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Convergence Rate to Stationary Solutions for Boltzmann Equation with External Force****

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Abstract For the Boltzmann equation with an external force in the form of the gradient of a potential function in space variable, the stability of its stationary solutions as local Maxwellians was studied by S. Ukai et al. (2005) through the energy method. Based on this stability analysis and some techniques on analyzing the convergence rates to stationary solutions for the compressible Navier-Stokes equations, in this paper, we study the convergence rate to the above stationary solutions for the Boltzmann equation which is a fundamental equation in statistical physics for non-equilibrium rarefied gas. By combining the dissipation from the viscosity and heat conductivity on the fluid components and the dissipation on the non-fluid component through the celebrated H-theorem, a convergence rate of the same order as the one for the compressible Navier-Stokes is obtained by constructing some energy functionals.

Keywords Convergence rate, Boltzmann equation with external force, Energy functionals, Stationary solutions
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1 Introduction

Consider the Boltzmann equation with an external force in the form of the gradient of a potential function of space variables

$$f_t + \xi \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_\xi f = Q(f, f), \quad |\Phi(x)| \to 0, \quad \text{as } |x| \to +\infty, \tag{1.1}$$

with initial data

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.2}$$

Here $f(t, x, \xi)$ is the distribution function of the particles at time $t \geq 0$ located at $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ with velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, and $\Phi(x)$ denotes the potential of the external force. The short-range interaction between particles is given by the standard Boltzmann collision operator $Q(f, g)$ for the hard-sphere model

$$Q(f,g)(\xi) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2_+} (f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi)) |(\xi - \xi_*) \cdot \Omega| \, d\xi_* d\Omega.$$

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Here $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0\}$, and $\xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\,\Omega,\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\,\Omega$, are the relations between velocities ξ', ξ'_* after and the velocities ξ, ξ_* before an elastic collision coming from the conservation of momentum and energy.

Since the potential of force depends only on the space variables, the local Maxwellian given by

$$\overline{\mathbf{M}} \equiv \overline{\mathbf{M}}(x,\xi) = \frac{\rho_1}{(2\pi R\bar{\theta})^{\frac{3}{2}}} \exp\Big(-\frac{\Phi(x) + \frac{|\xi|^2}{2}}{R\bar{\theta}}\Big) = \mathbf{M}_{[\bar{\rho}(x),0,\bar{\theta}]}(x,\xi),$$
$$\bar{\rho}(x) = \rho_1 \exp\Big(-\frac{\Phi(x)}{R\bar{\theta}}\Big),$$

is the stationary solution to (1.1) in the form of local Maxwellian. Here $R > 0$ is the gas constant and $\rho_1 > 0, \bar{\theta} > 0$ are any given constants.

Based on the decomposition of the solution and the equation around the local Maxwellian (cf. [13]), and the techniques used for the Navier-Stokes equations for the equilibrium gas, the nonlinear stability of the above stationary solutions was proved in [24] by using the celebrated H-theorem for the Boltzmann equation and the energy method. In this paper, we will study the convergence rate in time of the solution to the stationary solution for small initial perturbation. This is a continuation of the stability analysis and can be viewed as a generalization of the corresponding convergence rate results for the fluid dynamics, such as Navier-Stokes equations to the non-equilibrium gas. In the following discussion, we will use the notations from [5, 24] for consistence. For convenience, we state the stability result from [24] as follows.

Theorem 1.1 *Assume that* $f_0(x,\xi) \geq 0$ *and* $N \geq 4$*. There exist a global Maxwellian* $\mathbf{M}_-(\xi)$ *and two sufficiently small constants* $\varepsilon > 0$ *and* $\lambda_0 > 0$ *such that if*

$$\begin{aligned} \lambda &\equiv \|\Phi(x)\|_{L^2(\mathbf{R}^3)} + \sum_{1\leq|\alpha|\leq N+1} \|\partial^\alpha \Phi(x)\|_{L^3(\mathbf{R}^3)} < \lambda_0, \\ \mathcal{E}(f_0) &= \sum_{|\alpha|+|\beta|\leq N} \Big\| \frac{\partial^\alpha\partial^\beta(f_0(x,\xi) - \overline{\mathbf{M}}(x,\xi))}{\sqrt{\mathbf{M}_-}(\xi)} \Big\|_{L^2_{x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)} \leq \varepsilon, \end{aligned} \tag{1.3}$$

then there exists a unique global classical solution $f(t,x,\xi) \in \mathbf{H}^N_{x,\xi}(\mathbf{R}^+)$ *to the Cauchy problem* (1.1)-(1.2) *which satisfies* $f(t,x,\xi) \geq 0$*, and*

$$\begin{aligned} &\sum_{|\gamma|\leq N} \int_{\mathbf{R}^3} |\partial^\gamma(\rho - \bar{\rho}(x), u, \theta - \bar{\theta})(t,x)|^2 dx \\ &+ \sum_{|\gamma|+|\beta|\leq N} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\partial^\gamma\partial^\beta \mathbf{G}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx + \sum_{1\leq|\gamma|\leq N} \int_0^t \int_{\mathbf{R}^3} |\partial^\gamma(\rho - \bar{\rho}(x), u, \theta)|^2 dx d\tau \\ &+ \sum_{|\gamma|+|\beta|\leq N} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta \mathbf{G}|^2}{\mathbf{M}_-(\xi)} d\xi dx d\tau \leq \delta \quad \textit{for some} \quad \delta > 0. \end{aligned} \tag{1.4}$$

Consequently

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}^3} \sum_{|\gamma|+|\beta|\leq N-3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma\partial^\beta(f(t,x,\xi) - \overline{\mathbf{M}}(x,\xi))|^2}{\mathbf{M}_-(\xi)} d\xi = 0. \tag{1.5}$$

Here and in the sequel, for the derivatives on t, x and ξ, we use the following notations

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \beta = (\beta_1, \beta_2, \beta_3), \quad \gamma = (\gamma_0, \alpha)$$

for $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\partial^\beta = \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\xi_3}^{\beta_3}$, $\partial^\gamma = \partial_t^{\gamma_0}\partial^\alpha$, respectively.

As shown in (1.5), the solution to the Boltzmann equation converges to the local Maxwellian time asymptotically. Indeed, as shown in the following main theorem in this paper, a convergence rate of the order of $(1+t)^{-\frac{1}{4}}$ in sup-norm can be obtained. Notice that this convergence rate is not optimal, but it is the same as the one in [5] for the corresponding problem on the compressible Navier-Stokes equation. The improvement of this convergence rate is not in the scope of this paper and will be pursued in the future study.

Theorem 1.2 *Under the assumptions of Theorem* 1.1*, if we further assume that the index of the Sobolev space* $N \geq 5$ *for the initial condition, then*

$$\sup_{x\in\mathbf{R}^3} \sum_{|\gamma|+|\beta|\leq N-3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma\partial^\beta(f(t,x,\xi)-\overline{\mathbf{M}}(x,\xi))|^2}{\mathbf{M}_-(\xi)} d\xi \leq O(1)(1+t)^{-\frac{1}{2}}. \tag{1.6}$$

Remark 1.1 As one can see later in the proof of Theorem 1.2, for $\gamma_0 > 0$, the decay rate in (1.6) can be improved to $(1+t)^{-1}$.

