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Existence of the Discrete Travelling Waves for a Relaxing Scheme

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Abstract—The existence of a discrete travelling wave is proved for the relaxing scheme. The main idea is to change the original scheme such that the resulting scheme is monotonic to which Jennings' result can be applied. The equivalence of the resulting scheme and the original one is shown when $\eta = 1/q$. The $u-$component of the discrete travelling wave thus obtained is a discrete shock for a monotone conservative difference scheme, which approximates the corresponding conservation law.

Keywords—Relaxing scheme, Existence, Discrete travelling wave.

1. INTRODUCTION

We study a relaxing scheme of the form

$$\begin{aligned} &u_j^{n+1} - u_j^n + \frac{\lambda}{2}\left(v_{j+1}^n - v_{j-1}^n\right) - \frac{\mu}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) = 0, \\ &v_j^{n+1} - v_j^n + \frac{a\lambda}{2}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{\mu}{2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) = -k\left(v_j^n - f\left(u_j^n\right)\right), \end{aligned} \tag{1.1}$$

where $\lambda = (\Delta t/\Delta x)$, $\mu = \sqrt{a}\lambda$, $k = (\Delta t/\varepsilon)$, and $(u_j^n, v_j^n) = (u(j\Delta x, n\Delta t),\ v(j\Delta x, n\Delta t))$, and $(\Delta x, \Delta t)$ are the numerical approximation solution and the grid sizes of the space-time. Equation (1.1) is introduced in [1] as the first-order approximation to the system

$$\begin{aligned} u_t + v_x &= 0, \qquad x \in R^1, \\ v_t + au_x &= -\frac{1}{\varepsilon}(v - f(u)), \end{aligned} \tag{1.2}$$

which approximates scalar conservation laws $u_t + f(u)_x = 0$ when the relaxation rate ε is small. The study of the existence and stability of discrete shock waves is important in understanding

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the convergence behavior of numerical shock computations. Jennings [2] proved the existence of discrete shock waves for the general first-order monotone scheme, see also [3,4] for system case. The stability of travelling waves of (1.2) and discrete travelling waves (DTW) of (1.1) was studied in [5,6]. Let $(U,V)(x-st)$ be a travelling wave solution of (1.2) connecting (u_+,v_+) to (u_-,v_-), its existence is ensured by the subcharacteristic condition (cf. [7])

$$-\sqrt{a} < f'(u) < \sqrt{a}, \qquad \text{for all } u, \tag{1.3}$$

the Rankine-Hugoniot condition

$$-s\left(u_+ - u_-\right) + f\left(u_+\right) - f\left(u_-\right) = 0, \tag{1.4}$$

and the generalized entropy condition

$$Q(u) \equiv f(u) - f\left(u_\pm\right) - s\left(u - u_\pm\right) \begin{cases} < 0, & \text{for } u_+ < u < u_-, \\ > 0, & \text{for } u_- < u < u_+, \end{cases} \tag{1.5}$$

here, $v_\pm = f(u_\pm)$. From now on, we impose the CFL condition $\mu < 1$.

A DTW connecting $(u_\pm, v_\pm)$ is a special numerical solution of the difference scheme such that

$$\left(u_j^{n+1}, v_j^{n+1}\right) = \left(u_{j-\eta}^n, v_{j-\eta}^n\right), \tag{1.6}$$

where $\eta = s\lambda$. Since η is not necessarily an integer, the minimal domain for (1.6) is $\mathcal{L}_\eta = \{m\eta + n \mid \eta = s\lambda;\ m,n \in Z\}$.

From (1.1) and (1.6), the one-parameter discrete travelling wave (u_x, v_x), $x \in \mathcal{L}_\eta$, defined by (1.6) will satisfy

$$\begin{aligned} &u_{x-\eta} - u_x + \frac{\lambda}{2}\left(v_{x+1} - v_{x-1}\right) - \frac{\mu}{2}\left(u_{x+1} - 2u_x + u_{x-1}\right) = 0, \\ &v_{x-\eta} - v_x + \frac{a\lambda}{2}\left(u_{x+1} - u_{x-1}\right) - \frac{\mu}{2}\left(v_{x+1} - 2v_x + v_{x-1}\right) = -k\left(v_x - f\left(u_x\right)\right), \end{aligned} \tag{1.7}$$

and $\lim_{x\to\pm\infty}(u_x,v_x) = (u_\pm, v_\pm)$. First, we consider the case when $\eta = 1/q$, $q \in Z\backslash\{0\}$, and prove the following theorem. Without loss of generality, we assume $u_+ < u_-$ hereafter.

