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ScienceDirect
J. Differential Equations 242 (2007) 40–71
Journal of Differential Equations
www.elsevier.com/locate/jde

Existence and stability of planar diffusion waves for 2-D Euler equations with damping

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Received 21 February 2007; revised 16 June 2007

Available online 24 August 2007

Abstract

To study the non-linear stability of a non-trivial profile for a multi-dimensional systems of gas dynamics, the combination of the Green function on estimating the lower order derivatives and the energy method for the higher order derivatives is shown to be not only useful but sometimes maybe also essential. In this paper, we study the stability of a planar diffusion wave for the isentropic Euler equations with damping in two-dimensional space. By introducing an approximate Green function for the linearized equations around the planar diffusion wave and by applying the energy method, we prove the global existence and the L_2 convergence rate of the solution when the initial data is a small perturbation of the planar diffusion wave. The decay rates of the perturbation and its lower order spatial derivatives obtained are optimal in the L_2 norm. Furthermore, the constructed approximate Green function in this paper can be used for the pointwise and the L_p estimates of the solutions concerned. In fact, the approach by combining of the Green function and energy method can be applied to other system especially when the derivatives of the coefficients in the system have certain time decay properties.

Keywords: 2-D Euler equations; Frictional damping; Approximate Green function; Energy method; Convergence rates

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[1] The research was supported in part by National Natural Science Foundation of China 10531020.

[2] The research was supported in part by the RGC Competitive Earmarked Research Grant of Hong Kong CityU 1087/99P.

0022-0396/$ – see front matter © 2007 Published by Elsevier Inc.
doi:10.1016/j.jde.2007.07.002

1. Introduction

Even though there are extensive studies on the stability of non-linear profiles for the system of Euler equations with frictional damping in one-dimensional space, there are much less results on the multi-dimensional problems. This paper is concerned with the stability of planar waves for the two-dimensional isentropic Euler equations with frictional damping. That is, we consider the system

$$\begin{cases} \rho_t + (\rho u_1)_x + (\rho u_2)_y = 0, \\ (\rho u_1)_t + (\rho u_1^2)_x + (\rho u_1 u_2)_y + P(\rho)_x = -\kappa \rho u_1, \\ (\rho u_2)_t + (\rho u_1 u_2)_x + (\rho u_2^2)_y + P(\rho)_y = -\kappa \rho u_2, \end{cases} \tag{1.1}$$

with initial data

$$(\rho, u_1, u_2)(x, y, 0) = \big(\rho_0(x, y), u_{10}(x, y), u_{20}(x, y)\big). \tag{1.2}$$

In the following discussion, we assume that the initial data is a small perturbation of a planar diffusion wave with small wave strength. Here $\rho(x, y, t)$, $u(x, y, t) = (u_1, u_2)(x, y, t)$, and $P = P(\rho(x, y, t))$ represent the density, velocity and pressure respectively, and $\kappa > 0$ is the constant frictional damping coefficient. As for most of the physical cases, we assume the pressure $P(\rho)$ is a smooth function in a neighborhood of a constant state ρ^* with $P'(\rho) > 0$ and the ρ under consideration is in this neighborhood. Moreover, we assume that the initial data $\rho(x, y, 0)$ satisfies

$$\lim_{x \to \pm\infty} \rho(x, y, 0) = \rho_\pm, \tag{1.3}$$

where $\rho(x, y, 0) > 0$, and $\rho_\pm > 0$ are two constants with $\rho_- \neq \rho_+$.

To define the planar diffusion wave, let us first consider the one-dimensional diffusion equation,

$$\partial_t \phi = \kappa^{-1} P(\phi)_{xx}, \tag{1.4}$$

which can be derived from the Euler equations with frictional damping in one-dimensional case by imposing the Darcy's laws, cf. [8]. Then a planar diffusion wave $\phi(x, y, t)$ is a one-dimensional profile in two-dimensional space. Let $\varphi(x/\sqrt{1+t})$ be the self-similar solution of Eq. (1.4) connecting two end states $\rho_\pm$ at $x = \pm\infty$. Then the planar wave considered in the following is defined by $\phi(x, y, t) = \varphi(x/\sqrt{1+t})$.

For simplicity, we assume that the initial velocity $(u_1(x, y, 0), u_2(x, y, 0))$ satisfies

$$\lim_{x \to \pm\infty} (u_1, u_2)(x, y, 0) = 0, \tag{1.5}$$

which implies that there is no mass flux coming from $x = \pm\infty$. This assumption can be removed in a way similar to the argument for one-dimensional problem because of the exponential decay of the momentum at $x = \pm\infty$ induced by the linear frictional damping.

For later use, we need to introduce some notations for the one-dimensional problem. Consider (1.1) and (1.2) in one space dimension:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2)_x + P(\rho)_x = -\kappa \rho u_1, \\ (\rho, u_1)(x, 0) = (\tilde{\rho}, \tilde{u}_1)(x, 0). \end{cases} \tag{1.6}$$

Denote the solution of (1.6) by $(\tilde{\rho}, \tilde{u}_1)(x,t)$. When

$$\lim_{x\to\pm\infty} \tilde{\rho}(x,0)=\rho_\pm, \qquad \lim_{x\to\pm\infty} \tilde{u}_1(x,0)=0,$$

the time-asymptotic behavior of $(\tilde{\rho}, \tilde{u}_1)(x,t)$ has been well studied which is shown to be a non-linear profile governed by Darcy's law, cf. [8,16,18] and references therein. Roughly speaking, the solution $\tilde{\rho}(x,t)$ converges to the diffusion wave $\varphi(x/\sqrt{1+t})$ up to a constant shift in x.

In this paper, we will generalize the one-dimensional result to the case when the space dimension is two. Notice that even though the non-linear profile is one-dimensional, it is in a two-dimensional space and any perturbation can generate "waves" propagating in all direction in the plane. Therefore, the analysis cannot be closed by simply choosing a shift of the diffusion wave according to the initial perturbation. Indeed, it seems that the sole use of the energy method does not yield global existence and stability when the initial data is a perturbation of this profile. The main result of this paper shows that the solution of the Cauchy problem (1.1) and (1.2) converges to the planar diffusion wave $\phi(x,y,t)$ defined above with a shift only in x direction which is determined by initial perturbation.

In the following analysis, we do not compare the solution to the problem (1.1)–(1.2) directly with the planar diffusion $\phi(x,y,t)$. Instead, we will compare it with the solution to one-dimensional problem (1.6). For this, let us first assume the initial density $\tilde{\rho}(x,0)$ in (1.6) satisfies the following condition without loss of generality,

$$\int_{-\infty}^{+\infty} \big(\tilde{\rho}(x,0)-\varphi(x)\big)\,dx=0. \tag{1.7}$$

As for the two-dimensional problem, we can choose a shift function $\delta_0(y)$ such that the initial density function satisfies

$$\int_{-\infty}^{+\infty} \big(\rho(x,y,0)-\varphi\big(x+\delta_0(y)\big)\big)\,dx=0. \tag{1.8}$$

Note that $\delta_0(y)$ is uniquely determined by $\rho(x,y,0)$ and $\varphi(x)$ by

$$\delta_0(y)=\frac{1}{\rho_+-\rho_-}\int_{-\infty}^{\infty} \big(\rho(x,y,0)-\phi(x)\big)\,dx,$$

where $\rho_- \neq \rho_+$ is used. Moreover, we assume

$$\lim_{y\to\pm\infty}\int_{-\infty}^{+\infty} \big(\rho(x,y,0)-\varphi(x)\big)\,dx=\delta_*, \tag{1.9}$$

where δ_* is a constant. This implies that

$$\lim_{y\to\pm\infty} \delta_0(y)=\delta_*(\rho_+-\rho_-).$$

Notice that for problem in one-dimensional space, a shift can be defined for all time to satisfy the conservation of mass so that the anti-derivative of the perturbation of the density function can be introduced. For problem in two-dimensional space, a shift as a function of only t and y cannot be defined to satisfy the conservation of mass so that the technique of taking anti-derivative will not be applied here.

On the other hand, under some integrability condition imposed on the initial perturbation, the shift at $t = \infty$ is δ_* up to a diffusion profile which decays like $(1+t)^{-\frac{1}{2}}$ for the problem considered here. Hence, the combination of the approximate Green function and the energy method can be applied as follows.

We will show that the solution (ρ, u_1, u_2) of (1.1) with (1.2) satisfies

$$\lim_{t\to\infty}\big((\rho, u_1, u_2) - (\bar\rho, \bar u_1, \bar u_2)\big) = \lim_{t\to\infty}\big((\rho, u_1, u_2) - \big(\tilde\rho(x+\delta_*), \tilde u_1(x+\delta_*), 0\big)\big) = 0,$$

where

$$\begin{aligned} \bar\rho(x, y, t) &= \tilde\rho\big(x + \delta(y, t), t\big), \\ \bar u_1(x, y, t) &= \tilde u_1\big(x + \delta(y, t), t\big), \\ \bar u_2(x, y, t) &= 0, \end{aligned} \tag{1.10}$$

with

$$\delta(y, t) = \delta_* + e^{-\kappa t}\big(\delta_0(y/\sqrt{1+t}) - \delta_*\big).$$

Then the one-dimensional result implies that $\bar\rho(x, y, t)$ behaves almost like a planar diffusion wave with a shift $\delta(y, t)$.

Remark 1.1. The exponential decay function $e^{-\kappa t}$ chosen in the definition of $\delta(y, t)$ is for the simplicity of the analysis. In fact, as mentioned earlier, the time asymptotic shift is determined by δ_* up to a function decay like $(1+t)^{-\frac{1}{2}}$. If we include such a function in the final decay estimates stated in Theorem 1.1, simple calculation shows that the conclusion is not changed. This is true because the planar wave considered here is a diffusion profile and its spatial derivatives decay in time. Here, the part other than δ_* in $\delta(y, t)$ is used when we apply the Green function to the linearized system in particular for the term at $t = 0$. This contrasts to the stationary shock profile considered in [4] where the precise decay of the shift is needed because the spatial derivative of the shock profile does not have any decay in time.

We now point out the main difference between the study in this paper and the related results in the previous works. Firstly, one of the main difficulties in the study comes from the non-existence of the Lagrangian coordinates in the multi-dimensional space. For example, in the energy estimates, some new techniques are needed to deal with the convection terms in the velocity equations and the non-symmetry of the x and y coordinates because the planar wave is only in x direction. For the one-dimensional problem, cf. [1,8,17], the anti-derivative of the quantity $\rho - \bar\rho$ was used to close the a priori energy estimate. However, for the two-dimensional problem, a direct generalization of the one-dimensional idea leads to the implicitly defined shift depending on the solution instead of the initial data so that it does not give a clear picture of the large time behavior, cf. [6]. To overcome this difficulty, instead of taking anti-derivative of the perturbation to the density function, we combine the energy method with the Green function to prove the global existence of solution and give the convergence rates to the planar wave time asymptotically.

If the Darcy's law in the y-direction is imposed so that the third equation in (1.1) becomes

$$(\rho u_2)_t + (\rho u_1 u_2)_x + \left(\rho u_2^2\right)_y = 0, \tag{1.11}$$

then the three equations in the system (1.1) can be reduced to two equations. In this case, the non-linear terms on the right-hand sides of the equations after linearization can be integrated with respect to x so that the energy method can be applied in a more straightforward way, cf. [6]. In this paper, we do not impose this extra condition, instead, we study the full Euler equations with damping.

Another difficulty comes from the fact that the background non-linear profile is now a function of time and space which is different from the case of a constant state considered in [20]. Here the linearized equations have variable coefficients and therefore the Fourier analysis approach used in [20] cannot be applied directly. In [18], we used an approximate Green function to study the L_p decay estimates for the perturbation of non-constant states in one-dimensional space. The analysis used in [18] depends on the L_2 decay rate obtained by energy method. However, for the two-dimensional problem, how to close the a priori energy estimate only by energy method itself is not clear. Here, we improve previous methods on the construction of the approximate Green function for linear differential equations with variable coefficients when the derivatives of the coefficients have some decay properties in time. By using this approximate Green function to evaluate the lower order energy estimates, we succeed in obtaining the desired estimates by combining them with the energy method for higher order estimates.

