Methods and Applications of Analysis
4 (3) 1997, pp. 339–355

ISSN 1073–2772

UNIFORM L_1 BOUNDEDNESS OF SOLUTIONS OF HYPERBOLIC CONSERVATION LAWS

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ABSTRACT. In this paper, we study the L_1 stability of perturbation of constant states for 2×2 systems of conservation laws. We introduce a nonlinear functional which is equivalent to the L_1 norm of the difference between the constant state and the approximate solutions consisting of elementary waves and is non-increasing in time for the limiting weak solution. This yields the L_1 stability of the constant states. The approximate solutions are constructed with the aid of wave tracing for the random choice method. Our functional reveals the aspects of nonlinear wave behavior, particularly the coupling of waves pertaining to different characteristic families, which affects the L_1 norm of the solutions.

1. Introduction

Consider the initial-value problem for the 2×2 system of conservation laws,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad t \geq 0, \ -\infty < x < \infty, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty, \tag{1.2}$$

where u and $f(u)$ are 2-vectors. We assume that the system is strictly hyperbolic, i.e., the matrix $\partial f(u)/\partial u$ has real and distinct eigenvalues $\lambda_1(u) < \lambda_2(u)$ for all u under consideration, with the corresponding right eigenvectors $r_i(u)$, $i = 1, 2$. Each characteristic field is assumed to be either linearly degenerate or genuinely nonlinear (see Lax [6]), i.e., $r_i(u) \cdot \nabla\lambda_i(u) \equiv 0$ or $r_i(u) \cdot \nabla\lambda_i(u) \neq 0$, $i = 1, 2$.

The purpose of the present paper is to study the L_1 boundedness of solutions of the initial-value problem (1.1) and (1.2). Our approach is to find a nonlinear functional, which is equivalent to the L_1 norm of the weak solutions and is non-increasing in time. This functional reveals the aspects of the nonlinear wave behavior which affects the L_1 norm of the weak solution.

Our analysis is based on the approximation of general solutions by the solutions of Riemann problems through the random choice method, Glimm [3], and the wave tracing method, Liu [7]. In Section 2, we describe a slightly modified version of the wave tracing scheme and the analysis of wave interaction and cancellation. In each small time strip in the wave tracing scheme, the approximate solutions are reduced to a linear superposition of nonlinear waves propagating with constant strengths and speeds. Therefore, the family of these simplified approximate solutions in the small time strip can be viewed as a linear superposition of a family of step functions. A

Received February 18, 1997, revised May 29, 1997.

1991 *Mathematics Subject Classification*: 35L65.

Key words and phrases: L_1 boundedness, hyperbolic conservation laws, weak solutions, Glimm's scheme.

functional for the original approximate solutions is introduced and is equivalent to the L_1 norm of these approximate solutions. The functional depends on the wave location and strengths at a given time t. We show that there exists a corresponding functional for the simplified approximate, linearly superposed solutions. The later functional is shown to be the sum of a non-increasing functional plus an error term in each small time strip. The total sum of the error terms for any fixed time T will be shown to approach zero as the grid size goes to zero. Hence by comparing these two functionals across a set containing a countable number of times, it follows that the first functional is also a sum of a non-increasing functional and has an error which approaches zero for any fixed time T as the approximate solutions tend to the weak solution. Since the functional is equivalent to the L_1 norm of the difference between the approximate solutions and the constant state at any time t, the L_1 stability of the constant state follows.

In recent years, there has been much progress on the well-posedness of the weak solutions for systems of hyperbolic conservation laws. In [1], Bressan et al. develop a new algorithm for 2×2 systems based on the wave front tracking method, which yields a Cauchy sequence of approximate solutions converging to the exact weak solution. By homotopically deforming one solution to the other, they construct a Riemann semigroup and show that the limit functions depend continuously on the initial data in the L_1 norm. In [2], the result is extended to a general $n \times n$ system of hyperbolic conservation laws which is either genuinely nonlinear or linearly degenerate. Our new approach is to find a non-increasing functional which depends only on the wave strengths at any time t and is equivalent to the L_1 norm for the difference between the weak solutions. This would yield some understanding of the direct effect of nonlinear waves coupling on the $L_1(x)$ difference of two solutions. We have succeeded for 2×2 conservation laws when one of the solutions is a constant state. The general situation is being pursued by the authors.

2. Wave tracing

The following theorems hold for general $n \times n$ systems (1.1), but we state them only for the 2×2 system.

Choose any mesh lengths r and s, $\frac{r}{s}$ bounded, which satisfies the Courant-Friedrich-Lewy condition:

$$\frac{r}{s} \geq \sup_{i=1,2} |\lambda_i(u)|,$$

for all u under consideration. The approximate solution $u_r(x,t;a_m)$ is constructed inductively according to a prechosen random sequence $\{a_m\}_{m=1}^{\infty}$, $0 < a_m < 1$, in the following way: Set

$$u_r(x,0;a_m) = u_0(ir) \qquad \text{for } ir < x < (i+1)r.$$

Suppose that $u_r(x,t;a_m)$ is defined for $t < js$. Then, we set

$$u_r(x,js;a_m) = u_r((i+1+a_j)r - 0, js - 0; a_m), \qquad ir < x < (i+1)r$$

for any $i = 0,1,2,\ldots$. Thus, $u_r(x,js;a_m)$ is a step function of x with possible discontinuities at (ir,js). We then define $u_r(x,t;a_m)$, $js < t < (j+1)s$, by resolving these discontinuities, so that in the zone $js < t < (j+1)s$, the approximate solution is exact and consists of elementary waves issued from (ir, js). When u is in a small neighborhood of a constant state, the existence of a solution resolving the discontinuity was proved by Lax [5] using the implicit function theorem.

For simplicity, we assume that the system (1.1) is genuinely nonlinear. The cases when one or two of the characteristic fields are linearly degenerate can be treated similarly. The rarefaction curve $R_i(u_0)$ is the integral curve of $r_i(u)$ through u_0, and the shock curve $S_i(u_0)$ is the Rankine-Hugoniot curve which is tangent to $R_i(u_0)$ at u_0, i.e., for any $u \in S(u_0)$,

$$(u - u_0)\sigma(u, u_0) = f(u) - f(u_0),$$

for some $\sigma(u, u_0)$, the shock speed, which satisfies

$$\lim_{u \to u_0} \sigma(u, u_0) = \lambda_i(u).$$

These curves are divided into

$$R_i^{\pm}(u_0) = \{u \in R_i(u_0) | \lambda_i \gtrless \lambda_i(u_0)\},$$
$$S_i^{\pm}(u_0) = \{u \in S_i(u_0) | \sigma(u, u_0) \gtrless \lambda_i(u_0)\}.$$

A state u can be connected to u_0 on the left by an i-rarefaction (or i-shock) wave if $u \in R_i^+(u_0)$ (or $u \in S_i^-(u_0)$). The shock wave (u, u_0), $u \in S_i^-(u_0)$, is stable in the sense of Lax: $\lambda_i(u_0) > \sigma(u, u_0) > \lambda_i(u)$. The Riemann problem (u_l, u_r) is an initial-value problem for (1.1) with two constant initial states:

$$u(x, 0) = \begin{cases} u_l, & \text{for } x < 0, \\ u_r, & \text{for } x > 0. \end{cases}$$

It is solved by finding vectors u_i, $i = 0, 1, 2$, $u_0 = u_l$, $u_2 = u_r$, $u_1 \in S_1^-(u_0) \cup R_1^+(u_0)$, so that the solution consists of elementary i-waves (u_{i-1}, u_i), $i = 1, 2$. For any parameter μ_i along $S_i^- \cup R_i^+$, we set the strength of the i-wave in (u_l, u_r) as

$$(u_l, u_r)_i = \mu_i(u_i) - \mu_i(u_{i-1}), \qquad i = 1, 2.$$

We always choose μ_i so that shock waves have negative strengths and rarefaction waves have positive strengths. The following theorem on wave interaction is due to Glimm [3].