The Theorem 1.2 can be viewed as an improvement of the Theorem 1.1 where the convergence rate to the stationary solutions is given explicitly. As pointed out in [1], to find out the convergence rate is the one way to understand the relevant time scale for the equilibrium process, which in turn is important in modelling and judging the feasibility of numerical simulations.

For the convergence rate study, Desvillettes and Villani in [6] studied the time-decay rate to a global Maxwellian of large data solutions to Boltzmann type equations without force in a bounded domain. Even though the assumptions they imposed *in general* are a priori and difficult to be verified at present time, their impressive analysis does lead to an almost exponential decay rate for the Boltzmann equation with soft potentials and the Landau equation. On the other hand, based on the energy method developed in [9] for the Boltzmann equation without force with space periodic data, a proof of an almost exponential decay rate for solutions near a global Maxwellian to the Vlasov-Maxwell-Boltzmann system, the relativistic Landau-Maxwell system, the Boltzmann equation with soft potentials and the Landau equation is given in [20]. In the above analysis, the interpolation between the Sobolev norms with or without any weight functions plays an important role. And this is why only algebraic decay is obtained by assuming that the initial data has extra regularity. Notice also that in the above analysis, the Poincaré inequality plays an essential role and hence the techniques developed there can not be applied directly to the study of the corresponding problem for the Boltzmann equation with external force in the whole space.

For the Cauchy problem without external force, it is well known that both the compressible Navier-Stokes equations and the Boltzmann equation have the same convergence rate to the corresponding constant stationary solutions, namely, $(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}$, $1 \leq q \leq 2$, which is achieved by initial data that are sufficiently close to the stationary solutions both in some L^2 Sobolev space and the space L^q (see [3, 4, 10, 11, 17, 18, 22]). On the other hand, for the Cauchy problem with external force, the convergence rate to the corresponding space-dependent stationary solutions for the compressible Navier-Stokes equations was first given in [5], which is of order $(1+t)^{-\frac{1}{4}}$ for initial data sufficiently close to the relevant stationary solutions in some Soblev space. This convergence rate was improved in [19] to $(1+t)^{-\frac{1-\kappa}{2}}$ for any small positive number κ, under the additional assumption that initial data are also close to the stationary solutions in the space $L^{\frac{6}{5}}$. Finally, in [2], the same rate as for the force-free case, namely, $(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}$, is announced for the Boltzmann equation with external force. Notice that the decay in [19] corresponds to the one in [2] with $q = \frac{6}{5}$ so that $\frac{3}{2}(\frac{1}{q} - \frac{1}{2}) = \frac{1}{2}$. The method of proof in [2] relies largely on the spectral analysis of the linearized Boltzmann operator.

Our method is quite different. Based on the techniques used in [5] for the convergence rate and the stability analysis in [24] through the energy method, we succeed in obtaining the convergence rate for the Boltzmann equation with external force. The analysis is a combination of the equation (2.4) for the non-fluid part $\mathbf{G}(t, x, \xi)$ through the microscopic H-theorem (2.7) with the corresponding time-decay estimates on the fluid part $(\rho(t, x) - \bar{\rho}(x), u(t, x), \theta(t, x) - \bar{\theta})$ to obtain the time-decay estimates on $f(t, x, \xi) - \overline{\mathbf{M}}(x, \xi)$.

The rest of this paper is arranged as follows. Some preliminaries for the proof of Theorem 1.2 will be given in Section 2, while the proof of the main result Theorem 1.2 will be given in Section 3 for the case when $N = 5$. The case when $N > 5$ can be proved similarly.

2 Preliminaries

For the use in the proof of Theorem 1.2, we list some basic Sobolev inequalities and some estimates from [5] and [24] in this section. Interested readers please refer to these two references for the proof. First is a lemma from [5] which is a Gronwall type inequality leading to the convergence rate.

Lemma 2.1 *Let $f(t) \in C^1([t_0, \infty))$ such that $f(t) \geq 0$, $A = \int_{t_0}^{\infty} f(t)dt < +\infty$ and $f'(t) \leq a(t)f(t)$ for all $t \geq t_0$. Here $a(t) \geq 0$, $B = \int_{t_0}^{\infty} a(t)dt < +\infty$. Then*

$$f(t) \leq \frac{(t_0 f(t_0) + 1)e^{A+B} - 1}{t}, \quad \forall\, t \geq t_0.$$

As in [5], the following Sobolev type inequalities will also be used.

Lemma 2.2 *For $f(x) \in H^1(\mathbf{R}^3)$, we have*

$$\|f(x)\|_{L^6(\mathbf{R}^3)} \leq C_0 \|\nabla_x f(x)\|, \tag{2.1}$$

where C_0 is a positive constant independent of f. Consequently, for $f(x) \in H^2(\mathbf{R}^3)$, $g(x) \in H^1(\mathbf{R}^3)$, $h(x) \in L^2(\mathbf{R}^3)$ and any $\varepsilon > 0$, we have

$$\|f(x)\|_{L^\infty(\mathbf{R}^3)} \leq C_0 \|\nabla_x f(x)\|_1, \tag{2.2}$$

where C_0 is again a positive constant independent of f, so that

$$\begin{aligned} &\int_{\mathbf{R}^3} f(x) \cdot g(x) \cdot h(x) dx \leq \varepsilon \|\nabla_x f(x)\|^2 + C_\varepsilon \|g(x)\|_1^2 \|h(x)\|^2, \\ &\int_{\mathbf{R}^3} f(x) \cdot g(x) \cdot h(x) dx \leq \varepsilon \|g(x)\|^2 + C_\varepsilon \|\nabla_x f(x)\|_1^2 \|h(x)\|^2. \end{aligned} \tag{2.3}$$