We mention that the basic estimates for the wave equations with dissipation were obtained in the paper [15]. And there are works on some semi-linear systems, such as Jin–Xin model and works on problems related to vacuum to Euler equations with damping. Since they are not related to the problem considered in this paper, we will not refer them here. Furthermore, the combination of the energy method and the Green function (or the spectral analysis) has been applied to the study on some hyperbolic–parabolic systems such as Navier–Stokes equations, and some kinetic equations such as Boltzmann equation, cf. [2,3]. The main point of this paper is to apply it to a hyperbolic system in multi-dimensions. Some related results especially on the stability of solution profiles for the hyperbolic–parabolic systems can be found in [7,10–14,19,22]. The global existence of classical solutions to the hyperbolic systems in multi-space dimensions was discussed in [9,14].

Throughout this paper we denote the generic constants by C. $W^{s,p}(\mathbf{R}^n)$, $s \in \mathbf{Z}_+$, $p \in [1,\infty]$, denotes the usual Sobolev space with the norm

$$\|f\|_{W^{s,p}} := \sum_{|\alpha|=0}^{s} \left\|\partial^\alpha f\right\|_{L_p}.$$

In particular, $W^{s,2} = H^s$.

Finally, as in [5,6], set

$$\begin{aligned}
V(x,y,t) &= \rho(x,y,t) - \bar\rho(x,y,t) = \rho(x,y,t) - \tilde\rho\big(x+\delta(y,t),t\big),\\
U_1(x,y,t) &= u_1(x,y,t) - \bar u_1(x,y,t) = u_1(x,y,t) - \tilde u_1\big(x+\delta(y,t),t\big),\\
U_2(x,y,t) &= u_2(x,y,t).
\end{aligned}$$

Denote

$$\nu(x,y,0)=\int_{-\infty}^{x} V(x',y,0)\,dx', \qquad \nu_t(x,y,0)=\int_{-\infty}^{x} V_t(x',y,0)\,dx'. \tag{1.12}$$

From (1.7) and (1.8), we further assume that

$$(\nu,\nu_t)(x,y,0)\in L_2\big(\mathbf{R}^2\big). \tag{1.13}$$

By using above notations, we can now state the main result in this paper as follows.

Theorem 1.1. *Let $(\bar\rho,\bar u)(x,y,t)(x,y,t)$ be defined in* (1.10) *as a planar diffusion wave with a shift $\delta(y,t)$. Assume that the initial data $(\rho,u_1,u_2)(x,y,0)$ satisfy that*

$$(\nu,\nu_t)(x,y,0)\in L_2\big(\mathbf{R}^2\big), \quad (\rho-\bar\rho,u_1-\bar u_1,u_2)(x,y,0)\in H^k\big(\mathbf{R}^2\big) \quad (k\geqslant 3), \tag{1.14}$$

with smallness assumption

$$|\rho_+-\rho_-|+\big\|(\rho-\bar\rho,u_1-\bar u_1,u_2)(\cdot,0)\big\|_{H^k}+\big\|\nu(\cdot,0)\big\|_{L_2}+\big\|\nu_t(\cdot,0)\big\|_{L_2}<\epsilon_0, \tag{1.15}$$

where $\epsilon_0>0$ is a small constant. Then there exists a unique classical solution $(\rho,u_1,u_2)\in C([0,\infty),H^k)\cap C^1((0,\infty),H^{k-1})$ to the system (1.1) *globally defined in time. Moreover, for $|\alpha|\leqslant k-3$*

$$\big\|\partial^\alpha_{x,y}(\rho-\bar\rho)(\cdot,\cdot,t)\big\|_{L_2}\leqslant C(1+t)^{-\frac{|\alpha|+1}{2}}, \qquad \big\|\partial^\alpha_{x,y}(u_1-\bar u_1,u_2)(\cdot,\cdot,t)\big\|_{L_2}\leqslant C(1+t)^{-\frac{|\alpha|+2}{2}}. \tag{1.16}$$

Remark 1.2. To compare the decay rates given in (1.16) for the perturbation in the two-dimensional space to the previous works on the one-dimensional space, we can find out that these decay rates are consistent in the L_2 norm together with those derivatives in x variable with order not larger than $k-3$. That is, the decay rates represent the diffusion structure of the system with extra $\frac{1}{2}$ power decay in time for extra spatial differentiation. Thus, we call these decays rates optimal. Under stronger assumption on the perturbation, stronger decay rates could be obtained. However, this will not be discussed in this paper.

The rest of the paper is arranged as follows. In Section 2, we will reformulate the system around the planar wave. The properties of the diffusion wave with the shift are given in Section 3. In Section 4, we will study the Green function for a linear system with a parameter by using Fourier analysis. The L_2 estimates on the solution of (1.1) around the planar wave by using the approximate Green function are presented in Section 5. Finally, the estimates on the higher order estimates by using energy method are given in Section 6. Then the existence and the time-asymptotic behavior of the solution to (1.1) follow from these estimates.

2. Reduced system

In this section, we will derive the equations for a small perturbation of the non-linear planar wave. First, we rewrite (1.1) and (1.6) as follows

$$\begin{cases} \rho_t + (\rho u_1)_x + (\rho u_2)_y = 0, \\ (u_1)_t + u_1 (u_1)_x + u_2 (u_1)_y + \rho^{-1} P(\rho)_x = -\kappa u_1, \\ (u_2)_t + u_1 (u_2)_x + u_2 (u_2)_y + \rho^{-1} P(\rho)_y = -\kappa u_2, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (u_1)_t + u_1 (u_1)_x + \rho^{-1} P(\rho)_x = -\kappa u_1. \end{cases} \tag{2.2}$$

Here, we assume the initial data $\rho(x, y, 0)$ and $u_j(x, y, 0)$, $j = 1, 2$ satisfy

$$V(x, y, 0), U_j(x, y, 0) \in L_2.$$

For simplicity, we denote $U = (U_1, U_2)$, $u = (u_1, u_2)$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)$. Then from (2.1) and (1.10), we have

$$\begin{aligned} \rho_t + (\rho u_1)_x + (\rho u_2)_y = {} & -R_\rho + \big(V_t + (\bar{\rho} + V) \operatorname{div} U\big) \\ & + (U \cdot \nabla)\bar{\rho} + V \operatorname{div} \bar{u} + \big((\bar{u} + U) \cdot \nabla\big) V, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R_\rho &= -\tilde{\rho}_1'\big(x + \delta(y, t), t\big)\delta_t(y, t) \\ &= -\big(\tilde{\rho}\big(x + \delta(y, t), t\big)\delta_t(y, t)\big)_x. \end{aligned}$$

And (2.1) and (2.3) give

$$\begin{aligned} V_t + (\bar{\rho} + V) \operatorname{div} U &= R_\rho - (U \cdot \nabla)(\bar{\rho} + V) - V \operatorname{div} \bar{u} - (\bar{u} \cdot \nabla) V \\ &= R_\rho - (U \cdot \nabla)(\bar{\rho} + V) - V(\bar{u}_1)_x - V_x(\bar{u}_1). \end{aligned} \tag{2.4}$$

Similarly, the equations for U_1 and U_2 can be obtained as follows

$$\begin{aligned} & (U_1)_t + (\bar{\rho} + V)^{-1}\big(P(\bar{\rho} + V) - P(\bar{\rho})\big)_x + \kappa U_1 \\ & \quad = \big((u_1)_t + (\bar{\rho} + V)^{-1} P(\bar{\rho} + V)_x + \kappa u_1\big) - \big(\bar{u}_t + (\bar{\rho} + V)^{-1} P(\bar{\rho})_x + \kappa \bar{u}_1\big) \\ & \quad = \frac{P(\bar{\rho})_x V}{\bar{\rho}(\bar{\rho} + V)} + R_u - R_1, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} & (U_2)_t + (\bar{\rho} + V)^{-1}\big(P(\bar{\rho} + V) - P(\bar{\rho})\big)_y + \kappa U_2 \\ & \quad = \big((u_2)_t + (\bar{\rho} + V)^{-1} P(\bar{\rho} + V)_y + \kappa u_2\big) - (\bar{\rho} + V)^{-1} P(\bar{\rho})_y \\ & \quad = \frac{P(\bar{\rho})_y V}{\bar{\rho}(\bar{\rho} + V)} - R_2 - \bar{\rho}^{-1} P(\bar{\rho})_y, \end{aligned} \tag{2.6}$$

where

$$R_u = \big(\tilde{u}_1\big(x+\delta(y,t),t\big)\delta_t(y,t)\big)_x,$$
$$R_1 = U_1(\bar{u}_1+U_1)_x + \bar{u}_1(U_1)_x + U_2(\bar{u}_1+U_1)_y,$$
$$R_2 = (\bar{u}_1+U_1)(U_2)_x + U_2(U_2)_y.$$

In summary, the equations for the perturbation functions (V, U_1, U_2) are:

$$V_t + (\bar{\rho}+V)\operatorname{div} U = Q, \tag{2.7}$$

and

$$\begin{aligned}(U_1)_t + (\bar{\rho}+V)^{-1}\big(\mathcal{P}(V,\bar{\rho})V\big)_x + \kappa U_1 = H_1,\\ (U_2)_t + (\bar{\rho}+V)^{-1}\big(\mathcal{P}(V,\bar{\rho})V\big)_y + \kappa U_2 = H_2,\end{aligned} \tag{2.8}$$

where $\mathcal{P}(V,\bar{\rho}) = \int_0^1 P'(\bar{\rho}+\theta V)\,d\theta$ and

$$\begin{aligned}Q &= R_\rho - (U\cdot\nabla)(\bar{\rho}+V) - V(\bar{u}_1)_x - (\bar{u}_1)V_x,\\ H_1 &= R_u + \frac{P(\bar{\rho})_x V}{\bar{\rho}(\bar{\rho}+V)} - R_1,\\ H_2 &= -\frac{P(\bar{\rho})_y}{\bar{\rho}} + \frac{P(\bar{\rho})_y V}{\bar{\rho}(\bar{\rho}+V)} - R_2.\end{aligned} \tag{2.9}$$

Moreover, by using (2.7) and (2.8), the equation for $V(x,y,t)$ can be written as

$$V_{tt} - \triangle\big(\mathcal{P}(V,\bar{\rho})V\big) + \kappa V_t = \tilde{Q}(V,U,\bar{\rho},\bar{u}_1), \tag{2.10}$$

where

$$\begin{aligned}\tilde{Q}(V,U,\bar{\rho},\bar{u}) = {} & \big[(R_\rho)_t + \kappa(R_\rho)\big] - P(\bar{\rho})_{yy} + (R_{\rho u})_x\\ & + \big[\big(\bar{\rho}\big(2\bar{u_1}U_1 + U_1^2\big) + V(\bar{u}_1+U_1)^2\big)_{xx}\\ & + 2\big((\bar{\rho}+V)(\bar{u}_1+U_1)U_2\big)_{xy} + \big((\bar{\rho}+V)(U_2)^2\big)_{yy}\big],\end{aligned} \tag{2.11}$$

and

$$R_{\rho u} = -\big((\rho\tilde{u}_1)\big(x+\delta(y,t),t\big)\delta_t(y,t)\big)_x.$$

By linearizing the system (2.10) and (2.8) around $(\bar{\rho},\bar{u})$, we have the following system

$$V_{tt} - \triangle\big(a(x,y,t)V\big) + \kappa V_t = F(W,\bar{\rho},\bar{u}), \tag{2.12}$$

and

$$U_t + \bar{\rho}^{-1}\nabla\big(a(x,y,t)V\big) + \kappa U = \bar{H}(W,\bar{\rho},\bar{u}). \tag{2.13}$$