Theorem 2.1. *For any nearby states u_l, u_m, u_r, there exist bounds $O(1)$ depending only on the system* (1.1), *such that*

$$(u_l, u_r)_i = (u_l, u_m)_i + (u_m, u_r)_i + O(1)Q(u_l, u_m, u_r), \qquad i = 1, 2, \tag{2.1}$$

where Q measures the potential amount of interaction and is defined as follows:

A j-wave on the left interacts with a k-wave on the right if either $j > k$ or $j = k$ and at least one of the waves is a shock wave. We set

$$Q(u_l, u_m, u_r) = \sum_{j,k} (u_l, u_m)_j (u_m, u_r)_k,$$

the summation being over all interacting waves.

For any given sequence $\{a_m\}_{m=1}^{\infty}$, the (x, t)-plane consists of diamonds $\Delta_{i,j}$ with center (ir, js) and vertices $\big((i + a_{j-1})r, (j-1)s\big)$, $\big((i - 1 + a_j)r, js\big)$, $\big((i + 1 + a_j)r, js\big)$, and $\big((i + a_{j+1})r, (j+1)s\big)$. An I-curve is a space-like curve connecting vertices of the above form. An I-curve J_2 is an immediate successor of the I-curve J_1 if J_1 and J_2

pass through the same vertices except two and J_2 lies toward larger than J_1. The waves entering each diamond Δ are solutions of two Riemann problems, say, (u_l, u_m) and (u_m, u_r). We denote by $Q(\Delta) \equiv Q(u_l, u_m, u_r)$ the amount of interaction in Δ and $C_i(\Delta) = \frac{1}{2}(|(u_l, u_m)_i| + |(u_m, u_r)_i| - |(u_l, u_m)_i + (u_m, u_r)_i|)$ the amount of cancellation in Δ. Thus, (2.1) can be rewritten as

$$|(u_l, u_r)_i| = |(u_l, u_m)_i| + |(u_m, u_r)_i| + C_i(\Delta) + O(1)Q(\Delta).$$

The following theorem on wave interaction and cancellation is due to Glimm [3] and Glimm-Lax [4].

Theorem 2.2. *Suppose that the initial data* (1.2) *has sufficiently small total variation* TV. *Then,*

(a) *total variation* $\{u_r(\cdot, t)| - \infty < x < \infty\} \le 2\,TV$, *for all* t,

(b) $Q = \sum_\Delta Q(\Delta) \le 2(TV)^2$,

(c) $C = \sum_\Delta C(\Delta) \le TV + Q$,

(d) $F(J_2) - F(J_1) \le -\sum_\Delta (C(\Delta) + Q(\Delta))$

where J_2 *is an immediate successor of* J_1, *and* Δ *is the diamond between* J_1 *and* J_2 *in* (d), *and* $F(J) = L(J) + KQ(J)$ *where*

$$Q(J) = \sum \{\, |ab| \;\big|\; a \text{ and } b \text{ are strengths of interacting waves crossing } J\,\},$$
$$L(J) = \sum \{\, |a| \;\big|\; a \text{ is the strength of wave crossing } J\,\},$$

and K *is a large constant.*

In our discussion, we need to assume that the random sequence $\mathbf{a} = \{a_m\}_{m=1}^{\infty}$ is equidistributed in $(0,1)$, that is,

$$\lim_{k\to\infty} \frac{B(\mathbf{a}, k, I)}{k} = \mu(I)$$

for any subinterval I of $(0,1)$. Here $B(a, k, I)$ denotes the number m, $1 \le m \le k$, with $a_m \in I$, and $\mu(I)$ is the length of I.

For any small $\epsilon > 0$, let $N = \frac{1}{\epsilon}$ and M be large integers. We divide the interval $(0,1)$ into N subintervals with equal length ϵ. Let $\{I_i\}_{i=1}^{N}$ be the collection of unions of any such subintervals. We set

$$\delta = \sup_{1\le p\le N, 1\le i\le 2N} \left(\frac{B(a_{m-(p-1)M}, M, I_i)}{M} - \mu(I_i) \right), \tag{2.2}$$

which tends to zero as $M \to \infty$ for any fixed N.

For any fixed time T, we choose M such that

$$(N-1)Ms < T \le NMs.$$

Note that for each fixed N,

$$M \to \infty, \quad NMs \le 2T, \quad \text{and} \quad Ms \le T\epsilon,$$

as the mesh length $s \to 0$.

We will partition the elementary waves in the following way. Consider elementary waves (u_{k-1}, u_k), issued from (ir, js). If (u_{k-1}, u_k) is a k-shock wave, $k = 1, 2$, then we choose any vectors $y_0, y_1, \dots, y_l$, $y_0 = u_{k-1}$, $y_l = u_k$, $y_h \in S_k^-(u_{k-1})$, $\lambda_k(y_h) < \lambda_k(y_{h-1})$, $h = 1, 2, \dots, l$, and set

$$v_k^h(i,j) = y_h - y_{h-1}, \tag{2.3}$$

$$\lambda_k^h(i,j) = \sigma(u_{k-1}, u_k). \tag{2.4}$$

If (u_{k-1}, u_k) is a k-rarefaction wave, we choose vectors $y_0, y_1, \dots, y_l$, $y_0 = u_{k-1}$, $y_l = u_k$, $y_h \in R_k^+(u_{k-1})$, $\lambda_k(y_h) > \lambda_k(y_{h-1})$, $h = 1, 2, \dots, l$, and set

$$v_k^h(i,j) = y_h - y_{h-1}, \tag{2.5}$$

$$\lambda_k^h(i,j) = \lambda(h_{h-1}). \tag{2.6}$$

In the latter case, we require that

$$|\lambda_k^h(y_h) - \lambda_k^h(y_{h-1})| \leq \epsilon, \qquad h = 1, 2, \dots, l,$$

and, to make sure that $\{v_k^h(i,j)\}$ is not further partitioned at $t = (j+1)s$, we also require that

$$a_{j+1} \notin (\lambda_k(y_{h-1}), \lambda_k(y_h)), \quad h = 1, 2, \dots, l.$$

The following theorem is due to Liu [7].

Theorem 2.3. *Suppose that the total variation TV of the initial data (1.2) is small. Then for any equidistributed sequence $\{a_m\}_{m=1}^{\infty}$ in $(-1,1)$, there exists a partition of elementary waves $\{v_k^h(i,j), \lambda_k^h(i,j)\}$ which satisfies (2.2)–(2.5). Moreover, $\{v_k^h(i,j), \lambda_k^h(i,j)\}$ is a disjoint union of $\{\tilde{v}_k^h(i,j), \tilde{\lambda}_k^h(i,j)\}$ and $\{\tilde{\tilde{v}}_k^h(i,j), \tilde{\tilde{\lambda}}_k^h(i,j)\}$, so that for any j, $(p-1)M \leq j \leq pM$, $p \in \{1, 2, \dots, N\}$, the following hold:*

(i) $\sum_{i,h,k} |\tilde{\tilde{v}}_k^h(i,j)| \leq [Q(\Lambda_p) + C(\Lambda_p)]$ and there is a one-to-one correspondence between $\{\tilde{v}_k^h(i,j), \tilde{\lambda}_k^h(i,j)\}$ and $\{\tilde{v}_k^h(i,(p-1)M), \tilde{\lambda}_k^h(i,(p-1)M)\}$:

$$(\tilde{v}_k^h(i,(p-1)M), \tilde{\lambda}_k^h(i,(p-1)M)) \leftrightarrow (\tilde{v}_k^h(i_j,j), \tilde{\lambda}_k^h(i_j,j)),$$

such that

(ii) $\sum_{i,h,k} |\tilde{v}_k^h(i,(p-1)M) - \tilde{v}_k^h(i_j,j)| = O(1)Q(\Lambda_p)$,

(iii) $\sum_{i,h,k} |\tilde{v}_k^h(i,(p-1)M)| \cdot \max\limits_{(p-1)M \leq j \leq pM} |\tilde{\lambda}_k^h(i_j,j) - \tilde{\lambda}_k^h(i,(p-1)M)| = O(1)Q(\Lambda_p)$,

(iv) $|i - i_j| \leq |j - (p-1)M|$.

