Commun. Math. Phys. 351, 95–153 (2017)
Digital Object Identifier (DOI) 10.1007/s00220-017-2844-7

Communications in
Mathematical Physics

The Vlasov–Maxwell–Boltzmann System Near Maxwellians in the Whole Space with Very Soft Potentials

Renjun Duan[1], **Yuanjie Lei**[2], **Tong Yang**[3,4], **Huijiang Zhao**[5,6]

[1] Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.
E-mail: rjduan@math.cuhk.edu.hk
[2] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China. E-mail: leiyuanjie@hust.edu.cn
[3] Department of Mathematics, Jinan Unviersity, Guangdong, People's Republic of China
[4] Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong.
E-mail: matyang@cityu.edu.hk
[5] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China.
E-mail: hhjjzhao@whu.edu.cn
[6] Computational Science Hubei Key Laboratory, Wuhan University, Wuhan 430072, People's Republic of China

Received: 3 February 2016 / Accepted: 5 January 2017
Published online: 14 February 2017 –

Abstract: Since the work by Guo (Invent Math 153(3):593–630, 2003), it has remained an open problem to establish the global existence of perturbative classical solutions around a global Maxwellian to the Vlasov–Maxwell–Boltzmann system with the whole range of soft potentials. This is mainly due to the complex structure of the system, in particular, the degenerate dissipation at large velocity, the velocity-growth of the nonlinear term induced by the Lorentz force, and the regularity-loss of the electromagnetic fields. This paper solves this problem in the whole space provided that initial perturbation has sufficient regularity and velocity-integrability.

Contents

1. Introduction

The motion of dilute ionized plasmas consisting of two-species particles (e.g., electrons and ions) under the influence of binary collisions and the self-consistent electromagnetic field can be modelled by the Vlasov–Maxwell–Boltzmann system (cf. [3, Chapter 19] as well as [21, Chapter 6.6])

$$\begin{aligned}\partial_t F_+ + v\cdot\nabla_x F_+ + \frac{e_+}{m_+}\left(E+\frac{v}{c}\times B\right)\cdot\nabla_v F_+ &= Q(F_+,F_+)+Q(F_+,F_-),\\ \partial_t F_- + v\cdot\nabla_x F_- - \frac{e_-}{m_-}\left(E+\frac{v}{c}\times B\right)\cdot\nabla_v F_- &= Q(F_-,F_+)+Q(F_-,F_-).\end{aligned} \quad (1.1)$$

The electromagnetic field $[E,B]=[E(t,x),B(t,x)]$ satisfies the Maxwell equations

$$\begin{aligned}\partial_t E - c\nabla_x\times B &= -4\pi\int_{\mathbb{R}^3} v\,(e_+F_+ - e_-F_-)\,dv,\\ \partial_t B + c\nabla_x\times E &= 0,\\ \nabla_x\cdot E &= 4\pi\int_{\mathbb{R}^3}(e_+F_+ - e_-F_-)\,dv,\\ \nabla_x\cdot B &= 0.\end{aligned} \quad (1.2)$$

Here $\nabla_x = (\partial_{x_1},\partial_{x_2},\partial_{x_3})$, $\nabla_v = (\partial_{v_1},\partial_{v_2},\partial_{v_3})$. The unknown functions $F_\pm = F_\pm(t,x,v)\geq 0$ are the number density functions for the ions (+) and electrons (−) with position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ at time $t\geq 0$, respectively, $e_\pm$ and $m_\pm$ the magnitudes of their charges and masses, and c the speed of light.

Let $F(v)$, $G(v)$ be two number density functions for two types of particles with masses $m_\pm$ and diameters $\sigma_\pm$, then $Q(F,G)(v)$ is defined as (cf. [3])

$$\begin{aligned}Q(F,G)(v) &= \frac{(\sigma_+ + \sigma_-)^2}{4}\int_{\mathbb{R}^3\times\mathbb{S}^2}|u-v|^\gamma\,\mathbf{b}\left(\frac{\omega\cdot(v-u)}{|u-v|}\right)\\ &\quad\times\left\{F(v')G(u')-F(v)G(u)\right\}d\omega du\\ &\equiv Q_{gain}(F,G)-Q_{loss}(F,G).\end{aligned}$$

Here $\omega\in\mathbb{S}^2$ and $\mathbf{b}$, the angular part of the collision kernel, satisfies the Grad cutoff assumption (cf. [9])

$$0\leq \mathbf{b}(\cos\theta)\leq C|\cos\theta| \quad (1.3)$$

for some positive constant $C>0$. The deviation angle $\pi-2\theta$ satisfies $\cos\theta=\omega\cdot(v-u)/|v-u|$. Moreover, for $m_1,m_2\in\{m_+,m_-\}$,

$$v' = v-\frac{2m_2}{m_1+m_2}[(v-u)\cdot\omega]\omega,\quad u' = u+\frac{2m_1}{m_1+m_2}[(v-u)\cdot\omega]\omega,$$

which denote velocities (v',u') after a collision of particles having velocities (v,u) before the collision and vice versa. Notice that the above identities follow from the conservation of momentum m_1v+m_2u and energy $\frac{1}{2}m_1|v|^2+\frac{1}{2}m_2|u|^2$.

The exponent $\gamma\in(-3,1]$ in the kinetic part of the collision kernel is determined by the potential of intermolecular force, which is classified into the soft potential case when $-3<\gamma<0$, the Maxwell molecular case when $\gamma=0$, and the hard potential case when $0<\gamma\leq 1$ which includes the hard sphere model with $\gamma=1$ and $\mathbf{b}(\cos\theta)=C|\cos\theta|$

for some positive constant $C > 0$. For the soft potentials, the case $-2 \le \gamma < 0$ is called the moderately soft potentials while $-3 < \gamma < -2$ is called the very soft potentials, cf. [29] by Villani. The importance and the difficulty in studying the very soft potentials can be also found in that review paper.

The main purpose of this work is to construct global classical solutions to the Vlasov–Maxwell–Boltzmann system (1.1), (1.2) for the whole range of soft potentials, in particular, the very soft case when $-3 < \gamma < -2$, near global Maxwellians

$$\mu_+(v) = \frac{n_0}{e_+}\left(\frac{m_+}{2\pi k_B T_0}\right)^{\frac{3}{2}} \exp\left(-\frac{m_+|v|^2}{2k_B T_0}\right),$$
$$\mu_-(v) = \frac{n_0}{e_-}\left(\frac{m_-}{2\pi k_B T_0}\right)^{\frac{3}{2}} \exp\left(-\frac{m_-|v|^2}{2k_B T_0}\right),$$

in the whole space $\mathbb{R}^3$, where $k_B > 0$ is the Boltzmann constant, $n_0 > 0$ and $T_0 > 0$ are constant reference number density and temperature, respectively, and the reference bulk velocities have been chosen to be zero. We consider the Cauchy problem with prescribed initial data

$$F_\pm(0, x, v) = F_{0,\pm}(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x), \tag{1.4}$$

which satisfy the compatibility conditions

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0,+} - F_{0,-})dv, \quad \nabla_x \cdot B_0 = 0.$$

We remark that the angular non-cutoff case was considered in [6] based on the argument (cf. [16]) that the energy dissipations include an extra velocity differentiation due to the angular non-cutoff assumption. As will be explained later, the techniques used in [6] cannot be applied to the cutoff very soft case under consideration in the paper. The basic motivation here is to develop new strategies to deal with such a case.

We assume in the paper that all the physical constants are chosen to be one. Under such assumption, accordingly we normalize the above Maxwellians as

$$\mu = \mu_-(v) = \mu_+(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

To study the stability problem around μ, we define the perturbation $f_\pm = f_\pm(t, x, v)$ by

$$F_\pm(t, x, v) = \mu + \mu^{1/2} f_\pm(t, x, v).$$

Then, the Cauchy problem (1.1), (1.2), (1.4) is reformulated as

$$\begin{cases} \partial_t f_\pm + v \cdot \nabla_x f_\pm \pm (E + v \times B) \cdot \nabla_v f_\pm \mp E \cdot v\mu^{1/2} \mp \frac{1}{2} E \cdot v f_\pm + L_\pm f = \Gamma_\pm(f, f), \\ \partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} v\mu^{1/2}(f_+ - f_-)dv, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \int_{\mathbb{R}^3} \mu^{1/2}(f_+ - f_-)dv, \quad \nabla_x \cdot B = 0 \end{cases} \tag{1.5}$$

with initial data

$$f_\pm(0,x,v) = f_{0,\pm}(x,v), \quad E(0,x) = E_0(x), \quad B(0,x) = B_0(x) \tag{1.6}$$

satisfying the compatibility conditions

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} \mu^{1/2}(f_{0,+} - f_{0,-})dv, \quad \nabla_x \cdot B_0 = 0. \tag{1.7}$$

Here, as in [14], for later use, setting $f = [f_+, f_-]$, the first equation of (1.5) can also be written as

$$\partial_t f + v\cdot\nabla_x f + q_0(E + v\times B)\cdot\nabla_v f - E\cdot v\mu^{1/2}q_1 + Lf = \frac{q_0}{2}E\cdot vf + \Gamma(f,f), \tag{1.8}$$

where $q_0 = \mathrm{diag}(1,-1)$, $q_1 = [1,-1]$, and the linearized collision operator $L = [L_+, L_-]$ and the nonlinear collision operator $\Gamma = [\Gamma_+, \Gamma_-]$ are to be given later on.

We are now ready to state the main theorem in this paper.

Theorem 1.1. *Let* $-3 < \gamma < -1$ *and* (1.3) *hold. Assume* $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$. *Take* $1/2 \leq \varrho < 3/2$ *and* $0 < q \ll 1$. *There exist some integer* $N > 0$ *and* $l_0^* > 0$ *such that if*

$$\sum_{|\alpha|+|\beta|\leq N} \left\| \langle v\rangle^{l_0^*-|\beta|} e^{q\langle v\rangle^2} \partial^\alpha_\beta f_0 \right\| + \|f_0\|_{L^2_v(\dot{H}^{-\varrho})} + \|(E_0,B_0)\|_{H^N \bigcap \dot{H}^{-\varrho}}$$

is sufficiently small, then the Cauchy problem (1.5), (1.6), (1.7) *admits a unique global solution* $[f(t,x,v), E(t,x), B(t,x)]$ *satisfying* $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$.

In the next section, the statement of the above theorem will be given more precisely in Theorem 2.1 as well as Theorem 2.2 for the time decay property. Basically the result shows that as long as initial data is small with enough regularity, one can establish the global existence of small amplitude classical solutions for the full range of cutoff intermolecular interactions with $-3 < \gamma \leq 1$. Note that the case $-1 \leq \gamma \leq 1$ is a trivial consequence of [6]; details for that case will be briefly discussed in Sect. 2.2. Here, the bound in the Sobolev space of negative index is used for obtaining the time decay of solutions that is needed to close the a priori estimates. The general technique of adopting the negative Sobolev estimates to treat the time-decay of the Boltzmann equation and other types of dissipative equations was first introduced in [18]. For the case of the whole space, compared to L^1_x initial data used in [6], the space $\dot{H}^{-\varrho}_x$ is much more convenient to deduce the fast enough time-decay rates in terms of only the pure energy method and the interpolation inequalities.

The proof of Theorem 1.1 is based on a subtle time-weighted energy method. For this, in addition to the existing analytic techniques used in [6,16], we develop a new approach to deal with the weighted estimates involving both the negative power time-weight and the time-velocity dependent $w_{\ell-|\beta|,\kappa}(t,v)$ weight.

The rest of this paper is organized as follows. In Sect. 2, we explain the difficulty in studying the case when $-3 < \gamma < -1$, particularly including the very soft potential case, and give a complete statement of the main results. In Sect. 3, we list some basic lemmas for later use. The proof of the main results will be given in Sect. 4. For clear presentation, the proofs of several technical lemmas and estimates used in Sect. 4 will be given in the “Appendix”.

2. Main Results

In this section, we will first review the previous approaches for studying the global existence of classical solutions to Valsov–Maxwell–Boltzmann equations, and then we will point out the difficulties in studying the very soft potentials and give the complete statements of the main results.

First of all, we recall some basic facts concerning the collision operators and the macro-micro decomposition. L, Γ in (1.8) are respectively defined by

$$Lf = [L_+ f, L_- f], \quad \Gamma(f, g) = [\Gamma_+(f, g), \Gamma_-(f, g)]$$

with

$$L_\pm f = -\mu^{-1/2}\left\{Q\left(\mu, \mu^{1/2}(f_\pm + f_\mp)\right) + 2Q\left(\mu^{1/2} f_\pm, \mu\right)\right\},$$
$$\Gamma_\pm(f, g) = \mu^{-1/2}\left\{Q\left(\mu^{1/2} f_\pm, \mu^{1/2} g_\pm\right) + Q\left(\mu^{1/2} f_\pm, \mu^{1/2} g_\mp\right)\right\}.$$

For the linearized collision operator L, it is well known (cf. [14]) that it is non-negative and the null space $\mathcal{N}$ of L is spanned by

$$\mathcal{N} = \text{span}\left\{[1, 0]\mu^{1/2}, [0, 1]\mu^{1/2}, [v_i, v_i]\mu^{1/2}(1 \le i \le 3), [|v|^2, |v|^2]\mu^{1/2}\right\}.$$

Moreover, under Grad's angular cutoff assumption (1.3), it is easy to see that L can be decomposed as

$$Lf = \nu f - Kf \tag{2.1}$$

with the collision frequency $\nu(v)$ and the nonlocal integral operator $K = [K_+, K_-]$ being defined by

$$\nu(v) = 2Q_{loss}(1, \mu) = 2\int_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\gamma \mathbf{b}\left(\frac{\omega\cdot(v-u)}{|v-u|}\right)\mu(u)d\omega du \sim (1+|v|)^\gamma, \tag{2.2}$$

and

$$\begin{aligned}(K_\pm f)(v) &= \mu^{-\frac{1}{2}}\left\{2Q_{gain}\left(\mu^{\frac{1}{2}} f_\pm, \mu\right) - Q\left(\mu, \mu^{\frac{1}{2}}(f_\pm + f_\mp)\right)\right\}\\ &= \int_{\mathbb{R}^3\times\mathbb{S}^2} |u-v|^\gamma \mathbf{b}\left(\frac{\omega\cdot(v-u)}{|v-u|}\right)\mu^{\frac{1}{2}}(u)\\ &\quad\times\left\{2\mu^{\frac{1}{2}}(u')f_\pm(v') - \mu^{\frac{1}{2}}(v')(f_\pm + f_\mp)(u')\right.\\ &\quad\left.+\mu^{\frac{1}{2}}(v)(f_\pm + f_\mp)(u)\right\}d\omega du, \end{aligned} \tag{2.3}$$

respectively.

Define $\mathbf{P}$ as the orthogonal projection from $L^2(\mathbb{R}^3_v) \times L^2(\mathbb{R}^3_v)$ to $\mathcal{N}$. Then for any given function $f(t, x, v) \in L^2(\mathbb{R}^3_v) \times L^2(\mathbb{R}^3_v)$, one has

$$\begin{aligned}\mathbf{P}f &= a_+(t, x)[1, 0]\mu^{1/2} + a_-(t, x)[0, 1]\mu^{1/2}\\ &\quad+\sum_{i=1}^{3} b_i(t, x)[1, 1]v_i\mu^{1/2} + c(t, x)[1, 1](|v|^2 - 3)\mu^{1/2}\end{aligned}$$

with

$$a_{\pm} = \int_{\mathbb{R}^3} \mu^{1/2} f_{\pm} dv, \quad b_i = \frac{1}{2} \int_{\mathbb{R}^3} v_i \mu^{1/2} (f_+ + f_-) dv,$$
$$c = \frac{1}{12} \int_{\mathbb{R}^3} (|v|^2 - 3) \mu^{1/2} (f_+ + f_-) dv.$$

Therefore, we have the following macro-micro decomposition with respect to the given global Maxwellian $\mu(v)$, cf. [15],

$$f(t, x, v) = \mathbf{P} f(t, x, v) + \{\mathbf{I} - \mathbf{P}\} f(t, x, v),$$

where $\mathbf{I}$ denotes the identity operator, and $\mathbf{P} f$ and $\{\mathbf{I} - \mathbf{P}\} f$ are called the macroscopic and the microscopic component of $f(t, x, v)$, respectively.

Under the Grad's angular cutoff assumption (1.3), by [14, Lemma 1], L is locally coercive in the sense that

$$- \langle f, Lf \rangle \geq \sigma_0 \, |\{\mathbf{I} - \mathbf{P}\} f|_{\nu}^2 \equiv \sigma_0 \left\| \sqrt{\nu} \{\mathbf{I} - \mathbf{P}\} f \right\|_{L^2(\mathbb{R}^3_v)}^2, \quad \nu(v) \sim (1 + |v|)^{\gamma} \tag{2.4}$$

holds for some positive constant $\sigma_0 > 0$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^3_v) \times L^2(\mathbb{R}^3_v)$.

2.1. Existing approaches. For the problem on the construction of solutions to the Cauchy problem (1.5), (1.6), (1.7), the local existence and uniqueness of solution $[f_+(t, x, v), f_-(t, x, v), E(t, x), B(t, x)]$ in certain weighted Sobolev space to be specified later can be obtained by combining the arguments used in [14] and [16]. To extend the local solution $[f_+(t, x, v), f_-(t, x, v), E(t, x), B(t, x)]$ to be global in time, one needs to deduce certain a priori estimates in some function spaces. In general, the main difficulties in this step lies in:

- How to control the possible velocity-growth induced by the nonlinearity of the system (1.8)?
- How to control the convection term $v \cdot \nabla_x f$ in the weighted energy estimates?

The nonlinear energy method developed in [12,15,23,24] for the Boltzmann equation provides an effective approach in the perturbative framework; see also the recent progress [11]. The main idea in those work is to decompose the solution into the macroscopic part and the microscopic part and then rewrite the original equation as the combination of an equation satisfied by the microscopic part which contains the macroscopic part as source term and a system satisfied by the macroscopic part with the microscopic part as source term. In the perturbative framework, the dissipative mechanism on the microscopic part is the coercive estimate (2.4) of the linearized Boltzmann collision operator or its weighted variants, while for the macroscopic part, the corresponding mechanism comes from the dissipation of the compressible Navier-Stokes type system. The corresponding approach to treat the case of non-cutoff cross sections was developed in [2] and [10].

However, as pointed out in [16] and [6], when one applies the energy method to some complex systems such as the Vlasov–Maxwell–Boltzmann system (1.1), (1.2), in addition to the difficulty caused by the nonlinear collision operator mentioned above, additional difficulties are encountered:

- How to control the corresponding nonlinear terms induced by the Lorentz force, such as the terms $(E + v \times B) \cdot \nabla_v f$ and $E \cdot vf$, that can lead to velocity growth at the rate of the first order $|v|$?
- How to cope with the regularity loss of the electromagnetic field $[E(t, x), B(t, x)]$?

For the hard sphere model, the coercive estimate (2.4) of L is sufficient to control the nonlinear terms related to the Lorentz force provided that the electromagnetic field $[E(t, x), B(t, x)]$ is suitably small and thus satisfactory global well-posedness theory for the Vlasov–Maxwell–Boltzmann system (1.1), (1.2) for the hard sphere model has been established, cf. [7,14,17,20,25] and the references therein. But for the corresponding problem involving cutoff non-hard sphere intermolecular interactions with $\gamma < 1$, the story is quite different. One can not use the coercive estimate (2.4) of L to absorb the nonlinear terms related to the Lorentz force which yield the velocity growth at the rate of the first order $|v|$. Thus it is interesting and important to find out how to construct global classical solutions near Maxwellians to the Vlasov–Maxwell–Boltzmann system (1.1), (1.2) for cutoff non-hard sphere cases. Certainly, the same applies to the Vlasov–Poisson–Landau system and the Vlasov–Maxwell–Landau system containing the Coulomb potential, cf. [16,28,30] and [4,26], respectively; and the Vlasov–Poisson–Boltzmann system for non-hard sphere interactions cf. [5,8,32].

Particularly, a breakthrough was made in Guo's work [16] on the two-species Vlasov–Poisson–Landau system in a periodic box, that leads to the subsequent works for the Vlasov–Poisson–Landau system in the whole space mentioned above. The main ideas can be outlined as follows:

- An exponential weight of electric potential $e^{\mp\phi}$ is used to cancel the growth of the velocity in the nonlinear term $\mp\frac{1}{2}\nabla_x\phi \cdot vf_\pm$.
- A velocity weight
$$\overline{w}_{l-|\alpha|-|\beta|}(v) = \langle v\rangle^{-(\gamma+1)(l-|\alpha|-|\beta|)}, \quad \langle v\rangle = \sqrt{1+|v|^2}, \quad l \geq |\alpha| + |\beta|$$
is used to compensate the weak dissipation of the linearized Landau kernel $\mathcal{L}$ for the case of $-3 \leq \gamma < -2$;
- The decay of the electric field $\phi(t, x)$ is used to close the energy estimate.

However, since the Lorentz force $E + v \times B$ is not of the potential form, the argument in [16] can not be directly adopted to study the Vlasov–Maxwell–Boltzmann system (1.1), (1.2). For this, a time-velocity weighted energy method is introduced in [8] by using the following weight $\widetilde{w}_{\ell,|\beta|}(t, v)$ function:

$$\widetilde{w}_{\ell-|\beta|} \equiv \widetilde{w}_{\ell-|\beta|}(t, v) = \langle v\rangle^{-\gamma(\ell-|\beta|)} e^{\frac{q\langle v\rangle^2}{(1+t)^\vartheta}}, \quad 0 < q \ll 1, \quad |\beta| \leq \ell, \quad 0 < \vartheta \leq \frac{1}{4}. \tag{2.5}$$

Here it is worth pointing out that, unlike the weight function $\overline{w}_{l-|\alpha|-|\beta|}(v)$, the algebraic factor $\widetilde{w}^a_{\ell-|\beta|}(v) = \langle v\rangle^{-\gamma(\ell-|\beta|)}$ in (2.5) varies only with the order of the $v-$derivatives to represent the fact that the dissipative effect of the cutoff linearized Boltzmann collision operator L is "weaker" than that of the linearized Landau collision operator $\mathcal{L}$.

2.2. Difficulties for very soft potentials. To illustrate the main ideas used in [6,8] for $-1 \leq \gamma \leq 1$, and the problem to be studied in this paper, we first introduce the following general weight function

$$w_{\ell-|\beta|,\kappa} \equiv w_{\ell-|\beta|,\kappa}(t, v) = \langle v\rangle^{\kappa(\ell-|\beta|)} e^{\frac{q\langle v\rangle^2}{(1+t)^\vartheta}}, \quad \kappa \geq 0, \quad 0 < q \ll 1, \tag{2.6}$$

where the precise range of the parameter ϑ will be specified later. It is easy to see that

$$w_{\ell-|\beta|,-\gamma}(t,v) \equiv \widetilde{w}_{\ell-|\beta|}(t,v).$$

Since for cutoff non-hard sphere intermolecular interactions, the macroscopic part can be controlled as for the case of hard sphere model, the main difficulty for the case of non-hard sphere model is to control the microscopic component $\{\mathbf{I}-\mathbf{P}\}f(t,x,v)$ suitably. The idea for that purpose is to use the following two types of dissipative mechanisms:

- The first one is the dissipative term

$$D^L_{|\alpha|,\ell-|\beta|,\kappa} \equiv \left\| w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu \equiv \left\|\sqrt{\nu}w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{L^2(\mathbb{R}^3_v\times\mathbb{R}^3_x)}$$

  from the coercive estimate of the linearized collision operator L;
- The second type is the extra dissipative term

$$D^W_{|\alpha|,\ell-|\beta|,\kappa} \equiv (1+t)^{-1-\vartheta}\left\| w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2_{L^2(\mathbb{R}^3_v\times\mathbb{R}^3_x)}$$

  induced by the weight function $w_{\ell-|\beta|,\kappa}(t,v)$.

The most difficult terms to be studied are:

- The term

$$I^{lt}_{|\alpha|,\ell-|\beta|,\kappa} \equiv \left(\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f, w^2_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right) \tag{2.7}$$

  related to the linear transport term $v\cdot\nabla_x f$;
- The terms containing the electromagnetic field $[E(t,x),B(t,x)]$, i.e.

$$I^E_{|\alpha|,\ell-|\beta|,\kappa} \equiv \sum_{|\alpha_1|\geq 1}\left(\partial^{\alpha_1}E\cdot\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f, w^2_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right), \tag{2.8}$$

  and

$$I^B_{|\alpha|,\ell-|\beta|,\kappa} \equiv \sum_{|\alpha_1|\geq 1}\left(\left(v\times\partial^{\alpha_1}B\right)\cdot\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f, w^2_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right). \tag{2.9}$$

  Here $(\cdot,\cdot)$ denotes the standard $L^2(\mathbb{R}^3_v\times\mathbb{R}^3_x)\times L^2(\mathbb{R}^3_v\times\mathbb{R}^3_x)$ inner product in $\mathbb{R}^3_v\times\mathbb{R}^3_x$.

To deduce the desired estimates on the above terms, the main ingredients used in [6,8] can be summarized as follows:

- A time-velocity weighted energy method is introduced basing on the weight function $\widetilde{w}_{\ell-|\beta|}(t,v) = w_{\ell-|\beta|,-\gamma}(t,v)$. An advantage of this weight function is that the term $I^{lt}_{|\alpha|,\ell-|\beta|,-\gamma}$ related to the linear transport term $v\cdot\nabla_x f$ can be controlled suitably. In fact,

$$\widetilde{w}^2_{\ell-|\beta|} = \widetilde{w}_{\ell-|\beta|}\times\widetilde{w}_{\ell-|\beta-e_i|}\times\langle v\rangle^\gamma,$$

  then

$$\left|I^{lt}_{|\alpha|,\ell-|\beta|,-\gamma}\right| \leq \varepsilon\left\|\widetilde{w}_{\ell-|\beta|}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu + C(\varepsilon)\left\|\widetilde{w}_{\ell-|\beta-e_i|}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu,$$

that, by a suitable linear combination with respect to $|\alpha|$, can be further controlled by the dissipation

$$\sum_{|\alpha|+|\beta|\leq N} D^L_{|\alpha|,\ell-|\beta|,-\gamma}$$

induced by the linearized Boltzmann collision operator L.

