Global Existence to Boltzmann Equation with External Force in Infinite Vacuum

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Abstract

In this paper, we give a condition on the bicharacteristic which guarantees the global existence of the mild solution to the Boltzmann equation with an external force for the hard-sphere model and potentials with angular cutoff in infinite vacuum. This generalizes the previous results to the case when the force can have arbitrary strength. The constructive condition on the bicharacteristic is used to obtain the point-wise estimates on the collision operator so that the global existence comes from the contraction mapping theorem.

Key words: Boltzmann equation; external force; mild solution; global existence.

1. Introduction

The purpose of this paper is to study the global existence of mild solutions to the initial value problem for the Boltzmann equation with an external force for the hard-sphere model and some angle cutoff potential. Let $f = f(t, x, v)$ be the density distribution function of the interacting gas particles at time $t \geq 0$ and position $x \in \mathbf{R}^3$ with velocity $v \in \mathbf{R}^3$ for rarefied gases. In the presence of external forces depending only on space and time variables, the evolution of f is described by the Boltzmann equation

$$f_t + v \cdot \nabla_x f + E(t, x) \cdot \nabla_v f = Q[f], \tag{1.1}$$

with initial data

$$f(0, x, v) = f_0(x, v), \tag{1.2}$$

where $E(t, x) \in \mathbf{R}^3$ is an external force and Q is a nonlinear collision operator capturing the binary collisions between particles whose specific form will be given below.

Let (v, v_*) and (v', v'_*) be velocities before and after the collision respectively. Under the assumption of elastic collision, the conservation of the momentum and energy:

$$v + v_* = v' + v'_*,$$
$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

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yields

$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega, \tag{1.3}$$

where $\omega \in S^2_+ = \{\omega \in S^2 : (v - v_*) \cdot \omega \geq 0\}$. Moreover, the collision term $Q[f]$ takes the form

$$Q[f](v) = \frac{1}{\varepsilon} \int_{\mathbf{R}^3 \times S^2_+} B(|v - v_*|, \theta)\{f(v')f(v'_*) - f(v)f(v_*)\}\, dv_* d\omega, \tag{1.4}$$

where ε is the Knudsen number proportional to the mean free path. For simplicity of notation, we sometimes use $f(v)$ to denote $f(t, x, v)$ without any ambiguity. In (1.4), the function $B(|v - v_*|, \theta)$ is the collision cross section with

$$\theta = \cos^{-1}\left(\frac{(v - v_*) \cdot \omega}{|v - v_*|}\right) \in \left[0, \frac{\pi}{2}\right] \tag{1.5}$$

being the scattering angle between $v - v_*$ and ω. The definition of B depends on the physics of collision. In fact, for the inverse power interaction potential, B takes the form of

$$B(|v - v_*|, \theta) = b_\gamma(\theta)|v - v_*|^\gamma, \quad -3 < \gamma < 1, \tag{1.6}$$

with $\gamma = 0$ corresponding to the Maxwellian molecules, $\gamma > 0$ corresponding to the hard interaction and $\gamma < 0$ corresponding to the soft interaction. Moreover, the hard sphere model satisfies $B(|v - v_*|, \theta) = \sigma|v - v_*| \cos\theta$ with σ being the radius of the hard sphere.

For later use, as in [10] we denote $u = v - v_*$, $u_\| = (u \cdot \omega)\omega$ and $u_\perp = u - u_\|$ so that

$$v' = v - u_\|, \quad v'_* = v - u_\perp. \tag{1.7}$$

Then the collision term $Q[f]$ becomes

$$Q[f](v) = \frac{1}{\varepsilon} \int_{\mathbf{R}^3 \times S^2_+} B(|u|, \theta)\{f(v')f(v'_*) - f(v)f(v - u)\}\, du d\omega, \tag{1.8}$$

where v' and v'_* are given by (1.7). Furthermore, let's denote the collision operator

$$Q(f, g) = Q^+(f, g) - Q^-(f, g) \tag{1.9}$$

with the gain term Q^+ and the loss term Q^- given by

$$Q^+(f, g)(t, x, v) = \frac{1}{\varepsilon} \int_{\mathbf{R}^3 \times S^2_+} B(|u|, \theta) f(t, x, v') g(t, x, v'_*)\, du d\omega \tag{1.10}$$

and

$$Q^-(f, g)(t, x, v) = \frac{1}{\varepsilon} \int_{\mathbf{R}^3 \times S^2_+} B(|u|, \theta) f(t, x, v) g(t, x, v - u)\, du d\omega. \tag{1.11}$$

Clearly,

$$Q[f] = Q(f, f). \tag{1.12}$$

For any fixed point (t, x, v) in $\mathbf{R}^+ \times \mathbf{R}^3 \times \mathbf{R}^3$, we now consider the bicharacteristic equations of (1.1) in $\mathbf{R}^3 \times \mathbf{R}^3$:

$$\begin{cases} \dfrac{dX}{ds} = V, \quad \dfrac{dV}{ds} = E(s, X), \\ (X, V)|_{s=t} = (x, v). \end{cases} \tag{1.13}$$

Suppose that the above ODE system have smooth solutions globally in time denoted by

$$[X(s; t, x, v), V(s; t, x, v)] \tag{1.14}$$

for any $(t,x,v)\in\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3$. Then integrating (1.1) along the bicharacteristic, we obtain the representation of the mild solution to the Boltzmann equation:

$$\begin{aligned} f(t,x,v) &= f_0(X(0;t,x,v),V(0;t,x,v)) \\ &\quad + \int_0^t Q(f,f)(s,X(s;t,x,v),V(s;t,x,v))\,ds. \end{aligned} \tag{1.15}$$

In fact, the mild solution can be defined as follows.