Here and in the sequel, $\|\cdot\|$ and $\|\cdot\|_s$ denote the standard $L^2(\mathbf{R}^3)$ and $H^s(\mathbf{R}^3)$ norm respectively. In the following, we will list some basic energy estimates on the solution to the Cauchy problem (1.1)-(1.2) from [24] for the small perturbation of the stationary Maxwellian solution. As in [13], we first decompose the solution $f(t, x, \xi)$ of (1.1) as the sum of the fluid part $\mathbf{M}(t, x, \xi)$ and the non-fluid part $\mathbf{G}(t, x, \xi)$ so that $\mathbf{G}(t, x, \xi)$ solves

$$\mathbf{G}_t + \mathbf{P}_1(\xi \cdot \nabla_x \mathbf{G} + \xi \cdot \nabla_x \mathbf{M}) - \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} = L_{\mathbf{M}} \mathbf{G} + Q(\mathbf{G}, \mathbf{G}), \tag{2.4}$$

while the macroscopic variables $(\rho(t,x), u(t,x), \theta(t,x))$ solve the following fluid-type system

$$
\begin{aligned}
&\rho_t + \mathrm{div}_x m = 0,\\
&m_{it} + \sum_{j=1}^{3}(u_i m_j)_{x_j} + p_{x_i} - \bar{p}_{x_i} + (\rho-\bar{\rho})\Phi_{x_i}\\
&= \sum_{j=1}^{3}\Big[\mu(\theta)\Big(u_{ix_j} + u_{jx_i} - \frac{2}{3}\delta_{ij}\mathrm{div}_x u\Big)\Big]_{x_j} - \int_{\mathbf{R}^3}\psi_i(\xi\cdot\nabla_x\Theta)d\xi, \quad i=1,2,3,\\
&\Big[\rho\Big(\frac{1}{2}|u|^2 + \mathbf{E}\Big)\Big]_t + \sum_{j=1}^{3}\Big(u_j\Big(\rho\Big(\frac{1}{2}|u|^2+\mathbf{E}\Big)+p\Big)\Big)_{x_j} + m\cdot\nabla_x\Phi\\
&= \sum_{i,j=1}^{3}\Big\{\mu(\theta)u_i\Big(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\Big)\Big\}_{x_j} + \sum_{j=1}^{3}(\kappa(\theta)\theta_{x_j})_{x_j} - \int_{\mathbf{R}^3}\psi_4(\xi\cdot\nabla_x\Theta)d\xi.
\end{aligned}
\tag{2.5}
$$

Here

$$L_{\mathbf{M}}g = L_{[\rho,u,\theta]}g \equiv Q(\mathbf{M}+g,\mathbf{M}+g) - Q(g,g)$$

is the linearized collision operator and $\Theta = L_{\mathbf{M}}^{-1}(\mathbf{G}_t + \mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G}) - \nabla_x\Phi\cdot\nabla_\xi\mathbf{G} - Q(\mathbf{G},\mathbf{G}))$.

It is well known that the null space of the operator $L_{\mathbf{M}}$, denoted by $\mathcal{N}$, is spanned by the five collision invariants $\{\mathbf{M}, \mathbf{M}\xi, \mathbf{M}|\xi|^2\}$. The next lemma from [7] contains an inequality for the $L^2(\mathbf{R}^3)$ estimate on the nonlinear collision operator $Q(f,f)$.

Lemma 2.3 *There exists a positive constant $C>0$ such that*

$$\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}Q(f,g)^2}{\widetilde{\mathbf{M}}}d\xi \le C\Big\{\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)f^2}{\widetilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{g^2}{\widetilde{\mathbf{M}}}d\xi + \int_{\mathbf{R}^3}\frac{f^2}{\widetilde{\mathbf{M}}}d\xi\cdot\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)g^2}{\widetilde{\mathbf{M}}}d\xi\Big\}, \tag{2.6}$$

where $\widetilde{\mathbf{M}}$ is any Maxwellian such that the above integrals are well defined, and $\nu_{\mathbf{M}}(\xi)\sim 1+|\xi|$ as $|\xi|\to+\infty$ for hard sphere model.

Based on Lemma 2.3, the following lemma and corollary from [14] follow from the Cauchy inequality.

Lemma 2.4 *If $\frac{\theta}{2} < \tilde{\theta} < \theta$, then there exist two positive constants $\bar{\sigma} = \bar{\sigma}(u,\theta;\tilde{u},\tilde{\theta})$ and $\eta_0 = \eta_0(u,\theta;\tilde{u},\tilde{\theta})$ such that if $|u-\tilde{u}| + |\theta-\tilde{\theta}| < \eta_0$, then for $h(\xi)\in\mathcal{N}^{\perp}$,*

$$-\int_{\mathbf{R}^3}\frac{hL_{\mathbf{M}}h}{\widetilde{\mathbf{M}}}d\xi \ge \bar{\sigma}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)h^2}{\widetilde{\mathbf{M}}}d\xi. \tag{2.7}$$

Here $\mathbf{M}\equiv\mathbf{M}_{[\rho,u,\theta]}(t,x,\xi)$ and $\widetilde{\mathbf{M}}(t,x,\xi) = \widetilde{\mathbf{M}}_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}(t,x,\xi)$.

Corollary 2.1 *Under the assumptions in Lemma 2.4, we have for $h(\xi)\in\mathcal{N}^{\perp}$,*

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}|L_{\mathbf{M}}^{-1}h|^2 d\xi \le \sigma^{-2}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}h^2(\xi)}{\mathbf{M}}d\xi,\\
&\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\widetilde{\mathbf{M}}}|L_{\mathbf{M}}^{-1}h|^2 d\xi \le \bar{\sigma}^{-2}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}h^2(\xi)}{\widetilde{\mathbf{M}}}d\xi.
\end{aligned}
\tag{2.8}
$$

Throughout this paper, we choose positive constants ρ_- and θ_- such that

$$\frac{\bar{\theta}}{2} < \theta_- < \bar{\theta}, \quad |\rho_- - \bar{\rho}| + |\theta_- - \bar{\theta}| < \eta_0. \tag{2.9}$$

It is easy to see that if $\mathbf{M}(t,x,\xi)$ is a small perturbation of $\overline{\mathbf{M}}(x,\xi)$, the Lemmas 2.3–2.4 and Corollary 2.1 hold for such chosen ρ_- and θ_- when $\widetilde{\mathbf{M}} \equiv \mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}$. And we will also use the following notations:

$$\begin{aligned} \phi(t) &= \sum_{1\le|\gamma|\le 5} \|\partial^\gamma(\rho(t,x)-\bar\rho(x), u(t,x), \theta(t,x))\|^2, \\ \psi(t) &= \sum_{|\gamma|+|\beta|\le 5} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx. \end{aligned} \tag{2.10}$$

Finally, before concluding this section, we give an estimate on $\Theta(t,x,\xi)$ appearing in the right hand side of (2.5) in the following lemma.

Lemma 2.5 *Let $k>0$ be any fixed integer. We have for $|\gamma|\le 2$ that*

$$\begin{aligned} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} |\xi|^k|\partial^\gamma\Theta_t|^2 d\xi dx \le{}& O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)(|\partial^\gamma \mathbf{G}_{tt}|^2 + |\nabla_x\partial^\gamma \mathbf{G}_t|^2)}{\mathbf{M}} d\xi dx \\ &+ O(1)(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}} d\xi dx \\ &+ O(1)\psi(t)\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2. \end{aligned} \tag{2.11}$$

Notice that (2.11) can be proved by using Corollary 2.1, Lemmas 2.2–2.4 as in [24] so that we omit the details for brevity.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 for the case when $N=5$. The case when $N>5$ can be discussed similarly. The proof is based on the energy estimates obtained in [24] and the analytic techniques used in [5] for the estimation on convergence rate. The proof is arranged as follows. Since the potential force depends only on space variable, any differentiation with respect to time on the equation goes to the solution f. Then by using Lemma 2.1, we will first obtain the decay estimates on $\partial^\gamma\partial^\beta\partial_t f(t,x,\xi)$ for $|\gamma|+|\beta|\le 3$ which contains at least one order differentiation with respect to time. From the decay estimate involving differentiation with respect to time to those involving only space variables, we then use the equations (2.4) and (2.5) for the time evolution of the non-fluid and fluid components in the solution to yield the time decay estimates on $\partial_x^\alpha(f(t,x,\xi)-\overline{\mathbf{M}}(x,\xi))$ for $1\le|\alpha|\le 4$. Finally, the conclusion of Theorem 1.2 follows from these time decay estimates and the Sobolev inequality (2.2).