On the other hand, this approach leads to an additional difficulty on estimating the nonlinear term $(E + v \times B) \cdot \nabla_v f_\pm$ that requires a restriction on the range of the parameter γ. In fact, to control the term $I^B_{|\alpha|,\ell-|\beta|,-\gamma}$, by

$$\widetilde{w}^2_{\ell-|\beta|}(t,v) = \widetilde{w}_{\ell-|\beta|}(t,v) \times \widetilde{w}_{\ell-|\beta|-1}(t,v) \times \langle v\rangle^{-\gamma},$$

we can have

$$\begin{aligned}\left|I^B_{\alpha|,\ell-|\beta|,-\gamma}\right| \leq \sum_{0<\alpha_1\leq\alpha} \int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \langle v\rangle^{1-\gamma} \left|\partial^{\alpha_1} B\right| \left|\widetilde{w}_{\ell-|\beta|}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_\pm\right| \\ \times \left|\widetilde{w}_{\ell-|\beta|-1}\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f_\pm\right| dvdx, \qquad (2.10)\end{aligned}$$

which can be controlled by the dissipation

$$\sum_{|\alpha|+|\beta|\leq N} D^W_{|\alpha|,\ell-|\beta|,-\gamma}$$

induced by the exponential factor of the weight function $w_{\ell-|\beta|,-\gamma}(t,v)$ *only when*

$$1-\gamma \leq 2, \quad i.e. \quad \gamma \geq -1,$$

and $\partial^{\alpha_1}B(t,x)$ decays sufficiently fast.

Thus, up to now, the existing approaches for the construction of global classical solutions to the Vlasov–Maxwell–Boltzmann system (1.1), (1.2) near Maxwellians is limited to the case when $-1 \leq \gamma \leq 1$. And the purpose of this paper is to introduce a new approach for the whole range soft potential, that is, to include the case when $-3 < \gamma < -1$.

To continue, we first introduce some notations used throughout the paper.

- C and $O(1)$ denote some positive constants (generally large) and κ, δ and λ are used to denote some positive constants (generally small), where C, $O(1)$, κ, δ, and λ may take different values in different places;
- $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$;
- The multi-indices $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$ will be used to record spatial and velocity derivatives, respectively. And $\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}$. Similarly, the notation ∂^α will be used when $\beta = 0$ and likewise for ∂_β. The length of α is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $\alpha' \leq \alpha$ means that no component of α' is greater than the corresponding component of α, and $\alpha' < \alpha$ means that $\alpha' \leq \alpha$ and $|\alpha'| < |\alpha|$. And it is convenient to write $\nabla^k_x = \partial^\alpha$ with $|\alpha| = k$;
- $\langle\cdot,\cdot\rangle$ is used to denote the $L^2_v \times L^2_v$ inner product in $\mathbb{R}^3_v$, with the L^2 norm $|\cdot|_{L^2}$. For notational simplicity, $(\cdot,\cdot)$ denotes the $L^2 \times L^2$ inner product either in $\mathbb{R}^3_x \times \mathbb{R}^3_v$ or in $\mathbb{R}^3_x$ with the $L^2 \times L^2$ norm $\|\cdot\|$;

- χ_Ω is the standard indicator function of the set Ω;
- $\|f(t,\cdot,\cdot)\|_{L^p_xL^q_v} = \left(\int_{\mathbb{R}^3_x}\left(\int_{\mathbb{R}^3_v}|f(t,x,v)|^q dv\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$, and others like $\|f(t,\cdot,\cdot)\|_{L^p_xH^q_v}$ can be defined similarly;
- $B_C\subset\mathbb{R}^3$ denotes the ball of radius C centered at the origin, and $L^2(B_C)\times L^2(B_C)$ stands for the space $L^2\times L^2$ over B_C and likewise for other spaces. Recall that $\nu(v)\sim(1+|v|^2)^{\frac{\gamma}{2}}$, we set $|f|_\nu^2\equiv\int_{\mathbb{R}^3}|f|^2\nu(v)dv$ and for each $l\in\mathbb{R}$, $L^2_l(\mathbb{R}^3_v)\times L^2_l(\mathbb{R}^3_v)$ denotes the weighted function space with norm

$$|f|^2_{L^2_l}\equiv\int_{\mathbb{R}^3_v}|f(v)|^2\langle v\rangle^{2l}dv,\quad \langle v\rangle=\sqrt{1+|v|^2}.$$

$H^k_l(\mathbb{R}^3_v)\times H^k_l(\mathbb{R}^3_v)$ with the norm $|f|_{H^k_l}$ etc. can be defined similarly;
- For $s\in\mathbb{R}$,

$$\left(\Lambda^s g\right)(t,x,v)=\int_{\mathbb{R}^3}|\xi|^s\hat{g}(t,\xi,v)e^{2\pi ix\cdot\xi}d\xi=\int_{\mathbb{R}^3}|\xi|^s\mathcal{F}[g](t,\xi,v)e^{2\pi ix\cdot\xi}d\xi$$

with $\hat{g}(t,\xi,v)\equiv\mathcal{F}[g](t,\xi,v)$ being the Fourier transform of $g(t,x,v)$ with respect to x. The homogeneous Sobolev space $\dot{H}^s\times\dot{H}^s$ is the Banach space consisting of all g satisfying $\|g\|_{\dot{H}^s}<+\infty$, where

$$\|g(t)\|_{\dot{H}^s}\equiv\left\|\left(\Lambda^s g\right)(t,x,v)\right\|_{L^2_{x,v}}=\left\||\xi|^s\hat{g}(t,\xi,v)\right\|_{L^2_{\xi,v}}.$$

For an integer $N\geq 0$ and $\ell\in\mathbb{R}$, the parameter ϑ is suitably chosen so that

$$\begin{cases}0<\vartheta\leq\min\left\{\frac{\gamma-2\varrho\gamma+4\varrho+2}{4-4\gamma},\frac{2\varrho\gamma-3\gamma-4\varrho-6}{8\gamma-4}\right\}, & \text{when}\quad \varrho\in[\frac{1}{2},\frac{3}{2})\text{ and }N_0\geq 5,\\ 0<\vartheta\leq\min\left\{\frac{\gamma-2\varrho\gamma+4\varrho+2}{4-4\gamma},\frac{\varrho\gamma-2\gamma-2\varrho-2}{4\gamma-2}\right\}, & \text{when}\quad \varrho\in(1,\frac{3}{2})\text{ and }N_0=4.\end{cases}\tag{2.11}$$

Note that those strictly positive upper bounds for the choice of ϑ above are due to derivation of estimates (3.29) and (3.35) to be used in the later proof. Define the energy functional $\overline{\mathcal{E}}_{N,\ell,\kappa}(t)$ and the corresponding energy dissipation rate functional $\overline{\mathcal{D}}_{N,\ell,\kappa}(t)$ of a given function $f(t,x,v)$ with respect to the weight function $w_{\ell-|\beta|,\kappa}(t,v)$ defined by (2.6) as follows:

$$\overline{\mathcal{E}}_{N,\ell,\kappa}(t)\sim\mathcal{E}_{N,\ell,\kappa}(t)+\left\|\Lambda^{-\varrho}(f,E,B)\right\|^2,$$

and

$$\overline{\mathcal{D}}_{N,\ell,\kappa}(t)\sim\mathcal{D}_{N,\ell,\kappa}(t)+\left\|\Lambda^{1-\varrho}(a,b,c,E,B)\right\|^2$$
$$+\left\|\Lambda^{-\varrho}(a_+-a_-,E)\right\|^2+\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu,$$

respectively. Here

$$\mathcal{E}_{N,\ell,\kappa}(t)\sim\sum_{|\alpha|+|\beta|\leq N}\left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta f\right\|^2+\|(E,B)\|^2_{H^N_x},$$

and

$$\mathcal{D}_{N,\ell,\kappa}(t) \sim \sum_{1\le|\alpha|\le N} \left\|\partial^\alpha(a_\pm, b, c)\right\|^2 + \sum_{|\alpha|+|\beta|\le N} \left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \|a_+ - a_-\|^2$$
$$+\|E\|^2_{H^{N-1}_x} + \|\nabla_x B\|^2_{H^{N-2}_x} + (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|\le N} \left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2.$$

Moreover, we also need to define $\mathcal{E}_N(t)$, the energy functional without weight, $\mathcal{E}^k_{N_0}(t)$, the high order energy functional without weight, and $\mathcal{E}^k_{N_0,\ell,\kappa}(t)$, the high order energy functional with respect to the weight function $w_{\ell-|\beta|,\kappa}(t,v)$, as follows:

$$\mathcal{E}_N(t) \sim \sum_{k=0}^{N} \left\|\nabla^k(f,E,B)\right\|^2,$$
$$\mathcal{E}^k_{N_0}(t) \sim \sum_{|\alpha|=k}^{N_0} \left\|\partial^\alpha(f,E,B)\right\|^2,$$

and

$$\mathcal{E}^k_{N_0,\ell,\kappa}(t) \sim \sum_{\substack{|\alpha|+|\beta\le N_0,\\ |\alpha|\ge k}} \left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta f\right\|^2 + \sum_{|\alpha|=k}^{N_0} \left\|\partial^\alpha(E,B)\right\|^2,$$

respectively. The corresponding energy dissipation rate functionals $\mathcal{D}_N(t)$, $\mathcal{D}^k_{N_0}(t)$, and $\mathcal{D}^k_{N_0,\ell,\kappa}(t)$ are given by

$$\mathcal{D}_N(t) \sim \|(E, a_+ - a_-)\|^2 + \sum_{1\le|\alpha|\le N-1} \left\|\partial^\alpha(\mathbf{P}f, E, B)\right\|^2$$
$$+ \sum_{|\alpha|=N} \left\|\partial^\alpha\mathbf{P}f\right\|^2 + \sum_{|\alpha|\le N} \left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,$$
$$\mathcal{D}^k_{N_0}(t) \sim \left\|\nabla^k(E, a_+ - a_-)\right\|^2 + \sum_{k+1\le|\alpha|\le N_0-1} \left\|\partial^\alpha(\mathbf{P}f, E, B)\right\|^2$$
$$+ \sum_{|\alpha|=N_0} \left\|\partial^\alpha\mathbf{P}f\right\|^2 + \sum_{k\le|\alpha|\le N_0} \left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,$$

and

$$\mathcal{D}^k_{N_0,\ell,\kappa}(t) \sim \left\|\nabla^k(E, a_+ - a_-)\right\|^2 + \sum_{k+1\le|\alpha|\le N_0-1} \left\|\partial^\alpha(\mathbf{P}f, E, B)\right\|^2$$
$$+ \sum_{\substack{|\alpha|+|\beta\le N_0,\\ |\alpha|\ge k}} \left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2$$
$$+ \sum_{|\alpha|=N_0} \left\|\partial^\alpha\mathbf{P}f\right\|^2 + (1+t)^{-1-\vartheta} \sum_{\substack{|\alpha|+|\beta\le N_0,\\ |\alpha|\ge k}} \left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2,$$

respectively.

2.3. Main results and ideas. With the above preparation, the precise statement concerning the global in time solvability of the Cauchy problem (1.5), (1.6), (1.7) can be stated as follows.

Theorem 2.1. *Suppose that*

(i) $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$, $\frac{1}{2} \leq \varrho < \frac{3}{2}$, $-3 < \gamma < -1$. *Let*

$$\begin{cases} N_0 \geq 5, & N = 2N_0 - 1, \quad \text{when} \quad \varrho \in [\frac{1}{2}, 1], \\ N_0 \geq 4, & N = 2N_0, \qquad\;\; \text{when} \quad \varrho \in (1, \frac{3}{2}); \end{cases} \tag{2.12}$$

(ii) *The parameter ϑ is chosen to satisfy* (2.11) *and we take* $\sigma_{N,0} = \frac{1+\epsilon_0}{2}$, $\sigma_{n,0} = 0$ *with* $n \leq N-1$, $\sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)$ *when* $0 \leq j \leq n$ *and* $1 \leq n \leq N$;

(iii) *There exists a positive constants $\widetilde{l}$ which depends only on γ and N_0 such that*

(a) $\widetilde{l}_1 \geq \frac{\gamma}{2} + \frac{(1-2\gamma)\sigma_{N_0,N_0}}{2+\varrho}$, $\widetilde{\ell}_2 \geq \frac{\gamma}{2} + \frac{2(1-2\gamma)\sigma_{N,N}}{3+2\varrho}$, *and* $\widetilde{\ell}_3 \geq \frac{\gamma}{2} + \frac{(1-2\gamma)\sigma_{N-1,N-1}}{2+\varrho}$,

(b) $l_1 \geq N$, $l_1^* \geq \max\left\{\widetilde{\ell}_2 - \frac{\gamma}{2},\ \widetilde{\ell}_3 - \frac{\gamma}{2} - \gamma l_1\right\}$, $l_0 \geq l_1^* + \frac{5}{2}$, $l_0^* \geq \widetilde{\ell}_1 - \frac{\gamma}{2} - \gamma(l_0 + l^*)$ *with* $l^* = \frac{3}{2} - \frac{\widetilde{l}}{\gamma}$.

If we assume further that

$$\begin{aligned} Y_0 = & \sum_{|\alpha|+|\beta| \leq N_0} \left\| w_{l_0^*-|\beta|,1} \partial_\beta^\alpha f_0 \right\| \\ & + \sum_{N_0+1 \leq |\alpha|+|\beta| \leq N} \left\| w_{l_1^*-|\beta|,1} \partial_\beta^\alpha f_0 \right\| + \|(E_0, B_0)\|_{H^N \bigcap \dot{H}^{-\varrho}} + \|f_0\|_{\dot{H}^{-\varrho}} \end{aligned}$$

is sufficiently small, then the Cauchy problem (1.5), (1.6), (1.7) *admits a unique global solution* $[f(t,x,v), E(t,x), B(t,x)]$ *satisfying* $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$.

Remark 2.1. Several remarks concerning Theorem 2.1 are given.

- As mentioned before, although only the case of $-3 < \gamma < -1$ is studied in this paper, the case of $-1 \leq \gamma \leq 1$ is much simpler and similar result holds. Thus, the current work provides a satisfactory well-posedness theory for the Cauchy problem of the two-species Vlasov–Maxwell–Boltzmann system (1.5), (1.6), (1.7) in the perturbative framework for the whole range of the cutoff intermolecular interactions.
- Since in the proof of Lemma 4.3, N is assumed to satisfy $N > \frac{5}{3}N_0 - \frac{5}{3}$, while in the proof of Lemma 3.5, N is further required to satisfy $N \geq 2N_0 - 2 + \varrho$. Putting these assumptions together, we can take $N = 2N_0 - 1$ for $\varrho \in [\frac{1}{2}, 1]$ and $N = 2N_0$ for $\varrho \in (1, \frac{3}{2})$.
- The minimal regularity index, i.e., the lower bound on the parameter N, we imposed on the initial data is $N = 9$, $N_0 = 5$ for $\varrho \in [\frac{1}{2}, 1]$ and $N = 8$, $N_0 = 4$ for $\varrho \in (1, \frac{3}{2})$.
- The precise value of the parameter $\widetilde{l}$ will be specified in the proof of Lemma 4.3.

Note that Theorem 1.1 is an immediate consequence of Theorem 2.1. The next result is concerned with the temporal decay estimates on the global solution $[f(t,x,v), E(t,x), B(t,x)]$ obtained in Theorem 2.1.

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have*

(1) *Taking* $k = 0, 1, 2, \ldots, N_0 - 2$, *it follows that*

$$\mathcal{E}^k_{N_0}(t) \lesssim Y_0^2(1+t)^{-(\varrho+k)}. \tag{2.13}$$

(2) *Let* $0 \leq i \leq k \leq N_0 - 3$ *be an integer. Take* $l_{0,k} \geq N_0$ *with* $l_{0,k-1} \geq l_{0,k} + 3$ *for* $2 \leq k \leq N_0 - 3$. *Further take* l_0 *and* l^* *respectively as* $l_0 = l_{0,0} = l_{0,1} \geq \max\left\{\chi_{k\geq 2}(l_{0,k} + 3k - 3), l_1^* + \frac{5}{2}\right\}$ *and* $l^* = \frac{k+2}{2} - \frac{\widetilde{l}}{\gamma}$ *in Theorem 1.1. Then it follows that*

$$\mathcal{E}^k_{N_0, l_{0,k}+\frac{i}{2}, -\gamma}(t) \lesssim Y_0^2(1+t)^{-k-\varrho+i}, \quad i = 0, 1, \ldots, k + [\varrho]. \tag{2.14}$$

Here and in the sequel $[\varrho]$ *denotes the greatest integer less than* ϱ.

(3) *When* $N_0 + 1 \leq |\alpha| \leq N - 1$,

$$\|\partial^\alpha f\|^2 \lesssim Y_0^2(1+t)^{-\frac{(N-|\alpha|)(N_0-2+\varrho)}{N-N_0}}. \tag{2.15}$$

Remark 2.2. In Theorem 2.2, we notice that the highest index k of $\mathcal{E}^k_{N_0}(t)$ is $N_0 - 2$ while the highest index of $\mathcal{E}^k_{N_0,\ell,-\gamma}(t)$ is $N_0 - 3$. The reason is that the highest order $\|\partial^\alpha E\|^2$ appearing in (3.9) does not belong to the corresponding dissipation rate $\mathcal{D}^k_{N_0,\ell,-\gamma}(t)$.

Now we present the main ideas in the proof. To overcome the difficulties pointed out before for the case when $-3 < \gamma < -1$, the main observation is that two sets of time-velocity weighted energy estimates should be performed simultaneously as explained in the following.

(i) First of all, when estimating $I^B_{|\alpha|,\ell-|\beta|,\kappa}$ defined by (2.9) for $\kappa = -\gamma$, there are some error terms with higher weight when $-3 < \gamma < -1$, cf. (2.10) that can not be controlled. However, as long as the solution $[f(t,x,v), E(t,x), B(t,x)]$ constructed up to $t = T > 0$ satisfies the a priori assumption

$$\begin{aligned}
X(t) = &\sup_{0\leq s\leq t}\left\{\mathcal{E}_N(s) + \overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(s) + \mathcal{E}_{N-1,l_1,-\gamma}(s)\right\}\\
&+ \sup_{0\leq s\leq t}\left\{\sum_{N_0+1\leq n\leq N}\ \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j, 1\leq j\leq n}} (1+s)^{-\sigma_{n,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right.\\
&+ \sum_{N_0+1\leq n\leq N-1}\ \sum_{|\alpha|=n}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2 + \sum_{|\alpha|=N}(1+s)^{-\frac{1+\epsilon_0}{2}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2\\
&+ \sum_{1\leq n\leq N_0}\ \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j, 1\leq j\leq n}} (1+s)^{-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\left.+ \sum_{1\leq n\leq N_0}\ \sum_{|\alpha|=n}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2 + \left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right\} \leq M,
\end{aligned} \tag{2.16}$$

where $M > 0$ is sufficiently small, then one can obtain

$$\frac{d}{dt}\overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t) + \overline{\mathcal{D}}_{N_0,l_0+l^*,-\gamma}(t)$$

$$\lesssim \left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_1}} \widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t) + \sum_{|\alpha|=N_0} \varepsilon \|\partial^\alpha E\|^2,$$

$$\frac{d}{dt}\mathcal{E}_N(t) + \mathcal{D}_N(t)$$

$$\lesssim \left(\|E\|_{L_x^\infty} + \left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}\right)^{\frac{1}{\theta_2}} \widetilde{\mathcal{D}}_{N,l_1^*,1}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t),$$

and

$$\frac{d}{dt}\mathcal{E}_{N-1,l_1,-\gamma}(t) + \mathcal{D}_{N-1,l_1,-\gamma}(t)$$

$$\lesssim \left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}} \widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) + \sum_{|\alpha|=N-1} \left\|\partial^\alpha E\right\| \left\|\mu^\delta \partial^\alpha f\right\|,$$

where $\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$ are defined by (3.16) and (3.18) respectively.

Notice that θ_i $(i = 1, 2, 3)$ can be chosen sufficiently small as long as $l_j^*(j = 0, 1)$ is taken sufficiently large. Thus, one deduce some uniform-in-time estimates based on the above three differential inequalities provided that

(i1) The electromagnetic field $[E(t,x), B(t,x)]$ has certain temporal decay estimate and $\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \in L^1(\mathbb{R}^+)$;

(i2) There are some upper bound estimates on $\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$. For example, even if we can not deduce uniform-in-time bounds on $\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$, it suffices to show that the possible time increasing upper bounds on $\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$ are independent of the choices of the parameters l_j^* $(j = 0, 1)$ but depend only on N and N_0.

To achieve (i1), first of all, under the assumption of (2.16) with $M > 0$ sufficiently small, we can deduce that

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t) + \mathcal{D}^k_{N_0}(t) \le 0, \quad k = 0, 1, \ldots, N_0 - 2$$

and

$$\frac{d}{dt}\mathcal{E}^k_{N_0,\ell,-\gamma}(t) + \mathcal{D}^k_{N_0,\ell,-\gamma}(t) \lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2, \qquad k = 0, 1$$

hold for any $0 \le t \le T$.

From these two differential inequalities, by using the interpolation technique as in [18,30], we can deduce a temporal decay rate of $\mathcal{E}^k_{N_0}(t)$, from which one can further obtain the temporal decay rates of $\mathcal{E}^k_{N_0,l_0,-\gamma}(t)$ with $\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \in L^1(\mathbb{R}^+)$.

(ii) To deduce the estimates stated in (i2), we need the second set of time-velocity weighted energy estimates with the weight function $w_{\ell-|\beta|,1}(t,v)$ for some ℓ that is sufficiently large. In this case, since

$$w^2_{\ell-|\beta|,1}(t,v) = w_{\ell-|\beta|,1}(t,v) \times w_{\ell-|\beta|-1,1}(t,v) \times \langle v\rangle,$$

$$w^2_{\ell-|\beta|,1}(t,v) = w_{\ell-|\beta|,1}(t,v) \times w_{\ell-|\beta|+1,1}(t,v) \times \langle v\rangle^{-1},$$

we can deduce that for all $-3 < \gamma < -1$, the terms (2.8) and (2.9) can be controlled by the extra dissipative term (2.7) provided that the electromagnetic field $[E(t,x), B(t,x)]$ has certain temporal decay estimates. On the other hand, the term (2.7) related to the linear transport term $v \cdot \nabla_x f$ can only be bounded as

$$I^{lt}_{|\alpha|,\ell-|\beta|,1} \lesssim \eta \left\| w_{\ell-|\beta|,1}\partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 + C_\eta \left\| w_{\ell-|\beta-e_i|,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2.$$

Hence, it leads to how to control

$$\left\| w_{\ell-|\beta-e_i|,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2. \tag{2.17}$$

For (2.17), observe that

- Since $\frac{\gamma}{2} < -\frac{\gamma}{2} - 1 < 2$ holds for all $-3 < \gamma < -1$, it does not lead to the increase of the weight if we neglect the fact $(1+t)^{-1-\vartheta}$ in the extra dissipative term $D^W_{|\alpha|,\ell-|\beta|,1}$ given by (2.7);
- The order of the derivative with respect to x increases by one in (2.17) so that the corresponding temporal decay rate in L^2-norm increases $\frac{1}{2}$, cf. [6,7].

Therefore, motivated in [19] for deducing the temporal decay estimates on solutions to some nonlinear equations of regularity-loss type, we set different time increase rate $\sigma_{n,j}$ for

$$\sum_{|\alpha|+|\beta|=n, |\beta|=j} \left\| w_{l_1^*-j,1}\partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right\|^2,$$

where

$$\sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1+\gamma)}{\gamma-2}(1+\vartheta).$$

Thus, one can deduce that

$$\begin{aligned} &\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j, 1\le j\le n}} (1+t)^{-\sigma_{n,j}} \left\| w_{\ell-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2 \\ &\lesssim \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j, 1\le j\le N_0}} \left\{ (1+t)^{-\sigma_{n,j-1}-1-\vartheta} \left\| w_{\ell-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\} f\langle v\rangle \right\|^2 \right. \\ &\quad \left. + (1+t)^{-\sigma_{n,j-1}} \left\| w_{\ell-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 \right\}. \end{aligned}$$

Once the above argument is substantiated, we can then close the a priori assumption (2.16) and the global solvability result follows. And this will be given in detail in the following sections.

3. Proofs of the Main Results

The proofs of Theorems 2.1 and 2.2 will be given in this section. To illustrate the main ideas of the proof clearly and to make the presentation easy to follow, we will just state some key estimates first and then use them to prove our main results. The complete proofs of these key estimates will be given in the next section. To simplify the presentation, we divide this section into a few parts.

3.1. Preliminaries. In this subsection, for later use we collect several basic estimates on the linearized Boltzmann collision operator L and the nonlinear term Γ for cutoff potentials, whose one-species version can be found in [8,27].

The first lemma concerns the coercivity estimate (2.4) on the linearized collision operators L together with its weighted version with respect to the weight $w_{\ell,\kappa}(t,v)$ given by (2.6).

Lemma 3.1. *Let* $-3<\gamma<0$*, one has*

$$\langle Lf, f\rangle \geq |\{\mathbf{I}-\mathbf{P}\}f|_\nu^2. \tag{3.1}$$

Moreover, let $|\beta|>0$*, for* $\eta>0$ *small enough and any* $\ell\in\mathbb{R}$, $\kappa\geq 0$, $0<q\ll 1$, $\vartheta\in\mathbb{R}$*, there exists* $C_\eta>0$ *such that*

$$\left\langle w_{\ell,\kappa}^2\partial_\beta Lf, \partial_\beta f\right\rangle \geq \left|w_{\ell,\kappa}\partial_\beta f\right|_\nu^2-\eta\sum_{|\beta'|<|\beta|}\left|w_{\ell,\kappa}\partial_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right|_\nu^2-C_\eta\left|\chi_{\{|v|\leq 2C_\eta\}}f\right|^2 \tag{3.2}$$

holds.