Definition 1.1 *A nonnegative function $f(t,x,v)\in C\left([0,T);L^1_+(\mathbf{R}^3\times\mathbf{R}^3)\right)$ is a mild solution to (1.1) with a nonnegative initial data f_0 if and only if f satisfies the integral equation (1.15) for all $t\in[0,T)$ and a.e. $(x,v)\in\mathbf{R}^3\times\mathbf{R}^3$.*

The function spaces on $\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3$ for the solutions constructed later can be defined in the following. For any $p>0$, $q>0$, let $S_{p,q}$ be the completion of the set consisting of the continuous functions of compact support with respect to the norm

$$|||f||| = \sup_{t,x,v}\left(1+|X(0;t,x,v)|^2\right)^p \exp\left\{q|V(0;t,x,v)|^2\right\}|f(t,x,v)|. \tag{1.16}$$

And for any fixed time t, define the norm

$$\|f(t)\| = \sup_{x,v}\left(1+|X(0;t,x,v)|^2\right)^p \exp\left\{q|V(0;t,x,v)|^2\right\}|f(t,x,v)|, \tag{1.17}$$

in particular,

$$\|f_0\|_{p,q} = \sup_{x,v}(1+|x|^2)^p e^{q|v|^2}|f_0(x,v)|. \tag{1.18}$$

Throughout this paper, the assumptions on p, q, the collision kernel and the external force can be summarized as follows.

(A1) $p>\frac{1}{2}$ and $q>0$;

(A2) The collision kernel B takes the inverse power interaction (1.6) with $-2<\gamma\le 1$ and the following angular cutoff condition:

$$\int_0^{\frac{\pi}{2}} b_\gamma(\theta)(1+\tan\theta)\,d\theta \le b_0, \tag{1.19}$$

where b_0 is a positive constant;

(A3) The external force $E(t,x)$ is C^0 in (t,x). Furthermore, for any fixed point $(t,x,v)\in\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3$, the first order ODE system (1.13) has global smooth solutions (1.14) satisfying the following constructive condition:

$$\begin{cases} X(0;s,X(s;t,x,v),V(s;t,x,v)-\xi) = X(0;t,x,v)+\alpha_1(s;t,x,v)\xi, \\ V(0;s,X(s;t,x,v),V(s;t,x,v)-\xi) = V(0;t,x,v)-\alpha_2(s;t,x,v)\xi, \end{cases} \tag{1.20}$$

for any $s\in\mathbf{R}^+$ and $\xi\in\mathbf{R}^3$, where $\alpha_1(s;t,x,v),\alpha_2(s;t,x,v)\in C^1(s)$ satisfy the following inequalities

$$\begin{cases} \alpha_i(s;t,x,v)>0, \quad i=1,2, \\ \alpha(s;t,x,v)\equiv \alpha_1'(s;t,x,v)\alpha_2(s;t,x,v)-\alpha_1(s;t,x,v)\alpha_2'(s;t,x,v)>0, \\ (\alpha_2(s;t,x,v))^{\gamma+1}\alpha(s;t,x,v)\ge\alpha_0>0, \end{cases} \tag{1.21}$$

where α_0 is a positive constant independent of s and (t,x,v). Here and in the sequel $\alpha_i'(s;t,x,v)$ represent the derivative with respect to s.

Now we can state the main result of this paper as follows.

Theorem 1.2. *Under the assumptions (A1)-(A3), there is a sufficiently small positive constant* $\delta_0 > 0$ *such that if* $0 \le f_0(x,v)$ *and* $\|f_0\|_{p,q} \le \varepsilon\delta_0$*, then there exists a unique global in time mild solution* $f(t,x,v)$ *to the initial value problem (1.1) and (1.2) satisfying* $|||f||| \le 2\varepsilon\delta_0$.

Remark 1.3. For the moment let's discuss the assumptions (A1)-(A3). First, (A1) means that initial data decays with algebraic rate in the space variable x, and decays with exponential rate in the velocity variable v. Then (A2) holds for the collision kernels satisfying the inverse power law with an angular cutoff assumption and the hard sphere model. Finally, (A3) is a constructive assumption on the external forces which can be satisfied by forces without decay or smallness assumption on their strength. In the following, we will give two examples of external forces satisfying (A3). It should be pointed that in general the assumption (A3) is difficult to verify in realistic situation. But in any case this is the first step to deal with the Boltzmann equation with the external force which can have arbitrary strength. The proof of Theorem 1.2 for the case when the external force is more general than the assumption (A3) is left to the future study.

Example 1.4. Let $E(t,x) \equiv E(t)$. For any fixed (t,x,v), the bicharacteristic equations (1.13) have solutions

$$\begin{cases} X(s;t,x,v) = x + v(s-t) + \displaystyle\int_t^s \int_t^\eta E(\tau)\,d\tau d\eta, \\ V(s;t,x,v) = v + \displaystyle\int_t^s E(\tau)\,d\tau. \end{cases}$$

Hence,

$$\begin{cases} X(0;t,x,v) = x - vt - \displaystyle\int_0^t \int_t^\eta E(\tau)\,d\tau d\eta, \\ V(0;t,x,v) = v - \displaystyle\int_0^t E(\tau)\,d\tau. \end{cases}$$

Therefore, (A3) holds with

$$\alpha_1(s;t,x,v) = s \quad \text{and} \quad \alpha_2(s;t,x,v) = 1,$$

when $0 < \alpha_0 \le 1$ and $-2 < \gamma \le 1$.