Here, $a(x,y,t)=P'(\bar\rho)$, $\bar H=(\bar H_1,\bar H_2)^\tau$ and

$$\begin{aligned}
F(W,\bar\rho,\bar u) &= \big[(R_{\rho u})_t+\kappa R_\rho+(R_m)_x-P(\rho)_{yy}\big],\\
&\quad+\big[\big(\bar\rho\big(2\bar u_1U_1+U_1^2\big)+V(\bar u_1+U_1)^2\big)_{xx}\\
&\quad+2\big((\bar\rho+V)(\bar u_1+U_1)U_2\big)_{xy}+\big((\bar\rho+V)(U_2)^2\big)_{yy}\big]\\
&\quad+\big[\big(\mathcal{P}_1(\bar\rho,V)V^2\big)_{xx}+\big(\mathcal{P}_1(\bar\rho,V)V^2\big)_{yy}\big],\\
\bar H_1(W,\bar\rho,\bar u) &= R_u-\bar\rho^{-1}\big(\mathcal{P}_1(\bar\rho,V)V^2\big)_x-\frac{P(\bar\rho+V)_xV}{\bar\rho(\bar\rho+V)}-R_1,\\
\bar H_2(W,\bar\rho,\bar u) &= -\frac{P(\bar\rho)_y}{\bar\rho}-\frac{P(\bar\rho+V)_yV}{\bar\rho(\bar\rho+V)}-\bar\rho^{-1}\big(\mathcal{P}_1(\bar\rho,V)V^2\big)_y-R_2,
\end{aligned}\tag{2.14}$$

where

$$\mathcal{P}_1(\bar\rho,V)=\int_0^1\left(\int_0^{\theta_1}P''(\bar\rho+\theta_2V)\,d\theta_2\right)d\theta_1.$$

Finally, we denote that

$$\begin{aligned}
\mathcal{M}_1(t) &= \sup_{0\leqslant s\leqslant t,\,|\alpha|\leqslant k}(1+s)^{\frac{\nu(|\alpha|)+1}{2}}\big\|\partial^\alpha_{x,y}V(\cdot,s)\big\|_{L_2},\\
\mathcal{M}_2(t) &= \sup_{0\leqslant s\leqslant t,\,|\alpha|\leqslant k}(1+s)^{\frac{\mu(|\alpha|)+2}{2}}\big\|\partial^\alpha_{x,y}U(\cdot,s)\big\|_{L_2},
\end{aligned}$$

where

$$\nu(l)=\begin{cases} l, & l\leqslant k-2,\\ l-\epsilon, & l=k-1,\\ l-1-\epsilon, & l=k,\end{cases}\qquad \mu(l)=\begin{cases} l, & l\leqslant k-3,\\ l-\epsilon, & l=k-2,\\ l-1-\epsilon, & l=k-1,\\ l-2-\epsilon, & l=k,\end{cases}$$

with $\epsilon>0$ being sufficiently small. To prove Theorem 1.1, it suffices to prove the following two propositions.

Proposition 2.1. *If initial data $(\rho,u)(x,y,0)$ satisfy the conditions of Theorem* 1.1*, then for $|\alpha|\leqslant k-1$*

$$\begin{aligned}
\big\|\partial^\alpha(\rho-\bar\rho)(\cdot,t)\big\|^2_{L_2} &\leqslant C\big(E+\mathcal{M}^3\big)(1+t)^{-(\nu(|\alpha|)+1)},\\
\big\|\partial^\alpha(u-\bar u)(\cdot,t)\big\|^2_{L_2} &\leqslant C\big(E+\mathcal{M}^3+\mathcal{M}_1^2\big)(1+t)^{-(\mu(|\alpha|)+2)},
\end{aligned}\tag{2.15}$$

where $E>0$ is a small constant which depends on the small constant $\epsilon_0>0$ given in (1.15)*, $\mathcal{M}=\max(\mathcal{M}_1,\mathcal{M}_2)$.*

Proposition 2.2. *If initial data* $(\rho, u)(x, y, 0)$ *satisfy the conditions of Theorem* 1.1*, then for* $|\alpha| = k$

$$\begin{aligned} \big\|\partial^\alpha(\rho-\bar\rho)(\cdot,t)\big\|_{L_2}^2 &\leqslant C\big(E+\mathcal{M}^3\big)(1+t)^{-(\nu(|\alpha|)+1)}, \\ \big\|\partial^\alpha(u-\bar u)(\cdot,t)\big\|_{L_2}^2 &\leqslant C\big(E+\mathcal{M}^3\big)(1+t)^{-(\mu(|\alpha|)+2)}. \end{aligned} \tag{2.16}$$

In fact, Propositions 2.1 and 2.2 imply that

$$\mathcal{M}^2 \leqslant C\big(E+\mathcal{M}^3\big). \tag{2.17}$$

Then, if E is sufficiently small, by continuity argument we have $\mathcal{M} \leqslant CE$. Therefore, we have, for $|\alpha| \leqslant k$

$$\begin{aligned} \big\|\partial^\alpha(\rho-\bar\rho)(\cdot,t)\big\|_{L_2}^2 &\leqslant CE(1+t)^{-(\nu(|\alpha|)+1)}, \\ \big\|\partial^\alpha(u-\bar u)(\cdot,t)\big\|_{L_2}^2 &\leqslant CE(1+t)^{-(\mu(|\alpha|)+2)}, \end{aligned} \tag{2.18}$$

which gives Theorem 1.1. The Propositions 2.1 and 2.2 will be proved by using the approximate Green function in Section 5 and by the energy method in Section 6 respectively.

3. The diffusion wave

For preparation, we firstly give some estimates on $(\bar\rho, \bar u_1)(x, y, t)$. Set

$$\psi(x,t) = -\kappa^{-1}P\big(\varphi(x/\sqrt{1+t})\big)_x, \tag{3.1}$$

where φ is defined in Section 1. The one-dimensional diffusion profile (φ, ψ) has the following properties, cf. [8,18].

Lemma 3.1. *For the functions* φ *and* ψ*, if for any integer* N*,*

$$\begin{aligned} \sup_{x>0}\big|\varphi(x)-\rho_+\big| + \sup_{x<0}\big|\varphi(x)-\rho_-\big| &\leqslant C|\rho_+-\rho_-|\big(1+x^2\big)^{-N}, \\ \big|\partial_x^h\varphi(x)\big| &\leqslant C|\rho_+-\rho_-|\big(1+x^2\big)^{-N}, \end{aligned} \tag{3.2}$$

we have

$$\begin{aligned} \sup_{x>0}\big|\varphi(x/\sqrt{1+t})-\rho_+\big| + \sup_{x<0}\big|\varphi(x/\sqrt{1+t})-\rho_-\big| &\leqslant C|\rho_+-\rho_-|B_N(x,t), \\ \big|\partial_t^l\partial_x^h\varphi(x/\sqrt{1+t})\big| &\leqslant C|\rho_+-\rho_-|(1+t)^{-(2l+h)/2}B_N(x,t), \end{aligned} \tag{3.3}$$

for $l+h \geqslant 1$*; and*

$$\big\|(\varphi_x,\psi)(t)\big\|_{H^k}^2 \leqslant C\big\|(\varphi_x,\psi)(0)\big\|_{H^k}^2, \tag{3.4}$$

where $B_N(x,t) = (1+\frac{x^2}{1+t})^{-N}$.

The following theorem is from [21].

Theorem 3.1. *For a small positive constant $E_\rho > 0$ and any integer N, if the initial data $(\tilde{\rho}, \tilde{u}_1)(x, 0)$ satisfies*

$$\left\| \int_{-\infty}^{x} \big(\tilde{\rho}(x,0) - \varphi(x)\big)\,dx \right\|_{L_2} + \big\|\tilde{\rho}(\cdot,0) - \varphi(\cdot)\big\|_{H^m} + \big\|\tilde{u}_1(\cdot,0) - \psi(\cdot,0)\big\|_{H^m} \leqslant E_\rho,$$

for $m \geqslant 2$, and $|\rho_+ - \rho_-| \leqslant E_\rho$, then

$$\big\|(\tilde{\rho} - \varphi)(t)\big\|_{H^m} + \big\|(\tilde{u}_1 - \psi)(t)\big\|_{H^m} \leqslant C E_\rho.$$

Moreover, if

$$\left| \int_{-\infty}^{x} \big(\tilde{\rho}(x,0) - \varphi(x)\big)\,dx \right| + \sum_{h \leqslant m} \big|\partial_x^h\big(\tilde{\rho}(x,0) - \varphi(x)\big)\big| + \sum_{h \leqslant m-1} \big|\partial_x^h\big(\tilde{u}_1(x,0) - \psi(x,0)\big)\big| \leqslant E_\rho\big(1 + x^2\big)^{-N/2},$$

then we have

$$\begin{aligned} \big|\partial_x^k\big(\tilde{\rho}(x,t) - \varphi(x/\sqrt{1+t})\big)\big| &\leqslant C E_\rho (1+t)^{-(k+2)/2} B_N(x,t), \\ \big|\partial_x^l\big(\tilde{u}_1(x,t) - \psi(x,t)\big)\big| &\leqslant C E_\rho (1+t)^{-(l+3)/2} B_N(x,t), \end{aligned} \tag{3.5}$$

where $k \leqslant m$, $l \leqslant m-1$.

In the following, we assume that for $h \geqslant 0$,

$$\big|\partial_y^h\big(\delta_0(y) - \delta_*\big)\big| \leqslant C\big(1 + y^2\big)^{-N}. \tag{3.6}$$

Combining Theorem 3.1 and Lemma 3.1 yields the following theorem.

Theorem 3.2. *If $\varphi(x)$ and $\delta_0(y)$ satisfy the conditions* (3.2) *and* (3.6) *respectively, then*

$$\begin{aligned} \big|\partial_t^l \partial_x^h \tilde{\rho}\big| &\leqslant C E_\rho (1+t)^{-(2l+h)/2} B_N(x,t) \quad (l + h \geqslant 1), \\ \big|\partial_t^l \partial_x^h \tilde{u}_1\big| &\leqslant C E_\rho (1+t)^{-(1+2l+h)/2} B_N(x,t), \end{aligned} \tag{3.7}$$

and for $m \geqslant 1$,

$$\begin{aligned} \big|\partial_t^l \partial_x^h \partial_y^m \bar{\rho}_y\big| &\leqslant C E_\rho e^{-\kappa t} B_N(x,y,t), \\ \big|\partial_t^l \partial_x^h \partial_y^m (\bar{u}_1)_y\big| &\leqslant C E_\rho e^{-\kappa t} B_N(x,y,t), \end{aligned} \tag{3.8}$$

where

$$B_N(x,y,t)=\left(1+\frac{x^2+y^2}{1+t}\right)^{-N}. \tag{3.9}$$

By using Lemma 3.1 and Theorem 3.1, it is easy to see that

$$\begin{aligned}
&\left\|\partial^\alpha(\bar\rho_x,\bar u_1)(y,t)\right\|_{L_2(R)}\leqslant CE_\rho(1+t)^{-\frac{1+|\alpha|}{2}-\frac14},\\
&\left\|\partial^\alpha(\bar\rho_x,\bar u_1)(y,t)\right\|_{L_\infty(R)}\leqslant CE_\rho(1+t)^{-\frac{1+|\alpha|}{2}}.
\end{aligned}\tag{3.10}$$

By noticing the definition of $\delta(y,t)$, we have

$$\begin{aligned}
&\left\|\partial^\alpha\big(\bar\rho_y,(\bar u_1)_y\big)(t)\right\|_{L_2(R^2)}\leqslant CE_\rho e^{-\kappa t},\\
&\left\|\partial^\alpha\big(\bar\rho_y,(\bar u_1)_y\big)(t)\right\|_{L_\infty(R^2)}\leqslant CE_\rho e^{-\kappa t}.
\end{aligned}\tag{3.11}$$

4. Green function

In this section, we will study the Green function with a parameter by using Fourier analysis. As usual, Fourier transformation to the variable $z\in\mathbf{R}^n$ is

$$\hat f(\xi,t)\equiv(\mathcal{F}f)(\xi,t)=\int\limits_{\mathbf{R}^n} f(z,t)e^{-iz\xi}\,dz,$$

and the inverse Fourier transform to the variable ξ is

$$f(z,t)\equiv\big(\mathcal{F}^{-1}\hat f\big)(z,t)=(2\pi)^{-n}\int\limits_{\mathbf{R}^n}\hat f(\xi,t)e^{iz\xi}\,d\xi.$$

Consider the first equation in (2.12), that is,

$$V_{tt}-\triangle\big(a(x,y,t)V\big)+\kappa V_t=F(V,U,\bar\rho,\bar m), \tag{4.1}$$

where $a(x,y,t)=P'(\bar\rho(x,y,t))$. We assume that $0<C_0<a=a(x,y,t)=P'(\bar\rho(x,y,t))<C_1$ with constants C_1 and C_0.