Proof. For the estimate (3.1), the case for the hard sphere model has been proved in [14], while for general cutoff soft potentials, recall that L can be decomposed as in (2.1) with the collision frequency $\nu(v)$ and the nonlocal integral operator K being defined by (2.2) and (2.3) respectively, one can deduce by using the argument employed in Lemma 2 of [13] for one-species linearized Boltzmann collision operator with cutoff that the operator K can be decomposed into a "small part" K_s and a "compact part" K_c, therefore (3.1) follows by repeating the argument used in Lemma 3 of [13].

As to (3.2), it can be proved by a straightforward modification of the argument used in Lemma 2 of [27], we thus omit the details for brevity. □

The second lemma is concerned with the corresponding weighted estimates on the nonlinear term Γ. For this purpose, similar to that of [27], we can get that

$$\begin{aligned}\partial_\beta^\alpha\Gamma_\pm(g_1,g_2) &\equiv \sum C_\beta^{\beta_0\beta_1\beta_2}C_\alpha^{\alpha_1\alpha_2}\Gamma_\pm^0\left(\partial_{\beta_1}^{\alpha_1}g_1, \partial_{\beta_2}^{\alpha_2}g_2\right)\\ &\equiv \sum C_\beta^{\beta_0\beta_1\beta_2}C_\alpha^{\alpha_1\alpha_2}\int_{\mathbb{R}^3\times\mathbb{S}^2}|v-u|^\gamma\mathbf{b}(\cos\theta)\partial_{\beta_0}[\mu(u)^{\frac12}]\\ &\quad\times\left\{\partial_{\beta_1}^{\alpha_1}g_{1\pm}(v')\partial_{\beta_2}^{\alpha_2}g_{2\pm}(u')+\partial_{\beta_1}^{\alpha_1}g_{1\pm}(v')\partial_{\beta_2}^{\alpha_2}g_{2\mp}(u')\right.\\ &\quad\left.-\partial_{\beta_1}^{\alpha_1}g_{1\pm}(v)\partial_{\beta_2}^{\alpha_2}g_{2\pm}(u)-\partial_{\beta_1}^{\alpha_1}g_{1\pm}(v)\partial_{\beta_2}^{\alpha_2}g_{2\mp}(u)\right\}d\omega du,\end{aligned} \tag{3.3}$$

where $g_i(t,x,v)=[g_{i+}(t,x,v), g_{i-}(t,x,v)]$ $(i=1,2)$ and the summations are taken for all $\beta_0+\beta_1+\beta_2=\beta$, $\alpha_1+\alpha_2=\alpha$. From which one can deduce that

Lemma 3.2. *Assume* $\kappa\geq 0$, $\ell\geq 0$*. Let* $-3<\gamma<0$, $N\geq 4$, $g_i=g_i(t,x,v)=[g_{i+}(t,x,v), g_{i-}(t,x,v)]$ $(i=1,2,3)$, $\beta_0+\beta_1+\beta_2=\beta$ *and* $\alpha_1+\alpha_2=\alpha$*, we have the following results:*

(i) *When* $|\alpha_1|+|\beta_1|\leq N$*, we have*

$$\begin{aligned}\left\langle w_{\ell,\kappa}^2\Gamma_\pm^0\left(\partial_{\beta_1}^{\alpha_1}g_1, \partial_{\beta_2}^{\alpha_2}g_2\right), \partial_\beta^\alpha g_3\right\rangle \lesssim& \sum_{m\leq 2}\left\{\left|\nabla_v^m\left\{\mu^\delta\partial_{\beta_1}^{\alpha_1}g_1\right\}\right|+\left|w_{\ell,\kappa}\partial_{\beta_1}^{\alpha_1}g_1\right|\right\}\\ &\times\left|w_{\ell,\kappa}\partial_{\beta_2}^{\alpha_2}g_2\right|_{L_\nu^2}\left|w_{\ell,\kappa}\partial_\beta^\alpha g_3\right|_{L_\nu^2}\end{aligned} \tag{3.4}$$

or

$$\left\langle w_{\ell,\kappa}^2 \Gamma_{\pm}^0\left(\partial_{\beta_1}^{\alpha_1} g_1, \partial_{\beta_2}^{\alpha_2} g_2\right), \partial_\beta^\alpha g_3\right\rangle \lesssim \sum_{m\leq 2}\left\{\left|\nabla_v^m\left\{\mu^\delta \partial_{\beta_2}^{\alpha_2} g_2\right\}\right| + \left|w_{\ell,\kappa}\partial_{\beta_2}^{\alpha_2} g_2\right|\right\} \\ \times \left|w_{\ell,\kappa}\partial_{\beta_1}^{\alpha_1} g_1\right|_{L_v^2}\left|w_{\ell,\kappa}\partial_\beta^\alpha g_3\right|_{L_v^2}. \tag{3.5}$$

(ii) *Set* $\varsigma(v) = \langle v\rangle^{-\gamma} \equiv \nu(v)^{-1},\ l \geq 0$, *it holds that*

$$\begin{aligned} \left|\varsigma^l \Gamma(g_1, g_2)\right|_{L_v^2}^2 &\lesssim \sum_{|\beta|\leq 2}\left|\varsigma^{l-|\beta|}\partial_\beta g_1\right|_{L_v^2}^2\left|\varsigma^l g_2\right|_{L_v^2}^2, \\ \left|\varsigma^l \Gamma(g_1, g_2)\right|_{L_v^2}^2 &\lesssim \textstyle\sum_{|\beta|\leq 2}\left|\varsigma^l g_1\right|_{L_v^2}^2\left|\varsigma^{l-|\beta|}\partial_\beta g_2\right|_{L_v^2}^2. \end{aligned} \tag{3.6}$$

Proof. Although the definition of $\Gamma_{\pm}^0(g_1, g_2)$ in (3.3) is a little different from $\Gamma^0(g_1, g_2)$ of [27], one can still deduce (3.4) and (3.5) by employing the similar argument used to yield the estimates stated in Lemma 3 of [27], we thus omit its proof for simplicity. As for (3.6), it can also be proved by repeating the argument used in Lemma 2.4 of [31]. This completes the proof of Lemma 3.2. □

In what follows, we will collect some analytic tools which will be used in this paper. The first one is on the Sobolev interpolation among the spatial regularity.

Lemma 3.3. (cf. [1,18]) *Let* $2 \leq p < \infty$ *and* $k, \ell, m \in \mathbb{R}$, *then we have*

$$\left\|\nabla^k f\right\|_{L^p} \lesssim \left\|\nabla^\ell f\right\|_{L^2}^\theta \left\|\nabla^m f\right\|_{L^2}^{1-\theta}.$$

Here $0 \leq \theta \leq 1$ *and* ℓ *satisfy*

$$\frac{1}{p} - \frac{k}{3} = \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta + \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta).$$

Moreover, we have that

$$\left\|\nabla^k f\right\|_{L^\infty} \lesssim \left\|\nabla^\ell f\right\|_{L^2}^\theta \left\|\nabla^m f\right\|_{L^2}^{1-\theta},$$

where $0 \leq \theta \leq 1$ *and* ℓ *satisfy*

$$-\frac{k}{3} = \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta + \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta), \quad \ell \leq k+1, \quad m \geq k+2.$$

The second one is concerned with the $L^p - L^q$ estimate on the operator $\Lambda^{-\varrho}$.

Lemma 3.4. *Let* $0 < \varrho < 3,\ 1 < p < q < \infty,\ \frac{1}{q} + \frac{\varrho}{3} = \frac{1}{p}$, *then*

$$\|\Lambda^{-\varrho} f\|_{L^q} \lesssim \|f\|_{L^p}.$$

3.2. Some a priori estimates. In this subsection, we will deduce some a priori estimates on the solutions $[f(t,x,v), E(t,x), B(t,x)]$ to the Cauchy problem (1.5) and (1.6) under some additional assumptions imposed on $[f(t,x,v), E(t,x), B(t,x)]$. For this purpose, we suppose that the Cauchy problem (1.5) and (1.6) admits a unique local solution $[f(t,x,v), E(t,x), B(t,x)]$ defined on the time interval $0 \leq t \leq T$ for some $0 < T < \infty$. We now turn to deduce certain a priori estimates on $[f(t,x,v), E(t,x), B(t,x)]$. The first result is concerned with the temporal decay estimates on the energy functional $\mathcal{E}^k_{N_0}(t)$ for $k = 0, 1, 2, \ldots, N_0 - 2$:

Lemma 3.5. *Let N_0 and N satisfy (2.12), $n \geq \frac{2}{3}N_0 - \frac{5}{3}$, and take $k = 0, 1, 2, \ldots, N_0 - 2$, then one has*

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t) + \mathcal{D}^k_{N_0}(t) \leq 0, \quad 0 \leq t \leq T \tag{3.7}$$

provided that there exists a positive constant $\widetilde{l}$ whose precise range will be specified in the proof of Lemma 4.3 such that

(H_1) $\max\left\{\sup_{0\leq\tau\leq T}\mathcal{E}_{N_0+n}(\tau), \sup_{0\leq\tau\leq T}\mathcal{E}_{N-1,N-1,-\gamma}(\tau), \sup_{0\leq\tau\leq T}\overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(\tau)\right\}$ *is sufficiently small.*

Furthermore, as a consequence of (3.7), we can get that

$$\mathcal{E}^k_{N_0}(t) \lesssim \max\left\{\sup_{0\leq\tau\leq t}\overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau), \sup_{0\leq\tau\leq t}\mathcal{E}_{N_0+k+\varrho}(\tau)\right\}(1+t)^{-(k+\varrho)} \tag{3.8}$$

holds for $0 \leq t \leq T$.

Proof. First notice that under the smallness assumption (H_1), one can deduce that

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t) + \mathcal{D}^k_{N_0}(t) \leq 0,$$

which is an immediate consequence of Lemmas 4.3 and 4.4 whose proofs are complicated and thus are postponed to the next section.

Now we turn to compare the difference between $\mathcal{E}^k_{N_0}(t)$ and $\mathcal{D}^k_{N_0}(t)$. To this end, for the macroscopic component $\mathbf{P}f(t,x,v)$ and the electromagnetic field $[E(t,x), B(t,x)]$ one has by Lemma 3.3 that

$$\left\|\nabla^k(\mathbf{P}f, B)\right\| \leq \left\|\nabla^{k+1}(\mathbf{P}f, B)\right\|^{\frac{k+\varrho}{k+\varrho+1}} \left\|\Lambda^{-\varrho}(\mathbf{P}f, B)\right\|^{\frac{1}{k+\varrho+1}}$$

and

$$\left\|\nabla^{N_0}(E, B)\right\| \lesssim \left\|\nabla^{N_0-1}(E, B)\right\|^{\frac{k+\varrho}{k+\varrho+1}} \left\|\nabla^{N_0+k+\varrho}(E, B)\right\|^{\frac{1}{k+\varrho+1}},$$

while for the microscopic component $\{\mathbf{I}-\mathbf{P}\}f(t,x,v)$, we have by employing the Hölder inequality that

$$\begin{aligned}\sum_{k\leq|\alpha|\leq N_0}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\| &\leq \sum_{k\leq|\alpha|\leq N_0}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^{\frac{k+\varrho}{k+\varrho+1}}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma(k+\varrho)}{2}}\right\|^{\frac{1}{k+\varrho+1}} \\ &\leq \sum_{k\leq|\alpha|\leq N_0}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^{\frac{k+\varrho}{k+\varrho+1}}\left\|w_{\frac{k+\varrho}{2},-\gamma}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{k+\varrho+1}}.\end{aligned}$$

Therefore, we arrive at

$$\mathcal{E}^k_{N_0}(t) \le \left\{\mathcal{D}^k_{N_0}(t)\right\}^{\frac{k+\varrho}{k+\varrho+1}} \left\{\max\left\{\sup_{0\le\tau\le t} \overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau), \sup_{0\le\tau\le t} \mathcal{E}_{N_0+k+\varrho}(\tau)\right\}\right\}^{\frac{1}{k+\varrho+1}},$$

which combing with (3.7) yields that

$$\frac{d}{dt}\mathcal{E}^k_{N_0}(t) + \left\{\max\left\{\sup_{0\le\tau\le t} \overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau), \sup_{0\le\tau\le t} \mathcal{E}_{N_0+k+\varrho}(\tau)\right\}\right\}^{-\frac{1}{k+\varrho}} \left\{\mathcal{E}^k_{N_0}(t)\right\}^{1+\frac{1}{k+\varrho}} \le 0.$$

Solving the above inequality directly gives

$$\mathcal{E}^k_{N_0}(t) \lesssim \max\left\{\sup_{0\le\tau\le t} \overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau), \sup_{0\le\tau\le t} \mathcal{E}_{N_0+k+\varrho}(\tau)\right\}(1+t)^{-(k+\varrho)}.$$

Here we have used the fact that

$$\mathcal{E}^k_{N_0}(0) \lesssim \sup_{0\le\tau\le t}\left\{\overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau)\right\}.$$

This completes the proof of Lemma 3.5. □

Based on the above lemma, we can further obtain the temporal time decay of $\mathcal{E}^k_{N_0,\ell,-\gamma}(t)$ as in the following lemma.

Lemma 3.6. *Let $\ell \ge N_0$, $n \ge \frac{2}{3}N_0 - \frac{5}{3}$ and suppose that*

$$(H_2) \qquad \max\left\{\sup_{0\le\tau\le t} \mathcal{E}_{N_0+n}(\tau), \sup_{0\le\tau\le t} \mathcal{E}_{N_0,\ell-\frac{\widetilde{l}}{\gamma}}(\tau)\right\} \text{ is sufficiently small}$$

with $\widetilde{l}$ being given in Lemma 3.5, then the following estimates

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^k_{N_0,\ell,-\gamma}(t) + \mathcal{D}^k_{N_0,\ell,-\gamma}(t) \lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2 + \chi_{k\ge2} \sum_{\substack{1\le|\alpha'|\le k-1,\\ |\alpha|+|\beta|=N_0}} \left\|\partial^{\alpha'}(E,B)\right\|^2_{L^\infty_x} \\ \times \left\|w_{\ell-|\beta|-1,-\gamma}\partial^{\alpha-\alpha'}_{\beta+e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{3\gamma}{2}}\right\|^2 \qquad (3.9)\end{aligned}$$

hold for any $0\le t\le T$ and $k=0,1,\ldots,N_0-3$. Therefore, letting $l_{0,k}\ge N_0$ with $l_0 = l_{0,0} = l_{0,1}$ and $l_{0,k-1}\ge l_{0,k}+3$ for $2\le k\le N_0-3$, one has

$$\begin{aligned}\mathcal{E}^k_{N_0,l_{0,k}+\frac{i}{2},-\gamma}(t) \lesssim \max\left\{\sup_{0\le\tau\le t} \overline{\mathcal{E}}_{N_0,N_0+\frac{k+1+\varrho}{2},-\gamma}(\tau), \sup_{0\le\tau\le t} \mathcal{E}_{N_0+k+1+\varrho}(\tau)\right\} \\ \times(1+t)^{-k-\varrho+i}, \quad i=0,1,\ldots,k+[\varrho]. \qquad (3.10)\end{aligned}$$

Proof. We omit the proof of (3.9) as it is similar to the one of (3.7). Here, we point out that the main difference for proving (3.7) and (3.9):

- The term $\sum_{|\alpha|=N_0}\|\partial^\alpha E\|^2$ appears when we deal with the term $\sum_{|\alpha|=N_0}\left(\partial^\alpha E\cdot v\mu^{\frac{1}{2}},\right.$ $\left.w^2_{\ell,-\gamma}\partial^\alpha f\right)$;
- To deduce the desired estimates on

$$\sum_{\substack{1\le|\alpha_1|\le k-1,\\ |\alpha|=k,|\alpha|+|\beta|=N_0}}\left(\partial^{\alpha_1}E\cdot\nabla_v\partial_\beta^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f,w^2_{\ell-|\beta|,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\right)$$

  and

$$\sum_{\substack{1\le|\alpha_1|\le k-1,\\ |\alpha|=k,|\alpha|+|\beta|=N_0}}\left(\left(v\times\partial^{\alpha_1}B\right)\cdot\nabla_v\partial_\beta^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f,w^2_{\ell-|\beta|,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\right),$$

  one has to encounter the term

$$\sum_{\substack{1\le|\alpha_1|\le k-1,\\ |\alpha|=k,|\alpha|+|\beta|=N_0}}\left\|\partial^{\alpha_1}(E,B)\right\|^2_{L^\infty_x}\left\|\langle v\rangle^{1-\frac{3\gamma}{2}}w_{\ell-|\beta|-1,-\gamma}\partial^{\alpha-\alpha_1}_{\beta+e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|^2.$$

With (3.9) in hand, we now turn to prove (3.10). For the case $k=0,1$, the last term on the right hand side of (3.9) disappears, we have by replacing the parameter ℓ in (3.9) by $l_0+\frac{i}{2}(i=0,1,\dots,k+[\varrho])$ and then by multiplying the resulting inequality by $(1+t)^{k+\varrho-i+\epsilon}$ that

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{k+\varrho-i+\epsilon}\mathcal{E}^k_{N_0,l_0+\frac{i}{2},-\gamma}(t)\right\}+(1+t)^{k+\varrho-i+\epsilon}\mathcal{D}^k_{N_0,l_0+\frac{i}{2},-\gamma}(t)\\&\lesssim\sum_{|\alpha|=N_0}(1+t)^{k+\varrho-i+\epsilon}\|\partial^\alpha E\|^2+(1+t)^{k+\varrho-i-1+\epsilon}\mathcal{E}^k_{N_0,l_0+\frac{i}{2},-\gamma}(t).\end{aligned}\tag{3.11}$$

Here ϵ is taken as a sufficiently small positive constant.

By replacing the parameter ℓ in (3.9) by $l_0+\frac{k+[\varrho]+1}{2}$, it holds that

$$\frac{d}{dt}\mathcal{E}^k_{N_0,l_0+\frac{k+[\varrho]+1}{2},-\gamma}(t)+\mathcal{D}^k_{N_0,l_0+\frac{k+[\varrho]+1}{2},-\gamma}(t)\lesssim\sum_{|\alpha|=N_0}\|\partial^\alpha E\|^2.\tag{3.12}$$

By using the relation between the energy functional $\mathcal{E}^k_{N_0,l_0,-\gamma}(t)$ and its corresponding dissipation functional $\mathcal{D}^k_{N_0,l_0,-\gamma}(t)$, we deduce by a proper linear combination of (3.11) and (3.12) that

$$\begin{aligned}&\frac{d}{dt}\left\{\sum_{i=0}^{k+[\varrho]}C_i(1+t)^{k+\varrho-i+\epsilon}\mathcal{E}^k_{N_0,l_0+\frac{i}{2},-\gamma}(t)+C_{k+[\varrho]+1}\mathcal{E}^k_{N_0,l_0+\frac{k+[\varrho]+1}{2},-\gamma}(t)\right\}\\&\quad+\sum_{i=0}^{k+[\varrho]}(1+t)^{k+\varrho-i+\epsilon}\mathcal{D}^k_{N_0,l_0+\frac{i}{2},-\gamma}(t)+\mathcal{D}^k_{N_0,l_0+\frac{k+[\varrho]+1}{2},-\gamma}(t)\\&\lesssim\sum_{|\alpha|=N_0}(1+t)^{k+\varrho+\epsilon}\|\partial^\alpha E\|^2+(1+t)^{k+\varrho-1+\epsilon}\\&\quad\times\left\{\left\|\nabla^k(\mathbf{P}f,B)\right\|^2+\left\|\nabla^{N_0}B\right\|^2\right\}.\end{aligned}\tag{3.13}$$

On the other hand, Lemma 3.5 tells us that

$$\sum_{|\alpha|=N_0}(1+t)^{k+\varrho+\epsilon}\left\|\partial^\alpha E\right\|^2+(1+t)^{k+\varrho-1+\epsilon}\left\{\left\|\nabla^k(\mathbf{P}f,E,B)\right\|^2+\left\|\nabla^{N_0}B\right\|^2\right\}$$
$$\lesssim \max\left\{\sup_{0\le\tau\le t}\overline{\mathcal{E}}_{N_0,N_0+\frac{k+1+\varrho}{2},-\gamma}(\tau),\ \sup_{0\le\tau\le t}\mathcal{E}_{N_0+k+1+\varrho}(\tau)\right\}(1+t)^{-1+\epsilon}. \tag{3.14}$$

Plugging (3.14) into (3.13) and taking the time integration, one can get that

$$\sum_{i=0}^{k+[\varrho]}(1+t)^{k+\varrho-i+\epsilon}\mathcal{E}^k_{N_0,l_0+\frac{i}{2},-\gamma}(t)+\mathcal{E}^k_{N_0,l_0+\frac{k+[\varrho]+1}{2},-\gamma}(t)$$
$$+\int_0^t\left\{\sum_{i=0}^{k+[\varrho]}(1+\tau)^{k+\varrho-i+\epsilon}\mathcal{D}^k_{N_0,l_0+\frac{i}{2},-\gamma}(\tau)+\mathcal{D}^k_{N_0,l_0+\frac{k+[\varrho]+1}{2},-\gamma}(\tau)\right\}d\tau$$
$$\lesssim \max\left\{\sup_{0\le\tau\le t}\overline{\mathcal{E}}_{N_0,N_0+\frac{k+1+\varrho}{2},-\gamma}(\tau),\ \sup_{0\le\tau\le t}\mathcal{E}_{N_0+k+1+\varrho}(\tau)\right\}(1+t)^{\epsilon},$$

and the estimate (3.10) with the case $k=0,1$ follows by multiplying the above inequality by $(1+t)^{-\epsilon}$ where we take $l_0=l_{0,1}=l_{0,1}$.

As to the case of $2\le k\le N_0-3$, noticing that $\gamma\in(-3,-1)$, let $l_{0,k}\ge N_0$ and $l_{0,k-1}\ge l_{0,k}+3$, $l_0=l_{0,0}=l_{0,1}$, (3.10) with the case $2\le k\le N_0-3$ follows by using induction in k. Thus the proof of Lemma 3.6 is complete. □

The above two lemmas are for the temporal time decay estimates on $\mathcal{E}^k_{N_0}(t)$ and $\mathcal{E}^k_{N_0,\ell,-\gamma}(t)$ respectively which are based on the following two assumptions:

- $n>\frac{2}{3}N_0-\frac{5}{3}$ and $N_0+n\le N$. It is easy to see that if N_0 and N are suitably chosen such that (2.12) holds, one can be able to find such an index n;
- The assumptions (H_1) and (H_2) hold, that is, both

$$\max\left\{\sup_{0\le\tau\le t}\mathcal{E}_N(\tau),\ \sup_{0\le\tau\le t}\mathcal{E}_{N-1,N-1,-\gamma}(\tau),\ \sup_{0\le\tau\le t}\overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(\tau)\right\}$$

and

$$\max\left\{\sup_{0\le\tau\le t}\overline{\mathcal{E}}_{N_0,l_0+\frac{k+2}{2}-\frac{\widetilde{l}}{\gamma},-\gamma}(\tau),\ \sup_{0\le\tau\le t}\mathcal{E}_N(\tau)\right\}$$

are assumed to be small.

Set

$$l^*=\frac{k+2}{2}-\frac{\widetilde{l}}{\gamma}, \tag{3.15}$$

the above computation tells us that to guarantee the validity of the assumptions imposed in Lemmas 3.5 and 3.6, we need to control $\overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t)$, $\mathcal{E}_N(t)$, and $\mathcal{E}_{N-1,N-1,-\gamma}(t)$ suitably. To this end, we only outline the main ideas to yield these estimates and since the proofs are quite complicated, we leave the details to the next section. In fact, as pointed out in the introduction, if we perform the weighted energy estimate with respect to the weight function $w_{\ell-|\beta|,-\gamma}$, it is easy to see that the corresponding term $I^{lt}_{|\alpha|,\ell-|\beta|,-\gamma}$ defined by (2.7) related to the linear transport term $v\cdot\nabla_x f$ can be controlled suitably. In fact, due to

$$w^2_{\ell-|\beta|,-\gamma}(t,v)=w_{\ell-|\beta|,-\gamma}(t,v)\times w_{\ell-|\beta-e_i|,-\gamma}(t,v)\times\langle v\rangle^{\gamma},$$

the above term can be controlled by

$$I^{lt}_{|\alpha|,\ell-|\beta|,-\gamma}\lesssim\left\|w_{\ell-|\beta-e_i|,-\gamma}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\nu}+\varepsilon\left\|w_{\ell-|\beta|,-\gamma}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\nu}.$$

On the other hand, since

$$w^2_{\ell-|\beta|,-\gamma}(t,v)=w_{\ell-|\beta|,-\gamma}(t,v)\times w_{\ell-|\beta+e_i|,-\gamma}(t,v)\times\langle v\rangle^{-\gamma},$$

one can deduce that for $\gamma<-1$, the terms (2.8) and (2.9) containing the electromagnetic field $[E(t,x),B(t,x)]$ can not be controlled by the extra dissipation term

$$(1+t)^{-1-\vartheta}\left\|w_{\ell-|\beta|,-\gamma}(t,v)\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2$$

induced by the weight $w_{\ell-|\beta|,-\gamma}$.