THEOREM 1. *Let $f(u)$ satisfy (1.3)–(1.5). Suppose that $\eta = 1/q$ and the relaxation rate ε is sufficiently small. Then, for each $u_0 \in (u_+, u_-)$, there is a unique function (u_x, v_x) which is continuous on $\mathcal{L}_\eta$. u_x takes the value u_0 at $x = 0$ and satisfies (1.7) with $(u_{\pm\infty}, v_{\pm\infty}) = (u_\pm, v_\pm)$, where $v_\pm = f(u_\pm)$. Furthermore, the u-component of the DTW thus obtained is a discrete shock for a monotone conservative difference scheme which approximates $u_t + f(u)_x = 0$.*

2. EXISTENCE OF DTW

In this section, we will prove Theorem 1, and thus, establish the existence of DTW to the system (1.1) for the case $\eta = 1/q$. The proof will be divided into the following five steps.

STEP 1. DETERMINE u_x. From the first equation of (1.7), we have

$$\frac{\lambda}{2}\left(v_{x+1} - v_{x-1}\right) = u_x - u_{x-\eta} + \frac{\mu}{2}\left(u_{x+1} - 2u_x + u_{x-1}\right). \tag{2.1}$$

Rewrite the second equation of (1.7) into the following two forms:

$$v_{x+1-\eta} - v_{x+1} \frac{a\lambda}{2}\left(u_{x+2} - u_x\right) - \frac{\mu}{2}\left(v_{x+2} - 2v_{x+1} + v_x\right) = -k\left(v_{x+1} - f\left(u_{x+1}\right)\right), \tag{2.2}$$

and

$$v_{x-1-\eta}-v_{x-1}+\frac{a\lambda}{2}\left(u_x-u_{x-2}\right)-\frac{\mu}{2}\left(v_x-2v_{x-1}+v_{x-2}\right)=-k\left(v_{x-1}-f\left(u_{x-1}\right)\right). \tag{2.3}$$

Then, by substituting (2.1) into $\lambda/2\{(2.2)-(2.3)\}$, and eliminating the terms of $v_x's$, we obtain

$$\begin{aligned}(u_{x-2\eta}-2u_{x-\eta}+u_x)-\mu^2\left(u_{x+1}-2u_x+u_{x-1}\right)+k\left[(u_{x-\eta}-u_x)+\frac{\lambda}{2}\left(f\left(u_{x+1}\right)-f\left(u_{x-1}\right)\right)\right]\\ =\mu\left[(2u_x-u_{x+1}-u_{x-1})-(2u_{x-\eta}-u_{x+1-\eta}-u_{x-1-\eta})\right]+\frac{k\mu}{2}\left(u_{x+1}-2u_x+u_{x-1}\right),\end{aligned} \tag{2.4}$$

which is a scheme for u_x only, where $x\in\mathcal{L}_\eta$.

STEP 2. CHANGE TO A MONOTONE SCHEME. The existence theorem in [2] can only be applied to the monotone and conservative scheme. In order to have a monotone scheme, we are going to change our scheme to a monotonic one by using an iteration based on the shift operator. To this end, we rewrite (2.4) as

$$G_x^{(0)}\equiv G_x^{(0)}\left(u_{x-1-\eta},u_{x-1},u_{x-2\eta},u_{x-\eta},u_x,u_{x+1-\eta},u_{x+1}\right)=0, \tag{2.5}$$

where

$$\begin{aligned}G_x^{(0)}\equiv\ &\mu u_{x-1-\eta}+\frac{k\mu}{2}u_{x-1}+\frac{k\lambda}{2}f\left(u_{x-1}\right)-\mu\left(1-\mu\right)u_{x-1}-u_{x-2\eta}-\left(k+2\mu-2\right)u_{x-\eta}\\ &+\left(k-k\mu-1+2\mu-2\mu^2\right)u_x+\mu u_{x+1-\eta}+\frac{k\mu}{2}u_{x+1}-\frac{k\lambda}{2}f\left(u_{x+1}\right)-\mu\left(1-\mu\right)u_{x+1}.\end{aligned}$$

