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OPTIMAL CONVERGENCE RATES OF LANDAU EQUATION WITH EXTERNAL FORCING IN THE WHOLE SPACE*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

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Abstract In this paper, we combine the method of constructing the compensating function introduced by Kawashima and the standard energy method for the study on the Landau equation with external forcing. Both the global existence of solutions near the time asymptotic states which are local Maxwellians and the optimal convergence rates are obtained. The method used here has its own advantage for this kind of studies because it does not involve the spectrum analysis of the corresponding linearized operator.

Key words compensating function; energy method; Landau equation; optimal convergence rates

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1 Introduction

In this paper, we are concerned with the Landau equation under the influence of external forcing and source. That is, consider

$$\partial_t F + v\cdot\nabla_x F - (\nabla_x\Phi + E)\cdot\nabla_v F = Q(F,F) + S \tag{1.1}$$

with initial condition $F(0,x,v) = F_0(x,v)$. Here, $F(t,x,v)$ is the distribution function of particles at time $t\geq 0$, located at $x=(x_1,x_2,x_3)\in\mathbf{R}^3$ with velocity $v=(v_1,v_2,v_3)\in\mathbf{R}^3$. The external force takes the form of $-(\nabla_x\Phi+E)$ with given functions $\Phi=\Phi(x)$ and $E=E(t,x)$. In the following discussion, $E(t,x)$ is assumed to decay in time so that the time asymptotic

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state is governed only by the potential $\Phi(x)$. Moreover, the source term $S = S(t,x,v)$ is also assumed to be a given function.

Q is the usual bilinear collision operator defined by

$$\begin{aligned} Q(F,G)(v) &= \nabla_v \cdot \Big\{ \int_{\mathbf{R}^3} \phi(v-v')[F(v')\nabla_v G(v) - G(v)\nabla_{v'} F(v')]\mathrm{d}v' \Big\} \\ &= \sum_{i,j=1}^{3} \partial_i \int_{\mathbf{R}^3} \phi^{ij}(v-v')[F(v')\partial_j G(v) - G(v)\partial_j F(v')]\mathrm{d}v', \end{aligned} \tag{1.2}$$

where $\partial_i = \partial_{v_i}$. The non-negative matrix ϕ is given by

$$\phi^{ij}(v) = \Big\{\delta_{ij} - \frac{v_i v_j}{|v|^2}\Big\}|v|^{2+\gamma}.$$

Here, γ is a parameter leading to the standard classification of the hard potential ($\gamma > 0$), Maxwellian molecule ($\gamma = 0$) or soft potential ($\gamma < 0$), cf. [1]. In particular, $\gamma = -3$ corresponds to the Coulomb interaction in plasma physics. The following study is restricted to the case $\gamma \geq -2$.

Throughout this paper, we consider solutions which are perturbations near a local Maxwellian. Without loss of generality, define the perturbation $f(t,x,v)$ by $F = M + M^{1/2}f$, where the local Maxwellian is $M(v) = \exp\big(-\Phi(x) - \frac{|v|^2}{2}\big)$. Then equation (1.1) for the perturbation $f(t,x,v)$ is

$$\partial_t f + v\cdot\nabla_x f - (\nabla_x\Phi + E)\cdot\nabla_v f + \frac{1}{2}v\cdot E f + \mathrm{e}^{-\Phi}Lf = \mathrm{e}^{-\Phi/2}\Gamma(f,f) + \widetilde{S} \tag{1.3}$$

with $f(0,x,v) = f_0(x,v)$. Denote $\mu(v) = \exp(-|v|^2/2)$, then $M = \mu(v)\mathrm{e}^{-\Phi}$. The linearized collision operator L is defined by

$$Lf \equiv -\mu^{-1/2}[Q(\mu,\mu^{1/2}f) + Q(\mu^{1/2}f,\mu)] = -Af - Kf,$$

where $Af = \mu^{-1/2}Q(\mu,\mu^{1/2}f)$, $Kf = \mu^{-1/2}Q(\mu^{1/2}f,\mu)$, and the bilinear collision operator $\Gamma(f,g)$ and the source term $\widetilde{S}$ are given by

$$\begin{aligned} \Gamma(f,g) &= \mu^{-1/2}Q(\mu^{1/2}f,\mu^{1/2}g), \\ \widetilde{S} &= \mathrm{e}^{\Phi/2}\mu^{-1/2}S - \mathrm{e}^{-\Phi/2}\mu^{1/2}v\cdot E. \end{aligned} \tag{1.4}$$

By H-theorem, L is non-negative and self-adjoint on $L^2(\mathbf{R}^3_v)$ with domain

$$D(L) = \{f \in L^2(\mathbf{R}^3_v) \,\big|\, Lf \in L^2(\mathbf{R}^3_v)\}.$$

Furthermore, set $\psi_1(v) = 1$, $\psi_{j+1}(v) = v_j$, $j = 1,2,3$ and $\psi_5(v) = |v|^2$. Then the null space of L is the five dimensional space ($j = 1,2,3$)

$$\mathcal{N} = \mathrm{span}\{\psi_1\sqrt{\mu}, \psi_{j+1}\sqrt{\mu}, \psi_5\sqrt{\mu}\}. \tag{1.5}$$

For any fixed (t,x) and any function $f(t,x,v)$, we define $\mathbf{P}$ as the projection in $L^2(\mathbf{R}^3_v)$ to the null space $\mathcal{N}$. Then decompose $f(t,x,v)$ uniquely by

$$f(t,x,v) = \mathbf{P}f + (\mathbf{I}-\mathbf{P})f. \tag{1.6}$$

Here, $\mathbf{P}f$ and $(\mathbf{I}-\mathbf{P})f$ are called the macroscopic and microscopic components of the function f, respectively.

In this paper, the following notations are used. Let α and β be $\alpha=[\alpha_1,\alpha_2,\alpha_3]$ and $\beta=[\beta_1,\beta_2,\beta_3]$, respectively. Denote

$$\partial^\alpha_\beta \equiv \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

If each component of β is not greater than corresponding one of $\overline{\beta}$, we use the standard notation $\beta\le\overline{\beta}$. And $\beta<\overline{\beta}$ means that $\beta\le\overline{\beta}$ and $|\beta|<|\overline{\beta}|$. $C^{\overline{\beta}}_\beta$ is the usual binomial coefficient. For the study of the optimal time decay rates, the space $Z_q=L^2(\mathbf{R}^3_v;L^q(\mathbf{R}^3_x))$ is used with its norm defined by

$$\|f\|_{Z_q}=\Big(\int_{\mathbf{R}^3}\Big(\int_{\mathbf{R}^3}|f(x,v)|^q\mathrm{d}x\Big)^{\frac{2}{q}}\mathrm{d}v\Big)^{\frac12}.$$

We will use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbf{R}^3_v$, and $(\cdot,\cdot)$ for the one in $\mathbf{R}^3_x\times\mathbf{R}^3_v$. $|\cdot|_2$ denotes the L^2 norm in $\mathbf{R}^3_v$ and $\|\cdot\|$ to denote the L^2 norms in $\mathbf{R}^3_x\times\mathbf{R}^3_v$. For the weight function in v, we use $w=w(v)=(1+|v|)^{\gamma+2}$. From now on, C denotes a generic positive constant which may vary from line to line.

We now come back to the Landau equation. Note that the collision frequency is given by

$$\sigma^{ij}(v)=\int_{\mathbf{R}^3}\phi^{ij}(v-v')\mu(v')\mathrm{d}v'. \tag{1.7}$$

Accordingly, the following weighted L^2 norms will be used:

$$|g|^2_\sigma=\sum_{1\le i,j\le3}\int_{\mathbf{R}^3}\{\sigma^{ij}\partial_i g\partial_j g+\sigma^{ij}v_iv_jg^2\}\mathrm{d}v,$$

$$\|g\|^2_\sigma=\sum_{1\le i,j\le3}\int_{\mathbf{R}^3\times\mathbf{R}^3}\{\sigma^{ij}\partial_i g\partial_j g+\sigma^{ij}v_iv_jg^2\}\mathrm{d}v\mathrm{d}x.$$

For the energy estimates, the instant energy functional for a solution $f(t,x,v)$, denoted by $\mathcal{E}_l(t)$ has the equivalent relation with some $l\ge0$ representing the index in the weight in v,

$$\mathcal{E}_l(t)\sim\widetilde{\mathcal{E}}_l(t)=\sum_{|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta f(t)\|^2. \tag{1.8}$$

In addition, the corresponding dissipation functional denoted by $\mathcal{D}_l(t)$ satisfies

$$\mathcal{D}_l(t)\sim\widetilde{\mathcal{D}}_l(t)=\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f(t)\|^2+\sum_{|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\|^2_\sigma. \tag{1.9}$$

In the following discussion, we will choose the Sobolev space index $N\ge8$. Notice that from [1, 9], there exists $C>0$ such that

$$C|w^{1/2}g|_2\le|g|_\sigma. \tag{1.10}$$

Notice that $\mathcal{E}_l(t)$ and $\mathcal{D}_l(t)$ do not contain the temporal derivatives and $\mathcal{D}_l(t)$ does not contain the zero-th order derivative of the macroscopic component in the solution.

With the above preparation, the main results of this paper can be stated as follows.

Theorem 1.1 Let $F_0(x,v)=M+\sqrt{M}f_0(x,v)\ge0$ and $\gamma\ge-1$. Suppose that

(H_0) The functions $E=E(t,x)$, $S=S(t,x,v)$ and $f_0=f_0(x,v)$ satisfy

$$E\in C_b^0(\mathbf{R}^+;H^N(\mathbf{R}_x^3)),\quad S\in C_b^0(\mathbf{R}^+;H^N(\mathbf{R}_x^3\times\mathbf{R}_v^3)),\quad f_0\in H^N(\mathbf{R}_x^3\times\mathbf{R}_v^3);$$

(H_1) There are constants $\varepsilon>0$ and $l\geq 0$ such that $\Phi(x)$, $E(t,x)$ and f_0 are bounded in the sense that $\tilde{\mathcal{E}}_l(0)\leq\varepsilon$, and

$$\|\Phi(x)\|_{L^2(\mathbf{R}^3)}+\sum_{1\leq|\alpha|\leq N+1}\|\partial^\alpha\Phi(x)\|_{L^3(\mathbf{R}^3)}\leq\varepsilon, \tag{1.11}$$

$$\sum_{|\alpha|\leq N}\|(1+|x|)\partial^\alpha E(t,x)\|_{L^\infty_{t,x}}+\||x|E(t,x)\|_{L^\infty_t(L^2_x)}\leq\varepsilon. \tag{1.12}$$

Moreover $E(t,x)$ and $S(t,x)$ decay in time like

$$\sum_{|\alpha|\leq N}\|\partial^\alpha E(t,x)\|^2+\sum_{|\alpha|+\beta|\leq N}\|w^{l-1/2}\partial^\alpha_\beta(\mu^{-1/2}S(t,x))\|^2\leq\varepsilon(1+t)^{-1-\varrho}, \tag{1.13}$$

where ϱ is any positive constant. Then there is a constant $\varepsilon_1>0$ such that for any $\varepsilon\leq\varepsilon_1$, the Cauchy problem of the Landau equation (1.3) has a unique global classical solution $f(t,x,v)$ with $F(t,x,v)=M+\sqrt{M}f(t,x,v)\geq 0$ for (1.1), which satisfies

$$\mathcal{E}_l(t)+\int_0^t\mathcal{D}_l(s)\mathrm{d}s\leq C\varepsilon. \tag{1.14}$$

If we further assume $l\geq 1$, $\varrho=\frac{3}{2}$ and

$$(\mathrm{H}_2)\qquad \|f_0\|_{Z_1}+\|\nabla_x\Phi\|_{L^{6/5}(\mathbf{R}^3)}\leq\varepsilon, \tag{1.15}$$

$$\|E(t)\|_{L^1_x}+\|\mu^{-1/2}S\|_{Z_1}\leq\varepsilon(1+t)^{-5/4}, \tag{1.16}$$

we have the optimal time decay of the solution to its time asymptotic state

$$\|f(t)\|^2=\|\mathbf{P}f(t)\|^2+\|(\mathbf{I}-\mathbf{P})f(t)\|^2\leq C\varepsilon(1+t)^{-3/2}, \tag{1.17}$$

$$\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{P}f(t)\|^2+\sum_{|\alpha|+|\beta|\leq N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\|^2\leq C\varepsilon(1+t)^{-5/2}. \tag{1.18}$$

Note that if $E(t,x)=0$ and $S(t,x,v)=0$, equation (1.3) becomes

$$\partial_t f+v\cdot\nabla_x f-\nabla_x\Phi\cdot\nabla_v f+\mathrm{e}^{-\Phi}Lf=\mathrm{e}^{-\Phi/2}\Gamma(f,f) \tag{1.19}$$

with $f(0,x,v)=f_0(x,v)$. Corresponding to Theorem 1.1, we have the following theorem about the existence and optimal convergence rate which holds for larger range of γ.