To overcome such a difficulty, our main trick is to use the interpolation method for v to bound these terms by $\left\|\nabla^2(E,B)\right\|^{\frac{1}{\theta_1}}_{H^{N_0-2}_x}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$ with

$$\begin{aligned}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)\sim&\sum_{1\le n\le N_0}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}\left\|w_{l_0^*-j,1}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\nu}\\&+\sum_{1\le n\le N}\sum_{|\alpha|=n}\left\|w_{l_0^*,1}\partial^{\alpha}f\right\|^2_{\nu}+\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\nu}\end{aligned}\tag{3.16}$$

and some other similar terms. In fact, for $\overline{\mathcal{E}}_{N_0,\ell,-\gamma}(t)$, we can deduce that

Lemma 3.7. *Let* $N_0\ge 3$, $\ell\ge N_0$, $\widetilde{\ell}_1>\frac{1}{2}-\frac{1}{2}\gamma$, $\theta_1=\frac{1-2\gamma}{2\widetilde{l}_1-\gamma}$ *and* $l_0^*\ge\widetilde{\ell}_1-\frac{\gamma}{2}-\gamma\ell$, *then one has*

$$\begin{aligned}\frac{d}{dt}\overline{\mathcal{E}}_{N_0,\ell,-\gamma}(t)+\overline{\mathcal{D}}_{N_0,\ell,-\gamma}(t)\lesssim&\|E\|^{\frac{2-\gamma}{1-\gamma}}_{L^\infty_x}\sum_{|\alpha|+|\beta|\le N_0}\left\|w_{\ell-|\beta|,-\gamma}\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\&+\left\|\nabla^2(E,B)\right\|^{\frac{1}{\theta_1}}_{H^{N_0-2}_x}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)\\&+\sum_{|\alpha|=N_0}\varepsilon\|\partial^{\alpha}E\|^2\end{aligned}\tag{3.17}$$

provided that

(H_3) $\qquad\overline{\mathcal{E}}_{N_0,\ell}(t)$ *is sufficiently small.*

Note that $\varepsilon > 0$ is an arbitrary small constant, and for brevity of presentation, here and in the sequel the dependence of coefficient constants on ε similarly as on the right of (3.17) is skipped, since the order of those terms are strictly higher than that of the quadratic term.

Similar to the definition of $\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$ given in Lemma 3.7, for $m = N-1$ or N, $\widetilde{\mathcal{D}}_{m,l_1^*,1}(t)$ is defined by

$$\widetilde{\mathcal{D}}_{m,l_1^*,1}(t) \sim \sum_{N_0+1\leq n\leq m} \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\leq j\leq n}} \left\| w_{l_1^*-j,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \sum_{N_0+1\leq n\leq m}\sum_{|\alpha|=n} \left\| w_{l_1^*,1}\partial^\alpha f\right\|_\nu^2. \tag{3.18}$$

Here we emphasize that for the functional $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$ or $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$, the differentiation order in x and v starts from N_0+1, i.e. $|\alpha|+|\beta|\geq N_0+1$. We have the following two lemmas for $\mathcal{E}_N(t)$ and $\mathcal{E}_{N-1,\ell,-\gamma}(t)$ respectively:

Lemma 3.8. *Assume $N_0\geq 3$, $N_0+1\leq N\leq 2N_0$, $\widetilde{l}_2 > \frac{1}{2}-\frac{1}{2}\gamma$, $\theta_2 = \frac{1-2\gamma}{2\widetilde{\ell}_2-\gamma}$, $l_0 \geq \frac{3}{2}-\frac{1}{\gamma}$, and $l_1^* \geq \widetilde{\ell}_2 - \frac{\gamma}{2}$, we can deduce that*

$$\frac{d}{dt}\mathcal{E}_N(t)+\mathcal{D}_N(t) \lesssim \left(\|E\|_{L_x^\infty}+\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}\right)^{\frac{1}{\theta_2}} \widetilde{\mathcal{D}}_{N,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \tag{3.19}$$

provided that

(H_4) $\qquad\qquad$ *$\mathcal{E}_N(t)$ is sufficiently small.*

Lemma 3.9. *Take $N_0\geq 3$, $N_0+1\leq N\leq 2N_0$, $\widetilde{l}_3 > \frac{1}{2}-\frac{1}{2}\gamma$, $\theta_3=\frac{1-2\gamma}{2\widetilde{\ell}_3-\gamma}$, $l_1\geq N$, $l_0\geq l_1+\frac{5}{2}$, and $l_1^*\geq \widetilde{\ell}_3-\frac{\gamma}{2}-\gamma l_1$, one has*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{N-1,l_1,-\gamma}(t)+\mathcal{D}_{N-1,l_1,-\gamma}(t)\\
&\lesssim \|E\|_{L_x^\infty}^{\frac{2-\gamma}{1-\gamma}} \sum_{|\alpha|+|\beta|\leq N-1}\left\|w_{l_1-|\beta|,-\gamma}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\quad+\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\\
&\quad+\sum_{|\alpha|=N-1}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|,
\end{aligned}\tag{3.20}$$

where we have used the assumption that

(H_5) $\qquad\qquad$ *$\mathcal{E}_{N-1,l_1}(t)$ is sufficiently small.*

Lemmas 3.7, 3.8 and 3.9 together with the fact $\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\in L^1(\mathbb{R}^+)$ which is a direct consequence of the estimates (3.10) imply that to deduce the desired estimates on $\overline{\mathcal{E}}_{N_0,\ell,-\gamma}(t)$, $\mathcal{E}_N(t)$, and $\mathcal{E}_{N-1,l_1,-\gamma}(t)$, one needs to bound $\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)$ and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$ suitably. To this end, we have to perform the weighted energy estimates by replacing the weight $w_{\ell-|\beta|,-\gamma}$ by $w_{\ell-|\beta|,1}$ and in such a case, as explained in the introduction, the terms $I^E_{|\alpha|,\ell-|\beta|,1}$ and $I^B_{|\alpha|,\ell-|\beta|,1}$ corresponding to (2.8) and (2.9) can be controlled by the corresponding extra dissipation rate $D^W_{|\alpha|,\ell-|\beta|,1}$ given by (2.7) induced by the exponential factor of the weight $w_{\ell-|\beta|,1}(t,v)$ provided that the electromagnetic field $[E(t,x),B(t,x)]$ enjoys certain temporal decay estimates. However, compared with the weighted energy estimate with respect to the weight $w_{\ell,-\gamma}$, the linear term $I^{lt}_{|\alpha|,\ell-|\beta|,1}$ defined by (2.7) leads to a new difficult term

$$\left\| w_{\ell-|\beta-e_i|,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2,$$

which can not be controlled directly by combining the dissipative effects $D^L_{|\alpha|,\ell-|\beta|,1}$ induced by the linearized collision operator L.

Motivated by the argument developed in [19] to deduce the temporal decay estimates on solutions to some nonlinear equations of regularity-loss type, we want to design different time increase rate $\sigma_{n,j}$ for

$$\sum_{|\alpha|+|\beta|=n,|\beta|=j}\left\| w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2,$$

where $\sigma_{n,j}-\sigma_{n,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)$. For result in this direction, we have the following two lemmas whose proof will be given in the next section. The first one is concerned with the case of $N_0+1\le n\le N$.

Lemma 3.10. *Assume $N_0\ge 4$, $\sigma_{n,j}-\sigma_{n,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)$, $l_1^*\ge N$, and $l_0\ge l_1^*+\frac{5}{2}$, one can get that*

$$\begin{aligned}
&\sum_{N_0+1\le n\le N}\frac{d}{dt}\left\{\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\| w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right.\\
&\left.+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2\right\}\\
&+\sum_{N_0+1\le n\le N}\left\{\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\| w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu\right.\\
&\left.+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2_\nu\right\}\\
&+\sum_{N_0+1\le n\le N}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-1-\vartheta-\sigma_{n,j}}\left\| w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{N_0+1\le n\le N}\sum_{|\alpha|=n}(1+t)^{-1-\vartheta-\sigma_{n,0}}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2\\
\lesssim&\sum_{|\alpha|\le N-1}\left\{\left\|\nabla^{|\alpha|+1}f\right\|^2_\nu+\|\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu+\left\|\nabla^{|\alpha|}E\right\|^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&+(1+t)^{-2\sigma_{N,0}}\left\|\nabla^N E\right\|^2+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)+\mathcal{E}_N(t)\mathcal{D}_N(t)\\
&+\sum_{\substack{N_0+1\leq n\leq N,\\ 0\leq j\leq n}}(1+t)^{-\sigma_{n,j}}E^n_{tri,j}(t)+\eta\sum_{\substack{N_0+1\leq n\leq N,\\ 1\leq j\leq n}}\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,|\beta'|<j}}(1+t)^{-\sigma_{n,j}}\\
&\times\left\|w_{l_1^*-|\beta'|,1}\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu,
\end{aligned}
\tag{3.21}
$$

where $E^n_{tri,j}(t)$ is defined by

$$
\begin{aligned}
E^n_{tri,j}(t)\sim&\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j}}\|E\|^{\frac{2-\gamma}{1-\gamma}}_{L^\infty}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j}}\sum_{\substack{|\alpha-\alpha_1|+j+m\geq N_0+1,\\ 1\leq|\alpha_1|\leq N_0-2,m\leq 1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\left\|w_{l_1^*-j-m,1}\nabla^m_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j}}\sum_{\substack{|\alpha-\alpha_1|+j+m\geq N_0+1,\\ N_0-1\leq|\alpha_1|\leq N_0,m\leq 1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}B\right\|^2\left\|w_{l_1^*-j-m,1}\nabla^m_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2_{L^2_vL^\infty_x}\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j}}\sum_{\substack{|\alpha-\alpha_1|+j+m\geq N_0+1,\\ 1\leq|\alpha_1|\leq N_0-2,m\leq 1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}E\right\|^2_{L^\infty_x}\left\|w_{l_1^*-j-m,1}\nabla^m_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j}}\sum_{\substack{|\alpha-\alpha_1|+j+m\geq N_0+1,\\ N_0-1\leq|\alpha_1|\leq N_0,m\leq 1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}E\right\|^2\left\|w_{l_1^*-j-m,1}\nabla^m_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{L^2_vL^\infty_x}\\
&+\max\left\{\mathcal{E}_{N_0,l_0,-\gamma}(t),\mathcal{E}_{\overline{m},\overline{m},-\gamma}(t)\right\}\sum_{\substack{|\alpha'|+|\beta'|\leq n,\\ |\beta'|\leq|\beta|=j}}\\
&\times\left\|w_{l_1^*-|\beta'|,1}\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu.
\end{aligned}
\tag{3.22}
$$

Similar to Lemma 3.10, we can also get for the case of $1\leq n\leq N_0$ that

Lemma 3.11. *Under the assumptions of Lemma 3.10, for $l_0^*\geq N_0$, we have*

$$
\begin{aligned}
&\sum_{1\leq n\leq N_0}\frac{d}{dt}\Bigg\{\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\leq j\leq n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2+(1+t)^{-\sigma_{0,0}}\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{1\le n\le N_0}\left\{\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right.\\
&\left.+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2\right\}+(1+t)^{-\sigma_{0,0}}\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&+\sum_{1\le n\le N_0}\left\{\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-1-\vartheta-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\right.\\
&\left.+\sum_{|\alpha|=n}(1+t)^{-1-\vartheta-\sigma_{n,0}}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\|^2\right\}+(1+t)^{-1-\vartheta-\sigma_{0,0}}\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
\lesssim&\sum_{|\alpha|\le N_0-1}\left\{\left\|\nabla^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\nabla^{|\alpha|}E\right\|^2+\left\|\nabla^{N_0}E\right\|^2\right\}+\mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t)\\
&+\sum_{\substack{0\le n\le N_0,\\0\le j\le n}}(1+t)^{-\sigma_{n,j}}F_{tri,j}^n(t)+\eta\sum_{\substack{1\le n\le N_0,\\1\le j\le n,}}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,|\beta'|<j}}(1+t)^{-\sigma_{n,j}}\\
&\times\left\|w_{l_0^*-|\beta'|,1}\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,
\end{aligned}\tag{3.23}
$$

where $F_{tri,j}^n(t)$ is defined by

$$
\begin{aligned}
F_{tri,j}^n(t)\sim&\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j}}\|E\|_{L^\infty}^{\frac{2-\gamma}{1-\gamma}}\left\|w_{l_0^*-j,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j}}\sum_{\substack{1\le|\alpha_1|\le\min\{n-j,N_0-2\},\\m\le1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}B\right\|_{L_x^\infty}^2\left\|w_{l_0^*-m-j,1}\nabla_v^m\partial_\beta^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j}}\sum_{\substack{N_0-1\le|\alpha_1|\le N_0,\\m\le1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}B\right\|^2\left\|w_{l_0^*-m-j,1}\nabla_v^m\partial_\beta^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|_{L_v^2L_x^\infty}^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j}}\sum_{\substack{1\le|\alpha_1|\le\min\{n-j,N_0-2\},\\m\le1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}E\right\|_{L_x^\infty}^2\left\|w_{l_0^*-m-j,1}\nabla_v^m\partial_\beta^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j}}\sum_{\substack{N_0-1\le|\alpha_1|\le N_0,\\m\le1}}(1+t)^{1+\vartheta}\\
&\times\left\|\partial^{\alpha_1}E\right\|^2\left\|w_{l_0^*-m-j,1}\nabla_v^m\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2L_x^\infty}^2
\end{aligned}
$$

$$+\left(\mathcal{E}_{N_0,0}(t)+\left\|w_{l_0^*,1}f\right\|^2_{L^2_vH^2_x}\right)\sum_{\substack{|\alpha'|+|\beta'|\le n,\\|\alpha'|\ge 1,|\beta'|\le j}}\left\|w_{l_0^*-|\beta'|,1}\partial^{\alpha'}_{\beta'}f\right\|^2_\nu$$
$$+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j}}\sum_{1\le|\alpha_1|+|\beta_1|\le n-1}\left\|w_{l_0^*,1}\partial^{\alpha_1}_{\beta_1}f\right\|^2_{L^2_vL^3_x}\left\|w_{l_0^*,1}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\right\|^2_{L^2_vL^6_x}. \tag{3.24}$$

3.3. The proof of Theorem 2.1. We now prove Theorem 2.1 in this subsection. For this purpose, suppose that the Cauchy problem (1.5) and (1.6) admits a unique local solution $[f(t,x,v), E(t,x), B(t,x)]$ defined on the time interval $0\le t\le T$ for some $0<T<\infty$ and $f(t,x,v)$ satisfies the a priori assumption (2.16), where the parameters $\overline{m}$, N_0, N, l_0, l_1, and l^*, l_0^*, l_1^*, $\sigma_{n,j}$ are given in Theorem 2.1 and M is a sufficiently small positive constant. Then to use the continuity argument to extend such a solution step by step to a global one, one only need to deduce certain uniform-in-time energy type estimates on $f(t,x,v)$ such that the a priori assumption (2.16) can be closed, which is the main result of the following lemma.

Lemma 3.12. *Assume that*

- *The assumptions of Lemma 3.10 hold;*
- *ϑ is chosen to satisfy (2.11), N_0 and N satisfy (2.12);*
- *$\sigma_{N,0}=\frac{1+\epsilon_0}{2}$, $\sigma_{n,0}=0$ for $n\le N-1$;*
- *$\widetilde{\ell}_1\ge\frac{\gamma}{2}+\frac{(1-2\gamma)\sigma_{N_0,N_0}}{2+\varrho}$, $\widetilde{\ell}_2\ge\frac{\gamma}{2}+\frac{2(1-2\gamma)\sigma_{N,N}}{3+2\varrho}$ and $\widetilde{\ell}_3\ge\frac{\gamma}{2}+\frac{(1-2\gamma)\sigma_{N-1,N-1}}{2+\varrho}$;*
- *$l_1\ge N$, $l_1^*\ge\max\left\{\widetilde{\ell}_2-\frac{\gamma}{2},\ \widetilde{\ell}_3-\frac{\gamma}{2}-\gamma l_1\right\}$, $l_0\ge l_1^*+\frac{5}{2}$, $l_0^*\ge\widetilde{\ell}_1-\frac{\gamma}{2}-\gamma(l_0+l^*)$ with $l^*=\frac{3}{2}-\frac{\widetilde{l}}{\gamma}$ with $\widetilde{l}$ being given in Lemma 3.5;*
- *The a priori assumption (2.16) holds for some sufficiently small $M>0$.*

Then it holds that

$$\begin{aligned}
&\mathcal{E}_N(t)+\overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t)+\mathcal{E}_{N-1,l_1,-\gamma}(t)\\
&\quad+\sum_{N_0+1\le n\le N}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\quad+\sum_{N_0+1\le n\le N-1}\sum_{|\alpha|=n}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2\\
&\quad+\sum_{|\alpha|=N}(1+t)^{-(1+\epsilon_0)/2}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2\\
&\quad+\sum_{1\le n\le N_0}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\quad+\sum_{1\le n\le N_0}\sum_{|\alpha|=n}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2+\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\lesssim Y_0^2
\end{aligned}\tag{3.25}$$

for all $0\le t\le T$.

Proof. Before proving (3.25), we first point out that if the assumptions stated in Lemma 3.12 hold, especially the a priori assumption (2.16) is satisfied and the parameters such as ϑ, ϱ, N_0, N, $\sigma_{n,j}$, $\widetilde{l}_1$, $\widetilde{l}_2$, $\widetilde{l}_3$, l_1, l_1^*, l_0, and l_* satisfy the conditions listed in Lemma 3.12, then all the conditions listed in Lemmas 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, and 3.11 are satisfied, and based on the results obtained in these lemmas, we can deduce that:

(i) If we take

$$\sigma_{n,0} = \begin{cases} \frac{1+\epsilon_0}{2}, & n = N, \\ 0, & n \leq N-1 \end{cases}$$

and notice that

$$\sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1+\gamma)}{\gamma-2}(1+\vartheta),$$

we can deduce that

$$\max_{N_0+1\leq n\leq N, 0\leq j\leq n}\{\sigma_{n,j}\} = \sigma_{N,N}, \quad \max_{N_0+1\leq n\leq N-1, 0\leq j\leq n}\{\sigma_{n,j}\} = \sigma_{N-1,N-1},$$
$$\max_{0\leq n\leq N_0, 0\leq j\leq n}\{\sigma_{n,j}\} = \sigma_{N_0,N_0};$$

(ii) If we choose $\widetilde{\ell}_2 \geq \frac{\gamma}{2} + \frac{2(1-2\gamma)\sigma_{N,N}}{3+2\varrho}$ and $l_1^* \geq \widetilde{\ell}_2 - \frac{\gamma}{2}$, then we can deduce that $\theta_2 = \frac{1-2\gamma}{2\widetilde{\ell}_2-\gamma} \leq \frac{3+2\varrho}{4\sigma_{N,N}}$. Consequently, we have from Lemma 3.5 that

$$\begin{aligned}
&\left(\|E\|_{L^\infty} + \left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}\right)^{\frac{1}{\theta_2}} \widetilde{\mathcal{D}}_{N,l_1^*,1}(t)\\
&\lesssim \left(\|E\|_{L^\infty} + \left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}\right)^{\frac{1}{\theta_2}} (1+t)^{\sigma_{N,N}}(1+t)^{-\sigma_{N,N}}\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)\\
&\lesssim X(t)^{\frac{1}{2\theta_2}}(1+t)^{-\left(\frac{3}{4}+\frac{\varrho}{2}\right)\frac{1}{\theta_2}}(1+t)^{\sigma_{N,N}}(1+t)^{-\sigma_{N,N}}\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)\\
&\lesssim X(t)^{\frac{1}{2\theta_2}}(1+t)^{-\sigma_{N,N}}\widetilde{\mathcal{D}}_{N,l_1^*,1}(t);
\end{aligned} \tag{3.26}$$

(iii) If we take $\widetilde{\ell}_3 \geq \frac{\gamma}{2} + \frac{(1-2\gamma)\sigma_{N-1,N-1}}{2+\varrho}$ and $l_1^* \geq \widetilde{\ell}_3 - \frac{\gamma}{2} - \gamma l_1$, then $\theta_3 = \frac{1-2\gamma}{2\widetilde{l}_3-\gamma} \leq \frac{2+\varrho}{2\sigma_{N-1,N-1}}$ and we have from Lemma 3.5 that

$$\begin{aligned}
&\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}} \widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)\\
&\lesssim X(t)^{\frac{1}{2\theta_3}}(1+t)^{-\left(1+\frac{\varrho}{2}\right)\frac{1}{\theta_1}}(1+t)^{\sigma_{N-1,N-1}}(1+t)^{-\sigma_{N-1,N-1}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)\\
&\lesssim X(t)^{\frac{1}{2\theta_3}}(1+t)^{-\sigma_{N-1,N-1}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t);
\end{aligned} \tag{3.27}$$

(iv) For $\widetilde{\ell}_1 \geq \frac{\gamma}{2} + \frac{(1-2\gamma)\sigma_{N_0,N_0}}{2+\varrho}$ and $l_0^* \geq \widetilde{\ell}_1 - \frac{\gamma}{2} - \gamma(l_0 + l^*)$, it is easy to see that $\theta_1 = \frac{1-2\gamma}{2\widetilde{\ell}_1-\gamma} \leq \frac{2+\varrho}{2\sigma_{N_0,N_0}}$ and consequently we have from Lemma 3.5 that

$$\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_1}} \widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t) \lesssim X(t)^{\frac{1}{2\theta_1}}(1+t)^{-\sigma_{N_0,N_0}}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t); \tag{3.28}$$

(v) Since $N_0 \geq 4$, by (3.8), we take $0 < \vartheta \leq \frac{\gamma-2\varrho\gamma+4\varrho+2}{4-4\gamma}$ such that

$$
\begin{aligned}
&\|E\|_{L^\infty_x}^{\frac{2-\gamma}{1-\gamma}} \sum_{|\alpha|+|\beta|\leq N-1 \ or \ N_0} \left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2 \\
&\lesssim \|\nabla_x E\|^{\frac{2-\gamma}{2(1-\gamma)}} \left\|\nabla^2_x E\right\|^{\frac{2-\gamma}{2(1-\gamma)}} \sum_{|\alpha|+|\beta|\leq N-1 \ or \ N_0} \left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2 \\
&\lesssim \left\{\sup_{0\leq\tau\leq t} \overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau),\ \sup_{0\leq\tau\leq t} \mathcal{E}_{N_0+k+\varrho}(\tau)\right\}^{\frac{2-\gamma}{2(1-\gamma)}} \sum_{|\alpha|=N} (1+t)^{-\left(\frac{3}{4}+\frac{\varrho}{2}\right)\frac{2-\gamma}{1-\gamma}} \\
&\quad \times \sum_{|\alpha|+|\beta|\leq N-1 \ or \ N_0} \left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2 \\
&\lesssim X(t)^{\frac{2-\gamma}{2(1-\gamma)}} \sum_{|\alpha|=N} (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|\leq N-1 \ or \ N_0} \left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2.
\end{aligned}
\tag{3.29}
$$

With the above preparations in hand, we now turn to prove (3.25). To this end, we first multiply (3.19) by $(1+t)^{-\epsilon_0}$ and get by employing (3.26) that

$$
\begin{aligned}
&\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_N(t)\right\} + \epsilon_0(1+t)^{-1-\epsilon_0}\mathcal{E}_N(t) + (1+t)^{-\epsilon_0}\mathcal{D}_N(t) \\
&\lesssim (1+t)^{-\epsilon_0}\left(\|E\|_{L^\infty} + \left\|\nabla^2(E,B)\right\|_{H^{N_0-2}}\right)^{\frac{1}{\theta_2}} \widetilde{\mathcal{D}}_{N,l_1^*,1}(t) + (1+t)^{-\epsilon_0}\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \\
&\lesssim (1+t)^{-\epsilon_0}X(t)^{\frac{1}{2\theta_2}}(1+t)^{-\sigma_{N,N}}\widetilde{\mathcal{D}}_{N,l_1^*,1}(t) + (1+t)^{-\epsilon_0}\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t).
\end{aligned}
\tag{3.30}
$$

It is worth pointing out that the term $\epsilon_0(1+t)^{-1-\epsilon_0}\mathcal{E}_N(t)$ on the left hand side of the above inequality can be used to control the term $\sum_{|\alpha|=N}(1+t)^{-2\sigma_{N,0}}\|\partial^\alpha E\|^2$ on the right hand of (3.23).

Secondly, plugging (3.26) into (3.19) gives

$$
\frac{d}{dt}\mathcal{E}_N(t) + \mathcal{D}_N(t) \lesssim X(t)^{\frac{1}{2\theta_2}}(1+t)^{-\sigma_{N,N}}\widetilde{\mathcal{D}}_{N,l_1^*,1}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t). \tag{3.31}
$$

Thirdly, by combing (3.27), (3.29) with (3.20), one has

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{N-1,l_1,-\gamma}(t) + \mathcal{D}_{N-1,l_1,-\gamma}(t) \\
&\lesssim X(t)^{\frac{1}{2\theta_3}}(1+t)^{-\sigma_{N-1,N-1}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) + \sum_{|\alpha|=N-1}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|.
\end{aligned}
\tag{3.32}
$$

Thus if l_1^* is suitably chosen such that $l_1^* \geq \max\left\{\widetilde{\ell}_2 - \frac{\gamma}{2},\ \widetilde{\ell}_3 - \frac{\gamma}{2} - \gamma l_1\right\}$, then the estimates (3.31) and (3.32) hold and from these we can deduce that

- If we choose $l_1 \geq N$, then once we deduce the estimate on $\mathcal{E}_{N-1,l_1,-\gamma}(t)$, the estimate on $\mathcal{E}_{N-1,N-1,-\gamma}(t)$ follows immediately;

- A sufficient condition to control the term $\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)$ which appears on the right hand side of (3.31), (3.32), and (3.21) is to show that $\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \in L^1(\mathbb{R}^3)$. In fact Lemma 3.6 provides us with such a nice estimate provided that $\sup_{0\le\tau\le t} \overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(\tau)$ is sufficiently small.

