \tag{4.3}$$

The following lemma is about the pointwise estimates on Z^ϕ.

Lemma 4.2. *Under the assumptions A1 and A2, for any ξ, ξ_*, it holds that*

$$|Z^\phi(\xi,\xi_*)| \le Cb_1\|\phi\|_{L\infty(\mathbb{R}^3)}|\xi-\xi_*|^\gamma. \tag{4.4}$$

Furthermore, for any η, η_*, it holds that for $\gamma = 0$,

$$|Z^\phi(\xi,\xi_*) - Z^\phi(\eta,\eta_*)| \le C(b_1+b_2)\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)}(|\xi-\eta| + |\xi_*-\eta_*|), \tag{4.5}$$

and for $0 < \gamma \le 1$,

$$\begin{aligned}|Z^\phi(\xi,\xi_*) - Z^\phi(\eta,\eta_*)| \le{}& C(b_1+b_2)\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)}|\xi-\xi_*|^\gamma(|\xi-\eta| + |\xi_*-\eta_*|)\\ &+Cb_1\|\phi\|_{L^\infty(\mathbb{R}^3)}(|\xi-\eta|^\gamma + |\xi_*-\eta_*|^\gamma).\end{aligned} \tag{4.6}$$

Proof. By A1 and A2, (4.4) and (4.5) follow from straightforward calculations. For (4.6), Z^ϕ is rewritten as

$$Z^\phi(\xi,\xi_*) = |\xi-\xi_*|^\gamma Y^\phi(\xi,\xi_*),$$

where

$$Y^\phi(\xi,\xi_*) := \int\limits_\omega b(\cos\theta)\phi(\xi - (\xi-\xi_*)\cdot\omega\omega)d\omega.$$

Notice that $Y^\phi(\xi,\xi_*)$ enjoys the same estimate as in (4.5) and

$$||\xi-\xi_*|^\gamma - |\eta-\eta_*|^\gamma| \le ||\xi-\xi_*| - |\eta-\eta_*||^\gamma \le |\xi-\eta|^\gamma + |\xi_*-\eta_*|^\gamma.$$

Hence (4.6) holds. This completes the proof of the lemma. □

We now show that the velocity averaged functions given by (4.1), (4.2) and (4.3) are regular in the space variable.

Lemma 4.3. *Let the conditions A1, A2 hold. For any $|h| \le 1$ and each $i = 1, 2, 3$, we have*

$$\begin{aligned}\iint\limits_{[0,T]\times\mathbb{R}^3} dtdx \left|\int\limits_{\mathbb{R}^3} S_i^Q(t,x+h,\xi)\phi(\xi)d\xi - \int\limits_{\mathbb{R}^3} S_i^Q(t,x,\xi)\phi(\xi)d\xi\right|^2&\\ \le CK_T(u)^2\|\phi\|^2_{W^{1,\infty}(\mathbb{R}^3)}P_\gamma(h),&\end{aligned} \tag{4.7}$$

where

$$K_T(u) = N_T(u)\left(1 + \|u\|_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\right) \tag{4.8}$$

with $N_T(u)$ defined by (3.1), and $P_\gamma(h)$ is defined by

$$P_\gamma(h) = \begin{cases} |h|^{\frac{2}{5}}, & if\ \gamma = 0,\\ |h|^{\frac{2\gamma}{3+2\gamma}}, & if\ 0 < \gamma \le 1. \end{cases} \tag{4.9}$$

Proof. Firstly, consider the case of $i = 1$. Rewrite the averaged function (4.1) by

$$\int_{\mathbb{R}^3} S_1^Q(t,x,\xi)\phi(\xi)d\xi = J_1 + J_2, \tag{4.10}$$

$$J_1 = \iint_{\xi,\xi_*} \mathbf{M}(\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*) \iint_{\eta,\eta_*} Z(\eta,\eta_*)\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*), \tag{4.11}$$

$$J_2 = \iint_{\xi,\xi_*} \mathbf{M}(\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*) \iint_{\eta,\eta_*} [Z(\xi,\xi_*) - Z(\eta,\eta_*)]\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*), \tag{4.12}$$

where $\psi(\cdot)$ is a standard mollifier, and $\psi_\epsilon(\cdot) = \frac{1}{\epsilon^3}\psi(\frac{\cdot}{\epsilon})$, with $\epsilon > 0$ to be determined later. Furthermore, J_1 can be rewritten as

$$J_1 = \iint_{\eta,\eta_*} Z(\eta,\eta_*)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)(t,x)\int_\xi \mathbf{M}(\xi)\psi_\epsilon(\xi-\eta).$$