Here the bounds of $O(1)$ are independent of i, j, r, and Λ_p is the zone $(p-1)Ms \leq t \leq pMs$. Furthermore, the approximate solutions converge to an exact weak solution of (1.1) and (1.2) as $s \to 0$.

In order to estimate the derivative of the functional to be introduced in the next section, we now define a new, simpler wave pattern in Λ_p, $p = 1, 2, \dots, N$. Choose a subset of $\{\tilde{v}_k^h(i,j)\}$, $(p-1)M \leq j \leq pM$, denoted by $\{\bar{v}_k^h(i,j)\}$, with corresponding wave speed $\{\bar{\lambda}_k^h(i,j)\}$ satisfying the following conditions:

(i) there is a mapping from $\{\bar{v}_k^h(i,j), \bar{\lambda}_k^h(i,j)\}$ to $\{\tilde{v}_k^h(i,(p-1)M), \tilde{\lambda}_k^h(i,(p-1)M)\}$:

$$(\bar{v}_k^h(i,j), \bar{\lambda}_k^h(i,j)) \to (\tilde{v}_k^h(i_j,(p-1)M), \tilde{\lambda}_k^h(i_j,(p-1)M)),$$

such that

(ii) if $\hat{i}_j < i_j$, then $\hat{i} \leq i$, and the equal sign holds only when $j = pM$,

(iii) $\sum_{i,h,k} |\tilde{v}_k^h(i_j,(p-1)M) - \bar{v}_k^h(i,j)| = O(1)(Q(\Lambda_p) + C(\Lambda_p))$,

(iv) $\sum_{i,h,k} \max_{(p-1)M\le j\le pM} |\bar{v}_k^h(i,j)||\bar{\lambda}_k^h(i,j) - \bar{\lambda}_k^h(i_j,(p-1)M)| = O(1)(Q(\Lambda_p) + C(\Lambda_p))$,

(v) $|i-i_j| \le |j-(p-1)M|$, and

(vi) let $\{v_k^h(i,j),\ \lambda_k^h(i,j)\}\backslash\{\bar{v}_k^h(i,j),\ \bar{\lambda}_k^h(i,j)\} = \{\tilde{\bar{v}}_k^h(i,j),\ \tilde{\bar{\lambda}}_k^h(i,j)\}$, then

$$\sum_{i,h,k} |\tilde{\bar{v}}_k^h(i,j)| \le O(1)[Q(\Lambda_p) + C(\Lambda_p)]$$

where the bounds $O(1)$ are independent of i, j, and r. The existence of $\{\bar{v}_k^h(i,j),\ \bar{\lambda}_k^h(i,j)\}$ can be obtained by eliminating the waves in $\{\tilde{v}_k^h(i,j),\ \tilde{\lambda}_k^h(i,j)\}$ which are cancelled in Λ_p.

We denote by $l(\bar{v}_k^h(i,j))$ the straight line connecting the grid points of $\{\bar{v}_k^h(i,j)\}$ when $j=(p-1)M$ and $j=pM$, and its slope by $\lambda^*(\bar{v}_k^h(i,j))$. Our simplified approximate wave pattern in Λ_p, denoted by $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$, is a linear superposition of a family of step functions with the property that across each straight line $l(\bar{v}_k^h(i,j))$ the difference of the right and left states is $\{\bar{v}_k^h(i,(p-1)M)\}$.

According to the partition of the elementary waves and Theorem 2.3, we know that, for $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$, the following estimate holds:

$$\sum_{i,h,k} |v_k^h(i,(p-1)M| \max_{(p-1)M\le j\le pM} |\lambda_k^h(i_j,j) - \lambda^*(v_k^h(i,j))|$$
$$\le O(1)(Q(\Lambda_p) + C(\Lambda_p) + TV\delta)$$

where δ is defined in (2.1). The term $O(1)TV\delta$ is due to the equidistribution of the random sequence $\{a_m\}_{m=1}^\infty$ for $(p-1)M \le m \le pM$ and the wave speed $\lambda_k^h(i,j)$ being replaced by the slope of $l(v_k^h(i,j))$.

3. Functionals and the main theorem

Without loss of generality, we assume that the constant state is zero and the initial data $u_0(x)$ belong to $L_1(R)$. As is well known, there exists a pair of Riemann invariants to the 2×2 system (1.1). The i-th Riemann invariant is constant along the curve $R_j^+(u_0)$, $i,j=1,2,\ i\ne j$. We will use the i-th Riemann invariant to measure the strength of the i-th wave, and we use α and β to denote the 1-wave and 2-wave, respectively. Without any ambiguity, α and β denote the waves as well as their strengths.

In the Glimm scheme, the waves are partitioned using the wave tracing method as shown in Section 2. For any time t, the Riemann invariants are step functions of the spatial coordinate x. For convenience, the secondary strength of an i-wave measured by the j-th Riemann invariant, $j\ne i$, is denoted by $\hat{\alpha}$ and $\hat{\beta}$, respectively. Since the rarefaction curve $R^+(u_0)$ and the shock curve $S^-(u_0)$ have second contact continuity at $u=u_0$, we have

$$|\hat{\alpha}| = \begin{cases} 0, & \alpha\ge 0,\\ O(1)|\alpha|^3, & \alpha<0,\end{cases} \qquad |\hat{\beta}| = \begin{cases} 0, & \beta\ge 0,\\ O(1)|\beta|^3, & \beta<0.\end{cases} \tag{3.1}$$

Before defining the functionals which are equivalent to the L_1 norm of the approximate solutions, we now introduce some notations. For any time $0\le t\le T$, we denote the 1-

and 2-waves from left to right by $\alpha_i(t)$ and $\beta_i(t)$, respectively, and their corresponding x coordinates are denoted by $x_{\alpha_i(t)}$ and $x_{\beta_i(t)}$. Let $l_{i,j}(t) = (x_{\alpha_i(t)}, x_{\alpha_{i+1}(t)}) \cap (x_{\beta_i(t)}, x_{\beta_{i+1}(t)})$ and, without ambiguity, also its length. On each $l_{i,j}(t)$, the Riemann invariants are independent of x, and its 1st and 2nd Riemann invariants are denoted by $\gamma_{i,j}^-(t)$ and $\gamma_{i,j}^+(t)$, respectively. Now we define the following functionals for $u_r(x,t)$, $0 \le t \le T$:

$$\begin{aligned}
L(t) &= \sum_{(i,j)} \left(|\gamma_{i,j}^-(t)| + |\gamma_{i,j}^+(t)|\right) l_{i,j}(t),\\
Q_d(t) &= \sum_{(i,j)} |\gamma_{i,j}^+(t)| \Big(\sum_{k \ge i+1} |\alpha_k(t)| \Big) l_{i,j}(t) + \sum_{(i,j)} |\gamma_{i,j}^-(t)| \Big(\sum_{k \le j} |\beta_k(t)| \Big) l_{i,j}(t),\\
Q_s(t) &= \sum_{(i,j)} (|\gamma_{i,j}^-(t)|^2 + |\gamma_{i,j}^+(t)|^2) l_{i,j}(t),\\
H(t) &= L(t)(1 + k_1 F(t)) + k_2(Q_s(t) + Q_d(t))
\end{aligned}$$

where the summations are over all possible (i,j) at time t, $F(t)$ is the Glimm functional defined in Theorem 2.2, and k_1, k_2 are positive constants to be chosen later.