Now we turn to estimate $\overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t)$ and for this purpose, we first notice from (3.15) that since $k=1$, l^* is now taken as $l^* = \frac{3}{2} - \frac{\tilde{l}}{\gamma}$, then for $l_0 \ge l_1^* + \frac{5}{2}$, we have by replacing ℓ in the estimate (3.17) with $l_0 + l^*$ and the estimate (3.28) that

$$\frac{d}{dt}\overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t)+\overline{\mathcal{D}}_{N_0,l_0+l^*,-\gamma}(t) \lesssim X(t)^{\frac{1}{2\theta_1}}(1+t)^{-\sigma_{N_0,N_0}}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)+\sum_{|\alpha|=N_0}\varepsilon\left\|\partial^\alpha E\right\|^2, \tag{3.33}$$

where we have used the estimate (3.29).

Taking a proper linear combination of (3.31), (3.32), (3.33), (3.21), (3.23), and (3.30) and by using the smallness of $X(t)$ and ε, we can deduce by taking the time integration from 0 to t to the resulting differential inequality that

$$\begin{aligned}
&\mathcal{E}_N(t) + \overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t) + \mathcal{E}_{N-1,l_1,-\gamma}(t) \\
&\quad + \sum_{N_0+1\le n\le N}\ \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\le j\le n}} (1+t)^{-\sigma_{n,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \\
&\quad + \sum_{N_0+1\le n\le N-1}\ \sum_{|\alpha|=n}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2 \\
&\quad + \sum_{|\alpha|=N}(1+t)^{-\frac{1+\epsilon_0}{2}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2 + \sum_{1\le n\le N_0}\ \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}} \\
&\quad \times \left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \\
&\quad + \sum_{1\le n\le N_0}\ \sum_{|\alpha|=n}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2 + \left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \\
&\lesssim Y_0^2.
\end{aligned}$$

Here we have used the following estimate

$$\begin{aligned}
&\sum_{\substack{N_0+1\le n\le N,\\ 0\le j\le n}}(1+t)^{-\sigma_{n,j}}E^n_{tri,j}(t) + \sum_{\substack{0\le n\le N_0,\\ 0\le j\le n}}(1+t)^{-\sigma_{n,j}}F^n_{tri,j}(t) \\
&\lesssim \sum_{N_0+1\le|\alpha|+|\beta|\le N} X(t)(1+t)^{-\sigma_{|\alpha|+|\beta|,|\beta|}}\widetilde{\mathcal{D}}^{|\alpha|,|\beta|}_{l_1^*,1}(t) \\
&\quad + \sum_{0\le|\alpha|+|\beta|\le N_0} X(t)(1+t)^{-\sigma_{|\alpha|+|\beta|,|\beta|}}\widetilde{\mathcal{D}}^{|\alpha|,|\beta|}_{l_0^*,1}(t) \\
&\quad + \sum_{|\alpha|+|\beta|\le N} X(t)^{\frac{2-\gamma}{2(1-\gamma)}}(1+t)^{-\sigma_{|\alpha|+|\beta|,|\beta|}}\widetilde{\mathcal{D}}^{|\alpha|,|\beta|}_{l_0^*\ or\ l_1^*,1}(t) + X(t)D_{N_0}(t),
\end{aligned} \tag{3.34}$$

provided that the parameters ϑ, ϱ, N, and N_0 satisfy the conditions listed in Lemma 3.12. Here to state briefly, we use $\widetilde{\mathcal{D}}^{|\alpha|,|\beta|}_{\ell,1}(t)$ to denote

$$(1+t)^{-1-\vartheta}\left\|w_{\ell-|\beta|,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2+\left\|w_{\ell-|\beta|,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu.$$

Without loss of generality, we only verify the estimate (3.34) for the term

$$(1+t)^{-\sigma_{n,j}}\sum_{\substack{|\alpha|+|\beta|=n,\\|\alpha_1|=1,|\beta|=j,}}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\left\|w_{l_0^*-1-j,1}\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2$$

since the other terms can be estimated in a similar way. In such a case, Lemma 3.5 tells us that

$$\sum_{|\alpha_1|=1}\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\lesssim\begin{cases}X(t)(1+t)^{-\frac52-\varrho},\ N_0\geq 5,\\ X(t)(1+t)^{-2-\varrho},\ N_0=4\end{cases}$$

which implies

$$\sum_{|\alpha_1|=1}\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\lesssim X(t)(1+t)^{-2-2\vartheta-\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)} \tag{3.35}$$

if the parameters ϑ and ϱ are suitably chosen such that

$$\begin{cases}0<\vartheta\leq\frac{2\varrho\gamma-3\gamma-4\varrho-6}{8\gamma-4},\quad \varrho\in[\frac12,\frac32),\ N_0\geq 5,\\ 0<\vartheta\leq\frac{\varrho\gamma-2\gamma-2\varrho-2}{4\gamma-2},\quad \varrho\in(1,\frac32),\ N_0=4.\end{cases}$$

Now due to

$$\sigma_{n,j}-\sigma_{n,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta),$$

we can get from the estimate (3.35) that

$$\begin{aligned}&(1+t)^{-\sigma_{n,j}}\sum_{\substack{|\alpha|+|\beta|=n,\\|\alpha_1|=1,|\beta|=j,}}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\left\|w_{l_0^*-1-j,1}\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\&\lesssim X(t)(1+t)^{-\sigma_{n,j+1}-1-\vartheta}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j+1,}}\left\|w_{l_0^*-1-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2,\end{aligned}$$

that is exactly what we wanted.

Finally, Lemma 3.5 implies that

$$\begin{cases}N=2N_0-1,\quad \text{when}\quad \varrho\in[\frac12,1],\\ N=2N_0,\quad\quad\ \ \text{when}\quad \varrho\in(1,\frac32).\end{cases}$$

Thus the proof of Lemma 3.12 is complete. □

Now we turn to prove Theorem 2.1. To this end, recall the definition of the $X(t)-$norm. Lemma 3.12 tells that for the local solution $[f(t,x,v), E(t,x), B(t,x)]$ to the Cauchy problem (1.5) and (1.6) defined on the time interval $[0,T]$ for some $0 < T \leq +\infty$, if

$$X(t) \leq M, \quad \forall t \in [0,T],$$

then there exists a sufficiently small positive constant $\delta_0 > 0$ such that if

$$M \leq \delta_0^2,$$

there exists a positive constant $\overline{C} > 0$ such that

$$X(t) \leq \overline{C}^2 Y_0^2$$

holds for all $0 \leq t \leq T$.

Thus if the initial perturbation Y_0 is assumed to be sufficiently small such that

$$Y_0 \leq \frac{\delta_0}{\overline{C}},$$

then the global existence follows by combining the local solvability result with the continuation argument in the usual way. This completes the proof of Theorem 2.1. □

3.4. The proof of Theorem 2.2. Based on Theorem 2.1 and by taking $k = 0, 1, 2, \ldots, N_0-2$, we can get firstly from Lemma 3.5 that

$$\mathcal{E}_{N_0}^k(t) \lesssim Y_0^2(1+t)^{-(k+\varrho)},$$

that gives (2.13).

As to (2.14), as long as one takes l_0 and l^* respectively as

$$l_0 = l_{0,0} = l_{0,1} \geq \max\left\{\chi_{k\leq 2}(l_{0,k}+3k-3), l_1^* + \frac{5}{2}\right\},$$

and $l^* = \frac{k+2}{2} - \frac{\widetilde{l}}{\gamma}$ in Theorem 1.1, then (2.14) follows from Lemma 3.6.

Finally, to prove (2.15), we have by the interpolation method with respect to space derivative x for $N_0+1 \leq |\alpha| \leq N-1$ and by using the time decay of $\left\|\nabla^{N_0} f\right\|$ and the bound of $\left\|\nabla^N f\right\|$ that

$$\left\|\partial^\alpha f\right\|^2 \lesssim \left\|\nabla^N f\right\|^{\frac{|\alpha|-N_0}{N-N_0}} \left\|\nabla^{N_0} f\right\|^{\frac{N-|\alpha|}{N-N_0}} \lesssim Y_0^2(1+t)^{-\frac{(N-|\alpha|)(N_0-2+\varrho)}{N-N_0}}.$$

This is (2.15) and the proof of Theorem 2.2 is complete. □

Acknowledgements. RJD was supported by the General Research Fund (Project No. 409913) from RGC of Hong Kong and a Direct Grant (No. 4053117) from CUHK. YJL was supported by a grant from the National Natural Science Foundation of China under contract 11601169. TY was supported by the General Research Fund of Hong Kong, CityU No. 11302215. HJZ was supported by three grants from the National Natural Science Foundation of China under contracts 10925103, 11271160 and 11671309, respectively. This work was also supported by a grant from the National Natural Science Foundation of China under contract 11261160485 and the Fundamental Research Funds for the Central Universities. The authors appreciate many valuable and helpful comments by the anonymous referees.

4. Appendix

We will complete the proofs of some lemmas and estimates used in the previous section.

4.1. The proof of the key estimate in Lemma 3.5. First of all, the following lemmas are for proving (3.7).

Lemma 4.1. *Assume* $-3 < \gamma < -1$, N_0 *and* N *satisfying* (2.12) *and* $n \geq \frac{2}{3}N_0 - \frac{5}{3}$, *there exist a positive integer* $\overline{m}$ *satisfying* $N_0 + 1 \leq \overline{m} \leq N - 1$ *and a sufficiently large number* $\widetilde{l}$, *which both depend only* γ *and* N_0, *such that when* $1 \leq k \leq N_0 - 2$,

$$\left|\left(\nabla^k((v\times B)\cdot\nabla_v f), \nabla^k f\right)\right| \lesssim \max\left\{\mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\} \times\left(\left\|\nabla^{k+1}B\right\|^2 + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \|\nabla^{k+1}f\|_\nu^2\right) + \varepsilon\left(\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{k+1}f\right\|_\nu^2\right), \tag{4.1}$$

when $k = N_0 - 1$, *it holds that*

$$\left|\left(\nabla^k\left((v\times B)\cdot\nabla_v f\right), \nabla^k f\right)\right| \lesssim \max\left\{\mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\} \times\left(\left\|\nabla^{N_0-1}B\right\|^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2, \tag{4.2}$$

and as for $k = N_0$, *it follows that*

$$\begin{aligned}\sum_{k=N_0}&\left|\left(\nabla^k((v\times B)\cdot\nabla_v f), \nabla^k f\right)\right| \\ &\lesssim \max\left\{\mathcal{E}_{N_0+n}(t), \mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\} \\ &\quad\times\left(\left\|\nabla^{N_0-1}B\right\|^2 + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2 + \left\|\nabla^{N_0}f\right\|_\nu^2\right) \\ &\quad+\varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H^1_xL^2_\nu}^2 + \left\|\nabla^{N_0}f\right\|_\nu^2\right).\end{aligned} \tag{4.3}$$

Proof. To obtain (4.1), by using the macro-micro decomposition, one has

$$\begin{aligned}&\left|\left(\nabla^k((v\times B)\cdot\nabla_v f), \nabla^k f\right)\right| \\ &\lesssim \sum_{1\leq j\leq k}\left|\left(\left(v\times\nabla^j B\right)\cdot\nabla_v\nabla^{k-j}f, \nabla^k f\right)\right| \\ &= \underbrace{\sum_{1\leq j\leq k}\left|\left(\left(v\times\nabla^j B\right)\cdot\nabla_v\nabla^{k-j}\mathbf{P}f, \nabla^k f\right)\right|}_{I_{B,1}}\end{aligned}$$

$$+\underbrace{\sum_{1\le j\le k}\left|\left(\left(v\times\nabla^j B\right)\cdot\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f,\nabla^k\mathbf{P}f\right)\right|}_{I_{B,2}}$$

$$+\underbrace{\sum_{1\le j\le k}\left|\left(\left(v\times\nabla^j B\right)\cdot\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f,\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_{B,3}}.$$

Applying the interpolation method with respect to space derivative x, so we deduce from Lemma 3.3 that

$$\begin{aligned}
I_{B,1}+I_{B,2}&\lesssim\sum_{1\le j\le k}\left\|\nabla^j B\right\|_{L^3_x}\left\|\nabla^{k-j}\left(\mu^\delta f\right)\right\|\left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\|\\
&\lesssim\sum_{1\le j\le k}\left\|\Lambda^{-\frac12}B\right\|^{\frac{2k-2j+1}{2k+3}}\left\|\nabla^{k+1}B\right\|^{\frac{2j+2}{2k+3}}\left\|\Lambda^{-\frac12}\left(\mu^\delta f\right)\right\|^{\frac{2j+2}{2k+3}}\\
&\quad\times\left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\|^{\frac{2k-2j+1}{2k+3}}\left\|\nabla^{k+1}\left(\mu^\delta f\right)\right\|\\
&\lesssim\overline{\mathcal{E}}_{0,0,-\gamma}(t)\left(\left\|\nabla^{k+1}B\right\|^2+\left\|\nabla^{k+1}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{k+1}f\right\|_\nu^2.
\end{aligned}$$

As for $I_{3,3}$, when $j=k$, taking $L^6-L^3-L^2$ type inequality and applying Lemma 3.3, one has

$$\begin{aligned}
I_{B,3}&\lesssim\left\|\nabla^k B\right\|_{L^6_x}\left\|\nabla_v\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{1-\frac{\gamma}{2}}\right\|_{L^2_vL^3_x}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|_{L^2_vL^2_x}\\
&\lesssim\overline{\mathcal{E}}_{2,\frac52-\frac1\gamma,-\gamma}(t)\left\|\nabla^{k+1}B\right\|^2+\varepsilon\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}$$

While for the case $1\le j\le k-1$, by the similar virtue of the estimates on $I_{B,3}$ for $j=k$, one also has

$$\begin{aligned}
I_{B,3}&\lesssim\sum_{1\le j\le k-1}\left\|\nabla^j B\right\|_{L^\infty_x}\left\|\nabla_v\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|\\
&\lesssim\sum_{1\le j\le k-1}\left\|\Lambda^{-\varrho}B\right\|^{\frac{2k-2j-1}{2(k+1+\varrho)}}\left\|\nabla^{k+1}B\right\|^{\frac{2j+2\varrho+3}{2(k+1+\varrho)}}\left\|\nabla_v^{m_{1j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{1j}+1}}\\
&\quad\times\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{1j}j}{(m_{1j}+1)(k+\varrho)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{1j}(k-j+\varrho)}{(m_{1j}+1)(k+\varrho)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|\\
&\lesssim\sum_{1\le j\le k-1}\left\|\Lambda^{-\varrho}B\right\|^{\frac{2k-2j-1}{2(k+1+\varrho)}}\left\|\nabla^{k+1}B\right\|^{\frac{2j+2\varrho+3}{2(k+1+\varrho)}}\left\|\nabla_v^{m_{1j}+1}\nabla^{k-j}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{1}{m_{1j}+1}}\\
&\quad\times\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|^{\frac{m_{1j}j}{(m_{1j}+1)(k+\varrho)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\left\langle v\right\rangle^{\frac{\gamma}{2}}\right\|^{\frac{m_{1j}(k-j+\varrho)\beta_j}{(m_{1j}+1)(k+\varrho)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^{\beta_j}\\
&\quad\times\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\left\langle v\right\rangle^{l_{1j}}\right\|^{\frac{m_{1j}(k-j+\varrho)(1-\beta_j)}{(m_{1j}+1)(k+\varrho)}}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{l_{2j}}\right\|^{1-\beta_j}
\end{aligned}$$

$$\lesssim \max\left\{\overline{\mathcal{E}}_{k+m_1,1+m_1,-\gamma}(t), \overline{\mathcal{E}}_{k,-\frac{\hat{l}_2}{\gamma},-\gamma}(t)\right\}\left(\left\|\nabla^{k+1}B\right\|^2+\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right)$$
$$+\varepsilon\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.$$

Here we have used the fact that there exists a positive constant $\beta_j\in(0,1)$ such that

$$\frac{2j+2\varrho+3}{2(k+1+\varrho)}+\frac{m_{1j}(k-j+\varrho)\beta_j}{(m_{1j}+1)(k+\varrho)}+\beta_j=2$$

holds for $1\le j\le k-1$. A necessary and sufficient condition to guarantee the existence of such β_j is

$$\frac{2j+2\varrho+3}{2(k+1+\varrho)}+\frac{m_{1j}(k-j+\varrho)}{(m_{1j}+1)(k+\varrho)}+1>2,\quad 1\le j\le k-1,$$

from which one can deduce that $m_{1j}>\frac{2k^2+2\varrho k-2jk-k-2j\varrho-\varrho}{2k\varrho+3k+2\varrho^2+3\varrho-2j}$ holds for $1\le j\le k-1$. Noticing that $\frac{1}{2}\le\varrho<\frac{3}{2}$, it is easy to see that we can take

$$m_1=\max_{1\le j\le k-1}m_{1j}=\frac{2k}{2\varrho+3}+\frac{2\varrho-3}{2\varrho+3}>\frac{2k^2+2\varrho k-3k-3\varrho}{2k\varrho+3k+2\varrho^2+3\varrho-2}.$$

Consequently, $m_{1j}+1+k-j\le k+m_1=\frac{2\varrho+5}{2\varrho+3}k+\frac{2\varrho-3}{2\varrho+3}\le\frac{2\varrho+5}{2\varrho+3}N_0-\frac{2\varrho+13}{2\varrho+3}$ with $N_0\ge4$.

Moreover, since $\hat{l}_{1j}$ and $\hat{l}_{2j}$ satisfy respectively $\frac{\gamma}{2}\beta_j+\hat{l}_{1j}(1-\beta_j)=0$ and $\frac{\gamma}{2}\beta_j+\hat{l}_{2j}(1-\beta_j)=1$ with $0<\beta_j<1$, one can deduce that, $\hat{l}_{1j}=\frac{\gamma}{2}-\frac{\gamma}{2(1-\beta_j)}$ and $\hat{l}_{2j}=\frac{\gamma}{2}-\frac{\gamma-2}{2(1-\beta_j)}$ from which we can see that $\hat{l}_{2j}>\hat{l}_{1j}$ where

$$\beta_j=\frac{(4k+1+2\varrho-2j)(m_{1j}+1)(k+\varrho)}{(k+2+2\varrho)(k+\varrho+2km_{1j}+2\varrho m_{1j}-jm_{1j})}.$$

Here we take $\hat{l}_2=\max_{1\le j\le k-1}\left\{\hat{l}_{2j}\right\}$.

Consequently, if we take $\overline{m}=k+m_1$ and $\widetilde{l}\ge\max\left\{\hat{l}_2,\frac{1}{2}-\frac{1}{\gamma}\right\}$, (4.1) follows by collecting the above estimates. As well as case $k\le N_0-2$, for $k=N_0-1$, there exist a positive integer $\overline{m}$ and a sufficiently large number $\widetilde{l}$ such that

$$\sum_{k=N_0-1}\left|\left(\nabla^k((v\times B)\cdot\nabla_v f),\nabla^k f\right)\right|$$
$$\lesssim\max\left\{\mathcal{E}_{\overline{m},\overline{m},-\gamma}(t),\overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\}$$
$$\times\left(\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0-1}f\right\|_\nu^2\right)+\varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2.$$

With regard to the case $k=N_0$, compared with the above cases, we only notice that if we take $n>\frac{2}{3}N_0-\frac{5}{3}$,

$$
\begin{aligned}
&\left|\left(\left(v \times \nabla^{N_0} B\right) \cdot \nabla_v \{\mathbf{I}-\mathbf{P}\} f, \nabla^{N_0} f\right)\right| \\
&\lesssim \left\|\nabla^{N_0+n} B\right\|^{\frac{1}{1+n}} \left\|\nabla^{N_0-1} B\right\|^{\frac{n}{1+n}} \left\|\nabla_x \nabla_v^{m_2+1}\{\mathbf{I}-\mathbf{P}\} f\right\|_{H_x^1 L_v^2}^{\frac{1}{1+m_2}} \left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_{H_x^1 L_v^2}^{\frac{m_2(N_0-3)}{(1+m_2)(N_0-2+\varrho)}} \\
&\quad \times \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\hat{l}_3}\right\|_{H_x^1 L_v^2}^{\frac{m_2\beta(1+\varrho)}{(1+m_2)(N_0-2+\varrho)}} \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\frac{\gamma}{2}}\right\|_{H_x^1 L_v^2}^{\frac{m_2(1-\beta)(1+\varrho)}{(1+m_2)(N_0-2+\varrho)}} \\
&\quad \times \left\|\nabla^{N_0} f \langle v\rangle^{\hat{l}_4}\right\|^{1-\beta} \left\|\nabla^{N_0} f \langle v\rangle^{\frac{\gamma}{2}}\right\|^{\beta} \\
&\lesssim \max\left\{\mathcal{E}_{N_0+n}(t), \mathcal{E}_{3+m_2,3+m_2,-\gamma}(t), \overline{\mathcal{E}}_{N_0, N_0-\frac{\hat{l}_4}{\gamma},-\gamma}(t)\right\} \\
&\quad \times \left(\|\nabla^{N_0-1} B\|^2 + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\} f\right\|_{H_x^1 L_v^2}^2\right) \\
&\quad + \varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\} f\right\|_{H_x^1 L_v^2}^2 + \left\|\nabla^{N_0} f\right\|_\nu^2\right).
\end{aligned}
$$

Here we need to ask $\frac{n}{1+n} + \frac{m_2(1+\varrho)\beta}{(1+m_2)(N_0-2+\varrho)} + \beta = 2$ which deduce that $m_2 > \frac{N_0-2+\varrho}{\varrho n+n+3-N_0}$, we can get $\hat{l}_3 = \frac{\gamma}{2} - \frac{\gamma}{2(1-\beta)}$ and $\hat{l}_4 = \frac{\gamma}{2} - \frac{\gamma-2}{2(1-\beta)}$ from $\frac{\gamma}{2} \cdot \beta + \hat{l}_3(1-\beta) = 0$ and $\frac{\gamma}{2} \cdot \beta + \hat{l}_4(1-\beta) = 1$. We can choose m_2 suitably such that $3+m_2 \leq 3+\frac{N_0-2+\varrho}{(\varrho+1)n+3-N_0}$. The other terms can be estimated as well as (4.1). Consequently, if we take suitable numbers $\overline{m}$ and $\widetilde{l}$, we also have

$$
\begin{aligned}
&\sum_{k=N_0} \left|\left(\nabla^k((v \times B) \cdot \nabla_v f), \nabla^k f\right)\right| \\
&\lesssim \max\left\{\mathcal{E}_{N_0+n}(t), \mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0, N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\} \\
&\quad \times \left(\|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\} f\|_\nu^2 + \|\nabla^{N_0-1} f\|_\nu^2 + \|\nabla^{N_0} f\|_\nu^2\right) \\
&\quad + \varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\} f\right\|_{H_x^1 L_v^2}^2 + \left\|\nabla^{N_0} f\right\|_\nu^2\right).
\end{aligned}
$$

Thus we have completed the proof of this lemma. □

By repeating the argument used to prove Lemma 4.1, we can also obtain that

Lemma 4.2. *Under the assumptions of Lemma 4.1, we have estimates on the terms containing E and $\Gamma(f, f)$ as follows. For $k \leq N_0 - 2$, it holds that*

$$
\begin{aligned}
&\left|\left(\nabla^k(v \cdot E f), \nabla^k f\right)\right| + \left|\left(\nabla^k(E \cdot \nabla_v f), \nabla^k f\right)\right| + \left|\left(\nabla^k \Gamma(f, f), \nabla^k f\right)\right| \\
&\lesssim \max\left\{\mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0, N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\} \left(\left\|\nabla^{k+1} E\right\|^2 + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 + \|\nabla^{k+1} f\|_\nu^2\right) \\
&\quad + \varepsilon\left(\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 + \left\|\nabla^{k+1} f\right\|_\nu^2\right). \qquad (4.4)
\end{aligned}
$$

For $k = N_0 - 1$*, it holds that*

$$\left|\left(\nabla^k(v\cdot Ef),\nabla^k f\right)\right| + \left|\left(\nabla^k(E\cdot\nabla_v f),\nabla^k f\right)\right| + \left|\left(\nabla^k\Gamma(f,f),\nabla^k f\right)\right|$$
$$\lesssim \max\left\{\mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\}\left(\left\|\nabla^{N_0-1}E\right\|^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2. \tag{4.5}$$

For $k = N_0$*, it holds that*

$$\left|\left(\nabla^k(v\cdot Ef),\nabla^k f\right)\right| + \left|\left(\nabla^k(E\cdot\nabla_v f),\nabla^k f\right)\right| + \left|\left(\nabla^k\Gamma(f,f),\nabla^k f\right)\right|$$
$$\lesssim \max\left\{\mathcal{E}_{N_0+n}(t), \mathcal{E}_{\overline{m},\overline{m},-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\}\left(\left\|\nabla^{N_0-1}E\right\|^2 + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right.$$
$$\left. + \left\|\nabla^{N_0-1}f\right\|_\nu^2 + \left\|\nabla^{N_0}f\right\|_\nu^2\right) + \varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_{H_x^1L_\nu^2}^2 + \left\|\nabla^{N_0}f\right\|_\nu^2\right). \tag{4.6}$$

Remark 4.1. Comparing the proofs of the above two lemmas, we all take suitably numbers $\overline{m}$ satisfying $N_0 - 1 \le \overline{m} \le N - 1$ and $\widetilde{l}$. In fact, by complex calculation as well as (4.3), we obtain

$$\overline{m} = \max\left\{\chi_{N_0\ge 4}\left\{\frac{2\varrho+5}{2\varrho+3}N_0 - \frac{6}{2\varrho+3}\right\},\ \chi_{N_0\ge 6}\left\{\frac{2\varrho+7}{2\varrho+3}N_0 - \frac{8\varrho+38}{2\varrho+3}\right\},\right.$$
$$\left.\chi_{N_0\ge 7}\left\{\frac{2\varrho+9}{2\varrho+5}N_0 - \frac{4\varrho+36}{2\varrho+5}\right\},\ \chi_{N_0\ge 8}\left\{\frac{2\varrho+11}{2\varrho+7}N_0 - \frac{4\varrho+44}{2\varrho+9}\right\}\right\},$$

Thus we can choose $\overline{m} = N - 1$ without generality if N satisfies (2.12). Since the computation of accurate value of $\widetilde{l}$ is too complicated but standard, we claim that there exists a finite number $\widetilde{l}$ satisfying the above three lemmas.