In fact, notice that for this case when external forces depend only on the time t, the Boltzmann equation with external forces can be rewritten to the Boltzmann equation without forces by the following transformation of independent variables, cf. [7]:

$$\begin{cases} \tilde{t} = t, \\ \tilde{x} = x - \displaystyle\int_0^t \int_0^\eta E(\tau)\,d\tau d\eta, \\ \tilde{v} = v - \displaystyle\int_0^t E(\tau)\,d\tau. \end{cases} \tag{1.22}$$

Therefore, the existence of mild and classical solutions and L^1 stability around vacuum to the Boltzmann equation without forces, cf. [12, 24, 11], can be applied to this case without any difficulty.

Example 1.5. Let $E(t,x)=a^2x+E_0(t)$, with $a>0$ being a constant. For any fixed (t,x,v), the bicharacteristic equations (1.13) have solutions

$$\begin{cases} X(s;t,x,v)=\dfrac{ax+v}{2a}e^{a(s-t)}+\dfrac{ax-v}{2a}e^{-a(s-t)}-\dfrac{E_2(t)}{2a}e^{a(s-t)}+\dfrac{E_3(t)}{2a}e^{-a(s-t)}+E_1(s),\\ V(s;t,x,v)=\dfrac{ax+v}{2}e^{a(s-t)}-\dfrac{ax-v}{2}e^{-a(s-t)}-\dfrac{E_2(t)}{2}e^{a(s-t)}-\dfrac{E_3(t)}{2}e^{-a(s-t)}+E_1'(s),\end{cases}$$

where $E_1(s)$ is some special solution to the second order linear ODE:

$$\frac{d^2E_1(s)}{ds^2}=a^2E_1(s)+E_0(s),$$

and $(E_2(t),E_3(t))$ are defined by

$$\begin{cases} E_2(t)=E_1'(t)+aE_1(t),\\ E_3(t)=E_1'(t)-aE_1(t).\end{cases}$$

Hence,

$$\begin{cases} X(0;t,x,v)=\dfrac{ax+v}{2a}e^{-at}+\dfrac{ax-v}{2a}e^{at}-\dfrac{E_2(t)}{2a}e^{-at}+\dfrac{E_3(t)}{2a}e^{at}+E_1(0),\\ V(0;t,x,v)=\dfrac{ax+v}{2}e^{-at}-\dfrac{ax-v}{2}e^{at}-\dfrac{E_2(t)}{2}e^{-at}-\dfrac{E_3(t)}{2}e^{at}+E_1'(0).\end{cases}$$

Straightforward calculation shows that

$$\alpha_1(s;t,x,v)=\frac{1}{2a}(e^{as}-e^{-as})\ \text{ and }\ \alpha_2(s;t,x,v)=\frac{1}{2}(e^{as}+e^{-as}).$$

Thus,

$$\alpha(s;t,x,v)=\alpha_1'(s;t,x,v)\alpha_2(s;t,x,v)-\alpha_1(s;t,x,v)\alpha_2'(s;t,x,v)\equiv 1>0$$

and

$$\alpha_2(s;t,x,v)\geq 1.$$

If $0<\alpha_0\leq 1$ and $-1\leq\gamma\leq 1$, then (A3) holds.

Notice that the positive coefficient in front of x in the force term of Example 1.5 implies that the bicharacteristic curves go to infinity in space as time tends to infinity. If the coefficient is negative, then the bicharacteristic is oscillating in the space of (X,V) and there is no known existence results on this interesting case for the nonlinear Boltzmann equation. However, for the linearized Boltzmann equation, the case when $E(t,x)=-x$ was studied by Tabata [20] using the semigroup approach. This result was later generalized in [19] to the case of linearized Boltzmann equation with an unbounded external force potential that is spherically symmetric and satisfies some differential inequalities.

Now we compare Theorems 1.2 with the previous related work. First, if $p>3/2$, the bound on the initial data f_0 implies the total mass satisfies

$$\int_{\mathbf{R}^3\times\mathbf{R}^3}|f_0(x,v)|\,dxdv\leq C\varepsilon\delta_0,$$

where C is a generic positive constant. This requires that the mean free path is sufficient large if total mass is finite because δ_0 is sufficiently small. This is exactly the requirement on the Boltzmann equation without forces in infinite vacuum considered by Illner and Shinbrot in [12]. On the other hand, as in [2], if $1/2 < p \leq 3/2$, then the initial total mass can take infinity, which is a very interesting case not included in the well-known paper [6] about the large initial theory in L^1-norm. For the method of proof, as in [10, 24], Theorem 1.2 is obtained by using the contraction mapping theorem. Precisely, we obtain the following estimates

$$\begin{cases} |||\mathbf{T}f||| \leq C\|f_0\|_{p,q} + C|||f|||^2, \\ |||\mathbf{T}f - \mathbf{T}g||| \leq C(|||f||| + |||g|||)|||f-g|||, \end{cases} \tag{1.23}$$

where $\mathbf{T}$ is a mapping from $S_{p,q}$ to $S_{p,q}$ defined by (3.18) in Section 3. In order to prove (1.23), we need to control the time integration of the collision term $Q(f,g)$ along the bicharacteristic:

$$\int_0^t Q(f,g)(s, X(s;t,x,v), V(s;t,x,v))\, ds.$$

The estimation on this integral is based on the constructive assumption (A3) and is given in Lemma 3.1 of Section 3.