We now construct an approximate Green function $G(x,y,t;x',y',s)$ for the homogeneous part of (4.1) so that $G(x,y,t;x',y',s)$ satisfies the basic requirement

$$G(x,y,t;x',y',t)=0,\qquad G_t(x,y,t;x',y',t)=\delta(x'-x,y'-y), \tag{4.2}$$

where δ is the Dirac function. Multiplying (4.1) whose variables are now changed to (x',y',s) by G and integrating over the region $(x',y',s)\in\mathbf{R}^2\times(0,t)$, (4.2) gives

$$
\begin{aligned}
&V(x,y,t)\\
&\quad=\int_{\mathbf{R}^2} G_s(x,y,t;x',y',0)V(x',y',0)\,dx'\,dy'\\
&\qquad-\int_{\mathbf{R}^2} G(x,y,t;x',y',0)(\kappa V+V_t)(x',y',0)\,dx'\,dy'\\
&\qquad+\int_0^t\int_{\mathbf{R}^2} G(x,y,t;x',y',s)F(x',y',s)\,dx'\,dy'\,ds\\
&\qquad+\int_0^t\int_{\mathbf{R}^2} \big((G_{ss}-a\triangle G-\kappa G_s)(x,y,t;x',y',s)\big)V(x',y',s)\,dx'\,dy'\,ds.
\end{aligned}
\tag{4.3}
$$

If $a(x',y',s)$ is a constant and G is the Green function of the homogeneous part of (4.1), then we know that the last integral in (4.3) is equal to zero. However, when $a(x',y',s)$ is not a constant, it is difficult to give an explicit expression of the Green function. Instead, we will try to minimize the value of $G_{ss}-a\triangle G-\kappa G_s$ by choosing a function G called the approximate Green function. For this purpose, we first consider the following linear partial differential equation with a parameter μ

$$\partial_{tt}V-\mu\triangle V+\kappa V_t=0. \tag{4.4}$$

We denote the Green function of (4.4) by $G^\sharp(\mu;x,y,t)$, i.e.,

$$
\begin{cases}
(G^\sharp_{tt}-\mu\triangle G^\sharp+\kappa G^\sharp_t)(\mu;x,y,t)=0,\\
G^\sharp(\mu;x,y,0)=0,\quad G^\sharp_t(\mu;x,y,0)=\delta(x,y).
\end{cases}
\tag{4.5}
$$

After taking the Fourier transformation to the variable (x,y) in (4.5), we obtain the following ordinary differential equation for $\hat{G}^\sharp$,

$$
\begin{cases}
\big(\hat{G}^\sharp_{tt}+\mu|\xi|^2\hat{G}^\sharp(\xi,t)+\kappa\hat{G}^\sharp_t\big)(\mu;\xi,t)=0,\\
\hat{G}^\sharp(\mu;\xi,0)=0,\quad \hat{G}^\sharp_t(\mu;\xi,0)=1,
\end{cases}
\tag{4.6}
$$

where $\xi=(\xi_1,\xi_2)$ corresponds to (D_x,D_y), and μ is a bounded parameter satisfying $C_0<\mu<C_1$ with C_0,C_1 being two positive constants. By direct calculation, we have

$$\hat{G}^\sharp(\mu;\xi,t)=\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}, \tag{4.7}$$

where

$$\lambda_\pm(\xi)\equiv\frac{1}{2}\big(-\kappa\pm\sqrt{\kappa^2-4\mu|\xi|^2}\big). \tag{4.8}$$

In the following, we are going to obtain some properties of the Green function $G^\sharp(\mu; x, y, t)$. Let

$$\chi(\xi) = \begin{cases} 1, & |\xi| < \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases} \tag{4.9}$$

be a smooth cut-off function, with ε being sufficiently small. Set

$$\hat{G}^\sharp_L(\xi, t) = \chi \hat{G}^\sharp(\mu; \xi, t), \qquad \hat{G}^\sharp_H(\xi, t) = (1-\chi)\hat{G}^\sharp(\mu; \xi, t).$$

We will first prove the following lemma:

Lemma 4.1. *For any fixed ε, there exist positive constants b and C, such that*

$$\begin{aligned} \left|\xi^\alpha \hat{G}^\sharp_L(\mu; \xi, t)\right| &\leqslant C(1+t)^{-\frac{|\alpha|}{2}}, \\ \left|\xi^\alpha \big(\hat{G}^\sharp_L\big)_t(\mu; \xi, t)\right| &\leqslant C(1+t)^{-\frac{|\alpha|+2}{2}}, \end{aligned} \tag{4.10}$$

and for $|\beta| \leqslant 1$,

$$\left|\xi^\beta \hat{G}^\sharp_H(\mu; \xi, t)\right| + \left|\big(\hat{G}^\sharp_H\big)_t(\mu; \xi, t)\right| \leqslant Ce^{-bt}. \tag{4.11}$$

Proof. For $|\xi| \leqslant \epsilon$ being sufficiently small, we have

$$\lambda_+(\xi) = -\frac{\mu}{\kappa}|\xi|^2 + O\big(|\xi|^4\big),$$

and

$$\lambda_-(\xi) = -\kappa + \frac{\mu}{\kappa}|\xi|^2 + O\big(|\xi|^4\big).$$

Thus

$$\begin{aligned} e^{\lambda_+(\xi)t} &= e^{-(\mu|\xi|^2)t/\kappa}\big(1 + O\big(|\xi|^4\big)t\big), \\ e^{\lambda_-(\xi)t} &= e^{-\kappa t} e^{(\mu|\xi|^2)t/\kappa}\big(1 + O\big(|\xi|^4\big)t\big), \end{aligned}$$

and

$$\lambda_+ - \lambda_- = 1/\kappa + O\big(|\xi|^2\big).$$

Since $e^{\lambda_-(\xi)t}$ is exponentially decay, we can only consider $e^{\lambda_+(\xi)t}$. It is easy to see the first inequality in (4.10) holds. Noticing $\lambda_+(\xi) = O(|\xi|^2)$, the second inequality in (4.10) follows.

For any fixed ε, we can choose m sufficiently large such that $\frac{\kappa}{2\sqrt{\mu m}} \leqslant \varepsilon$. When $\varepsilon \leqslant |\xi| \leqslant \frac{\kappa}{2\sqrt{\mu}}$, we have

$$\lambda_+(\xi) \leqslant -\frac{1}{2}\Big(\kappa - \sqrt{\kappa^2 - \big(\kappa^2/m\big)}\Big) \leqslant -\frac{\kappa}{4m}.$$

This implies that

$$\left|(1-\chi)(\xi)e^{\lambda_+(\xi)t}\right| \leqslant Ce^{-\kappa t/4m}.$$

If $\frac{\kappa}{2\sqrt{\mu}} \leqslant |\xi|$, then $\operatorname{Re}\sqrt{\kappa^2-4\mu|\xi|^2}=0$. Thus, we have

$$\left|(1-\chi)(\xi)e^{\lambda_+(\xi)t}\right| \leqslant e^{-\kappa t/2}\left|e^{\sqrt{\kappa^2-4\mu|\xi|^2}/2}\right| \leqslant Ce^{-\kappa t/2}.$$

As for $e^{\lambda_-(\xi)t}$, since $\operatorname{Re}\lambda_-(\xi) \leqslant -\frac{\kappa}{2}$, we have

$$\left|(1-\chi)(\xi)e^{\lambda_-(\xi)t}\right| \leqslant Ce^{-\kappa t/2}.$$

If $||\xi| - \frac{\kappa}{2\sqrt{\mu}}| \geqslant \delta_0 > 0$, then

$$|\lambda_+ - \lambda_-| = \left|\sqrt{\kappa^2-4\mu|\xi|^2}\right| \geqslant \sqrt{2\sqrt{\mu}k\delta_0},$$

and for $|\beta| \leqslant 1$

$$\left|\xi^\beta(\lambda_+-\lambda_-)^{-1}\right| = \left|\xi^\beta\left(\sqrt{\kappa^2-4\mu|\xi|^2}\right)^{-1}\right| \leqslant C\left(\sqrt{2\sqrt{\mu}k\delta_0}\right)^{-1}.$$

Thus, we have

$$\left|\xi^\beta\hat{G}_H^\sharp\right| + \left|\left(\hat{G}_H^\sharp\right)_t\right| \leqslant Ce^{-bt}.$$

If $||\xi| - \frac{\kappa}{2\sqrt{\mu}}| \leqslant \delta_0$, by denoting $\tau_1 = \sqrt{\kappa^2-4\mu|\xi|^2}$, then we have

$$\left|\xi^\beta\frac{e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\right| \leqslant C\sup_{0\leqslant s\leqslant 1}\left(te^{-\kappa t/2}e^{t\operatorname{Re}((\frac{1}{2}+s)\tau_1)}\right) \leqslant Ce^{-\kappa t/4},$$

and

$$\begin{aligned}\left|\left(\hat{G}_H^\sharp\right)_t(\mu;\xi,t)\right| &= \left|\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t}-\lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\right| \\ &\leqslant \left(\left|\lambda_+(\xi)\frac{e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\right| + \left|e^{\lambda_-(\xi)}\right|\right) \leqslant Ce^{-\kappa t/4}.\end{aligned}$$

Thus, for any fixed ε, (4.11) holds. □

Lemma 4.2. *For any fixed ε, there exist positive constants b and C, such that for $h \leqslant 2$*

$$\begin{gathered}\left|\xi^\alpha\partial_\mu^h\hat{G}_L^\sharp(\mu;\xi,t)\right| \leqslant C(1+t)^{-\frac{|\alpha|}{2}}, \\ \left|\xi^\alpha\partial_\mu^h\left(\hat{G}_L^\sharp\right)_t(\mu;\xi,t)\right| \leqslant C(1+t)^{-\frac{|\alpha|+2}{2}}, \\ \left|\partial_\mu\hat{G}_H^\sharp(\mu;\xi,t)\right| \leqslant Ce^{-bt},\end{gathered} \tag{4.12}$$

and for $|\beta| = 1$,

$$\left|\left(\xi^\beta|\xi|^{-2}\right)\partial_\mu^2 \hat{G}_H^\sharp(\mu;\xi,t)\right| + \left|\left(\xi^\beta|\xi|^{-2}\right)\partial_\mu\left(\hat{G}_H^\sharp\right)_t(\mu;\xi,t)\right| \leqslant Ce^{-bt}. \tag{4.13}$$

Proof. We write $\hat{G}^\sharp(\mu;\xi,t) = \hat{E}^+(\mu;\xi,t) + \hat{E}^-(\mu;\xi,t)$, where

$$\hat{E}^+(\mu;\xi,t) = \eta_0 e^{\lambda_+ t}, \qquad \hat{E}^-(\mu;\xi,t) = -\eta_0 e^{\lambda_- t},$$

with $\eta_0(\mu;\xi) = (\lambda_+(\mu,\xi) - \lambda_-(\mu,\xi))^{-1}$. Since

$$\partial_\mu \lambda_\pm = \mp|\xi|^2\eta_0, \qquad \partial_\mu \eta_0 = 2|\xi|^2\eta_0^3,$$

it is easy to see that

$$\begin{aligned}
&\partial_\mu\left(\hat{E}^+ + \hat{E}^-\right) = 2|\xi|^2\eta_0^2\left(\hat{E}^+ + \hat{E}^-\right) - t|\xi|^2\eta_0\left(\hat{E}^+ - \hat{E}^-\right)\\
&\partial_\mu^2\left(\hat{E}^+ + \hat{E}^-\right) = \left(6\eta_0^4|\xi|^4 + \eta_0^2|\xi|^4t^2\right)\left(\hat{E}^+ + \hat{E}^-\right) - 6\eta_0^3|\xi|^4 t\left(\hat{E}^+ - \hat{E}^-\right).
\end{aligned} \tag{4.14}$$

