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Existence of global smooth solutions for Euler equations with symmetry (II)

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1. Introduction

Consider the compressible Euler equations governing the gas flow surrounding a solid ball with mass M and frictional damping in n dimensions,

$$\begin{aligned} &\tilde{\rho}_t + \nabla\cdot(\tilde{\rho}\mathbf{u}) = 0, \\ &(\tilde{\rho}\mathbf{u})_t + \nabla\cdot\tilde{\rho}(\mathbf{u}\otimes\mathbf{u}) + \nabla P(\tilde{\rho}) = -M\tilde{\rho}\frac{\mathbf{x}}{|\mathbf{x}|^n} - 2\alpha\tilde{\rho}\mathbf{u}, \end{aligned} \tag{1.1}$$

where $\tilde{\rho}, \mathbf{u}, P$ and M are the density, velocity, pressure and mass of the gas, respectively, $n \geq 3$ is the dimension of $\mathbf{x}$, and $\alpha > 0$ is the frictional constant. In the following discussion, we assume the pressure satisfies the γ law and $1<\gamma<3$, i.e. $P = K^2\tilde{\rho}^\gamma$, K is a positive constant. We will study the existence and non-existence of global smooth solutions for the initial boundary problem of Eq. (1.1).

Firstly, we will show that regular solutions cannot be global if the initial density has compact support. This is a generalization of Theorem 2.1 in [7] to the case when there is an extra term due to the external force caused by the mass of the ball. In fact, the authors in [7] have noted that the Theorem 2.1 there is valid for any value of α by studying an ordinary differential equation of $H(t)$ as shown in Section 2 of this paper.

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For Euler equations without damping in multi-dimensional space, Sideris [12] gave sufficient conditions for non-global existence of C^1 solutions when $\inf \rho_0(\mathbf{x}) > 0$, where $\rho_0(\mathbf{x})$ is the initial density. The non-existence of C^1 solutions in [12] is related to the shock formation. In Section 2, we will study the singularity of the solutions at the vacuum states when the solutions contain no shocks. Thus, the time when the regular solutions blow up in our discussion is before the time when shock forms.

When $\inf \rho_0(\mathbf{x}) = 0$, Makino et al. [10] proved the non-global existence of regular solutions by assuming the initial data $(\rho_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x}))$ to be of compact support, where $\mathbf{u}_0(\mathbf{x})$ is the initial velocity. For the Euler–Poisson equations governing gaseous stars, Makino and Perthame [9] proved the non-global existence of tame solutions under the condition of spherical symmetry. Local existence of tame or regular solutions for these two systems was proved by Makino et al. [8, 10] by using Kato's [3] theory for quasilinear symmetric hyperbolic system.

Secondly, we study the global existence of smooth solutions of the initial boundary value problem of Eq. (1.1) with spherical symmetry. Even though the global existence of regular solutions for Cauchy and initial boundary value problem of one-dimensional quasilinear hyperbolic systems has been extensively investigated (see [2,4–6,14]), much less is known for systems in high dimensions. To study the problem with spherical symmetry, where the system reduces to a one-dimensional system with singular source terms, is an initial stage to achieve this goal in high dimensions. The singularity is at $r = 0$ or ∞, where r is the radius. In Section 3, we study Eq. (1.1) with spherical symmetry and damping outside a core region of ball with mass M. Some sufficient conditions are given for the global existence of smooth solutions when the frictional damping is sufficiently large. The study is a generalization of the one in [13] to the case when $M \neq 0$, which is based on technical estimation of the C^1-norm of the solutions.

2. No global existence

In this section, we consider the initial-boundary value problem of Eq. (1.1) with initial and boundary conditions given as follows:

$$\begin{aligned} &\tilde{\rho}(\mathbf{x},0) = \tilde{\rho}_0(\mathbf{x}) \geq 0, \qquad \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \text{when } |\mathbf{x}| \geq 1, \\ &\mathbf{u} \cdot \mathbf{x} = 0, \quad \text{when } |\mathbf{x}| = 1. \end{aligned} \tag{2.1}$$

We consider the regular solution of Eqs. (1.1) and (2.1) defined as follows.

Definition 2.1. A solution of Eqs. (1.1) and (2.1) is called a regular solution in $[0,T) \times \Omega$, if

(i) $(\tilde{\rho}, \mathbf{u}) \in C^1([0,T) \times \Omega)$, $\tilde{\rho} \geq 0$,

(ii) $\tilde{\rho}^{\gamma-1} \in C^1([0,T) \times \Omega)$,

and

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\alpha \mathbf{u} - \frac{M\mathbf{x}}{|\mathbf{x}|^n}$$

holds on the exterior of the support of $\tilde{\rho}$. Here $\Omega = \mathbf{R}^n \backslash B_1$, $B_1 = \{\mathbf{x} \colon |\mathbf{x}| < 1\}$.

Theorem 2.2. *Let $(\tilde{\rho}(\mathbf{x},t),\mathbf{u}(\mathbf{x},t))$ be a regular solution of Eqs.* (1.1) *and* (2.1) *on* $0\le t<T$. *If the support of the initial data $\tilde{\rho}_0(\mathbf{x})$ has compact support and $\tilde{\rho}_0(\mathbf{x})\not\equiv 0$ and the mass M is sufficiently small, then T is finite.*

Remark 2.3. *The following proof is similar to that of Makino et al. for the case when* $\alpha=0$, $M=0$ [10] *and Liu and Yang for the case* $\alpha>0$, $M=0$ [7]. *We generalize their proof to the case when* $\alpha>0$, $M>0$.