Then one has

$$\iint_{t,x} |\tau_h J_1 - J_1|^2 = \iint_{t,x} \left| \iint_{\eta,\eta_*} Z(\eta,\eta_*)\left[(\tau_h - Id)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\right](t,x)\int_\xi \mathbf{M}(\xi)\psi_\epsilon(\xi-\eta)\right|^2.$$

From Lemma 4.2, we have

$$\begin{aligned} |Z(\eta,\eta_*)| &\le C\|\psi\|_{L^\infty(\mathbb{R}^3)}|\eta-\eta_*|^\gamma \\ &\le C\|\psi\|_{L^\infty(\mathbb{R}^3)}(|\eta-\xi|^\gamma + |\xi_*-\eta_*|^\gamma + |\xi|^\gamma + |\xi_*|^\gamma) \\ &\le C\|\psi\|_{L^\infty(\mathbb{R}^3)}(2\epsilon^\gamma + |\xi|^\gamma + |\xi_*|^\gamma), \end{aligned}$$

for any $|\xi-\eta| \le \epsilon$ and any $|\xi_*-\eta_*| \le \epsilon$. Therefore, we obtain

$$\iint_{t,x} |\tau_h J_1 - J_1|^2 \le Cb_1^2\|\phi\|^2_{L^\infty(\mathbb{R}^3)}(\epsilon^{2\gamma}J_{11} + J_{12} + J_{13}), \tag{4.13}$$

where

$$J_{11} = \iint_{t,x}\left|\int_{\eta_*}\left|\left[(\tau_h - Id)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\right](t,x)\right|\iint_{\xi,\eta}\mathbf{M}(\xi)\psi_\epsilon(\xi-\eta)\right|^2,$$

$$J_{12} = \iint_{t,x}\left|\int_{\eta_*}\left|\left[(\tau_h - Id)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\right](t,x)\right|\iint_{\xi,\eta}|\xi|^\gamma\mathbf{M}(\xi)\psi_\epsilon(\xi-\eta)\right|^2,$$

$$J_{13} = \iint_{t,x}\left|\int_{\eta_*}\left|\left[(\tau_h - Id)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}|\xi|^\gamma u)\right](t,x)\right|\iint_{\xi,\eta}\mathbf{M}(\xi)\psi_\epsilon(\xi-\eta)\right|^2.$$

In the following, we will only estimate J_{11} because J_{12} and J_{13} can be estimated similarly. In J_{11}, the integral over η,ξ is bounded uniformly in ϵ, that is,

$$\iint_{\eta,\xi}\mathbf{M}(\xi)\psi_\epsilon(\xi-\eta) = \int_\xi \mathbf{M}(\xi) \le C.$$

Let $\lambda > 0$ be a small constant to be determined later. Then it follows from the Hölder inequality and Lemma 2.5 that

$$\begin{aligned} J_{11} &\le \iint\limits_{t,x} \left| \int\limits_{\eta_*} \left| \left[(\tau_h - Id)\rho_{\psi_\epsilon(\cdot - \eta_*)}(\sqrt{\mathbf{M}}u) \right](t,x) \right| \right|^2 \\ &\le \iint\limits_{t,x} \int\limits_{\eta_*} \exp(-\lambda|\eta_*|^2) \int\limits_{\eta_*} \exp(\lambda|\eta_*|^2) \left| \left[(\tau_h - Id)\rho_{\psi_\epsilon(\cdot - \eta_*)}(\sqrt{\mathbf{M}}u) \right](t,x) \right|^2 \\ &\le \frac{|h|}{\lambda^{3/2}} \int\limits_{\eta_*} \exp(\lambda|\eta_*|^2) \left\| \rho_{\psi_\epsilon(\cdot - \eta_*)e_\lambda} \left(\frac{\sqrt{\mathbf{M}}u}{e_\lambda} \right) \right\|^2_{L^2([0,T];\dot{H}^{1/2}(\mathbb{R}^3))}. \end{aligned}$$