As the first step, we now deduce the decay estimates on $\partial^\gamma\partial^\beta\partial_t f(t,x,\xi)$ for $|\gamma|+|\beta|\le 3$. For this, we first consider the estimates on the fluid part $\partial^\gamma\partial_t(\rho,u,\theta)(t,x)$ for $|\gamma|\le 2$ which are summarized in the following lemma.

Lemma 3.1 *Let*

$$H_0^\gamma(t) = \int_{\mathbf{R}^3}\Big[\frac{\bar\theta}{3\bar\rho}|\partial^\gamma\rho_t|^2 + \frac{\rho}{2}|\partial^\gamma u_t|^2 + \frac{\rho}{2\bar\theta}|\partial^\gamma\theta_t|^2\Big](t,x)dx.$$

We have for $|\gamma|\le 2$ *that*

$$\begin{aligned}&\frac{d}{dt}H_0^\gamma(t)+C_0\int_{\mathbf{R}^3}[|\nabla_x\partial^\gamma u_t|^2+|\nabla_x\partial^\gamma\theta_t|^2]dx\\ &\le O(1)(\phi(t)+\psi(t))\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2+O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}(\rho,u,\theta)_t\|^2\\ &\quad+O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\Big[(|\partial^\gamma\mathbf{G}_{tt}|^2+|\nabla_x\partial^\gamma\mathbf{G}_t|^2)+(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le 3}|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2\Big]d\xi dx. \quad (3.1)\end{aligned}$$

Here $C_0>0$ *is a constant.*

Proof To estimate $\partial^\gamma\rho_t$, by applying $\partial^\gamma\partial_t$ to $(2.5)_1$, multiplying it by $\frac{2\bar\theta}{3\bar\rho}\partial^\gamma\rho_t$, and then integrating the final equation with respect to x over $\mathbf{R}^3$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbf{R}^3}\frac{\bar\theta}{3\bar\rho}|\partial^\gamma\rho_t|^2dx+\int_{\mathbf{R}^3}\Big[\frac{2\bar\theta}{3}\partial^\gamma\rho_t\mathrm{div}_x(\partial^\gamma u_t)+\frac{2\bar\theta}{3\bar\rho}\partial^\gamma\rho_t\nabla_x\bar\rho(x)\cdot\partial^\gamma u_t\Big]dx\\ &=-\sum_{0<\alpha'\le\alpha}C_\alpha^{\alpha'}\int_{\mathbf{R}^3}\frac{2\bar\theta}{3\bar\rho}\partial^{\alpha'}\bar\rho\,\mathrm{div}_x(\partial^{\alpha-\alpha'}\partial^{\gamma_0}u_t)\partial^\gamma\rho_tdx\\ &\quad-\int_{\mathbf{R}^3}\frac{2\bar\theta}{3\bar\rho}\partial^\gamma\rho_t\partial^\gamma[(\rho-\bar\rho)\mathrm{div}_xu_t+\rho_t\mathrm{div}_xu+\nabla_x\rho_t\cdot u+\nabla_x(\rho-\bar\rho)\cdot u_t]dx\\ &\quad-\sum_{0<\alpha'\le\alpha}C_\alpha^{\alpha'}\int_{\mathbf{R}^3}\frac{2\bar\theta}{3\bar\rho}\partial^\gamma\rho_t\nabla_x\partial^{\alpha'}\bar\rho\cdot\partial^{\alpha-\alpha'}\partial^{\gamma_0}u_tdx:=\sum_{j=1}^6\mathrm{I}_j. \quad (3.2)\end{aligned}$$

Here and in the sequel, the notation I_j representing the corresponding term in the summation without any ambiguity.

I_j $(j=1,2,3,4,5,6)$ will be estimated term by term in the following. For I_1 and I_6, we have from Lemma 2.2 that

$$\begin{aligned}|\mathrm{I}_1|+|\mathrm{I}_6|&\le O(1)|\alpha|\sum_{0<\alpha'\le\alpha}\|\partial^\gamma\rho_t\|[\|\partial^{\alpha'}\bar\rho(x)\|_{L^\infty}\|\nabla_x\partial^{\alpha-\alpha'}\partial^{\gamma_0}u_t\|+\|\nabla_x\partial^{\alpha'}\bar\rho\|_{L^3}\|\partial^{\alpha-\alpha'}\partial^{\gamma_0}u_t\|_{L^6}]\\ &\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}(\rho,u)_t\|^2. \quad (3.3)\end{aligned}$$

For I_2, we have from (2.3) that

$$\begin{aligned}\mathrm{I}_2&=-\int_{\mathbf{R}^3}\frac{2\bar\theta}{3\bar\rho}\partial^\gamma\rho_t(\rho-\bar\rho)\mathrm{div}_x(\partial^\gamma u_t)dx-\sum_{0<\gamma'\le\gamma}C_\gamma^{\gamma'}\int_{\mathbf{R}^3}\frac{2\bar\theta}{3\bar\rho}\partial^\gamma\rho_t\partial^{\gamma'}(\rho-\bar\rho)\mathrm{div}_x(\partial^{\gamma-\gamma'}u_t)dx\\ &\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}u_t\|^2+O(1)\phi(t)\|\partial^\gamma\rho_t\|^2. \quad (3.4)\end{aligned}$$

In the above estimate, we have used $(2.3)_2$ twice when $f=\rho-\bar\rho, g=\mathrm{div}_x(\partial^\gamma u_t), h=\partial^\gamma\rho_t$, and $f=\partial^\gamma\rho_t, g=\partial^{\gamma'}(\rho-\bar\rho), h=\mathrm{div}_x(\partial^{\gamma-\gamma'}u_t)$ respectively.

Similar estimates hold for $\mathrm{I}_3,\mathrm{I}_4$ and I_5 which give

$$\sum_{j=3}^5|\mathrm{I}_j|\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}\rho_t\|^2+O(1)\phi(t)\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u)_t\|^2. \quad (3.5)$$

Substituting (3.3), (3.4), and (3.5) into (3.2) gives

$$\frac{d}{dt}\int_{\mathbf{R}^3}\frac{\bar{\theta}}{3\bar{\rho}}|\partial^\gamma\rho_t|^2dx+\int_{\mathbf{R}^3}\Big[\frac{2\bar{\theta}}{3}\partial^\gamma\rho_t\mathrm{div}_x(\partial^\gamma u_t)+\frac{2\bar{\theta}}{3\bar{\rho}}\partial^\gamma\rho_t\nabla_x\bar{\rho}(x)\cdot\partial^\gamma u_t\Big]dx$$
$$\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}(\rho,u)_t\|^2+O(1)\phi(t)\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u)_t\|^2, \tag{3.6}$$

which completes the estimate on $\partial^\gamma\rho_t$.