In Section 2, we have defined a simplified wave pattern $\{\bar{v}_k^h(i,(p-1)M), \lambda^*(\bar{v}_k^h(i,j))\}$. For these approximate solutions in the region Λ_p, we define the corresponding functional, $\bar{L}(t)$, $\bar{Q}_d(t)$, $\bar{Q}_s(t)$, and $\bar{H}(t)$ by

$$\begin{aligned}
\bar{L}(t) &= \sum_{(i,j)} (|\bar{\gamma}_{i,j}^-(t)| + |\bar{\gamma}_{i,j}^+(t)|) \bar{l}_{i,j}(t),\\
\bar{Q}_d(t) &= \sum_{(i,j)} |\bar{\gamma}_{i,j}^+(t)| \Big(\sum_{k \ge i+1} |\bar{\alpha}_k(t)| \Big) \bar{l}_{i,j}(t) + \sum_{(i,j)} |\bar{\gamma}_{i,j}^-(t)| \Big(\sum_{k \le j} |\bar{\beta}_k(t)| \Big) \bar{l}_{i,j}(t),\\
\bar{Q}_s(t) &= \sum_{(i,j)} (|\bar{\gamma}_{i,j}^-(t)|^2 + |\bar{\gamma}_{i,j}^+(t)|^2) \bar{l}_{i,j}(t),\\
\bar{H}(t) &= \bar{L}(t)\big(1 + k_1 F((p-1)Ms)\big) + k_2\big(\bar{Q}_s(t) + \bar{Q}_d(t)\big)
\end{aligned}$$

where $(p-1)Ms \le t \le pMs$, $p = 1, 2, \ldots, N$, and the summations are over all possible (i,j) at time t. In the above, we have used $\bar{\alpha}_i(t)$, $\bar{\beta}_i(t)$, $\bar{l}_{i,j}(t)$, and $\bar{\gamma}_{i,j}^\pm$ to denote the corresponding 1-wave, 2-wave, x interval, and the Riemann invariants on $\bar{l}_{i,j}(t)$ in the simplified wave pattern, respectively. Notice that the form of $\bar{H}(t)$ differs slightly from that of $H(t)$ in that the Glimm functional $F(t)$ in $\bar{H}(t)$ is fixed at the beginning of the time $(p-1)Ms$.

Now we can state the main lemma in this paper; its proof will be given in Section 4:

Main Lemma. *Suppose that the total variation of the initial data TV is sufficiently small, and $u_0(x) \in L_1(R)$. Then there exist constants k_1 and k_2 independent of T and s such that*

$$\bar{H}(pMs - 0) - \bar{H}((p-1)Ms + 0) \le ce(\Lambda_p)Ms, \qquad p = 1, 2, \ldots, N, \tag{3.2}$$

where $e(\Lambda_p) = Q(\Lambda_p) + C(\Lambda_p) + \delta + \epsilon$.

Hereafter c denotes the generic positive constant which is independent of T and s. In order to prove a similar estimate for the functional $H(t)$ for the approximate solutions constructed by the wave tracing method, we need the following lemma.

Lemma 3.1. *Under the conditions of the Main Lemma, there exists a sequence* $\{\varepsilon_i\}_{i=1}^N$ *such that*

$$H(pMs) - \bar{H}(pMs - 0) \le \varepsilon_p, \tag{3.3}$$

$$\bar{H}((p-1)Ms + 0) - H((p-1)Ms) \le \varepsilon_p, \qquad p = 1, 2, \ldots, N, \tag{3.4}$$

and

$$\sum_{i=1}^{N} \varepsilon_i \to 0, \quad \text{as} \quad N \to \infty$$

where $\varepsilon_p = (Q(\Lambda_p) + C(\Lambda_p))Ms$.

Proof. We prove (3.3) first by estimating $H(pMs) - \bar{H}(pms - 0)$ as the sum of I_1, I_2, and I_3 below. The mostly linear part is

$$\begin{aligned} I_1 &= L(pMs)(1 + k_1 F(pMs)) - \bar{L}(pMs - 0)\big(1 + k_1 F((p-1)Ms)\big) \\ &= \big(L(pMs) - \bar{L}(pMs - 0)\big)\big(1 + k_1 F((p-1)Ms)\big) \\ &\quad + k_1 L(pMs)\big(F(pMs) - F((p-1)Ms)\big). \end{aligned} \tag{3.5}$$

By Theorem 2.3 and the definition of $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$ in Λ_p, the difference between $L(pMs)$ and $\bar{L}(pMs - 0)$ is due to the wave family $\{v_k^h(i,pM),\ \lambda_k^h(i,pM))\}$ minus $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$ at $t = pMs$. By the discussion toward the end of Section 2, the total strength of the remaining waves is controlled by $O(1)(Q(\Lambda_p) + C(\Lambda_p))$, and, at time $t = pM - 0$, the x-coordinate of any wave $\bar{v}_k^h(i,pM) \in \{\bar{v}_k^h(i,j)\}$ is the same as the one for $\bar{v}_k^h(i,pM) \in \{v_k^h(i,j)\}$. Due to the finite propagation of the hyperbolic waves, the x-interval of each of the remaining waves in $\{v_k^h(i,j)\}\backslash\{\bar{v}_k^h(i,(p-1)M)\}$ at time $t = pMs$ which affect the $L(pMs)$ is controlled by $O(1)Ms$. Therefore,

$$L(pMs) - \bar{L}(pMs - 0) \le c(Q(\Lambda_p) + C(\Lambda_p))Ms. \tag{3.6}$$

By Theorem 2.2 (d), we have

$$F(pMs) - F((p-1)Ms) \le -(Q(\Lambda_p) + C(\Lambda_p)). \tag{3.7}$$

Hence, combining (3.6) and (3.7) yields

$$I_1 \le c\big(Q(\Lambda_p) + C(\Lambda_p)\big)Ms - k_1\big(Q(\Lambda_p) + C(\Lambda_p)\big)L(pMs). \tag{3.8}$$

Next we estimate the difference between $Q_d(pMs)$ and $\bar{Q}_d(pMs - 0)$. We denote the finer partitions containing all $l_{p,q} \cap \bar{l}_{k,l}$ by $\tilde{l}_{i,j}$, and the corresponding Riemann invariants of $\{v_k^h(i,pM),\ \lambda_k^h(i,pM)\}$ and $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$ on $\tilde{l}_{i,j}$ by $\tilde{\gamma}_{i,j}^{\pm}$ and $\tilde{\bar{\gamma}}_{i,j}^{\pm}$, respectively. Then

$$\begin{aligned} I_2 &= Q_d(pMs) - \bar{Q}_d(pMs - 0) \\ &= \sum_{(i,j)} \Big\{ |\gamma_{i,j}^-| \Big(\sum_{k \le j} |\beta_k| \Big) + |\gamma_{i,j}^+| \Big(\sum_{k \ge i+1} |\alpha_k| \Big) \Big\} l_{i,j} \\ &\quad - \sum_{(k,l)} \Big\{ |\bar{\gamma}_{l,k}^-| \Big(\sum_{p \le k} |\bar{\beta}_k| \Big) + |\bar{\gamma}_{l,k}^+| \Big(\sum_{p \ge l+1} |\bar{\alpha}_k| \Big) \Big\} \bar{l}_{l,k} \\ &= \sum_{(i,j)} \Big\{ \sum_{k \le j} |\tilde{\beta}_k| (|\tilde{\gamma}_{i,j}^-| - |\tilde{\bar{\gamma}}_{i,j}^-|) + \sum_{k \ge i+1} |\tilde{\alpha}_k| (|\tilde{\gamma}_{i,j}^+| - |\tilde{\bar{\gamma}}_{i,j}^+|) \Big\} \tilde{l}_{i,j} \end{aligned}$$

$$+\sum_{(i,j)}\left\{|\tilde{\bar{\gamma}}_{i,j}^{-}|\tilde{l}_{i,j}\left(\sum_{k\le j}|\tilde{\beta}_k|-\sum_{k\le j}|\tilde{\bar{\beta}}_k|\right)+|\tilde{\gamma}_{i,j}^{+}|\tilde{l}_{i,j}\left(\sum_{k\ge i+1}|\tilde{\alpha}_k|-\sum_{k\ge i+1}|\tilde{\bar{\alpha}}_k|\right)\right\}. \tag{3.9}$$