Proof. Let $\Omega(t)=\operatorname{supp}\tilde{\rho}(\mathbf{x},t)$, $S(t)=\partial\Omega(t)$. By Eq. (1.1) and the definition of regular solution, for any $\mathbf{x}\in S(t_0)$, there exists $\mathbf{x}_0\in S(0)$ and a curve $\mathbf{x}(t)$ connecting $\mathbf{x}_0$ and $\mathbf{x}$ such that

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}=\mathbf{u}(\mathbf{x}(t),t),\quad \mathbf{x}(t)\in S(t),\ 0\le t\le t_0.$$

Let $L=\langle\mathbf{x},\mathbf{x}\rangle$, $I=\langle\mathbf{u},\mathbf{u}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes inner product in $\mathbf{R}^n$. Then we have

$$\frac{\mathrm{d}L}{\mathrm{d}t}=2\langle\mathbf{u},\mathbf{x}\rangle,$$

$$\frac{\mathrm{d}^2L}{\mathrm{d}t^2}=2\left\langle\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t},\mathbf{x}\right\rangle+2\langle\mathbf{u},\mathbf{u}\rangle=-\alpha\frac{\mathrm{d}L}{\mathrm{d}t}-\frac{2M}{|\mathbf{x}|^{n-2}}+2I \tag{2.2}$$

and

$$\frac{dI}{\mathrm{d}t}=2\left\langle\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t},\mathbf{u}\right\rangle=-2\alpha I-2M\left\langle\mathbf{u},\frac{\mathbf{x}}{|\mathbf{x}|^n}\right\rangle. \tag{2.3}$$

It follows from Eqs. (2.2) and (2.3) that

$$\frac{\mathrm{d}^2L}{\mathrm{d}t^2}=-\alpha\frac{\mathrm{d}L}{\mathrm{d}t}-\frac{1}{\alpha}\frac{\mathrm{d}I}{\mathrm{d}t}-\frac{2M}{|\mathbf{x}|^{n-2}}-\frac{M}{\alpha}L^{-n/2}\frac{\mathrm{d}L}{\mathrm{d}t}. \tag{2.4}$$

Integrating Eq. (2.4), we have

$$\frac{\mathrm{d}L}{\mathrm{d}t}\le-\alpha L-\frac{1}{\alpha}I+\frac{2M}{\alpha(n-2)}L^{1-n/2}+C_0, \tag{2.5}$$

where C_0 is a constant. Thus

$$\frac{\mathrm{d}L}{\mathrm{d}t}\le-\alpha L+\frac{2M}{\alpha(n-2)}L^{1-n/2}+C_0. \tag{2.6}$$

By letting $L_1=L^{n/2}$, we have

$$\frac{\mathrm{d}L_1}{\mathrm{d}t}\le-\frac{\alpha n}{2}L_1+\frac{C_0 n}{2}L_1^{1-2/n}+\frac{Mn}{\alpha(n-2)},$$

which implies that there exists a constant L_1^* depending only on the initial data, such that

$$1\le L_1\le L_1^*. \tag{2.7}$$

Now as in [10], let

$$H(t)=\frac{1}{2}\int_{\Omega(t)}\tilde{\rho}(\mathbf{x},t)|\mathbf{x}|^2\,\mathrm{d}\mathbf{x}. \tag{2.8}$$

From Eq. (1.1), we have

$$H'(t)=\frac{1}{2}\int_{\Omega(t)}\tilde{\rho}\mathbf{u}\cdot\mathbf{x}\,\mathrm{d}\mathbf{x}$$

and

$$H''(t)=\int_{\Omega(t)}(\tilde{\rho}|\mathbf{u}|^2+nP)\,\mathrm{d}\mathbf{x}-\alpha H'(t)+\int_{D_0}P\,\mathrm{d}S-M\int_{\Omega(t)}\frac{\tilde{\rho}}{|\mathbf{x}|^{n-2}}\,\mathrm{d}\mathbf{x}, \tag{2.9}$$

where $D_0=\{\mathbf{x}\colon\ |\mathbf{x}|=1\}$.

Let $m=\int_{\Omega(t)}\tilde{\rho}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$ be the total mass. Then we have from the Hölder inequality that

$$m=\int_{\Omega(t)}\tilde{\rho}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\le\left(\int_{\Omega(t)}\tilde{\rho}^{\gamma}\,\mathrm{d}\mathbf{x}\right)^{1/\gamma}\left(\int_{\Omega(t)}\mathrm{d}\mathbf{x}\right)^{1/\gamma'},$$

where $1/\gamma+1/\gamma'=1$. Thus there exists a constant $V^*>0$ such that

$$\left(\int_{\Omega(t)}\tilde{\rho}^{\gamma}\mathrm{d}\mathbf{x}\right)^{1/\gamma}\ge m(V^*)^{-1/\gamma'}.$$

Thus,

$$H''(t)+\alpha H'(t)\ge(nK^2m^{\gamma-1}(V^*)^{-\gamma/\gamma'}-M)m.$$

When M is sufficiently small, there exists a constant $\eta>0$, such that

$$H''(t)+\alpha H'(t)\ge\eta.$$

Integrating the above inequality yields

$$H(t)\ge H(0)+\frac{\eta}{\alpha}\left(t+\frac{1}{\alpha}\mathrm{e}^{-\alpha t}-\frac{1}{\alpha}\right)-\frac{H'(0)}{\alpha}(\mathrm{e}^{-\alpha t}-1).$$

Thus,

$$H(t)\to\infty\quad\text{as } t\to\infty,$$

which is a contradiction to Eqs. (2.7) and (2.8). Hence, T must be finite and this completes the proof of Theorem 2.2. □