There have been extensive studies on the mathematical aspects on the Boltzmann equation, see monographs [2, 4, 5]. For the Boltzmann equation in the absence of an external force in infinite vacuum, the global existence of mild solutions to (1.1) was first given by Illner-Shinbrot [12] following the work on local existence by Kaniel-Shinbrot in [13]. For perturbation of a global Maxwellian, Ukai [23], Nishida-Imai [15], Shizuta-Asano [18] and Ukai-Asano [25] and others showed the global existence of solutions to the initial or initial boundary value problem of the Boltzmann equation in various situations. For other interesting issues, such as large data existence theory, stability and convergence to the Maxwellian, see [6, 14, 17, 21] and references therein.

However, there are fewer works done for the Boltzmann equation with an external force. Glikson [8, 9] obtained the unique local existence of solutions to the initial value problem for sufficiently small initial data. When the initial data was arbitrary large, the local existence of solutions to the initial and initial boundary value problem was obtained by Asano [1]. And a general framework on the global existence solutions in infinite vacuum is given in [3]. Recently, Guo [10] proved the global existence of classical solutions with small amplitude to the initial value problem for the Boltzmann equation with an external force and a "soft" potential when the external force decays in time. Moreover, the global existence and stability of the stationary solutions to time independent potential force was obtained by Ukai-Yang-Zhao in [26] through the energy method. Our result is new in the sense that a constructive condition is given for the global existence of mild solutions in infinite vacuum when external force can be arbitrary large.

The rest of this paper is organized as follows. In Section 2, we give some preliminary estimates for later use. In Section 3, we study the integration of the collision term to obtain the global in time estimate. And then the existence of the mild solution to the initial value problem (1.1) and (1.2) follows from the contraction mapping theorem. In Section 4, based on the same idea as the last two sections and more technique inequalities, we extend Theorem 1.2 without the full proof for the expositive brevity to the case of the initial data with polynomial decay in velocity variable as well as position variable.

2. Preliminaries

In this section, we give some preliminary lemmas which will be used in the proof of the global existence of solutions in the next section. First we borrow three lemmas from

[11, 16] for the completeness of the paper. Interested readers please refer to these two papers for their proofs.

Lemma 2.1. *For any $z \in \mathbf{R}^3$, $\eta \in \mathbf{R}$ and $(u_\parallel, u_\perp) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $u_\parallel \cdot u_\perp = 0$, we have*

$$|z + \eta u_\parallel|^2 + |z + \eta u_\perp|^2 = |z|^2 + |z + \eta(u_\parallel + u_\perp)|^2. \tag{2.1}$$

Lemma 2.2 *For any $p > \frac{1}{2}$ and $(z, u) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $u \neq 0$, we have*

$$\int_0^\infty (1 + |z + \eta u|^2)^{-p} \, d\eta \leq \frac{4p}{|u|(2p-1)}. \tag{2.2}$$

Moreover, for any $q > 0$, $-2 < \gamma \leq 1$ and $z \in \mathbf{R}^3$, we have

$$\int_{\mathbf{R}^3} |u|^{\gamma-1} \exp\{-q|z-u|^2\} \, du \leq I^1_{\gamma,q}, \tag{2.3}$$

where

$$I^1_{\gamma,q} = \frac{4\pi}{\gamma+2} + \frac{\pi}{q^{3/2}}, \tag{2.4}$$

is a positive constant depending only on γ and q.

Lemma 2.3 *For any $p > 0$, $z \in \mathbf{R}^3$, $s \in \mathbf{R}^+$ and $(u_\parallel, u_\perp) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $u_\parallel \cdot u_\perp = 0$, we have that*

$$\begin{aligned}(1+ &|z + su_\parallel|^2)^{-p}(1 + |z + su_\perp|^2)^{-p} \leq (1 + |z|^2)^{-p} \\ &\left\{(1 + |z + su_\parallel|^2)^{-p} + (1 + |z + su_\perp|^2)^{-p} + (1 + |z + s(u_\parallel + u_\perp)|^2)^{-p}\right\}.\end{aligned} \tag{2.5}$$

In order to control the integration of the collision term $Q(f,g)$ along the bicharacteristic

$$\int_0^t \int_{\mathbf{R}^3 \times S^2_+} (\cdots)(s, X(s;t,x,v), V(s;t,x,v)) \, dud\omega ds,$$

we consider the following two integrals:

$$\begin{aligned}I^{2,1}(z_1, z_2, t, x, v) = \int_0^\infty \int_{\mathbf{R}^3 \times S^2_+} & b_\gamma(\theta)|u|^\gamma (1 + |z_1 + \alpha_1(s;t,x,v)u|^2)^{-p} \\ & \times \exp\{-q|z_2 - \alpha_2(s;t,x,v)u|^2\} \, dud\omega ds,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}I^{2,2}(z_1, z_2, t, x, v) = \int_0^\infty \int_{\mathbf{R}^3 \times S^2_+} & b_\gamma(\theta)|u|^\gamma \{(1 + |z_1 + \alpha_1(s;t,x,v)u_\parallel|^2)^{-p} \\ & + (1 + |z_1 + \alpha_1(s;t,x,v)u_\perp|^2)^{-p}\} \\ & \times \exp\{-q|z_2 - \alpha_2(s;t,x,v)u|^2\} \, dud\omega ds,\end{aligned} \tag{2.7}$$

for any $(z_1, z_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ and $(t, x, v) \in \mathbf{R}^+ \times \mathbf{R}^3 \times \mathbf{R}^3$. Here $\alpha_1(s;t,x,v)$ and $\alpha_2(s;t,x,v)$ satisfy the assumption (A3). The estimates on (2.6) and (2.7) are given in the following lemma.