The sum of the first two terms of (3.9) is controlled by $cTV(Q(\Lambda_p)+C(\Lambda_p))Ms$ as discussed in estimating I_1. For the sum of the last two terms, the difference between $\sum_{k\le j}|\tilde{\beta}_k|$, $\sum_{k\ge i+1}|\tilde{\alpha}_k|$ and $\sum_{k\le j}|\tilde{\bar{\beta}}_k|$, $\sum_{k\ge i+1}|\tilde{\bar{\alpha}}_k|$ is due to replacing $\{v_k^h(i,j),\ \lambda_k^h(i,j)\}$ by $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$ at $t=pMs$. Since the total wave strengths of the difference between $\{v_k^h(i,j),\ \lambda_k^h(i,j)\}$ and $\{\bar{v}_k^h(i,(p-1)M),\ \lambda^*(\bar{v}_k^h(i,j))\}$ at $t=pMs$ is controlled by $O(1)(Q(\Lambda_p)+C(\Lambda_p))$, the sum of the last two terms is controlled by $O(1)(Q(\Lambda_p)+C(\Lambda_p))L(pMs)$. Therefore,

$$I_2\le cTV(Q(\Lambda_p)+C(\Lambda_p))Ms+c(Q(\Lambda_p)+C(\Lambda_p))L(pMs). \tag{3.10}$$

Since $Q_s(t)$ and $\bar{Q}_s(t)$ depend on $\gamma_{i,j}^{\pm}$, $l_{i,j}$ and $\bar{\gamma}_{i,j}^{\pm}$, $\bar{l}_{i,j}$, respectively, as discussed in estimating $L(pMs)-\bar{L}(pMs-0)$, we have

$$\begin{aligned}I_3&=Q_s(pMs)-\bar{Q}_s(pMs-0)\\&\le cTV(Q(\Lambda_p)+C(\Lambda_p))Ms.\end{aligned} \tag{3.11}$$

Combining (3.8), (3.10), and (3.11), we have

$$\begin{aligned}H(pMs)-\bar{H}(pMs-0)\le{}& c(1+k_2TV)(Q(\Lambda_p)+C(\Lambda_p))Ms\\&+(ck_2-k_1)(Q(\Lambda_p)+C(\Lambda_p))L(pMs).\end{aligned}$$

Therefore, if we choose k_1 and k_2 satisfying

$$k_1>ck_2,\qquad k_2TV<c, \tag{3.12}$$

then

$$H(pMs)-\bar{H}(pMs-0)\le\varepsilon_p$$

where $\varepsilon_p=c(Q(\Lambda_p)+C(\Lambda_p))Ms$.

Similarly, we can show that

$$\bar{H}((p-1)Ms+0)-H((p-1)Ms)\le\varepsilon_p,\qquad p=1,2,\dots,N. \tag{3.13}$$

From Theorem 2.2,

$$\begin{aligned}\sum_{i=1}^{N}\varepsilon_i&=c(Q(\Lambda_p)+C(\Lambda_p))Ms\\&\le c\big(4(TV)^2+TV\big)\frac{2T}{N}.\end{aligned}$$

Hence as $N\to\infty$, we have $\sum_{i=1}^{N}\varepsilon_i\to 0$. This proves Lemma 3.1. □

We now state and prove our main theorem.

Theorem 3.1. *Under the hypotheses of the Main Lemma, for the exact weak solution of* (1.1) *constructed by the wave tracing method, there exists a constant G independent of time such that*

$$\|u(x,t)\|_{L_1}\le G\|u(x,s)\|_{L_1}$$

for any s, $0\le s\le t<\infty$.

Proof. Without loss of generality, we will show that $\|u(x,T)\|_{L_1} \le G\|u(x,0)\|_{L_1}$ for any time T. By the Main Lemma and Lemma 3.1, we have

$$\bar{H}(pMs-0) - \bar{H}((p-1)Ms+0) \le ce(\Lambda_p)Ms, \tag{3.14}$$

$$H(pMs) - \bar{H}(pMs-0) \le \varepsilon_p, \tag{3.15}$$

$$\bar{H}((p-1)Ms+0) - H((p-1)Ms) \le \varepsilon_p, \qquad p = 1,2,\dots,N. \tag{3.16}$$

Summing up (3.14)–(3.16), we have

$$H(pMs) - H((p-1)Ms) \le 2\varepsilon_p + ce(\Lambda_p)MS, \qquad p = 1,2,\dots,N. \tag{3.17}$$

Summing (3.17) with respect to p from 1 to N yields

$$H(T) \le H(0) + 2\sum_{i=1}^{N} \varepsilon_i + c(Q+C)Ms + c(\delta+\epsilon)T.$$

As $N \to \infty$, by the definition of δ and ϵ, we have $\delta \to 0$ and $\epsilon \to 0$. Hence by Lemma 3.1, we have, for any fixed T,

$$2\sum_{i=1}^{N} \varepsilon_i + c(Q+C)Ms + c(\delta+\epsilon)T \to 0$$

as $N \to \infty$.

Note that for any fixed M and N, the functional $H(t)$ is equivalent to the L_1 norm of the approximate solutions $\{u_r(x,t)\}$ constructed by the wave tracing method. Moreover, by Theorem 2.3, the approximate solutions $\{u_r(x,t)\}$ converge to an exact solution locally in the L_1 norm. Consequently, there exists a constant G independent of T and s such that

$$\|u(x,T)\|_{L_1} \le G\|u(x,0)\|_{L_1}.$$

This completes the proof of the Theorem. □

4. Proof of the Main Lemma

The following proof of the Main Lemma stated in the last section contains the main idea of the construction of our nonlinear functional $H(t)$.

Proof of the Main Lemma. In the simplified approximate wave pattern $\{\bar{v}_k^h(i,(p-1)M), \lambda^*(\bar{v}_k^h(i,j))\}$ in Λ_p, $p = 1,2,\dots,N$, defined in Section 2, the approximate solutions are linear superpositions of a family of approximate nonlinear waves represented by straight lines. By the definition of $l(\bar{v}_k^h(i,j))$, they will not cross each other with the same k, $k = 1,2$, inside the region Λ_p. And $\bar{H}(t)$ is a continuous functional with respect to $t \in ((p-1)Ms, pMs)$, $p = 1,2,\dots,N$.

Since the initial data $u_0(x)$ is in $L_1(x)$, we can assume for convenience that $u_0(x)$ is zero outside a large interval $[-X, X]$. Thus in the region Λ_p, there are finite intersection points between $l(\bar{v}_k^h(i,j))$. Hence $\frac{d}{dt}\bar{H}(t)$ is well-defined except for a set of a finite number of times for any fixed N and s. In the following, we show that when N is sufficiently large, there exists an upper bound for $\frac{d}{dt}\bar{H}(t)$ where it is defined, whence an upper bound for $\bar{H}(pMs-0) - \bar{H}((p-1)Ms+0)$ follows.

Following the notation of Section 3, we consider the region $(p-1)Ms < t < pMs$ where $\frac{d}{dt}\bar{H}(t)$ exists.

Step 1. Estimate $\frac{d}{dt}\bar{L}(t)$.