By the similar argument as the one for Lemma 4.1, we have (4.12) and (4.13). Since

$$\partial_\mu^2\left(\hat{E}^+ + \hat{E}^-\right) = O\left(|\xi|\right), \qquad \partial_t\partial_\mu\left(\hat{E}^+ + \hat{E}^-\right) = O\left(|\xi|\right),$$

when $|\xi| \to \infty$, the factor $(|\xi|^{-2}\xi)$ is needed in (4.13). Thus the lemma is proved. □

Now we define the approximate Green function by

$$G(x,y,t;x',y',s) = G^\sharp\left(a\left(x',y',\sigma(t,s)\right); x-x', y-y', t-s\right), \tag{4.15}$$

with $a(x',y',\sigma(t,s)) = P'(\bar{\rho}(x',y',\sigma(t,s)))$. Here $\sigma(t,s) \in C^2([2,\infty] \times [0,\infty])$ and

$$\sigma(t,s) = \begin{cases} s, & s > t/2,\\ t/2, & s \leqslant t/2 - 1.\end{cases}$$

Moreover, we can choose $\sigma(t,s)$ so that when $s \in (t/2-1, t/2)$, we have

$$\sum_{1\leqslant l_1+l_2\leqslant 3}\left|\partial_t^{l_1}\partial_s^{l_2}\sigma(t,s)\right| \leqslant C.$$

When $t > 2$, we have $\sigma^{-1}(t,s) \leqslant C(1+t)^{-1}$. Thus, we have

$$(1+t)\left|\partial_s a\left(x',y',\sigma(t,s)\right)\right| + (1+t)^2\left|\partial_s^2 a\left(x',y',\sigma(t,s)\right)\right| \leqslant C. \tag{4.16}$$

Denote

$$\begin{aligned}
&G_L(x,y,t;x',y',s) = \chi(D_{x,y})G(x,y,t;x',y',s),\\
&G_H(x,y,t;x',y',s) = \left(1-\chi(D_{x,y})\right)G(x,y,t;x',y',s),
\end{aligned} \tag{4.17}$$

where $\chi(D_{x,y})$ is the pseudo-differential operator with symbol $\chi(\xi)$ given in (4.9). Denote

$$T_G f(x,y) = \int_{\mathbf{R}^2} G^\sharp(a; x-x', y-y', \tau) f(x',y')\,dx'\,dy',$$

$$T_{G_L} f(x,y) = \int_{\mathbf{R}^2} G_L^\sharp(a; x-x', y-y', \tau) f(x',y')\,dx'\,dy',$$

$$T_{G_H} f(x,y) = \int_{\mathbf{R}^2} G_H^\sharp(a; x-x', y-y', \tau) f(x',y')\,dx'\,dy'.$$

Now we will give the key proposition in this section.

Proposition 4.1. *Suppose there are two positive constants C_0 and C_1 such that $C_0 \leqslant a(x',y',\sigma) \leqslant C_1$, then for $|\beta| + l \leqslant 2$ and $l \leqslant 1$, we have*

$$\begin{aligned}
&\big\|\partial_{x,y}^\alpha \partial_s^l \partial_{x',y'}^\beta T_{G_L} f(x',y')\big\|_{L_2} \leqslant C(1+t-s)^{-(2l+|\alpha|+|\beta|+1)/2}\big\|f(\cdot,\cdot)\big\|_{L_2},\\
&\big\|\partial_{x,y}^\alpha A\big(l+|\beta|, D\big)\partial_s^l \partial_{x',y'}^\beta T_{G_H} f(x',y')\big\|_{L_2} \leqslant Ce^{-b(t-s)/4}\big\|f(\cdot,\cdot)\big\|_{L_2},
\end{aligned} \tag{4.18}$$

where $A(l+|\beta|, D)$ is a pseudo-differential operator with symbol

$$\sigma(A) = \begin{cases} (1-\chi(|\xi|))\frac{\xi}{|\xi|^2}, & l+|\beta| = 2,\\ 1, & l+|\beta| < 2,\end{cases}$$

for $l + |\beta| \leqslant 2$.

Proof. If $a(x',y',\sigma)$ is constant, then the proposition follows directly from Lemmas 4.1, 4.2 and Plancherel's theorem. Here, by Fourier transform, we have

$$\begin{aligned}
&\mathcal{F}(T_G f)(\xi, t-s)\\
&\quad= \int_{\mathbf{R}^2} e^{-i(x\xi_1+y\xi_2)}\bigg(\int_{\mathbf{R}^2} G^\sharp(a; x-x', y-y', t-s) f(x',y')\,dx'\,dy'\bigg)\,dx\,dy\\
&\quad= \int_{\mathbf{R}^2} e^{-i(x'\xi_1+y'\xi_2)}\hat{G}\big(a(x',y',\sigma),\xi,t-s\big) f(x',y')\,dx'\,dy'.
\end{aligned} \tag{4.19}$$

Denote

$$(\tilde{T}_{\hat{G}} f)(x',y',t-s) = \hat{G}\big(a(x',y',\sigma),\xi,\tau\big) f(x',y').$$

Then $\mathcal{F}(T_G f)(\xi,\tau) = \mathcal{F}(\tilde{T}_{\hat{G}} f)(\xi,\tau)$. Here T_G is a singular integral operator, and $\tilde{T}_{\hat{G}}$ is a multiplier operator. By Plancherel's theorem and (4.15), we have

$$\|T_G f\|_{L_2} = \big\|\mathcal{F}(T_G f)\big\|_{L_2} = \big\|\mathcal{F}(\tilde{T}_{\hat{G}} f)\big\|_{L_2} = \|\tilde{T}_{\hat{G}} f\|_{L_2}.$$

Then

$$\|T_G f\|_{L_2} \leqslant \sup_{\xi,a}\bigl|\hat{G}(a,\xi,t-s)\bigr| \cdot \|f\|_{L_2}.$$

In general, we have

$$\bigl\|\partial_{x,y}^{\alpha}\partial_s^l\partial_a^h T_G f\bigr\|_{L_2} \leqslant \sup_{\xi,a}\bigl|\xi^{\alpha}\partial_s^l\partial_a^h\hat{G}(a,\xi,t-s)\bigr| \cdot \|f\|_{L_2}.$$

By noticing $C_0 \leqslant a(x',y',\sigma) \leqslant C_1$, we have

$$\begin{aligned}
&\bigl\|\partial_{x,y}^{\alpha}\partial_s^l\partial_a^h T_{G_L} f(x',y')\bigr\|_{L_2} \leqslant C(1+t-s)^{-(1+2l+|\alpha|)/2}\bigl\|f(\cdot,\cdot)\bigr\|_{L_2},\\
&\bigl\|\partial_{x,y}^{\alpha}A(l+h,D)\partial_s^l\partial_a^h T_{G_H} f(x',y')\bigr\|_{L_2} \leqslant Ce^{-b\tau/4}\bigl\|f(\cdot,\cdot)\bigr\|_{L_2}.
\end{aligned} \tag{4.20}$$

For the case when $|\beta| = 1$ and $l = 1$, since

$$\partial_{x,y}^{\alpha}\partial_s\partial_{x',y'}^{\beta}T_G = \bigl(\partial_{x,y}^{\alpha}\partial_s\partial_a T_G\bigr)\bigl(\partial_{x',y'}^{\beta}a\bigr) + (-1)\partial_{x,y}^{\alpha}\partial_s\partial_{x,y}^{\beta}T_G,$$

and $|\partial_{x',y'}^{\beta}a| \leqslant C(1+\sigma)^{-|\beta|/2} \leqslant C(1+(t-s))^{-|\beta|/2}$, (4.18) follows from (4.20) by using Lemmas 4.1 and 4.2. Similar argument gives the case when $|\beta| = 2$ and $l = 0$ so that the proposition is proved. □

5. The L_2 estimate by approximate Green function

In this section, we will establish some L_2 estimates for $|\alpha| \leqslant k-1$ by using the approximate Green function, that is, we will prove the Proposition 2.1.

We firstly estimate the derivatives of each term on the right-hand side of (4.3). Set

$$\begin{aligned}
I_1^{\alpha} &= \int_{\mathbf{R}^2} \partial_{x,y}^{\alpha}G_s(x,y,t;x',y',0)V(x',y',0)\,dx'\,dy',\\
I_2^{\alpha} &= -\int_{\mathbf{R}^2} \partial_{x,y}^{\alpha}G(x,y,t;x',y',0)(\kappa V+V_t)(x',y',0)\,dx'\,dy',\\
I_3^{\alpha} &= \int_0^t\int_{\mathbf{R}^2} \partial_{x,y}^{\alpha}G(x,y,t;x',y',s)F(x',y',s)\,dx'\,dy'\,ds,\\
I_4^{\alpha} &= \int_0^t\int_{\mathbf{R}^2} \partial_{x,y}^{\alpha}R_G(x,y,t;x',y',s)V(x',y',s)\,dx'\,dy'\,ds,
\end{aligned}$$

where

$$R_G \equiv G_{ss}(x,y,t;x',y',s) - a\,\triangle_{x',y'}G(x,y,t;x',y',s) - \kappa G_s(x,y,t;x',y',s).$$

Since

$$\big(G^\sharp_{4,4} - a \triangle_{x,y} G^\sharp + \kappa G^\sharp_4\big)\big(a(x',y',s); x-x', y-y', t-s\big) = 0, \tag{5.1}$$

we have

$$\begin{aligned} R_G = {} & \big[G^\sharp_{1,1}\big(a_s(x',y',\sigma)^2 - 2G^\sharp_{1,4}a_s(x',y',\sigma) + G^\sharp_1 a_{ss}(x',y',\sigma) - \kappa G^\sharp_1 a_s(x',y',\sigma) \\ & + a(x',y',s)\big(G^\sharp_{1,1}\big(a^2_{x'} + a^2_{y'}\big)(x',y',\sigma) + G^\sharp_1 \triangle_{x',y'} a(x',y',\sigma) \\ & - 2\big(G^\sharp_{1,2}a_{x'}(x',y',\sigma) + G^\sharp_{1,3}a_{y'}(x',y',\sigma)\big)\big) \triangle_{x,y} G^\sharp\big)\big] \\ & + \big[\big(a(x',y',\sigma) - a(x',y',\sigma)\big) \triangle_{x,y} G^\sharp\big] \\ = {} & R^1_G + R^2_G. \end{aligned} \tag{5.2}$$

Here R^1_G comes from the dependence of a on x', y', t and s, while R^2_G comes from the definition of $\sigma(t,s)$ which is different when $s \leqslant \frac{t}{2}$.

