

# BOUNDED SOLUTIONS OF THE BOLTZMANN EQUATION IN THE WHOLE SPACE

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**ABSTRACT.** We construct bounded classical solutions of the Boltzmann equation in the whole space without specifying any limit behaviors at the spatial infinity and without assuming the smallness condition on initial data. More precisely, we show that if the initial data is non-negative and belongs to a uniformly local Sobolev space in the space variable with Maxwellian type decay property in the velocity variable, then the Cauchy problem of the Boltzmann equation possesses a unique non-negative local solution in the same function space, both for the cutoff and non-cutoff collision cross section with mild singularity. The known solutions such as solutions on the torus (space periodic solutions) and in the vacuum (solutions vanishing at the spatial infinity), and solutions in the whole space having a limit equilibrium state at the spatial infinity are included in our category.

## 1. INTRODUCTION

Consider the Boltzmann equation,

$$(1.1) \quad f_t + v \cdot \nabla_x f = Q(f, f),$$

where  $f = f(t, x, v)$  is the density distribution function of particles with position  $x \in \mathbb{R}^3$  and velocity  $v \in \mathbb{R}^3$  at time  $t$ . The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*,$$

which is well-defined for suitable functions  $f$  and  $g$  specified later. Notice that the collision operator  $Q(\cdot, \cdot)$  acts only on the velocity variable  $v \in \mathbb{R}^3$ . In the following discussion, we will use the  $\sigma$ -representation, that is, for  $\sigma \in \mathbb{S}^2$ ,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which give the relations between the pre- and post- collisional velocities.

It is well known that the Boltzmann equation is a fundamental equation in statistical physics. For the mathematical theories on this equation, we refer the readers to [10, 11, 12, 13, 25], and the references therein also for the physics background.

In addition to the special bilinear structure of the collision operator, the cross-section  $B(v - v_*, \sigma)$  is a function of only  $|v - v_*|$  and  $\theta$  where

$$\cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

$B$  varies with different physical assumptions on the particle interactions and it plays an important role in the well-posedness theory for the Boltzmann equation. In fact, except for

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the hard sphere model, for most of the other molecular interaction potentials such as the inverse power laws, the cross section  $B(v - v_*, \sigma)$  has a non-integrable angular singularity. For example, if the interaction potential obeys the inverse power law  $r^{-(p-1)}$  for  $2 < p < \infty$ , where  $r$  denotes the distance between two interacting molecules, the cross-section behaves like

$$B(|v - v_*|, \sigma) \sim |v - v_*|^\gamma \theta^{-2-2s},$$

with

$$-3 < \gamma = \frac{p-5}{p-1} < 1, \quad 0 < s = \frac{1}{p-1} < 1.$$

As usual, the hard and soft potentials correspond to  $p > 5$  and  $2 < p < 5$  respectively, and the Maxwellian potential corresponds to  $p = 5$ .

The main consequence of the non-integrable singularity of  $B$  at  $\theta = 0$  is that it makes the collision operator  $Q$  behave like a pseudo differential operator, as pointed out by many authors, e.g. [2, 18, 20, 23]. To avoid this difficulty, Grad [13] introduced an assumption to cutoff this singularity. This was a substantial step made in the study of the Boltzmann equation (1.1) and is now called Grad's angular cutoff assumption.

One of the main issues in the study of (1.1) is the existence theory of the solutions. Many authors have developed various methods for constructing local and global solutions for different situations. Among them, the Cauchy problem has been studied most extensively for both cutoff and non-cutoff cases.

An essential observation here is that so far, all solutions for the Cauchy problem have been constructed so as to satisfy one of the following three spatial behaviors at infinity;  $x$ -periodic solutions (solutions on the torus, [13, 14, 21]), solutions approaching an equilibrium ([4, 5, 6, 7, 15, 19, 22]) and solutions approaching 0 (solutions near vacuum, [1, 3, 9, 12]). Notice that the solutions constructed in [16] are also solutions approaching an global equilibrium.

However, it is natural to wonder if there are any other solutions behaving differently at  $x$ -infinity. In fact the aim of the present paper is to show that the admissible limit behaviors are not restricted to the above three behaviors. More precisely, we show that the Cauchy problem admits a very large solution space which includes not only the solutions of the above three types but also the solutions having no specific limit behaviors such that almost periodic solutions and perturbative solutions of arbitrary bounded functions which are not necessarily equilibrium state. This will be done for both cutoff and non-cutoff cases without the smallness condition on initial data.

The method developed in this paper works for local existence theory. The global existence in the same solution space is a big open issue and is our current subject. Also the present method works for the Landau equation but since the extension is straightforward, the detail is omitted.

Our assumption on the cross section is as follows. For the non-cutoff case we assume as usual that  $B$  takes the form

$$(1.2) \quad B(v - v_*, \sigma) = \Phi(|v - v_*|)b(\cos \theta),$$

in which it contains a kinetic part

$$\Phi(|v - v_*|) = \Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma,$$

and a factor related to the collision angle with singularity,

$$b(\cos \theta) \approx K \theta^{-2-2s} \quad \text{when } \theta \rightarrow 0+,$$

for some constant  $K > 0$ . For the cutoff case we assume that  $b$  takes the form (2.2) or is bounded by it.

In order to introduce our working function spaces, set

$$\partial_\beta^\alpha = \partial_x^\alpha \partial_v^\beta, \quad \alpha, \beta \in \mathbb{N}^3.$$

Let  $\phi_1 = \phi_1(x)$  be a smooth cutoff function

$$(1.3) \quad \phi_1 \in C_0^\infty(\mathbb{R}^3), \quad 0 \leq \phi_1(x) \leq 1, \quad \phi_1(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Our function space is the uniformly local Sobolev space with respect to the space variable and the usual Sobolev space with respect to the velocity variable with weight. More precisely, let  $k, \ell \in \mathbb{N}$  and  $W_\ell = (1 + |v|^2)^{\ell/2}$  be a weight function. We define

$$(1.4) \quad \begin{aligned} H_{ul}^{k,\ell}(\mathbb{R}^6) &= \{g \mid \|g\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \\ &= \sum_{|\alpha+\beta| \leq k} \sup_{a \in \mathbb{R}^3} \int_{\mathbb{R}^6} |\phi_1(x-a) W_\ell \partial_\beta^\alpha g(x,v)|^2 dx dv < +\infty\}. \end{aligned}$$

We will set  $H_{ul}^k(\mathbb{R}^6) = H_{ul}^{k,0}(\mathbb{R}^6)$ .

The uniformly local Sobolev space was first introduced by Kato in [17] as a space of functions of  $x$  variable, and was used to develop the local existence theory on the quasi-linear symmetric hyperbolic systems without specifying the limit behavior at infinity.

This space could be defined also by the cutoff function  $\phi_R(x) = \phi(x/R)$  for any  $R > 0$ , but the choice of  $R$  is not a matter. Indeed let  $R > 1$ . Then we see that

$$\begin{aligned} \|g\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)} &\leq \sum_{|\alpha+\beta| \leq k} \sup_{a \in \mathbb{R}^3} \int_{\mathbb{R}^6} |\phi_R(x-a) W_\ell \partial_\beta^\alpha g(x,v)|^2 dx dv \\ &\leq \sum_{|\alpha+\beta| \leq k} \sum_{j \in \mathbb{Z}^3, |j| \leq R} \sup_{a \in \mathbb{R}^3} \int_{\mathbb{R}^6} |\phi_1(x-a-j) W_\ell \partial_\beta^\alpha g(x,v)|^2 dx dv \\ &\leq CR^3 \|g\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}. \end{aligned}$$

The case  $0 < R < 1$  can be proved similarly. In the sequel, therefore, we fix  $R = 1$ .

This space shares many important properties with the usual Sobolev space such as the Sobolev embedding and hence it is contained in the space of bounded functions if  $k > 3$ . An important difference from the usual Sobolev space is that no limit property is specified at  $x$ -infinity for the space (1.4).

We shall consider the solutions satisfying the Maxwellian type exponential decay in the velocity variable. More precisely, set  $\langle v \rangle = (1 + |v|^2)^{1/2}$ . For  $k \in \mathbb{N}$ , our function space of initial data will be

$$\mathcal{E}_0^k(\mathbb{R}^6) = \left\{ g \in \mathcal{D}'(\mathbb{R}_{x,v}^6); \exists \rho_0 > 0 \text{ s.t. } e^{\rho_0 \langle v \rangle^2} g \in H_{ul}^k(\mathbb{R}_{x,v}^6) \right\},$$

while the function space of solutions will be, for  $T > 0$ ,

$$\begin{aligned} \mathcal{E}^k([0, T] \times \mathbb{R}_{x,v}^6) &= \left\{ f \in C^0([0, T]; \mathcal{D}'(\mathbb{R}_{x,v}^6)); \exists \rho > 0 \right. \\ &\quad \left. \text{s.t. } e^{\rho \langle v \rangle^2} f \in C^0([0, T]; H_{ul}^k(\mathbb{R}_{x,v}^6)) \right\}. \end{aligned}$$

Our main result is stated as follows.

**Theorem 1.1.** *Assume that the cross section  $B$  takes the form (1.2) with  $0 < s < 1/2$ ,  $\gamma > -3/2$  and  $2s + \gamma < 1$ . If the initial data  $f_0$  is non-negative and belongs to the function*

space  $\mathcal{E}_0^{k_0}(\mathbb{R}^6)$  for some  $k_0 \in \mathbb{N}, k_0 \geq 4$ , then, there exists  $T_* > 0$  such that the Cauchy problem

$$(1.5) \quad \begin{cases} f_t + v \cdot \nabla_x f = Q(f, f), \\ f|_{t=0} = f_0, \end{cases}$$

admits a non-negative unique solution in the function space  $\mathcal{E}^{k_0}([0, T_*] \times \mathbb{R}^6)$ .

**Remark 1.2.** For the cutoff case, if  $\gamma > -3/2$ , the same theorem and the same proof are valid because our assumption is that the cross-section  $b$  is given by (2.2) or is bounded by it. In the sequel, therefore, we consider the non-cutoff case only.

Before closing this section we give some comparison of this paper and our recent paper [3]. First, we shall compare the existence results. Both papers solve the same modified Cauchy problem (2.1). Thus, we shall compare Theorem 2.1 of this paper and Theorem 4.1 of [3].