3. Global existence with spherical symmetry

Now, we are going to study the global existence of smooth solutions for Eq. (1.1) with spherical symmetry outside a core region. That is, we look for solutions of the form

$$\tilde{\rho} = \tilde{\rho}(|\mathbf{x}|, t), \qquad \mathbf{u} = \frac{\mathbf{x}}{|\mathbf{x}|} \cdot u(|\mathbf{x}|, t).$$

By denoting $r = |\mathbf{x}|$, Eq. (1.1) becomes

$$\begin{aligned} &\tilde{\rho}_t + \frac{1}{r^{n-1}}(r^{n-1}\tilde{\rho}u)_r = 0, \\ &\tilde{\rho}(u_t + uu_r) + P_r = -\frac{M\tilde{\rho}}{r^{n-1}} - 2\alpha\tilde{\rho}u, \end{aligned} \tag{3.1}$$

where $P(\tilde{\rho}) = K^2\tilde{\rho}^\gamma$, $1 < \gamma < 3$. We consider system (3.1) with the following initial and boundary conditions:

$$\tilde{\rho}(r,0) = \tilde{\rho}_0(r), \quad u(r,0) = u_0(r), \quad u(1,t) = 0. \tag{3.2}$$

Let $\rho = r^{n-1}\tilde{\rho}$, then we can rewrite Eq. (3.1) as

$$\begin{aligned} &\rho_t + (\rho u)_r = 0, \\ &u_t + uu_r + \frac{K^2\gamma\rho^{\gamma-2}}{r^{(n-1)(\gamma-1)}}\rho_r = -\frac{M}{r^{n-1}} - 2\alpha u + \frac{K^2\gamma(n-1)\rho^{\gamma-1}}{r^n r^{(n-1)(\gamma-2)}}. \end{aligned} \tag{3.3}$$

In the following discussion we use the Lagrangian coordinates as follows:

$$\tau = t, \quad \xi = \int_1^r \rho(r,t)\,\mathrm{d}r. \tag{3.4}$$

Then $\xi > 0$ as long as $\rho > 0$ for $r > 1$, and Eq. (3.3) becomes

$$\begin{aligned} &\rho_\tau + \rho^2 u_\xi = 0, \\ &u_\tau + \frac{K^2\gamma\rho^{\gamma-1}}{r^{(n-1)(\gamma-1)}}\rho_\xi = -\frac{M}{r^{n-1}} - 2\alpha u + \frac{K^2\gamma(n-1)\rho^{\gamma-1}}{r^n r^{(n-1)(\gamma-2)}}. \end{aligned} \tag{3.5}$$

Now we let $\sigma = \rho^{-1}$, and replace (ξ, τ) by (x, t). Eq. (3.5) can be rewritten as

$$\begin{aligned} &\sigma_t - u_x = 0, \\ &u_t - \frac{K^2\gamma\sigma^{-\gamma-1}}{r^{(n-1)(\gamma-1)}}\sigma_x = -\frac{M}{r^{n-1}} - 2\alpha u + \frac{K^2\gamma(n-1)\sigma^{1-\gamma}}{r^n r^{(n-1)(\gamma-2)}} \end{aligned} \tag{3.6}$$

with the following initial and boundary conditions:

$$\sigma(x,0) = \sigma_0(x), \quad u(x,0) = u_0(x), \quad u(0,t) = 0.$$

By transformation (3.4), we have

$$r = 1 + \int_0^x \sigma(s,t)\,\mathrm{d}s, \quad x \geq 0. \tag{3.7}$$

To symmetrize system (3.6), we use the following Riemann invariants:

$$w = u + \frac{2K\sqrt{\gamma}}{\gamma - 1} r^{-(1/2)(n-1)(\gamma-1)} \sigma^{-(\gamma-1)/2},$$
$$z = u - \frac{2K\sqrt{\gamma}}{\gamma - 1} r^{-(1/2)(n-1)(\gamma-1)} \sigma^{-(\gamma-1)/2}. \tag{3.8}$$

By using $r_t = u$ and $r_x = \sigma$, Eq. (3.6) becomes

$$w_t + \mu w_x = -\frac{M}{r^{n-1}} - \alpha(w+z) - \frac{(n-1)(\gamma-1)}{8r}(w-z)(w+z),$$
$$z_t + \lambda z_x = -\frac{M}{r^{n-1}} - \alpha(w+z) + \frac{(n-1)(\gamma-1)}{8r}(w-z)(w+z) \tag{3.9}$$

with the corresponding initial and boundary conditions given by

$$w(x,0) = w_0(x), \quad z(x,0) = z_0(x), \quad w(0,t) + z(0,t) = 0.$$

Here $-\lambda = \mu = K\sqrt{\gamma} A r^{n-1}(w-z)^{(\gamma+1)/(\gamma-1)}$ are the two characteristic speeds and

$$A = \left(\frac{\gamma - 1}{4K\sqrt{\gamma}}\right)^{(\gamma+1)/(\gamma-1)}.$$

For later use, we list some equations as follows:

$$\frac{\mathrm{d}r}{\mathrm{d}_\mu t} = \frac{1}{2}(w+z) + \frac{\gamma-1}{4}(w-z),$$
$$\frac{\mathrm{d}r}{\mathrm{d}_\lambda t} = \frac{1}{2}(w+z) - \frac{\gamma-1}{4}(w-z),$$
$$\frac{\mathrm{d}(w-z)}{\mathrm{d}_\mu t} = -2K\sqrt{\gamma} A r^{n-1}(w-z)^{(\gamma+1)/(\gamma-1)} z_x - \frac{(n-1)(\gamma-1)}{4r}(w-z)(w+z),$$
$$\frac{\mathrm{d}(w-z)}{\mathrm{d}_\lambda t} = -2K\sqrt{\gamma} A r^{n-1}(w-z)^{(\gamma+1)/(\gamma-1)} w_x - \frac{(n-1)(\gamma-1)}{4r}(w-z)(w+z),$$
$$\frac{\mathrm{d}(w+z)}{\mathrm{d}_\mu t} = -\frac{2M}{r^{n-1}} - 2\alpha(w+z) + 2K\sqrt{\gamma} A r^{n-1}(w-z)^{(\gamma+1)/(\gamma-1)} z_x,$$
$$\frac{\mathrm{d}(w+z)}{\mathrm{d}_\lambda t} = -\frac{2M}{r^{n-1}} - 2\alpha(w+z) - 2K\sqrt{\gamma} A r^{n-1}(w-z)^{(\gamma+1)/(\gamma-1)} w_x, \tag{3.10}$$