Theorem 1.2 Let $F_0(x,v)=M+\sqrt{M}f_0(x,v)\geq 0$ and $\gamma\geq -2$. If we assume (1.11) and $\tilde{\mathcal{E}}_l(0)\leq\varepsilon$ for some small constant $\varepsilon>0$, then the Cauchy problem of the Landau equation (1.19) has a unique global classical solution $f(t,x,v)$ with $F(t,x,v)=M+\sqrt{M}f(t,x,v)\geq 0$, which satisfies

$$\mathcal{E}_l(t)+\int_0^t\mathcal{D}_l(s)\mathrm{d}s\leq C\mathcal{E}_l(0). \tag{1.20}$$

If we further assume (1.15) and $l\geq 1$, we have the optimal time decay:

$$\|f(t)\|^2=\|\mathbf{P}f(t)\|^2+\|(\mathbf{I}-\mathbf{P})f(t)\|^2\leq C\varepsilon(1+t)^{-3/2}, \tag{1.21}$$

$$\sum_{1\le|\alpha|\le N}\|\partial^\alpha \mathbf{P}f(t)\|^2+\sum_{|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\|^2\le C\varepsilon(1+t)^{-5/2}. \tag{1.22}$$

Finally in the introduction, we review some related works to the study in this paper. Recently, there is some progress on the L^2 energy methods for the Boltzmann equation. One was initiated by Liu-Yu [16] and developed by Liu-Yang-Yu [15] in terms of the macro-micro decomposition around the local Maxwellian defined by the solution. The other was proposed by Guo [9, 10, 11] about the decomposition around a global Maxwellian. On the other hand, instead of using the spectrum analysis initiated by Grad and finished by Ukai and others, Kawashima [13] introduced the compensating function method which is based on the Fourier transform for the Botlzamnn equation. With these methods, the authors in [24] study the nonlinear stability and the optimal time decay of the solutions for the relativistic Boltzmann and Landau equations near a uniform equilibrium by combining the Kawashima's compensating function method and the macro-micro decomposition of the solution. The main observation is that the energy estimate on the macroscopic component can be obtained by improving the Kawashima's compensating function method [13]. And the advantage is not only to obtain the global existence of the solution without time derivative, but also to obtain the optimal time decay to the equilibrium. As a further study by using the method introduced in [24], in this paper, we consider the Landau equation with external forcing.

As for Landau equation, there have been intensive investigations, cf. [1, 2, 9, 12, 14, 17, 21] and references therein. More precisely, Desvillettes and Villani [2] proved the global existence and uniqueness of classical solutions for spatially homogeneous Landau equation for hard potentials and a large class of initial data. Degond and Lemou [1] studied the spectral properties and the dispersion relation of linearized Landau operator. Guo [9] constructed global classical solutions near a global Maxwellian in a periodic box. Hsiao and Yu extended Guo's results to the whole space in [12]. It is shown in [14] that (1.19) has a unique global solution and it converges to the steady state with a rate $O((1+t)^{-1/4})$.

We should also mention that recently Duan-Ukai-Yang-Zhao [6] introduced a method by combining the spectrum analysis and energy method for the study on the optimal time decay rate for the Boltzmann equation and systems of fluid dynamics such as Navier-Stokes equations, cf. also [5]. Some of the ideas from this work will be used here, however, we do not need to consider the spectrum of the corresponding linearized equation because the compensating function coming from the Fourier transform will show to be enough. We believe that the method used here can be applied to the study on some more complicated systems in the kinetic theories.

2 Basic Estimates

For later use, we derive some estimates on the the bilinear collision operator $\Gamma(f,f)$.

Lemma 2.1 For any $g_1=g_1(x,v)$ and $g_2=g_2(x,v)$ with the following norms well defined, we have

$$\|\Gamma(g_1,g_2)\|_{Z_1}\le C\sum_{|\beta_1|\le 2}\|\partial_{\beta_1}g_1\|\cdot\sum_{|\beta_2|\le 2}\|w\partial_{\beta_2}g_2\|. \tag{2.1}$$

Proof By (1.2), $\Gamma(g_1,g_2)$ can be written as

$$\Gamma(g_1,g_2)=\mu^{-1/2}(v)Q(\mu^{1/2}g_1,\mu^{1/2}g_2)$$

$$
\begin{aligned}
&= \mu^{-1/2}(v)\partial_i \int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu^{1/2}(v)\mu^{1/2}(v')\Big\{g_1(v')\partial_j g_2(v) - \partial_j g_1(v')g_2(v)\Big\}\mathrm{d}v' \\
&\quad +\mu^{-1/2}(v)\partial_i \int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu^{1/2}(v)\mu^{1/2}(v)\Big\{\frac{v'_j}{2} - \frac{v_j}{2}\Big\} g_2(v')g_1(v)\mathrm{d}v' \\
&= \mu^{-1/2}(v)\partial_i \int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu^{1/2}(v')\mu^{1/2}(v)\Big\{g_1(v')\partial_j g_2(v) - \partial_j g_1(v')g_2(v)\Big\}\mathrm{d}v' \\
&= \Big(\partial_i - \frac{v_i}{2}\Big)\int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu^{1/2}(v')\Big\{g_1(v')\partial_j g_2(v) - \partial_j g_1(v')g_2(v)\Big\}\mathrm{d}v' \\
&= \partial_i \int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu^{1/2}(v')\Big\{g_1(v')\partial_j g_2(v) - \partial_j g_1(v')g_2(v)\Big\}\mathrm{d}v' \\
&\quad -\frac{1}{2}\int_{\mathbf{R}^3} \phi^{ij}(v-v')v'_i\mu^{1/2}(v')\Big\{g_1(v')\partial_j g_2(v) - \partial_j g_1(v')g_2(v)\Big\}\mathrm{d}v',
\end{aligned} \tag{2.2}
$$

where we have used the fact that

$$
\sum_{i=1}^{3}\phi^{ij}(v-v')(v'_i - v_i) = \sum_{j=1}^{3}\phi^{ij}(v-v')(v'_j - v_j) = 0.
$$

Since $|\mu^{1/2}(v')| + |\partial_i\mu^{1/2}(v')| \le C\mu^{1/4}(v')$, the Cauchy-Schwartz inequality implies

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}^3} \phi^{ij}(v-v')\partial_i(\mu^{1/2}(v')g_1(v'))\mathrm{d}v'\Big| \\
&\le \int_{\mathbf{R}^3} \phi^{ij}(v-v')[\mu^{1/2}(v') + |\partial_i\mu^{1/2}(v')|][|g_1(v')| + |\partial_i g_1(v')|]\mathrm{d}v' \\
&\le C\Big(\int_{\mathbf{R}^3} |\phi^{ij}(v-v')|^2\mu^{1/2}(v')\mathrm{d}v'\Big)^{1/2}[|g_1|_2 + |\partial_i g_1|_2] \\
&\le C(1+|v|)^{\gamma+2}[|g_1|_2 + |\partial_i g_1|_2].
\end{aligned} \tag{2.3}
$$

Thus,

$$
\begin{aligned}
&\Big|\|\partial_i \int_{\mathbf{R}^3} \phi^{ij}(v-v')\mu^{1/2}(v')\Big\{g_1(v')\partial_j g_2(v) - \partial_j g_1(v')g_2(v)\Big\}\mathrm{d}v'\|_{L^1_x}\Big|_2 \\
&\le C\sum_{|\beta_1|\le 2}\|\partial_{\beta_1}g_1\|\sum_{|\beta_2|\le 2}\Big|(1+|v|)^{\gamma+2}\|\partial_{\beta_2}g_2(v)\|_{L^2_x}\Big|_2 \\
&\le C\sum_{|\beta_1|\le 2}\|\partial_{\beta_1}g_1\|\sum_{|\beta_2|\le 2}\|w\partial_{\beta_2}g_2\|.
\end{aligned}
$$

This gives the first part of (2.2). Similar argument leads to the proof of the second part of (2.2) and we skip the detail. This completes the proof of the lemma.

Lemma 2.2 For any $l \ge 1$, it holds that

$$
\sum_{|\alpha|\le 1}\|\partial^\alpha\Gamma(f,f)\|^2 \le C\widetilde{\mathcal{E}}_l(t)\overline{\mathcal{E}}_l(t), \tag{2.4}
$$

where

$$
\overline{\mathcal{E}}_l(t) = \sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f(t)\|^2 + \sum_{|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\|^2.
$$

Note that $\overline{\mathcal{E}}_l(t)$ is different from $\widetilde{\mathcal{D}}_l(t)$ by the norms on the microscopic component.

Proof By (2.2), we have

$$\begin{aligned}\partial^\alpha\Gamma(f,f) &= \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\mu^{-1/2}(v)Q(\mu^{1/2}\partial^{\alpha_1}f,\mu^{1/2}\partial^{\alpha_2}f)\\
&= \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\partial_i\int_{\mathbf{R}^3}\phi^{ij}(v-v')\mu^{1/2}(v')\Big\{\partial^{\alpha_1}f(v')\partial_j^{\alpha_2}f(v)-\partial_j^{\alpha_1}f(v')\partial^{\alpha_2}f(v)\Big\}\mathrm{d}v'\\
&\quad -\sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\frac{1}{2}\int_{\mathbf{R}^3}\phi^{ij}(v-v')v_i'\mu^{1/2}(v')\\
&\quad \cdot\Big\{\partial^{\alpha_1}f(v')\partial_j^{\alpha_2}f(v)-\partial_j^{\alpha_1}f(v')\partial^{\alpha_2}f(v)\Big\}\mathrm{d}v'.\end{aligned}\tag{2.5}$$

By using (2.3), we have

$$|\partial^\alpha\Gamma(f,f)|_2\le C\sum_{\alpha_1+\alpha_2=\alpha}\Big\{\sum_{|\beta_1|\le 2}|\partial_{\beta_1}^{\alpha_1}f|_2\sum_{|\beta_2|\le 2}|(1+|v|)^{\gamma+2}\partial_{\beta_2}^{\alpha_2}f|_2\Big\}.$$

Thus,

$$\begin{aligned}\sum_{|\alpha|=1}\|\partial^\alpha\Gamma(f,f)\|^2 &\le C\sum_{|\beta_1|\le 2}\Big||\partial_{\beta_1}f|_2\Big|_{L_x^\infty}^2\sum_{|\alpha_2|=1,|\beta_2|\le 2}\|(1+|v|)^{\gamma+2}\partial_{\beta_2}^{\alpha_2}f\|^2\\
&\quad +C\sum_{|\beta_2|\le 2}\Big|(1+|v|)^{\gamma+2}|\partial_{\beta_2}f|_2\Big|_{L_x^\infty}^2\sum_{|\alpha_1|=1,|\beta_1|\le 2}\|\partial_{\beta_1}^{\alpha_1}f\|^2\\
&\le C\Big[\sum_{|\alpha|+|\beta|\le N}\|\partial_\beta^\alpha f\|^2\Big]\cdot\Big[\sum_{1\le|\alpha|,|\alpha|+|\beta|\le N}\|(1+|v|)^{\gamma+2}\partial_\beta^\alpha f\|^2\Big]\\
&\le C\widetilde{\mathcal{E}}_l(t)\overline{\mathcal{E}}_l(t).\end{aligned}$$

Similar argument leads to the proof of the case when $|\alpha|=0$. This completes the proof of the lemma.

We now recall two basic estimates from [9] in the following two lemmas.

Lemma 2.3 The linearized operator L has the properties that $\langle Lg,h\rangle=\langle g,Lh\rangle$, $\langle Lg,g\rangle\ge 0$. And $Lg=0$ if and only if $g=\mathbf{P}g$. Furthermore, there exists a $\delta>0$ such that

$$\langle Lg,g\rangle\ge\delta|(\mathbf{I}-\mathbf{P})g|_\sigma^2.\tag{2.6}$$

Lemma 2.4 For any $\eta>0$ and $l\ge 0$, there exist constants $C_\eta>0$ and $C>0$ such that

$$\langle w^{2l}\partial_\beta Lg,\partial_\beta g\rangle\ge|w^l\partial_\beta g|_\sigma^2-\eta\sum_{\beta_1\le\beta}|w^l\partial_{\beta_1}g|_\sigma^2-C_\eta|g|_\sigma^2,\tag{2.7}$$

and

$$\begin{aligned}&|\langle w^{2l}\partial_\beta^\alpha\Gamma(f,g),\partial_\beta^\alpha h\rangle|\\
&\le C\Big[\sum_{\alpha_1\le\alpha,\beta_1+\beta_2\le\beta}|w^l\partial_{\beta_1}^{\alpha_1}f|_2|w^l\partial_{\beta_2}^{\alpha-\alpha_1}g|_\sigma+|w^l\partial_{\beta_1}^{\alpha_1}f|_\sigma|w^l\partial_{\beta_2}^{\alpha-\alpha_1}g|_2\Big]|w^l\partial_\beta^\alpha h|_\sigma,\end{aligned}\tag{2.8}$$

where $|\alpha|+|\beta|\le N$.

In the following, we prove one more weighted estimate on the nonlinear collision operator.

Lemma 2.5 Let $l \geq 0$ and $|\alpha| + |\beta| \leq N$. Then for any $\eta > 0$, there is some constant $C_\eta > 0$ such that

$$\Big|\Big(w^{2l}\partial^\alpha_\beta \Gamma(f,f), \partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\Big)\Big| \leq \eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma + C_\eta\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t). \tag{2.9}$$

Proof By using the decomposition (1.6), we have

$$\Gamma(f,f) = \Gamma(\mathbf{P}f,\mathbf{P}f) + \Gamma(\mathbf{P}f,(\mathbf{I}-\mathbf{P})f) + \Gamma((\mathbf{I}-\mathbf{P})f,\mathbf{P}f) + \Gamma((\mathbf{I}-\mathbf{P})f,(\mathbf{I}-\mathbf{P})f).$$

By (2.8), we obtain

$$\begin{aligned}
&\Big|\Big(w^{2l}\partial^\alpha_\beta \Gamma(\mathbf{P}f,\mathbf{P}f), \partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\Big)\Big| \\
\leq\ & C\sum_{\alpha_1\leq\alpha,\beta_1+\beta_2\leq\beta}\int_{\mathbf{R}^3}[|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f|_2|w^l\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f|_\sigma \\
&+|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f|_\sigma|w^l\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f|_2]|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f|_\sigma \mathrm{d}x.
\end{aligned}\tag{2.10}$$

Notice that there exists $C > 1$ such that

$$|g|^2_\sigma \geq C^{-1}|w^{1/2}g|_2, \qquad |w^l\partial_\beta\mathbf{P}f|^2_\sigma \leq C|\mathbf{P}f|^2. \tag{2.11}$$

We only consider the first term in (2.10) because the second term can be estimated similarly. If $|\alpha_1| + |\beta_1| \leq N/2$, for any $\eta > 0$, we easily get

$$\begin{aligned}
&\int_{\mathbf{R}^3}|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f|_2|w^l\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f|_\sigma|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f|_\sigma \mathrm{d}x \\
\leq\ & \eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma + \eta\sum_{|\alpha'|\leq 1}\|w^l\nabla_x\partial^{\alpha'}\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\|^2\|w^l\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\|^2_\sigma \\
\leq\ & \eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma + \eta\sum_{|\alpha'|\leq 1}\|\nabla_x\partial^{\alpha'}\partial^{\alpha_1}\mathbf{P}f\|^2\|\partial^{\alpha-\alpha_1}\mathbf{P}f\|^2,
\end{aligned}$$

which is bounded by the right hand side of (2.9). The discussion on the case when $|\alpha_1| + |\beta_1| \geq N/2$ is similar.