Lemma 2.4 *Under the assumptions (A1)-(A3), it holds that*

$$\sup I^{2,i}(z_1, z_2, t, x, v) \leq I^2_{\gamma,p,q}, \quad i = 1, 2, \tag{2.8}$$

where

$$I^2_{\gamma,p,q} = \frac{8\pi p b_0 I^1_{\gamma,q}}{\alpha_0(2p-1)}, \tag{2.9}$$

is a positive constant depending only on γ, p, q, α_0 *and* b_0.

Proof. For $i = 1$, fix $(z_1, z_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ and $(t, x, v) \in \mathbf{R}^+ \times \mathbf{R}^3 \times \mathbf{R}^3$. Since $\alpha_2(s; t, x, v) > 0$, we let

$$\alpha_2(s; t, x, v)u = \bar{u},$$

to obtain

$$\begin{aligned} I^{2,1}(z_1, z_2, t, x, v) = \int_0^\infty \int_{\mathbf{R}^3 \times S^2_+} & b_\gamma(\theta)|\bar{u}|^\gamma (\alpha_2(s; t, x, v))^{-\gamma-3} \\ & \left(1 + \left|z_1 + \frac{\alpha_1(s; t, x, v)}{\alpha_2(s; t, x, v)}\bar{u}\right|^2\right)^{-p} \exp\{-q|z_2 - \bar{u}|^2\}\, d\bar{u}d\omega ds. \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} \frac{d}{ds}\left(\frac{\alpha_1(s; t, x, v)}{\alpha_2(s; t, x, v)}\right) &= \frac{\alpha_1'(s; t, x, v)\alpha_2(s; t, x, v) - \alpha_1(s; t, x, v)\alpha_2'(s; t, x, v)}{(\alpha_2(s; t, x, v))^2} \\ &= \frac{\alpha(s; t, x, v)}{(\alpha_2(s; t, x, v))^2} > 0, \end{aligned}$$

by the assumption (A3), we can let variable

$$\eta = \frac{\alpha_1(s; t, x, v)}{\alpha_2(s; t, x, v)},$$

to have

$$\begin{aligned} & I^{2,1}(z_1, z_2, t, x, v) \\ & \leq \int_0^\infty \int_{\mathbf{R}^3 \times S^2_+} b_\gamma(\theta)|\bar{u}|^\gamma \frac{1}{(\alpha_2(s; t, x, v))^{\gamma+1}\alpha(s; t, x, v)} \\ & \qquad \times (1 + |z_1 + \eta\bar{u}|^2)^{-p} \exp\{-q|z_2 - \bar{u}|^2\}\, d\bar{u}d\omega d\eta \\ & \leq \frac{1}{\alpha_0}\int_0^\infty \int_{\mathbf{R}^3 \times S^2_+} b_\gamma(\theta)|\bar{u}|^\gamma (1 + |z_1 + \eta\bar{u}|^2)^{-p} \exp\{-q|z_2 - \bar{u}|^2\}\, d\bar{u}d\omega d\eta. \end{aligned} \tag{2.11}$$

Then it follows from the Lemma 2.2 and the assumption (A2) that

$$\begin{aligned} I^{2,1}(z_1, z_2, t, x, v) &\leq \frac{4p}{\alpha_0(2p-1)} \int_{\mathbf{R}^3 \times S^2_+} b_\gamma(\theta)|\bar{u}|^{\gamma-1} \exp\{-q|z_2 - \bar{u}|^2\}\, d\bar{u}d\omega \\ &= \frac{8\pi p}{\alpha_0(2p-1)} \int_0^{\frac{\pi}{2}} b_\gamma(\theta)\sin\theta\, d\theta \int_{\mathbf{R}^3} |\bar{u}|^{\gamma-1} \exp\{-q|z_2 - \bar{u}|^2\}\, d\bar{u} \\ &\leq \frac{8\pi p b_0 I^1_{\gamma,q}}{\alpha_0(2p-1)}. \end{aligned} \tag{2.12}$$

This completes the proof for $i = 1$.