Based on the above three lemmas and Remark 4.1, it is straightforward to obtain

Lemma 4.3. *Let* N_0 *and* N *satisfying* (2.12)*, then there exists a positive constant* $\widetilde{l}$*, which depends only on* N_0*,* ϱ *and* γ*, such that:*

(1) *For* $k = 0, 1, \ldots, N_0 - 2$*, it holds that*

$$\frac{d}{dt}\left(\left\|\nabla^k f\right\|^2 + \left\|\nabla^k(E,B)\right\|^2\right) + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2$$
$$\lesssim \max\left\{\mathcal{E}_{N-1,N-1,-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\}\left(\left\|\nabla^{k+1}(E,B)\right\|^2 + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right.$$
$$\left. + \left\|\nabla^{k+1}f\right\|_\nu^2\right) + \varepsilon\left(\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{k+1}f\right\|_\nu^2\right). \tag{4.7}$$

(2) *If* $k = N_0 - 1$*, it follows that*

$$\frac{d}{dt}\left(\left\|\nabla^{N_0-1}f\right\|^2 + \left\|\nabla^{N_0-1}(E,B)\right\|^2\right) + \left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2$$
$$\lesssim \max\left\{\mathcal{E}_{N-1,N-1,-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\widetilde{l}}{\gamma},-\gamma}(t)\right\}\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2\right.$$
$$\left. + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0-1}f\right\|_\nu^2\right) + \varepsilon\left\|\nabla^{N_0-1}f\right\|_\nu^2. \tag{4.8}$$

(3) *As for* $k = N_0 \geq 4$, *if we let* $n > \frac{2}{3}N_0 - \frac{5}{3}$, *one has*

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^{N_0} f\right\|^2 + \left\|\nabla^{N_0}(E, B)\right\|^2\right) + \left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\quad\lesssim \max\left\{\mathcal{E}_{N_0+n}(t), \mathcal{E}_{N-1,N-1,-\gamma}(t), \overline{\mathcal{E}}_{N_0,N_0-\frac{\tilde{l}}{\gamma},-\gamma}(t)\right\} \\
&\qquad\times\left(\left\|\nabla^{N_0-1}(E, B)\right\|^2 + \left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right. \\
&\qquad\left.+\left\|\nabla^{N_0-1} f\right\|_{H^1_x L^2_\nu}^2\right) + \varepsilon\left(\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{N_0-1} f\right\|_{H^1_x L^2_\nu}^2\right).
\end{aligned} \tag{4.9}$$

Proof. To prove (4.7), we apply ∇^k to (1.5), multiply the resulting identity by $\nabla^k f$, and further integrate it with respect to x and v over $\mathbb{R}^3_x \times \mathbb{R}^3_v$. Then, for $k \leq N_0 - 2$, (4.7) follows by recalling (4.1), (4.4) and the coercive property of the linear operator L. Similarly, for $k = N_0 - 1 \geq 2$, (4.2) and (4.5) imply (4.8). Regarding the last case $k = N_0 \geq 4$, one has (4.9) by combing (4.3) and (4.6). Thus the proof of Lemma 4.3 is complete. □

The next lemma is concerned with the macro dissipation $\mathcal{D}_{N,mac}(t)$ defined by

$$\mathcal{D}_{N,mac}(t) \sim \|\nabla_x(a_\pm, b, c)\|^2_{H^{N-1}_x} + \|a_+ - a_-\|^2 + \|E\|^2_{H^{N-1}_x} + \|\nabla_x B\|^2_{H^{N-2}_x}.$$

Lemma 4.4. *For the macro dissipation estimates on* $f(t, x, v)$, *we have the following results:*

(i) *For* $k = 0, 1, 2 \ldots, N_0 - 2$, *there exist interactive energy functionals* $G^k_f(t)$ *satisfying*

$$G^k_f(t) \lesssim \left\|\nabla^k(f, E, B)\right\|^2 + \left\|\nabla^{k+1}(f, E, B)\right\|^2 + \left\|\nabla^{k+2} E\right\|^2$$

such that

$$\begin{aligned}
&\frac{d}{dt}G^k_f(t) + \left\|\nabla^k(E, a_+ - a_-)\right\|^2_{H^1_x L^2_v} + \left\|\nabla^{k+1}(\mathbf{P}f, B)\right\|^2 \\
&\quad\lesssim \overline{\mathcal{E}}_{N_0-1,0,-\gamma}(t)\left(\left\|\nabla^{k+1}(E, B)\right\|^2 + \left\|\nabla^{k+1} f\right\|_\nu^2\right) + \left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 \\
&\qquad+\left\|\nabla^{k+1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2 + \left\|\nabla^{k+2}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2;
\end{aligned}$$

(ii) *For* $k = N_0 - 1$, *there exists an interactive energy functional* $G^{N_0-1}_f(t)$ *satisfying*

$$G^{N_0-1}_f(t) \lesssim \left\|\nabla^{N_0-2}(f, E, B)\right\|^2 + \left\|\nabla^{N_0-1}(f, E, B)\right\|^2 + \left\|\nabla^{N_0}(f, E)\right\|^2$$

such that

$$\frac{d}{dt}G_f^{N_0-1}(t)+\left\|\nabla^{N_0-2}(E,a_+-a_-)\right\|^2_{H^1_xL^2_v}+\left\|\nabla^{N_0-1}B\right\|^2+\left\|\nabla^{N_0}\mathbf{P}f\right\|^2$$
$$\lesssim\overline{\mathcal{E}}_{N_0,0,-\gamma}(t)\left(\left\|\nabla^{N_0-1}(E,B)\right\|^2+\left\|\nabla^{N_0-1}f\right\|^2_\nu\right)+\left\|\nabla^{N_0-2}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu$$
$$+\left\|\nabla^{N_0-1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu+\left\|\nabla^{N_0}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu;$$

(iii) *There exists an interactive energy functional $\mathcal{E}_N^{int}(t)$ satisfying*

$$\mathcal{E}_N^{int}(t)\lesssim\sum_{|\alpha|\le N}\left\|\partial^\alpha(f,E,B)\right\|^2$$

such that

$$\frac{d}{dt}\mathcal{E}_N^{int}(t)+\mathcal{D}_{N,mac}(t)\lesssim\sum_{|\alpha|\le N}\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu+\mathcal{E}_N(t)\mathcal{D}_N(t)$$

holds for any $t\in[0,T]$.

Proof. Since the procedure of the proof is almost the same as the proof of Lemma 3.5 in [22, page 3742], we omit it for brevity. □

4.2. The estimate in the negative indexed space. Our first result in this subsection is concerned with the estimate on $\|[f,E,B](t)\|_{\dot{H}^{-\varrho}}$.

Lemma 4.5. *For $\varrho\in[\frac{1}{2},\frac{3}{2})$, it holds that*

$$\frac{d}{dt}\left(\left\|\Lambda^{-\varrho}f\right\|^2+\left\|\Lambda^{-\varrho}(E,B)\right\|^2\right)+\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu$$
$$\lesssim\left(\overline{\mathcal{E}}_{0,0,-\gamma}(t)\right)^{1/2}\left(\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}}(E,B)\right\|^2+\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}}f\right\|^2_\nu\right)$$
$$+\overline{\mathcal{E}}_{2,\frac{3}{2}-\frac{1}{\gamma},-\gamma}(t)\left\|\Lambda^{\frac{3}{2}-\varrho}(f,E,B)\right\|^2. \tag{4.10}$$

Proof. We have by taking Fourier transform of (1.5) with respect to x, multiplying the resulting identity by $|\xi|^{-2s}\overline{\hat{f}}_\pm$ with $\overline{\hat{f}}_\pm$ being the complex conjugate of $\hat{f}_\pm$, and integrating the final result with respect to ξ and v over $\mathbb{R}^3_\xi\times\mathbb{R}^3_v$ that

$$\Big(\partial_t\hat{f}_\pm+v\cdot\mathcal{F}[\nabla_xf_\pm]\pm\mathcal{F}[(E+v\times B)\cdot\nabla_vf_\pm]\mp\frac{1}{2}v\cdot\mathcal{F}[Ef_\pm]\mp\hat{E}$$
$$\cdot v\mu^{\frac{1}{2}}+\mathcal{F}[L_\pm f]-\mathcal{F}[\Gamma_\pm(f,f)]\mid|\xi|^{-2\varrho}\hat{f}\Big)$$
$$=0. \tag{4.11}$$

Recall that throughout this paper, $\mathcal{F}[g](t,\xi,v)=\hat{g}(t,\xi,v)$ denotes the Fourier transform of $g(t,x,v)$ with respect to x.

Equation (4.11) together with Lemma 3.1 yield

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\Lambda^{-\varrho} f\right\|^2+\left\|\Lambda^{-\varrho}(E, B)\right\|^2\right)+\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\sigma^2 \\
&\lesssim \underbrace{\sum_{\pm}\left|\left(\mathcal{F}[E \cdot \nabla_v f_{\pm}] \mid |\xi|^{-2\varrho} \hat{f}_{\pm}\right)\right|}_{I_1}+\underbrace{\sum_{\pm}\left|\left(\mathcal{F}[v \times B \cdot \nabla_v f_{\pm}] \mid |\xi|^{-2\varrho} \hat{f}_{\pm}\right)\right|}_{I_2} \\
&\quad+\underbrace{\sum_{\pm}\left|\left(v \cdot \mathcal{F}[E f_{\pm}] \mid |\xi|^{-2\varrho} \hat{f}_{\pm}\right)\right|}_{I_3}+\underbrace{\sum_{\pm}\left|\left(\mathcal{F}[\Gamma_{\pm}(f, f)] \mid |\xi|^{-2\varrho} \hat{f}_{\pm}\right)\right|}_{I_4}.
\end{aligned} \tag{4.12}
$$

To estimate I_i $(i=1,2,3)$, we have from Lemma 3.1, Lemmas 3.3 and 3.4 that

$$
\begin{aligned}
I_1 &\lesssim \left\|\Lambda^{-\varrho}\left(E \cdot \mu^\delta f\right)\right\|\left\|\Lambda^{-\varrho}\left(\mu^\delta f\right)\right\|+\left\|\Lambda^{-\varrho}\left(E \cdot \nabla_v f\langle v\rangle^{-\frac{\gamma}{2}}\right)\right\|\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\
&\lesssim \left\|E \cdot \mu^\delta f\right\|_{L_v^2 L_x^{\frac{6}{3+2\varrho}}}\left\|\Lambda^{-\varrho}\left(\mu^\delta f\right)\right\|+\left\|E \cdot \nabla_v f\langle v\rangle^{-\frac{\gamma}{2}}\right\|_{L_x^{\frac{6}{3+2\varrho}} L_v^2}\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\
&\lesssim \left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}} E\right\|\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}}\left(\mu^\delta f\right)\right\|\left\|\Lambda^{-\varrho}\left(\mu^\delta f\right)\right\|+\left\|\Lambda^{\frac{3}{2}-\varrho} E\right\|^2\left\|\nabla_v f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^2 \\
&\quad+\varepsilon\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 \\
&\lesssim \left(\overline{\mathcal{E}}_{0,0,-\gamma}(t)\right)^{1/2}\left(\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}} E\right\|^2+\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}} f\right\|_\nu^2\right)+\overline{\mathcal{E}}_{1, \frac{3}{2},-\gamma}(t)\left\|\Lambda^{\frac{3}{2}-\varrho} E\right\|^2 \\
&\quad+\varepsilon\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 .
\end{aligned}
$$

For I_2 and I_3, we have by repeating the argument used in deducing the estimate on I_1 that

$$
\begin{aligned}
I_2+I_3 \lesssim &\left(\overline{\mathcal{E}}_{0,0,-\gamma}(t)\right)^{1/2}\left(\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}}(E, B)\right\|^2+\left\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}} f\right\|_\nu^2\right) \\
&+\overline{\mathcal{E}}_{1, \frac{3}{2}-\frac{1}{\gamma},-\gamma}(t)\left\|\Lambda^{\frac{3}{2}-\varrho}(E, B)\right\|^2+\varepsilon\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2 .
\end{aligned}
$$

I_4 can be bounded from Lemma 3.2 by

$$
\begin{aligned}
I_4 &=\left(\mathcal{F}[\Gamma(f, f)],|\xi|^{-2\varrho} \mathcal{F}\{\mathbf{I}-\mathbf{P}\} f\right) \\
&\lesssim \left\|\Lambda^{-\varrho}\left(\langle v\rangle^{-\frac{\gamma}{2}} \Gamma(f, f)\right)\right\|\left\|\Lambda^{-\varrho}\left(\langle v\rangle^{\frac{\gamma}{2}}\{\mathbf{I}-\mathbf{P}\} f\right)\right\| \\
&\lesssim \left\|\Lambda^{\frac{3}{2}-\varrho} f\right\|\|f\|_{L_x^2 H_v^2}\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu \\
&\lesssim \overline{\mathcal{E}}_{2,2,-\gamma}(t)\left\|\Lambda^{\frac{3}{2}-\varrho} f\right\|^2+\varepsilon\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\} f\right\|_\nu^2
\end{aligned}
$$

Substituting the estimates on $I_i(i=1,2,3,4)$ into (4.12) yields

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\Lambda^{-\varrho}f\right\|^2+\left\|\Lambda^{-\varrho}(E,B)\right\|^2\right)+\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\&\lesssim\left(\overline{\mathcal{E}}_{0,0,-\gamma}(t)\right)^{1/2}\left(\left\|\Lambda^{\frac34-\frac{\varrho}{2}}(E,B)\right\|^2+\left\|\Lambda^{\frac34-\frac{\varrho}{2}}f\right\|_\nu^2\right)\\&\quad+\overline{\mathcal{E}}_{2,\frac32-\frac1\gamma,-\gamma}(t)\left\|\Lambda^{\frac32-\varrho}(f,E,B)\right\|^2.\end{aligned}$$

Thus we complete the proof of Lemma 4.5. □

Applying the argument of Lemma 3.2 in [22, page 3727] and Lemma 3.3 in [22, page 3731], we easily have the following lemma:

Lemma 4.6. *Let $\varrho\in[\frac12,\frac32)$, there exists an interactive functional $G_{E,B}(t)$ satisfying*

$$|G_f^{-\varrho}(t)|\lesssim\left\|\Lambda^{1-\varrho}(f,E,B)\right\|^2+\left\|\Lambda^{-\varrho}(f,E,B)\right\|^2+\|\Lambda^{2-\varrho}E\|^2$$

such that

$$\begin{aligned}&\frac{d}{dt}G_f^{-\varrho}(t)+\left\|\Lambda^{1-\varrho}(E,B)\right\|^2+\left\|\Lambda^{-\varrho}E\right\|^2+\left\|\Lambda^{-\varrho}(a_+-a_-)\right\|_{H^1}^2\\&\lesssim\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{1-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\Lambda^{2-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\overline{\mathcal{E}}_{1,0,-\gamma}(t)\overline{\mathcal{D}}_{2,0,-\gamma}(t)\end{aligned}\tag{4.13}$$

holds for any $0\leq t\leq T$.

4.3. The proof of Lemmas 3.7, 3.8 and 3.9. We first give the proof of Lemma 3.9 as it is the most difficult one among those three lemmas. The standard energy estimate on $\partial^\alpha f$ with $1\leq|\alpha|\leq N-1$ weighted by the time-velocity dependent function $w_{\ell,-\gamma}=w_{\ell,-\gamma}(t,v)$ gives

$$\begin{aligned}&\frac{d}{dt}\sum_{1\leq|\alpha|\leq N-1}\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|^2+\sum_{1\leq|\alpha|\leq N-1}\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|_\nu^2\\&\quad+\frac{1}{(1+t)^{1+\vartheta}}\left\|w_{\ell,-\gamma}\partial^\alpha f\langle v\rangle\right\|^2\\&\lesssim\sum_{1\leq|\alpha|\leq N-1}\left\|\partial^\alpha f\right\|_\nu^2+\sum_{1\leq|\alpha|\leq N-1}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|\\&\quad+\underbrace{\sum_{1\leq|\alpha|\leq N-1}\left|\left(\partial^\alpha(v\times B)\cdot\nabla_v f),w_{\ell,-\gamma}^2\partial^\alpha f\right)\right|}_{J_1}\\&\quad+\underbrace{\sum_{1\leq|\alpha|\leq N-1}\left|\left(\partial^\alpha(E\cdot vf+E\cdot\nabla_v f),w_{\ell,-\gamma}^2\partial^\alpha f\right)\right|}_{J_2}\\&\quad+\underbrace{\sum_{1\leq|\alpha|\leq N-1}\left|\left(\partial^\alpha\Gamma(f,f),w_{\ell,-\gamma}^2\partial^\alpha f\right)\right|}_{J_3}.\end{aligned}$$

For J_1, we deduce that

$$
\begin{aligned}
J_1 \lesssim & \underbrace{\|\partial^\alpha B\| \left\| w_{\ell-1,-\gamma} \nabla_v f \langle v\rangle^{1-\frac{3\gamma}{2}} \right\|_{L^2_v L^\infty_x} \|w_{\ell,-\gamma}\partial^\alpha f\|_\nu}_{J_{1,1}} \\
& + \underbrace{\sum_{1\le|\alpha-\alpha_1|\le N_0-2} \|\partial^{\alpha_1} B\|_{L^6_x} \left\| w_{\ell-1,-\gamma} \nabla_v \partial^{\alpha-\alpha_1} f \langle v\rangle^{1-\frac{3\gamma}{2}} \right\|_{L^2_v L^3_x} \|w_{\ell,-\gamma}\partial^\alpha f\|_\nu}_{J_{1,2}} \\
& + \underbrace{\sum_{|\alpha-\alpha_1|=N_0-1} \|\partial^{\alpha_1} B\|_{L^\infty_x} \left\| w_{\ell-1,-\gamma} \nabla_v \partial^{\alpha-\alpha_1} f \langle v\rangle^{1-\frac{3\gamma}{2}} \right\| \|w_{\ell,-\gamma}\partial^\alpha f\|_\nu}_{J_{1,3}} \\
& + \underbrace{\sum_{\substack{|\alpha-\alpha_1|\ge N_0,\\ |\alpha_1|=N_0-1}} \|\partial^{\alpha_1} B\|_{L^6_x} \left\| w_{\ell-1,-\gamma} \nabla_v \partial^{\alpha-\alpha_1} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\frac12-\frac{\gamma}{2}} \right\|_{L^2_v L^3_x} \left\| \langle v\rangle^{\frac12-\frac{\gamma}{2}} w_{\ell,-\gamma}\partial^\alpha f\right\|}_{J_{1,4}} \\
& + \underbrace{\sum_{\substack{|\alpha-\alpha_1|\ge N_0,\\ 2\le|\alpha_1|\le N_0-1}} \int_{\mathbb{R}^3} |\partial^{\alpha_1} B| \left| w_{\ell-1,-\gamma} \nabla_v \partial^{\alpha-\alpha_1} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\frac12-\frac{\gamma}{2}} \right| \left| w_{\ell,-\gamma}\partial^\alpha f \langle v\rangle^{\frac12-\frac{\gamma}{2}}\right| dx}_{J_{1,5}} \\
& + \underbrace{\sum_{|\alpha-\alpha_1|\ge N_0} \int_{\mathbb{R}^3} |\partial^{\alpha_1} B| \left| w_{\ell-1,-\gamma} \nabla_v \partial^{\alpha-\alpha_1} \mathbf{P} f \langle v\rangle^{1-\frac{3\gamma}{2}} \right| |w_{\ell,-\gamma}\partial^\alpha f|_{L^2_v} dx}_{J_{1,6}}
\end{aligned}
$$

The first three term and the last term can be bounded by

$$
\sum_{i=1}^{3} J_{1,i} + J_{1,6} \lesssim \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) + \mathcal{E}_{N_0}(t)\mathcal{D}_{N-1}(t) + \varepsilon\mathcal{D}_{N-1,\ell,-\gamma}(t)
$$

where we take $l_0 \ge \ell + \frac{3}{2} - \frac{1}{\gamma}$ such that $w_{\ell,-\gamma}\langle v\rangle^{1-\frac{3\gamma}{2}} \le w_{l_0,-\gamma}$.

As for the last two terms $J_{1,4}$ and $J_{1,5}$, we only estimate $J_{1,4}$ since $J_{1,5}$ can be obtained in a similar way,

$$
\begin{aligned}
J_{1,4} \lesssim & \sum_{\substack{|\alpha-\alpha_1|\ge N_0,\\ |\alpha_1|=N_0-1}} \|\partial^{\alpha_1} B\|_{L^6_x} \left\| w_{\ell-1,-\gamma} \partial^{\alpha-\alpha_1} \nabla_v \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\frac{\gamma}{2}} \right\|^{1-\theta_3}_{L^2_v L^3_x} \\
& \times \left\| w_{\ell-1,-\gamma} \partial^{\alpha-\alpha_1} \nabla_v \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\widetilde{\ell}_3} \right\|^{\theta_3}_{L^2_v L^3_x} \\
& \times \left\| w_{\ell,-\gamma}\partial^\alpha f \langle v\rangle^{\frac{\gamma}{2}} \right\|^{1-\theta_3} \left\| w_{\ell,-\gamma}\partial^\alpha f \langle v\rangle^{\widetilde{\ell}_3} \right\|^{\theta_3} \\
\lesssim & \sum_{\substack{|\alpha-\alpha_1|\ge N_0,\\ |\alpha_1|=N_0-1}} \left\| \nabla^{N_0} B \right\|^{\frac{1}{\theta_3}} \left\| w_{\ell-1,-\gamma} \partial^{\alpha-\alpha_1} \nabla_v \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\widetilde{\ell}_3} \right\|_{L^2_v L^3_x} \left\| w_{\ell,-\gamma}\partial^\alpha f \langle v\rangle^{\widetilde{\ell}_3} \right\|
\end{aligned}
$$

$$+\sum_{\substack{|\alpha-\alpha_1|\geq N_0,\\|\alpha_1|=N_0-1}}\varepsilon\left(\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|_\nu^2+\left\|w_{\ell-1,-\gamma}\partial^{\alpha-\alpha_1}\nabla_v\{\mathbf{I}-\mathbf{P}\}f\right\|_{L_v^2L_x^3}^2\right)$$
$$\lesssim\left\|\nabla^{N_0}B\right\|^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\varepsilon\mathcal{D}_{N-1,\ell,-\gamma}(t).$$

where $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$ is given in (3.18) with $m=N-1$ and θ_3 satisfies that $\frac{1}{2}-\frac{1}{2}\gamma=\frac{\gamma}{2}(1-\theta_3)+\widetilde{\ell}_3\theta_3$ which yields that $\theta_3=\frac{1-2\gamma}{2\widetilde{\ell}_3-\gamma}$. Meanwhile l_1^* satisfies that $\widetilde{\ell}_3+(-\gamma)(\ell-1)\leq l_1^*-1+\frac{\gamma}{2}$ which deduce that $l_1^*\geq\widetilde{\ell}_3+(-\gamma)(\ell-1)+1-\frac{\gamma}{2}$. Notice that $\gamma\in(-3,-1)$, we can take $l_1^*\geq\widetilde{\ell}_3-\frac{\gamma}{2}-\gamma\ell$. Consequently,

$$J_1\lesssim\left\|\nabla^2B\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)$$
$$+\mathcal{E}_{N_0}(t)\mathcal{D}_{N-1}(t)+\varepsilon\mathcal{D}_{N-1,\ell,-\gamma}(t).$$

By the virtue of the estimates on J_1, we also have

$$J_2\lesssim\left\|\nabla^2E\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)+\mathcal{E}_{N_0}(t)\mathcal{D}_{N-1}(t)$$
$$+\|E\|_{L_x^\infty}^{\frac{2-\gamma}{1-\gamma}}\sum_{1\leq|\alpha|\leq N-1}\left\|w_{\ell,-\gamma}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2+\varepsilon\mathcal{D}_{N-1,\ell,-\gamma}(t)$$

Applying Lemma 3.2 gives

$$J_3\lesssim(\mathcal{E}_{N-1,\ell,-\gamma}(t)+\varepsilon)\mathcal{D}_{N-1,\ell,-\gamma}(t).$$

Collecting the above estimates gives the desired weighted energy type estimates on the derivatives of $f(t,x,v)$ with respect to the $x-$variables only as follows

$$\begin{aligned}&\frac{d}{dt}\sum_{1\leq|\alpha|\leq N-1}\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|^2+\sum_{1\leq|\alpha|\leq N-1}\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|_\nu^2\\&\quad+\frac{1}{(1+t)^{1+\vartheta}}\left\|w_{\ell,-\gamma}\partial^\alpha f\langle v\rangle\right\|_\nu^2\\&\lesssim\sum_{1\leq|\alpha|\leq N-1}\left\|\partial^\alpha f\right\|_\nu^2+\sum_{1\leq|\alpha|\leq N-1}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|+\{\mathcal{E}_{N-1,\ell,-\gamma}(t)+\varepsilon\}\mathcal{D}_{N-1,\ell,-\gamma}(t)\\&\quad+\|E\|_{L_x^\infty}^{\frac{2-\gamma}{1-\gamma}}\sum_{1\leq|\alpha|\leq N-1}\left\|w_{\ell,-\gamma}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\&\quad+\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t).\end{aligned}\tag{4.14}$$

After applying $\{\mathbf{I}-\mathbf{P}\}$ to the equation (1.8), one can get the weighted energy estimate on $\{\mathbf{I}-\mathbf{P}\}f$

$$\frac{d}{dt}\left\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\left\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\frac{1}{(1+t)^{1+\vartheta}}\left\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2$$
$$\lesssim\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|E\|_{L_x^\infty}^{\frac{2-\gamma}{1-\gamma}}\left\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2+\|E\|^2+\|\nabla_x f\|_\nu^2$$
$$+\left(\mathcal{E}_{N-1,\ell,-\gamma}(t)+\varepsilon\right)\mathcal{D}_{N-1,\ell,-\gamma}(t). \tag{4.15}$$

As to the weighted energy estimate on $\{\mathbf{I}-\mathbf{P}\}\partial_\beta^\alpha f$ with $|\alpha|+|\beta|\le N-1$, $|\beta|\ge 1$, applying the similar trick as (4.14), we also deduce

$$\frac{d}{dt}\sum_{m=1}^{N-1}C_m\sum_{\substack{|\beta|=m,\\|\alpha|+|\beta|\le N-1}}\left\|w_{\ell-|\beta|,-\gamma}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2$$
$$+\sum_{\substack{|\alpha|+|\beta|\le N-1,\\|\beta|\ge 1}}\left\{\left\|w_{\ell-|\beta|,-\gamma}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right.$$
$$\left.+\frac{1}{(1+t)^{1+\vartheta}}\left\|w_{\ell-|\beta|,-\gamma}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\right\}$$
$$\lesssim\sum_{|\alpha|\le N-2}\left(\left\|\nabla_x^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\partial^\alpha E\right\|^2\right)$$
$$+\|E\|_{L_x^\infty}^{\frac{2-\gamma}{1-\gamma}}\sum_{|\alpha|+|\beta|\le N-1}\left\|w_{\ell,-\gamma}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2$$
$$+\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)$$
$$+\left(\mathcal{E}_{N-1,\ell,-\gamma}(t)+\varepsilon\right)\mathcal{D}_{N-1,\ell,-\gamma}(t). \tag{4.16}$$

Here we used the fact that $\left((v\times B)\cdot\partial_\beta^\alpha\nabla_v\{\mathbf{I}-\mathbf{P}\}f,\, w^2_{\ell-|\beta|,-\gamma}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)=0.$

Therefore, recalling (3.18), a proper linear combination of (3.19), (4.14), (4.15) and (4.16) gives (3.20) by taking $l_0\ge l_1+\frac{5}{2}$, $\theta_3=\frac{1-2\gamma}{2\widetilde{\ell}_3-\gamma}$, $l_1^*\ge\widetilde{\ell}_3-\frac{\gamma}{2}-\gamma\ell$ and further by replacing ℓ with $l_1\ge N$. This proves Lemma 3.9.