By applying Lemma 2.4 again, we have

$$\begin{aligned}
&\Big|\Big(w^{2l}\partial^\alpha_\beta \Gamma(\mathbf{P}f,(\mathbf{I}-\mathbf{P})f), \partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\Big)\Big| \\
\leq\ & C\sum_{\alpha_1\leq\alpha,\beta_1+\beta_2\leq\beta}\int_{\mathbf{R}^3}[|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f|_2|w^l\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f|_\sigma \\
&+|w^l\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f|_2|w^l\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f|_\sigma]|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f|_\sigma \mathrm{d}x.
\end{aligned}$$

Again, we only consider the first term because the second term can be estimated similarly.

If $|\alpha_1| + |\beta_1| \leq N/2$, by using the Sobolev inequality, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^3}|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f|_2|w^l\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f|_\sigma|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f|_\sigma \mathrm{d}x \\
\leq\ & \eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma + \eta\sum_{|\alpha'|\leq 2}\|w^l\partial^{\alpha'}\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\|^2\|w^l\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\|^2_\sigma \\
\leq\ & \eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma + \eta\sum_{|\alpha'|\leq 2}\|\partial^{\alpha'}\partial^{\alpha_1}\mathbf{P}f\|^2\|w^l\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\|^2_\sigma.
\end{aligned}$$

If $|\alpha-\alpha_1|+|\beta_2|\le N/2$, we also have

$$\begin{aligned}&\int_{\mathbf{R}^3}|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f|_2|w^l\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f|_\sigma|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f|_\sigma\mathrm{d}x\\ &\le\eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma+\eta\sum_{|\alpha'|\le2}\|w^l\partial^{\alpha'}\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\|w^l\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\|^2\\ &\le\eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma+\eta\sum_{|\alpha'|\le2}\|w^l\partial^{\alpha'}\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\|\partial^{\alpha_1}\mathbf{P}f\|^2.\end{aligned}$$

Both are bounded by the right hand side of (2.9).

Since the other cases can be estimated similarly, we complete the proof of this lemma.

3 Energy Estimates and Compensating Function

In this section, we will first consider the energy estimate on the microscopic component. For this, we have the following equation from (1.3)

$$\begin{aligned}&[\partial_t+v\cdot\nabla_x+\mathrm{e}^{-\Phi}L](\mathbf{I}-\mathbf{P})f\\ &=\mathrm{e}^{-\Phi/2}\Gamma(f,f)+(\nabla_x\Phi+E)\cdot\nabla_vf-\frac12v\cdot Ef+\widetilde{S}-[\partial_t+v\cdot\nabla_x]\mathbf{P}f.\end{aligned}\tag{3.1}$$

In fact, the estimate on the microscopic component can be obtained by applying the standard energy method because of the dissipation of the linearized collision operator on the microscopic component.

Lemma 3.1 Under the assumptions (H_0) and (H_1), for any $l\ge0$, we have

$$\begin{aligned}&\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2+\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma\Big]\\ &\le C\varepsilon(1+t)^{-1-\varrho}+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\sum_{|\alpha|\le N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).\end{aligned}\tag{3.2}$$

Proof By applying ∂^α_β ($\beta\ne0$) on (3.1), we obtain

$$\begin{aligned}&[\partial_t+v\cdot\nabla_x]\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f+\sum_{|\beta_1|=1}C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f+\mathrm{e}^{-\Phi}\partial^\alpha_\beta L(\mathbf{I}-\mathbf{P})f\\ &+\sum_{\alpha_1<\alpha}C^{\alpha_1}_\alpha\partial^{\alpha-\alpha_1}\mathrm{e}^{-\Phi}\partial^{\alpha_1}_\beta L(\mathbf{I}-\mathbf{P})f=\partial^\alpha_\beta(\mathrm{e}^{-\Phi/2}\Gamma(f,f))+(\nabla_x\Phi+E)\cdot\nabla_v\partial^\alpha_\beta f\\ &+\sum_{\alpha_1<\alpha}C^{\alpha_1}_\alpha(\partial^{\alpha-\alpha_1}\nabla_x\Phi+\partial^{\alpha-\alpha_1}E)\cdot\nabla_v\partial^{\alpha_1}_\beta f-\sum_{\alpha_1\le\alpha,\beta_1\le\beta}\frac12C^{\alpha_1}_\alpha C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\partial^{\alpha_1}E\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\\ &+\partial^\alpha_\beta\widetilde{S}-\sum_{\beta_1<\beta}\partial_{\beta-\beta_1}C^{\beta_1}_\beta v\cdot\nabla_x\partial^\alpha_{\beta_1}\mathbf{P}f-[\partial_t+v\cdot\nabla_x]\partial^\alpha_\beta\mathbf{P}f.\end{aligned}\tag{3.3}$$

We take the inner product of (3.3) over $\mathbf{R}^3\times\mathbf{R}^3$ with $w^{2l}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f$. The first term on the left hand side is $\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2$.

By Hölder inequality, the second term on the left hand side of the inner product is bounded by

$$\eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2+C_\eta\sum_{|\beta_1|=1}\|w^l\nabla_x\partial^\alpha_{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f\|^2.\tag{3.4}$$

Notice that the assumption (1.11) and the Sobolev imbedding theorem imply that

$$\sup_{x\in\mathbf{R}^3}\sum_{|\alpha|\le N-1}|\partial^\alpha\Phi(x)|\le C\varepsilon. \tag{3.5}$$

Then by Lemma 2.4, for any $\eta>0$, the third term on the left hand side in the inner product satisfies

$$\begin{aligned}&(w^{2l}\mathrm{e}^{-\Phi}\partial^\alpha_\beta L(\mathbf{I}-\mathbf{P})f,\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)\\ \ge{}&\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma-\eta\sum_{\beta_1\le\beta}\|w^l\partial^\alpha_{\beta_1}(\mathbf{I}-\mathbf{P})f\|^2_\sigma-C_\eta\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma.\end{aligned}$$

In the following, we only consider the case when $|\alpha-\alpha_1|=1$ for the last term on the left hand side of (3.3) because the other cases can be estimated similarly. By Lemma 2.4, we obtain

$$\begin{aligned}&|(w^{2l}\partial^{\alpha-\alpha'}\Phi\mathrm{e}^{-\Phi}\partial_\beta L[(\partial^{\alpha_1}(\mathbf{I}-\mathbf{P})f],\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\\ \le{}&C\sum_{|\alpha-\alpha_1|=1,\beta_1\le\beta}\int_{\mathbf{R}^3}|\partial^{\alpha-\alpha_1}\Phi|\cdot|w^l\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f|_\sigma|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f|_\sigma\mathrm{d}x\\ \le{}&C\varepsilon\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\varepsilon\sum_{\alpha_1<\alpha,\beta_1\le\beta}\|w^l\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f\|^2_\sigma.\end{aligned}$$

For the nonlinear collision operator, by (1.11) and (3.5), Lemma 2.5 gives

$$|(w^{2l}\partial^\alpha_\beta(\mathrm{e}^{-\Phi/2}\Gamma(f,f)),\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\le\eta\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C_\eta\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).$$

For the second term on the right hand side of (3.3), recalling (H_0), (H_1) and $|\beta|\ge1$, we apply the decomposition on f as (1.6) to obtain

$$\begin{aligned}&|(w^{2l}(\nabla_x\Phi+E)\cdot\nabla_v\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f,\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\\ ={}&|((\nabla_x\Phi+E)\cdot\nabla_v w^{2l}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f,\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\\ \le{}&C(\|\nabla_x\Phi\|_{L^\infty}+\|E\|_{L^\infty_{t,x}})\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2\le C\varepsilon\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2;\\ &|(w^{2l}(\nabla_x\Phi+E)\cdot\nabla_v\partial^\alpha_\beta\mathbf{P}f,\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\\ \le{}&(\|\nabla_x\Phi\|_{L^3(\mathbf{R}^3)}+\||x|E\|_{L^\infty_{t,x}})(\|w^l\nabla_v\nabla_x\partial^\alpha_\beta\mathbf{P}f\|^2+\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2)\\ \le{}&C\varepsilon\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\varepsilon\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2,\end{aligned}$$

where we have the Hardy inequality $\|\frac{g}{|x|}\|\le C\|\nabla_x g\|$ and Hölder inequality. Similarly, when $|\alpha-\alpha_1|\le N/2$, the third term on the right hand side of (3.3) satisfies

$$\begin{aligned}&|(w^{2l}(\partial^{\alpha-\alpha_1}\nabla_x\Phi+\partial^{\alpha-\alpha_1}E)\cdot\nabla_v\partial^{\alpha_1}_\beta(\mathbf{I}-\mathbf{P})f,\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\\ \le{}&C(\|\partial^{\alpha-\alpha_1}\nabla_x\Phi\|_{L^\infty_x}+\|\partial^{\alpha-\alpha_1}E\|_{L^\infty_{t,x}})(\|w^l\nabla_v\partial^{\alpha_1}_\beta(\mathbf{I}-\mathbf{P})f\|^2+\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2)\\ \le{}&C\varepsilon\sum_{|\alpha_1|+|\beta_1|\le N}\|w^l\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f\|^2+C\varepsilon\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2;\\ &|(w^{2l}(\partial^{\alpha-\alpha_1}\nabla_x\Phi+\partial^{\alpha-\alpha_1}E)\cdot\nabla_v\partial^{\alpha_1}_\beta\mathbf{P}f,\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f)|\\ \le{}&C(\|\partial^{\alpha-\alpha_1}\nabla_x\Phi\|_{L^3(\mathbf{R}^3)}+\||x|\partial^{\alpha-\alpha_1}E\|_{L^\infty_{t,x}})(\|w^l\nabla_x\nabla_v\partial^{\alpha_1}_\beta\mathbf{P}f\|^2+\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2)\\ \le{}&C\varepsilon\sum_{1\le|\alpha_1|\le N}\|\partial^{\alpha_1}\mathbf{P}f\|^2+C\varepsilon\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2.\end{aligned}$$

Similarly, when $|\alpha-\alpha_1|\geq N/2$, this term has the same bound. Thus, we have

$$\begin{aligned}&|(w^{2l}(\partial^{\alpha-\alpha_1}\nabla_x\Phi+\partial^{\alpha-\alpha_1}E)\cdot\nabla_v\partial^{\alpha_1}_{\beta}f,\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)|\\ \leq{}& C\varepsilon\sum_{|\alpha_1|+|\beta_1|\leq N}\|w^l\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f\|^2+C\varepsilon\sum_{1\leq|\alpha_1|\leq N}\|\partial^{\alpha_1}\mathbf{P}f\|^2+C\varepsilon\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2.\end{aligned}$$

For the fourth term on the right hand side of (3.3), by using the decomposition (1.6) again, the condition (H_1) and the relation $|g|_\sigma\geq C|(1+|v|)^{1/2}g|_2$, we get

$$\begin{aligned}&\Big|(w^{2l}\partial_{\beta_1}v\cdot\partial^{\alpha_1}E\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f,\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)\Big|\\ \leq{}& C\|\partial^{\alpha_1}E\|_{L^\infty_{t,x}}(\|w^l(1+|v|)^{1/2}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f\|^2+\|w^l(1+|v|)^{1/2}\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2)\\ \leq{}& C\varepsilon\|w^l\partial^{\alpha-\alpha_1}_{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\varepsilon\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma;\\ &\Big|(w^{2l}\partial_{\beta_1}v\cdot\partial^{\alpha_1}E\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\mathbf{P}f,\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)\Big|\\ \leq{}& C\||x|\partial^{\alpha_1}E\|_{L^\infty_{t,x}}\|\nabla_x\partial^{\alpha-\alpha_1}\mathbf{P}f\|^2+C\varepsilon\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\ \leq{}& C\varepsilon\|\nabla_x\partial^{\alpha-\alpha_1}\mathbf{P}f\|^2+C\varepsilon\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma,\end{aligned}$$

where we have used $|\beta|\geq 1$ and the Hardy inequality.