where we have used the notation $\mathrm{d}/\mathrm{d}_\mu t = \partial/\partial t + \mu(\partial/\partial x)$ and $\mathrm{d}/\mathrm{d}_\lambda t = \partial/\partial t + \lambda(\partial/\partial x)$.

From now on, we assume the initial and boundary conditions satisfy

$$\begin{aligned} &w_0(x), \quad z_0(x) \in C_b^1[0,\infty), \\ &(r_0(x))^{n/2-1}(w_0(x)-z_0(x)) \geq \delta \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &w_0(0)+z_0(0)=0, \\ &\mu_0(0)w_{0x}(0)+\lambda_0(0)z_{0x}(0)=-2M, \end{aligned} \tag{3.12}$$

where δ is a positive constant, and $\mu_0(x)$ and $\lambda_0(x)$ are the given characteristic speeds at $t=0$.

Remark 3.1. *There exists a special solution of Eq.* (3.9) *satisfying Eqs.* (3.11) *and* (3.12), *i.e.*

$$w(x,t) = -z(x,t) = M_1(r(x,t))^{1-n/2},$$

where $M_1 = 2\sqrt{M}/\sqrt{(n-2)(\gamma-1)}$.

Now, we are ready to prove the following lemma on the C^0-norm of the solutions inside the smoothness region of the solutions.

Lemma 3.2. *Under the conditions of Eqs.* (3.11) *and* (3.12), *if*

$$|(r_0(x))^{n/2-1}w_0(x)|, |(r_0(x))^{n/2-1}z_0(x)| \leq M_1, \tag{3.13}$$

then there exists a positive constant α_0 *depending only on* M_1 *and* δ *such that when* $\alpha > \alpha_0$, *we have*

$$|(r(x,t))^{n/2-1}w(x,t)|, |(r(x,t))^{n/2-1}z(x,t)| \leq M_1, \tag{3.14}$$

$$w(x,t)-z(x,t) > 0, \tag{3.15}$$

where (x,t) *is inside the smoothness region of the solutions and* M_1 *is the constant defined in the above Remark* 3.1.

Proof. We now prove Eq. (3.15) first. By Eq. (3.10), we have

$$\frac{\mathrm{d}(w-z)}{\mathrm{d}_\lambda t} = -D(x,t)(w-z), \tag{3.16}$$

where

$$D(x,t) = 2K\sqrt{\gamma}Ar^{n-1}(w-z)^{2/(\gamma-1)}w_x + \frac{(n-1)(\gamma-1)}{4r}(w+z).$$

Integrating Eq. (3.16) along the λ-characteristic curve, we have

$$(w-z)(x,t) \geq \delta(r_0(x))^{1-n/2}\exp\left(-\int_0^t D(x(\tau),\tau)\,\mathrm{d}\tau\right) > 0,$$

as long as the solution is smooth.

To prove Eq. (3.14), we introduce the following transformations:

$$
\begin{aligned}
w(x,t) &= \bar{w}(x,t) + M_1(r(x,t))^{1-n/2},\\
z(x,t) &= \bar{z}(x,t) - M_1(r(x,t))^{1-n/2}.
\end{aligned}
\tag{3.17}
$$

Then Eq. (3.9) becomes

$$
\begin{aligned}
\bar{w}_t + \mu\bar{w}_x &= -A(x,t)(\bar{w}+\bar{z}) + br^{-n/2}\bar{w},\\
\bar{z}_t + \lambda\bar{z}_x &= -B(x,t)(\bar{w}+\bar{z}) - br^{-n/2}\bar{z},
\end{aligned}
\tag{3.18}
$$

where

$$
\begin{aligned}
A(x,t) &= \alpha + \frac{(n-1)(\gamma-1)}{8r}(\bar{w}-\bar{z})\\
&\qquad + \frac{(3-\gamma)(n-2)-2(n-1)(\gamma-1)}{8}M_1 r^{-n/2},\\
B(x,t) &= \alpha - \frac{(n-1)(\gamma-1)}{8r}(\bar{w}-\bar{z})\\
&\qquad - \frac{(3-\gamma)(n-2)-2(n-1)(\gamma-1)}{8}M_1 r^{-n/2},
\end{aligned}
$$

$$
b = \tfrac{1}{2}\sqrt{(n-2)(\gamma-1)M}. \tag{3.19}
$$

Let

$$
\begin{aligned}
\bar{w} &= \left(W + \frac{N}{L}(x + C\mathrm{e}^t)\right)\mathrm{e}^{bt},\\
\bar{z} &= -\left(Z + \frac{N}{L}(x + C\mathrm{e}^t)\right)\mathrm{e}^{bt},
\end{aligned}
\tag{3.20}
$$

where C and L are two positive constants to be determined later. A smooth solution exists at least locally under conditions (3.11) and (3.12), cf. [1]. We use N to denote the bound for $|w(x,t)|$ and $|z(x,t)|$ of the local solution. Using Eq. (3.20), the system (3.18) for $(\bar{w}(x,t),\bar{z}(x,t))$ can be written as follows:

$$
\begin{aligned}
&W_t + \mu W_x + \frac{N}{L}(C\mathrm{e}^t + \mu) + \frac{N}{L}(x + C\mathrm{e}^t)b\left(1 - r^{-n/2}\right)\\
&\quad = -A(x,t)(W-Z) - bW + br^{-n/2}W,\\
&Z_t + \lambda Z_x + \frac{N}{L}(C\mathrm{e}^t + \lambda) + \frac{N}{L}(x + C\mathrm{e}^t)b(1 + r^{-n/2})\\
&\quad = B(x,t)(W-Z) - bZ - br^{-n/2}Z.
\end{aligned}
\tag{3.21}
$$