Without any ambiguity, we still call the discontinuity across $l(\bar{v}_k^h(i,j))$ the k-wave, $k=1,2$, denoted by α_i and β_i, respectively. On each $l_{i,j}$, the Riemann invariants $\gamma_{i,j}^{\pm}$ can be represented by $\{\alpha_i\}$ and $\{\beta_i\}$ through telescoping:

$$\gamma_{i,j}^- = \sum_{k\le i}\alpha_k + \sum_{k\le j}\hat{\beta}_k, \quad \gamma_{i,j}^+ = \sum_{k\le j}\beta_k + \sum_{k\le i}\hat{\alpha}_k. \tag{4.1}$$

Between two consecutive 1-waves α_i and α_{i+1}, we denote the 2-waves from left to right by $\beta_{i,j}$, $j=1,2,\dots,n_{\alpha_i}$. Similarly the 1-waves between two consecutive 2-waves β_i and β_{i+1} are denoted by $\alpha_{i,j}$, $j=1,2,\dots,n_{\beta_i}$. When $j=0$, there are no such waves. Now we define a sequence of waves $\{\hat{\bar{\alpha}}_i\}$ and $\{\hat{\bar{\beta}}_i\}$ on $l(\bar{v}_1^h(i,j))$ and $l(\bar{v}_2^h(i,j))$, respectively, as follows: $\{\hat{\bar{\alpha}}_i\}$ is defined inductively from left to right; $\hat{\bar{\alpha}}_i$ is an approximate 1-wave on the same line $l(\bar{v}_1^h(i,j))$ as α_i with left state $\sum_{k=1}^{i-1}\hat{\bar{\alpha}}_k$ where $\hat{\bar{\alpha}}_i = \alpha_i + \sum_{j=1}^{n_{\alpha_i}}\hat{\beta}_{i,j}$. Similarly, $\hat{\bar{\beta}}_i$ is an approximate 2-wave on the same line $l(\bar{v}_2^h(i,j))$ as β_i with left state $\sum_{k=1}^{i-1}\hat{\bar{\beta}}_k$ where $\hat{\bar{\beta}}_i = \beta_i + \sum_{j=1}^{n_{\beta_i}}\hat{\alpha}_{i,j}$. For convenience, we introduce some more notation: We denote the approximate waves in $\{\bar{v}_k^h(i,(p-1)M), \lambda^*(\bar{v}_k^h(i,j))\}$ on the left and right of $l_{i,j}$ by $\theta_{i,j}^l$ and $\theta_{i,j}^r$, respectively. These waves $\{\theta_{i,j}^l\}$ and $\{\theta_{i,j}^r\}$ are denoted, from left to right, by $\{\theta^p\}$, and the Riemann invariants of the right state of θ^p are denoted by $\gamma_p^{\pm}$. In particular, the $\gamma_{i,j}^+$ of the left state of α_i is denoted by $\gamma_{\alpha_i}^+$, and $\gamma_{i,j}^-$ of the right state of β_i is denoted by $\gamma_{\beta_i}^-$.

By the definition of $\bar{L}(t)$, we have

$$\frac{d}{dt}\bar{L}(t) = \frac{d}{dt}\bar{L}^-(t) + \frac{d}{dt}\bar{L}^+(t)$$

where $\frac{d}{dt}\bar{L}^{\pm}(t) = \sum_{(i,j)}|\gamma_{i,j}^{\pm}|(\lambda(\theta_{i,j}^r) - \lambda(\theta_{(i,j)}^l))$.

Now we estimate the first term on the right-hand side:

$$\begin{aligned}
\frac{d}{dt}\bar{L}^-(t) &= \sum_p |\gamma_p^-|(\lambda(\theta_{i,j}^r) - \lambda(\theta_{i,j}^l)) \\
&= \sum_i \Bigg\{ \sum_{j=1}^{n_{\alpha_{i-1}}} \left|\sum_{k=1}^{i-1}\hat{\bar{\alpha}}_k + \alpha_i + \sum_{k=1}^{j}\hat{\beta}_{i,k}\right| (\lambda^*(\beta_{i,j+1}) - \lambda^*(\beta_{i,j})) \\
&\quad + \left|\sum_{k=1}^{i}\hat{\bar{\alpha}}_k\right| (\lambda^*(\alpha_{i+1}) - \lambda^*(\beta_{i,n_{\alpha_i}})) + \left|\sum_{k=1}^{i-1}\hat{\bar{\alpha}}_k + \alpha_i\right| (\lambda^*(\beta_{i,1}) - \lambda^*(\alpha_i)) \Bigg\} \\
&\le \sum_i \Bigg\{ \sum_{j=1}^{n_{\alpha_{i-1}}} \left|\sum_{k=1}^{i}\hat{\bar{\alpha}}_k\right| (\lambda^*(\beta_{i,j+1}) - \lambda^*(\beta_{i,j})) + \sum_{k=j+1}^{n_{\alpha_i}} |\hat{\beta}_{i,k}||\lambda^*(\beta_{i,k+1}) - \lambda^*(\beta_{i,k})| \\
&\quad + \left|\sum_{k=1}^{i}\hat{\bar{\alpha}}_k\right| (\lambda^*(\alpha_{i+1}) - \lambda^*(\beta_{i,n_{\alpha_i}}) + \left|\sum_{k=1}^{i}\hat{\bar{\alpha}}_k\right| (\lambda^*(\beta_{i,1}) - \lambda^*(\alpha_i)) \\
&\quad + \sum_{j=1}^{n_{\alpha_i}} |\hat{\beta}_{i,j}||\lambda^*(\beta_{i,1}) - \lambda^*(\alpha_i)| \Bigg\}
\end{aligned}$$

$$\leq \sum_i \left\{ \left| \sum_{k=1}^{i} \hat{\bar{\alpha}}_k \right| (\lambda^*(\alpha_{i+1}) - \lambda^*(\alpha_i)) + O(1) \sum_{j=1}^{n_{\alpha_i}} |\hat{\beta}_{i,j}| \right\}$$

$$\leq \sum_i \left| \sum_{k=1}^{i} \hat{\bar{\alpha}}_k \right| (\lambda^*(\alpha_{i+1}) - \lambda^*(\alpha_i)) + O(1) \sum_i |\hat{\beta}_i|. \tag{4.2}$$

Consider the scalar conservation laws

$$r_t + \lambda_1(r, 0) r_x = 0. \tag{4.3}$$

If a wave of (4.3) has left state $\sum_{k=1}^{i-1} \hat{\bar{\alpha}}_k$ and right state $\sum_{k=1}^{i} \hat{\bar{\alpha}}_k$, we denote this wave by $[\sum_{k=1}^{i-1} \hat{\bar{\alpha}}_k, \sum_{k=1}^{i} \hat{\bar{\alpha}}_k]$ with $\mu_i(\alpha)$ as its corresponding wave speed. Since $\sum_{k=1}^{\infty} \hat{\bar{\alpha}}_k = 0$, by the L_1 contraction of scalar conservation laws [8], we have

$$\sum_i \left| \sum_{k=1}^{i} \hat{\bar{\alpha}}_k \right| (\mu_{i+1}(\alpha) - \mu_i(\alpha)) \leq c\,TV\epsilon. \tag{4.4}$$

The error term $cTV\epsilon$ is due to partitioning the rarefaction waves into small sub-waves with strength of order ϵ.

By the construction of $\{\bar{v}_k^h(i, (p-1)M,\ \lambda^*(\bar{v}_k^h(i,j))\}$, we have

$$|\lambda^*(\alpha_i) - \mu_i(\alpha)| \leq c(|\gamma_{\alpha_i}^+| + |\alpha_i|^2 + Q(\Lambda_p) + C(\Lambda_p) + \delta) + c\Big(\sum_i |\hat{\beta}_i| + \sum_{l \in S(t)} |\beta_l|\Big) \tag{4.5}$$

where the last summation is over all 2-waves β_l which cross $l(\alpha_i)$ between $(p-1)Ms \leq \tau \leq t$. Therefore,

$$\begin{aligned}
\frac{d}{dt}\bar{L}^-(t) &\leq \sum_i \mu_i(\alpha)\left(-\left| \sum_{k=1}^{i+1} \hat{\bar{\alpha}}_k \right| + \left| \sum_{k=1}^{i} \hat{\bar{\alpha}}_k \right| \right) \\
&\quad + \sum_i |\lambda^*(\alpha_i) - \mu_i(\alpha)| \cdot \left| \left| \sum_{k=1}^{i+1} \hat{\bar{\alpha}}_k \right| - \left| \sum_{k=1}^{i} \hat{\bar{\alpha}}_k \right| \right| + O(1) \sum_i |\hat{\beta}_i| \\
&\leq c \sum_i \Big(|\gamma_{\alpha_i}^+| + |\alpha_i|^2 + \sum_i |\hat{\beta}_i| + Q(\Lambda_p) + C(\Lambda_p) + \delta + \sum_{l \in S(t)} |\beta_l| \Big) \\
&\qquad \times \Big(|\alpha_i| + \sum_i |\hat{\beta}_i| \Big) + O(1) \sum_i |\hat{\beta}_i| + c\,TV\epsilon \\
&\leq c \sum_i |\alpha_i||\gamma_{\alpha_i}^+| + c \sum_i (|\hat{\beta}_i| + |\hat{\alpha}_i|) + ce(\Lambda_p).
\end{aligned} \tag{4.6}$$