This together with Lemma 2.1 give

$$\begin{aligned} J_{11} \le \frac{C_s|h|}{\lambda^{3/2}} \int\limits_{\eta_*} \exp(\lambda|\eta_*|^2) &\left(\iint\limits_{x,\xi_*} \left| \frac{\sqrt{\mathbf{M}}(\xi_*)u_0(x,\xi_*)}{e_\lambda(\xi_*)} \right|^2 \psi_\epsilon(\xi_* - \eta_*)^2 e_\lambda(\xi_*)^2 \langle \xi_* \rangle^{2s} \right. \\ &\left. + \iiint\limits_{t,x,\xi_*} \left| (\partial_t + \xi_* \cdot \nabla_x) \frac{\sqrt{\mathbf{M}}(\xi_*)u(t,x,\xi_*)}{e_\lambda(\xi_*)} \right|^2 \psi_\epsilon(\xi_* - \eta_*)^2 e_\lambda(\xi_*)^2 \langle \xi_* \rangle^{2s} \right). \end{aligned} \tag{4.14}$$

Notice that

$$\exp(\lambda|\eta_*|^2) \le \exp(2\lambda|\xi_* - \eta_*|^2 + 2\lambda|\xi_*|^2) \le \exp(2\lambda\epsilon^2 + 2\lambda|\xi_*|^2),$$

holds for any $|\xi_* - \eta_*| \le \epsilon$, and

$$\int\limits_{\eta_*} \psi_\epsilon(\xi_* - \eta_*)^2 = \int\limits_{\mathbb{R}^3} \psi_\epsilon(\xi)^2 d\xi \le \frac{C_\psi}{\epsilon^3},$$

where C_ψ is a constant depending only on ψ. Putting these estimates into (4.14) yields

$$\begin{aligned} J_{11} \le \frac{C_s C_\psi \exp(2\lambda\epsilon^2)|h|}{\lambda^{3/2}\epsilon^3} &\left(\iint\limits_{x,\xi_*} \left| \frac{\sqrt{\mathbf{M}}(\xi_*)u_0(x,\xi_*)}{e_{2\lambda}(\xi_*)} \right|^2 e_\lambda(\xi_*)^2 \langle \xi_* \rangle^{2s} \right. \\ &\left. + \iiint\limits_{t,x,\xi_*} \left| (\partial_t + \xi_* \cdot \nabla_x) \frac{\sqrt{\mathbf{M}}(\xi_*)u(t,x,\xi_*)}{e_{2\lambda}(\xi_*)} \right|^2 e_\lambda(\xi_*)^2 \langle \xi_* \rangle^{2s} \right). \end{aligned}$$

As for the estimates on $I_1(\xi)$, $I_2(\xi)$ given in (3.3) and (3.4), we can suitably choose some small constant $\lambda > 0$ such that

$$\begin{aligned} J_{11} &\le \frac{C_s C_\psi \exp(2\lambda\epsilon^2)|h|}{\lambda^{3/2}\epsilon^3} \Big[M^2_{\lambda,s,\infty} \|u_0\|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)} \\ &\quad + b_1^2 M^2_{\lambda,s,\infty} T \|u\|_{L^\infty([0,T];L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))} \left(1 + \|u\|_{L^\infty[0,T] \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi} \right) \Big] \\ &\le C_\psi N_T(u)^2 \frac{\exp(2\lambda\epsilon^2)|h|}{\epsilon^3}, \end{aligned}$$

where $N_T(u)$ is defined by (3.1). Similarly, it holds that

$$J_{12}, J_{13} \le C_\psi N_T(u)^2 \frac{\exp(2\lambda\epsilon^2)|h|}{\epsilon^3}.$$

Therefore, (4.13) gives

$$\iint\limits_{t,x} |\tau_h J_1 - J_1|^2 \le C_\psi \|\phi\|^2_{L^\infty(\mathbb{R}^3)} N_T(u)^2 \frac{\exp(2\lambda\epsilon^2)(\epsilon^{2\gamma}+1)|h|}{\epsilon^3}. \tag{4.15}$$