The solution space in [3] is the usual weighted Sobolev space  $H_\ell^k(\mathbb{R}_{x,v}^6)$ , so that Theorem 4.1 of [3] gives solutions vanishing at  $x$ -infinity (solutions near vacuum). And it is easy to see that even if the space is replaced by  $H_\ell^k(\mathbb{T}_x^3 \times \mathbb{R}_v^3)$ , the proof of [3] is still valid and gives rise to  $x$ -periodic solutions (solutions on torus). The same space was used also in [14]. On the other hand, it is clear that the space  $H_{ul}^{k,\ell}(\mathbb{R}^6)$  defined by (1.4), the locally uniform Sobolev space with respect to  $x$ -variables, contains, as its subset, not only the spaces  $H_\ell^k(\mathbb{R}^6)$  and  $H_\ell^k(\mathbb{T}^3 \times \mathbb{R}^3)$  but also the set of functions having the form  $G = \mu + \mu^{1/2}g$ . If  $\mu$  is a global Maxwellian, then we have well-known perturbative solutions of equilibrium as in [5, 6, 7, 22], but more generally  $G$  can be any bounded functions. Hence, Theorem 2.1 gives, for example, almost periodic solutions, solutions having different limits at  $x$ -infinity like shock profile solutions which attain different equilibrium at the right and left infinity, and bounded solution behaving in more general way at  $x$ -infinity. Thus the present paper extends extensively the function space of admissible solutions. This is an essential difference between the two papers.

Another big difference is the collision cross section. The present paper deals with the original kinetic factor  $\Phi(z) = |z|^\gamma$  without regularizing the singularity at  $z = 0$  whereas [3] considers only a regularized one of the form  $\Phi(z) = (1 + |v|^2)^{\gamma/2}$ . This regularization simplifies drastically the estimates of collision operator  $Q$ . For example the proof of Proposition 4.4 with non-regularized kinetic factor is far more subtle than (2.1.2) of [3]. Also the case  $\gamma < 0$  should be handled separately from the case  $\gamma \geq 0$  if the kinetic factor is not regularized, and the range of admissible values of  $\gamma$  is restricted in this paper.

The rest of this paper is organized as follows. In the next section we first rewrite the Cauchy problem (1.5) by the one involving the wight function of the time-dependent Maxwellian type and approximate it by introducing the cutoff cross section. After establishing the upper bounds of the cutoff collision operator, we introduce linear iterative Cauchy problems and show that the iterative solutions converge to the solutions to the cutoff Cauchy problem. In Section 3 we derive the a priori estimates satisfied by the solutions to the cutoff Cauchy problem uniformly with respect to the cutoff parameter. The estimates thus obtained are enough to conclude the local existence and hence to lead to Theorem 1.1. The last section is devoted to the proof of Lemma 3.3 which is essential for the uniform estimate established in Section 3.

## 2. CONSTRUCTION OF APPROXIMATE SOLUTIONS

As will be seen later (Lemma 2.2 and Theorem 4.4), the non-linear collision operator induces a weight loss, which implies that it cannot be Lipschitz continuous so that the usual

iteration procedure is not valid for constructing local solutions. This difficulty can be overcome, however, by introducing weight functions in  $v$  of time-dependent Maxwellian type, developed previously in [3, 23, 24]. Indeed it compensates the weight loss by producing an extra gain term of one order higher weight in the velocity variable at the expense of the loss of the decay order of the time dependent Maxwellian-type weight.

**2.1. Modified Cauchy Problem.** More precisely, we set, for any  $\kappa, \rho > 0$ ,

$$T_0 = \rho/(2\kappa),$$

and put

$$\mu_\kappa(t) = \mu(t, v) = e^{-(\rho - \kappa t)(1 + |v|^2)},$$

and

$$f = \mu_\kappa(t)g, \quad \Gamma^t(g, g) = \mu_\kappa(t)^{-1}Q(\mu_\kappa(t)g, \mu_\kappa(t)g)$$

for  $t \in [0, T_0]$ . Then the Cauchy problem (1.5) is reduced to

$$(2.1) \quad \begin{cases} g_t + v \cdot \nabla_x g + \kappa(1 + |v|^2)g = \Gamma^t(g, g), \\ g|_{t=0} = g_0. \end{cases}$$

Define

$$\begin{aligned} \mathcal{M}^{k, \ell}([0, T[\times \mathbb{R}^6)) &= \{g \mid \|g\|_{\mathcal{M}^{k, \ell}([0, T[\times \mathbb{R}^6))} \\ &= \sum_{|\alpha + \beta| \leq k} \sup_{a \in \mathbb{R}^3} \int_{]0, T[\times \mathbb{R}^6} |\phi_0(x - a) W_\ell \partial_\beta^\alpha g(x, v)|^2 dt dx dv < +\infty\}. \end{aligned}$$

Our existence theorem can be stated as follows

**Theorem 2.1.** *Assume that  $0 < s < 1/2$ ,  $\gamma > -3/2$  and  $2s + \gamma < 1$ . Let  $\kappa, \rho > 0$  and let  $g_0 \in H_{ul}^{k, \ell}(\mathbb{R}^6)$ ,  $g_0 \geq 0$  for some  $k \geq 4$  and  $\ell \geq 3$ . Then there exists  $T_* \in ]0, T_0]$  such that the Cauchy problem (2.1) admits a unique non-negative solution satisfying*

$$g \in C^0([0, T_*]; H_{ul}^{k, \ell}(\mathbb{R}^6)) \cap \mathcal{M}^{k, \ell+1}([0, T_*[\times \mathbb{R}^6)).$$

The strategy of proof is in the same spirit as in [3]. That is, first, approximate the non-cutoff cross-section by a family of cutoff cross-sections and construct the corresponding solutions by a sequence of iterative linear equations. Then the existence of solutions to these approximate linear equations and by obtaining a uniform estimate on these solutions with respect to the cutoff parameter in the uniformly local Sobolev space, the compactness argument will lead to the convergence of the approximate solutions to the desired solution for the original problem.

**2.2. Cutoff Approximation.** Recall that the cross-section takes the form (1.2). For  $0 < \varepsilon \ll 1$ , we approximate (cutoff) the cross-section by

$$(2.2) \quad b_\varepsilon(\cos \theta) = \begin{cases} b(\cos \theta), & \text{if } |\theta| \geq 2\varepsilon, \\ b(\cos \varepsilon), & \text{if } |\theta| \leq 2\varepsilon. \end{cases}$$

Denote the corresponding cutoff cross-section by  $B_\varepsilon = \Phi(v - v_*)b_\varepsilon(\cos \theta)$  and the collision operator by  $\Gamma_\varepsilon^t(g, g)$ . We shall establish a upper weighted estimate on the cutoff collision operator in the uniformly local Sobolev space  $H_{ul}^{k, \ell}(\mathbb{R}^6)$ .

**Lemma 2.2.** *Let  $\gamma > -3/2$ . Then for any  $\varepsilon > 0$ ,  $k \geq 4$ ,  $l \geq 0$ , and for any  $U, V$  belonging to  $H_{ul}^{k, \ell}(\mathbb{R}^6)$ , it holds that*

$$\Gamma_\varepsilon^t(U, V) \in H_{ul}^{k, \ell}(\mathbb{R}^6)$$

with

$$(2.3) \quad \|\Gamma_\varepsilon^t(U, V)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)} \leq C \|U\|_{H_{ul}^{k,\ell+\gamma^+}(\mathbb{R}^6)} \|V\|_{H_{ul}^{k,\ell+\gamma^+}(\mathbb{R}^6)}, \quad 0 \leq t \leq T_0,$$

for some  $C > 0$  depending on  $\varepsilon, k, \ell$  as well as  $\rho, \kappa$ .

*Proof.* First, for simplicity of notations, denote  $\mu_\kappa(t)$  by  $\mu(t)$  without any confusion. By using the collisional energy conservation,

$$|v'_*|^2 + |v'|^2 = |v_*|^2 + |v|^2,$$

we have  $\mu_*(t) = \mu^{-1}(t) \mu'_*(t) \mu'(t)$ . Then for suitable functions  $U, V$ , it holds that

$$(2.4) \quad \begin{aligned} & \Gamma_\varepsilon^t(U, V)(v) \\ &= \mu^{-1}(t, v) \iint_{\mathbb{R}_*^3 \times \mathbb{S}_\sigma^2} B_\varepsilon(v - v_*, \sigma) (\mu'_*(t) U'_* \mu'(t) V' - \mu_*(t) U_* \mu(t) V) dv_* d\sigma \\ &= \iint_{\mathbb{R}_*^3 \times \mathbb{S}_\sigma^2} B_\varepsilon(v - v_*, \sigma) \mu_*(t) (U'_* V' - U_* V) dv_* d\sigma \\ &= \mathcal{I}_\varepsilon(U, V, \mu(t)). \end{aligned}$$

Then we have by the Leibniz formula in the  $x$  variable and by the translation invariance property in the  $v$  variable that for any  $\alpha, \beta \in \mathbb{N}^3$ ,

$$\partial_\beta^\alpha \Gamma_\varepsilon^t(U, V) = \sum_{\substack{\alpha_1 + \alpha_2 = \alpha; \\ \beta_1 + \beta_2 + \beta_3 = \beta}} C_{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3} \mathcal{I}_\varepsilon(\partial_x^{\alpha_1} \partial_v^{\beta_1} U, \partial_x^{\alpha_2} \partial_v^{\beta_2} V, \partial_v^{\beta_3} \mu(t)).$$

Next, recall the cutoff function  $\phi_1$  in (1.3) and set  $\phi_2(x) = \phi_1(x/2)$ , that is,

$$\phi_2 \in C_0^\infty(\mathbb{R}^3), \quad 0 \leq \phi_2(x) \leq 1, \quad \phi_2(x) = \begin{cases} 1, & |x| \leq 2, \\ 0, & |x| \geq 4. \end{cases}$$

Since  $\phi_1(x-a) = \phi_1(x-a)\phi_2(x-a)$  holds, we see that for  $a \in \mathbb{R}^3$ ,

$$\begin{aligned} & \phi_1(x-a) \partial_\beta^\alpha \Gamma_\varepsilon^t(U, V) \\ &= \sum_{\substack{\alpha_1 + \alpha_2 = \alpha; \\ \beta_1 + \beta_2 + \beta_3 = \beta}} C_{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3} \mathcal{I}_\varepsilon(\phi_1(x-a) \partial_{\beta_1}^{\alpha_1} U, \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} V, \partial_v^{\beta_3} \mu(t)). \end{aligned}$$

To prove (2.6), put

$$\begin{aligned} g_1 &= \phi_1(x-a) \partial_{\beta_1}^{\alpha_1} U, & h_2 &= \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} V, & \mu_3(t) &= \partial_v^{\beta_3} \mu(t), \\ \mathcal{I}_\varepsilon(g_1, h_2, \mu_3(t)) &= \mathcal{I}_\varepsilon^+ - \mathcal{I}_\varepsilon^-. \end{aligned}$$

Throughout this section, we often use the estimates

$$\mu(t, v), \quad |\mu_3(t)| = |\partial_v^{\beta_3} \mu(t, v)| \leq C_{\rho, k} e^{-\rho(v)^2/4}, \quad t \in [0, T_0], \quad v \in \mathbb{R}^3.$$

We compute  $\mathcal{I}_\varepsilon^+$  as follows.