Similarly, we have

$$\frac{d}{dt}\bar{L}^+(t) \leq c \sum_i |\beta_i||\gamma_{\beta_i}^-| + c \sum_i (|\hat{\alpha}_i| + |\hat{\beta}_i|) + ce(\Lambda_p). \tag{4.7}$$

Combining (4.5) and (4.6), we have

$$\frac{d}{dt}\bar{L}(t) \leq c \sum_i \big(|\alpha_i||\gamma_{\alpha_i}^+| + |\beta_i||\gamma_{\beta_i}^-| + |\hat{\alpha}_i| + |\hat{\beta}_i| \big) + ce(\Lambda_p). \tag{4.8}$$

Step 2. Estimate $\frac{d}{dt}\bar{Q}_d(t)$.

By the definition of $\bar{Q}_d(t)$, we have

$$\frac{d}{dt}\bar{Q}_d(t) = \frac{d}{dt}\bar{Q}_d^-(t) + \frac{d}{dt}\bar{Q}_d^+(t) \tag{4.9}$$

where $\bar{Q}_d^+(t) = \sum_{(i,j)} |\gamma_{i,j}^+|(\sum_{k\geq i+1}|\alpha_k|)l_{i,j}$ and $\bar{Q}_d^-(t) = \sum_{(i,j)} |\gamma_{i,j}^-|(\sum_{k\leq j}|\beta_k|)l_{i,j}$. We now estimate $\frac{d}{dt}\bar{Q}_d^-(t)$; $\frac{d}{dt}\bar{Q}_d^+(t)$ is similar.

$$\begin{aligned}\frac{d}{dt}\bar{Q}_d^-(t) &= \sum_{(i,j)} |\gamma_{i,j}^-|\Big(\sum_{k\leq j}|\beta_k|)(\lambda(\theta_{i,j}^r) - \lambda(\theta_{i,j}^l)\Big)\\ &= \sum_l |\beta_l|\Big(\sum_{(i,j),j\geq l}|\gamma_{i,j}^-|(\lambda(\theta_{i,j}^r) - \lambda(\theta_{i,j}^l)\Big).\end{aligned}$$

If we denote the first 1-wave on the right of β_p by α_{p_β} and use the estimates (4.1), (4.4), and (4.5), we have

$$\begin{aligned}\frac{d}{dt}\bar{Q}_d^-(t) \leq \sum_p |\beta_p|\Big\{&\sum_{i\geq p_\beta}\Big|\sum_{k=1}^{i}\hat{\alpha}_k\Big|(\lambda^*(\alpha_{i+1}) - \lambda^*(\alpha_i))\\ &+\Big|\sum_{k=1}^{p_\beta-1}\hat{\alpha}_k\Big|(\lambda(\alpha_{p_\beta}) - \lambda(\beta_p)) + O(1)\sum_{j\geq p_\beta}|\hat{\beta}_j|\Big\}\\ \leq \sum_p |\beta_p|\Big\{&\sum_{i\geq p_\beta}\Big|\sum_{k=1}^{i}\hat{\alpha}_k\Big|(\mu_{i+1}(\alpha) - \mu_i(\alpha)) + |\gamma_{\beta_p}^-|(\lambda(\alpha_{p_\beta}) - \lambda(\beta_p))\\ &+O(1)\Big(\sum_{j\geq p}|\hat{\beta}_j| + \sum_{i\geq p_\beta}|\alpha_i||\gamma_{\alpha_i}^+| + \sum_i|\hat{\alpha}_i| + e(\Lambda_p)\Big)\Big\}.\end{aligned} \tag{4.10}$$

Let $\mu^*(\alpha)$ be the wave speed of the wave $[0, \sum_{k=1}^{p_\beta-1}\hat{\alpha}_k]$ of the scalar conservation law (4.3). If it is a rarefaction wave, then we partition it into small waves with strengths less than ϵ; the criterion for partitioning is the same as the one used in Section 2. Let

$$J_p(\alpha) = \begin{cases}\Big|\sum_{k=1}^{p_\beta-1}\hat{\alpha}\Big|(\mu_{p_\beta}(\alpha) - \mu^*(\alpha)), & \text{if } \sum_{k=1}^{p_\beta-1}\hat{\alpha} < 0,\\ \sum_j |\gamma_j^*|(\mu_{j+1}^*(\alpha) - \mu_j^*(\alpha)), & \text{if } \sum_{k=1}^{p_\beta-1}\hat{\alpha} > 0\end{cases}$$

where the summation on j is over all sub-waves partitioned, and γ_j^* is the corresponding first Riemann invariant. Here we have used the fact that the characteristic fields are genuinely nonlinear. Without loss of generality, we assume that $r_1\cdot\nabla\lambda_1 > 0$. By the L_1 contraction of the scalar conservation law, we have

$$\sum_{i\geq p_\beta}\Big(\Big|\sum_{k=1}^{i}\hat{\alpha}_k\Big|(\mu_{i+1}) - \mu(\alpha_i)\Big) + J_p(\alpha) \leq cTV\epsilon. \tag{4.11}$$

Here the error term $cTV\epsilon$ also is due to partitioning the rarefaction waves into sub-waves with strengths of order ϵ. Using (4.8) and (4.9), we have

$$\frac{d}{dt}\bar{Q}_d^-(t) \le \sum_p |\beta_p|\{-J_p(\alpha) + |\gamma_{\beta_p}^-|(\lambda(\alpha_{p_\beta}) - \lambda(\beta_p))\}$$
$$+ cTV\Big(\sum_i (|\hat{\alpha}_i| + |\hat{\beta}_i| + |\alpha_i||\gamma_{\alpha_i}^+|) + e(\Lambda_p)\Big). \tag{4.12}$$

System (1.1) is strictly hyperbolic, $\lambda(\alpha_{p_\beta}) - \lambda(\beta_p) \le -c_1$ for some positive constant c_1 independent of T and s. Thus, by the definition of $J_p(\alpha)$, we have

$$|J_p(\alpha)| \le TV\Big|\sum_{k=1}^{p_\beta - 1} \hat{\alpha}_k\Big|$$
$$\le TV\big(|\gamma_{\beta_i}^-| + \sum_i |\hat{\beta}_i|\big). \tag{4.13}$$

This, along with (4.12), yields

$$\frac{d}{dt}\bar{Q}_d^-(t) \le -2c_2 \sum_i |\beta_i||\gamma_{\beta_i}^-| + cTV\sum_i |\alpha_i||\gamma_{\alpha_i}^+|$$
$$+ cTV\sum_i (|\hat{\alpha}_i| + |\hat{\beta}_i|) + cTVe(\Lambda_p) \tag{4.14}$$

where c_2 is a positive constant independent of T and s. Similarly,

$$\frac{d}{dt}\bar{Q}_d^+(t) \le -2c_2 \sum_i |\alpha_i||\gamma_{\alpha_i}^+| + cTV\sum_i |\beta_i||\gamma_{\beta_i}^-|$$
$$+ cTV\sum_i (|\hat{\alpha}_i| + |\hat{\beta}_i|) + cTVe(\Lambda_p). \tag{4.15}$$

From (4.14) and (4.15), we have

$$\frac{d}{dt}\bar{Q}_d(t) \le -c_2 \sum_i \big(|\beta_i||\gamma_{\beta_i}^-| + |\alpha_i||\gamma_{\alpha_i}^+|\big)$$
$$+ cTV\sum_i (|\hat{\alpha}_i| + |\hat{\beta}_i|) + cTVe(\Lambda_p), \tag{4.16}$$

provided that TV is sufficiently small, $TV < \frac{c_2}{c}$.