For k very large, that is, ε being sufficiently small for given grid sizes, $G_x^{(0)}$ is monotone increasing function of each of its arguments except for $u_{x-2\eta}$ and $u_{x-\eta}$; therefore, (2.5) cannot be written directly as a monotone scheme. To overcome this difficulty, we consider $\eta=1/q$ as the grid size in the space direction, and transform $G_x^{(0)}=0$ into $\bar{G}_x=0$ such that $\bar{G}_x$ depends on every argument explicitly defined on every grid point. And $\bar{G}_x$ has only one term with a negative coefficient. To this end, we multiply $G_{x-\eta}^{(0)}$ by a positive constant θ_0 and add it to $G_x^{(0)}$, i.e.,

$$G_x^{(1)}=G_x^{(0)}+\theta_0G_{x-\eta}^{(0)}, \tag{2.6}$$

then the coefficients of $u_{x-3\eta}$, $u_{x-2\eta}$, and $u_{x-\eta}$ are $-\theta_0$, $-\theta_0k+2(1-\mu)\theta_0-1$ and $[\theta_0(1-\mu)-1]\,k+2(1-\mu)(1+\theta_0\mu)-\theta_0$, respectively. We choose $\theta_0>(1/1-\mu)$ such that there are still two terms with negative coefficients (here the coefficient means the derivative of $G_x^{(1)}$ with respect to its corresponding argument). Repeating the same process yields the following induction formulae:

$$G_x^{(m+1)}=G_x^{(m)}+\theta_mG_{x-\eta}^{(m)},\qquad m=1,2,\dots q-4, \tag{2.7}$$

where θ_m is chosen so that there are always two negative terms in $G_x^{(m+1)}$; precisely, we have

$$\begin{aligned}\frac{\partial G^{(m)}}{\partial u_{x-(m+2)\eta}}&=-\prod_{i=0}^{m-1}\theta_i,\\ \frac{\partial G^{(m)}}{\partial u_{x-(m+1)\eta}}&=-\left(\prod_{i=0}^{m-1}\theta_i\right)k-\Theta_m+2\left(1-\mu\right)\prod_{i=0}^{m-1}\theta_i,\\ \frac{\partial G^{(m)}}{\partial u_{x-m\eta}}&=\left[\left(1-\mu)(\prod_{i=0}^{m-1}\theta_i\right)-\Theta_m\right]k-O(1)>0,\\ \frac{\partial G^{(m)}}{\partial u_{x+l\eta}}&>0,\qquad \text{for } l=-(q+m+1),\dots,-q \quad\text{and}\quad -m+1,\dots,0\\ &\qquad\quad \text{and}\quad q-m-1,\dots,q,\end{aligned}$$

here, $O(1)$ depends only on θ_i and μ and $\Theta_m = \sum_{0\le i_1<i_2<\cdots<i_{m-1}\le m-1} \theta_{i_1}\theta_{i_2}\dots\theta_{i_{m-1}}$. We now change the negative term $u_{x-1+\eta}$ in $G_x^{(q-3)}$ to be positive and introduce a new positive term $u_{x+\eta}$, by setting