Similarly, one can obtain the corresponding estimates on $u(t,x)$ and $\theta(t,x)$. That is, there exists a positive constant $d_0>0$ such that

$$\frac{d}{dt}\int_{\mathbf{R}^3}\frac{\rho}{2}|\partial^\gamma u_t|^2dx+d_0\int_{\mathbf{R}^3}|\nabla_x\partial^\gamma u_t|^2dx$$
$$-\int_{\mathbf{R}^3}\Big[\frac{2\bar{\theta}}{3}\partial^\gamma\rho_t\mathrm{div}_x(\partial^\gamma u_t)+\frac{2\bar{\rho}}{3}\partial^\gamma\theta_t\mathrm{div}_x(\partial^\gamma u_t)+\frac{2\bar{\theta}}{3\bar{\rho}}\partial^\gamma\rho_t\nabla_x\bar{\rho}\cdot\partial^\gamma u_t\Big]dx$$
$$\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}(\rho,u)_t\|^2+O(1)\phi(t)\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2$$
$$-\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\partial^\gamma u_t\cdot\psi(\xi\cdot\nabla_x\partial^\gamma\Theta_t)d\xi dx, \tag{3.7}$$

$$\frac{d}{dt}\int_{\mathbf{R}^3}\frac{\rho}{2\bar{\theta}}|\partial^\gamma\theta_t|^2dx+d_0\int_{\mathbf{R}^3}|\nabla_x\partial^\gamma\theta_t|^2dx+\int_{\mathbf{R}^3}\frac{2\bar{\rho}}{3}\partial^\gamma\theta_t\mathrm{div}_x(\partial^\gamma u_t)dx$$
$$\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le 2}\|\nabla_x\partial^{\gamma'}(u,\theta)_t\|^2+O(1)\phi(t)\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2$$
$$-\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\theta_t}{\bar{\theta}}\partial^\gamma[(\psi_4-\xi\cdot u)(\xi\cdot\nabla_x\Theta_t)]d\xi dx. \tag{3.8}$$

By (3.6)–(3.8), (3.1) follows from Lemma 2.5 and this completes the proof of the lemma.

Next, we estimate the non-fluid part $\mathbf{G}(t,x,\xi)$ through the equation (2.4).

Lemma 3.2 *For each $|\gamma|+|\beta|\le 3$, the following estimate holds*

$$\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx+\frac{\bar{\sigma}}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx$$
$$\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx+O(1)\|\nabla_x\partial^\gamma(u,\theta)_t\|^2$$
$$+O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}\Big[|\nabla_x\partial^\gamma\mathbf{G}_t|^2+\sum_{|\beta'|=|\beta|-1}|\nabla_x\partial^\gamma\partial^{\beta'}\mathbf{G}_t|^2\Big]d\xi dx$$
$$+O(1)(\phi(t)+\psi(t))\sum_{|\gamma'|\le 3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2. \tag{3.9}$$

Proof Applying $\partial^\gamma\partial^\beta\partial_t$ to (2.4) and multiplying it by $\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}$, we have by integrating the final equation with respect to ξ and x over $\mathbf{R}^3\times\mathbf{R}^3$ that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx$$
$$=-\frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}^2}\mathbf{M}_td\xi dx-\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\partial^\gamma\partial^\beta[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})]_td\xi dx$$

$$-\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\partial^\gamma\partial^\beta[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G})]_t d\xi dx+\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\partial^\alpha[\nabla_x\Phi(x)\cdot\nabla_\xi\partial^\beta\partial^{\gamma_0}\mathbf{G}_t]d\xi dx$$

$$+\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\partial^\gamma\partial^\beta(L_{\mathbf{M}}\mathbf{G})_t d\xi dx+\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\partial^\gamma\partial^\beta(Q(\mathbf{G},\mathbf{G}))_t d\xi dx:=\sum_{j=7}^{12}\mathrm{I}_j. \quad (3.10)$$

It is straightforward to have

$$|\mathrm{I}_7|\le O(1)(\lambda_0+\delta)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx, \quad (3.11)$$

$$|\mathrm{I}_8|\le\mu\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx+O(1)\|\nabla_x\partial^\gamma(u,\theta)_t\|^2+O(1)\phi(t)\sum_{|\gamma'|\le3}\|\partial^{\gamma'}(\rho,u\theta)_t\|^2$$
$$+O(1)(\lambda_0+\delta)\sum_{|\gamma'|\le2}\|\nabla_x\partial^{\gamma'}(u,\theta)_t\|^2. \quad (3.12)$$

From now on, $\mu>0$ denotes a sufficiently small positive constant.

For I_9, since

$$\partial^\gamma\partial^\beta[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G})]_t=\xi\cdot\nabla_x\partial^\gamma\partial^\beta\mathbf{G}_t+\sum_{|\beta'|=|\beta|-1}C_\beta^{\beta'}\partial^{\beta-\beta'}(\xi)\cdot\nabla_x\partial^\gamma\partial^{\beta'}\mathbf{G}_t$$
$$-\sum_{j=0}^4\partial^\gamma\{\langle\xi\cdot\nabla_x\mathbf{G}_t,\chi_j\rangle\partial^\beta\chi_j\}$$
$$-\sum_{j=0}^4\partial^\gamma\Big\{\Big\langle\xi\cdot\nabla_x\mathbf{G},\Big(\frac{\chi_j}{\mathbf{M}}\Big)_t\mathbf{M}\Big\rangle\partial^\beta\chi_j+\langle\xi\cdot\nabla_x\mathbf{G},\chi_j\rangle\partial^\gamma\chi_{jt}\Big\},$$

we have

$$|\mathrm{I}_9|\le\mu\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx+O(1)\sum_{|\beta'|=|\beta|-1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\gamma\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx$$
$$+O(1)(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx$$
$$+O(1)(\psi(t)+\phi(t))\sum_{|\gamma'|\le3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2+O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx. \quad (3.13)$$

And I_{10} satisfies

$$\mathrm{I}_{10}=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\Big[\nabla_x\Phi(x)\cdot\nabla_\xi\partial^\gamma\partial^\beta\mathbf{G}_t+\sum_{0<\alpha'\le\alpha}C_\alpha^{\alpha'}\nabla_x\partial^{\alpha'}\Phi(x)\cdot\nabla_\xi\partial^{\alpha-\alpha'}\partial^\beta\partial^{\gamma_0}\mathbf{G}_t\Big]d\xi dx$$
$$\le O(1)(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx. \quad (3.14)$$

Finally, by Lemma 2.4, I_{11} and I_{12} are estimated as

$$\mathrm{I}_{11}=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\partial^\beta\mathbf{G}_t}{\mathbf{M}}\partial^\gamma\partial^\beta(L_{\mathbf{M}}\mathbf{G}_t+2Q(\mathbf{M}_t,\mathbf{G}))d\xi dx$$
$$\le-\bar\sigma\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx+O(1)(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx$$
$$+O(1)(\psi(t)+\phi(t))\sum_{|\gamma'|\le3}\|\partial^{\gamma'}(\rho,u,\theta)_t\|^2, \quad (3.15)$$

$$\mathrm{I}_{12}\le \mu\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx+O(1)(\lambda_0+\delta)\sum_{|\gamma'|+|\beta'|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\partial^{\beta'}\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx. \quad (3.16)$$

By combining (3.10)–(3.16) and choosing $\mu>0$ to be sufficiently small, we have (3.9) and then complete the proof of the lemma.