By using (1.4), (3.5) and the condition (H_1), we have

$$\begin{aligned}&\Big|(w^{2l}\partial^{\alpha}_{\beta}\widetilde{S},\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)\Big|\\ \leq{}& C_\eta\|w^{l-1/2}\partial^{\alpha}_{\beta}(\mathrm{e}^{\Phi/2}\mu^{-1/2}S)\|^2+C_\eta\|\partial^{\alpha}(\mathrm{e}^{-\Phi/2}E)\|^2+2\eta\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\ \leq{}& 2C_\eta\varepsilon(1+t)^{-1-\varrho}+2\eta\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma.\end{aligned}$$

Since $|\beta|\geq 1$, we have

$$\Big|(w^{2l}\partial_{\beta-\beta_1}v\cdot\nabla_x\partial^{\alpha}_{\beta_1}\mathbf{P}f,\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)\Big|\leq C\|\nabla_x\partial^{\alpha}\mathbf{P}f\|^2+\eta\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma.$$

And for any $\eta>0$, the last term on the right hand side of (3.3) satisfies

$$|(w^{2l}[\partial_t+v\cdot\nabla_x]\partial^{\alpha}_{\beta}\mathbf{P}f,\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)|\leq\eta\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C_\eta[\|\partial_t\partial^{\alpha}\mathbf{P}f\|^2+\|\nabla_x\partial^{\alpha}\mathbf{P}f\|^2].$$

Hence, based on the above inequality, the last term on the right hand side of (3.3) satisfies

$$\begin{aligned}&(w^{2l}[\partial_t+v\cdot\nabla_x]\partial^{\alpha}_{\beta}\mathbf{P}f,\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f)\\ \leq{}&\eta\|w^l\partial^{\alpha}_{\beta}(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C_\eta[\varepsilon(1+t)^{-1-\varrho}+\sum_{1\leq|\alpha|\leq N}\|\partial^{\alpha}\mathbf{P}f\|^2+\sum_{1\leq|\alpha|\leq N}\|\partial^{\alpha}(\mathbf{I}-\mathbf{P})f\|^2].\end{aligned}\tag{3.6}$$

On the other hand, applying ∂^α ($|\alpha|\leq N-1$) on (3.1) gives

$$\begin{aligned}[\partial_t+v\cdot\nabla_x]\partial^{\alpha}\mathbf{P}f={}&\partial^{\alpha}(\mathrm{e}^{-\Phi/2}\Gamma(f,f))-\partial^{\alpha}((\nabla_x\Phi+E)\cdot\nabla_vf)-\frac{1}{2}\partial^{\alpha}(v\cdot Ef)+\partial^{\alpha}\widetilde{S}\\ &-[\partial_t+v\cdot\nabla_x]\partial^{\alpha}(\mathbf{I}-\mathbf{P})f-\sum_{\alpha_1\leq\alpha}\partial^{\alpha_1}\mathrm{e}^{-\Phi}L[\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f].\end{aligned}\tag{3.7}$$

Since

$$\langle\mathbf{P}\partial_t\partial^{\alpha}f,\partial^{\alpha}\Gamma(f,f)-[\partial_t+L]\partial^{\alpha_1}(\mathbf{I}-\mathbf{P})f\rangle=0,$$

by multiplying $\partial_t\partial^\alpha \mathbf{P}f$ to (3.7) and then integrating over $\mathbf{R}^3\times\mathbf{R}^3$, (H_1) implies that

$$\begin{aligned}&\Big|\Big(-\frac{1}{2}\partial^\alpha(v\cdot Ef)-\partial^\alpha((\nabla_x\Phi+E)\cdot\nabla_v f),\partial_t\partial^\alpha\mathbf{P}f\Big)\Big|\\&\le \eta\|\partial_t\partial^\alpha\mathbf{P}f\|^2+C_\eta\|\partial^{\alpha_1}(\nabla_x\Phi+E)\cdot\partial^{\alpha-\alpha_1}f\|^2\\&\le \eta\|\partial_t\partial^\alpha\mathbf{P}f\|^2+C_\eta\varepsilon\|\nabla_x\partial^{\alpha-\alpha_1}f\|^2,\end{aligned}$$

where we have the Hardy inequality and Hölder inequality.

On the other hand, by using (1.4), (3.5) and the condition (H_1), we have

$$\Big|\Big(\partial^\alpha\widetilde{S},\partial_t\partial^\alpha\mathbf{P}f\Big)\Big|\le\eta\|\partial_t\partial^\alpha\mathbf{P}f\|^2+C_\eta\varepsilon(1+t)^{-1-\varrho}.$$

Thus, we can obtain from (3.7) that

$$\|\partial_t\partial^\alpha\mathbf{P}f\|^2\le C[\varepsilon(1+t)^{-1-\varrho}+\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2]. \tag{3.8}$$

By combining the above inequalities, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2+\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\&\le C\varepsilon(1+t)^{-1-\varrho}+C(\eta+\varepsilon)\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\varepsilon\sum_{1\le|\beta'|,|\alpha'|+|\beta'|\le N}\|w^l\partial^{\alpha'}_{\beta'}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\&\quad+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\sum_{|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\&\quad+C_\eta\sum_{|\beta_1|=1}\|w^l\nabla_x\partial^\alpha_{\beta-\beta_1}(\mathbf{I}-\mathbf{P})f\|^2+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).\end{aligned} \tag{3.9}$$

For any $0<|\beta|\le N$ and any $l\ge0$, by taking $\eta>0$ and $\varepsilon>0$ small enough, the summation of (3.9) over $|\alpha|+|\beta|\le N$ by a suitable linear combination gives (3.2). And this completes the proof of the lemma.

For the decay rate estimate, we also need the following weighted energy estimate.

Lemma 3.2 Under the assumptions (H_0) and (H_1), for any $l\ge0$ and small ε, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha f\|^2+\|w^l(\mathbf{I}-\mathbf{P})f\|^2\Big]+\sum_{|\alpha|\le N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\&\le C\varepsilon(1+t)^{-1-\varrho}+C\varepsilon\sum_{1\le|\beta'|,|\alpha'|+|\beta'|\le N}\|w^l\partial^{\alpha'}_{\beta'}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\&\quad+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).\end{aligned} \tag{3.10}$$

Proof We apply ∂^α on equation (1.3) with $1\le|\alpha|\le N$ to have

$$\begin{aligned}&[\partial_t+v\cdot\nabla_x]\partial^\alpha f+\mathrm{e}^{-\Phi}L\partial^\alpha f+\sum_{\alpha_1<\alpha}C^{\alpha_1}_\alpha\partial^{\alpha-\alpha_1}\mathrm{e}^{-\Phi}L[\partial^{\alpha_1}(\mathbf{I}-\mathbf{P})f]-(\nabla_x\Phi+E)\cdot\nabla_v\partial^\alpha f\\&-\sum_{0<\alpha_1\le\alpha}C^{\alpha_1}_\alpha(\partial^{\alpha_1}\nabla_x\Phi+\partial^{\alpha_1}E)\cdot\nabla_v\partial^{\alpha-\alpha_1}f+\frac12\sum_{\alpha_1\le\alpha}C^{\alpha_1}_\alpha v\cdot\partial^{\alpha_1}E\partial^{\alpha-\alpha_1}f\\&=\partial^\alpha(\mathrm{e}^{-\Phi/2}\Gamma(f,f))+\partial^\alpha\widetilde{S}.\end{aligned} \tag{3.11}$$

In the inner product of (3.11) over $\mathbf{R}^3\times\mathbf{R}^3$ with $w^{2l}\partial^\alpha f$, the first term on the left hand side is equal to $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial^\alpha f\|^2$. The other terms can be estimated as follows.

First, (3.5) and Lemma 2.4 imply that $(\mathrm{e}^{-\Phi}L\partial^\alpha f, w^{2l}\partial^\alpha f)\ge\frac{1}{2}\|w^l\partial^\alpha f\|_\sigma^2-C\|\partial^\alpha f\|_\sigma^2$. For the third term on the left hand side of (3.11), it suffices to consider the case $|\alpha-\alpha_1|=1$ because the other cases are easier. For this, by Lemma 2.4, we obtain

$$\begin{aligned}&|(w^{2l}\partial^{\alpha-\alpha'}\Phi\mathrm{e}^{-\Phi}L[(\partial^{\alpha_1}(\mathbf{I}-\mathbf{P})f],\partial^\alpha f))|\\ \le{}& C\sum_{|\alpha-\alpha_1|=1}\int_{\mathbf{R}^3}|\partial^{\alpha-\alpha_1}\Phi|\cdot|w^l\partial^{\alpha_1}(\mathbf{I}-\mathbf{P})f|_\sigma|w^l\partial^\alpha f|_\sigma\mathrm{d}x\\ \le{}& C\varepsilon\|w^l\partial^\alpha f\|_\sigma^2+C\varepsilon\sum_{\alpha_1<\alpha}\|w^l\partial^{\alpha_1}(\mathbf{I}-\mathbf{P})f\|_\sigma^2.\end{aligned}$$

By using the condition (H_1), the fourth term on the left hand side of (3.11) is bounded by

$$|((\nabla_x\Phi+E)\cdot\nabla_v\partial^\alpha f, w^{2l}\partial^\alpha f)|\le C\varepsilon\|w^l\partial^\alpha f\|_\sigma^2.$$

For the fifth term on the left hand side of (3.11), we can apply the decomposition (1.6) on f and use condition (H_1) to obtain

$$\begin{aligned}&|((\partial^{\alpha_1}\nabla_x\Phi+\partial^{\alpha_1}E)\cdot\nabla_v\partial^{\alpha-\alpha_1}\mathbf{P}f, w^{2l}\partial^\alpha f)|\\ \le{}& C(\|\partial^{\alpha_1}\nabla_x\Phi\|_{L^3(\mathbf{R}^3)}+\||x|\partial^{\alpha_1}E\|_{L^\infty_{t,x}})\|w^l\nabla_x\partial^{\alpha-\alpha_1}\mathbf{P}f\|\cdot\|w^l\partial^\alpha f\|\\ \le{}& C\varepsilon\|\nabla_x\partial^{\alpha-\alpha_1}\mathbf{P}f\|^2+C\varepsilon\|w^l\partial^\alpha f\|^2;\\ &|((\partial^{\alpha_1}\nabla_x\Phi+\partial^{\alpha_1}E)\cdot\nabla_v\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f, w^{2l}\partial^\alpha f)|\\ \le{}& C(\|\partial^{\alpha_1}\nabla_x\Phi\|_{L^\infty}+\|\partial^{\alpha_1}E\|_{L^\infty_{t,x}})\|w^l\nabla_v\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\|\cdot\|w^l\partial^\alpha f\|\\ \le{}& C\varepsilon\|w^l\nabla_v\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\|^2+C\varepsilon\|w^l\partial^\alpha f\|^2,\end{aligned}$$

when $|\alpha_1|\le N/2$. On the other hand, when $|\alpha_1|\ge N/2$, we have

$$\begin{aligned}&|((\partial^{\alpha_1}\nabla_x\Phi+\partial^{\alpha_1}E)\cdot\nabla_v\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f, w^{2l}\partial^\alpha f)|\\ \le{}& C(\|\partial^{\alpha_1}\nabla_x\Phi\|_{L^3}+\||x|\partial^{\alpha_1}E\|_{L^\infty_{t,x}})\|w^l\nabla_v\nabla_x\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\|\cdot\|w^l\partial^\alpha f\|\\ \le{}& C\varepsilon\|w^l\nabla_x\nabla_v\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\|^2+C\varepsilon\|w^l\partial^\alpha f\|^2,\end{aligned}$$

where we have used the Hardy inequality.

We thus have the following estimate on the fifth term on the left hand side of (3.11) as

$$\begin{aligned}&|((\partial^{\alpha_1}\nabla_x\Phi+\partial^{\alpha_1}E)\cdot\nabla_v\partial^{\alpha-\alpha_1}f, w^{2l}\partial^\alpha f)|\\ \le{}& C\varepsilon\|\nabla_x\partial^{\alpha-\alpha_1}\mathbf{P}f\|^2+C\varepsilon\sum_{|\alpha_1|+|\beta_1|\le N}\|w^l\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f\|^2+C\varepsilon\|w^l\partial^\alpha f\|^2.\end{aligned}$$

The similar argument yields that the sixth term satisfies

$$\begin{aligned}&\left|(v\cdot\partial^{\alpha_1}E\partial^{\alpha-\alpha_1}f, w^{2l}\partial^\alpha f)\right|\\ \le{}& C\varepsilon\sum_{\alpha_1\ne0}\|\nabla_x\partial^{\alpha-\alpha_1}\mathbf{P}f\|^2+C\varepsilon\sum_{\alpha_1\ne0}\|w^l\nabla_x\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\|_\sigma^2+C\varepsilon\|w^l\partial^\alpha f\|_\sigma^2.\end{aligned}$$

By using (1.4), (3.5) and the condition (H_1), we have

$$\begin{aligned}|(\partial^\alpha\widetilde{S}, w^{2l}\partial^\alpha f)|&\le C_\eta\|w^{l-1/2}\partial^\alpha(\mathrm{e}^{\Phi/2}\mu^{-1/2}S)\|^2+C_\eta\|\partial^\alpha(\mathrm{e}^{-\Phi/2}E)\|^2+2\eta\|w^l\partial^\alpha f\|_\sigma^2\\ &\le 2C_\eta\varepsilon(1+t)^{-1-\varrho}+2\eta\|w^l\partial^\alpha f\|_\sigma^2.\end{aligned}$$

Moreover, by (2.9), (1.11) and (3.5), the nonlinear collision operator satisfies

$$|(\partial^\alpha(\mathrm{e}^{-\Phi/2}\Gamma(f,f)),w^{2l}\partial^\alpha f)|\le\eta\|w^l\partial^\alpha f\|_\sigma^2+C_\eta\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).$$

Notice that there exists a constant $C>0$ such that

$$\|w^l\partial^\alpha f\|_\sigma^2\le C(\|\partial^\alpha\mathbf{P}f\|^2+\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2).$$

By combining the above estimates and taking $\eta>0$ and $\varepsilon>0$ small enough, we have from (3.11) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha f\|^2+\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2\\&\le C\varepsilon(1+t)^{-1-\varrho}+C\varepsilon\sum_{1\le|\beta'|,|\alpha'|+|\beta'|\le N}\|w^l\partial_{\beta'}^{\alpha'}(\mathbf{I}-\mathbf{P})f\|_\sigma^2\\&\quad+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2+C_\eta\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).\end{aligned}\tag{3.12}$$

If we take the inner product of (3.1) with $w^{2l}(\mathbf{I}-\mathbf{P})f$, by the smallness assumption on the external force and the source term, the standard energy estimation leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^l(\mathbf{I}-\mathbf{P})f\|^2+\|w^l(\mathbf{I}-\mathbf{P})f\|_\sigma^2\\&\le C\varepsilon(1+t)^{-1-\varrho}+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2+C_\eta\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).\end{aligned}\tag{3.13}$$

Finally, a suitable combination of (3.12) and (3.13) gives (3.10) and this completes the proof of the lemma.

The inequalities (3.2) and (3.10) give the estimate of the microscopic component in Section 4.

As the second part in this section, we will recall the compensating function introduced by Kawashima [13] for the Boltzmann equation. For this, consider

$$[\partial_t+v\cdot\nabla_x+L]\partial^\alpha f=g,\tag{3.14}$$

where g is given and $|\alpha|\le N$.