Now we turn to the estimates on the following integral for J_2:

$$\iint\limits_{t,x} |\tau_h J_2 - J_2|^2 = \iint\limits_{t,x} \Bigg| \iint\limits_{\xi,\xi_*} \mathbf{M}(\xi)\sqrt{\mathbf{M}(\xi_*)}[u(t,x+h,\xi_*) - u(t,x,\xi_*)] \times \iint\limits_{\eta,\eta_*} [Z(\xi,\xi_*) - Z(\eta-\eta_*)]\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*)\Bigg|^2. \tag{4.16}$$

When $\gamma = 0$, by (4.5), we have

$$\begin{aligned}
&\iint\limits_{\eta,\eta_*} |Z(\xi,\xi_*) - Z(\eta-\eta_*)|\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) \\
&\le Cb_1\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)} \left(\iint\limits_{\eta,\eta_*} |\xi-\eta|\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) + \iint\limits_{\eta,\eta_*} |\xi_*-\eta_*|\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) \right) \\
&\le Cb_1\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)} \left(\int\limits_\eta |\xi-\eta|\psi_\epsilon(\xi-\eta) \int\limits_{\eta_*} \psi_\epsilon(\xi_*-\eta_*) + \int\limits_\eta \psi_\epsilon(\xi-\eta) \int\limits_{\eta_*} |\xi_*-\eta_*|\psi_\epsilon(\xi_*-\eta_*) \right) \\
&\le C_\psi b_1 \|\phi\|_{W^{1,\infty}(\mathbb{R}^3)}\epsilon.
\end{aligned}$$

When $0 < \gamma \le 1$, similarly, it follows from (4.6) that

$$\begin{aligned}
&\iint\limits_{\eta,\eta_*} |Z(\xi,\xi_*) - Z(\eta-\eta_*)|\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) \\
&\le Cb_1\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)}|\xi-\xi_*|^\gamma \iint\limits_{\eta,\eta_*} (|\xi-\eta| + |\xi_*-\eta_*|)\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) \\
&\quad + Cb_1\|\phi\|_{L^\infty(\mathbb{R}^3)} \iint\limits_{\eta,\eta_*} (|\xi-\eta|^\gamma + |\xi_*-\eta_*|^\gamma)\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) \\
&\le C_\psi b_1\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)}(|\xi-\xi_*|^\gamma\epsilon + \epsilon^\gamma).
\end{aligned}$$

Thus, for $0 \le \gamma \le 1$, it holds that

$$\iint\limits_{\eta,\eta_*} |Z(\xi,\xi_*) - Z(\eta-\eta_*)|\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*) \le C_\psi b_1\|\phi\|_{W^{1,\infty}(\mathbb{R}^3)}(|\xi-\xi_*|^\gamma\epsilon + \delta_\gamma\epsilon^\gamma),$$

where $\delta_\gamma = 0$ if $\gamma = 0$ and $\delta_\gamma = 1$ if $0 < \gamma \leq 1$. Putting the above estimate into (4.16) gives

$$\iint_{t,x} |\tau_h J_2 - J_2|^2 \leq C_\psi b_1^2 \|\phi\|^2_{W^{1,\infty}(\mathbb{R}^3)} \epsilon^2$$
$$\times \iint_{t,x} \left| \iint_{\xi,\xi_*} |\xi - \xi_*|^\gamma \mathbf{M}(\xi)\sqrt{\mathbf{M}(\xi_*)}\,(|u(t,x+h,\xi_*)| + |u(t,x,\xi_*)|) \right|^2$$
$$+ C_\psi b_1^2 \|\phi\|^2_{W^{1,\infty}(\mathbb{R}^3)} \delta_\gamma \epsilon^{2\gamma} \iint_{t,x} \left| \iint_{\xi,\xi_*} \mathbf{M}(\xi)\sqrt{\mathbf{M}(\xi_*)}(|u(t,x+h,\xi_*)| + |u(t,x,\xi_*)|) \right|^2.$$