$$\begin{aligned} |W_\ell \mathcal{I}_\varepsilon^+| &\leq C \iint |v - v_*|^\gamma |\mu_3(t, v_*)| \frac{W_\ell}{(W_\ell)_* (W_\ell)'} |(W_\ell g_1)'_*| |(W_\ell h_2)'| dv_* d\sigma \\ &\leq C \left[ \iint |v - v_*|^{2\gamma} |\mu_3(t, v_*)|^2 dv_* d\sigma \right]^{1/2} \\ &\quad \times \left[ \iint |(W_\ell g_1)'_* (W_\ell h_2)'|^2 dv_* d\sigma \right]^{1/2} \\ &\leq C \left[ \iint W_{2\gamma} |(W_\ell g_1)'_* (W_\ell h_2)'|^2 dv_* d\sigma \right]^{1/2}, \end{aligned}$$

where we have used

$$\frac{W_l}{(W_l)'_* (W_l)'} \leq 1$$

which comes from the energy conservation, and an elementary inequality

$$(2.5) \quad \begin{aligned} & \iint |v-v_*|^\gamma |\mu_3(t, v_*)| dv_* d\sigma \\ & \leq C \iint |v-v_*|^\gamma e^{-\rho|v_*|^2/4} dv_* \leq C \langle v \rangle^\gamma, \quad \gamma > -3, \end{aligned}$$

with some  $\rho > 0$ . Since the change of variables

$$(v, v_*, \sigma) \rightarrow (v', v'_*, \sigma'), \quad \sigma' = (v - v_*)/|v - v_*|,$$

has a unit Jacobian, and since  $W_\gamma \leq (W_{\gamma^+})'_*(W_{\gamma^+})'$  holds, we get

$$\begin{aligned} \|W_\ell \mathcal{T}_\varepsilon^+\|_{L^2(\mathbb{R}^6)}^2 & \leq C \iiint |(W_{\ell+\gamma^+} g_1)'_*(W_{\ell+\gamma^+} h_2)'|^2 dv_* d\sigma dv dx \\ & \leq C \int \|W_{\ell+\gamma^+} g_1\|_{L^2(\mathbb{R}_v^3)}^2 \|W_{\ell+\gamma^+} h_2\|_{L^2(\mathbb{R}_v^3)}^2 dx. \end{aligned}$$

If  $|\alpha_1 + \beta_1| \leq 2$ , by virtue of the Sobolev embedding and by the assumption  $k \geq 4$ , we have

$$\begin{aligned} \|W_\ell \mathcal{T}_\varepsilon^+\|_{L^2(\mathbb{R}^6)} & \leq C \|W_{\ell+\gamma^+} g_1\|_{L^\infty(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \|W_{\ell+\gamma^+} h_2\|_{L^2(\mathbb{R}_{x,v}^6)} \\ & \leq C \|\phi_1(x-a) W_{\ell+\gamma^+} \partial_{\beta_1}^{\alpha_1} U\|_{H_x^2(L_v^2)} \|\phi_2(x-a) W_{\ell+\gamma^+} \partial_{\beta_2}^{\alpha_2} V\|_{L^2(\mathbb{R}_{x,v}^6)}. \end{aligned}$$

Taking the supremum of both sides with respect to  $a \in \mathbb{R}^3$  and since  $\phi_1$  and  $\phi_2$  define the equivalent norms, we have

$$\|\mathcal{T}_\varepsilon^+\|_{H_{ul}^{0,\ell}(\mathbb{R}^6)} \leq C \|U\|_{H_{ul}^{k,\ell+\gamma^+}(\mathbb{R}^6)} \|V\|_{H_{ul}^{k,\ell+\gamma^+}(\mathbb{R}^6)}.$$

The computation is similar when  $|\alpha_2 + \beta_2| > 2$  for which  $|\alpha_1 + \beta_1| \leq k - |\alpha_2 + \beta_2| \leq k - 3$ . Also, the estimate of  $\mathcal{T}_\varepsilon^-$  can be done similarly but more straightforwardly. This completes the proof of the lemma. In the below we will use the following estimates which comes directly from the above proof.  $\square$

**Lemma 2.3.** *Let  $\gamma > -3/2$ . Then for any  $\varepsilon > 0$ ,  $k \geq 4$ ,  $l \geq 0$ , and for any  $U, V$  belonging to  $H_{ul}^{k,\ell}(\mathbb{R}^6)$ , it holds that*

$$(2.6) \quad \begin{aligned} & \|\phi_1(x-a) W_\ell \partial_\beta^\alpha \Gamma_\varepsilon^{t,\pm}(U, V)\|_{L^2(\mathbb{R}^6)} \\ & \leq C \sum \left( \|\phi_1(x-a) W_{\ell+\gamma^+} \partial_{\beta_1}^{\alpha_1} U\|_{H_x^2(L_v^2)} \|\phi_2(x-a) W_{\ell+\gamma^+} \partial_{\beta_2}^{\alpha_2} V\| \right. \\ & \quad \left. + \|\phi_1(x-a) W_{\ell+\gamma^+} \partial_{\beta_1}^{\alpha_1} U\|_{L^2(\mathbb{R}^6)} \|\phi_2(x-a) W_{\ell+\gamma^+} \partial_{\beta_2}^{\alpha_2} V\|_{H_x^2(L_v^2)} \right). \end{aligned}$$

for some  $C > 0$  depending on  $\varepsilon, k, \ell$  as well as  $\rho, \kappa$ .

We now study the following Cauchy problem for the cutoff Boltzmann equation

$$(2.7) \quad \begin{cases} g_t + v \cdot \nabla_x g + \kappa \langle v \rangle^2 g = \Gamma_\varepsilon^t(g, g), \\ g|_{t=0} = g_0, \end{cases}$$

for which we shall obtain uniform estimates in weighted Sobolev spaces. We first prove the existence of weak solutions.

**Theorem 2.4.** *Assume that  $-3/2 < \gamma \leq 1$ . Let  $k \geq 4$ ,  $l \geq 0$ ,  $\varepsilon > 0$  and  $D_0 > 0$ . Then, there exists  $T_\varepsilon \in ]0, T_0]$  such that for any initial data  $g_0$  satisfying*

$$(2.8) \quad g_0 \in H_{ul}^{k,\ell}(\mathbb{R}^6), \quad g_0 \geq 0, \quad \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)} \leq D_0,$$

*the Cauchy problem (2.7) admits a unique solution  $g^\varepsilon$  having the property*

$$g^\varepsilon \in C^0(]0, T_\varepsilon[; H_{ul}^{k,\ell}(\mathbb{R}^6)), \quad g \geq 0, \quad \|g^\varepsilon\|_{L^\infty(]0, T_\varepsilon[; H_{ul}^{k,\ell}(\mathbb{R}^6))} \leq 2D_0.$$

*Moreover, this solution enjoys a moment gain in the sense that*

$$g^\varepsilon \in \mathcal{M}^{k,\ell+1}(]0, T_\varepsilon[ \times (\mathbb{R}^6)).$$

**Remark 2.5.** (1) *Notice that we do not assume  $g_0 \in H_{ul}^{k,\ell+1}(\mathbb{R}^6)$ . The moment gain will be essentially used below to control the weight loss of the collision operator shown in Lemma 2.2.*

(2) *The regularity of  $g^\varepsilon$  with respect to  $t$  variable follows directly from the equation (2.7).*

*Proof of Theorem 2.4.* We prove the existence of non-negative solutions by successive approximation that preserves the non-negativity, which is defined by using the usual splitting of the collision operator (2.4) into the gain (+) and loss (-) terms,

$$\begin{aligned} \Gamma_\varepsilon^{t,+}(g, h) &= \iint_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} B_\varepsilon(v - v_*, \sigma) \mu_*(t) g'_* h' dv_* d\sigma, \\ \Gamma_\varepsilon^{t,-}(g, h) &= h L_\varepsilon(g), \\ L_\varepsilon(g) &= \iint_{\mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2} B_\varepsilon(v - v_*, \sigma) \mu(t, v_*) g_* dv_* d\sigma. \end{aligned}$$

Evidently, Lemma 2.2 applies to  $\Gamma_\varepsilon^{t,\pm}$ , and in view of (2.5), the linear operator  $L_\varepsilon$  satisfies

$$\begin{aligned} |\partial_\beta^\alpha L_\varepsilon(g)(t, x, v)| &\leq C \sum_{\beta_1 + \beta_2 = \beta} \int |v - v_*|^\gamma |(\partial_{\beta_1} \mu)(t, v_*)| |(\partial_{\beta_2}^\alpha g)(t, x, v_*)| dv_* \\ (2.9) \quad &\leq C \left[ \int |v - v_*|^{2\gamma} e^{-\rho|v_*|^2/4} dv_* \right]^{1/2} \sum_{\beta_2 \leq \beta} \left[ \int |(\partial_{\beta_2}^\alpha g)(t, x, v_*)|^2 dv_* \right]^{1/2} \\ &\leq C \langle v \rangle^\gamma \|\partial_x^\alpha g\|_{H^{|\beta|}(\mathbb{R}_v^3)}, \quad t \in [0, T_0], \end{aligned}$$

for a constant  $C > 0$  depending on  $\varepsilon$ .

We now define a sequence of approximate solutions  $\{g^n\}_{n \in \mathbb{N}}$  by

$$(2.10) \quad \begin{cases} g^0 = g_0; \\ \partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \kappa \langle |v| \rangle^2 g^{n+1} \\ \quad \quad \quad = \Gamma_\varepsilon^{t,+}(g^n, g^n) - \Gamma_\varepsilon^{t,-}(g^n, g^{n+1}), \\ g^{n+1}|_{t=0} = g_0. \end{cases}$$

Actually, in view of (2.9) we can consider the mild form

$$(2.11) \quad \begin{aligned} g^{n+1}(t, x, v) &= e^{-\kappa \langle |v| \rangle^2 t - V^n(t,0)} g_0(x - tv, v) \\ &\quad + \int_0^t e^{-\kappa \langle |v| \rangle^2 (t-s) - V^n(t,s)} \Gamma_\varepsilon^{s,+}(g^n, g^n)(s, x - (t-s)v, v) ds, \end{aligned}$$

where

$$V^n(t, s) = \int_s^t L_\varepsilon(g^n)(s, x - (t-s)v, v) ds.$$

We shall first notice that (2.11) defines well an approximate sequence of solutions. In fact, it follows from Lemma 2.2 that if

$$(2.12) \quad \begin{aligned} g_0 &\in H_{ul}^{k,\ell}(\mathbb{R}^6), \quad g_0 \geq 0, \\ g^n &\in L^\infty(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6)), \quad g^n \geq 0, \end{aligned}$$

for any  $T \in ]0, T_0[$ , then the mild form (2.11) determines  $g^{n+1}$  in the function class

$$(2.13) \quad g^{n+1} \in L^\infty(]0, T[; H_{ul}^{k,\ell-\gamma^+}(\mathbb{R}^6)), \quad g^{n+1} \geq 0,$$

and solves (2.10). Thus  $g^{n+1}$  exists and is non-negative. However, it appears to have a loss of weight in the velocity variable. We shall now show that the term  $\kappa\langle v \rangle^2 g^{n+1}$  in (2.10) not only recovers this weight loss but also creates higher moments. To see this, introduce the space and norm defined by

$$\begin{aligned} Y &= L^\infty(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6)) \cap \mathcal{M}^{k,\ell+1}(]0, T[\times \mathbb{R}^6), \\ \|g\|_Y^2 &= \|g\|_{L^\infty(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2 + \kappa \|g\|_{\mathcal{M}^{k,\ell+1}(]0, T[\times \mathbb{R}^6)}^2. \end{aligned}$$

**Lemma 2.6.** *Assume that  $-3/2 < \gamma \leq 1$  and let  $k \geq 4, l \geq 0, \varepsilon > 0$ . Then, there exist positive numbers  $C_1, C_2$  such that if  $\rho > 0, \kappa > 0$  and if  $g_0$  and  $g^n$  satisfy (2.12) with some  $T \leq T_0$ , the function  $g^{n+1}$  given by (2.11) enjoys the properties*

$$\begin{aligned} g^{n+1} &\in Y \\ \|g^{n+1}\|_Y &\leq e^{C_1 K_n T} \left( \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{C_2}{\kappa} \|g^n\|_{L^4(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6))}^4 \right), \end{aligned}$$

where  $K_n$  is a positive constant depending on  $\|g^n\|_{L^\infty(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6))}$  and  $\kappa$ .