Let $v\cdot\xi$ ($\xi\in\mathbf{R}^3$) be the symbol of the streaming operator $v\cdot\nabla_x$. Note that $v\cdot\xi$ is a linear operator in $L^2(\mathbf{R}^3)$ from $\mathcal{N}$ onto $\widetilde{W}$ as the subspace of $L^2(\mathbf{R}^3)$ spanned by the thirteen functions(moments) $\varphi_j\mu^{1/2}$, $j=1,2,\cdots,13$, namely, $\widetilde{W}=\mathrm{span}\{\varphi_j\mu^{1/2}|j=1,\cdots,13\}$. Here,

$$\varphi_1=1;\quad\varphi_{j+1}=v_j\ \ (j=1,2,3);\ \ \varphi_{j+4}=v_j^2\ \ (j=1,2,3);$$

$$\varphi_8=v_1v_2;\quad\varphi_9=v_2v_3;\ \ \varphi_{10}=v_3v_1;\ \ \varphi_{j+10}=|v|^2v_j\ \ (j=1,2,3).$$

Notice that any collision invariant is a linear combination of $\psi_k=\varphi_k$, $k=1,\cdots,4$, and $\psi_5=\varphi_5+\varphi_6+\varphi_7$. Thus, we can rewrite the thirteen moments as:

$$\psi_1=1;\quad\psi_{j+1}=v_j\ \ (j=1,2,3);\ \ \psi_5=|v|^2;\ \ \psi_{j+4}=v_j^2\ \ (j=2,3);$$

$$\psi_8=v_1v_2;\quad\psi_9=v_2v_3;\ \ \psi_{10}=v_3v_1;\ \ \psi_{j+10}=|v|^2v_j\ \ (j=1,2,3).$$

Denote an orthogonal basis for this 13 dimensional space $\widetilde{W}$ by e_j, $1 \le j \le 13$ as in [7, 13]. Let

$$\{\psi_k \mu^{1/2}\}_{k=1}^{13} A_{13\times 13} = [e_j]_{j=1}^{13},$$

where $\det A \neq 0$ and $[e_j]_{j=1}^{5}$ is the orthogonal basis of the null space of L.

Moreover, let P_0 be the orthogonal projection from $L^2(\mathbf{R}^3)$ onto $\widetilde{W}$:

$$P_0 f = \sum_{k=1}^{13} \langle f, e_k\rangle e_k. \tag{3.15}$$

Set $W_k = \langle \partial^\alpha f, e_k\rangle$. Then (3.14) implies that $\partial_t W + \sum_j V^j \partial_{x_j} W + \overline{L} W = \overline{g} + R$, where V^j $(j = 1, 2, 3)$ and $\overline{L}$ are symmetric matrices

$$\overline{L} = \{\langle L[e_l], e_k\rangle\}_{k,l=1}^{13}, \quad V(\xi) = \sum_{j=1}^{3} V^j \xi_j = \{\langle (v\cdot\xi) e_k, e_l\rangle\}_{k,l=1}^{13}, \tag{3.16}$$

$\overline{g}$ is the vector with components $\langle g, e_j\rangle$, and R contains the terms having $(\mathbf{I} - P_0)f$. Write $\mathcal{R}z$ for the real part of $z \in \mathbf{C}$ and $W = [W_I, W_{II}]^T, W_I = [W_1, \cdots, W_5]^T, W_{II} = [W_6, \cdots, W_{13}]^T$.

With the above preparation and notations, the following definition of compensating function comes from [13].

Definition 3.3 $S(\omega)$ with $\omega \in \mathbf{S}^2$ is called a compensating function for (3.14) if

(i) $S(\cdot)$ is a C^∞ function on $\mathbf{S}^2$ with value taken in the space of bounded linear operators on $L^2(\mathbf{R}^3)$ satisfying $S(-\omega) = -S(\omega)$ for all $\omega \in \mathbf{S}^2$.

(ii) $iS(\omega)$ is self-adjoint on $L^2(\mathbf{R}^3)$ for all $\omega \in \mathbf{S}^2$.

(iii) There exists $c_0 > 0$ such that for all $\partial^\alpha f \in D(L)$ and $\omega \in \mathbf{S}^2$,

$$\mathcal{R}\langle S(\omega)(v\cdot\omega)\partial^\alpha f, \partial^\alpha f\rangle + \langle L[\partial^\alpha f], \partial^\alpha f\rangle \ge c_0(|\partial^\alpha \mathbf{P} f|_2^2 + |\partial^\alpha(\mathbf{I} - \mathbf{P})f|_\sigma^2).$$

The construction of the compensating function for (3.14) needs the following lemma proved in [7, 13].

Lemma 3.4 There exist three 13×13 real constant skew-symmetric matrices R^j $(j = 1, 2, 3)$ and positive constants C_1 and C_2 such that $R(\omega) \equiv \sum\limits_{j=1}^{3} R^j \omega_j$, satisfies

$$\mathcal{R}\langle\langle R(\omega)V(\omega)W, W\rangle\rangle \ge C_1|W_I|^2 - C_2|W_{II}|^2$$

for all $W \in \mathbf{C}^{13}$. Here $\langle\langle\cdot,\cdot\rangle\rangle$ is the inner product on $\mathbf{C}^{13}$.

Then a compensating function for the equation (3.14) can be defined as follows. For any given $\omega \in \mathbf{S}^2$, set $R(\omega) \equiv \{r_{ij}(\omega)\}_{i,j=1}^{13}$ and define

$$S(\omega)\partial^\alpha f \equiv \sum_{k,\ell=1}^{13} \lambda r_{k\ell}(\omega)\langle \partial^\alpha f, e_\ell\rangle e_k \quad \text{for some } \lambda > 0, \quad \partial^\alpha f \in L^2(\mathbf{R}^3). \tag{3.17}$$

The following lemma showing that $S(\omega)$ is a compensating function was proved in [13] and we include the proof here for the convenience of the readers. Notice that here we also include the dissipation on the microscopic component in $\|\cdot\|_\sigma$-norm as in the Definition 3.3 for the analysis in this paper.

Lemma 3.5 If $\lambda > 0$ is small enough, the operator $S(\omega)$ defined (3.17) is a compensating function for (3.14). Moreover, $S(\omega)$: $L^2(\mathbf{R}^3) \to \widetilde{W}$.

Proof Firstly, recall that $[e_k]_{k=1}^{13}$ is an orthogonal basis of $\widetilde{W}$ and $R(\omega) \equiv \sum\limits_{j=1}^{3} R^j \omega_j$, where R^j is a constant 13×13 real skew-symmetric matrix. Thus $S(\cdot)$ is $C^\infty(S^2)$ and $S(-\omega) = -S(\omega)$. This verifies (i) of the definition. The conditions (ii) and (iii) can be checked as follows. Let $g_1,\ g_2 \in L^2(\mathbf{R}_v^3)$. Then

$$\langle S(\omega) g_1, g_2\rangle = \sum_{k,\ell=1}^{13} \lambda r_{k\ell}(\omega) \langle g_1, e_\ell\rangle \overline{\langle g_2, e_k\rangle}. \tag{3.18}$$

Write $w = \{w_k\}_{k=1}^{13} = \{\langle g_1, e_k\rangle\}_{k=1}^{13};\ u = \{u_k\}_{k=1}^{13} = \{\langle g_2, e_k\rangle\}_{k=1}^{13}$. Then

$$\langle S(\omega) g_1, g_2\rangle = \lambda\langle\langle R(\omega) w, u\rangle\rangle.$$

Hence

$$\langle iS(\omega) g_1, g_2\rangle = \lambda\langle\langle iR(\omega) w, u\rangle\rangle.$$

Since $R(\omega)$ is skew symmetric, $iS(\omega)$ is self-adjoint so that condition (ii) holds.

For the condition (iii), let $f \in D(L)$. From (3.17), we have

$$\langle S(\omega)(v\cdot\omega)\partial^\alpha f, \partial^\alpha f\rangle \equiv \sum_{k,\ell=1}^{13} \lambda r_{k\ell}(\omega)\langle (v\cdot\omega)\partial^\alpha f, e_\ell\rangle \overline{\langle \partial^\alpha f, e_k\rangle}.$$

We substitute the decomposition $\partial^\alpha f = P_0\partial^\alpha f + (\mathbf{I} - P_0)\partial^\alpha f$ into the right hand side of (3.19), where P_0 is defined in (3.15). Then

$$\langle S(\omega)(v\cdot\omega)\partial^\alpha f, \partial^\alpha f\rangle = \lambda\langle\langle R(\omega)V(\omega)W, W\rangle\rangle + \sum_{k,\ell=1}^{13} \lambda r_{k\ell}\langle (v\cdot\omega)(\mathbf{I} - P_0)\partial^\alpha f, e_\ell\rangle \overline{\langle \partial^\alpha f, e_k\rangle},$$

where $V(\omega)$ is the matrix defined in (3.16) and W is the vector in $\mathbf{C}^{13}$ whose $k-$th component is $\langle \partial^\alpha f, e_k\rangle$. By virtue of Lemma 3.4 and the exponential decay of e_k in v, we have

$$\mathcal{R}\lambda\langle\langle R(\omega)V(\omega)W, W\rangle\rangle \geq C_1\ \lambda|W_I|^2 - \lambda C_2|W_{II}|^2 \geq C_1\lambda|\mathbf{P}\partial^\alpha f|_2^2 - C_2\lambda|(\mathbf{I}-\mathbf{P})\partial^\alpha f|_\sigma^2.$$

It is obvious that there exists some positive constant C_3 such that

$$|\langle (v\cdot\omega)(\mathbf{I} - P_0)\partial^\alpha f, e_\ell\rangle| \leq C_3|(\mathbf{I} - P_0)\partial^\alpha f|_\sigma \leq C_3|(\mathbf{I}-\mathbf{P})\partial^\alpha f|_\sigma.$$

Therefore, we obtain

$$\begin{aligned}\mathcal{R}\langle S(\omega)(v\cdot\omega)\partial^\alpha f, \partial^\alpha f\rangle &\geq \lambda[C_1|\partial^\alpha \mathbf{P} f|_2^2 - C_2|\partial^\alpha(\mathbf{I}-\mathbf{P})f|_\sigma^2] - C_3\lambda|\partial^\alpha f|_\sigma|(\mathbf{I}-\mathbf{P})\partial^\alpha f|_\sigma \\ &\geq \lambda(C_1-\varepsilon)|\partial^\alpha\mathbf{P}f|_2^2 - \lambda C_\varepsilon|(\mathbf{I}-\mathbf{P})\partial^\alpha f|_\sigma^2,\end{aligned} \tag{3.19}$$

for any $\varepsilon > 0$ small enough where $C_\varepsilon > 0$ is a constant depending on ε. Here, we have used the fact that $|\partial^\alpha f|_\sigma^2 \leq C(|(\mathbf{I}-\mathbf{P})\partial^\alpha f|_\sigma^2 + |\partial^\alpha \mathbf{P} f|_2^2)$. Recall

$$\langle L[\partial^\alpha f], \partial^\alpha f\rangle \geq \delta_0|(\mathbf{I}-\mathbf{P})\partial^\alpha f|_\sigma^2 \quad \text{for some}\ \ \delta_0 > 0. \tag{3.20}$$

Then a suitable combination of (3.19) and (3.20) shows that there exists $c_0 > 0$ such that

$$\mathcal{R}\langle S(\omega)(v\cdot\omega)\partial^\alpha f, \partial^\alpha f\rangle + \langle L[\partial^\alpha f], \partial^\alpha f\rangle \geq c_0(|\partial^\alpha \mathbf{P} f|_2^2 + |\partial^\alpha(\mathbf{I}-\mathbf{P})f|_\sigma^2).$$

And this completes the proof of the lemma.

In the following, we will use the above compensating function to estimate the macroscopic component of the solution to (1.3).

Lemma 3.6 Under the assumptions (H_0) and (H_1), for the equation (1.3), we have the following estimate:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha f\|^2 - \kappa\sum_{1\leq|\alpha|\leq N-1}\int_{\mathbf{R}^3}|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}), \widehat{\partial^\alpha f}\rangle \mathrm{d}\xi\Big]\\ &+\delta_2\sum_{2\leq|\alpha|\leq N}\|\mathbf{P}\partial^\alpha f\|^2 + \delta_1\sum_{1\leq|\alpha|\leq N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2\\ &\leq C\|\nabla_x \mathbf{P} f\|^2 + C\varepsilon(1+t)^{-1-\varrho} + C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{|\alpha|\leq N}\|\partial^\alpha f\|^2 - \kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f}, \hat{f}\rangle \mathrm{d}\xi\Big]\\ &+\delta_2\sum_{1\leq|\alpha|\leq N}\|\mathbf{P}\partial^\alpha f\|^2 + \delta_1\sum_{|\alpha|\leq N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2\\ &\leq C\varepsilon(1+t)^{-1-\varrho} + C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),\end{aligned}\tag{3.22}$$

where $\kappa > 0$ is small.