Then Hölder inequality yields

$$\iint_{t,x} |\tau_h J_2 - J_2|^2 \leq C_\psi \|\phi\|^2_{W^{1,\infty}(\mathbb{R}^3)} N_T(u)^2 (\epsilon^2 + \delta_\gamma \epsilon^{2\gamma}). \tag{4.17}$$

Now we can complete the proof for (4.7) when $i = 1$. In fact, combining (4.15) and (4.17) gives

$$\iint_{[0,T]\times\mathbb{R}^3} dtdx \left| \int_{\mathbb{R}^3} S_1^Q(t,x+h,\xi)\phi(\xi)d\xi - \int_{\mathbb{R}^3} S_1^Q(t,x,\xi)\phi(\xi)d\xi \right|^2$$
$$\leq 2 \iint_{t,x} |\tau_h J_1 - J_1|^2 + 2 \iint_{t,x} |\tau_h J_2 - J_2|^2$$
$$\leq C \|\phi\|^2_{W^{1,\infty}(\mathbb{R}^3)} N_T(u)^2 \left[\frac{\exp(2\lambda\epsilon^2)(\epsilon^{2\gamma}+1)|h|}{\epsilon^3} + (\epsilon^2 + \delta_\gamma \epsilon^{2\gamma}) \right],$$

for any $\epsilon > 0$. Hence, (4.7) for the case of $i = 1$ follows by taking $\epsilon = |h|^{1/5}$ if $\gamma = 0$, and $\epsilon = |h|^{1/(3+2\gamma)}$ if $0 < \gamma \leq 1$ for $|h| \leq 1$. The same argument leads to (4.7) when $i = 2$.

Finally, we consider the case when $i = 3$ in (4.7) whose proof needs some modification on the above proof for the case when $i = 1$. In fact, similar to (4.10), (4.11) and (4.12), the velocity averaged function (4.3) of S_3^Q can be rewritten as

$$\int_{\mathbb{R}^3} S_3^Q(t,x,\xi)\phi(\xi)d\xi = \widetilde{J}_1 + \widetilde{J}_2,$$

where

$$\widetilde{J}_1 = \iint_{\xi,\xi_*} \sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*) \iint_{\eta,\eta_*} Z(\eta,\eta_*)\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*)$$
$$= \iint_{\eta,\eta_*} Z(\eta,\eta_*)\rho_{\psi_\epsilon(\cdot-\eta)}(\sqrt{\mathbf{M}}u)(t,x)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)(t,x),$$

and

$$\widetilde{J}_2 = \iint_{\xi,\xi_*} \sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*) \iint_{\eta,\eta_*} [Z(\xi,\xi_*) - Z(\eta,\eta_*)]\psi_\epsilon(\xi-\eta)\psi_\epsilon(\xi_*-\eta_*).$$

Notice that

$$\begin{aligned}&\tau_h\left[\rho_{\psi_\epsilon(\cdot-\eta)}(\sqrt{\mathbf{M}}u)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\right]-\rho_{\psi_\epsilon(\cdot-\eta)}(\sqrt{\mathbf{M}}u)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\\&=\tau_h\left[\rho_{\psi_\epsilon(\cdot-\eta)}(\sqrt{\mathbf{M}}u)\right]\left(\tau_h\left[\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\right]-\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u)\right)\\&\quad+\left(\tau_h\left[\rho_{\psi_\epsilon(\cdot-\eta)}(\sqrt{\mathbf{M}}u)\right]-\rho_{\psi_\epsilon(\cdot-\eta)}(\sqrt{\mathbf{M}}u)\right)\rho_{\psi_\epsilon(\cdot-\eta_*)}(\sqrt{\mathbf{M}}u),\end{aligned}$$

and

$$\iint\limits_{\eta,\xi}\sqrt{\mathbf{M}(\xi)}|u(t,x+h,\xi)|\psi_\epsilon(\xi-\eta)\le\|u\|_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\iint\limits_{\eta,\xi}\sqrt{\mathbf{M}(\xi)}\psi_\epsilon(\xi-\eta).$$