The following corollary is a direct consequence of (3.9).

Corollary 3.1 *Under the assumptions in Theorem* 1.2, *one has*

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{\beta>0,|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\Big\}+\frac{\bar\sigma}{2}\sum_{\beta>0,|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\\ \le&O(1)(\lambda_0+\delta)\sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx+O(1)\sum_{|\gamma|\le 2}\|\nabla_x\partial^\gamma(u,\theta)_t\|^2\\ &+O(1)\sum_{|\gamma|\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx+O(1)(\phi(t)+\psi(t))\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2, \quad (3.17)\end{aligned}$$

and for $j=0,1,2$,

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{|\gamma|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\Big\}+\frac{\bar\sigma}{2}\sum_{|\gamma|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\\ \le&O(1)(\lambda_0+\delta)\sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx+O(1)(\phi(t)+\psi(t))\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2\\ &+O(1)\sum_{|\gamma|=j}\Big(\|\nabla_x\partial^\gamma(u,\theta)_t\|^2+\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\Big). \quad (3.18)\end{aligned}$$

Remark 3.1 Similar argument shows that if $\mathbf{M}$ is replaced by $\mathbf{M}_-$, then the estimates (3.17) and (3.18) still hold.

By suitably choosing positive constants $\lambda_j^1>0$, $\lambda_j^0>0$ $(j=0,1,2)$ with $\lambda_j^0>0$ being sufficiently large, $\lambda_0^1=1$ and λ_j^1 $(j\ge 1)$ being suitably large, and denoting

$$H_1(t)=\sum_{j=0}^{2}\lambda_j^1\Big(\lambda_j^0\sum_{|\gamma|=j}H_0^\gamma(t)+\sum_{|\gamma|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\Big), \quad (3.19)$$

we have from (3.1) and (3.18) that there exists a positive constant $C_1>0$ such that

$$\begin{aligned}&\frac{d}{dt}H_1(t)+C_1\sum_{|\gamma|\le 2}\int_{\mathbf{R}^3}\Big[|\nabla_x\partial^\gamma(u,\theta)_t|^2+\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}}d\xi\Big]dx\\ \le&O(1)(\lambda_0+\delta)\sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx+O(1)(\phi(t)+\psi(t))\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2\\ &+O(1)(\lambda_0+\delta)\sum_{|\gamma|\le 2}\|\nabla_x\partial^\gamma\rho_t\|^2+O(1)\sum_{|\gamma|=2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}}(|\nabla_x\partial^\gamma\mathbf{G}_t|^2+|\partial^\gamma\mathbf{G}_{tt}|^2)d\xi dx. \quad (3.20)\end{aligned}$$

Besides (3.17) and (3.20), we still need the estimate on $\nabla_x\partial^\gamma\rho_t$ and the estimate on the third order derivatives of $\mathbf{G}_t$ with respect to t and x. For $\nabla_x\partial^\gamma\rho_t$, we can apply the conservation laws and the estimate is given in the following lemma.

Lemma 3.3 *For each* $|\gamma| \le 2$, *we have*

$$\int_{\mathbf{R}^3} \frac{2\theta}{3\rho} |\nabla_x \partial^\gamma \rho_t|^2 dx \le O(1) \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi) |\nabla_x \partial^\gamma \mathbf{G}_t|^2}{\mathbf{M}} d\xi dx + \frac{d}{dt} \int_{\mathbf{R}^3} \partial^\gamma \rho_t \mathrm{div}_x (\partial^\gamma u_t) dx$$
$$+ O(1) \sum_{|\gamma'| \le 2} \|\nabla_x \partial^{\gamma'} (u, \theta)_t\|^2 + O(1)\phi(t) \sum_{|\gamma'| \le 3} \|\partial^{\gamma'} (\rho, u, \theta)_t\|^2$$
$$+ O(1)(\lambda_0 + \delta) \sum_{|\gamma'| < |\gamma|} \|\nabla_x \partial^{\gamma'} \rho_t\|^2. \tag{3.21}$$

Proof Notice that the conservation laws give

$$\frac{2\theta}{3} \nabla_x (\rho - \bar{\rho}) = -\rho u_t - \rho (u \cdot \nabla_x) u - \frac{2}{3} (\rho - \bar{\rho}) \nabla_x \theta - \nabla_x \bar{\rho} (\theta - \bar{\theta})$$
$$- \bar{\rho} \nabla_x \theta - (\rho - \bar{\rho}) \nabla_x \Phi(x) - \int_{\mathbf{R}^3} \psi (\xi \cdot \nabla_x \mathbf{G}) d\xi.$$

By applying $\partial^\gamma \partial_t$ to the above equation, and then multiplying it by $\frac{\nabla_x \partial^\gamma \rho_t}{\rho}$ before integrating the resulting equation with respect to x over $\mathbf{R}^3$, a similar argument for obtaining (3.1) leads to (3.21). Hence, we omit the details of the calculation for brevity.

To obtain the estimates on the third order derivatives on $\mathbf{G}_t$ with respect to t and x, we need to use the original system (1.1) to avoid the appearance of the fourth order derivatives of $\mathbf{G}_t$. Note that $\partial_t f(t, x, \xi)$ solves

$$f_{tt} + \xi \cdot \nabla_x f_t - \nabla_x \Phi(x) \cdot \nabla_\xi f_t = L_{\mathbf{M}} \mathbf{G}_t + 2Q(\mathbf{M}_t, \mathbf{G}) + [Q(\mathbf{G}, \mathbf{G})]_t.$$

By using the the orthogonal property on the derivatives on the fluid components and the non-fluid components, that is,

$$\int_{\mathbf{R}^3} \frac{\mathbf{P}_0 (\partial^\gamma \mathbf{M}_t) \partial^\gamma (L_{\mathbf{M}} \mathbf{G})}{\mathbf{M}} d\xi = 0, \tag{3.22}$$

we have the following lemma on $\partial^\gamma f_t$. Since the proof is straightforward, we also omit it for brevity.