If we consider system (3.21) in the region $[0,L]\times\mathbf{R}^+$, then the initial and boundary conditions have the following properties:

$$\begin{aligned}
&W(x,0)=\bar{w}(x,0)-\frac{N}{L}(x+C)<0,\\
&Z(x,0)=-\bar{z}(x,0)-\frac{N}{L}(x+C)<0,\\
&W(L,t)=\bar{w}(L,t)\mathrm{e}^{-bt}-N-\frac{N}{L}C\mathrm{e}^{t}<0,\\
&Z(L,t)=-\bar{z}(L,t)\mathrm{e}^{-bt}-N-\frac{N}{L}C\mathrm{e}^{t}<0.
\end{aligned}\tag{3.22}$$

From Eqs. (3.21) and (3.22), we claim that

$$W(x,t)<0,\quad Z(x,t)<0,\quad (x,t)\in[0,L]\times[0,T].\tag{3.23}$$

If not, we let $\bar{t}=\sup_t\{t\colon W(x,\tau)<0,\ Z(x,\tau)<0,\ \forall x\in[0,L],\ \tau\in(0,t)\}$, then $0<\bar{t}\le T<+\infty$. By the continuity of $W(x,t)$ and $Z(x,t)$, there exists $(\bar{x},\ \bar{t})$ with $0\le\bar{x}<L$, such that one of the following cases holds:

Case 1: When $\bar{x}\in(0,L)$,

$$W(\bar{x},\bar{t})=0,\quad Z(\bar{x},\bar{t})\le 0,\quad \left.\frac{\partial W}{\partial t}\right|_{(\bar{x},\bar{t})}\ge 0,\quad \left.\frac{\partial W}{\partial x}\right|_{(\bar{x},\bar{t})}=0,$$

or

$$Z(\bar{x},\bar{t})=0,\quad W(\bar{x},\bar{t})\le 0,\quad \left.\frac{\partial Z}{\partial t}\right|_{(\bar{x},\bar{t})}\ge 0,\quad \left.\frac{\partial Z}{\partial x}\right|_{(\bar{x},\bar{t})}=0.$$

Case 2: When $\bar{x}=0$,

$$W(\bar{x},\bar{t})=Z(\bar{x},\bar{t})=0,\quad \left.\frac{\partial Z}{\partial t}\right|_{(\bar{x},\bar{t})}\ge 0,\quad \left.\frac{\partial Z}{\partial x}\right|_{(\bar{x},\bar{t})}\le 0.$$

For the above two cases, if we assume the following *a priori* estimate:

$$|(r(x,t))^{n/2-1}w(x,t)|,\ |(r(x,t))^{n/2-1}z(x,t)|\le M_1,\tag{3.24}$$

then for sufficiently large α, we have

$$A(x,t)>0,\quad B(x,t)>0.$$

By applying the "maximum principle", cf. [13, 14], when $C>\sup\mu$ for all w,z under consideration, we have a contradiction. Therefore, Eq. (3.23) holds.

Equality (3.20) and inequality (3.23) imply that

$$\bar{w}(x,t)<\left(\frac{N}{L}(x+C\mathrm{e}^{t})\right)\mathrm{e}^{bt},$$

$$\bar{z}(x,t)>-\left(\frac{N}{L}(x+C\mathrm{e}^{t})\right)\mathrm{e}^{bt}.$$

Since L can be arbitrary, letting $L \to \infty$ yields

$$\bar{w}(x,t) \le 0, \quad \bar{z}(x,t) \ge 0.$$

Hence by Eq. (3.17), we have

$$w(x,t) \le M_1(r(x,t))^{1-n/2}, \quad z(x,t) \ge -M_1(r(x,t))^{1-n/2}. \tag{3.25}$$

It is easy to get Eq. (3.14) from Eqs. (3.15) and (3.25). This completes the Proof of Lemma 3.2. □

Now, we estimate the derivatives of $w(x,t)$, $z(x,t)$ with respect to x.

Let $P(x,t) = w_x(x,t)$, $Q(x,t) = z_x(x,t)$. By differentiating Eq. (3.9) with respect to x, we have the following system for P, Q:

$$\begin{aligned} P_t + \mu P_x = & -\alpha(P+Q) - K\sqrt{\gamma}A\frac{\gamma+1}{\gamma-1}r^{n-1}(w-z)^{2/(\gamma-1)}(P-Q)P \\ & - \frac{(n-1)(\gamma-1)}{4r}(w-z)P - \frac{(n-1)(\gamma-1)}{8r}(w-z)(P+Q) \\ & - \frac{(n-1)(\gamma-1)}{8r}(w+z)(P-Q) + \frac{(n-1)M}{r^{2n-1}}B(w-z)^{-2/(\gamma-1)} \\ & + \frac{(n-1)(\gamma-1)}{8r^{n+1}}B(w-z)^{-(3-\gamma)/(\gamma-1)}(w+z), \end{aligned}$$