Thus, $\iint|\tau_h\widetilde{J}_1-\widetilde{J}_1|^2dtdx$ can be estimated in a way similar to the first two cases. On the other hand, the estimate on $\iint|\tau_h\widetilde{J}_2-\widetilde{J}_2|^2dtdx$ follows by noticing that

$$\begin{aligned}&\left|\tau_h[\sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*)]-\sqrt{\mathbf{M}(\xi)}u(t,x,\xi)\sqrt{\mathbf{M}(\xi_*)}u(t,x,\xi_*)\right|\\&\quad\le\|u\|_{L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\sqrt{\mathbf{M}(\xi)}\sqrt{\mathbf{M}(\xi_*)}(|u(t,x+h,\xi_*)|+|u(t,x,\xi_*)|).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.4. *Let $\alpha_0(\gamma)$ be defined in (1.6). For each $i=1,2,3$ and for any $0<\alpha<\alpha_0(\gamma)$, we have*

$$S_i^Q\in L^2([0,T]\times B_{R_\xi};H^\alpha(B_{R_x})).$$

Furthermore,

$$\|S_i^Q\|_{L^2([0,T]\times B_{R_\xi};H^\alpha(B_{R_x}))}\le\frac{CK_T(u)\langle R\rangle^{3/2}}{\sqrt{\alpha_0(\gamma)-\alpha}},$$

where $K_T(u)$ is given by (4.8).

Proof. Let $0<|h|\le 1$. Take the mollifier $\psi_\delta(\cdot)$ with $\delta>0$ to be determined later. Write S_i^Q as

$$S_i^Q=(S_i^Q-\psi_\delta*_\xi S_i^Q)+\psi_\delta*_\xi S_i^Q.$$

Then

$$\begin{aligned}&\iiint\limits_{[0,T]\times\mathbb{R}^3_x\times B_{R_\xi}}dtdxd\xi|\tau_hS_i^Q-S_i^Q|^2\\&\le 4\iiint\limits_{[0,T]\times\mathbb{R}^3_x\times B_{R_\xi}}dtdxd\xi|S_i^Q-\psi_\delta*_\xi S_i^Q|^2+2\int\limits_{B_{R_\xi}}d\xi\iint\limits_{[0,T]\times\mathbb{R}^3_x}dtdx|\tau_h(\psi_\delta*_\xi S_i^Q)-\psi_\delta*_\xi S_i^Q|^2.\end{aligned}$$

It follows from Lemmas 2.6, 4.1 and 4.3 that

$$\begin{aligned}&\iiint\limits_{[0,T]\times\mathbb{R}^3_x\times B_{R_\xi}}dtdxd\xi|\tau_hS_i^Q-S_i^Q|^2\le C\|S_i^Q\|^2_{L^2([0,T]\times\mathbb{R}^3_x;\dot H^1(\mathbb{R}^3_\xi))}\delta^2\\&+C_\psi K_T(u)^2P_\gamma(h)\int\limits_{B_{R_\xi}}\|\psi_\delta(\xi-\cdot)\|_{W^{1,\infty}(\mathbb{R}^3)}d\xi\le C_\psi K_T(u)^2\langle R\rangle^3\left(\delta^2+\frac{P_\gamma(h)}{\delta^8}\right),\end{aligned}$$

where $P_\gamma(h)$ is defined by (4.9). We choose $\delta = |h|^{1/25}$ if $\gamma = 0$, and $\delta = |h|^{\gamma/5(3+2\gamma)}$ if $0 < \gamma \le 1$ to have

$$\iiint_{[0,T]\times\mathbb{R}^3_x\times B_{R_\xi}} dtdxd\xi |\tau_h S_i^Q - S_i^Q|^2 \le C_\psi K_T(u)^2 \langle R\rangle^3 |h|^{2\alpha_0(\gamma)}.$$

Then, for any fixed $\alpha \in (0, \alpha_0(\gamma))$, we have

$$\begin{aligned}\iiiint_{[0,T]\times B_{R_x}\times B_{R_h}\times B_{R_\xi}} \frac{|\tau_h S_i^Q - S_i^Q|^2}{|h|^{3+2\alpha}} &\le \int_{|h|\le 1} \frac{C_\psi K_T(u)^2\langle R\rangle^3}{|h|^{3+2[\alpha-\alpha_0(\gamma)]}} + \int_{|h|\le R} \|S_i^Q\|_{L^2([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \\ &\le \frac{C_\psi K_T(u)^2\langle R\rangle^3}{\alpha_0(\gamma)-\alpha} + CN_T(u)^2\langle R\rangle^3 \le \frac{C_\psi K_T(u)^2\langle R\rangle^3}{\alpha_0(\gamma)-\alpha}.\end{aligned}$$

This completes the proof of the lemma.