We have from (4.3)

$$\partial^\alpha_{x,y} V(x,y,t) = I^\alpha_1 + I^\alpha_2 + I^\alpha_3 + I^\alpha_4. \tag{5.3}$$

By using Propositions 4.1 and 4.2, it is straightforward to show that

$$\begin{aligned} \big\|I^\alpha_1\big\|_{L_2} & \leqslant C(1+t)^{-(|\alpha|+2)/2}\|V_0\|_{L_2} + Ce^{-bt}\sum_{|\beta|\leqslant|\alpha|}\big\|\partial^\beta_{x,y}V_0\big\|_{L_2} \\ & \leqslant C\big((1+t)^{-\frac{|\alpha|+2}{2}}\|V_0\|_{L_2} + e^{-bt}\|V_0\|_{H^k}\big). \end{aligned} \tag{5.4}$$

For I^α_2, set

$$\tilde{\nu}(x',y') = \nu_t(x',y',0) + \kappa\nu(x',y',0),$$

where $\nu(x',y',s)$ is given in (1.12). Then

$$\begin{aligned} \big|I^\alpha_2\big| \leqslant & \left|\int\limits_{\mathbf{R}^2} \partial^\alpha_{x,y}(G_L)_{x'}(x,y,t;x',y',0)\tilde{\nu}(x',y')\,dx'\,dy'\right| \\ & + \left|\int\limits_{\mathbf{R}^2} \partial^\alpha_{x,y}(G_H)_{x'}(x,y,t;x',y',0)\tilde{\nu}(x',y')\,dx'\,dy'\right|. \end{aligned}$$

Also by using Propositions 4,1 and 4.2, we have

$$\begin{aligned} \big\|I^\alpha_2\big\|_{L_2} & \leqslant C(1+t)^{-(|\alpha|+1)/2}\|\tilde{\nu}\|_{L_2} + Ce^{-bt}\sum_{|\beta|\leqslant|\alpha|}\|\partial^\beta_{x,y}\tilde{\nu}\|_{L_2} \\ & \leqslant C(1+t)^{-\frac{|\alpha|+1}{2}}\big(\|\nu_0\|_{H^k} + \big\|(\nu_t)_0\big\|_{H^k}\big). \end{aligned} \tag{5.5}$$

Before considering I_3^α, we first estimate each term in F of (2.14) respectively. Set $\vartheta(x) \in C^\infty$ and

$$\vartheta(x) = \begin{cases} \rho_+, & x > 1, \\ \rho_-, & x < -1, \end{cases} \qquad \vartheta_1(x) = \begin{cases} \vartheta(x), & x \in (-1, 1), \\ 0, & x \in R \setminus (-1, 1). \end{cases}$$

Denote

$$r_\rho = -\big(\bar{\rho}(x, y, t) - \vartheta(x)\big)\delta_t(y, t) - \vartheta_1(x)\delta_t(y, t).$$

Then

$$R_\rho = \big(\bar{\rho}(x, y, t)\delta_t(y, t)\big)_x = \big(r_\rho(x, y, t)\big)_x,$$

and for $|\alpha| \geqslant 0$, we have

$$\big|\partial_{x,y}^\alpha r_\rho(x, y, t)\big| \leqslant C E_\rho e^{-\kappa t} B_N(x, y, t).$$

We rewrite F as

$$F = \big(F^1\big)_x + \big(F^2\big)_y + \big(F^3\big)_{xx} + \big(F^4\big)_{xy} + \big(F^5\big)_{yy}, \tag{5.6}$$

where

$$\begin{aligned} &F^1 = \kappa r_\rho + (r_\rho)_t + (R_{\rho u}), \qquad F^2 = -P(\bar{\rho})_y, \\ &F^3 = \big(\bar{\rho}\big(2\bar{u}_1 U_1 + U_1^2\big) + V(\bar{u}_1 + U_1)^2\big) + \big(\mathcal{P}_1(\bar{\rho}, V)V^2\big), \\ &F^4 = 2\big((\bar{\rho} + V)(\bar{u}_1 + U_1)U_2\big), \\ &F^5 = \big((\bar{\rho} + V)(U_2)^2\big) + \big(\mathcal{P}_1(\bar{\rho}, V)V^2\big). \end{aligned} \tag{5.7}$$

Hence, we obtain

$$\begin{aligned} \big|I_3^\alpha\big| &= \left| \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha G(x, y, t; x', y', s) F(x', y', s)\, dx'\, dy'\, ds \right| \\ &\leqslant \sum_{j=1}^6 |\Pi_j|, \end{aligned}$$

where

$$\Pi_1 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_{x'} G_L(x, y, t; x', y', s) F^1(x', y', s)\, dx'\, dy'\, ds,$$

$$\Pi_2 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_{y'} G_L(x, y, t; x', y', s) F^2(x', y', s)\, dx'\, dy'\, ds,$$

$$\varPi_3 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^{\alpha} \partial_{x'x'} G_L(x,y,t;x',y',s) F^3(x',y',s)\,dx'\,dy'\,ds,$$

$$\varPi_4 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^{\alpha} \partial_{x'y'} G_L(x,y,t;x',y',s) F^4(x',y',s)\,dx'\,dy'\,ds,$$

$$\varPi_5 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^{\alpha} \partial_{y'y'} G_L(x,y,t;x',y',s) F^5(x',y',s)\,dx'\,dy'\,ds,$$

$$\varPi_6 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^{\alpha} G_H F\,dx'\,dy'\,ds.$$

It is easy to see that, for $l = 0, 1$, $|\alpha| \leqslant k$,

$$\left|\partial_t^l \partial_{x,y}^{\alpha} r_\rho\right| + \left|\partial_{x,y}^{\alpha} R_{\rho u}\right| + \left|\partial_{x,y}^{\alpha}\big(P(\bar\rho)_y\big)\right| \leqslant C E_\rho e^{-\kappa t} B_N(x,y,t).$$

Then, for $|\beta| \leqslant k$ and $j = 1, 2$, we have

$$\left\|\partial_{x',y'}^{\beta} F^j(\cdot,\cdot,s)\right\|_{L_2} \leqslant C E_\rho e^{-\kappa s}.$$

For F^j $(j = 3, 4, 5)$ and $|\beta| \leqslant k$ $(k > 3)$, by using Theorem 3.2 and the definition of $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$, we obtain

$$\begin{aligned}
&\left\|\partial_{x',y'}^{\beta} F^j(\cdot,\cdot,s)\right\|_{L_2} \\
&\quad \leqslant C \sum_{|\gamma|=|\beta|} \left(\left\|\partial^{\gamma} W^2\right\|_{L_2} + \left\|\partial^{\gamma}(\bar u_1 W)\right\|_{L_2}\right) \\
&\quad \leqslant C\big(E_\rho + \mathcal{M}_1^2 + \mathcal{M}_2^2\big)(1+s)^{-\frac{\nu(|\beta|)+2}{2}}.
\end{aligned}$$

Thus, by using Proposition 4.1 and above inequalities, we have for $|\alpha| \leqslant k-1$

$$\begin{aligned}
\sum_{l=1}^{5} \|\varPi_l\|_{L_2} &\leqslant C E_\rho \int_0^t (1+t-s)^{-(1+|\alpha|)/2} e^{-\kappa s}\,ds \\
&\quad + \left(\int_0^{t/2} (1+t-s)^{-\frac{|\alpha|+2}{2}} (1+s)^{-1}\big(\mathcal{M}^2 + \mathcal{M}E_\rho\big)\,ds \right.\\
&\quad \left. + \int_{t/2}^{t} (1+t-s)^{-1} (1+s)^{-\frac{\nu(|\alpha|)+2}{2}} \big(\mathcal{M}^2 + \mathcal{M}E_\rho\big)\,ds\right) \\
&\leqslant C\big(\mathcal{M}^2 + E_\rho\big)(1+t)^{-\frac{\nu(|\alpha|)+2}{2}} \ln(1+t) \leqslant C\big(\mathcal{M}^2 + E_\rho\big)(1+t)^{-\frac{|\alpha|+1}{2}}.
\end{aligned}$$

For Π_6 with $|\alpha| \leqslant k-1$, we have

$$\Pi_6 = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_{x'} G_H F^1\, dx'\, dy'\, ds + \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_{y'} G_H F^2\, dx'\, dy'\, ds$$
$$+ \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_{x'} G_H \partial_{x'} F^3\, dx'\, dy'\, ds + \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_{y'} G_H \big(\partial_{x'} F^4 + \partial_{y'} F^5\big)\, dx'\, dy'\, ds.$$

Then Proposition 4.1 yields

$$\|\Pi_6\|_{L_2} \leqslant C\big(E_\rho \mathcal{M} + \mathcal{M}^2\big) \int_0^t e^{-b(t-s)} (1+s)^{-\frac{\nu(|\alpha|+1)+2}{2}}\, ds$$
$$\leqslant C\big(E_\rho \mathcal{M} + \mathcal{M}^2\big)(1+t)^{-\frac{\nu(|\alpha|)+2}{2}}.$$

Thus for $|\alpha| \leqslant k-1$,

$$\big\| I_3^\alpha \big\|_{L_2} \leqslant C(1+t)^{-\frac{|\alpha|+1}{2}} \big(\mathcal{M}^2 + E_\rho\big). \tag{5.8}$$

Now, let us study the error term I_4^α which comes from the approximate Green function. Consider

$$J_1^\alpha = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha G_1^\sharp\big(a(x',y',\sigma), x-x', y-y', t-s\big) a_s(x',y',\sigma) V(x',y',s)\, dx'\, dy'\, ds,$$
$$J_2^\alpha = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha G_{1,4}^\sharp\big(a(x',y',\sigma), x-x', y-y', t-s\big) a_s(x',y',\sigma) V(x',y',s)\, dx'\, dy'\, ds,$$

and

$$J_3^\alpha = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha R_G^2(x,y,t;x',y',s) V(x',y',s)\, dx'\, dy'\, ds.$$

Firstly, J_1^α can be written as

$$J_1^\alpha = \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha (G_L)'_a(x,y,t;x',y',s) a_s(x',y',\sigma) V(x',y',s)\, dx'\, dy'\, ds$$
$$+ \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha (G_H)'_a(x,y,t;x',y',s) a_s(x',y',\sigma) V(x',y',s)\, dx'\, dy'\, ds$$
$$= J_{1,1}^\alpha + J_{1,2}^\alpha.$$

Notice that (4.16) gives

$$\left|a_s(x', y', \sigma)\right| \leqslant E_\rho (1+t)^{-1}.$$

Then by Propositions 4.1, for $|\alpha| \leqslant k-1$, we have

$$\begin{aligned}
\left\| J_{1,1}^\alpha \right\|_{L_2} &\leqslant C E_\rho \mathcal{M}_1(t) \left(\int_{t/2}^{t} (1+t-s)^{-1/2} (1+t)^{-1} (1+s)^{-(\nu(|\alpha|)/2)}\, ds \right. \\
&\quad \left. + \int_0^{t/2} (1+t-s)^{-(|\alpha|/2)} (1+t)^{-1} (1+s)^{-(1/2)}\, ds \right) \\
&\leqslant C\left(E_\rho + \mathcal{M}_1^2(t)\right)(1+t)^{-(\nu(|\alpha|)+1)/2},
\end{aligned}$$

and

$$\begin{aligned}
\left\| J_{1,2}^\alpha \right\|_{L_2} &\leqslant C\left(E_\rho + \mathcal{M}_1^2(t)\right) \int_0^t e^{-\kappa(t-s)/2} (1+t)^{-1} (1+s)^{-(\nu(|\alpha|)+1)/2}\, ds \\
&\leqslant C\left(E_\rho + \mathcal{M}_1^2(t)\right)(1+t)^{-(|\alpha|+1)/2}.
\end{aligned}$$

For J_2^α, similar to J_1^α, we have

$$\begin{aligned}
J_2^\alpha &= \int_0^t \int_{\mathbf{R}^2} \partial_{x,y}^\alpha \partial_t (G_L)'_a(x, y, t; x', y', s) a_s(x', y', \sigma) V(x', y', s)\, dx'\, dy'\, ds \\
&\quad + \int_0^t \int_{\mathbf{R}^2} a_s(x', y', \sigma)\left(\partial_{x,y}^\alpha A(1, D) \partial_t (G_H)'_a(x, y, t; x', y', s)\right) \cdot \left(\nabla V(x', y', s)\right) dx'\, dy'\, ds \\
&= J_{2,1}^\alpha + J_{2,2}^\alpha.
\end{aligned}$$

The proof of $J_{2,1}^\alpha$ is the same as $J_{1,1}^\alpha$. But for $J_{2,2}^\alpha$, we have

$$\begin{aligned}
\left\| J_{2,2}^\alpha \right\|_{L_2} &\leqslant C\left(E_\rho + \mathcal{M}_1^2(t)\right) \int_0^t e^{-\kappa(t-s)/2} (1+t)^{-1} (1+s)^{-(1+\nu(|\alpha|+1))/2}\, ds \\
&\leqslant C\left(E_\rho + \mathcal{M}_1^2(t)\right)(1+t)^{-(|\alpha|+1)/2}.
\end{aligned}$$

For J_3^α, since

$$\left|a(x', y', s) - a(x', y', \sigma)\right| \leqslant \int_s^\sigma \left|a_\tau(x', y', \tau)\right| d\tau \leqslant \begin{cases} C\vartheta(t, s), & s < t/2, \\ 0, & s \geqslant t/2, \end{cases}$$

where

$$\vartheta(t,s) = (1+t-s)(1+t)^{-1+1/h}(1+s)^{-1/h},$$

with any positive integer h. By using Proposition 4.1, we have

$$\begin{aligned}
\|J_3^\alpha\|_{L_2} &\leqslant CE_\rho \mathcal{M}_1(t)\Bigg(\int_0^{t/2}(1+t-s)^{-(2+|\alpha|)/2}\vartheta(t,s)(1+s)^{-(1/2)}\,ds \\
&\quad + \int_0^t e^{-\kappa(t-s)/2}\vartheta(t,s)(1+s)^{-(\nu(|\alpha|+1)+1)/2}\,ds\Bigg) \\
&\leqslant C\big(E_\rho + \mathcal{M}_1^2(t)\big)(1+t)^{-(\nu(|\alpha|)+1)/2}.
\end{aligned}$$