Now, for brevity let us modify the proof of Lemma 3.9 above so as to obtain Lemmas 3.7 and 3.8. To prove Lemma 3.7, similarly for deducing (4.14), (4.15) and (4.16), one can get

$$\frac{d}{dt}\sum_{1\le|\alpha|\le N_0}\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|^2+\sum_{1\le|\alpha|\le N_0}\left\|w_{\ell,-\gamma}\partial^\alpha f\right\|_\nu^2+\frac{1}{(1+t)^{1+\vartheta}}\left\|w_{\ell,-\gamma}\partial^\alpha f\langle v\rangle\right\|^2$$
$$\lesssim\sum_{1\le|\alpha|\le N_0}\left\|\partial^\alpha f\right\|_\nu^2+\sum_{1\le|\alpha|\le N_0}\left\|\partial^\alpha E\right\|\left\|\mu^\delta\partial^\alpha f\right\|$$
$$+\|E\|_{L_x^\infty}^{\frac{2-\gamma}{1-\gamma}}\sum_{1\le|\alpha|\le N_0}\left\|w_{\ell,-\gamma}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2$$
$$+\left\|\nabla^2(E,B)\right\|_{H_x^{N_0-2}}^{\frac{1}{\theta_1}}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)+\left(\mathcal{E}_{N_0,\ell,-\gamma}(t)+\varepsilon\right)\mathcal{D}_{N_0,\ell,-\gamma}(t), \tag{4.17}$$

and

$$
\begin{aligned}
&\frac{d}{dt}\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\|^2+\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\frac{1}{(1+t)^{1+\vartheta}}\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\|^2\\
&\lesssim \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|E\|_{L^\infty_x}^{\frac{2-\gamma}{1-\gamma}}\left\|w_{\ell,-\gamma}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2+\|E\|^2+\|\nabla f\|_\nu^2\\
&\quad+(\mathcal{E}_{N_0,\ell,-\gamma}(t)+\varepsilon)\mathcal{D}_{N_0,\ell,-\gamma}(t),
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\sum_{m=1}^{N_0}C_m\sum_{\substack{|\beta|=m,\\|\alpha|+|\beta|\le N_0}}\left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\quad+\sum_{\substack{|\alpha|+|\beta|\le N_0,\\|\beta|\ge 1}}\left(\left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right.\\
&\quad\left.+\frac{1}{(1+t)^{1+\vartheta}}\left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\right)\\
&\lesssim\sum_{|\alpha|\le N_0-1}\left(\left\|\nabla^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\partial^\alpha E\right\|^2\right)\\
&\quad+\|E\|_{L^\infty_x}^{\frac{2-\gamma}{1-\gamma}}\sum_{|\alpha|+|\beta|\le N_0}\left\|w_{\ell-|\beta|,-\gamma}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\quad+\left\|\nabla^2(E,B)\right\|_{H^{N_0-2}_x}^{\frac{1}{\theta_1}}\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)+\left(\mathcal{E}_{N_0,\ell,-\gamma}(t)+\varepsilon\right)\mathcal{D}_{N_0,\ell,-\gamma}(t),
\end{aligned}
\tag{4.19}
$$

where $\widetilde{\mathcal{D}}_{N_0,\ell,-\gamma}(t)$ is given in (3.16). Therefore, recalling (3.16), a proper linear combination of (3.7), (4.10), (4.13), (4.17), (4.18) and (4.19) yields the desired estimate (3.17) by taking $\theta_1=\frac{1-2\gamma}{2\widetilde{l}_1-\gamma}$ and $l_0^*\ge\widetilde{\ell}_1-\frac{\gamma}{2}-\gamma\ell$. This proves Lemma 3.7. Finally, Lemma 3.8 follows from modifying the proof of Lemma 3.9 in a straightforward way without considering any weight function; the details of the proof are omitted for brevity. □

4.4. The proof of Lemma 3.10. To prove Lemma 3.10, we firstly estimate the highest $N-$th order norm as follows:

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=N}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2+\sum_{|\alpha|=N}(1+t)^{-1-\vartheta}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2+\sum_{|\alpha|=N}\left\|w_{l_1^*,1}\partial^\alpha f\right\|_\nu^2\\
&\lesssim\sum_{|\alpha|=N}\left\|\partial^\alpha f\right\|_\nu^2+\sum_{|\alpha|=N}\left\|\partial^\alpha f\right\|_\nu\left\|\partial^\alpha E\right\|+\underbrace{\sum_{|\alpha|=N}\left(\partial^\alpha(v\times B\cdot\nabla_v f),w^2_{l_1^*,1}\partial^\alpha f\right)}_{R_1}
\end{aligned}
$$

$$+\underbrace{\sum_{|\alpha|=N}\left(\partial^\alpha(E\cdot\nabla_v f),\, w^2_{l_1^*,1}\partial^\alpha f\right)}_{R_2}+\underbrace{\sum_{|\alpha|=N}\left(\partial^\alpha(v\cdot Ef),\, w^2_{l_1^*,1}\partial^\alpha f\right)}_{R_3}$$

$$+\underbrace{\sum_{|\alpha|=N}\left(\partial^\alpha\Gamma(f,f),\, w^2_{l_1^*,1}\partial^\alpha f\right)}_{R_4}. \tag{4.20}$$

Applying macro-micro decomposition and Holder inequalities gives

$$R_1=\underbrace{\sum_{|\alpha-\alpha_1|\le N_0-2}\left\|\partial^{\alpha_1}B\right\|_{L^3_x}\left\|w_{l_1^*,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}+1}\right\|_{L^2_vL^6_x}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle^{\frac{\gamma}{2}}\right\|}_{R_{1,1}}$$

$$+\underbrace{\sum_{|\alpha-\alpha_1|=N_0-1}\left\|\partial^{\alpha_1}B\right\|_{L^\infty_x}\left\|w_{l_1^*,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}+1}\right\|\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle^{\frac{\gamma}{2}}\right\|}_{R_{1,2}}$$

$$+\underbrace{\sum_{\substack{|\alpha-\alpha_1|+1\ge N_0+1,\\ 1\le|\alpha_1|\le N_0-2}}\left\|\partial^{\alpha_1}B\right\|_{L^\infty_x}\left\|w_{l_1^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|}_{R_{1,3}}$$

$$+\underbrace{\sum_{\substack{|\alpha-\alpha_1|+1\ge N_0+1,\\ N_0-1\le|\alpha_1|\le N_0}}\left\|\partial^{\alpha_1}B\right\|\left\|w_{l_1^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|_{L^2_vL^\infty_x}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|}_{R_{1,4}}$$

$$+\underbrace{\sum_{\alpha_1\le\alpha}\int_{\mathbb{R}^3_x}\left|\partial^{\alpha_1}B\right|\left|\mu^\delta\partial^{\alpha-\alpha_1}f\right|\left|\mu^\delta\partial^\alpha f\right|dx}_{R_{1,5}}.$$

It is straightforward to compute that

$$R_{1,1}+R_{1,2}+R_{1,5}\lesssim\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)+\mathcal{E}_N(t)\mathcal{D}_N(t)+\varepsilon\sum_{|\alpha|=N}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2_\nu,$$

where l_0 satisfies that $l_1^*-|\beta|+1-\frac{\gamma}{2}\le-\gamma(l_0-|\beta|-1)$ so $l_0\ge\frac{l_1^*}{-\gamma}+\frac{3}{2}+|\beta|+\frac{|\beta|-1}{\gamma}$. Noticing that $\gamma\in(-3,-1)$, so we take $l_0\ge l_1^*+\frac{5}{2}$. As to $R_{1,3}$ and $R_{1,4}$, one deduce by Cauchy's inequality

$$R_{1,3}+R_{1,4}\lesssim\sum_{\substack{|\alpha-\alpha_1|+1\ge N_0+1,\\ 1\le|\alpha_1|\le N_0-2}}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\left\|w_{l_1^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2$$

$$+\sum_{\substack{|\alpha-\alpha_1|+1\ge N_0+1,\\ N_0-1\le|\alpha_1|\le N_0}}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}B\right\|^2\left\|w_{l_1^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2_{L^2_vL^\infty_x}$$

$$+\varepsilon(1+t)^{-1-\vartheta}\sum_{|\alpha|=N}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2.$$

Consequently

$$
\begin{aligned}
R_1 \lesssim & \sum_{\substack{|\alpha-\alpha_1|+1\geq N_0+1,\\ 1\leq|\alpha_1|\leq N_0-2}} (1+t)^{1+\vartheta} \left\|\partial^{\alpha_1} B\right\|_{L^\infty_x}^2 \left\| w_{l_1^*-1,1} \nabla_v \partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\langle v\rangle \right\|^2 \\
& + \sum_{\substack{|\alpha-\alpha_1|+1\geq N_0+1,\\ N_0-1\leq|\alpha_1|\leq N_0}} (1+t)^{1+\vartheta} \left\|\partial^{\alpha_1} B\right\|^2 \left\| w_{l_1^*-1,1} \nabla_v \partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\langle v\rangle \right\|_{L^2_v L^\infty_x}^2 \\
& + \mathcal{E}_N(t)\mathcal{D}_N(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \\
& + \varepsilon \sum_{|\alpha|=N} \left\{ (1+t)^{-1-\vartheta} \left\| w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2 + \left\| w_{l_1^*,1}\partial^\alpha f\right\|_\nu^2 \right\}.
\end{aligned}
$$

By exploiting the same argument used to estimate R_1, one also deduces

$$
\begin{aligned}
R_2+R_3 \lesssim & \sum_{\substack{|\alpha-\alpha_1|+m\geq N_0+1,\\ 1\leq|\alpha_1|\leq N_0-2, m\leq 1}} (1+t)^{1+\vartheta} \|\partial^{\alpha_1} E\|_{L^\infty_x}^2 \|w_{l_1^*-m,1} \nabla_v^m \partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\|^2 \\
& + \sum_{\substack{|\alpha-\alpha_1|+m\geq N_0+1,\\ N_0-1\leq|\alpha_1|\leq N_0, m\leq 1}} (1+t)^{1+\vartheta} \|\partial^{\alpha_1} E\|^2 \|w_{l_1^*-m,1} \nabla_v^m \partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\|_{L^2_v L^\infty_x}^2 \\
& + \sum_{|\alpha|=N} \|E\|_{L^\infty_x}^{\frac{2-\gamma}{1-\gamma}} \left\| w_{l_1^*,1}\partial^\alpha\{\mathbf{I}-\mathbf{P}\} f\langle v\rangle\right\|^2 + \mathcal{E}_N(t)\mathcal{D}_N(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) \\
& + \varepsilon \sum_{|\alpha|=N} \left\{ (1+t)^{-1-\vartheta} \left\| w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2 + \left\| w_{l_1^*,1}\partial^\alpha f\right\|_\nu^2 \right\}.
\end{aligned}
$$

For R_4, Lemma 3.2 tells us that

$$
\begin{aligned}
R_4 \lesssim & \underbrace{\sum_{|\alpha_1|\leq N_0-4, m\leq 2} \int_{\mathbb{R}^3} \left|\nabla_v^m\left(\mu^\delta \partial^{\alpha_1} f\right)\right| \left|w_{l_1^*,1}\partial^{\alpha-\alpha_1} f\right|_\nu \left|w_{l_1^*,1}\partial^\alpha f\right|_\nu dx}_{R_{4,1}} \\
& + \underbrace{\sum_{|\alpha_1|=N_0-3 \ or \ N_0-2, m\leq 2} \int_{\mathbb{R}^3} \left|\nabla_v^m\left(\mu^\delta \partial^{\alpha_1} f\right)\right| \left|w_{l_1^*,1}\partial^{\alpha-\alpha_1} f\right|_\nu \left|w_{l_1^*,1}\partial^\alpha f\right|_\nu dx}_{R_{4,2}} \\
& + \underbrace{\sum_{|\alpha_1|\geq N_0-2, m\leq 2} \int_{\mathbb{R}^3} \left|w_{l_1^*,1}\partial^{\alpha_1} f\right|_\nu \left|\nabla_v^m\left(\mu^\delta \partial^{\alpha-\alpha_1} f\right)\right| \left|w_{l_1^*,1}\partial^\alpha f\right|_\nu dx}_{R_{4,3}} \\
& + \underbrace{\sum_{|\alpha_1|\leq N_0} \int_{\mathbb{R}^3} \left|w_{l_1^*,1}\partial^{\alpha_1} f\right| \left|w_{l_1^*,1}\partial^{\alpha-\alpha_1} f\right|_\nu \left|w_{l_1^*,1}\partial^\alpha f\right|_\nu dx}_{R_{4,4}}
\end{aligned}
$$

$$
+ \underbrace{\sum_{N_0+1\le|\alpha_1|\le|\alpha|-1} \int_{\mathbb{R}^3} \left|w_{l_1^*,1}\partial^{\alpha_1} f\right|_\nu \left|w_{l_1^*,1}\partial^{\alpha-\alpha_1} f\right| \left|w_{l_1^*,1}\partial^{\alpha} f\right|_\nu dx}_{R_{4,5}}
$$

$$
+ \underbrace{\sum_{\alpha_1=\alpha} \int_{\mathbb{R}^3} \left|w_{l_1^*,1}\partial^{\alpha} f\right|_\nu \left|w_{l_1^*,1} f\right| \left|w_{l_1^*,1}\partial^{\alpha} f\right|_\nu dx}_{R_{4,6}} .
$$

For $R_{4,1}$, one obtains by the Hölder inequality that

$$
\begin{aligned}
R_{4,1} &\lesssim \sum_{\substack{|\alpha_1|\le N_0-4,\\ m\le 2}} \left\|\nabla_v^m \left(\mu^\delta \partial^{\alpha_1} f\right)\right\|^2_{L^\infty_x L^2_v} \left\|w_{l_1^*,1}\partial^{\alpha-\alpha_1} f\right\|^2_\nu + \varepsilon \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|_\nu \\
&\lesssim \mathcal{E}_{N_0,l_0,-\gamma}(t) \sum_{|\alpha_1|\le N_0-4} \left\|w_{l_1^*,1}\partial^{\alpha-\alpha_1} f\right\|^2_\nu + \varepsilon \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|^2_\nu .
\end{aligned}
$$

By using $L^2-L^\infty-L^2$, $L^\infty-L^2-L^2$, $L^6-L^3-L^2$ or $L^3-L^6-L^2$ type inequalities with respect to space derivative x, one also has

$$
\begin{aligned}
\sum_{i=2}^{6} R_{4,i} \lesssim{}& \max\left\{\mathcal{E}_{N_0,l_0,-\gamma}(t), \mathcal{E}_{N-1,N-1,-\gamma}(t)\right\} \\
&\times \sum_{1\le|\alpha_1|\le|\alpha|} \left\|w_{l_1^*,1}\partial^{\alpha_1} f\right\|^2_\nu + \varepsilon \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|^2_\nu .
\end{aligned}
$$

Consequently

$$
R_4 \lesssim \max\{\mathcal{E}_{N_0,l_0,-\gamma}(t), \mathcal{E}_{N-1,N-1,-\gamma}(t)\} \sum_{1\le|\alpha_1|\le|\alpha|} \left\|w_{l_1^*,1}\partial^{\alpha_1} f\right\|^2_\nu + \varepsilon \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|^2_\nu .
$$

Recalling (3.22) in the case when $n = N$ and $j = 0$, we refer to

$$
\begin{aligned}
\sum_{i=1}^{4} R_i \lesssim{}& E^N_{tri,0}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) + \mathcal{E}_N(t)\mathcal{D}_N(t) \\
&+ \sum_{|\alpha|=N} \left\|\partial^\alpha f\right\|_\nu \left\|\partial^\alpha E\right\| \\
&+ \varepsilon \sum_{|\alpha|=N} \left\{(1+t)^{-1-\vartheta} \left\|w_{l_1^*,1}\partial^{\alpha} f\langle v\rangle\right\|^2 + \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|^2_\nu\right\} .
\end{aligned}
$$

Collecting the above estimates into (4.20) yields

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=N} \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|^2 + \sum_{|\alpha|=N} (1+t)^{-1-\vartheta} \left\|w_{l_1^*,1}\partial^{\alpha} f\langle v\rangle\right\|^2 + \sum_{|\alpha|=N} \left\|w_{l_1^*,1}\partial^{\alpha} f\right\|^2_\nu \\
&\lesssim \sum_{|\alpha|=N} \left\|\partial^\alpha f\right\|^2_\nu + E^N_{tri,0}(t) + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) + \mathcal{E}_N(t)\mathcal{D}_N(t) + \sum_{|\alpha|=N} \left\|\partial^\alpha f\right\|_\nu \left\|\partial^\alpha E\right\| .
\end{aligned}
$$

When $|\alpha|+|\beta|=N$, $|\beta|=1$, one has

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left\|w_{l_1^*-1,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left\|w_{l_1^*-1,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&\quad+(1+t)^{-1-\vartheta}\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left\|w_{l_1^*-1,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\lesssim\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left\{\eta\left\|w_{l_1^*,1}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right\}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial_\beta^\alpha(v\cdot\{\mathbf{I}-\mathbf{P}\}f),\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)}_{R_5}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial_\beta^\alpha((v\times B)\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)}_{R_6}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial_\beta^\alpha(E\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)}_{R_7}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial_\beta^\alpha(v\cdot E\{\mathbf{I}-\mathbf{P}\}f),\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)}_{R_8}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial_\beta^\alpha\Gamma(f,f),\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)}_{R_9}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial_\beta^\alpha I_{mac}(t),\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)}_{R_{10}},
\end{aligned}
\tag{4.21}
$$

where $I_{mac}(t)$ is defined by

$$
\begin{aligned}
I_{mac}(t)=\{\mathbf{I}-\mathbf{P}\}q_1E\cdot v\mu^{1/2}+\mathbf{P}\left\{v\cdot\nabla_x f+q_0(E+v\times B)\cdot\nabla_v f-\frac{q_0}{2}E\cdot vf\right\}\\
-v\cdot\nabla_x\mathbf{P}f-q_0(E+v\times B)\cdot\nabla_v\mathbf{P}f+\frac{q_0}{2}E\cdot v\mathbf{P}f.
\end{aligned}
\tag{4.22}
$$

Unlike the corresponding linear term for the weight $w_{\ell,-\gamma}$, here R_5 can be dominated by

$$
\begin{aligned}
R_5&\lesssim\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left(\partial^{\alpha+e_i}\{\mathbf{I}-\mathbf{P}\}f,\,w_{l_1^*-1,1}^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\lesssim\sum_{|\alpha|+|\beta|=N,|\beta|=1}\left\|w_{l_1^*,1}\partial^{\alpha+e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|\left\|w_{l_1^*-1,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\| w_{l_1^*,1} \partial^{\alpha+e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\|^2 \\
&\quad + \varepsilon \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \right\|_{\nu}^2
\end{aligned}
$$

As for R_{10}, it is straightforward to compute that

$$
R_{10} \lesssim \|\nabla^{|\alpha|+1} f\|_{\nu}^2 + \|\nabla^{|\alpha|} E\|^2 + \varepsilon \left\| \nabla^{|\alpha|} \{\mathbf{I}-\mathbf{P}\} f \right\|_{\nu}^2 + \mathcal{E}_N(t)\mathcal{D}_N(t).
$$

Applying the similar trick as $R_1 \sim R_4$ gives

$$
\begin{aligned}
\sum_{i=6}^{9} R_i \lesssim E^N_{tri,1}(t) + \varepsilon \sum_{|\alpha|+|\beta|=N,|\beta|=1} &\left\{ (1+t)^{-1-\vartheta} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle \right\|^2 \right. \\
&\left. + \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \right\|_{\nu}^2 \right\},
\end{aligned}
$$

where $E^N_{tri,1}(t)$ is given in (3.22) with $n = N$ and $j = 1$. Thus plugging the estimates on $R_5 \sim R_{10}$ into (4.21) yields

$$
\begin{aligned}
&\frac{d}{dt} \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \right\|^2 + \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \right\|_{\nu}^2 \\
&\quad + (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle \right\|^2 \\
&\lesssim \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\| w_{l_1^*,1} \partial^{\alpha+e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\|^2 + \left\| \nabla^{|\alpha|+1} f \right\|_{\nu}^2 + \left\| \nabla^{|\alpha|} E \right\|^2 + E^N_{tri,1}(t) \\
&\quad + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t) + \mathcal{E}_N(t)\mathcal{D}_N(t) \\
&\quad + \sum_{|\alpha|+|\beta|=N,|\beta|=1} \left\{ \eta \left\| w_{l_1^*,1} \partial^{\alpha} \{\mathbf{I}-\mathbf{P}\} f \right\|_{\nu}^2 + \|\partial^{\alpha} \{\mathbf{I}-\mathbf{P}\} f\|_{\nu}^2 \right\}.
\end{aligned}
$$

Similarly, we can obtain that

$$
\begin{aligned}
&\frac{d}{dt} \sum_{|\alpha|+|\beta|=N,|\beta|=j} \left\| w_{l_1^*-j,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \right\|^2 + \sum_{|\alpha|+|\beta|=N,|\beta|=j} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \right\|_{\nu}^2 \\
&\quad + (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|=N,|\beta|=j} \left\| w_{l_1^*-1,1} \partial^{\alpha}_{\beta} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle \right\|^2 \\
&\lesssim \sum_{|\alpha|+|\beta|=N,|\beta|=j} \left\| w_{l_1^*-j+1,1} \partial^{\alpha+e_i}_{\beta-e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\|^2 \\
&\quad + \left\| \nabla^{|\alpha|+1} f \right\|_{\nu}^2 + \left\| \nabla^{|\alpha|} E \right\|^2 + E^N_{tri,j}(t)
\end{aligned}
$$

$$
+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)+\mathcal{E}_N(t)\mathcal{D}_N(t)
+\sum_{|\alpha|+|\beta|=N,|\beta'|<j}\left\{\eta\left\|w_{l_1^*-|\beta'|,1}\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right\},
$$

for $2\leq j\leq N$, where $E^N_{tri,j}(t)$ is defined in (3.22) with $n=N$. Taking summation over $0\leq j\leq N$, one deduces

$$
\begin{aligned}
&\frac{d}{dt}\left\{\sum_{\substack{|\alpha|+|\beta|=N,\\|\beta|=j,1\leq j\leq N}}(1+t)^{-\sigma_{N,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right.\\
&\left.\quad+\sum_{|\alpha|=N}(1+t)^{-\sigma_{N,0}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2\right\}\\
&\quad+\sum_{\substack{|\alpha|+|\beta|=N,\\|\beta|=j,1\leq j\leq N}}(1+t)^{-\sigma_{N,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\sum_{|\alpha|=N}(1+t)^{-\sigma_{N,0}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|_\nu^2\\
&\quad+\sum_{\substack{|\alpha|+|\beta|=N,\\|\beta|=j,1\leq j\leq N}}(1+t)^{-1-\vartheta-\sigma_{N,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\quad+\sum_{|\alpha|=N}(1+t)^{-1-\vartheta-\sigma_{N,0}}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2\\
&\lesssim\sum_{|\alpha|\leq N-1}\left\{\left\|\nabla^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\nabla^{|\alpha|}E\right\|^2\right\}+(1+t)^{-2\sigma_{N,0}}\left\|\nabla^N E\right\|^2\\
&\quad+\sum_{0\leq j\leq N}(1+t)^{-\sigma_{N,j}}E^N_{tri,j}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\\
&\quad+\mathcal{E}_N(t)\mathcal{D}_N(t)+\eta\sum_{\substack{|\alpha|+|\beta|=N,\\|\beta|=j,1\leq j\leq N,|\beta'|<j}}(1+t)^{-\sigma_{N,j}}\left\|w_{l_1^*-|\beta'|,1}\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+|\beta|=N,\\|\beta|=j,1\leq j\leq N}}(1+t)^{-\sigma_{N,j}}\left\|w_{l_1^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2\\
&\quad\lesssim\sum_{\substack{|\alpha|+|\beta|=N,\\|\beta|=j,1\leq j\leq N}}(1+t)^{-\sigma_{N,j-1}-\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)}\left\|w_{l_1^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^{\frac{4(1+\gamma)}{\gamma-2}}\\
&\qquad\times\left\|w_{l_1^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^{\frac{-2\gamma-8}{\gamma-2}}
\end{aligned}
$$

$$\lesssim \sum_{\substack{|\alpha|+|\beta|=N,\\ |\beta|=j,1\le j\le N}} \left\{(1+t)^{-\sigma_{N,j-1}-1-\vartheta}\left\|w_{l_1^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\right.$$
$$\left.+(1+t)^{-\sigma_{N,j-1}}\left\|w_{l_1^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma}{2}}\right\|^2\right\}, \tag{4.23}$$

where follows from the fact that $\sigma_{N,j}-\sigma_{N,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)$.