Lemma 3.4 *For* $|\gamma| = 3$, *there exists a positive constant* $d_1 > 0$ *such that*

$$\frac{d}{dt} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma f_t|^2}{\mathbf{M}} d\xi dx + d_1 \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi) |\partial^\gamma \mathbf{G}_t|^2}{\mathbf{M}} d\xi dx$$
$$\le O(1)(\lambda_0 + \delta) \sum_{|\gamma'| + |\beta'| \le 3} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^{\gamma'} \partial^{\beta'} \mathbf{G}_t|^2}{\mathbf{M}_-} d\xi dx + O(1)(\lambda_0 + \delta) \sum_{0 < |\gamma'| \le 3} \|\partial^{\gamma'} (\rho, u, \theta)_t\|^2. \tag{3.23}$$

Now, let $\lambda_2 > 0$ be a suitably large constant and define

$$H_2(t) = \lambda_2 \sum_{|\gamma| = 3} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma f_t|^2}{\mathbf{M}} d\xi dx + H_1(t) - O(1)(\lambda_0 + \delta) \sum_{|\gamma| \le 2} \int_{\mathbf{R}^3} \partial^\gamma \rho_t \mathrm{div}_x (\partial^\gamma u_t) dx$$
$$+ \sum_{\beta > 0, |\gamma| + |\beta| \le 3} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\gamma \partial^\beta \mathbf{G}_t|^2}{\mathbf{M}} d\xi dx. \tag{3.24}$$

By summarizing (3.17), (3.20), (3.21), and (3.23), we have that there exists a positive constant $C_2 > 0$ such that

$$\frac{d}{dt}H_2(t) + C_2\Big(\sum_{0<|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2 + \sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}}d\xi dx\Big)$$
$$\le O(1)(\lambda_0+\delta)\sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx + O(1)(\phi(t)+\psi(t))\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2. \quad (3.25)$$

It is clear from (3.25) that to close the energy estimates we need to have the energy estimates with respect to the weight $\mathbf{M}_-$. As usual in the previous works, the difference between the estimates with respect to $\mathbf{M}$ and $\mathbf{M}_-$ is that the orthogonal property (3.22) holds only with respect to $\mathbf{M}$. Without this orthogonal property, there is a corresponding error term in the form of $\sum_{1\le|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2$.

Therefore, similar calculation shows that if we set

$$\overline{H}_2(t) = \bar\lambda_2\sum_{|\gamma|=3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f_t|^2}{\mathbf{M}_-}d\xi dx + \overline{H}_1(t) - O(1)(\lambda_0+\delta)\sum_{|\gamma|\le 2}\int_{\mathbf{R}^3}\partial^\gamma\rho_t \mathrm{div}_x(\partial^\gamma u_t)dx$$
$$+ \sum_{\beta>0,|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx \quad (3.26)$$

with

$$\overline{H}_1(t) = \sum_{j=0}^{2}\bar\lambda_j^1\Big(\bar\lambda_j^0\sum_{|\gamma|=j}H_0^\gamma(t) + \sum_{|\gamma|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx\Big), \quad (3.27)$$

then there exists a positive constant $\overline{C}_2 > 0$ such that

$$\frac{d}{dt}\overline{H}_2(t) + \overline{C}_2\sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx$$
$$\le O(1)(\phi(t)+\psi(t))\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2 + O(1)\sum_{0<|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2. \quad (3.28)$$

Here $\bar\lambda_j^i$ $(i=0,1,\ j=0,1,2)$ and $\bar\lambda_2$ are some suitably chosen positive constants.

Combining (3.25) with (3.28), if we choose $\lambda_3 > 0$ sufficiently large and set

$$H_3(t) = \lambda_2 H_2(t) + \overline{H}_2(t), \quad (3.29)$$

then there exists a positive constant $C_3 > 0$ such that

$$\frac{d}{dt}H_3(t) + C_3\Big(\sum_{0<|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2 + \sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx\Big)$$
$$\le O(1)(\phi(t)+\psi(t))\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2. \quad (3.30)$$

Since

$$H_3(t) \sim \sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2 + \sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx,$$

we have the following theorem on the decay estimate by using Lemma 2.1.

Theorem 3.1 *Under the assumptions listed in Theorem* 1.2, *we have*

$$\sum_{|\gamma|\le 3}\|\partial^\gamma(\rho,u,\theta)_t\|^2+\sum_{|\gamma|+|\beta|\le 3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\partial^\beta\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dx\le O(1)(1+t)^{-1}. \tag{3.31}$$

In the following, we will use the equations (2.4), (2.5) and the decay estimates (3.31) to deduce a decay estimate on $\nabla_x\partial^\alpha(f-\overline{\mathbf{M}})$ $(1\le|\alpha|\le 3)$. Notice that some of the following estimates follow from the same arguments as those for the Lemmas 3.1–3.4, and we will omit the proofs for these estimates for brevity.

As usual, the first estimate on $\nabla_x\partial^\alpha(\rho-\bar\rho,u,\theta)$ can be given as follows.

Lemma 3.5 *For* $|\alpha|\le 3$, *we have*

$$\begin{aligned}\|\nabla_x\partial^\alpha(\rho-\bar\rho,u,\theta)\|^2\le{}&O(1)(1+t)^{-\frac12}+O(1)(\lambda_0+\delta)\sum_{|\alpha'|\le 3}\|\nabla_x\partial^{\alpha'}(\rho-\bar\rho,u,\theta)\|^2\\&+O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\Big(\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}+(\lambda_0+\delta)\sum_{|\alpha'|+|\beta'|\le 4}\frac{|\partial^{\alpha'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}\Big)d\xi dx.\end{aligned} \tag{3.32}$$

Proof By using the equation for $\nabla_x(\rho-\bar\rho)$ (after (3.21)), (3.31) gives

$$\begin{aligned}\|\nabla_x\partial^\alpha(\rho-\bar\rho)\|^2\le{}&O(1)(1+t)^{-\frac12}+O(1)(\lambda_0+\delta)\sum_{|\alpha'|\le 3}\|\nabla_x\partial^{\alpha'}(\rho-\bar\rho,u,\theta)\|^2\\&+O(1)\|\nabla_x\partial^\alpha\theta\|^2+O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx.\end{aligned} \tag{3.33}$$

For $\nabla_x\partial^\alpha(u,\theta)$, the equations for $u(t,x)$ and $\theta(t,x)$ are

$$\begin{aligned}&u_{it}+\sum_{j=1}^3u_ju_{ix_j}+\frac{2}{3\rho}(\rho\theta-\bar\rho\bar\theta)_{x_i}+\frac{\rho-\bar\rho}{\rho}\Phi_{x_i}\\&=-\int_{\mathbf{R}^3}\frac{\psi_i(\xi\cdot\nabla_x\Theta)}{\rho}d\xi+\frac{1}{\rho}\sum_{j=1}^3\Big\{\mu(\theta)\Big(u_{ix_j}+u_{jx_i}-\frac23\delta_{ij}\mathrm{div}_xu\Big)\Big\}_{x_j},\quad i=1,2,3,\\&\theta_t+\sum_{j=1}^3\Big(u_j\theta_{x_j}+\frac23\theta u_{jx_j}\Big)\\&=-\int_{\mathbf{R}^3}\frac{\psi_4-\xi\cdot u}{\rho}(\xi\cdot\nabla_x\Theta)d\xi+\frac{1}{\rho}\Big\{\sum_{j=1}^3(\kappa(\theta)\theta_{x_j})_{x_j}\\&\quad+\frac12\mu(\theta)\sum_{i,j=1}^3(u_{ix_j}+u_{jx_i})^2-\frac23\mu(\theta)(\mathrm{div}_xu)^2\Big\}.\end{aligned} \tag{3.34}$$