Thus, for $|\alpha| \leqslant k-1$, we obtain

$$\|I_4^\alpha\|_{L_2} \leqslant C\big(E_\rho + \mathcal{M}_1^2(t)\big)(1+t)^{-\frac{\nu(|\alpha|)+1}{2}}. \tag{5.9}$$

In summary, by (5.4), (5.5), (5.8) and (5.9), we obtain, for $|\alpha| \leqslant k-1$

$$\big\|\partial^\alpha V(t)\big\|_{L_2} \leqslant C\big(\|V_0\|_{H^k} + \|\nu_0\|_{L_2} + \big\|(\nu_t)_0\big\|_{L_2} + E_\rho + \mathcal{M}^2\big)(1+t)^{-\frac{\nu(|\alpha|)+1}{2}}. \tag{5.10}$$

Now, we come back to consider the equations in (2.13). It is easy to see that

$$U(x,y,t) = e^{-\kappa t}U(x,y,0) + \int_0^t e^{-\kappa(t-s)}\big(\bar\rho^{-1}\big(\nabla\big(a(x,y,s)V\big)\big) + \bar H(V,U,\rho,\bar u_1)\big)\,ds.$$

From (2.14), we obtain, for $|\alpha| \leqslant k-1$

$$\big\|\partial^\alpha \bar H(\cdot,\cdot,s)\big\|_{L_2} \leqslant C\big(E_\rho + \mathcal{M}^2\big)\big((1+s)^{-\frac{\mu(|\alpha|+1)+2}{2}} + (1+s)^{-\frac{\nu(|\alpha|)+1}{2}}\big).$$

Hence,

$$\big|\partial^\alpha\big(\bar\rho^{-1}\big(\nabla a(x,y,s)V\big)\big)\big| \leqslant C(E_\rho + \mathcal{M}_1)(1+s)^{-\frac{\nu(|\alpha|+1)+1}{2}}.$$

Thus, for $|\alpha| \leqslant k-1$,

$$\begin{aligned}
\big\|\partial^\alpha U(t)\big\|_{L_2} &\leqslant e^{-\kappa t}\Bigg(\big\|\partial^\alpha U(0)\big\|_{L_2} \\
&\quad + C\big(E_\rho + \mathcal{M}^2 + \mathcal{M}_1\big)\int_0^t e^{-\kappa(t-s)}\big((1+s)^{-\frac{\nu(|\alpha|+1)+1}{2}} + (1+s)^{-\frac{\mu(|\alpha|+1)+2}{2}}\big)\,ds\Bigg) \\
&\leqslant Ce^{-\kappa t}\big(\big\|\partial^\alpha U(0)\big\|_{L_2} + E_\rho + \mathcal{M}^2 + \mathcal{M}_1\big)(1+t)^{\frac{-\nu(|\alpha|+1)+1}{2}}.
\end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11) gives

$$\begin{aligned} \left\|\partial^\alpha V(t)\right\|_{L_2} &\leqslant C(1+t)^{-\frac{\nu|\alpha|+1}{2}}\left(E_0+E_\rho+\mathcal{M}^2\right), \\ \left\|\partial^\alpha U(t)\right\|_{L_2} &\leqslant C(1+t)^{-\frac{\mu|\alpha|+2}{2}}\left(E_0+E_\rho+\mathcal{M}^2+\mathcal{M}_1\right), \end{aligned} \tag{5.12}$$

where $|\alpha| \leqslant k-1$, and

$$E_0 = \max\left\{\|V_0\|_{H^k}^2, \left\|V_t(0)\right\|_{H^{k-1}}^2 \|U_0\|_{H^k}^2, \left\|\nu(0)\right\|_{L_2}^2, \left\|\nu_t(0)\right\|_{L_2}^2\right\}.$$

By denoting $E = \max\{E_\rho, E_0\}$, (5.12) gives Proposition 2.1.

6. The L_2 estimate by energy method

Finally, we will prove Proposition 2.2 for the higher order derivatives of (V, U) by using energy method. We will discuss the problem in the following two cases.

Case 1. Estimate on V.

Set $\theta^2(t) = (\Lambda + t)^{k-\epsilon}$ with small constant $\epsilon > 0$ and large constant $\Lambda > 0$. By taking $\partial^\alpha (|\alpha| = k-1)$ on (2.10) and integrating its inner product with $2\theta^2(t)\partial^\alpha V_t$ and $\lambda\theta^2(t)\partial^\alpha V$ over $\mathbf{R}^2 \times [0, t]$ respectively, where $\lambda > 0$ is a small constant, we have

$$\begin{aligned} &\left\|\theta(t)\partial^\alpha V_t(t)\right\|_{L_2}^2 - \left\|\partial^\alpha V_t(0)\right\|_{L_2}^2 + 2\kappa \int_0^t \left\|\theta(s)\partial^\alpha V_s\right\|_{L_2}^2 ds \\ &\quad + \left\|\theta(t)\left(\sqrt{\mathcal{P}}\partial^\alpha \nabla V\right)(t)\right\|_{L_2}^2 - \left\|\left(\sqrt{\mathcal{P}}\partial^\alpha \nabla V\right)(0)\right\|_{L_2}^2 \\ &\quad - \int_0^t \int_{\mathbf{R}^2} \left(\left(\left(\mathcal{P}\theta^2(s)\right)_s \left|\partial^\alpha \nabla V\right|^2\right) + \left(\theta^2(s)\right)_s \left|\partial^\alpha V_s\right|^2\right) dx\, dy\, ds + \mathcal{R}_1 \\ &= 2\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\left(\partial^\alpha \tilde{Q}\right)\left(\partial^\alpha V_s\right) dx\, dy\, ds, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &-\lambda \int_0^t \left\|\theta(s)\partial^\alpha V_s\right\|_{L_2}^2 ds + \lambda\theta^2(s) \int_{\mathbf{R}^2} \partial^\alpha V_s \partial^\alpha V\, dx\, dy\Big|_{s=0}^{s=t} \\ &\quad + \lambda \int_0^t \left\|\theta(s)\sqrt{\mathcal{P}}\partial^\alpha \nabla V\right\|_{L_2}^2 ds + \frac{\lambda\kappa}{2}\left(\left\|\theta(t)\partial^\alpha V(t)\right\|_{L_2}^2 - \left\|\partial^\alpha V(0)\right\|_{L_2}^2\right) \end{aligned}$$

$$-\lambda\int_0^t\int_{\mathbf{R}^2}\big(\theta^2(s)\big)_s\big(\big(\partial^\alpha V\partial^\alpha V_s\big)+\kappa\big|\partial^\alpha V\big|^2\big)\,dx\,dy\,ds+\mathcal{R}_2$$

$$=\lambda\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\alpha\tilde{Q}\big)\big(\partial^\alpha V\big)\,dx\,dy\,ds. \tag{6.2}$$

Here,

$$\mathcal{R}_1=2\sum_{\beta+\gamma=\alpha}\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\beta V\big)\big(\partial^\gamma\nabla\sqrt{\mathcal{P}}\big)\big(\partial^\alpha\nabla V_t\big)\,dx\,dy\,dt$$

$$+2\sum_{\beta+\gamma=\alpha,|\gamma|\geqslant 1}\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\beta\nabla V\big)\big(\partial^\gamma\sqrt{\mathcal{P}}\big)\big(\partial^\alpha\nabla V_s\big)\,dx\,dy\,dt,$$

$$\mathcal{R}_2=\lambda\sum_{\beta+\gamma=\alpha}\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\beta V\big)\big(\partial^\gamma\nabla\sqrt{\mathcal{P}}\big)\big(\partial^\alpha\nabla V\big)\,dx\,dy\,dt$$

$$+\lambda\sum_{\beta+\gamma=\alpha,|\gamma|\geqslant 1}\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\beta\nabla V\big)\big(\partial^\gamma\sqrt{\mathcal{P}}\big)\big(\partial^\alpha\nabla V\big)\,dx\,dy\,dt.$$

For $\mathcal{R}_1$, we only show how to estimate one of the terms with kth order derivatives of V, that is,

$$2\left|\sum_{\beta+\gamma=\alpha,|\gamma|=1}\left(\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\beta\nabla V\big)\big(\partial^\gamma\sqrt{\mathcal{P}}\big)\big(\partial^\alpha\nabla V_s\big)\,dx\,dy\,dt\right)\right|$$

$$\leqslant CE_\rho\left(\int_0^t\theta^2(s)(1+s)^{-2-|\alpha|+\epsilon-1}\mathcal{M}_1^2\,ds\right.$$

$$\left.+\int_0^t\theta^2(s)\Big(\sum_{|\beta|=k-2}\big\|\partial^\beta\triangle V(s)\big\|_{L_2}^2+\big\|\partial^\alpha V_t\big\|_{L_2}^2\Big)\,ds\right).$$

Similar argument gives

$$|\mathcal{R}_1|\leqslant CE_\rho\left(\mathcal{M}_1^2(t)+\int_0^t\theta^2(s)\Big(\sum_{|\beta|=k-2}\big\|\partial^\beta\triangle V(s)\big\|_{L_2}^2+\big\|\partial^\alpha V_t\big\|_{L_2}^2+\big\|\partial^\alpha\nabla V\big\|_{L_2}^2\Big)\,ds\right). \tag{6.3}$$

For $\mathcal{R}_2$, by noticing that

$$
\begin{aligned}
&2\sum_{\beta+\gamma=\alpha,|\gamma|=1}\left(\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\beta\nabla V\big)\big(\partial^\gamma\sqrt{\mathcal{P}}\big)\big(\partial^\alpha\nabla V\big)\,dx\,dy\,dt\right)\\
&\quad=-\sum_{\beta+\gamma=\alpha,|\gamma|=1}\left(\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\gamma\partial^\gamma\sqrt{\mathcal{P}}\big)\big|\partial^\beta\nabla V\big|^2\,dx\,dy\,dt\right)\\
&\quad=\sum_{\beta+\gamma=\alpha,|\gamma|=1}\left(\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\partial^\gamma\partial^\gamma\sqrt{\mathcal{P}}\big)\big(\partial^\beta V\big)\big(\partial^\beta\triangle V\big)\,dx\,dy\,dt\right),
\end{aligned}
$$

we have

$$
|\mathcal{R}_2|\leqslant CE_\rho\left(\mathcal{M}_1^2(t)+\int_0^t\theta^2(s)\left(\sum_{|\beta|=k-2}\big\|\partial^\beta\triangle V(s)\big\|_{L_2}^2+\big\|\partial^\alpha\nabla V\big\|_{L_2}^2\right)ds\right). \tag{6.4}
$$

For the other terms in (6.1) and (6.2), we only show how to estimate the terms which contain derivatives of U and V with order greater than $|\alpha|+1$. The estimation on other terms is easier. In order to estimate $\partial^\alpha\tilde{Q}$, denote

$$
\tilde{Q}_\alpha=2(\bar\rho+V)\big((\bar u+U)\cdot\nabla\big)\operatorname{div}\partial^\alpha U+\big((\bar u+U)\cdot\nabla\big)^2\partial^\alpha V. \tag{6.5}
$$

Since (2.4) implies

$$
\operatorname{div}U=-(\bar\rho+V)^{-1}\big(V_t+(\bar u+U)\cdot\nabla V+(U\cdot\nabla)\bar\rho+V\operatorname{div}\bar u-R_\rho\big), \tag{6.6}
$$