For the case of $i=2$, similar to (2.11) and (2.12), we have

$$
\begin{aligned}
& I^{2,2}(z_1,z_2,t,x,v) \\
\le\; & \frac{4p}{\alpha_0(2p-1)} \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|\bar{u}|^\gamma \left(\frac{1}{|\bar{u}_\parallel|}+\frac{1}{|\bar{u}_\perp|}\right) \exp\{-q|z_2-\bar{u}|^2\}\, d\bar{u}d\omega \\
=\; & \frac{8\pi p}{\alpha_0(2p-1)} \int_{\mathbf{R}^3}\int_0^{\frac{\pi}{2}} b_\gamma(\theta)|\bar{u}|^\gamma \left(\frac{1}{|\bar{u}|\cos\theta}+\frac{1}{|\bar{u}|\sin\theta}\right) \exp\{-q|z_2-\bar{u}|^2\}\sin\theta\, d\bar{u}d\theta \\
=\; & \frac{8\pi p}{\alpha_0(2p-1)} \int_0^{\frac{\pi}{2}} b_\gamma(\theta)(1+\tan\theta)\,d\theta \int_{\mathbf{R}^3} |\bar{u}|^{\gamma-1}\exp\{-q|z_2-\bar{u}|^2\}\,d\bar{u} \\
\le\; & \frac{8\pi p b_0 I^1_{\gamma,q}}{\alpha_0(2p-1)}.
\end{aligned} \tag{2.13}
$$

Hence, (2.12) and (2.13) yields the proof of Lemma 2.4.

3. Existence of the mild solution

In this section, we give the crucial estimate for the global existence of the solution by the contraction mapping theorem.

First, similar to (1.9)-(1.11), denote

$$N(f,g) = N^+(f,g) - N^-(f,g), \tag{3.1}$$

by

$$
\begin{aligned}
N^+(f,g)(t,x,v) &= \int_0^t Q^+(f,g)(s,X(s;t,x,v),V(s;t,x,v))ds \\
&= \frac{1}{\varepsilon}\int_0^t \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma f(s,X(s;t,x,v),V(s;t,x,v)-u_\parallel) \\
&\qquad \times g(s,X(s;t,x,v),V(s;t,x,v)-u_\perp)\,dud\omega ds,
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
N^-(f,g)(t,x,v) &= \int_0^t Q^-(f,g)(s,X(s;t,x,v),V(s;t,x,v))ds \\
&= \frac{1}{\varepsilon}\int_0^t \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma f(s,X(s;t,x,v),V(s;t,x,v)) \\
&\qquad \times g(s,X(s;t,x,v),V(s;t,x,v)-u)\,dud\omega ds.
\end{aligned} \tag{3.3}
$$

From (1.9), we have

$$N(f,g)(t,x,v) = \int_0^t Q(f,g)(s,X(s;t,x,v),V(s;t,x,v))\,ds. \tag{3.4}$$

Lemma 3.1. *Under the assumptions (A1)-(A3), it holds that*

$$|||N(f,g)||| \le \frac{1}{\varepsilon} I_{\gamma,p,q}|||f||| \times |||g|||, \tag{3.5}$$

where $I_{\gamma,p,q}$ is a positive constant.

Proof. We first estimate the loss term $N^-(f,g)$ in (3.1). Let (t,x,v) fixed. From the definition (1.17) of the norm $\|\cdot\|$ and the assumption (A3), we have for any s in $(0,t)$,

$$\begin{aligned}
&|f(s,\, X(s;t,x,v),V(s;t,x,v))|\\
&\quad\le \|f(s)\|(1+|X(0;s,X(s;t,x,v),V(s;t,x,v))|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;s,X(s;t,x,v),V(s;t,x,v))|^2\}\\
&\quad=\|f(s)\|(1+|X(0;t,x,v)|^2)^{-p}\exp\{-q|V(0;t,x,v)|^2\},
\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}
&|g(s,\, X(s;t,x,v),V(s;t,x,v)-u)|\\
&\quad\le \|g(s)\|(1+|X(0;s,X(s;t,x,v),V(s;t,x,v)-u)|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;s,X(s;t,x,v),V(s;t,x,v)-u)|^2\}\\
&\quad=\|g(s)\|(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u|^2\}.
\end{aligned}\tag{3.7}$$

Hence, from (3.3), we have

$$\begin{aligned}
&|N^-(f,g)(t,x,v)|\\
&\le\frac{1}{\varepsilon}\int_0^t\int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma\|f(s)\|\|g(s)\|(1+|X(0;t,x,v)|^2)^{-p}\\
&\quad\times\exp\{-q|V(0;t,x,v)|^2\}(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u|^2\}\,dud\omega ds\\
&\le\frac{1}{\varepsilon}(1+|X(0;t,x,v)|^2)^{-p}\exp\{-q|V(0;t,x,v)|^2\}|||f|||\times|||g|||\\
&\quad\times\int_0^\infty\int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u|^2\}\,dud\omega ds.
\end{aligned}\tag{3.8}$$

By Lemma 2.4, we have

$$\begin{aligned}
&|N^-(f,g)(t,x,v)|\\
&\le\frac{1}{\varepsilon}I^2_{\gamma,p,q}(1+|X(0;t,x,v)|^2)^{-p}\exp\{-q|V(0;t,x,v)|^2\}|||f|||\times|||g|||.
\end{aligned}\tag{3.9}$$

Multiplying (3.9) by $(1+|X(0;t,x,v)|^2)^p\exp\{q|V(0;t,x,v)|^2\}$ and taking the supremum with respect to (t,x,v) in $\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3$, we have by (1.16) that

$$|||N^-(f,g)|||\le\frac{1}{\varepsilon}I^2_{\gamma,p,q}|||f|||\times|||g|||.\tag{3.10}$$