$$\bar{G}_x = G_x^{(q-3)} + \bar{\theta} G_{x+\eta}^{(q-3)}. \tag{2.8}$$

Then,

$$\begin{aligned}
\frac{\partial \bar{G}}{\partial u_{x-(q-1)\eta}} &= \bar{\theta}\left\{\frac{k\mu}{2} - \mu(1-\mu) + \frac{k\lambda}{2} f'\right\} - \prod_{i=0}^{q-4}\theta_i > 0,\\
\frac{\partial \bar{G}}{\partial u_{x-(q-2)\eta}} &= -\bar{\theta}\left(\prod_{i=0}^{q-4}\theta_i\right)k - \left(\prod_{i=0}^{q-4}\theta_i\right)k - \Theta_{q-3} + 2(1-\mu)\prod_{i=0}^{q-4}\theta_i < 0,\\
\frac{\partial \bar{G}}{\partial u_{x-(q-3)\eta}} &= \left[(1-\mu)\left(\prod_{i=0}^{q-4}\theta_i\right) - \Theta_{q-3}\right]k - \bar{\theta}\left(\prod_{i=0}^{q-4}\theta_i\right)k - O(1) > 0,\\
\frac{\partial \bar{G}}{\partial u_{x+l\eta}} &> 0, \qquad \text{for } l = -2q+2,\dots,-q \quad \text{and} \quad -q+4,\dots,q+1,
\end{aligned}$$

provided ε is suitably small and $0 < \bar{\theta} < [(1-\mu)\prod_{i=0}^{q-4}\theta_i - \Theta_{q-3}](\prod_{i=0}^{q-4}\theta_i)^{-1}$. Furthermore, we multiply $\bar{G}_x = 0$ by $1/k$ first, and then add $\alpha u_{x-1+2\eta}$ on both sides of $\bar{G}_x/k = 0$, where $\alpha = (1+\bar{\theta})\prod_{i=0}^{q-4}(1+\theta_i)$. Then, by multiplying $1/\alpha$ on both sides of the resulting scheme, we obtain

$$u_{x-1+2\eta} = G\left(u_{x-2+2\eta}, u_{x-2+3\eta}, \dots, u_{x+1+\eta}\right), \tag{2.9}$$

which is a strict monotone scheme.

Step 3. Passage to a Conservative Scheme. Next we will prove the following lemma.

Lemma 1. *Suppose ε is suitably small, (2.9) can be written as the following conservative scheme:*

$$u_{y-\eta} = u_y - \theta\left\{g_y - g_{y-\eta}\right\}, \qquad y \in \mathcal{L}_\eta, \tag{2.10}$$

where $\theta = (\Delta t/\eta\Delta x)$, $g_y = g(u_{y+2-2\eta}, u_{y+2-3\eta}, \dots, u_{y-1})$ and $g(u,u,\dots,u) = f(u)$.

Proof. If we define a shift operator S_y as $S_y(g_x) = g_{x+y}$, then the process in Step 2 is equivalent to applying the operator P to $G_x^{(0)}/k = 0$, where

$$P = \alpha^{-1}\left(I + \bar{\theta}S_\eta\right)\left(I + \theta_{q-4}S_{-\eta}\right)\left(I + \theta_{q-5}S_{-\eta}\right)\dots\left(I + \theta_0 S_{-\eta}\right), \tag{2.11}$$

here, I is an identity operator and α is defined above. It is easy to see that, for any u independent of x, $P(u) = u$.

Multiplying $G_x^{(0)}$ by $1/k$ and rearranging the terms yields

$$\begin{aligned}
0 = -u_{x-\eta} + u_x - \lambda\Bigg\{&\frac{f(u_{x+1})}{2} + \frac{1}{k\lambda}\Bigg[(u_x - u_{x-\eta})\\
&+ \left(\mu - \frac{k\mu}{2} - \mu^2\right)(u_{x+1} - u_x) - \mu\left(u_{x+1-\eta} - u_{x-\eta}\right)\Bigg] - \frac{f(u_{x-1})}{2} - \frac{1}{k\lambda}\Bigg[(u_{x-\eta} - u_{x-2\eta})\\
&+ \left(\mu - \frac{k\mu}{2} - \mu^2\right)(u_x - u_{x-1}) - \mu\left(u_{x-\eta} - u_{x-1-\eta}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{2.12}$$

To have a conservative scheme, we rewrite (2.12) as

$$0 = -u_{x-\eta} + u_x - \theta\left\{g_x^{(0)} - g_{x-\eta}^{(0)}\right\}, \tag{2.13}$$

where

$$g_x^{(0)} = \eta \frac{f(u_{x+1}) + f(u_{x+1-\eta}) + \cdots + f(u_{x-1+\eta})}{2} + \frac{\eta}{k\lambda}(u_x - u_{x-\eta})$$
$$+ \frac{\eta}{k\lambda}\left(\mu - \frac{k\mu}{2} - \mu^2\right)(u_{x+1} - u_x + u_{x+1-\eta} - u_{x-\eta} + \cdots + u_{x+\eta} - u_{x-1+\eta})$$
$$- \frac{\eta\mu}{k\lambda}(u_{x+1-\eta} - u_{x-\eta} + u_{x+1-2\eta} - u_{x-2\eta} + \cdots + u_x - u_{x-1}).$$