*Proof.* Put

$$(2.14) \quad h_\ell^n = \phi_1(x-a) W_\ell \partial_\beta^\alpha g^n.$$

Differentiate (2.10) with respect to  $x, v$  and multiply by  $\phi_1(x-a)$  to deduce

$$\begin{aligned} \partial_t h_\ell^{n+1} + v \cdot \nabla_x h_\ell^{n+1} + \kappa \langle v \rangle^2 h_\ell^{n+1} &= G_1^+ - G_1^- + G_2 + G_3, \\ G_1^+ &= \phi_1(x-a) W_\ell \partial_\beta^\alpha \Gamma_\varepsilon^{t,+}(g^n, g^n), \\ G_1^- &= \phi_1(x-a) W_\ell \partial_\beta^\alpha \Gamma_\varepsilon^{t,-}(g^n, g^{n+1}), \\ G_2 &= -W_\ell [\phi_1(x-a) \partial_\beta, v \cdot \nabla_x] \partial^\alpha g^{n+1}, \\ G_3 &= -\kappa W_\ell \sum_{|\beta'|=1,2} C_{\beta'} \phi_1(x-a) (\partial_{\beta'} \langle v \rangle^2) \partial_{\beta-\beta'}^\alpha g^{n+1}. \end{aligned}$$

Let  $\chi_j \in C_0^\infty(\mathbb{R}^3)$ ,  $j \in \mathbb{N}$ , be the cutoff functions

$$\chi_j(v) = \begin{cases} 1, & |v| \leq j, \\ 0, & |v| \geq j+1. \end{cases}$$

Let  $S_N(D_x)$  be a mollifier defined by the Fourier multiplier

$$S_N(\xi) = 2^{-N} S(2^{-N} \xi), \quad S(\xi) \in \mathcal{S}(\mathbb{R}^3), \quad S(\xi) = 1 \quad (|\xi| \leq 1), \quad = 0 \quad (|\xi| \geq 1).$$

We remark that (2.13) does not necessarily imply  $h_{\ell+1}^{n+1}(t) \in L^2(\mathbb{R}^6)$ , but  $\chi_j h_{\ell+1}^{n+1}(t)$  does for all  $j \in \mathbb{N}$ . Hence, we can use  $\chi_j^2 S_N^2(D_x) h_{\ell+1}^{n+1}$  as a test function to get

$$(2.15) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|S_N(D_x) \chi_j h_{\ell+1}^{n+1}\|^2 + \kappa \|S_N(D_x) \chi_j h_{\ell+1}^{n+1}\|^2 \\ &= (G_1^+ - G_1^- + G_2 + G_3, S_N(D_x)^2 \chi_j^2 h_{\ell+1}^{n+1}). \end{aligned}$$

Here and in what follows, the norm  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$  are those of  $L^2(\mathbb{R}_{x,v}^6)$  unless otherwise stated. We shall evaluate the inner products on the right hand side. Observe that Lemma 2.3 gives, for  $t \in [0, T]$ , since  $\gamma \leq 1$  is assumed, and putting  $\tilde{h}_{\ell}^n = \phi_2(x-a) W_{\ell} \partial_{\beta}^{\alpha} g^n$ ,

$$\begin{aligned} \left| (G_1^+, S_N^2 \chi_j^2 h_{\ell+1}^{n+1}) \right| &= \left| (S_N \chi_j W_{-1} G_1^+, S_N \chi_j h_{\ell+1}^{n+1}) \right| \\ &\leq C \|W_{-1} G_1^+\| \|S_N \chi_j h_{\ell+1}^{n+1}\| \\ &\leq C \|h_{\ell-1+\gamma^+}^n\| \|\tilde{h}_{\ell-1+\gamma^+}^n\| \|S_N \chi_j h_{\ell+1}^{n+1}\| \\ &\leq C \|g^n\|_{H_{ul}^{k,\ell-1+\gamma^+}(\mathbb{R}^6)}^2 \|S_N \chi_j h_{\ell+1}^{n+1}\| \\ &\leq \frac{C}{\kappa} \|g^n\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^4 + \frac{\kappa}{4} \|S_N \chi_j h_{\ell+1}^{n+1}\|^2, \\ \left| (G_1^-, S_N^2 \chi_j^2 h_{\ell+1}^{n+1}) \right| &= \left| (S_N \chi_j W_{-1} G_1^-, S_N \chi_j h_{\ell+1}^{n+1}) \right| \\ &\leq C \|W_{-1} G_1^-\| \|S_N \chi_j h_{\ell+1}^{n+1}\| \\ &\leq C \|h_{\ell-1+\gamma^+}^n\| \|\tilde{h}_{\ell-1+\gamma^+}^{n+1}\| \|S_N \chi_j W_{\ell+1} h_{\ell+1}^{n+1}\| \\ &\leq C \|g^n\|_{H_{ul}^{k,\ell-1+\gamma^+}(\mathbb{R}^6)} \|g^{n+1}\|_{H_{ul}^{k,\ell-1+\gamma^+}(\mathbb{R}^6)} \|S_N \chi_j W_{\ell+1} h_{\ell+1}^{n+1}\| \\ &\leq \frac{C}{\kappa} \|g^n\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \|g^{n+1}\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{\kappa}{4} \|S_N \chi_j W_{\ell+1} h_{\ell+1}^{n+1}\|^2. \end{aligned}$$

Here  $C$  is a positive constants independent of  $\kappa$ , and we have used

$$\|h_{\ell}^n\| \leq C \|g^n\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}.$$

and similarly for  $\tilde{h}_{\ell}^n$ . The estimation on the remaining two inner products are more straightforward and can be given as follows. With some abuse of notation,

$$|G_2| \leq |v| W_{\ell} |\nabla_x \phi_1(x-a)| |\phi_2(x-a) \partial_{\beta}^{\alpha} g^{n+1}| + \phi_1(x-a) W_{\ell} \eta_{\beta} |\partial_{\beta-1}^{\alpha+1} g^{n+1}|$$

where  $\eta_{\beta} = 0$  for  $\beta = 0$  and  $= 1$  for  $|\beta| \geq 1$ , while

$$|G_3| \leq \kappa C W_{\ell} \sum_{|\beta'|=1,2} \phi_1(x-a) (|v|+1) |\partial_{\beta-\beta'}^{\alpha} g^{n+1}|.$$

Therefore we get

$$\begin{aligned} \left| (G_2 + G_3, S_N^2 \chi_j^2 h_{\ell+1}^{n+1}) \right| &= \left| W_{-1} (G_2 + G_3), S_N^2 \chi_j^2 W_1 h_{\ell+1}^{n+1} \right| \\ &\leq C \|W_{-1} (G_2 + G_3)\| \|S_N \chi_j h_{\ell+1}^{n+1}\| \\ &\leq C (1 + \kappa) \|g^{n+1}\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)} \|S_N \chi_j h_{\ell+1}^{n+1}\| \\ &\leq \frac{C' (1 + \kappa)^2}{\kappa} \|g^{n+1}\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{\kappa}{2} \|S_N \chi_j h_{\ell+1}^{n+1}\|^2. \end{aligned}$$

The constants  $C'$  are independent of  $\varepsilon$  and  $\kappa$ .

Putting together all the estimates obtained above in (2.15) yields

$$\begin{aligned} & \frac{d}{dt} \|S_N \mathcal{X}_j h_\ell^{n+1}\|^2 + \kappa \|S_N \mathcal{X}_j h_{\ell+1}^{n+1}\|^2 \\ & \leq \frac{C''}{\kappa} ((1 + \kappa)^2 + \|g^n\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2) \|g^{n+1}\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{C}{\kappa} \|g^n\|_{H_t^k(\mathbb{R}^6)}^4. \end{aligned}$$

The constants  $C, C''$  are independent of  $\varepsilon$  and  $\kappa$ . Integrate this over  $[0, t]$  to deduce

$$\begin{aligned} & \|S_N \mathcal{X}_j h_\ell^{n+1}(t)\|^2 + \kappa \int_0^t \|S_N \mathcal{X}_j h_{\ell+1}^{n+1}(\tau)\|^2 d\tau \\ & \leq \|S_N \mathcal{X}_j h_\ell^{n+1}(0)\|^2 + C_1 K_n \int_0^t \|g^{n+1}(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 d\tau + \frac{C_2}{\kappa} \int_0^t \|g^n(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^4 d\tau. \end{aligned}$$

where

$$K_n = \frac{1}{\kappa} \left( \|g^n\|_{L^\infty(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2 + (1 + \kappa)^2 \right),$$

and  $C_1 > 0$  is a constant independent of  $\varepsilon, \kappa$  while  $C_2$  is independent of  $\kappa$  but depends on  $\varepsilon$ . It is easy to see that we can now take the limit  $N \rightarrow \infty$  and  $j \rightarrow \infty$ , which results in

$$\begin{aligned} & \|h_\ell^{n+1}(t)\|^2 + \kappa \int_0^t \|h_{\ell+1}^{n+1}(\tau)\|^2 d\tau \\ & \leq \|h_\ell^{n+1}(0)\|^2 + C_1 K_n \int_0^t \|g^{n+1}(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 d\tau + \frac{C_2}{\kappa} \int_0^t \|g^n(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^4 d\tau. \end{aligned}$$

Sum up this for  $|\alpha + \beta| \leq k$  (see (2.14)) and take the speremum of the left hand side with respect to  $a \in \mathbb{R}^3$ . Knowing that the right hand side is independent of  $\alpha, \beta$  and also of  $a \in \mathbb{R}^3$ , we have

$$\begin{aligned} & \|g^{n+1}(t)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \kappa \|g^{n+1}\|_{\mathcal{M}^{k,\ell+1}(]0, t[ \times \mathbb{R}^6)}^2 \\ & \leq \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + C_1 K_n \int_0^t \|g^{n+1}(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 d\tau + \frac{C_2}{\kappa} \int_0^t \|g^n(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^4 d\tau. \end{aligned}$$

The Gronwall inequality then gives

$$\begin{aligned} (2.16) \quad & \|g^{n+1}(t)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \kappa \|g^{n+1}\|_{\mathcal{M}^{k,\ell+1}(]0, t[ \times \mathbb{R}^6)}^2 \\ & \leq e^{C_1 K_n t} \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{C_2}{\kappa} \int_0^t e^{C_1 K_n (t-\tau)} \|g^n(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^4 d\tau. \end{aligned}$$

for all  $t \in [0, T]$ .

