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Stability of Nonlinear Wave Patterns to the Bipolar Vlasov–Poisson–Boltzmann System

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Abstract

The main purpose of the present paper is to investigate the nonlinear stability of viscous shock waves and rarefaction waves for the bipolar Vlasov–Poisson–Boltzmann (VPB) system. To this end, motivated by the micro–macro decomposition to the Boltzmann equation in Liu and Yu (Commun Math Phys 246:133–179, 2004) and Liu et al. (Physica D 188:178–192, 2004), we first set up a new micro–macro decomposition around the local Maxwellian related to the bipolar VPB system and give a unified framework to study the nonlinear stability of the basic wave patterns to the system. Then, as applications of this new decomposition, the time-asymptotic stability of the two typical nonlinear wave patterns, viscous shock waves and rarefaction waves are proved for the 1D bipolar VPB system. More precisely, it is first proved that the linear superposition of two Boltzmann shock profiles in the first and third characteristic fields is nonlinearly stable to the 1D bipolar VPB system up to some suitable shifts without the zero macroscopic mass conditions on the initial perturbations. Then the time-asymptotic stability of the rarefaction wave fan to compressible Euler equations is proved for the 1D bipolar VPB system. These two results are concerned with the nonlinear stability of wave patterns for Boltzmann equation coupled with additional (electric) forces, which together with spectral analysis made in Li et al. (Indiana Univ Math J 65(2):665–725, 2016) sheds light on understanding the complicated dynamic behaviors around the wave patterns in the transportation of charged particles under the binary collisions, mutual interactions, and the effect of the electrostatic potential forces.

Contents

1. Introduction

It is an interesting and challenging problem to investigate the nonlinear wave phenomena and understand the dynamical behaviors of charged particles transport under the influence of external forces such as electrostatic potential, magnetic field, or electromagnetic fields, etc.. To begin with, we first investigate wave phenomena for the bipolar Vlasov–Poisson–Boltzmann system, which is used to simulate the transport of two dilute charged particles (for example ions and electrons) affected by the self-consistent electrostatic potential force [26]. In spatial three-dimensional space, the bipolar Vlasov–Poisson–Boltzmann system takes the form

$$\begin{cases} F_{At} + v \cdot \nabla_x F_A + \nabla_x \Pi \cdot \nabla_v F_A = Q(F_A, F_A) + Q(F_A, F_B), \\ F_{Bt} + v \cdot \nabla_x F_B - \nabla_x \Pi \cdot \nabla_v F_B = Q(F_B, F_A) + Q(F_B, F_B), \\ \Delta \Pi = \displaystyle\int (F_A - F_B)\, \mathrm{d}v, \end{cases} \tag{1.1}$$

where $v = (v_1, v_2, v_3) \in \mathbf{R}^3$, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}^+$ and $F_A = F_A(t, x, v)$, $F_B = F_B(t, x, v)$ are the density distribution function of two-species particles (for example ions and electrons) at time–space (t, x) with velocity v, and $\Pi = \Pi(x, t)$ is the electric field potential. For the hard sphere model, the collision operator $Q(f, g)$ takes the bilinear form

$$Q(f, g)(v) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2_+} \big(f(v') g(v'_*) - f(v) g(v_*) \big) \, |(v - v_*) \cdot \Omega)| \, \mathrm{d}v_* \, \mathrm{d}\Omega,$$

where the unit vector $\Omega \in \mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 : (v - v_*) \cdot \Omega \geqq 0\}$, (v, v_*) and (v', v'_*) are the two particle velocities before and after the binary elastic collision respectively, which together with the conservation laws of momentum and energy, satisfy the following relations:

$$v' = v - [(v - v_*) \cdot \Omega] \, \Omega, \quad v'_* = v_* + [(v - v_*) \cdot \Omega] \, \Omega, \quad \Omega \in \mathbf{S}^2.$$

In the case that the effect of electrons is neglected, the bipolar Vlasov–Poisson–Boltzmann (abbreviated as VPB for simplicity below) system (1.1) can be reduced to the unipolar VPB equations [26]

$$\begin{cases} F_{At} + v \cdot \nabla_x F_A + \nabla_x \Pi \cdot \nabla_v F_A = Q(F_A, F_A), \\ \Delta \Pi = \displaystyle\int F_A \, \mathrm{d}v - \bar{\rho}(x), \end{cases} \tag{1.2}$$

with $\bar{\rho}(x) > 0$ a given function representing the background doping profile.