Proof Let $\omega = \xi/|\xi|$ and take the Fourier transform in x of (3.14) to have

$$\partial_t(\widehat{\partial^\alpha f}) + i|\xi|(v\cdot\omega)\widehat{\partial^\alpha f} + L[\widehat{\partial^\alpha f}] = \hat{g}.\tag{3.23}$$

Hence,

$$\frac{1}{2}\partial_t|\widehat{\partial^\alpha f}|_2^2 + \langle L[\widehat{\partial^\alpha f}], \widehat{\partial^\alpha f}\rangle = \mathcal{R}\langle \widehat{\partial^\alpha f}, \hat{g}\rangle.\tag{3.24}$$

Applying $-i|\xi|S(\omega)$ to (3.23) gives

$$-i|\xi|S(\omega)\partial_t(\widehat{\partial^\alpha f}) + |\xi|^2 S(\omega)((v\cdot\omega)\widehat{\partial^\alpha f}) - i|\xi|S(\omega)L[\widehat{\partial^\alpha f}] = -i|\xi|S(\omega)\hat{g}.\tag{3.25}$$

By taking the inner product of the above equation with $\widehat{\partial^\alpha f}$, we have

$$\begin{aligned}&\mathcal{R}\langle -i|\xi|S(\omega)\partial_t(\widehat{\partial^\alpha f}), \widehat{\partial^\alpha f}\rangle + |\xi|^2\mathcal{R}\langle S(\omega)((v\cdot\omega)\widehat{\partial^\alpha f}), \widehat{\partial^\alpha f}\rangle\\ &= |\xi|\mathcal{R}\Big\{\langle iS(\omega)L[\widehat{\partial^\alpha f}], \widehat{\partial^\alpha f}\rangle - \langle iS(\omega)\hat{g}, \widehat{\partial^\alpha f}\rangle\Big\}.\end{aligned}\tag{3.26}$$

Since $iS(\omega)$ is self-adjoint, the first term is $-\frac{1}{2}\partial_t[|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}), \widehat{\partial^\alpha f}\rangle]$. (3.24) $\times$ $(1+|\xi|^2)$ + (3.26) $\times$ κ with a positive constant κ yields

$$\begin{aligned}&\partial_t\Big[\frac{(1+|\xi|^2)}{2}|\widehat{\partial^\alpha f}|_2^2 - \frac{\kappa|\xi|}{2}\langle iS(\omega)(\widehat{\partial^\alpha f}), \widehat{\partial^\alpha f}\rangle\Big] + (1+|\xi|^2 - \kappa|\xi|^2)\\ &\times\langle L[\widehat{\partial^\alpha f}], \widehat{\partial^\alpha f}\rangle + \kappa|\xi|^2\{\mathcal{R}\langle S(\omega)((v\cdot\omega)\widehat{\partial^\alpha f}), \widehat{\partial^\alpha f}\rangle + \langle L[\widehat{\partial^\alpha f}], \widehat{\partial^\alpha f}\rangle\}\\ &= (1+|\xi|^2)\mathcal{R}\langle \widehat{\partial^\alpha f}, \hat{g}\rangle + \kappa|\xi|\mathcal{R}\Big\{\langle iS(\omega)L[\widehat{\partial^\alpha f}], \widehat{\partial^\alpha f}\rangle - \langle iS(\omega)\hat{g}, \widehat{\partial^\alpha f}\rangle\Big\}.\end{aligned}\tag{3.27}$$

For $0<\kappa<1$, the second term on the left hand side of (3.27) is bounded from below by

$$(1+|\xi|^2-\kappa|\xi|^2)\langle L[\widehat{\partial^\alpha f}],\widehat{\partial^\alpha f}\rangle \geq (1-\kappa)(1+|\xi|^2)\cdot\delta_0|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}|_\sigma^2.$$

By Lemma 3.5, the third term on the left hand side of (3.27) is bounded by

$$\kappa|\xi|^2\{\mathcal{R}\langle S(\omega)((v\cdot\omega)\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle+\langle L[\widehat{\partial^\alpha f}],\widehat{\partial^\alpha f}\rangle\}\geq \kappa|\xi|^2\cdot c_0(|\mathbf{P}\widehat{\partial^\alpha f}|_2^2+|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}|_\sigma^2).$$

In addition, we have

$$S(\omega)L[\widehat{\partial^\alpha f}]=\sum_{k,\ell=1}^{13}\lambda r_{k\ell}(\omega)\langle L[\widehat{\partial^\alpha f}],e_\ell\rangle e_k=\sum_{k,\ell=1}^{13}\lambda r_{k\ell}(\omega)\langle L[(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}],e_\ell\rangle e_k,$$

and

$$|\langle L[(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}],e_\ell\rangle|\leq C|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}|_\sigma,$$

where we have used Lemma 2.4 and the fact that $Lg=\Gamma(\mu,g)+\Gamma(g,\mu)$.

Thus, the absolute value of the second term on the right hand side of (3.27) satisfies

$$\begin{aligned}
&c\kappa|\xi|\Big\{|\langle iS(\omega)L[\widehat{\partial^\alpha f}],\widehat{\partial^\alpha f}\rangle|+|\langle iS(\omega)\hat{g},\widehat{\partial^\alpha f}\rangle|\Big\}\\
\leq\ & c\kappa|\xi|\Big\{|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}|_\sigma|\widehat{\partial^\alpha f}|_2+\sum_{k,\ell=1}^{13}|\langle\hat{g},e_\ell\rangle|\cdot|\langle\widehat{\partial^\alpha f},e_k\rangle|\Big\}\\
\leq\ & c_\varepsilon\kappa|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}|_\sigma^2+\kappa\varepsilon|\xi|^2|\widehat{\partial^\alpha f}|_2^2+c_\varepsilon\sum_{\ell=1}^{13}|\langle\hat{g},e_\ell\rangle|^2.
\end{aligned}$$

If we choose κ, $\varepsilon>0$ small enough, the above inequalities imply that there exist δ_1, $\delta_2>0$ such that

$$\begin{aligned}
&\partial_t\Big[(1+|\xi|^2)|\widehat{\partial^\alpha f}|_2^2-\kappa|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle\Big]+\delta_1(1+|\xi|^2)|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}|_\sigma^2+\delta_2|\xi|^2|\mathbf{P}\widehat{\partial^\alpha f}|_2^2\\
\leq\ &(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle+c_\varepsilon\sum_{\ell=1}^{13}|\langle\hat{g},e_\ell\rangle|^2.
\end{aligned}\tag{3.28}$$

Integrating (3.28) over ξ and summing over $1\leq|\alpha|\leq N-1$ give

$$\begin{aligned}
&\partial_t\Big[\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha f\|^2-\kappa\sum_{1\leq|\alpha|\leq N-1}\int_{\mathbf{R}^3}|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle\mathrm{d}\xi\Big]\\
&+\delta_1\sum_{1\leq|\alpha|\leq N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+\delta_2\sum_{2\leq|\alpha|\leq N}\|\mathbf{P}\partial^\alpha f\|^2\\
\leq\ &\sum_{1\leq|\alpha|\leq N-1}\int_{\mathbf{R}^3}(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\mathrm{d}\xi+c_\varepsilon\sum_{1\leq|\alpha|\leq N-1}\sum_{\ell=1}^{13}\int_{\mathbf{R}^3}|\langle\hat{g},e_\ell\rangle|^2\mathrm{d}\xi.
\end{aligned}\tag{3.29}$$

Back to the Landau equation with external forcing, set

$$g=\partial^\alpha(\mathrm{e}^{-\Phi/2}\Gamma(f,f))+\partial^\alpha\widetilde{S}+\partial^\alpha((\nabla_x\Phi+E)\cdot\nabla_v f)-\frac{1}{2}v\cdot\partial^\alpha(Ef)-\partial^\alpha((\mathrm{e}^{-\Phi}-1)Lf).$$

We first consider the second term on the right hand side of (3.29). By (3.5) and the condition (H_1), we obtain

$$\begin{aligned}
&\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\langle\partial^\alpha(\widehat{\mathrm{e}^{-\Phi/2}\Gamma}(f,f)),e_\ell\rangle|^2\mathrm{d}\xi\\
=&\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\langle\partial^\alpha(\mathrm{e}^{-\Phi/2}\Gamma(f,f)),e_\ell\rangle|^2\mathrm{d}x\\
\le& C\sum_{1\le|\alpha|\le N-1}\sum_{\alpha'+\alpha''\le\alpha}\int_{\mathbf{R}^3}\{\Big|\partial^{\alpha'}f\Big|_2^2\Big|\partial^{\alpha''}f\Big|_\sigma^2+\Big|\partial^{\alpha'}f\Big|_\sigma^2\Big|\partial^{\alpha''}f\Big|_2^2]\mathrm{d}x\\
\le& C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbf{R}^3}|\langle\widehat{\partial^\alpha\widetilde{S}},e_\ell\rangle|^2\mathrm{d}\xi&=\int_{\mathbf{R}^3}|\langle\partial^\alpha\widetilde{S},e_\ell\rangle|^2\mathrm{d}x\\
&\le C\|w^{-1/2}\partial^\alpha(\mathrm{e}^{\Phi/2}\mu^{-1/2}S)\|^2+C\|\partial^\alpha(\mathrm{e}^{-\Phi/2}E)\|^2\\
&\le 2C\varepsilon(1+t)^{-1-\varrho},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\langle v\cdot\widehat{\partial^\alpha(E}f)-\partial^\alpha((\nabla_x\widehat{\Phi+E})\cdot\nabla_vf),e_\ell\rangle|^2\mathrm{d}\xi\\
\le& C\sum_{1\le|\alpha|\le N}\sum_{\alpha'\le\alpha}\int_{\mathbf{R}^3}(|\partial^{\alpha'}\nabla_x\Phi|^2+|\partial^{\alpha'}E|^2)|\partial^{\alpha-\alpha'}f|_2^2\mathrm{d}x\\
\le& C\varepsilon\sum_{1\le|\alpha|\le N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+C\varepsilon\sum_{1\le|\alpha|\le N}\|\mathbf{P}\partial^\alpha f\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\langle(\mathrm{e}^{-\widehat{\Phi}-1)}Lf,e_\ell\rangle|^2\mathrm{d}\xi\\
=&\sum_{1\le|\alpha|\le N-1}\sum_{\alpha'\le\alpha}\int_{\mathbf{R}^3}|\langle\partial^{\alpha-\alpha'}(\mathrm{e}^{-\Phi}-1)L\partial^{\alpha'}f,e_\ell\rangle|^2\mathrm{d}x\\
\le& C\varepsilon\sum_{1\le|\alpha|\le N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2.
\end{aligned}$$

For the first term on the right hand side of (3.29), we have

$$\Big|\int_{\mathbf{R}^3}(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\mathrm{d}\xi\Big|\le\Big|\int_{\mathbf{R}^3}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\mathrm{d}\xi\Big|+\Big|\int_{\mathbf{R}^3}|\xi|^2\langle\widehat{\partial^\alpha f},\hat{g}\rangle\mathrm{d}\xi\Big|. \tag{3.30}$$

By using (1.11), (3.5) and Lemma 2.5, we get

$$\begin{aligned}
\Big|\int_{\mathbf{R}^3}\langle\widehat{\partial^\alpha f},\partial^\alpha(\widehat{\mathrm{e}^{-\Phi/2}\Gamma}(f,f))\rangle\mathrm{d}\xi\Big|&=\Big|(\partial^\alpha(\mathrm{e}^{-\Phi/2}\Gamma(f,f)),\partial^\alpha f)\Big|\\
&=\Big|(\partial^\alpha(\mathrm{e}^{-\Phi/2}\Gamma(f,f)),(\mathbf{I}-\mathbf{P})\partial^\alpha f)\Big|\\
&\le\eta\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Big|\int_{\mathbf{R}^3}|\xi|^2\langle\widehat{\partial^\alpha f},\partial^\alpha(\mathrm{e}^{-\widehat{\Phi/2}\Gamma}(f,f))\rangle\mathrm{d}\xi\Big| &= \Big|\int_{\mathbf{R}^3}\langle\widehat{\partial^{\delta_i}\partial^\alpha}f,\partial^{\delta_i}\partial^\alpha(\mathrm{e}^{-\widehat{\Phi/2}}\Gamma(f,f))\rangle\mathrm{d}\xi\Big|\\ &= \Big|(\partial^{\delta_i}\partial^\alpha\Gamma(f,f),\partial^{\delta_i}\partial^\alpha f)\Big|\\ &\le \eta\|(\mathbf{I}-\mathbf{P})\partial^{\delta_i}\partial^\alpha f\|_\sigma^2+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),\end{aligned}$$

where $|\alpha|\le N-1$. The other terms in g can be estimated similarly by using the condition (H_1). In summary, we have

$$\begin{aligned}&\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\mathrm{d}\xi+c_\varepsilon\sum_{1\le|\alpha|\le N-1}\sum_{\ell=1}^{13}\int_{\mathbf{R}^3}|\langle\hat{g},e_\ell\rangle|^2\mathrm{d}\xi\\ &\le C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t)+C\varepsilon(1+t)^{-1-\varrho}+C(\varepsilon+\eta)\sum_{1\le|\alpha|\le N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+C(\varepsilon+\eta)\sum_{1\le|\alpha|\le N}\|\mathbf{P}\partial^\alpha f\|^2.\end{aligned}$$

If we take $\eta>0$ and $\varepsilon>0$ small enough, (3.29) and the above estimate give the desired estimate (3.21).

To prove (3.22), let $\alpha=0$ in (3.28) and set

$$g=\mathrm{e}^{-\Phi/2}\Gamma(f,f)+\widetilde{S}+(\nabla_x\Phi+E)\cdot\nabla_v f-\frac{1}{2}v\cdot Ef-(\mathrm{e}^{-\Phi}-1)Lf.$$

Multiplying the resulting equation by $(1+|\xi|^2)^{N-1}$ and integrating over ξ give

$$\begin{aligned}&\partial_t\Big[\int_{\mathbf{R}^3}(1+|\xi|^2)^N|\hat{f}|_2^2\mathrm{d}\xi-\kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle\mathrm{d}\xi\Big]\\ &+\delta_1\int_{\mathbf{R}^3}(1+|\xi|^2)^N|(\mathbf{I}-\mathbf{P})\hat{f}|_\sigma^2\mathrm{d}\xi+\delta_2\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|^2|\mathbf{P}\hat{f}|_2^2\mathrm{d}\xi\\ &\le\int_{\mathbf{R}^3}(1+|\xi|^2)^N\mathcal{R}\langle\hat{f},\hat{g}\rangle\mathrm{d}\xi+c_\varepsilon\sum_{\ell=1}^{13}\int_{\mathbf{R}^3}(1+|\xi|^2)^N|\langle\hat{g},e_\ell\rangle|^2\mathrm{d}\xi.\end{aligned}\tag{3.31}$$

By using Lemmas 2.4–2.5 and the assumptions in Theorem 1.1, the same argument as above implies that (3.22) holds. And this completes the proof of the lemma.