By applying ∂^α to (3.34) and multiplying it by $(\frac23\partial^\alpha u,\frac1\theta\partial^\alpha\theta)$ before integrating it with respect to x over $\mathbf{R}^3$, similar argument for proving the Lemma 3.1 shows that there exists some positive constant $C_4 > 0$ such that

$$C_4\|\nabla_x\partial^\alpha u\|^2 \le O(1)(1+t)^{-\frac{1}{2}} + O(1)(\lambda_0+\delta)\sum_{|\alpha'|\le 3}\|\nabla_x\partial^{\alpha'}(\rho-\bar\rho,u,\theta)\|^2 + O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}|\partial^\alpha\Theta|^2}{\mathbf{M}}d\xi dx - \int_{\mathbf{R}^3}\frac{2\bar\rho}{3}\nabla_x\partial^\alpha\theta\cdot\partial^\alpha u dx, \tag{3.35}$$

$$C_4\|\nabla_x\partial^\alpha\theta\|^2 \le O(1)(1+t)^{-\frac{1}{2}} + O(1)(\lambda_0+\delta)\sum_{|\alpha'|\le 3}\|\nabla_x\partial^{\alpha'}(\rho-\bar\rho,u,\theta)\|^2 + O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)^{-1}}{\mathbf{M}}(|\nabla_x\partial^\alpha\Theta|^2 + (\lambda_0+\delta)\sum_{\alpha'<\alpha}|\nabla_x\partial^{\alpha'}\Theta|^2)d\xi dx - \int_{\mathbf{R}^3}\frac{2\bar\rho}{3}\partial^\alpha\theta\ \mathrm{div}_x(\partial^\alpha u)dx. \tag{3.36}$$

By (3.33), (3.35) and (3.36), (3.32) follows immediately from Lemma 2.5 and (3.31). And this completes the proof of the lemma.

For the non-fluid component $\mathbf{G}$, the argument for proving the Corollary 3.1 gives the following estimates.

Lemma 3.6 *Under the assumptions in Theorem* 1.2*, we have*

$$\sum_{\beta>0,|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx \le O(1)(1+t)^{-1} + O(1)\sum_{|\alpha|\le 3}\|\nabla_x\partial^\alpha(u,\theta)\|^2 + O(1)(\lambda_0+\delta)\sum_{|\alpha|\le 3}\|\nabla_x\partial^\alpha(\rho-\bar\rho)\|^2 + O(1)(\lambda_0+\delta)\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx + O(1)\sum_{|\alpha|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx, \tag{3.37}$$

and for $j = 0,1,2,$

$$\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx \le O(1)(1+t)^{-1} + O(1)\sum_{|\alpha|=j}\|\nabla_x\partial^\alpha(u,\theta)\|^2 + O(1)(\lambda_0+\delta)\sum_{|\alpha|\le j}\|\nabla_x\partial^\alpha(\rho-\bar\rho,u,\theta)\|^2 + O(1)(\lambda_0+\delta)\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx + O(1)\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx. \tag{3.38}$$

Therefore, a suitably combination of (3.32) and (3.38) yields

$$\sum_{|\alpha|\le 3}\Big(\|\nabla_x\partial^\alpha(\rho-\bar\rho,u,\theta)\|^2 + \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big) \le O(1)(1+t)^{-\frac{1}{2}} + O(1)(\lambda_0+\delta)\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx + O(1)\sum_{|\alpha|=3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\nabla_x\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx. \tag{3.39}$$

The fourth order derivatives of $\mathbf{G}$ with respect to x is given in the following lemma.

Lemma 3.7 *For* $|\alpha| = 4$, *we have*

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx \le O(1)(1+t)^{-\frac{1}{2}} + O(1)(\lambda_0+\delta)\sum_{|\alpha'|\le 3}\|\nabla_x \partial^{\alpha'}(\rho-\bar{\rho},u,\theta)\|^2 + O(1)(\lambda_0+\delta)\sum_{|\alpha'|+|\beta'|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\alpha'}\partial^{\beta'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx. \tag{3.40}$$

Indeed, since $f - \overline{\mathbf{M}}$ solves

$$(f-\overline{\mathbf{M}})_t + \xi\cdot\nabla_x(f-\overline{\mathbf{M}}) - \nabla_x\Phi\cdot\nabla_\xi(f-\overline{\mathbf{M}}) = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G},\mathbf{G}),$$

(3.40) can be proved similar to (3.23) in Lemma 3.4. The only difference is that we need to use the following orthogonal properties

$$\begin{aligned}&\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma \mathbf{M}_{[\rho-\bar{\rho},u,\theta]})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi = 0,\\ &\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma(\bar{\rho}(\mathbf{M}_{[1,u,\theta]}-\mathbf{M}_{[1,0,\bar{\theta}]})))\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi = 0.\end{aligned} \tag{3.41}$$

Hence, the details are omitted.

Putting (3.37), (3.39), and (3.40) together, we have

$$\begin{aligned}&\sum_{|\alpha|\le 3}\|\nabla_x\partial^\alpha(\rho-\bar{\rho},u,\theta)\|^2 + \sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx\\ &\le O(1)(1+t)^{-\frac{1}{2}} + O(1)(\lambda_0+\delta)\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx.\end{aligned} \tag{3.42}$$

And the corresponding estimate with respect to the weight $\mathbf{M}_-$ is (cf. Lemma 3.6)

$$\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx \le O(1)(1+t)^{-1} + O(1)\sum_{|\alpha|\le 3}\|\nabla_x\partial^\alpha(\rho-\bar{\rho},u,\theta)\|^2. \tag{3.43}$$

When $\lambda_0 > 0, \delta > 0$ are sufficiently small, the combination of (3.42) and (3.43) gives the following theorem.

Theorem 3.2 *Under the assumptions listed in Theorem* 1.2, *we have*

$$\sum_{|\alpha|\le 3}\|\nabla_x\partial^\alpha(\rho-\bar{\rho},u,\theta)\|^2 + \sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu_{\mathbf{M}}(\xi)|\partial^\alpha\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx \le O(1)(1+t)^{-\frac{1}{2}}. \tag{3.44}$$

Finally, the statement in Theorem 1.2 follows directly from (3.44) and the Sobolev inequality (2.2).

Remark 3.2 Notice that even though the time decay estimates in (3.31) and (3.44) are for $H^4(\mathbf{R}^3)$, we need to use the boundedness of the solutions in $H^5(\mathbf{R}^3)$ obtained in the existence theory.

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