Then we apply the operator P to (2.13),

$$0 = -P(u_{x-\eta}) + P(u_x) - \theta\left\{P\left(g_x^{(0)}\right) - P\left(g_{x-\eta}^{(0)}\right)\right\}. \tag{2.14}$$

Since (2.9) is a monotone scheme and $u_{x-1+2\eta} = S_{-\eta}^{q-3} u_{x-\eta}$, (2.14) can be written as

$$u_{x-1+2\eta} = u_{x-1+3\eta} - \theta\left\{P\left(g_x^{(0)}\right) - \frac{1}{\theta}\left(P - S_{-\eta}^{q-3}\right)u_x - P\left(g_{x-\eta}^{(0)}\right) + \frac{1}{\theta}\left(P - S_{-\eta}^{q-3}\right)u_{x-\eta}\right\}.$$

Set $y = x - 1 + 3\eta$ and $P(g_x^{(0)}) - 1/\theta(P - S_{-\eta}^{q-3})u_x = g_y$, (2.10) follows immediately. On the other hand,

$$g_y(u,u,\ldots,u) = P\left(g_x^{(0)}(u,u,\ldots,u)\right) - \frac{1}{\theta}\left(P - S_{-\eta}^{q-3}\right)u = f(u),$$

where we have used $P(f(u)) = f(u)$ and $\eta = 1/q$. ∎

Now Jennings' result [2] can be applied to our case.

LEMMA 2. *Under the assumptions of Theorem 1, for each $u_0 \in (u_+, u_-)$, there is a unique function, continuous on $\mathcal{L}_\eta$, which takes on the value u_0 at $x = 0$, satisfies $\bar{G}_x = 0$, and has the limits $u_{\pm\infty} = u_\pm$. The solution u_x is a monotone function of $x \in \mathcal{L}_\eta$ and depends continuously at each value of x on u_0.*

STEP 4. EQUIVALENCE BETWEEN TWO SCHEMES. We next prove the solution u_x constructed above is also a solution of (2.5), precisely, we will prove the following lemma.

LEMMA 3. *Let u_x be a unique solution of $\bar{G}_x = 0$ determined in Lemma 2, then u_x satisfy $G_x^{(0)} = 0$, $\forall\, x \in \mathcal{L}_\eta$.*

PROOF. Since u_x is a solution of (2.9), we have

$$\bar{G}_x = G_x^{(q-3)} + \bar{\theta} G_{x+\eta}^{(q-3)} = 0, \qquad \forall\, x \in \mathcal{L}_\eta, \tag{2.15}$$

where $G_x^{(q-3)}$ is defined by the induction formulae (2.7). Thus,

$$G_x^{(q-3)} = \sum_{i=1}^{q-3} c_i G_{x-i\eta}^{(0)} + G_x^{(0)}, \tag{2.16}$$

where c_i is a positive constant depending only on $\{\theta_i\}_{i=1}^{q-3}$. By (2.5), we have $G_x^{(0)}(u_\pm, u_\pm, \ldots, u_\pm) = 0$, that is

$$\lim_{x\to\pm\infty} G_x^{(0)} = 0, \tag{2.17}$$

here the arguments of $G_x^{(0)}$ are substituted by the solution u_x of $\bar{G}_x = 0$. Combining (2.16) and (2.17), we obtain $\lim_{x\to\pm\infty} G_x^{(q-3)} = 0$.

From (2.15), by virtue of $\bar{\theta} < 1$,

$$\left|G_x^{(q-3)}\right| = \bar{\theta}\left|G_{x+\eta}^{(q-3)}\right| = \bar{\theta}^2\left|G_{x+2\eta}^{(q-3)}\right| = \cdots = \lim_{n\to\infty}\left|\bar{\theta}\right|^n\left|G_{x+n\eta}^{(q-3)}\right| = 0,$$

which holds for any $x \in \mathcal{L}_\eta$. So u_x is also a solution of $G_x^{(q-3)} = 0$. Repeating the same process, one can prove that u_x satisfies $G_x^{(m)} = 0$ $(m = 0, 1, \dots, q-2)$ for any $x \in \mathcal{L}_\eta$.