Now the proof of Lemma 2.6 is completed.  $\square$

We are now ready to prove the convergence of  $\{g^n\}_{n \in \mathbb{N}}$ . Fix  $\kappa > 0$ . Let  $D_0, g_0$  be as in Theorem 2.4 and introduce an induction hypothesis

$$(2.17) \quad \|g^n\|_{L^\infty(]0, T[; H_{ul}^{k,\ell}(\mathbb{R}^6))} \leq 2D_0.$$

for some  $T \in ]0, T_0]$ . Notice that the factor 2 can be any number  $> 1$ .

(2.17) is true for  $n = 0$  due to (2.8). Suppose that this is true for some  $n > 0$ . We shall determine  $T$  independent of  $n$ . A possible choice is given by

$$e^{C_1 K_0 T} = 2, \quad \frac{2^4 C_2}{\kappa} T D_0^2 = 1 \quad \text{where} \quad K_0 = \frac{1}{\kappa} (2D_0 + (1 + \kappa)^2)$$

or

$$(2.18) \quad T = \min \left\{ \frac{\log 2}{C_1 K_0}, \frac{\kappa}{2^4 C_2 D_0^2} \right\}.$$

In fact, (2.16) and (2.17) imply that  $g^{n+1} \in Y$  and

$$\begin{aligned} \|g^{n+1}\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 &\leq e^{C_1 K_0 T} \left( \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{C_2}{\kappa} T \|g^n\|_{L^\infty([0,T]; H_{ul}^{k,\ell}(\mathbb{R}^6))}^4 \right) \\ &\leq e^{C_1 K_0 T} \left( D_0^2 + \frac{C_2}{\kappa} T 2^4 D_0^4 \right) \leq 4D_0^2. \end{aligned}$$

That is, the induction hypothesis (2.17) is fulfilled for  $n+1$ , and hence holds for all  $n$ .

For the convergence, set  $w^n = g^n(t) - g^{n-1}(t)$ , for which (2.10) leads to

$$\begin{cases} \partial_t w^{n+1} + v \cdot \nabla_x w^{n+1} + \kappa(|v|)^2 w^{n+1} = \Gamma_\varepsilon^{t,+}(w^n, g^n) \\ \quad + \Gamma_\varepsilon^{t,+}(g^{n-1}, w^n) - \Gamma_\varepsilon^{t,-}(w^n, g^{n+1}) - \Gamma_\varepsilon^{t,-}(g^{n-1}, w^{n+1}), \\ w^{n+1}|_{t=0} = 0. \end{cases}$$

By the same computation as in (2.15), we get

$$\begin{aligned} \|w^{n+1}\|_Y^2 &\leq \frac{1}{2} C_2 e^{C_1 K_0 T} \frac{1}{\kappa} T \left\{ \|g^{n+1}\|_{L^\infty([0,T]; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2 + \|g^n\|_{L^\infty([0,T]; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2 \right. \\ &\quad \left. + \|g^{n-1}\|_{L^\infty([0,T]; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2 \right\} \|w^n\|_{L^\infty([0,T]; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2, \end{aligned}$$

with the same constants  $C_1, C_2$  and  $K_0$  as above. Then, (2.17) and (2.18) give

$$\|g^{n+1} - g^n\|_Y^2 \leq 2^4 C_2 D_0^2 \kappa^{-1} T \|g^n - g^{n-1}\|_{L^\infty([0,T]; H_{ul}^{k,\ell}(\mathbb{R}^6))}^2.$$

Finally, choose  $T$  smaller if necessary so that

$$2^4 C_2 D_0^2 \kappa^{-1} T \leq \frac{1}{4}.$$

Then, we have proved that for any  $n \geq 1$ ,

$$(2.19) \quad \|g^{n+1} - g^n\|_Y \leq \frac{1}{2} \|g^n - g^{n-1}\|_Y.$$

Consequently,  $\{g^n\}$  is a convergence sequence in  $Y$ , and the limit

$$g^\varepsilon \in Y,$$

is therefore a non-negative solution of the Cauchy problem (2.7). The estimate (2.19) also implies the uniqueness of solutions.

By means of the mild form (2.11), it can be proved also that for each  $n$ ,

$$g^n \in C^0([0, T]; H_{ul}^{k,\ell}(\mathbb{R}^6))$$

and hence so is the limit  $g^\varepsilon$ . The non-negativity of  $g^\varepsilon$  follows because  $g^n \geq 0$ . Now the proof of Theorem 2.4 is completed.

### 3. UNIFORM ESTIMATE

We now prove the convergence of approximation sequence  $\{g^\varepsilon\}$  as  $\varepsilon \rightarrow 0$ . The first step is to prove the uniform boundedness of this approximation sequence. Below, the constant  $C$  are various constants independent of  $\varepsilon > 0$ .

**Theorem 3.1.** *Assume that  $0 < s < 1$ ,  $\gamma > -3/2$ ,  $\gamma + 2s < 1$ . Let  $g_0 \in H_{ul}^{k,\ell}(\mathbb{R}^6)$ ,  $g_0 \geq 0$  for some  $k \geq 4$ ,  $l \geq 3$ . Then there exists  $T_* \in ]0, T_0]$  depending on  $\|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}$  but not on  $\varepsilon$  such that if for some  $0 < T \leq T_0$ ,*

$$(3.1) \quad g^\varepsilon \in C^0([0, T]; H_{ul}^{k,\ell}(\mathbb{R}^6)) \cap \mathcal{M}^{k,\ell+1}([0, T] \times \mathbb{R}^6)$$

is a non-negative solution of the Cauchy problem (2.7) and if  $T_{**} = \min\{T, T_*\}$ , then it holds that

$$(3.2) \quad \|g^\varepsilon\|_{L^\infty(]0, T_{**}[; H_{ul}^{k,\ell}(\mathbb{R}^6))} \leq 2\|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}.$$

**Remark 3.2.** The case  $T_* \leq T$  gives a uniform estimate of local solutions on the fixed time interval  $[0, T_*]$  while the case  $T < T_*$  gives an a priori estimate on the existence time interval  $[0, T]$  of local solutions. The latter is used for the continuation argument of local solutions, in Subsection 4.4 below.

In the following,  $\rho > 0$ ,  $\kappa > 0$  are fixed. Furthermore, recall  $T_0 = \rho/(2\kappa)$ . We start with a solution  $g^\varepsilon$  subject to (3.1) for some  $T \in ]0, T_0]$ . Put

$$(3.3) \quad h_\ell^{\alpha,\beta} = \phi_1(x-a)W_\ell \partial_\beta^\alpha g^\varepsilon$$

and take the  $L^2$  inner product of it and the equation for it. As before  $\|\cdot\|$  and  $(\cdot, \cdot)$  stand for the  $L^2(\mathbb{R}^6)$  norm and inner product respectively unless otherwise stated. Then we have

$$(3.4) \quad \frac{1}{2} \frac{d}{dt} \|h_\ell^{\alpha,\beta}\|^2 + \kappa \|h_{\ell+1}^{\alpha,\beta}\|^2 = (\Xi, h_\ell^{\alpha,\beta}),$$

where

$$\begin{aligned} \Xi &= \phi_1(x-a)W_\ell \partial_\beta^\alpha \Gamma(g^\varepsilon, g^\varepsilon) - [\phi_1(x-a)W_\ell \partial_\beta^\alpha, v \cdot \nabla_x]g^\varepsilon - \kappa[\phi_1(x-a)W_\ell \partial_\beta^\alpha, \langle v \rangle^2]g^\varepsilon \\ &= \Xi_1 + \Xi_2 + \Xi_3. \end{aligned}$$

We shall derive the estimates

**Lemma 3.3.** Assume that  $0 < s < 1/2$ ,  $\gamma \geq -3/2$ ,  $\gamma + 2s < 1$ . Then,

$$\begin{aligned} |(\Xi_1, h_\ell^{\alpha,\beta})| &\leq C \|g^\varepsilon\|_{H_{ul}^{k,\ell}}^2 \sum_{|\alpha'+\beta'|\leq k} \|h_{\ell+1}^{\alpha',\beta'}\|, \\ |(\Xi_2 + \Xi_3, h_\ell^{\alpha,\beta})| &\leq C(1 + \kappa + \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}) \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)} \|h_{\ell+1}^{\alpha,\beta}\|. \end{aligned}$$

This lemma will be proved in Section 4. Now we have

$$|(\Xi, h_\ell^{\alpha,\beta})| \leq C((1 + \kappa)^2 + \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2) \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{\kappa}{k^3} \sum_{|\alpha'+\beta'|\leq k} \|h_{\ell+1}^{\alpha',\beta'}\|^2$$

whence (3.4) yields

$$\frac{d}{dt} \|h_\ell^{\alpha,\beta}(t)\|^2 + 2\kappa \|h_{\ell+1}^{\alpha,\beta}\|^2 \leq \frac{C}{\kappa} ((1 + \kappa)^2 + \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2) \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{\kappa}{k^3} \sum_{|\alpha'+\beta'|\leq k} \|h_{\ell+1}^{\alpha',\beta'}\|^2,$$

and after integrating over  $]0, t[$ ,

$$\begin{aligned} &\|h_\ell^{\alpha,\beta}(t)\|^2 + \kappa \int_0^t \|h_{\ell+1}^{\alpha,\beta}(\tau)\|^2 d\tau \\ &\leq \|g^{\alpha,\beta}(0)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{C}{\kappa} \int_0^t ((1 + \kappa)^2 + \|g^\varepsilon(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2) \|g^\varepsilon(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 d\tau \\ &\quad + \frac{\kappa}{k^3} \sum_{|\alpha'+\beta'|\leq k} \int_0^t \|h_{\ell+1}^{\alpha',\beta'}(\tau)\|^2 d\tau. \end{aligned}$$

Take the supremum with respect to  $a \in \mathbb{R}^3$  (see (3.3)) and sum up over  $|\alpha + \beta| \leq k$  to deduce that

$$\begin{aligned} & \|g^\varepsilon(t)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \kappa \|g^\varepsilon\|_{\mathcal{M}^{k,\ell+1}([0,t] \times \mathbb{R}^6)}^2 \\ & \leq \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 + \frac{C}{\kappa} \int_0^t (1 + \|g^\varepsilon(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)})^2 \|g^\varepsilon(\tau)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 d\tau. \end{aligned}$$

Then the Gronwall type inequality gives for  $C_\kappa = C/\kappa$ ,

$$\|g^\varepsilon(t)\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \leq \frac{\|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 e^{C_\kappa t}}{1 - (e^{C_\kappa t} - 1) \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2},$$

as long as the denominator remains positive. We choose  $T_* > 0$  small enough such that

$$\frac{e^{C_\kappa T_*}}{1 - (e^{C_\kappa T_*} - 1) \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2} = 4.$$

Then

$$T_* = \frac{1}{C_\kappa} \log \left( 1 + \frac{3}{1 + 4 \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2} \right)$$

is independent of  $\varepsilon > 0$ , but depends on  $\|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}$  and the constant  $C$  which depends on  $\rho, \kappa, k$  and  $l$ . Now we have (3.2) for  $T_{**} = \min(T, T_*)$ .