When $N_0+1\le n\le N-1$, we can deduce similarly that

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2\right\}\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\le j\le n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_1^*,1}\partial^\alpha f\right\|_\nu^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\le j\le n}}(1+t)^{-1-\vartheta-\sigma_{n,j}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{|\alpha|=n}(1+t)^{-1-\vartheta-\sigma_{n,0}}\left\|w_{l_1^*,1}\partial^\alpha f\langle v\rangle\right\|^2\\
&\lesssim\sum_{|\alpha|\le n-1}\left\{\left\|\nabla^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\nabla^{|\alpha|}E\right\|^2\right\}\\
&\quad+(1+t)^{-2\sigma_{n,0}}\left\|\nabla^n E\right\|^2+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\\
&\quad+\sum_{0\le j\le n}(1+t)^{-\sigma_{n,j}}E^n_{tri,j}(t)+\mathcal{E}_N(t)\mathcal{D}_N(t)\\
&\quad+\eta\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\le j\le n,|\beta'|<j}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_1^*-|\beta'|,1}\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2.
\end{aligned}$$

Here we have used $\sigma_{n,j}-\sigma_{n,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)$ and recall that $E^n_{tri,j}(t)$ is given in (3.22). Taking summation over $N_0+1\le n\le N$ gives (3.21). This completes the proof of Lemma 3.10. □

4.5. The proof of Lemma 3.11. Similar to the proof of Lemma 3.10, we have firstly that

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=N_0}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2+\sum_{|\alpha|=N_0}(1+t)^{-1-\vartheta}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\|^2+\sum_{|\alpha|=N_0}\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2\\
&\lesssim\sum_{|\alpha|=N_0}\|\partial^\alpha f\|_\nu^2+\sum_{|\alpha|=N_0}\|\partial^\alpha f\|_\nu\|\partial^\alpha E\|+\underbrace{\sum_{|\alpha|=N_0}\left(\partial^\alpha((v\times B)\cdot\nabla_v f),w^2_{l_0^*,1}\partial^\alpha f\right)}_{H_1}
\end{aligned}$$

$$+\underbrace{\sum_{|\alpha|=N_0}\left(\partial^\alpha(E\cdot\nabla_v f),w^2_{l_0^*,1}\partial^\alpha f\right)}_{H_2}+\underbrace{\sum_{|\alpha|=N_0}\left(\partial^\alpha(v\cdot Ef),w^2_{l_0^*,1}\partial^\alpha f\right)}_{H_3}$$

$$+\underbrace{\sum_{|\alpha|=N_0}\left(\partial^\alpha\Gamma(f,f),w^2_{l_0^*,1}\partial^\alpha f\right)}_{H_4}. \tag{4.24}$$

Applying the Hölder inequality and the Sobolev inequality, one has

$$\begin{aligned}
H_1 \lesssim & \sum_{1\le|\alpha_1|\le N_0-2}\left\|\partial^{\alpha_1}B\right\|_{L^\infty_x}\left\|w_{l_0^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\| \\
&+\sum_{N_0-1\le|\alpha_1|\le N_0}\left\|\partial^{\alpha_1}B\right\|\left\|w_{l_0^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|_{L^2_vL^\infty_x}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\| \\
&+\sum_{\alpha_1\le\alpha}\int_{\mathbb{R}^3_x}\left|\partial^{\alpha_1}B\right|\left|\mu^\delta\partial^{\alpha-\alpha_1}f\right|_{L^2_v}\left|\mu^\delta\partial^\alpha f\right|_{L^2_v}dx \\
\lesssim & \sum_{1\le|\alpha_1|\le N_0-2}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}B\right\|^2_{L^\infty_x}\left\|w_{l_0^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2 \\
&+\sum_{N_0-1\le|\alpha_1|\le N_0}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}B\right\|^2\left\|w_{l_0^*-1,1}\nabla_v\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2_{L^2_vL^\infty_x} \\
&+\mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t)+\varepsilon(1+t)^{-1-\vartheta}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\|^2+\varepsilon\left\|\mu^\delta\partial^\alpha f\right\|^2.
\end{aligned}$$

In a similar way, we can also get that

$$\begin{aligned}
H_2+H_3 \lesssim & \sum_{1\le|\alpha_1|\le N_0-2,m\le1}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}E\right\|^2_{L^\infty_x}\left\|w_{l_0^*-m,1}\nabla_v^m\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \\
&+\sum_{N_0-1\le|\alpha_1|\le N_0,m\le1}(1+t)^{1+\vartheta}\left\|\partial^{\alpha_1}E\right\|^2\left\|w_{l_0^*-m,1}\nabla_v^m\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{L^2_vL^\infty_x} \\
&+\sum_{|\alpha|=N_0}\|E\|^{\frac{2-\gamma}{1-\gamma}}_{L^\infty}\left\|w_{l_0^*,1}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2+\varepsilon\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2_\nu \\
&+\varepsilon(1+t)^{-1-\vartheta}\left\|w_{l_0^*,1}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2+\mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t).
\end{aligned}$$

As for H_4, Lemma 3.2 suggests that

$$\begin{aligned}
H_4 \lesssim & \underbrace{\sum_{|\alpha_1|=0,m\le2}\int_{\mathbb{R}^3}\left|\nabla_v^m\left(\mu^\delta f\right)\right|\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,1}} \\
&+\underbrace{\sum_{1\le|\alpha_1|\le N_0-3,m\le2}\int_{\mathbb{R}^3}\left|\nabla_v^m\left(\mu^\delta\partial^{\alpha_1}f\right)\right|\left|w_{l_0^*,1}\partial^{\alpha-\alpha_1}f\right|_\nu\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,2}}
\end{aligned}$$

$$+\underbrace{\sum_{|\alpha_1|=N_0-2\ or\ N_0-1,m\le 2}\int_{\mathbb{R}^3}\left|w_{l_0^*,1}\partial^{\alpha_1}f\right|_\nu\left|\nabla_v^m\left(\mu^\delta\partial^{\alpha-\alpha_1}f\right)\right|\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,3}}$$

$$+\underbrace{\sum_{\alpha_1=\alpha,m\le 2}\int_{\mathbb{R}^3}\left|w_{l_0^*,1}\partial^{\alpha}f\right|_\nu\left|\nabla_v^m\left(\mu^\delta f\right)\right|\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,4}}$$

$$+\underbrace{\sum_{|\alpha_1|=0}\int_{\mathbb{R}^3}\left|w_{l_0^*,1}f\right|\left|w_{l_0^*,1}\partial^{\alpha}f\right|_\nu\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,5}}$$

$$+\underbrace{\sum_{1\le|\alpha_1|\le N_0-1}\int_{\mathbb{R}^3}\left|w_{l_0^*,1}\partial^{\alpha_1}f\right|\left|w_{l_0^*,1}\partial^{\alpha-\alpha_1}f\right|_\nu\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,6}}$$

$$+\underbrace{\sum_{\alpha_1=\alpha}\int_{\mathbb{R}^3}\left|w_{l_0^*,1}f\right|\left|w_{l_0^*,1}\partial^{\alpha}f\right|_\nu\left|w_{l_0^*,1}\partial^\alpha f\right|_\nu dx}_{H_{4,7}}.$$

By using $L^2-L^\infty-L^2$ or $L^\infty-L^2-L^2$ type inequalities with respect to space derivative x, one has

$$\begin{aligned}&H_{4,1}+H_{4,4}+H_{4,5}+H_{4,7}\\&\quad\lesssim\left\{\mathcal{E}_{N_0,0}(t)+\left\|w_{l_0^*,1}f\right\|^2_{L^2_vL^\infty_x}\right\}\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2+\varepsilon\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2,\end{aligned}$$

while employing $L^3-L^6-L^2$ or $L^6-L^3-L^2$ type inequalities gives

$$\begin{aligned}H_{4,2}+H_{4,3}+H_{4,6}\lesssim&\sum_{1\le|\alpha_1|\le N_0-3,m\le 2}\left\|\nabla_v^m\left(\mu^\delta\partial^{\alpha_1}f\right)\right\|_{L^3_x}\left\|w_{l_0^*,1}\partial^{\alpha-\alpha_1}f\right\|^2_{L^2_vL^6_x}\\&+\sum_{|\alpha_1|=N_0-2,m\le 2}\left\|w_{l_0^*,1}\partial^{\alpha_1}f\right\|^2_{L^2_vL^\infty_x}\left\|\nabla_v^m\left(\mu^\delta\partial^{\alpha-\alpha_1}f\right)\right\|^2\\&+\sum_{|\alpha_1|=N_0-1,m\le 2}\left\|w_{l_0^*,1}\partial^{\alpha_1}f\right\|^2_{L^2_vL^6_x}\left\|\nabla_v^m\left(\mu^\delta\partial^{\alpha-\alpha_1}f\right)\right\|^2_{L^2_vL^3_x}\\&+\sum_{1\le|\alpha_1|\le N_0-1}\left\|w_{l_0^*,1}\partial^{\alpha_1}f\right\|^2_{L^2_vL^3_x}\left\|w_{l_0^*,1}\partial^{\alpha-\alpha_1}f\right\|^2_{L^2_vL^6_x}+\varepsilon\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2.\end{aligned}$$

Consequently

$$\begin{aligned}H_4\lesssim&\left\{\mathcal{E}_{N_0,0}(t)+\left\|w_{l_0^*,1}f\right\|^2_{L^2_vL^\infty_x}\right\}\sum_{1\le|\alpha_1|\le N_0}\left\|w_{l_0^*,1}\partial^{\alpha_1}f\right\|_\nu^2\\&+\sum_{1\le|\alpha_1|\le N_0-1}\left\|w_{l_0^*,1}\partial^{\alpha_1}f\right\|^2_{L^3_x}\left\|w_{l_0^*,1}\partial^{\alpha-\alpha_1}f\right\|^2_{L^2_vL^6_x}+\varepsilon\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2.\end{aligned}$$

Recalling (3.24) with $n = N_0$, $j = 0$ for $F^{N_0}_{tri,0}(t)$, it follows by inserting the estimates on $H_1 \sim H_4$ into (4.24) that

$$\frac{d}{dt}\sum_{|\alpha|=N_0}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2+\sum_{|\alpha|=N_0}\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2+\sum_{|\alpha|=N_0}(1+t)^{-1-\vartheta}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\|^2$$
$$\lesssim \sum_{|\alpha|=N_0}\left\|\partial^\alpha f\right\|_\nu^2+F^{N_0}_{tri,0}(t)+\|\nabla^{N_0}E\|^2+\mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t).$$

When $|\alpha|+|\beta|=N_0$, $|\beta|=1$, one has

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left\|w_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left\|w_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&\quad+(1+t)^{-1-\vartheta}\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left\|w_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\lesssim \sum_{|\alpha|+|\beta|=N,|\beta|=1}\left\{\eta\left\|w_{l_0^*,1}\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\left\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\right\}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^\alpha_\beta(v\cdot\{\mathbf{I}-\mathbf{P}\}f),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)}_{H_5}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^\alpha_\beta((v\times B)\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)}_{H_6}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^\alpha_\beta(E\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)}_{H_7}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^\alpha_\beta(v\cdot E\{\mathbf{I}-\mathbf{P}\}f),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)}_{H_8}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^\alpha_\beta\Gamma(f,f),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)}_{H_9}\\
&\quad+\underbrace{\sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^\alpha_\beta I_{mac}(t),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)}_{H_{10}},
\end{aligned}\tag{4.25}$$

where $I_{mac}(t)$ is given in (4.22). H_5 can be dominated by

$$H_5 \lesssim \sum_{|\alpha|+|\beta|=N_0,|\beta|=1}\left(\partial^{\alpha+e_i}\{\mathbf{I}-\mathbf{P}\}f),\,w^2_{l_0^*-1,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)$$

$$
\begin{aligned}
&\lesssim \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*,1} \partial^{\alpha+e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\| \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{\frac{\gamma}{2}} \right\| \\
&\lesssim \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*,1} \partial^{\alpha+e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\|^2 \\
&\quad + \varepsilon \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 .
\end{aligned}
$$

Applying the same trick as $H_1 \sim H_4$ suggests that

$$
\begin{aligned}
\sum_{i=6}^{10} H_i \lesssim F_{tri,1}^{N_0}(t) + \varepsilon(1+t)^{-1-\vartheta} & \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle \right\|^2 \\
& + \left\| \nabla^{|\alpha|} E \right\|^2 + \left\| \nabla^{|\alpha|+1} f \right\|_\nu^2 + \varepsilon \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 ,
\end{aligned}
$$

where $F_{tri,1}^{N_0}(t)$ is given in (3.24) with $n = N_0$ and $j = 1$. Thus plugging the estimates on $H_5 \sim H_{10}$ into (4.25) yields

$$
\begin{aligned}
&\frac{d}{dt} \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|^2 + \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 \\
&\quad + (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle \right\|^2 \\
&\lesssim \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\| w_{l_0^*,1} \partial^{\alpha+e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\|^2 + \left\| \nabla^{|\alpha|+1} f \right\|_\nu^2 + \left\| \nabla^{|\alpha|} E \right\|^2 + F_{tri,1}^{N_0}(t) \\
&\quad + \mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t) + \sum_{|\alpha|+|\beta|=N_0,|\beta|=1} \left\{ \eta \left\| w_{l_0^*,1} \partial^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 + \left\| \partial^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 \right\}.
\end{aligned}
$$

Similarly, we can get for $2 \le j \le N_0$ that

$$
\begin{aligned}
&\frac{d}{dt} \sum_{|\alpha|+|\beta|=N_0,|\beta|=j} \left\| w_{l_0^*-j,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|^2 + \sum_{|\alpha|+|\beta|=N_0,|\beta|=j} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 \\
&\quad + (1+t)^{-1-\vartheta} \sum_{|\alpha|+|\beta|=N_0,|\beta|=j} \left\| w_{l_0^*-1,1} \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle \right\|^2 \\
&\lesssim \sum_{|\alpha|+|\beta|=N_0,|\beta|=j} \left\| w_{l_0^*-j+1,1} \partial_{\beta-e_i}^{\alpha+e_i} \{\mathbf{I}-\mathbf{P}\} f \langle v\rangle^{-\frac{\gamma}{2}-1} \right\|^2 + \left\| \nabla^{|\alpha|+1} f \right\|_\nu^2 + \left\| \nabla^{|\alpha|} E \right\|^2 + F_{tri,j}^{N_0}(t) \\
&\quad + \mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t) + \sum_{\substack{|\alpha|+|\beta|=N_0, \\ |\beta|=j,|\beta'|<j}} \left\{ \eta \left\| w_{l_0^*-|\beta'|,1} \partial_{\beta'}^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 + \left\| \partial^\alpha \{\mathbf{I}-\mathbf{P}\} f \right\|_\nu^2 \right\},
\end{aligned}
$$

where $F_{tri,j}^{N_0}(t)$ is given in (3.24) with $n = N_0$. Taking summation over $0 \le j \le N_0$, one deduces

$$
\begin{aligned}
&\frac{d}{dt}\Bigg\{\sum_{\substack{|\alpha|+|\beta|=N_0,\\|\beta|=j,1\leq j\leq N_0}}(1+t)^{-\sigma_{N_0,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\
&\quad+\sum_{|\alpha|=N_0}(1+t)^{-\sigma_{N_0,0}}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2\Bigg\}\\
&\quad+\sum_{\substack{|\alpha|+|\beta|=N_0,\\|\beta|=j,1\leq j\leq N_0}}(1+t)^{-\sigma_{N_0,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2+\sum_{|\alpha|=N_0}(1+t)^{-\sigma_{N,0}}\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2\\
&\quad+\sum_{\substack{|\alpha|+|\beta|=N_0,\\|\beta|=j,1\leq j\leq N_0}}(1+t)^{-1-\vartheta-\sigma_{N_0,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\quad+\sum_{|\alpha|=N_0}(1+t)^{-1-\vartheta-\sigma_{N_0,0}}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\|^2\\
&\lesssim\sum_{|\alpha|\leq N_0-1}\left\{\left\|\nabla^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\nabla^{|\alpha|}E\right\|^2+\left\|\nabla^{N_0}E\right\|^2\right\}+\mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t)\\
&\quad+\sum_{0\leq j\leq N_0}(1+t)^{-\sigma_{N_0,j}}F^{N_0}_{tri,j}(t)\\
&\quad+\eta\sum_{\substack{|\alpha|+|\beta|=N_0,\\|\beta|=j,1\leq j\leq N_0,|\beta'|<j}}(1+t)^{-\sigma_{N_0,j}}\left\|w_{l_0^*-|\beta'|,1}\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,
\end{aligned}
$$

where we have used the facts that

$$
\sigma_{N_0,j}-\sigma_{N_0,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta)
$$

and in a similar way as (4.23),

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+|\beta|=N_0,\\|\beta|=j,1\leq j\leq N_0}}(1+t)^{-\sigma_{N_0,j}}\left\|w_{l_0^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2\\
&\quad\lesssim\sum_{\substack{|\alpha|+|\beta|=N_0,\\|\beta|=j,1\leq j\leq N_0}}\Bigg\{(1+t)^{-\sigma_{N_0,j-1}-1-\vartheta}\left\|w_{l_0^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&\qquad+(1+t)^{-\sigma_{N_0,j-1}}\left\|w_{l_0^*-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\Bigg\}.
\end{aligned}
$$

By exploiting the same argument as before, we can get for $1\leq n\leq N_0-1$ that

$$
\begin{aligned}
&\frac{d}{dt}\Bigg\{\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\leq j\leq n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2\\
&\quad+(1+t)^{-\sigma_{0,0}}\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\Bigg\}+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\leq j\leq n}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{|\alpha|=n}(1+t)^{-\sigma_{n,0}}\left\|w_{l_0^*,1}\partial^\alpha f\right\|_\nu^2+(1+t)^{-\sigma_{0,0}}\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\
&+\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}(1+t)^{-1-\vartheta-\sigma_{n,j}}\left\|w_{l_0^*-j,1}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
&+\sum_{|\alpha|=n}(1+t)^{-1-\vartheta-\sigma_{n,0}}\left\|w_{l_0^*,1}\partial^\alpha f\langle v\rangle\right\|^2+(1+t)^{-1-\vartheta-\sigma_{0,0}}\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\\
\lesssim&\sum_{|\alpha|\le n-1}\left\{\left\|\nabla^{|\alpha|+1}f\right\|_\nu^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\left\|\nabla^{|\alpha|}E\right\|^2+\left\|\nabla^nE\right\|^2\right\}+\mathcal{E}_{N_0}(t)\mathcal{D}_{N_0}(t)\\
&+\sum_{0\le j\le n}(1+t)^{-\sigma_{n,j}}F^n_{tri,j}(t)+\eta\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n,|\beta'|<j}}(1+t)^{-\sigma_{n,j}}\left\|w_{l_0^*-|\beta'|,1}\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,
\end{aligned}
$$

where $F^n_{tri,j}(t)$ is given in (3.24). With the above estimates in hand, (3.23) follows by taking summation over $1\le n\le N_0$ and by using the energy estimates on $\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\|^2$. Thus we have completed the proof of Lemma 3.11. □

References

1. Adams, R.: Sobolev Spaces. Academic Press, New York (1985)
2. Alexandre, R., Morimoto, Y., Ukai, S., Yang, C.-J., Yang, T.: The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. J. Funct. Anal. **263**(3), 915–1010 (2012)
3. Chapman, S., Cowling, T.G.: The Mathematical Theory of Non-uniform Gases. An account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases, 3rd edn, prepared in co-operation with D. Burnett. Cambridge University Press, London (1970)
4. Duan, R.-J.: Global smooth dynamics of a fully ionized plasma with long-range collisions. Ann. Inst. Henri Poincar Anal. Non Linaire **31**(4), 751–778 (2014)
5. Duan, R.-J., Liu, S.-Q.: The Vlasov–Poisson–Boltzmann system without angular cutoff. Commun. Math. Phys. **324**(1), 1–45 (2013)
6. Duan, R.-J., Liu, S.-Q., Yang, T., Zhao, H.-J.: Stabilty of the nonrelativistic Vlasov–Maxwell–Boltzmann system for angular non-cutoff potentials. Kinet. Relat. Models **6**(1), 159–204 (2013)
7. Duan, R.-J., Strain, R.M.: Optimal large-time behavior of the Vlasov–Maxwell–Boltzmann system in the whole space. Commun. Pure Appl. Math. **24**(11), 1497–1546 (2011)
8. Duan, R.-J., Yang, T., Zhao, H.-J.: The Vlasov–Poisson–Boltzmann system for soft potentials. Math. Methods Models Appl. Sci. **23**(6), 979–1028 (2013)
9. Grad, H.: Asymptotic theory of the Boltzmann equation II. In: Laurmann, J.A. (ed.) Rarefied Gas Dynamics, pp. 26–59. Academic Press, New York (1963)
10. Gressman, P.T., Strain, R.M.: Global classical solutions of the Boltzmann equation without angular cutoff. J. Am. Math. Soc. **24**(3), 771–847 (2011)
11. Gualdani, M.P., Mischler, S., Mouhot, C.: Factorization for non-symmetric operators and exponential H-theorem. arXiv:1006.5523
12. Guo, Y.: The Landau equation in a periodic box. Commun. Math. Phys. **231**, 391–434 (2002)
13. Guo, Y.: Classical solutions to the Boltzmann equation formolecules with an angular cutoff. Arch. Ration. Mech. Anal. **169**, 305–353 (2003)
14. Guo, Y.: The Vlasov–Maxwell–Boltzmann system near Maxwellians. Invent. Math. **153**(3), 593–630 (2003)
15. Guo, Y.: The Boltzmann equation in the whole space. Indiana Univ. Math. J. **53**, 1081–1094 (2004)
16. Guo, Y.: The Vlasov–Poisson–Laudau system in a periodic box. J. Am. Math. Soc. **25**, 759–812 (2012)
17. Guo, Y., Strain, R.: Momentum regularity and stability of the relativistic Vlasov–Maxwell–Boltzmann system. Commun. Math. Phys. **310**(3), 649–673 (2012)
18. Guo, Y., Wang, Y.J.: Decay of dissipative equation and negative sobolev spaces. Commun. Partial Differ. Equ. **37**, 2165–2208 (2012)
19. Hosono, T., Kawashima, S.: Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system. Math. Models Methods Appl. Sci. **16**(11), 1839–1859 (2006)

20. Jang, J.: Vlasov–Maxwell–Boltzmann diffusive limit. Arch. Ration. Mech. Anal. **194**(2), 531–584 (2009)
21. Krall, N.A., Trivelpiece, A.W.: Principles of Plasma Physics. Vol. 1, McGraw-Hill, New York (1973)
22. Lei, Y.-J., Zhao, H.-J.: Negative Sobolev spaces and the two-species Vlasov–Maxwell–Landau system in the whole space. J. Funct. Anal. **267**(10), 3710–3757 (2014)
23. Liu, T.-P., Yang, T., Yu, S.-H.: Energy method for the Boltzmann equation. Phys. D **188**, 178–192 (2004)
24. Liu, T.-P., Yu, S.-H.: Boltzmann equation: micro–macro decompositions and positivity of shock profiles. Commun. Math. Phys. **246**, 133–179 (2004)
25. Strain, R.M.: The Vlasov–Maxwell–Boltzmann system in the whole space. Commun. Math. Phys. **268**(2), 543–567 (2006)
26. Strain, R.M., Guo, Y.: Stability of the relativistic Maxwellian in a collisional plasma. Commun. Math. Phys. **251**(2), 263–320 (2004)
27. Strain, R.M., Guo, Y.: Exponential decay for soft potentials near Maxwellian. Arch. Ration. Mech. Anal. **187**(2), 287–339 (2008)
28. Strain, R.M., Zhu, K.-Y.: The Vlasov–Poisson–Landau system in $\mathbb{R}^3_x$. Arch. Ration. Mech. Anal. **210**(2), 615–671 (2013)
29. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: Friedlander, S., Serre, D. (eds.) Handbook of Mathematical Fluid Dynamics, vol. I. North-Holland, Amsterdam, pp. 71–305 (2002)
30. Wang, Y.-J.: Golobal solution and time decay of the Vlasov–Poisson–Landau system in $\mathbb{R}^3_x$. SIAM J. Math. Anal. **44**(5), 3281–3323 (2012)
31. Xiao, Q.-H., Xiong, L.-J., Zhao, H.-J.: The Vlasov–Posson–Boltzmann system with angular cutoff for soft potential. J. Differ. Equ. **255**(6), 1196–1232 (2013)
32. Xiao, Q.-H., Xiong, L.-J., Zhao, H.-J.: The Vlasov–Posson–Boltzmann system for the whole range of cutoff soft potentials. J. Funct. Anal. **272**, 166–226 (2017)

Communicated by C. Mouhot