$$\begin{aligned} Q_t + \lambda Q_x = & -\alpha(P+Q) - K\sqrt{\gamma}A\frac{\gamma+1}{\gamma-1}r^{n-1}(w-z)^{2/(\gamma-1)}(Q-P)Q \\ & + \frac{(n-1)(\gamma-1)}{4r}(w-z)Q + \frac{(n-1)(\gamma-1)}{8r}(w-z)(P+Q) \\ & + \frac{(n-1)(\gamma-1)}{8r}(w+z)(P-Q) + \frac{(n-1)M}{r^{2n-1}}B(w-z)^{-2/(\gamma-1)} \\ & - \frac{(n-1)(\gamma-1)}{8r^{n+1}}B(w-z)^{-(3-\gamma)/(\gamma-1)}(w+z), \end{aligned} \tag{3.26}$$

where

$$B = \left(\frac{\gamma-1}{4K\sqrt{\gamma}}\right)^{-2/(\gamma-1)}.$$

Let

$$P = (w-z)^l r^\beta \overline{P}, \quad Q = (w-z)^l r^\beta \bar{Q}, \tag{3.27}$$

where $l = -(\gamma+1)/2(\gamma-1) < 0$ and β is a constant to be determined later. Then system (3.26) becomes

$$\begin{aligned}
\overline{P}_t + \mu\overline{P}_x &= -\alpha(\overline{P}+\overline{Q}) - K\sqrt{\gamma}A\frac{\gamma+1}{\gamma-1}r^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}\overline{P}^2 \\
&\quad - \frac{(3n+2\beta-3)(\gamma-1)}{8r}(w-z)\overline{P} - \frac{(n-1)(\gamma-1)}{8r}(w-z)\overline{Q} \\
&\quad - \frac{(n-1)\gamma+2\beta}{4r}(w+z)\overline{P} + \frac{(n-1)(\gamma-1)}{8r}(w+z)\overline{Q} \\
&\quad + \frac{(n-1)M}{r^{2n+\beta-1}}B(w-z)^{-(3-\gamma)/2(\gamma-1)} \\
&\quad + \frac{(n-1)(\gamma-1)}{8r^{n+\beta+1}}B(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z), \\
\overline{Q}_t + \lambda\overline{Q}_x &= -\alpha(\overline{P}+\overline{Q}) - K\sqrt{\gamma}A\frac{\gamma+1}{\gamma-1}r^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}\overline{Q}^2 \\
&\quad + \frac{(n-1)(\gamma-1)}{8r}(w-z)\overline{P} + \frac{(3n+2\beta-3)(\gamma-1)}{8r}(w-z)\overline{Q} \\
&\quad + \frac{(n-1)(\gamma-1)}{8r}(w+z)\overline{P} - \frac{(n-1)\gamma+2\beta}{4r}(w+z)\overline{Q} \\
&\quad + \frac{(n-1)M}{r^{2n+\beta-1}}B(w-z)^{-(3-\gamma)/2(\gamma-1)} \\
&\quad - \frac{(n-1)(\gamma-1)}{8r^{n+\beta+1}}B(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z). \qquad (3.28)
\end{aligned}$$

Let

$$\begin{aligned}
\overline{P} &= F + a_1r^{-\beta-n+1}(w-z)^m + a_2r^{-\beta-n}(w-z)^{m+1} \\
&\quad + a_3r^{-\beta-n}(w-z)^m(w+z) \\
\overline{Q} &= G + a_1r^{-\beta-n+1}(w-z)^m - a_2r^{-\beta-n}(w-z)^{m+1} \\
&\quad + a_3r^{-\beta-n}(w-z)^m(w+z), \qquad (3.29)
\end{aligned}$$

where $m = -(3-\gamma)/2(\gamma-1) < 0$. Then we have

$$\begin{aligned}
&F_t + \mu F_x \\
&\quad = -K\sqrt{\gamma}A\frac{\gamma+1}{\gamma-1}r^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}\overline{P}^2 - \alpha\overline{P} \\
&\qquad - \frac{(3n+2\beta-3)(\gamma-1)}{8r}(w-z)\overline{P} \\
&\qquad - \frac{(n-1)\gamma+2\beta}{4r}(w+z)\overline{P} + (2K\sqrt{\gamma}Ama_1-\alpha)\overline{Q}
\end{aligned}$$

$$+\left(2K\sqrt{\gamma}A(m+1)a_2 - 2K\sqrt{\gamma}Aa_3 - \frac{(n-1)(\gamma-1)}{8}\right)\frac{1}{r}(w-z)\overline{Q}$$

$$+\left(2K\sqrt{\gamma}Ama_3 + \frac{(n-1)(\gamma-1)}{8}\right)\frac{1}{r}(w+z)\overline{Q}$$

$$+\frac{(n-1)(\gamma-1)}{4}ma_1 r^{-\beta-n}(w-z)^{-(3-\gamma)/2(\gamma-1)}(w+z)$$

$$+\left(\frac{1}{2}(\beta+n-1)a_1 + 2\alpha a_3\right)r^{-\beta-n}(w-z)^{-(3-\gamma)/2(\gamma-1)}(w+z)$$

$$+\frac{\gamma-1}{4}(\beta+n-1)a_1 r^{-\beta-n}(w-z)^{(3\gamma-5)/2(\gamma-1)}$$

$$+\frac{\gamma-1}{4}(\beta+n)a_2 r^{-\beta-n-1}(w-z)^{(5\gamma-7)/2(\gamma-1)}$$

$$+\frac{(n-1)(\gamma-1)}{4}(m+1)a_2 r^{-\beta-n-1}(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z)$$

$$+\left(\frac{1}{2}(\beta+n)a_2 + \frac{\gamma-1}{4}(\beta+n)a_3\right)r^{-\beta-n-1}(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z)$$