From (3.2) and (3.4), we get also, for  $\kappa > 0$ ,

$$\kappa \|g^\varepsilon\|_{\mathcal{M}^{k,\ell+1}([0, T_{**}] \times \mathbb{R}^6)}^2 \leq 2 \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \left( 1 + 2CT_* (1 + 2 \|g_0\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2) \right).$$

We have proved Theorem 3.1.

Combine Theorems 2.4 and 3.1 and use the compactness argument as in Section 4.4 of [3], to conclude the existence part of Theorem 2.1. The uniqueness part comes from Theorem 4.1 of [8]. Now the main theorem 1.1 is proved by the help of Theorem 2.1, in the same manner as in Section 4.5 of [3].

#### 4. PROOF OF LEMMA 3.3

In the sequel, the notation  $A \lesssim B$  means that there is a constant  $C$  independent of  $A, B$  such that  $A \leq CB$ , and similarly for  $A \gtrsim B$ . We start with

4.1. **Estimate of  $\Xi_1$ .** Notice that

$$\begin{aligned} \Xi_1 &= \phi_1(x-a) W_\ell \partial_\beta^\alpha \Gamma(g^\varepsilon, g^\varepsilon) \\ &= \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 + \beta_3 = \beta}} C_{\beta_1, \beta_2, \beta_3}^{\alpha_1, \alpha_2} W_\ell \mathcal{T}_\varepsilon(\phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon, \phi_1(x-a) \partial_{\beta_1}^{\alpha_1} g^\varepsilon, \partial_{\beta_3} \mu(t)). \end{aligned}$$

Put

$$F = \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon, \quad G = \phi_1(x-a) \partial_{\beta_1}^{\alpha_1} g^\varepsilon, \quad M = \partial_{\beta_3} \mu,$$

and write

$$\begin{aligned} W_\ell \mathcal{T}_\varepsilon(F, G, M) &= W_\ell Q(MF, G) + W_\ell \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(M_* - M'_*) F'_* G' dv_* d\sigma \\ &= A_1 + A_2. \end{aligned}$$

**Estimate of  $A_1$ :**

(1) The case  $\alpha_1 = \alpha, \beta_1 = \beta$ : Notice that then  $\alpha_2 = \beta_2 = \beta_3 = 0$ . Write

$$A_1 = Q(MF, W_\ell G) + [W_\ell Q(MF, G) - Q(MF, W_\ell G)] = A_{10} + A_{11}.$$

Here,  $F = \phi_2(x-a)g^\varepsilon$ ,  $W_\ell G = h_\ell^{\alpha, \beta}$ , and  $M = \mu$ . We start with

**Lemma 4.1.** *It holds that*

$$(A_{10}, h_\ell^{\alpha, \beta}) = -\frac{1}{2}D + A_{101}$$

where

$$(4.5) \quad D = \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B(\mu F)_* ((h')_\ell^{\alpha, \beta} - h_\ell^{\alpha, \beta})^2 dx dv dv_* d\sigma \geq 0$$

while  $A_{101}$  enjoys the estimate that for  $0 < s < 1$  and  $-3/2 < \gamma \leq 1$ , and for any  $k \geq 2$  and  $\ell \geq 1$ ,

$$|A_{101}| \lesssim \|g^\varepsilon\|_{H_{ul}^{k, \ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha, \beta}\|.$$

*Proof.* Put  $H = h_\ell^{\alpha, \beta}$ . Then,

$$\begin{aligned} (A_{10}, h_\ell^{\alpha, \beta}) &= (Q(\mu F, W_\ell G), H) = (Q(\mu F, H), H) \\ &= \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B(\mu F)_* H(H' - H) dx dv dv_* d\sigma \\ &= -\frac{1}{2} \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B(\mu F)_* (H' - H)^2 dx dv dv_* d\sigma \\ &\quad + \frac{1}{2} \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B(\mu F)_* [(H')^2 - (H)^2] dx dv dv_* d\sigma \\ &= -\frac{1}{2}D + A_{101}. \end{aligned}$$

Thanks to the cancellation lemma in [2] we get with  $S(z) \lesssim |z|^\gamma$  for the inverse power potential ,

$$|A_{101}| \lesssim \iint_{\mathbb{R}^6} |\mu F(v_*)| |S(v_*) *_{\nu} H^2| dv_* dx.$$

Since

$$(4.6) \quad \int_{\mathbb{R}^3} |v - v_*|^\gamma (\mu F)_* dv_* \lesssim \langle v \rangle^\gamma \|F\|_{L_v^2}$$

holds true for  $\gamma > -3/2$ , we now have

$$|A_{101}| \lesssim \int_{\mathbb{R}^3} \|F\|_{L_v^2} \|H\|_{L_v^2} \|H\|_{L_{\gamma, v}^2} dx \lesssim \|F\|_{H_x^2(L_v^2)} \|H\| \|W_\gamma H\|$$

Hence since  $\gamma \leq 1$  is assumed, we get

$$|A_{101}| \lesssim \|\phi_2(x-a)g^\varepsilon\|_{H^2(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \|h_\ell^{\alpha, \beta}\| \|h_{\ell+1}^{\alpha, \beta}\| \lesssim \|g^\varepsilon\|_{H_{ul}^{k, \ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha, \beta}\|,$$

which proves the lemma.  $\square$

The estimate of  $A_{11}$  is stated as follows.

**Lemma 4.2.** *Let  $0 < s < 1$  and  $-5/2 < \gamma \leq 1$ . Then, for  $k \geq 0, \ell \geq 2$ ,*

$$\left| (A_{11}, h_\ell^{\alpha, \beta})_{L^2(\mathbb{R}^6)} \right| \leq \frac{1}{4}D + C \|g^\varepsilon\|_{H_{ul}^{k, \ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha, \beta}\|,$$

where  $D$  is defined by (4.5).

*Proof.* With  $W_\ell G = h_\ell^{\alpha, \beta} = H$ ,

$$\begin{aligned}
& (W_\ell Q(MF, G) - Q(MF, W_\ell G), H)_{L^2(\mathbb{R}_v^3)} \\
&= \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} B(W'_\ell - W_\ell)(\mu F)_* GH' dv dv_* d\sigma \\
&= \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} B(W'_\ell - W_\ell)(\mu F)_* GH dv dv_* d\sigma \\
&\quad + \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} B(W'_\ell - W_\ell)(\mu F)_* G(H' - H) dv dv_* d\sigma \\
&= K_1 + K_2.
\end{aligned}$$

For  $K_1$ , we use the Taylor formula of second order

$$W'_\ell - W_\ell = \nabla(W_\ell) \cdot (v' - v) + \frac{1}{2} \int_0^1 \nabla^2(W_\ell(v_\tau)) d\tau (v' - v)^{\otimes 2}$$

and write  $K_1 = K_{1,1} + K_{1,2}$  in terms of this decomposition. Note that

$$v' - v = \frac{|v - v_*|}{2} (\sigma - (k \cdot \sigma)k) + \frac{|v - v_*|}{2} (k \cdot \sigma - 1)k,$$

where  $k = (v - v_*)/|v - v_*|$ . Then, since it follows from the symmetry that the integral corresponding to the first term vanishes, we have, using (4.6),

$$\begin{aligned}
|K_{1,1}| &= \left| \iiint b(\cos \theta) |v - v_*|^{\gamma+1} (1 - \cos \theta) (\nabla(W_\ell) \cdot k) (\mu F)_* GH dv dv_* d\sigma \right| \\
&\lesssim \int \left\{ \int |v - v_*|^{\gamma+1} (\mu F)_* dv_* \right\} |W_{\ell-1} GH| dv \\
&\lesssim \|F\|_{L_v^2} \int W_{\ell+\gamma} |GH| dv \\
&\lesssim \|F\|_{L_v^2} \|W_\ell G\|_{L_v^2} \|W_\gamma H\|_{L_v^2}.
\end{aligned}$$

Next, let  $\ell \geq 2$ . Then, since the energy conservation property  $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$  implies

$$|(\nabla^2 W_\ell)(v_\tau)| \lesssim W'_{\ell-2} + W_{\ell-2} \lesssim (W_{\ell-2})_* + W_{\ell-2} \lesssim (W_{\ell-2})_* W_{\ell-2},$$

and again using (4.6), we get

$$\begin{aligned}
|K_{1,2}| &\lesssim \left| \iiint b(\cos \theta) \theta^2 |v - v_*|^{\gamma+2} ((W_{\ell-2})_* W_{\ell-2}) (\mu F)_* GH dv dv_* d\sigma \right| \\
&\lesssim \int \left\{ \int |v - v_*|^{\gamma+2} (W_{\ell-2} \mu F)_* dv_* \right\} |W_{\ell-2} GH| dv \\
&\lesssim \|F\|_{L_v^2} \int W_{\ell+\gamma} |GH| dv \\
&\lesssim \|F\|_{L_v^2} \|W_\ell G\|_{L_v^2} \|W_\gamma H\|_{L_v^2}.
\end{aligned}$$

Thus, by a similar computation of  $A_{101}$ , we conclude

$$\begin{aligned}
\int |K_1| dx &\leq \int |K_{1,1}| dx + \int |K_{1,2}| dx \\
&\lesssim \|F\|_{H_x^2(L_v^2)} \|W_\ell G\| \|W_\gamma H\| \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha,\beta}\|.
\end{aligned}$$

We shall estimate  $K_2$ . By the Cauchy-Schwarz inequality and again by (4.6),

$$\begin{aligned}
\left(\int |K_2| dx\right)^2 &\leq \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} B(\mu F)_* |H' - H|^2 dv dv_* d\sigma dx \\
&\quad \times \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} B(W'_\ell - W_\ell)^2 (\mu F)_* G^2 dv dv_* d\sigma dx \\
&\lesssim D \left( \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} b(\cos \theta) \theta^2 |v - v_*|^{\gamma+2} (W_{\ell-1}^2 \mu F)_* (W_{\ell-1} G)^2 dv dv_* d\sigma dx \right) \\
&\lesssim D \int \left( \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} (W_{\ell-1}^2 \mu F)_* dv_* \right) (W_{\ell-1} G)^2 dv dx \\
&\lesssim D \|F\|_{H_x^2(L_v^2)} \|W_\ell G\| \|W_{\ell+\gamma} G\| \\
&\lesssim D \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha,\beta}\|
\end{aligned}$$

where  $D$  is as in (4.5). We now obtain

$$\int |K_2| dx \leq \frac{1}{4} D + C \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha,\beta}\|.$$

This ends the proof of the lemma.  $\square$

**Estimate of  $A_1$  continued: (2) The case  $|\alpha_1 + \beta_1| \leq k - 1$ .** We shall establish

**Lemma 4.3.** *Let  $0 < s < 1/2$  and  $\gamma > \max\{-3, -2s - 3/2\}$ . Then, for  $k \geq 4, \ell \geq 0$  and for  $|\alpha_1 + \beta_1| \leq k - 1$ ,*

$$|(A_1, h_\ell^{\alpha,\beta})| \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}}^2 \|h_{\ell+1}^{\alpha,\beta}\|.$$

The proof is based on the following upper bound estimate of  $Q$  established in Proposition 2.9 of [8].