For the study on the optimal time decay, we need the following estimate on the solution operator of the linearized Landau equation:

$$\partial_t f+v\cdot\nabla_x f+Lf=0,\quad f(0,x,v)=f_0(x,v),\tag{3.32}$$

whose solution can be written as

$$f(t,x,v)=U(t,0)f_0(x,v),\tag{3.33}$$

with $U(t,s)$ being the solution operator. The solution operator has the following decay estimates.

Lemma 3.7 Let $k\ge k_1\ge 0$ and $f_0\in H^N\cap Z_q$. If $f(t,x,v)\in C^0([0,\infty);H^N)\cap C^1([0,\infty);H^{N-1})$ is a solution of (3.32), we have

$$\|\nabla_x^k U(t,0)f_0\|\le C(1+t)^{-\sigma_{q,m}}(\|\nabla_x^{k_1}f_0\|_{Z_q}+\|\nabla_x^k f_0\|),\tag{3.34}$$

for any integer $m = k - k_1 \geq 0$, where $q \in [1, 2]$ and $\sigma_{q,m} = \frac{3}{2}\big(\frac{1}{q} - \frac{1}{2}\big) + \frac{m}{2}$.

Proof By letting $|\alpha| = 0$ in (3.28) and $g = 0$, we have

$$\partial_t\Big[(1+|\xi|^2)|\hat{f}|_2^2 - \kappa|\xi|\langle iS(\omega)\hat{f}, \hat{f}\rangle\Big] + \delta_1(1+|\xi|^2)|(\mathbf{I}-\mathbf{P})\hat{f}|_\sigma^2 + \delta_2|\xi|^2|\mathbf{P}\hat{f}|_2^2 \leq 0. \tag{3.35}$$

Hence,

$$\partial_t E[f] + \delta\frac{|\xi|^2}{1+|\xi|^2}E[f] \leq 0, \tag{3.36}$$

where

$$E[f] = |\hat{f}(t,\xi,\cdot)|_2^2 - \frac{\kappa|\xi|}{1+|\xi|^2}\langle iS(\omega)\hat{f}(t,\xi,\cdot), \hat{f}(t,\xi,\cdot)\rangle.$$

Since $\kappa > 0$ is small enough and $S(\omega)$ is a bounded operator, it is clear that for small κ,

$$\frac{1}{2}|\hat{f}(t,\xi,\cdot)|_2^2 \leq E[f] \leq 2|\hat{f}(t,\xi,\cdot)|_2^2. \tag{3.37}$$

From (3.36) and (3.37), we have

$$|\hat{f}(t,\xi,\cdot)|_2^2 \leq c\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2.$$

Multiplying this by $|\xi|^{2k}$ and integrating over ξ give

$$\|\nabla_x^k f\|^2 = \int_{\mathbf{R}^3}|\xi|^{2k}|\hat{f}(t,\xi,\cdot)|_2^2\mathrm{d}\xi \leq c\int_{\mathbf{R}^3}|\xi|^{2k}\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2\mathrm{d}\xi. \tag{3.38}$$

Set

$$I_0 = \int_{|\xi|\leq 1}|\xi|^{2k}\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2\mathrm{d}\xi + \int_{|\xi|\geq 1}|\xi|^{2k}\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2\mathrm{d}\xi. \tag{3.39}$$

For the first term in I_0, we have

$$\begin{aligned}&\Big|\int_{|\xi|\leq 1}|\xi|^{2k}\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2\mathrm{d}\xi\Big| \leq \int_{|\xi|\leq 1}|\xi^{\alpha-\alpha'}|^2\mathrm{e}^{-\delta t\frac{|\xi|^2}{2}}|\xi^{\alpha'}\hat{f}_0(\xi)|_2^2\mathrm{d}\xi\\ &\leq \Big(\int_{|\xi|\leq 1}|\xi|^{2p'm}\mathrm{e}^{-\delta p't\frac{|\xi|^2}{2}}\mathrm{d}\xi\Big)^{1/p'}\Big(\int_{|\xi|\leq 1}|\xi^{\alpha'}\hat{f}_0(\xi)|_2^{2q'}\mathrm{d}\xi\Big)^{1/q'},\end{aligned}$$

where $|\alpha| = k$, $|\alpha'| = k_1$, $m = |\alpha - \alpha'|$ and $p' \in [1, \infty)$ with $\frac{1}{p'} + \frac{1}{q'} = 1$. Note that

$$\int_{|\xi|\leq 1}|\xi|^{2p'm}\mathrm{e}^{-\delta p't\frac{|\xi|^2}{2}}\mathrm{d}\xi \leq C(1+t)^{-3/2-p'm},$$

and

$$\Big(\int_{|\xi|\leq 1}|\xi^{\alpha'}\hat{f}_0(\xi)|_2^{2q'}\mathrm{d}\xi\Big)^{1/q'} \leq \|\partial^{\alpha'}f_0\|_{Z_q}^2, \quad \frac{1}{q} + \frac{1}{2q'} = 1.$$

Thus,

$$\int_{|\xi|\leq 1}|\xi|^{2k}\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2\mathrm{d}\xi \leq c(1+t)^{-3/2-p'm}\|\partial^{\alpha'}f_0\|_{Z_q}^2.$$

For the second term in I_0, we directly have $\int_{|\xi|\geq 1}|\xi|^{2k}\mathrm{e}^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}|\hat{f}_0(\xi)|_2^2\mathrm{d}\xi \leq c\mathrm{e}^{\frac{-\delta t}{2}}\|\partial^\alpha f_0\|^2$. Hence,

$$\begin{aligned}I_0 &\leq C(1+t)^{-3/2p'-m}\|\partial^{\alpha'}f_0\|_{Z_q}^2 + c\mathrm{e}^{\frac{-\delta t}{2}}\|\partial^\alpha f_0\|^2\\ &\leq C(1+t)^{-2\sigma_{q,m}}(\|\nabla_x^{k_1}f_0\|_{Z_q} + \|\nabla_x^k f_0\|)^2.\end{aligned}$$

Therefore, (3.34) follows from (3.38) and the above estimate on I_0. And this completes the proof of the lemma.

4 Optimal Time Decay Rate

In this section, we shall derive a refined energy estimate which is used in constructing global solutions and proving the optimal time decay. For this, we first recall the local existence result proved in [14] as follows.

Lemma 4.1 For any $l \geq 0$, under the conditions (H_0) and (H_1), if $\mathcal{E}_{\updownarrow}(\prime) \leq \varepsilon$, there exists $T^* = T^*(\varepsilon) > 0$ such that there is a unique solution $f(t,x,v)$ to (1.3) in $[0,T^*) \times \mathbf{R}^3 \times \mathbf{R}^3$ satisfying

$$\mathcal{E}_l(t) + \sum_{|\alpha| \leq N} \int_0^t \|w^l \partial^\alpha f(s)\|_\sigma^2 \mathrm{d}s \leq C\varepsilon.$$

Moreover, if $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$, then $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$.

We are now ready to prove Theorem 1.1 and we split the proof into two parts, namely for existence and optimal decay for clear presentation.

Proof of global existence By Lemma 3.6, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{|\alpha|\leq N}\|\partial^\alpha f\|^2 - \kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle \mathrm{d}\xi\Big] \\
&\quad +\delta_1\sum_{|\alpha|\leq N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2 + \delta_2\sum_{1\leq|\alpha|\leq N}\|\mathbf{P}\partial^\alpha f\|^2 \\
&\leq C\varepsilon(1+t)^{-1-\varrho} + C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),
\end{aligned} \tag{4.1}$$

where $\kappa > 0$ is a small constant. From Lemma 3.2, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{1\leq|\alpha|\leq N}\|w^l\partial^\alpha f\|^2 + \|w^l(\mathbf{I}-\mathbf{P})f\|^2\Big] + \sum_{|\alpha|\leq N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2 \\
&\leq C\varepsilon(1+t)^{-1-\varrho} + C\varepsilon\sum_{1\leq|\beta'|,|\alpha'|+|\beta'|\leq N}\|w^l\partial_{\beta'}^{\alpha'}(\mathbf{I}-\mathbf{P})f\|_\sigma^2 \\
&\quad +C\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{P}f\|^2 + C\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2 + C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),
\end{aligned} \tag{4.2}$$

where $\varepsilon > 0$ is also small constant. And by Lemma 3.1, we have

$$\begin{aligned}
&\sum_{1\leq|\beta|,|\alpha|+|\beta|\leq N}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\|^2 + \|w^l\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2\Big] \\
&\leq C\varepsilon(1+t)^{-1-\varrho} + C\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{P}f\|^2 + C\sum_{|\alpha|\leq N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2 + C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).
\end{aligned} \tag{4.3}$$

A suitable linear combination of (4.1), (4.2) and (4.3) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{|\alpha|\leq N}\|\partial^\alpha f\|^2 - \kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle \mathrm{d}\xi \\
&\quad + \sum_{1\leq|\alpha|\leq N}\|w^l\partial^\alpha f\|^2 + \|w^l(\mathbf{I}-\mathbf{P})f\|^2 + \sum_{1\leq|\beta|,|\alpha|+|\beta|\leq N}\|w^l\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\|^2\Big\} \\
&\quad +\Big\{\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{P}f\|^2 + \sum_{|\alpha|\leq N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2 + \sum_{|\alpha|\leq N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\sigma^2
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma\Big\}\\
&\le C\varepsilon(1+t)^{-1-\varrho}+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t).
\end{aligned}
\tag{4.4}
$$

On the other hand, the boundedness of the operator $S(\omega)$ implies

$$
\Big|\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat f,\hat f\rangle\mathrm{d}\xi\Big|\le C\sum_{|\alpha|\le N}\|\partial^\alpha f\|^2+C\sum_{|\alpha|\le N-1}\|\partial^\alpha f\|^2.
$$

Therefore, we can define the functionals in (1.8) and (1.9) as

$$
\begin{aligned}
\mathcal{E}_l(t)=&\sum_{|\alpha|\le N}\|\partial^\alpha f\|^2-\kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat f,\hat f\rangle\mathrm{d}\xi\\
&+\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha f\|^2+\|w^l(\mathbf{I}-\mathbf{P})f\|^2+\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2,
\end{aligned}
$$

and

$$
\mathcal{D}_l(t)=\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+\sum_{|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma+\sum_{|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma.
$$

Finally, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l(t)+\mathcal{D}_l(t)\le C\varepsilon(1+t)^{-1-\varrho}+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t)\le C\varepsilon(1+t)^{-1-\varrho}+C\mathcal{E}_l(t)\mathcal{D}_l(t).
\tag{4.5}
$$

If we assume $\mathcal{E}_l(0)\le\varepsilon$ for a sufficiently small constant $\varepsilon>0$, based on the local existence in Lemma 4.1, the global existence follows from the standard continuity argument.

The following proof on the optimal decay estimate is motivated by the work of Duan-Ukai-Yang-Zhao [6] on the Boltzmann equation by using an energy-spectrum approach.

Proof of optimal decay Take $\varrho=3/2$ in the assumption on the source term. By Lemma 3.6, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{1\le|\alpha|\le N}\|\partial^\alpha f\|^2-\kappa\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle\mathrm{d}\xi\Big]\\
&+\delta_2\sum_{2\le|\alpha|\le N}\|\mathbf{P}\partial^\alpha f\|^2+\delta_1\sum_{1\le|\alpha|\le N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|^2_\sigma\\
&\le C\|\nabla_x\mathbf{P}f\|^2+C\varepsilon(1+t)^{-5/2}+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),
\end{aligned}
\tag{4.6}
$$

where $\kappa>0$ is a small constant. From Lemma 3.2, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha f\|^2+\|w^l(\mathbf{I}-\mathbf{P})f\|^2\Big]+\sum_{|\alpha|\le N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\
&\le C\varepsilon(1+t)^{-5/2}+C\varepsilon\sum_{1\le|\beta'|,|\alpha'|+|\beta'|\le N}\|w^l\partial^{\alpha'}_{\beta'}(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\
&+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t),
\end{aligned}
\tag{4.7}
$$

where $\varepsilon > 0$ is a small constant. Lemma 3.1 implies that

$$\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2+\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma\Big]$$
$$\le C\varepsilon(1+t)^{-5/2}+C\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+\sum_{|\alpha|\le N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t). \tag{4.8}$$

In addition, the standard energy estimate gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{I}-\mathbf{P})f\|^2+\|(\mathbf{I}-\mathbf{P})f\|^2_\sigma$$
$$\le C\varepsilon(1+t)^{-5/2}+C\|\nabla_x\mathbf{P}f\|^2+C\|\nabla_x(\mathbf{I}-\mathbf{P})f\|^2_\sigma+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t). \tag{4.9}$$

A suitable linear combination of (4.6), (4.7), (4.8) and (4.9) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{1\le|\alpha|\le N}\|\partial^\alpha f\|^2-\kappa\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle\mathrm{d}\xi\\&+\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha f\|^2+\|w^l(\mathbf{I}-\mathbf{P})f\|^2+\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2+\|(\mathbf{I}-\mathbf{P})f\|^2\Big\}\\&+\Big\{\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f\|^2+\sum_{|\alpha|\le N}\|w^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma+\sum_{|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|^2_\sigma\\&+\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2_\sigma+\|(\mathbf{I}-\mathbf{P})f\|^2_\sigma\Big\}\\&\le C\varepsilon(1+t)^{-5/2}+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t)+C\|\nabla_x\mathbf{P}f\|^2.\end{aligned}$$

On the other hand, the boundedness of the operator $S(\omega)$ implies that

$$\Big|\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle\mathrm{d}\xi\Big|\le C\sum_{1\le|\alpha|\le N}\|\partial^\alpha f\|^2+C\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha f\|^2.$$

Therefore, we can define a functional $H_l(t)$ equivalent to $\overline{\mathcal{E}}_l(t)$ in Lemma 2.2 as

$$\begin{aligned}\overline{\mathcal{E}}_l(t)\sim H_l(t)=&\sum_{1\le|\alpha|\le N}\|\partial^\alpha f\|^2-\kappa\sum_{1\le|\alpha|\le N-1}\int_{\mathbf{R}^3}|\xi|\langle iS(\omega)(\widehat{\partial^\alpha f}),\widehat{\partial^\alpha f}\rangle\mathrm{d}\xi\\&+\sum_{1\le|\alpha|\le N}\|w^l\partial^\alpha f\|^2+\|w^l(\mathbf{I}-\mathbf{P})f\|^2\\&+\sum_{1\le|\beta|,|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|^2+\|(\mathbf{I}-\mathbf{P})f\|^2,\end{aligned}$$

which is bounded by $C\mathcal{D}_l(t)$.