Step 3. Estimate $\frac{d}{dt}\bar{Q}_s(t)$.

By the definition of $\bar{Q}_s(t)$, we have

$$\frac{d}{dt}\bar{Q}_s(t) = \frac{d}{dt}\bar{Q}_s^-(t) + \frac{d}{dt}\bar{Q}_s^+(t)$$

where $\bar{Q}_s^\pm(t) = \sum_{(i,j)} |\gamma_{i,j}^\pm|^2 l_{i,j}$. As before, we only estimate $\frac{d}{dt}\bar{Q}_s^-(t)$.

Following the notation used in estimating $\frac{d}{dt}\bar{L}(t)$, we have

$$\frac{d}{dt}\bar{Q}_s^-(t) = \sum_p |\gamma_p^-|^2(\lambda(\theta^{p+1}) - \lambda(\theta^p))$$
$$\le \sum_i \Big|\sum_{k=1}^{i} \hat{\alpha}_k\Big|^2 (\lambda^*(\alpha_{i+1}) - \lambda^*(\alpha_i)) + O(1)\,TV\sum_i |\hat{\beta}_i|$$

$$\le \sum_i \Big| \sum_{k=1}^{i} \hat{\bar{\alpha}}_k \Big|^2 (\mu_{i+1}(\alpha) - \mu_i(\alpha))$$
$$+ cTV \sum_i \big(|\alpha_i||\gamma^+_{\alpha_i}| + |\hat{\alpha}_i| + |\hat{\beta}_i| \big) + cTVe(\Lambda_p). \tag{4.17}$$

For an $n \times n$ system of conservation laws which is genuinely nonlinear and has a convex entropy and entropy flux, if its solution contains countably many shock waves and small total variation, then the derivative of the integral of the entropy over x with respect to time t is less than a negative constant times the sum of the total cubic shock strengths. It is easy to check that this is a consequence of Theorem 2.1 in Lax [6]. For the scalar equation (4.3), any convex function, in particular r^2, is an entropy function. Thus the summation $\sum_i |\sum_{k=1}^i \hat{\bar{\alpha}}_k|^2 (\mu_{i+1}(\alpha) - \mu_i(\alpha))$ can be used to control $\sum_i |\hat{\alpha}_i| = O(1) \sum_i |\alpha_i|^3$. For completeness, we state this as the following lemma.

Lemma 4.1. *Suppose that $u(x,t)$ is a solution of the scalar conservation law*

$$u_t + f(u)_x = 0,$$

with small total variation and a countable number of shock waves, denoted by $\{\zeta_i\}$. If $f(u)$ is convex, then

$$\frac{d}{dt} \int_{-\infty}^{\infty} u^2(x,t)dx \le -c_3 \sum_i |\zeta_i|^3 \tag{4.18}$$

where c_3 is a positive constant.

Proof. We denote the jump of any function g across the shock wave ζ_i by $[g]_i$ with the left and right values g_i^l and g_i^r, respectively, and the shock speed of ζ_i by s_i. Let q be the entropy flux corresponding to u^2, i.e., $q' = 2uf'$ and $' = \frac{d}{du}$, then we have

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{\infty} u^2(x,t)dx &= \sum_i (-s_i[u^2]_i + [q]_i) \\
&= \sum_i (-(u_i^r + u_i^l)[f]_i + [q]_i) \\
&= \sum_i \Big(-(u_i^r + u_i^l)(f_i'[u]_i + \frac{1}{2} f_i''[u]_i^2 + \frac{1}{6} f_i'''[u]_i^3) \\
&\quad + q_i'[u]_i + \frac{1}{2} q_i''[u]_i^2 + \frac{1}{6} q_i'''[u]_i^3 \Big) \\
&\quad + O(1) \sum_i [u]_i^4
\end{aligned}$$

where all the derivatives are evaluated at the left states. Since $q_i' = 2f_i'u_i^l$, $q_i'' = 2f_i''u_i^l + 2f_i'$ and $q_i''' = 2f_i'''u_i^l + 4f_i''$,

$$\frac{d}{dt} \int_{-\infty}^{\infty} u^2(x,t)dx = \frac{1}{6} \sum_i f_i''[u]_i^3 + 0(1) \sum_i [u]_i^4. \tag{4.19}$$

Since $f_i'' > 0$, we have $[u]_i < 0$ by the entropy condition for shocks, and so (4.18) follows from (4.19). □

By (4.15), Lemma 4.1, and (3.1), we have

$$\frac{d}{dt}\bar{Q}_s^-(t) \leq -2c_4 \sum_i |\hat{\alpha}_i| + cTVe(\Lambda_p) + cTV \sum_i \left(|\alpha_i||\gamma^+_{\alpha_i}| + |\beta_i||\gamma^-_{\beta_i}| + |\hat{\alpha}_i| + |\hat{\beta}_i|\right) \tag{4.20}$$

where c_4 is a constant independent of T and s. Similarly, we can prove

$$\frac{d}{dt}\bar{Q}_s^+(t) \leq -2c_4 \sum_i |\hat{\beta}_i| + cTVe(\Lambda_p) + cTV \sum_i \left(|\alpha_i||\gamma^+_{\alpha_i}| + |\beta_i||\gamma^-_{\beta_i}| + |\hat{\alpha}_i| + |\hat{\beta}_i|\right). \tag{4.21}$$

Combining (4.18) and (4.19) yields

$$\frac{d}{dt}\bar{Q}(t) \leq -c_4 \sum_i (|\hat{\alpha}_i| + |\hat{\beta}_i|) + cTV \sum_i (|\alpha_i||\gamma^+_{\alpha_i}| + |\beta_i||\gamma^-_{\beta_i}|) + cTVe(\Lambda_p) \tag{4.22}$$

where we have used the hypothesis that the total variation is small.

According to the estimates in Steps 1–3 and the fact that $F((p-1)Ms) \leq cTV$, we have

$$\begin{aligned}\frac{d}{dt}\bar{H}(t) &= \frac{d}{dt}\bar{L}(t)(1 + k_1F((p-1)Ms)) + k_2\left(\frac{d}{dt}\bar{Q}_s(t) + \frac{d}{dt}\bar{Q}_d(t)\right) \\ &\leq \left(c(1+k_1TV) - c_5k_2 + ck_2TV\right) \sum_i \left(|\alpha_i||\gamma^+_{\alpha_i}| + |\beta_i||\gamma^-_{\beta_i}| + |\hat{\alpha}_i| + |\hat{\beta}_i|\right) \\ &\quad + (c + ck_2 + cK_2TV)e(\Lambda_p)\end{aligned} \tag{4.23}$$

where $c_5 = \min\{c_2, c_4\}$. If we choose $TV < c_5/2c$, and

$$k_2 > \frac{2c}{c_5}(1 + k_1TV), \tag{4.24}$$

then

$$\frac{d}{dt}\bar{H}(t) \leq ce(\Lambda_p) \tag{4.25}$$

where $e(\Lambda_p) = Q(\Lambda_p) + C(\Lambda_p) + \delta + \epsilon$. It can be checked easily that there exist k_1 and k_2 satisfying (3.12) and (4.24) if the total variation TV is sufficiently small. Since $\bar{H}(t)$ is a continuous function of t in $(p-1)Ms \leq t \leq pMs$, we have

$$\begin{aligned}\bar{H}(pMs - 0) - \bar{H}((p-1)Ms + 0) &\leq \int_{(p-1)Ms}^{pMs} \frac{d}{dt}\bar{H}(t)dt \\ &\leq ce(\Lambda_p)Ms, \qquad p = 1, 2, \ldots, N.\end{aligned}$$

This proves the Main Lemma. □

Acknowledgements. The research of the first author is supported in part by Army Basic Research Grant DAAH 04-94-GS0045, ONR Grant N00014-95-1-0468, and NSF Grant DMS-9623025. The research of the second author is supported in part by the RGC Competitive Earmarked Research Grant 9040150.

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