**Proposition 4.4.** *Let  $0 < s < 1$  and  $\gamma > \max\{-3, -2s - 3/2\}$ . Then for any  $\ell \in \mathbb{R}$  and  $m \in [s - 1, s]$ ,*

$$\left| (Q(f, g), h) \right|_{L_v^2} \lesssim \left( \|f\|_{L_{\ell^+ + (\gamma+2s)^+}^1} + \|f\|_{L^2} \right) \|g\|_{H_{\ell^+ + (\gamma+2s)^+}^{\max\{s+m, (2s-1+\varepsilon)\}}} \|h\|_{H_{-\ell}^{s-m}},$$

for any  $\varepsilon > 0$ .

Admit this and put

$$F = \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon, \quad G = \phi_1(x-a) \partial_{\beta_1}^{\alpha_1} g^\varepsilon, \quad M = \partial_{\beta_3} \mu.$$

Recall the norm and inner product of  $L^2(\mathbb{R}^6)$  are denoted by  $\|\cdot\|$  and  $(\cdot, \cdot)$  respectively. Use the above theorem for  $m = s$  and  $\alpha = \ell - 1$  to deduce

$$\begin{aligned}
|(A_1, h_\ell^{\alpha,\beta})| &= |W_\ell Q(MF, G), h_\ell^{\alpha,\beta}| \\
&\lesssim \int_{\mathbb{R}^3} \left( \|MF\|_{L_{\ell-1+(\gamma+2s)^+}^1} + \|MF\|_{L_{\ell-2}^2} \right) \|G\|_{H_{\ell-1+(\gamma+2s)^+}^{2s}} \|W_\ell h_\ell^{\alpha,\beta}\|_{L_{-\ell+1}^2} dx \\
&\lesssim \int_{\mathbb{R}^3} \|F\|_{L_v^2} \|G\|_{H_{\ell^s}^2(\mathbb{R}_v^3)} \|h_{\ell+1}^{\alpha,\beta}\|_{L_v^2} dx
\end{aligned}$$

where  $\gamma + 2s < 1$  is assumed.

Suppose  $|\alpha_2 + \beta_2| \leq 2$ . Then,

$$\begin{aligned}
|(A_1, h_\ell^{\alpha,\beta})| &\lesssim \|\phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{H_x^2(L_v^2)} \|\phi_1(x-a) W_\ell \partial_{\beta_1}^{\alpha_1} g^\varepsilon\|_{L_x^2(H_v^{2s})} \|h_{\ell+1}^{\alpha,\beta}\| \\
&\lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}}^2 \|h_{\ell+1}^{\alpha,\beta}\|,
\end{aligned}$$

where  $|\alpha_1 + \beta_1| + 2s \leq |\alpha_1 + \beta_1| + 1 \leq k$  was taken into account.

On the other hand let  $|\alpha_2 + \beta_2| > 2$ . Then  $|\alpha_1 + \beta_1| \leq k - 1 - |\alpha_2 + \beta_2| \leq k - 4$  holds and

$$\begin{aligned} |(A_1, h_\ell^{\alpha, \beta})| &\lesssim \|\phi_2(x-a)\partial_{\beta_2}^{\alpha_2} g^\varepsilon\| \|\phi_1(x-a)W_\ell \partial_{\beta_1}^{\alpha_1} g^\varepsilon\|_{H_x^2(H_v^{2s})} \|h_{\ell+1}^{\alpha, \beta}\| \\ &\lesssim \|g^\varepsilon\|_{H_{ul}^{k, \ell}}^2 \|h_{\ell+1}^{\alpha, \beta}\|. \end{aligned}$$

Now the proof of Lemma 4.3 is complete.

**Estimate of  $A_2$ .** We shall prove

**Lemma 4.5.** *Let  $0 < s < 1/2$ ,  $\gamma > -3/2$ , and  $2s + \gamma < 1$ . Then for  $k > 3, \ell > 5/2$*

$$|(A_2, h_\ell^{\alpha, \beta})| \lesssim \|g^\varepsilon\|_{H_{ul}^{k, \ell}(\mathbb{R}^6)}.$$

*Proof.* Firstly, we write

$$\begin{aligned} A_2 &= W_\ell \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(M_* - M'_*) F'_* G' dv_* d\sigma \\ &= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(W_\ell - W'_\ell)(M_* - M'_*) F'_* G' dv_* d\sigma \\ &\quad + \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(M_* - M'_*) F'_*(W_\ell G)' dv_* d\sigma \end{aligned}$$

and hence, putting  $H = h_\ell^{\alpha, \beta}$ ,

$$\begin{aligned} |(A_2, h_\ell^{\alpha, \beta})| &\leq \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B|W_\ell - W'_\ell| |M_* - M'_*| |F'_*| |G'| |H| dv dv_* d\sigma dx \\ &\quad + \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B|M_* - M'_*| |F'_*| |(W_\ell G)'| |H| dv dv_* d\sigma dx \\ &= A_{21} + A_{22}. \end{aligned}$$

If  $\ell \geq 1$ ,

$$\begin{aligned} |W_\ell - W'_\ell| &\lesssim |v - v'| |W_{\ell-1} + W'_{\ell-1}| \lesssim |v' - v'_*| \theta |(W_{\ell-1})'_* + W'_{\ell-1}| \\ &\lesssim \theta |(W_\ell)'_* + W'_\ell| \lesssim \theta (W_\ell)'_* W'_\ell, \end{aligned}$$

and knowing that  $|M| \lesssim \mu(v)^{1/2}$ , we get

$$\begin{aligned} A_{21} &\leq \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} b(\cos \theta) \theta |v - v_*|^\gamma \mu_*^{1/2} |(W_\ell F)'_*| |(W_\ell G)'| |H| dv dv_* d\sigma dx \\ &\quad + \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} b(\cos \theta) \theta |v - v_*|^\gamma |(\mu^{1/2} W_\ell F)'_*| |(W_\ell G)'| |H| dv dv_* d\sigma dx \\ &= A_{211} + A_{212}. \end{aligned}$$

By the Schwarz inequality

$$\begin{aligned} A_{211}^2 &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-1-2s} |v - v_*|^{2\gamma} \mu_* |H|^2 dv dv_* d\sigma dx \\ &\quad \times \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-1-2s} |(W_\ell F)'_*|^2 |(W_\ell G)'|^2 dv dv_* d\sigma dx \\ &= J_1 J_2. \end{aligned}$$

Using (4.6) yields

$$J_1 \lesssim \iint \langle v \rangle^{2\gamma} |H|^2 dv dx \lesssim \|W_\gamma H\|^2 = \|h_{\ell+\gamma}^{\alpha,\beta}\|^2 \leq \|h_{\ell+1}^{\alpha,\beta}\|^2,$$

while by the change of variables  $(v, v_*) \rightarrow (v', v'_*)$

$$\begin{aligned} J_2 &\lesssim \iint |(W_\ell F)'_*|^2 |(W_\ell G)'|^2 dv dv_* dx \\ &= \int \|W_\ell F\|_{L_v^2}^2 \|W_\ell G\|_{L_v^2}^2 dx \\ &= \int \|W_\ell \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2}^2 \|h_\ell^{\alpha,\beta}\|_{L_v^2}^2 dx. \end{aligned}$$

Then, if  $|\alpha_1 + \beta_1| \leq 2$ ,

$$J_2 \lesssim \|W_\ell \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|^2 \|h_\ell^{\alpha,\beta}\|_{H_x^2(L_v^2)}^2 \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^3)}^4,$$

while if  $|\alpha_1 + \beta_1| > 2$  so that  $|\alpha_2 + \beta_2| \leq k - |\alpha_1 + \beta_1| \leq k - 2$ ,

$$J_2 \lesssim \|W_\ell \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{H_x^2(L_v^2)}^2 \|h_\ell^{\alpha,\beta}\|^2 \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^3)}^4.$$

In conclusion, we obtained

$$A_{211}^2 \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \|h_{\ell+1}^{\alpha,\beta}\|.$$

We turn to  $A_{212}$ . Notice by the change of variables  $(v, v_*) \rightarrow (v', v'_*)$  and by the Cauchy-Schwarz inequality and (4.6) that for  $\gamma > -3/2$ ,

$$\begin{aligned} A_{212} &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-1-2s} |v' - v'_*|^\gamma |(\mu^{1/2} W_\ell F)'_*| |(W_\ell G)'| |H| dv dv_* d\sigma dx \\ &= \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} \theta^{-1-2s} |v - v_*|^\gamma |(\mu^{1/2} W_\ell F)_*| |(W_\ell G)| |H'| dv dv_* d\sigma dx \\ &= \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-1-2s} \left( \int |v - v_*|^\gamma |(\mu^{1/2} W_\ell F)_*| dv_* \right) |(W_\ell G)| |H'| dv d\sigma dx \\ &\lesssim \int \|F\|_{L_v^2} \left\{ \iiint_{\mathbb{R}^3 \times \mathbb{S}^2} \theta^{-1-2s} \langle v \rangle^\gamma |(W_\ell G)| |H'| dv d\sigma \right\} dx. \end{aligned}$$