Next for the gain term $N^+(f,g)$, similar to (3.6) and (3.7), we have for any s in $(0,t)$,

$$\begin{aligned}
&|f(s,\, X(s;t,x,v),V(s;t,x,v)-u_\parallel)|\\
&\quad\le \|f(s)\|(1+|X(0;s,X(s;t,x,v),V(s;t,x,v)-u_\parallel)|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;s,X(s;t,x,v),V(s;t,x,v)-u_\parallel)|^2\}\\
&\quad=\|f(s)\|(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\parallel|^2)^{-p}\\
&\qquad\times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u_\parallel|^2\},
\end{aligned}\tag{3.11}$$

and

$$
\begin{aligned}
&|g(s,\,X(s;t,x,v),V(s;t,x,v)-u_\perp)| \\
&\quad \le \|g(s)\|(1+|X(0;s,X(s;t,x,v),V(s;t,x,v)-u_\perp)|^2)^{-p} \\
&\qquad\qquad \times \exp\{-q|V(0;s,X(s;t,x,v),V(s;t,x,v)-u_\perp)|^2\} \\
&\quad = \|g(s)\|(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\perp|^2)^{-p} \\
&\qquad\qquad \times \exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u_\perp|^2\}. \qquad (3.12)
\end{aligned}
$$

Putting (3.11) and (3.12) into (3.2), we have that

$$
\begin{aligned}
&|N^+(f,g)(t,x,v)| \\
&\le \frac{1}{\varepsilon}\int_0^t \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma \|f(s)\|\|g(s)\|(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\parallel|^2)^{-p} \\
&\qquad \times(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\perp|^2)^{-p} \\
&\qquad\quad \times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u_\parallel|^2 \\
&\qquad\qquad -q|V(0;t,x,v)-\alpha_2(s;t,x,v)u_\perp|^2\}\,dud\omega ds \\
&\le \frac{1}{\varepsilon}|||f|||\times|||g|||\int_0^\infty \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma (1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\parallel|^2)^{-p} \\
&\qquad \times(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\perp|^2)^{-p} \\
&\qquad\quad \times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u_\parallel|^2 \\
&\qquad\qquad -q|V(0;t,x,v)-\alpha_2(s;t,x,v)u_\perp|^2\}\,dud\omega ds. \qquad (3.13)
\end{aligned}
$$

By Lemmas 2.1 and 2.3, we have

$$
\begin{aligned}
&|N^+(f,g)(t,x,v)| \\
&\le \frac{1}{\varepsilon}|||f|||\times|||g|||\int_0^\infty \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma (1+|X(0;t,x,v)|^2)^{-p} \\
&\quad \times\{(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\parallel|^2)^{-p} \\
&\qquad +(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\perp|^2)^{-p} \\
&\qquad\quad +(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u|^2)^{-p}\} \\
&\quad \times\exp\{-q|V(0;t,x,v)|^2-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u|^2\}\,dud\omega ds \\
&\le \frac{1}{\varepsilon}(1+|X(0;t,x,v)|^2)^{-p}\exp\{-q|V(0;t,x,v)|^2\}|||f|||\times|||g||| \\
&\quad \times\int_0^\infty \int_{\mathbf{R}^3\times S^2_+} b_\gamma(\theta)|u|^\gamma \{(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\parallel|^2)^{-p} \\
&\qquad +(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u_\perp|^2)^{-p} \\
&\qquad\quad +(1+|X(0;t,x,v)+\alpha_1(s;t,x,v)u|^2)^{-p}\} \\
&\quad \times\exp\{-q|V(0;t,x,v)-\alpha_2(s;t,x,v)u|^2\}\,dud\omega ds. \qquad (3.14)
\end{aligned}
$$

Thus, it follows from Lemma 2.4 that

$$
\begin{aligned}
&|N^+(f,g)(t,x,v)| \\
&\le \frac{2}{\varepsilon} I^2_{\gamma,p,q}(1+|X(0;t,x,v)|^2)^{-p}\exp\{-q|V(0;t,x,v)|^2\}|||f|||\times|||g|||,
\end{aligned}
$$

That is,

$$
|||N^+(f,g)||| \le \frac{2}{\varepsilon} I^2_{\gamma,p,q}|||f|||\times|||g|||. \tag{3.15}
$$

Combining (3.10) and (3.15), we have

$$
|||N(f,g)||| \le |||N^+(f,g)||| + |||N^-(f,g)||| \le \frac{3}{\varepsilon} I^2_{\gamma,p,q}|||f|||\times|||g|||. \tag{3.16}
$$

Take $I_{\gamma,p,q} = 3I^2_{\gamma,p,q}$ and then it follows from (2.4) and (2.9) that

$$
I_{\gamma,p,q} = \frac{24\pi p b_0 I^1_{\gamma,q}}{\alpha_0(2p-1)} = \frac{24\pi p b_0}{\alpha_0(2p-1)}\left(\frac{4\pi}{\gamma+2}+\frac{\pi}{q^{3/2}}\right). \tag{3.17}
$$

Therefore, (3.16) yields (3.5) and this completes the proof of Lemma 3.1.

Finally, we prove Theorem 1.2. For this purpose, we define the mapping $\mathbf{T}: S_{p,q}\to S_{p,q}$ by

$$
\mathbf{T}f(t,x,v) = f_0(X(0;t,x,v),V(0;t,x,v)) + N(f,f)(t,x,v) \tag{3.18}
$$

for any $f\in S_{p,q}$. For the mapping $\mathbf{T}$, we have the following lemma.