Now we obtain a unique solution u_x of (2.5) which is a monotone function of x. And u_x, for any fixed x, is a continuous function of the value u_0. ∎

STEP 5. DETERMINE v_x. Finally, we determine the v-component of (1.7) based on the existence of u-component u_x. To this end, by summing the first equation of (1.7) over x from $y - 2N + 1$ to $y - 1$ with step size 2, then we have

$$v_{y-2N} = v_y + \frac{2}{\lambda} \sum_{x=y-2N+1}^{y-1} (u_{x-\eta} - u_x) + \sqrt{a} \sum_{x=y-2N+1}^{y-1} [(u_x - u_{x-1}) - (u_{x+1} - u_x)] .$$

Consider $\sum_{x=-\infty}^{y_0} (u_{x-x_0} - u_x)$, for any fixed $x_0 > 0$, since $u_{x-x_0} - u_x > 0$ and $\sum_{x=-\infty}^{y_0} (u_{x-x_0} - u_x) < u_- - u_+$, then $\sum_{x=-\infty}^{y_0} (u_{x-x_0} - u_x)$ converges. Hence, v_{y-2N} converges as $N \to \infty$ for any $y \in \mathcal{L}_\eta$, which implies that there exists $\bar{v}$ such that $\lim_{x\to-\infty} v_x = \bar{v}$. Let $x \to -\infty$ in (1.7), we yield $\bar{v} = f(u_-) = v_-$. Similarly, we have $\lim_{x\to+\infty} v_x = v_+ = f(u_+)$. Now, v_x can be expressed as

$$\begin{aligned} v_x = v_- &+ \frac{2}{\lambda} \sum_{m=-\infty}^{0} (u_{x+2m-1} - u_{x+2m-1-\eta}) \\ &+ \sqrt{a} \sum_{m=-\infty}^{0} (u_{x+2m} - 2u_{x+2m-1} + u_{x+2m-2}) . \end{aligned} \tag{2.18}$$

We can verify that (u_x, v_x) given by Lemma 2 and (2.18) is a solution of (1.7).

REMARK. When η is a general rational number, we can also construct a DTW which approximates the travelling wave of (1.2). That is, under the same conditions of Theorem 1, if $\eta = p/q$ for a given grid size $(\Delta x, \Delta t)$, we consider the system (1.1) for $\lambda = (1\Delta t/p\Delta x)$ and $k = (\Delta t/p\varepsilon)$, denoted by $S^{(1)}$. By Theorem 1, we know that there exists a DTW which is uniquely defined on the grid points $(j\Delta x, (n/p)\Delta t)$. Hence, we define the DTW on the grid points $(j\Delta x, n\Delta t)$ as the one for approximating (1.2). In fact, the DTW thus obtained is the one for the scheme $S^{(p)}$ which is the scheme by iterating $S^{(1)}$ p times. The u-component of this DTW is also a monotone discrete shock profile for scalar conservation law.

REFERENCES

1. S. Jin and Z. Xin, The relaxing schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* **48**, 555–563 (1995).
2. G. Jennings, Discrete shocks, *Comm. Pure Appl. Math.* **27**, 25–37 (1974).
3. B. Engquist and S. Osher, One-sided difference approximations for nonlinear conservation laws, *Math. Comp.* **36**, 321–351 (1981).
4. A. Majda and J. Ralston, Discrete shock profiles for systems of conservation laws, *Comm. Pure Appl. Math.* **32**, 445–483 (1979).
5. H.L. Liu, J. Wang and T. Yang, Stability for a relaxation model with a nonconvex flux (preprint).
6. H.L. Liu, J. Wang and T. Yang, Nonlinear stability and existence of stationary discrete travelling waves for the relaxing schemes (preprint).
7. T.P. Liu, Hyperbolic conservation laws with relaxation, *Commun. Math. Phys.* **108**, 153–175 (1987).