$$+((n-1)B + 2a_3)Mr^{-\beta-2n+1}(w-z)^{-(3-\gamma)/2(\gamma-1)}$$

$$+\frac{(n-1)(\gamma-1)}{8}Br^{-\beta-n-1}(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z)$$

$$+\left(\frac{(n-1)(\gamma-1)}{4}m + \frac{1}{2}(\beta+n)\right)a_3 r^{-\beta-n-1}(w-z)^{-(3-\gamma)/2(\gamma-1)}(w+z)^2,$$

$$G_t + \lambda G_x$$

$$= -K\sqrt{\gamma}A\frac{\gamma+1}{\gamma-1}r^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}\overline{Q}^2 - \alpha\overline{Q}$$

$$+\frac{(3n+2\beta-3)(\gamma-1)}{8r}(w-z)\overline{Q}$$

$$-\frac{(n-1)\gamma+2\beta}{4r}(w+z)\overline{Q} + (2K\sqrt{\gamma}Ama_1 - \alpha)\overline{P}$$

$$-\left(2K\sqrt{\gamma}A(m+1)a_2 - 2K\sqrt{\gamma}Aa_3 - \frac{(n-1)(\gamma-1)}{8}\right)\frac{1}{r}(w-z)\overline{P}$$

$$+\left(2K\sqrt{\gamma}Ama_3 + \frac{(n-1)(\gamma-1)}{8}\right)\frac{1}{r}(w+z)\overline{P}$$

$$+\frac{(n-1)(\gamma-1)}{4}ma_1 r^{-\beta-n}(w-z)^{-(3-\gamma)/2(\gamma-1)}(w+z)$$

$$
\begin{aligned}
&+\left(\frac{1}{2}(\beta+n-1)a_1+2\alpha a_3\right) r^{-\beta-n}(w-z)^{-(3-\gamma)/2(\gamma-1)}(w+z)\\
&-\frac{\gamma-1}{4}(\beta+n-1)a_1 r^{-\beta-n}(w-z)^{(3\gamma-5)/2(\gamma-1)}\\
&+\frac{\gamma-1}{4}(\beta+n)a_2 r^{-\beta-n-1}(w-z)^{(5\gamma-7)/2(\gamma-1)}\\
&-\frac{(n-1)(\gamma-1)}{4}(m+1)a_2 r^{-\beta-n-1}(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z)\\
&-\left(\frac{1}{2}(\beta+n)a_2+\frac{\gamma-1}{4}(\beta+n)a_3\right) r^{-\beta-n-1}(w-z)^{(3\gamma-5)/2(\gamma-1)}(w+z)\\
&+((n-1)B+2a_3)Mr^{-\beta-2n+1}(w-z)^{-(3-\gamma)/2(\gamma-1)}\\
&-\frac{(n-1)(\gamma-1)}{8}Br^{-\beta-n-1}(w-z)^{(3\gamma-5)2(\gamma-1)}(w+z)\\
&+\left(\frac{(n-1)(\gamma-1)}{4}m+\frac{1}{2}(\beta+n)\right) a_3 r^{-\beta-n-1}(w-z)^{-(3-\gamma)/2(\gamma-1)}(w+z)^2.
\end{aligned}
\tag{3.30}
$$

Now choose a_i, $i=1,2,3$, as follows:

$$
\begin{aligned}
&2K\sqrt{\gamma}Ama_1-\alpha=0,\\
&2K\sqrt{\gamma}A(m+1)a_2-2K\sqrt{\gamma}Aa_3-\frac{(n-1)(\gamma-1)}{8}=0,\\
&2K\sqrt{\gamma}Ama_3+\frac{(n-1)(\gamma-1)}{8}=0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
a_1&=-\frac{(\gamma-1)\alpha}{K\sqrt{\gamma}A(3-\gamma)}<0,\\
a_2&=\frac{(n-1)(\gamma-1)^2(\gamma+1)}{8K\sqrt{\gamma}A(3\gamma-5)(3-\gamma)},\\
a_3&=\frac{(n-1)(\gamma-1)^2}{8K\sqrt{\gamma}A(3-\gamma)}>0,
\end{aligned}
\tag{3.31}
$$

where $1<\gamma<\frac{5}{3}$ or $\frac{5}{3}<\gamma<3$. Then the system for F and G becomes

$$
\begin{aligned}
F_t+\mu F_x=&-K\sqrt{\gamma}\frac{\gamma+1}{\gamma-1}Ar^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}F^2\\
&+\left\{\frac{3\gamma-1}{3-\gamma}\alpha+\eta_1\right\}F\\
&-\left\{\frac{2\alpha^2(\gamma-1)^2}{K\sqrt{\gamma}A(3-\gamma)^2}+\theta_1\right\}r^{-\beta-n+1}(w-z)^{-(3-\gamma)/2(\gamma-1)},
\end{aligned}
$$

$$G_t + \lambda G_x = -K\sqrt{\gamma}\frac{\gamma+1}{\gamma-1}Ar^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}G^2 + \left\{\frac{3\gamma-1}{3-\gamma}\alpha + \eta_2\right\}G - \left\{\frac{2\alpha^2(\gamma-1)^2}{K\sqrt{\gamma}A(3-\gamma)^2} + \theta_2\right\}r^{-\beta-n+1}(w-z)^{-(3-\gamma)/2(\gamma-1)}, \qquad (3.32)$$

where η_i, θ_i are continuous and bounded functions of the corresponding variables, $i=1,2$.

Based on Lemma 3.2 and system (3.32), we are now ready to prove the following theorem.