□

Corollary 4.1. *Let* $0 < \alpha < \alpha_0(\gamma)$ *with* $\alpha_0(\gamma)$ *defined in* (1.6). *For each* $i = 1, 2, 3$, *we have*

$$S_i^Q, Q^+(f,f) \in L^2_{loc}(\mathbb{R}_+; H^\alpha_{loc}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)),$$
$$0 \le Q^+(f,f) \in L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi).$$

We are now ready to complete the proof for Theorem 1.1.

Proof of Theorem 1.1. By combining all estimates obtained above, we have

(i) the mild form of the solution:

$$f(t,x,\xi) = f_0(x-\xi t,\xi)\Gamma_1(t,x,\xi) + \Gamma_2(t,x,\xi),$$

where

$$\Gamma_1(t,x,\xi) = \exp\left(-\int_0^t Lf(\theta, x-\xi\theta,\xi)d\theta\right),$$
$$\Gamma_2(t,x,\xi) = \Gamma_1(t,x,\xi)\int_0^t Q^+(f,f)(s,x-\xi s,\xi)\exp\left(\int_0^s Lf(\theta,x-\xi\theta,\xi)d\theta\right)ds.$$

(ii) The regularization estimates:

$$0 \le Lf \in L^2([0,T]; H^{1/2}_{loc}(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)) \cap L^\infty([0,T]\times\mathbb{R}^3_x; L^\infty_{loc}(\mathbb{R}^3_\xi)),$$
$$0 \le Q^+(f,f) \in L^2([0,T]; H^\alpha_{loc}(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)) \cap L^\infty([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi),$$

for any $0 < \alpha < \alpha_0(\gamma)$ with $\alpha_0(\gamma)$ defined by (1.6).

As in [8], the above estimates are sufficient to give (1.5)-(1.7) by straightforward calculation so that we omit the details. As for the perturbation u, since

$$u(t,x,\xi) = u_0(x-\xi t,\xi)\Gamma_1(t,x,\xi) + \left[\sqrt{\mathbf{M}}\Gamma_1(t,x,\xi) + \frac{1}{\sqrt{\mathbf{M}}}\Gamma_2(t,x,\xi) - \sqrt{\mathbf{M}}\right],$$

(1.8)-(1.9) follow from the estimation on f. This completes the proof of Theorem 1.1.

Remark 4.1. *Here we claim that the nonnegative functions* $\Gamma_1(t,x,\xi)$ *and hence* $\widetilde{\Gamma}_1(t,x,\xi)$ *decay exponentially for large* t *uniformly in* x,ξ, *provided that the* L^∞*-norm of* u *over* $\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$ *is small enough. This means that the singular part vanishes with an exponential rate. In fact, notice that*

$$\int_0^t Lf(\theta,x-\xi\theta,\xi)d\theta = \nu(\xi)t + \int_0^t A *_\xi (\sqrt{\mathbf{M}}u)(\theta,x-\xi\theta,\xi)d\theta,$$

where the collision frequency $\nu(\xi) := A *_\xi \mathbf{M}$ *satisfies the condition, cf. [15, 25] that there is a positive constant* ν_0 *such that*

$$\nu_0(1+|\xi|)^\gamma \le \nu(\xi) \le \frac{1}{\nu_0}(1+|\xi|)^\gamma.$$

Furthermore, one can see

$$|A *_\xi (\sqrt{\mathbf{M}}u)| \le C\|u\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\nu(\xi).$$

Thus one has

$$0 \le \Gamma_1(t,x,\xi) \le \exp\Big(-\nu(\xi)(1-C\|u\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)})t\Big)$$

for any $t\ge 0, x\in\mathbb{R}^3, \xi\in\mathbb{R}^3$. *Therefore, the aforementioned claim follows.*

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