$\tilde{Q}_\alpha$ can be written as

$$
\tilde{Q}_\alpha=-2(\bar u+U)\cdot\nabla\partial^\alpha V_t-\big((\bar u+U)\cdot\nabla\big)^2\partial^\alpha V+\bar{Q}_\alpha,
$$

where we use $\bar{Q}_\alpha$ to denote the remainder which contains derivatives of U and V with order at most $|\alpha|+1=k$ and each term is a product of at least three functions of $\bar u$, ρ, U and(or) V. With this, we have the following expression of $\partial^\alpha\tilde{Q}$,

$$
\partial^\alpha\tilde{Q}=Q_1^\alpha+Q_2^\alpha+\bar{Q}_\alpha+\mathcal{R}_3,
$$

with

$$
\begin{aligned}
&Q_1^\alpha=\partial^\alpha\big((R_\rho)_t+\kappa R_\rho+R_{\rho u}-P(\rho)_{yy}\big),\\
&Q_2^\alpha=-2(\bar u+U)\cdot\nabla\partial^\alpha V_t-\big((\bar u+U)\cdot\nabla\big)^2\partial^\alpha V.
\end{aligned}
$$

Here, the term $\mathcal{R}_3$ is also used to denote the remainder which contains derivatives of U and V with order at most $|\alpha|+1=k$ and each term is a product of at least three functions of $\bar u$, ρ, U

and(or) V. It is easy to see that

$$\left|\int_0^t\int_{\mathbf{R}^2}\theta^2(s)Q_1^\alpha\partial^\alpha V_s\,dx\,dy\,ds\right|\leqslant C\left(E_\rho+E_\rho\int_0^t\left\|\theta(s)\partial^\alpha V_s(s)\right\|_{L_2}^2\right).$$

Integration by parts gives

$$\begin{aligned}
&\int_0^t\int_{\mathbf{R}^2}\theta^2(s)Q_2^\alpha\partial^\alpha V_s\,dx\,dy\,ds\\
&\quad=-\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big((\bar u+U)\cdot\nabla\left|\partial^\alpha V_s\right|^2\big)\,dx\,dy\,ds\\
&\qquad-\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big((\bar u+U)\cdot\nabla\big(\big((\bar u+U)\cdot\nabla\big)\partial^\alpha V\big)\big)\partial^\alpha V_s\,dx\,dy\,ds\\
&\quad=\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\operatorname{div}(\bar u+U)\big)\left|\partial^\alpha V_s\right|^2dx\,dy\,ds\\
&\qquad+\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\partial_s\left|\big((\bar u+U)\cdot\nabla\big)\partial^\alpha V\right|^2dx\,dy\,ds+\mathcal{R}_4,\\
&\quad=\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\big(\operatorname{div}(\bar u+U)\big)\left|\partial^\alpha V_s\right|^2dx\,dy\,ds\\
&\qquad+\theta^2(s)\left\|\big(l(\bar u+U)\cdot\nabla\big)\partial^\alpha V(s)\right\|_{L_2}^2\Big|_{s=0}^{s=t}\\
&\qquad-\int_0^t\int_{\mathbf{R}^2}\big(\theta^2(s)\big)_s\left|\big((\bar u+U)\cdot\nabla\big)\partial^\alpha V\right|^2dx\,dy\,ds+\mathcal{R}_4.
\end{aligned}$$

Here, $\mathcal{R}_4$ denotes the remainder which contains derivatives of U and V with order at most k. Then

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbf{R}^2}\theta^2(s)Q_2^\alpha\partial^\alpha V_s\,dx\,dy\,ds\right|\\
&\quad\leqslant C(E_\rho+\mathcal{M})\left(\mathcal{M}^2+\int_0^t\theta^2(s)\big(\left\|\partial^\alpha V_s(s)\right\|_{L_2}^2+\left\|\partial^\alpha\nabla V(s)\right\|_{L_2}^2\big)\,ds\right).
\end{aligned}$$

Combining the above inequalities gives

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\partial^\alpha\tilde{Q}\partial^\alpha V_s\,dx\,dy\,ds\right| \\
&\quad\leqslant C(E_\rho+\mathcal{M})\left(\mathcal{M}^2+\int_0^t\theta^2(s)\big(\big\|\partial^\alpha V_s(s)\big\|_{L_2}^2+\big\|\partial^\alpha\nabla V(s)\big\|_{L_2}^2\big)\,ds\right).
\end{aligned}\tag{6.7}$$

Similarly, we have

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbf{R}^2}\theta^2(s)\partial^\alpha\tilde{Q}\partial^\alpha V\,dx\,dy\,ds\right| \\
&\quad\leqslant C(E_\rho+\mathcal{M})\left(\mathcal{M}^2+\int_0^t\theta^2(s)\big(\big\|\partial^\alpha V_s(s)\big\|_{L_2}^2+\big\|\partial^\alpha\nabla V(s)\big\|_{L_2}^2\big)\,ds\right).
\end{aligned}\tag{6.8}$$

By noticing $(\theta^2(s))_s\leqslant\frac{k}{\Lambda}\theta^2(s)$, we have

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbf{R}^2}\big(\big(\big(\mathcal{P}\theta^2(s)\big)_s\big|\partial^\alpha\nabla V\big|^2\big)+\big(\theta^2(s)\big)_s\big|\partial^\alpha V_s\big|^2\big)\,dx\,dy\,ds\right| \\
&\quad\leqslant C\left(E_\rho+\frac{1}{\Lambda}\right)\left(\int_0^t\theta^2(s)\big(\big\|\partial^\alpha V_s(s)\big\|_{L_2}^2+\big\|\partial^\alpha\nabla V(s)\big\|_{L_2}^2\big)\,ds\right).
\end{aligned}\tag{6.9}$$

Therefore, by Cauchy inequality and choosing Λ sufficiently large and λ sufficiently small, we have

$$\begin{aligned}
&\theta^2(t)\big(\big\|\partial^\alpha V_t(t)\big\|_{L_2}^2+\big\|\nabla\partial^\alpha V(t)\big\|_{L_2}^2+\big\|\partial^\alpha V(t)\big\|_{L_2}^2\big) \\
&\qquad+\int_0^t\theta^2(s)\big(\big\|\partial^\alpha V_s\big\|_{L_2}^2+\big\|\nabla\partial^\alpha V\big\|_{L_2}^2\big)\,ds \\
&\quad\leqslant C\big(\big\|\partial^\alpha V_t(\cdot,0)\big\|_{L_2}^2+\big\|\partial^\alpha V(0)\big\|_{L_2}^2\big)+C\big(E\mathcal{M}+\mathcal{M}^3\big).
\end{aligned}\tag{6.10}$$

Case 2. Estimate on U.

For $|\alpha|=k$, by multiplying the first equation of (2.8) by $2\partial^\alpha(\partial^\alpha U_1)$ and the second equation by $2\theta(t)\partial^\alpha(\partial^\alpha U_2)$, integrating their sum over $\mathbf{R}^2\times[0,t]$ yields

$$\|\theta(t)\partial^\alpha U(t)\|_{L_2}^2 - \|\partial^\alpha U(0)\|_{L_2}^2 + 2\kappa \int_0^t \|\theta(s)\partial^\alpha U\|_{L_2}^2$$
$$+ 2\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\big(\partial^\alpha\big((\bar\rho + V)^{-1}\nabla(\mathcal{P}V)\big)\cdot\big(\partial^\alpha U\big)\big)\,dx\,dy\,ds$$
$$= 2\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\big(\partial^\alpha H\cdot\partial^\alpha U\big)\,dx\,dy\,ds, \tag{6.11}$$

where $H = (H_1, H_2)^\tau$. We can write

$$2\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\big(\partial^\alpha\big((\bar\rho + V)^{-1}\nabla(\mathcal{P}V)\big)\cdot\big(\partial^\alpha U\big)\big)\,dx\,dy\,ds$$
$$= -2\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\big(\partial^\alpha\big((\bar\rho + V)^{-1}(\mathcal{P}V)\big)\partial^\alpha(\operatorname{div} U)\big)\,dx\,dy\,ds + \mathcal{R}_5.$$

By using (6.6) in the first term on the right-hand side in above equation, we have

$$2\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\big(\partial^\alpha\big((\bar\rho + V)^{-1}\nabla(\mathcal{P}V)\big)\cdot\big(\partial^\alpha U\big)\big)\,dx\,dy\,ds = \theta(s)\big\|\big(h\partial^\alpha V\big)(s)\big\|_{L_2}^2\Big|_{s=0}^{s=t} + \mathcal{R}_5 + \mathcal{R}_6,$$

where $h^2 = ((\bar\rho + V))^{-2}\mathcal{P}(\bar\rho, V) > C_1 > 0$. By noticing $(\theta^2(s))_s \leqslant \frac{k}{\Lambda}\theta^2(s)$ again, we have

$$|\mathcal{R}_5 + \mathcal{R}_6| \leqslant C E_\rho \int_0^t \theta^2(s)\big(\|\partial^\alpha V(s)\|_{L_2}^2 + \|\partial^\alpha U(s)\|_{L_2}^2\big)\,ds + C\mathcal{M}^3(t).$$

To illustrate the estimation on the right-hand side of (6.2), we only consider

$$\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\partial^\alpha\big(U_1(U_2)_x\big)\partial^\alpha U_2\,dx\,dy\,ds = -\frac{1}{2}\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\partial_x(U_1)\big|\partial^\alpha U\big|^2\,dx\,dy\,ds + \mathcal{R}_7.$$

Here, we use $\mathcal{R}_7$ to denote the remainder which contains derivatives of U with order at most k. Then

$$\left|\int_0^t \int_{\mathbf{R}^2} \theta^2(s)\big(\partial^\alpha H\big)\cdot\big(\partial^\alpha U\big)\,dx\,dy\,ds\right| \leqslant C\big(E_\rho + \mathcal{M}^3\big).$$

The other terms in (6.11) can be estimated similarly. Therefore, we have

$$\begin{aligned}
&\theta^2(t)\left(\left\|\partial^\alpha U\right\|_{L_2}^2+\left\|\partial^\alpha V\right\|_{L_2}^2+\int_0^t\left\|\partial^\alpha U\right\|_{L_2}^2\right)\\
&\quad\leqslant C\left(\left\|\partial^\alpha U_0\right\|_{L_2}^2+\left\|\partial^\alpha V(0)\right\|_{L_2}^2\right)+C\left(E_\rho+\mathcal{M}^3\right)\\
&\qquad+C\int_0^t\theta^2(s)\left\|\partial^\alpha V(s)\right\|_{L_2}^2\,ds.
\end{aligned}\tag{6.12}$$

By combining the estimates in Case 1 when $|\alpha|=k-1$ and Case 2 when $|\alpha|=k$, we have

$$\begin{aligned}
&\sum_{|\alpha|=k-1}\theta^2(t)\left(\left\|\partial^\alpha(V_t,\nabla V)\right\|_{L_2}^2+\left\|\partial^\alpha\nabla U\right\|_{L_2}^2\right)\\
&\quad\leqslant C\sum_{|\alpha|=k-1}\left\|\partial^\alpha(V_t,\nabla V,\nabla U)_{t=0}\right\|_{L_2}^2+C\left(E_\rho+\mathcal{M}^3\right).
\end{aligned}\tag{6.13}$$

Since $\nu(k)=\mu(k)+1$ and $E=\max\{E_\rho,E_0\}$ with

$$E_0=\max\left\{\|V_0\|_{H^k}^2,\left\|V_t(0)\right\|_{H^{k-1}}^2\|U_0\|_{H^k}^2,\left\|\nu(0)\right\|_{L_2}^2,\left\|\nu_t(0)\right\|_{L_2}^2\right\},$$

(6.13) gives

$$\begin{aligned}
&\sum_{|\alpha|=k}\left\|\partial^\alpha V(t)\right\|_{L_2}^2\leqslant C(1+t)^{-(\nu(|\alpha|)+1)}\left(E+\mathcal{M}^3\right),\\
&\sum_{|\alpha|=k}\left\|\partial^\alpha U(t)\right\|_{L_2}^2\leqslant C(1+t)^{-(\mu(|\alpha|)+2)}\left(E+\mathcal{M}^3\right).
\end{aligned}\tag{6.14}$$

Thus, Proposition 2.2 holds.

Again the Theorem 1.1 is a direct consequence of Propositions 2.1 and 2.2.

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