Lemma 3.2. *For any $f,g\in S_{p,q}$, it holds that*

$$
\begin{cases}
|||\mathbf{T}f||| \le \|f_0\|_{p,q} + \dfrac{1}{\varepsilon} I_{\gamma,p,q}|||f|||^2, \\[2ex]
|||\mathbf{T}f-\mathbf{T}g||| \le \dfrac{1}{\varepsilon} I_{\gamma,p,q}(|||f|||+|||g|||)|||f-g|||,
\end{cases} \tag{3.19}
$$

where $I_{\gamma,p,q}$ is defined by (3.17).

Proof. Fix $f\in S_{p,q}$. Multiplying (3.18) by $(1+|X(0;t,x,v)|^2)^p\exp\{q|V(0;t,x,v)|^2\}$ and taking the supremum with respect to (t,x,v) over $\mathbf{R}^+\times\mathbf{R}^3\times\mathbf{R}^3$, by Lemma 3.1, we obtain the first estimate in (3.19). Then, notice that

$$
\mathbf{T}f-\mathbf{T}g = N(f-g,f) + N(g,f-g). \tag{3.20}
$$

The second estimate in (3.19) follows similarly.

Proof of Theorem 1.2. We only need to show that $\mathbf{T}$ has a fixed point by the contraction mapping theorem. In fact, let's denote the closed subset S_0 of $S_{p,q}$ by

$$
S_0 = \{f\in S_{p,q} : |||f||| \le 2\varepsilon\delta_0\}, \tag{3.21}
$$

where δ_0 is a sufficiently small positive constant such that

$$
\lambda_0 \equiv 4\delta_0 I_{\gamma,p,q} < 1. \tag{3.22}
$$

Let $\|f_0\|_{p,q} \le \varepsilon\delta_0$ and then we have from Lemma 3.2 that

$$\mathbf{T}f \in S_0, \quad \text{and} \quad |||\mathbf{T}f - \mathbf{T}g||| \le \lambda_0 |||f-g|||, \tag{3.23}$$

for any f, $g \in S_0$. Thus the mapping $\mathbf{T} : S_0 \to S_0$ is a contraction and hence has a fixed point f in $S_0 = \{f \in S_{p,q} : |||f||| \le 2\varepsilon\delta_0\}$. This implies that the initial value problem (1.1) and (1.2) has a unique solution f such that $|||f||| \le 2\varepsilon\delta_0$. It then follows from the same argument as the one in [24] that if $f_0(x,v) \ge 0$ then $f(t,x,v) \ge 0$. Hence, the proof of Theorem 1.2 is complete.

4. Conclusions

It is well-known that there exists a global mild solution to the Boltzmann equation without external forces [22] and with external forces integrable in time in some sense up to subtraction of a constant [3] both in the framework of the small perturbation of the vacuum, where the polynomial decay in velocity for the initial data is assumed. Hence it is a natural attempt to extend Theorem 1.2 to the case of polynomial decay in velocity for the initial data, which we can complete by using the same idea combined with more technique inequalities.

Along the same line as before, let's define the norm

$$|||f|||' = \sup_{t,x,v} (1+|X(0;t,x,v)|^2)^p (1+|V(0;t,x,v)|^2)^q |f(t,x,v)|$$

and

$$\|f_0\|'_{p,q} = \sup_{x,v} (1+|x|^2)^p (1+|v|^2)^q |f_0(x,v)|.$$

Furthermore the following assumptions are stated:
(A1)′ $p > \frac{1}{2}$ and $q > \frac{3}{2}$;

(A2)′ The collision kernel B takes the inverse power interaction (1.6) with $-2 < \gamma \le 1$ and the angular cutoff condition

$$\left| \frac{b_\gamma(\theta)}{\cos\theta} \right| \le b'_0,$$

where b'_0 is some positive constant.

Then we have the global existence result similar to Theorem 1.2.

Theorem 3.1. *Under the assumptions (A1)′, (A2)′ and (A3), there is a sufficiently small positive constant* $\delta'_0 > 0$ *such that if* $0 \le f_0(x,v)$ *and* $\|f_0\|'_{p,q} \le \varepsilon\delta'_0$*, then there exists a unique global in time mild solution* $f(t,x,v)$ *to the initial value problem (1.1) and (1.2) satisfying* $|||f|||' \le 2\varepsilon\delta'_0$.

The proof of Theorem 3.1 is based on the two known inequalities, which correspond to the inequality (2.3) when we consider the integration with respect to the velocity variable.

Lemma 3.2([2]). *Let $q > \frac{3}{2}$. Under the assumption (A2)′, the following integrals are bounded:*

$$\sup_v \int_{\mathbf{R}^3\times[0,2\pi]\times[0,\pi/2]} \frac{B(|u|,\theta)}{|u|\sin\theta\cos\theta}(1+|v-u|^2)^q\, dud\varepsilon d\theta \le I^3_{\gamma,q},$$

and

$$\sup_v \int_{\mathbf{R}^3\times[0,2\pi]\times[0,\pi/2]} \frac{B(|u|,\theta)}{|u|\sin\theta\cos\theta}\frac{(1+|v-u|^2)^q}{(1+|v-u_{\|}|^2)^q(1+|v-u_{\perp}|^2)^q}\, dud\varepsilon d\theta \le I^4_{\gamma,q},$$

where $I^3_{\gamma,q}$ and $I^4_{\gamma,q}$ are some constants depending only on γ and q.

By Lemma 3.2 and the same idea as Lemma 2.4, we can complete the proof of Theorem 3.1 and thus omit it for brevity.

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