Thus, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H_l(t)+\mathcal{D}_l(t)&\le C\varepsilon(1+t)^{-5/2}+C\widetilde{\mathcal{E}}_l(t)\widetilde{\mathcal{D}}_l(t)+C\|\nabla_x\mathbf{P}f\|^2\\&\le C\varepsilon(1+t)^{-5/2}+C\mathcal{E}_l(t)\mathcal{D}_l(t)+C\|\nabla_x\mathbf{P}f\|^2.\end{aligned}$$

For $\mathcal{E}_l(t)<\varepsilon$ with $\varepsilon>0$ being small enough, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H_l(t)+\mathcal{D}_l(t)\le C\varepsilon(1+t)^{-5/2}+C\|\nabla_x\mathbf{P}f\|^2,$$

which gives from the fact that $\overline{\mathcal{E}}_l(t) \le C\mathcal{D}_l(t)$ that,

$$\frac{\mathrm{d}}{\mathrm{d}t}H_l(t) + H_l(t) \le C\varepsilon(1+t)^{-5/2} + C\|\nabla_x \mathbf{P} f\|^2. \tag{4.10}$$

If we set

$$G = \mathrm{e}^{-\Phi/2}\Gamma(f,f) + \widetilde{S} + (\nabla_x\Phi + E)\cdot\nabla_v f - \frac{1}{2}v\cdot Ef - (\mathrm{e}^{-\Phi}-1)Lf,$$

where $\widetilde{S} = \mathrm{e}^{\Phi/2}\mu^{-1/2}S + \mathrm{e}^{\Phi/2}\mu^{1/2}v\cdot\Phi$, then equation (1.5) can be written as

$$\partial_t f + v\cdot\nabla_x f + Lf = G.$$

With (3.33), the solution to equation (1.3) can be written in the mild form

$$f(t) = U(t,0)f_0 + \int_0^t U(t,s)G(s)\mathrm{d}s.$$

By using Lemma 3.7, we can obtain

$$\|\nabla_x \mathbf{P} f(t)\| \le \|\nabla_x f(t)\| \le C\lambda_0(1+t)^{-5/4} + C\int_0^t (1+t-s)^{-5/4}(\|G(s)\|_{Z_1} + \|\nabla_x G(s)\|)\mathrm{d}s,$$

where $\lambda_0 = \|f_0\|_{Z_1} + \|\nabla_x f_0\|$.

If $l \ge 1$, by (3.5), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}\|\mathrm{e}^{-\Phi/2}\Gamma(f,f)\|_{Z_1}^2 &\le C\sum_{|\beta_1|\le 2}\|\partial_{\beta_1} f\|^2 \cdot \sum_{|\beta_2|\le 2}\|w\partial_{\beta_2} f\|^2 \\ &\le C\sum_{|\beta|\le 2}\|w\partial_\beta(\mathbf{I}-\mathbf{P})f\|^4 + C\|\mathbf{P}f\|^4 \le C\mathcal{E}_l(t)H_l(t) + C\|\mathbf{P}f\|^4,\end{aligned}$$

and

$$\|\nabla_x(\mathrm{e}^{-\Phi/2}\Gamma(f,f))\|^2 \le C\|\nabla_x\Gamma(f,f)\|^2 + C\|\nabla_x\Phi\Gamma(f,f)\|^2 \le C\mathcal{E}_l(t)H_l(t),$$

where we have used the fact that $\mathcal{E}_l(t) \sim \widetilde{\mathcal{E}}_l(t)$ and $\overline{\mathcal{E}}_l(t) \sim H_l(t)$.

By (1.11), (3.5) and the condition (H_2), we have

$$\begin{aligned}\|G(s)\|_{Z_1} &\le \|\mathrm{e}^{-\Phi/2}\Gamma(f,f)\|_{Z_1} + \|\widetilde{S}\|_{Z_1} + \frac{1}{2}\|v\cdot Ef\|_{Z_1} + \|(\nabla_x\Phi + E)\cdot\nabla_v f\|_{Z_1} \\ &\quad + \sup_{x\in\mathbf{R}^3}|\mathrm{e}^{-\Phi}-1|\cdot\|Lf\|_{Z_1} \\ &\le C\sqrt{\varepsilon}\sqrt{H_l(s)} + C\|\mathbf{P}f\|^2 + C\varepsilon(1+t)^{-5/4} + \||x|E\|_{L_t^\infty(L_x^2)}\|v\nabla_x f\| \\ &\quad + (\|\nabla_x\Phi\|_{L^{6/5}} + \||x|E\|_{L_t^\infty(L_x^2)})\|\nabla_x\nabla_v f\| + C\sup_{x\in\mathbf{R}^3}|\mathrm{e}^{-\Phi}-1|\sum_{|\beta|\le 2}\|w\partial_\beta(\mathbf{I}-\mathbf{P})f\| \\ &\le C\sqrt{\varepsilon}\sqrt{H_l(s)} + C\|\mathbf{P}f\|^2 + C\varepsilon(\|\nabla_v\nabla_x f\| + \|w\nabla_x f\| \\ &\quad + \sum_{|\beta|\le 2}\|w\partial_\beta(\mathbf{I}-\mathbf{P})f\|) + C\varepsilon(1+t)^{-5/4} \\ &\le C\sqrt{\varepsilon}\sqrt{H_l(s)} + C\|\mathbf{P}f\|^2 + C\varepsilon(1+t)^{-5/4} \\ &\quad + C\varepsilon\Big(\|\nabla_x\mathbf{P}f\| + \|\nabla_v\nabla_x(\mathbf{I}-\mathbf{P})f\| + \|w\nabla_x(\mathbf{I}-\mathbf{P})f\| + \sum_{|\beta|\le 2}\|w\partial_\beta(\mathbf{I}-\mathbf{P})f\|\Big),\end{aligned}$$

where we have used Lemma 2.1, the Hardy inequality and the fact that $\mathcal{E}_l(t) \le \varepsilon$.

Notice that by an argument similar to those for Lemma 2.1 and Lemma 2.2, we have

$$\|Lf\| \le C \sum_{|\beta| \le 2} \|w\partial_\beta(\mathbf{I}-\mathbf{P})f\|.$$

Similarly, for $l \ge 1$, it holds that

$$\begin{aligned}\|\nabla_x G(s)\| \le{}& C\sqrt{\varepsilon}\sqrt{H_l(s)} + C\sqrt{\varepsilon}(1+t)^{-5/4} + C\varepsilon\Big(\|\nabla_x \mathbf{P}f\| + \|\nabla_v\nabla_x(\mathbf{I}-\mathbf{P})f\| \\ &+ \|w\nabla_x(\mathbf{I}-\mathbf{P})f\| + \sum_{|\alpha|\le 1, |\beta|\le 2} \|w\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\|\Big).\end{aligned}$$

Define

$$M(t) = \sup_{0\le s\le t}\Big\{(1+s)^{5/2}H_l(s)\Big\}, \quad M_0(t) = \sup_{0\le s\le t}\Big\{(1+s)^{3/2}\|f(s)\|^2\Big\}. \tag{4.11}$$

Since $M(t)$ and $M_0(t)$ are non-decreasing, when $l \ge 1$, we have

$$\begin{aligned}\|G(s)\|_{Z_1} + \|\nabla_x G(s)\| &\le C\sqrt{\varepsilon}\sqrt{H_l(s)} + C\|\mathbf{P}f(s)\|^2 + C\sqrt{\varepsilon}(1+s)^{-5/4} \\ &\le C\sqrt{\varepsilon}(1+s)^{-5/4}\sqrt{M(s)} + C(1+s)^{-3/2}M_0(t) + C\sqrt{\varepsilon}(1+s)^{-5/4},\end{aligned}$$

for any $0 \le s \le t$. With this, we have

$$\begin{aligned}\|\nabla_x \mathbf{P}f(t)\| &\le \|\nabla_x f(t)\| \\ &\le C\lambda_0(1+t)^{-5/4} + C(\varepsilon + \sqrt{\varepsilon}\sqrt{M(t)} + M_0(t))\int_0^t (1+t-s)^{-5/4}(1+s)^{-5/4}\mathrm{d}s \\ &\le C(1+t)^{-5/4}(\lambda_0 + \sqrt{\varepsilon} + \sqrt{\varepsilon}\sqrt{M(t)} + M_0(t)).\end{aligned} \tag{4.12}$$

On the other hand, by the Gronwall inequality, (4.10) gives

$$H_l(t) \le \mathrm{e}^{-ct}H_l(0) + C\int_0^t \mathrm{e}^{-c(t-s)}\|\nabla_x \mathbf{P}f(s)\|^2\mathrm{d}s + C\varepsilon(1+t)^{-5/2}.$$

Then (4.12) yields

$$\begin{aligned}H_l(t) &\le \mathrm{e}^{-ct}H_l(0) + C\int_0^t \mathrm{e}^{-c(t-s)}(1+s)^{-5/2}\mathrm{d}s(\lambda_0^2 + \varepsilon + \varepsilon M(t) + M_0^2(t)) + C\varepsilon(1+t)^{-5/2} \\ &\le C(1+t)^{-5/2}(H_l(0) + \lambda_0^2 + \varepsilon + \varepsilon M(t) + M_0^2(t)).\end{aligned}$$

Hence, for any $t \ge 0$, $M(t) \le C(H_l(0) + \lambda_0^2 + \varepsilon + \varepsilon M(t) + M_0^2(t))$. That is, when $\varepsilon > 0$ is small enough, one has $M(t) \le C(H_l(0) + \lambda_0^2 + \varepsilon + M_0^2(t))$. By (4.11), this gives

$$H_l(t) \le C(1+t)^{-5/2}(H_l(0) + \lambda_0^2 + \varepsilon + M_0^2(t)). \tag{4.13}$$

By using Lemma 3.7, it holds that

$$\|f(t)\| \le C(1+t)^{-3/4}(\|f_0\|_{Z_1} + \|f_0\|) + C\int_0^t (1+t-s)^{-3/4}(\|G(s)\|_{Z_1} + \|G(s)\|)\mathrm{d}s.$$

Since

$$\begin{aligned}\|G(s)\| &\le \|\mathrm{e}^{-\Phi/2}\Gamma(f,f)\| + \|\widetilde{S}\| + \frac{1}{2}\|v\cdot Ef\| + \|(\nabla_x\Phi+E)\cdot\nabla_v f\| + \|(\mathrm{e}^{-\Phi}-1)Lf\| \\ &\le C\sqrt{\varepsilon}\sqrt{H_l(t)} + \|\mathbf{P}f\|^2 + C\sqrt{\varepsilon}(1+t)^{-5/4} \\ &\quad + C\varepsilon\Big(\|\nabla_x\mathbf{P}f\| + \|\nabla_v\nabla_x(\mathbf{I}-\mathbf{P})f\| + \|w\nabla_x(\mathbf{I}-\mathbf{P})f\| + \sum_{|\beta|\le 2}\|w\partial_\beta(\mathbf{I}-\mathbf{P})f\|\Big),\end{aligned}$$

we have

$$\begin{aligned}\|G(s)\|_{Z_1} + \|G(s)\| &\le C(\sqrt{\varepsilon}+\varepsilon)\sqrt{H_l(s)} + \|\mathbf{P}f\|^2 + C\sqrt{\varepsilon}(1+s)^{-5/4} \\ &\le C(\sqrt{\varepsilon}+\lambda_0+\sqrt{H_l(0)}+M_0(s))(1+s)^{-5/4} + (1+s)^{-3/2}M_0(s).\end{aligned}$$

Finally, it follows that

$$\begin{aligned}\|f(t)\| &\le C(1+t)^{-3/4}(\|f_0\|_{Z_1}+\|f_0\|) \\ &\quad + C\int_0^t (1+t-s)^{-3/4}(1+s)^{-5/4}\mathrm{d}s(\sqrt{\varepsilon}+\lambda_0+\sqrt{H_l(0)}+M_0(t)) \\ &\le C(1+t)^{-3/4}(\|f_0\|_{Z_1}+\|f_0\|+\sqrt{\varepsilon}+\lambda_0+\sqrt{H_l(0)}+M_0(t)).\end{aligned}\tag{4.14}$$

Notice that $\varepsilon>0$ can be taken small enough and

$$\|f_0\|_{Z_1}+\|f_0\|+\sqrt{\varepsilon}+\lambda_0+\sqrt{H_l(0)}\le C\sqrt{\varepsilon}.$$

From (4.14), we obtain $\sqrt{M_0(t)}\le C\sqrt{\varepsilon}+CM_0(t)$. Thus, when $M_0(0)\le C\varepsilon$, we have $M_0(t)\le C\varepsilon$, which implies

$$\|f(t)\|\le C\varepsilon(1+t)^{-3/4}. \tag{4.15}$$

From (4.13) and (4.15), we obtain $H_l(t)\le C\varepsilon(1+t)^{-5/2}$. Since

$$H_l(t)\sim\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{P}f(t)\|^2+\sum_{|\alpha|+|\beta|\le N}\|w^l\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\|^2,$$

(1.18) is proved. And this completes the proof of Theorem 1.1.

Proof of Theorem 1.2 In the proof of Theorem 1.1, we have used the fact that when $\gamma\ge-1$,

$$|g|_\sigma\ge C|w^{1/2}g|_2\ge C|(1+|v|)^{1/2}g|_2$$

to control the terms involving the multiplication of the velocity v. If $E(t,x)=0$ and $S(t,x,v)=0$, we do not have these terms in the equation. Therefore, the same argument for Theorem 1.1 holds when $\gamma\ge-2$ so that the Theorem 1.2 follows.

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