By the regular change of variable  $v \rightarrow v'$ ,

$$A_{212} \lesssim \int \|\phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2} \|h_{\ell+\gamma}^{\alpha_1,\beta_1}\|_{L_v^2} \|h_\ell^{\alpha,\beta}\|_{L_v^2} dx,$$

the last integral of which is bounded, by the Sobolev embedding, by

$$\|\phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{H_x^2(L_v^2)} \|h_{\ell+\gamma}^{\alpha_1,\beta_1}\| \|h_\ell^{\alpha,\beta}\| \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \|h_{\ell+\gamma}^{\alpha_1,\beta_1}\|$$

when  $|\alpha_2 + \beta_2| \leq 2$ , and by

$$\|\phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\| \|h_{\ell+\gamma}^{\alpha_1,\beta_1}\|_{H_x^2(L_v^2)} \|h_\ell^{\alpha,\beta}\| \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \sum_{|\alpha'+\beta'|\leq k} \|h_{\ell+\gamma}^{\alpha',\beta'}\|$$

when  $|\alpha_2 + \beta_2| > 2$  for which  $|\alpha_1 + \beta_1| \leq k - |\alpha_2 + \beta_2| \leq k - 2$ . Thus we obtained

$$|A_{21}| \lesssim \|g^\varepsilon\|_{H_{ul}^{k,\ell}(\mathbb{R}^6)}^2 \sum_{|\alpha'+\beta'|\leq k} \|h_{\ell+\gamma}^{\alpha',\beta'}\|.$$

In remains to evaluate  $A_{22}$ . First, an interpolation of the Taylor formula and the boundness  $|M| \leq C$  yields

$$|M_* - M'_*| \leq C|v_* - v'_*|^\lambda \leq C\theta^\lambda |v - v_*|^\lambda \leq C\theta^\lambda |v' - v'_*|^\lambda.$$

Since  $s \in (0, 1/2)$  and  $\gamma + 2s < 1$  are assumed, there is  $\lambda \in (0, 1)$  such that  $\lambda > 2s$ ,  $\gamma + \lambda < 1$ . Therefore after the change of variable  $(v, v_*) \rightarrow (v', v'_*)$ ,

$$\begin{aligned} A_{22} &= \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} B|M_* - M'_*|F'_*|(W_\ell G)'| |H| dv dv_* d\sigma dx \\ &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |v' - v'_*|^{\gamma+\lambda} |F'_*|(W_\ell G)'| |H| dv dv_* d\sigma dx \\ &= \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |v - v_*|^{\gamma+\lambda} |F_*|(W_\ell G)| |H'| dv dv_* d\sigma dx \end{aligned}$$

We shall check the two cases.

(1) The case  $\gamma + \lambda \geq 0$ : Then, noting  $|v - v_*|^{\gamma+\lambda} \lesssim \langle v \rangle^{\gamma+\lambda} \langle v_* \rangle^{\gamma+\lambda}$  and using the regular change of variable  $v \rightarrow v'$ ,

$$\begin{aligned} A_{22} &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |(W_{\gamma+\lambda} F)_*| |W_{\ell+\gamma+\lambda} G| |H'| dv dv_* d\sigma dx \\ &\lesssim \int \|W_1 F\|_{L_v^1} \left\{ \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |W_{\ell+1} G| |H'| dv d\sigma \right\} dx \\ &\lesssim \int \|W_{\ell_0} F\|_{L_v^2} \|W_{\ell+1} G\|_{L_v^2} \|H\|_{L_v^2} dx \\ &= \int \|W_{\ell_0} \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2} \|h_{\ell+1}^{\alpha_1, \beta_1}\|_{L_v^2} \|h_\ell^{\alpha, \beta}\|_{L_v^2} dx \end{aligned}$$

for  $\ell_0 > 1 + 3/2 = 5/2$ . By the Sobolev embedding applied separately as before to the cases  $|\alpha_1 + \beta_1| \leq 2$  and  $|\alpha_1 + \beta_1| > 2$ , We finally obtain

$$A_{22} \lesssim \|g_\ell^\varepsilon\|_{H_{ul}^{k, \ell}}^2 \sum_{|\alpha' + \beta'| \leq k} \|h_{\ell+1}^{\alpha', \beta'}\|.$$

(2) The case  $\gamma + \lambda < 0$ : We shall use the following split of integral,

$$\begin{aligned} A_{22} &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |v - v_*|^{\gamma+\lambda} |F_*|(W_\ell G)| |H'| dv dv_* d\sigma dx \\ &= \iiint_{|v-v_*| \geq 1} + \iiint_{|v-v_*| < 1} = A_{221} + A_{222}. \end{aligned}$$

The integral  $A_{221}$ , since  $|v - v_*|^{\gamma+\lambda} \leq 1$  holds, can be reduced to a special case of (1) with  $\gamma + \lambda = 0$ , implying

$$\begin{aligned} A_{221} &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |F_*|(W_\ell G)| |H'| dv dv_* d\sigma dx \\ &\lesssim \int \|F\|_{L_v^1} \left\{ \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |W_\ell G| |H'| dv d\sigma \right\} dx \\ &\lesssim \int \|W_{\ell_1} \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2} \|h_\ell^{\alpha_1, \beta_1}\|_{L_v^2} \|h_\ell^{\alpha, \beta}\|_{L_v^2} dx \\ &\lesssim \int \|W_{\ell_1} \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2} \|h_\ell^{\alpha_1, \beta_1}\|_{L_v^2} \|h_{\ell+1}^{\alpha, \beta}\|_{L_v^2} dx \end{aligned}$$

for  $\ell_1 > 3/2$ , and hence by the Sobolev embedding,

$$A_{221} \lesssim \|g_\ell^\varepsilon\|_{H_{ul}^{k, \ell}}^2 \|h_{\ell+1}^{\alpha, \beta}\|.$$

On the other hand

$$\begin{aligned} A_{222} &\lesssim \iiint_{\mathbb{R}^9 \times \mathbb{S}^2, |v-v_*|<1} \theta^{-2-2s+\lambda} |v-v_*|^{\gamma+\lambda} |F_*| |(W_\ell G)| |H'| dv dv_* d\sigma dx \\ &\lesssim \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} \left\{ \int_{\mathbb{R}^3, |v-v_*|<1} |v-v_*|^{\gamma+\lambda} |F_*| dv_* \right\} |(W_\ell G)| |H'| dv d\sigma dx. \end{aligned}$$

Clearly, if  $\gamma + \lambda > -3/2$ , by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\int_{\mathbb{R}^3, |v-v_*|<1} |v-v_*|^{\gamma+\lambda} |F_*| dv_* \\ &\leq \left( \int_{\mathbb{R}^3, |v-v_*|<1} |v-v_*|^{2(\gamma+\lambda)} dv_* \right)^{1/2} \|F\|_{L_v^1} \lesssim \|F\|_{L_v^1}, \end{aligned}$$

so that we get again by the Sobolev embedding

$$\begin{aligned} A_{222} &\lesssim \iiint_{\mathbb{R}^6 \times \mathbb{S}^2} \theta^{-2-2s+\lambda} |F|_{L_v^1} |W_\ell G| |H'| dv d\sigma dx \\ &\lesssim \int |W_{\ell_1} F|_{L_v^2} |W_\ell G|_{L_v^2} |H|_{L_v^2} dx \\ &\leq \int \|W_{\ell_1} \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2} \|h_\ell^{\alpha_1, \beta_1}\|_{L_v^2} \|h_\ell^{\alpha, \beta}\|_{L_v^2} dx \\ &\leq \int \|W_{\ell_1} \phi_2(x-a) \partial_{\beta_2}^{\alpha_2} g^\varepsilon\|_{L_v^2} \|h_\ell^{\alpha_1, \beta_1}\|_{L_v^2} \|h_{\ell+1}^{\alpha, \beta}\|_{L_v^2} dx \\ &\lesssim \|g_\ell^\varepsilon\|_{H_{ul}^{k, \ell}}^2 \|h_{\ell+1}^{\alpha, \beta}\|. \end{aligned}$$

Consequently, we have for both positive and negative  $\gamma + \lambda$

$$A_{22} \lesssim \|g_\ell^\varepsilon\|_{H_{ul}^{k, \ell}}^2 \sum_{|\alpha' + \beta'| \leq k} \|h_{\ell+1}^{\alpha', \beta'}\|.$$

Combine the above estimates to conclude the proof of Lemma 4.5.  $\square$

This completes the proof of the part of Lemma 3.3 for  $\Xi_1$ .

**4.2. Estimate of  $\Xi_2$ .** Since  $|\nabla \phi_1(x)| \leq C\phi_2(x)$ , we have

$$\begin{aligned} \Xi_2 &= [\phi_1(x-a) W_\ell \partial_\beta^\alpha, v \cdot \nabla_x] g^\varepsilon \\ &= \phi_1(x-a) W_\ell \nabla_x \partial_{\beta-1}^\alpha g^\varepsilon - (v \cdot \nabla_x \phi_1(x-a)) |W_\ell \partial_\beta^\alpha g^\varepsilon|, \\ |\Xi_2| &\leq \phi_1(x-a) W_\ell |\nabla_x \partial_{\beta-1}^\alpha g^\varepsilon| + \phi_2(x-a) W_{\ell+1} |\partial_\beta^\alpha g^\varepsilon|, \end{aligned}$$

whence

$$\begin{aligned} |(\Xi_2, h_\ell^{\alpha, \beta})| &\leq (\|\phi_1(x-a) W_{\ell-1} \nabla_x \partial_{\beta-1}^\alpha g^\varepsilon\| + \|\phi_2(x-a) W_\ell \partial_\beta^\alpha g^\varepsilon\|) \|h_{\ell+1}^{\alpha, \beta}\| \\ &\lesssim \|g^\varepsilon\|_{H_{ul}^{k, \ell}} \|h_{\ell+1}^{\alpha, \beta}\|, \end{aligned}$$

which implies Lemma 3.3 for  $\Xi_2$ .

4.3. **Estimate of  $\Xi_3$ .** It holds that

$$\begin{aligned} |\Xi_3| &\leq |\kappa[\phi_1(x-a)W_\ell\partial_\beta^\alpha, \langle v \rangle^2]g^\varepsilon| \\ &\lesssim \kappa\phi_1(x-a)W_\ell(|v|\partial_{\beta-1}^\alpha g^\varepsilon + |\partial_{\beta-2}^\alpha g^\varepsilon| \end{aligned}$$

whence follows

$$\begin{aligned} |(\Xi_3, h_\ell^{\alpha,\beta})| &\lesssim \kappa(\|\phi_1(x-a)W_{\ell-1}|v|\partial_{\beta-1}^\alpha g^\varepsilon\| + \|\phi_1(x-a)W_{\ell-1}\partial_{\beta-2}^\alpha g^\varepsilon\|) \|h_{\ell+1}^{\alpha,\beta}\| \\ &\lesssim \kappa\|g^\varepsilon\|_{H_{ul}^{k,\ell}} \|h_{\ell+1}^{\alpha,\beta}\|. \end{aligned}$$

This proves the part of Lemma 3.3 for  $\Xi_3$ , and hence we are done.

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