Theorem 3.3. *Under the conditions of Lemma* 3.2, *if* $1<\gamma<\frac{5}{3}$ or $\frac{5}{3}<\gamma<3$, *and* α *is sufficiently large*, *then there exists a constant* M_2 *depending only on* M_1 *and* δ, *such that if*

$$|M_1-\delta|\ll 1,$$

$$F(x,0), G(x,0) \geq M_2,$$

then there exists a global smooth solution for system (3.1) *and* (3.2).

Proof. Firstly, we assume *a priori* estimate

$$(r(x,t))^{n/2-1}(w(x,t)-z(x,t)) \geq \delta_1 > 0, \qquad (3.33)$$

where δ_1 is a positive constant depending only on M_1, M_2 and δ. By Lemma 3.2, we have

$$|\eta_i|, |\theta_i| < M_3, \quad i=1,2,$$

where M_3 is a positive constant depending only on M_1, M_2 and δ. When α is large enough and $1<\gamma<3$, we know that the equation

$$0 = -K\sqrt{\gamma}\frac{\gamma+1}{\gamma-1}Ar^{n+\beta-1}(w-z)^{(3-\gamma)/2(\gamma-1)}y^2 + \left\{\frac{3\gamma-1}{3-\gamma}\alpha + \eta_i\right\}y - \left\{\frac{2\alpha^2(\gamma-1)^2}{K\sqrt{\gamma}A(3-\gamma)^2} + \theta_i\right\}r^{-\beta-n+1}(w-z)^{-(3-\gamma)/2(\gamma-1)},$$

has two positive roots given by

$$y_i^1 = \left(\frac{2(\gamma-1)^2}{K\sqrt{\gamma}A(\gamma+1)(3-\gamma)}\alpha + \zeta_i^1\alpha^{1/2}\right)r^{-n-\beta+1}(w-z)^{-(3-\gamma)/2(\gamma-1)},$$

$$y_i^2 = \left(\frac{\gamma-1}{K\sqrt{\gamma}A(3-\gamma)}\alpha + \zeta_i^2\alpha^{1/2}\right)r^{-n-\beta+1}(w-z)^{-(3-\gamma)/2(\gamma-1)}, \quad i=1,2,$$

with $y_i^1 < y_i^2$, $i = 1, 2$, where $|\zeta_i^1|$ and $|\zeta_i^2|$ have an upper bound depending only on M_1, M_2 and δ. Therefore, when $1 < \gamma < \frac{5}{3}$ or $\frac{5}{3} < \gamma < 3$, if we choose

$$\beta = \frac{(n-2)(3-\gamma)}{4(\gamma-1)} - n + 1,$$

then $\sup y_i^1 < \inf y_i^2$, provided $\delta_1 - M_1$ is sufficiently small. Thus, there exists a constant $M_2 = \max\{\sup y_1^1, \sup y_2^1\}$ depending only on M_1 and δ, such that if

$$F(x,0), G(x,0) \geq M_2,$$

we have

$$M_2 \leq F(x,t), \quad G(x,t) \leq \sup\{F(x,0), G(x,0)\}. \tag{3.34}$$

Combining Lemma 3.2 and Eq. (3.34) yields the global existence of a smooth solution for Eqs. (3.1) and (3.2).

Now, it remains to verify the *a priori* estimate (3.33). To do this, let

$$H(x,t) = (r(x,t))^{n/2-1}(w(x,t) - z(x,t)). \tag{3.35}$$

Then, from Eqs. (3.10), (3.29) and (3.34), we have

$$-\frac{1}{2K\sqrt{\gamma}A} H^{-(3-\gamma)/2(\gamma-1)-1} \frac{\mathrm{d}}{\mathrm{d}_\lambda t} H + \left\{ \frac{(\gamma-1)\alpha}{K\sqrt{\gamma}A(3-\gamma)} + \eta_3 \right\} H^{-(3-\gamma)/2(\gamma-1)} \leq M_4, \tag{3.36}$$

where $\eta_3 \in C_b^1$ and M_4 is a positive constant depending only on M_1, M_2 and δ.

From Eq. (3.36), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}_\lambda t}\left\{ H^{-(3-\gamma)/2(\gamma-1)} \exp\left(\int_0^t (\alpha + \eta_4(\tau))\,\mathrm{d}\tau \right)\right\} \\ &\quad \leq M_5 \exp\left(\int_0^t (\alpha + \eta_4(\tau))\,\mathrm{d}\tau \right), \end{aligned} \tag{3.37}$$

where

$$\eta_4(\tau) = \frac{K\sqrt{\gamma}A(3-\gamma)}{\gamma-1}\eta_3(\tau),$$

and M_5 is a positive constant depending only on M_1, M_2 and δ.

When α is sufficiently large, we have

$$\alpha + \eta_4(\tau) > \frac{\alpha}{2}.$$

Hence,

$$\begin{aligned}
&\int_0^t \exp\left(\int_0^s (\alpha+\eta_4(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \le \frac{2}{\alpha}\int_0^t (\alpha+\eta_4(s))\exp\left(\int_0^s (\alpha+\eta_4(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad = \frac{2}{\alpha}\left\{\exp\left(\int_0^t (\alpha+\eta_4(\tau))\,\mathrm{d}\tau\right)-1\right\}. \qquad (3.38)
\end{aligned}$$

Integrating Eq. (3.37) and using Eq. (3.38), we have

$$H^{-(3-\gamma)/2(\gamma-1)} \le M_6,$$

where M_6 is a positive constant depending only on M_1, M_2 and δ. Therefore, when $1<\gamma<\frac{5}{3}$ or $\frac{5}{3}<\gamma<3$, we have

$$(r(x,t))^{n/2-1}(w(x,t)-z(x,t)) \ge \delta_1 > 0.$$

This completes the proof of Theorem 3.3. □

Remark 3.4. *When $\gamma=\frac{5}{3}$, the conclusion of Theorem* 3.3 *still holds.*

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