Both the bipolar VPB system (1.1) and unipolar VPB system can be viewed as the Boltzmann equations under the affect of the electrostatic potential force determined through the self-consistent Poisson equation related to the macroscopic density of charged particles. However, this bipolar or unipolar VPB system is by no means the simple extension of the Boltzmann equation. Indeed, there are complicated asymptotical behaviors, which are different completely from those of the Boltzmann equation, as has been observed and justified rigorously for the VPB systems (1.1)–(1.2) in [5,17,18] due to the combined effects such as the binary elastic collision between the particles of same species, the electrostatic potential force, and/or the mutual interactions among the charged particles of two different species. To be more precise, it was shown in [17] that for the unipolar VPB system for motion of one species, the influence of the electric field affects the spectrum structure of the linearized VPB system and causes the slower but optimal (compared with the Boltzmann equation) time-asymptotical convergence rate of global solution to the equilibrium state, and there is no wave pattern propagation (such as the shock profile and rarefaction wave compared with the Boltzmann equation) due to effect of the electric field [18]; one can refer to [5,17,18,43] and references therein for more details. On the other hand, however, a completely different dynamical phenomena/behaviors of global solution are observed for bipolar VPB system in [18]. Therein, it was shown that the linearized VPB system around the global Maxwellian consists of a decoupled system: one is the linear Boltzmann equation for the distribution function $F_1 = \frac{F_A+F_B}{2}$ of the average of the total charged particles which admits the wave modes at lower frequency, the other is the equation of unipolar VPB type for the neutral function $F_2 = \frac{F_A-F_B}{2}$ of the particles with different charge which admits spectral gap at lower frequency, and it causes strong neutrality in the sense that the neutral function F_2 and the electric field related the neutral function F_2 decay exponentially in time. In addition, the multi-dimensional pointwise diffusive properties similar to those of the Boltzmann equation are also shown in [18]. A natural problem follows then: can one observe the nonlinear wave pattern propagation and justify the combined influence of the electrostatic potential force and/or the mutual interactions among the charged particles for the bipolar VPB system?

The main purpose of the present paper is to investigate the nonlinear wave phenomena and understand the dynamical behaviors of charged particles transported under the influence of the electrostatic potential force. It is well-known that the Boltzmann equation is asymptotically equivalent to the compressible Euler equations as illustrated by the famous Hilbert expansions. The system of compressible Euler equations is a typical example of hyperbolic conservation laws system. There are three basic wave patterns to the hyperbolic conservation laws: two nonlinear waves, that is, the shock wave and rarefaction wave in the genuinely nonlinear field, and one linearly degenerate wave called contact discontinuity. Therefore, the Boltzmann equation has rich wave phenomena as for the macroscopic fluid mechanics, and there has been important progress on the nonlinear stability of these basic wave patterns of the Boltzmann equation; refer for instance to [2,14,15,21,22,24,35,42] and references therein. However, we should mention here that the pioneering study on the stability and positivity of viscous shock waves was first made by Liu and Yu

[21] in energy space with the zero total macroscopic mass condition based on the micro–macro decomposition proposed by LIU AND YU [21]. Furthermore, YU [42] made an important breakthrough to establish the stability of a single viscous shock profile without the zero mass condition by the elegant point-wise method based on the Green function around the shock profile. Then, the stability of the rarefaction wave is proved by LIU ET AL. [24] and the stability of the viscous contact wave, which is the viscous version of contact discontinuity by HUANG AND YANG [15] with the zero mass condition and HUANG ET AL. [14] without the zero mass condition. Recently, WANG AND WANG [35] proved the stability of superposition of two viscous shock profiles to the Boltzmann equation without the zero mass condition by the weighted characteristic energy method.

Therefore, due to the appearance of wave modes and the spectral gap of the linearized bipolar VPB system as shown in [18], it is natural and interesting to investigate and understand the nonlinear wave phenomena of the bipolar VPB system under the influence of the electric field and mutual interactions between charged particles. To this end, we first consider the nonlinear stability of viscous shock waves and the rarefaction wave for the bipolar VPB system. Nevertheless, compared with the stability analysis made for the Boltzmann equation, it is not straightforward to study the stability of viscous shock waves and the rarefaction wave under the influence of the electric field effect and the mutual interactions among the charged particles. Moreover, there is no generic framework made concerned with the stability of basic wave patterns to the bipolar VPB system as far as we know. To overcome these difficulties, for $F_1 = \frac{F_A+F_B}{2}$ satisfying the Boltzmann-type equation with the additional electric fields, we employ the micro–macro type decomposition as in [21,23] for the Boltzmann equation, while in the VPB type equation for $F_2 = \frac{F_A-F_B}{2}$, we introduce a new micro–macro type decomposition around the local Maxwellian with respect to F_1. More importantly, we can derive a new diffusion equation with the damping from the macroscopic part of F_2, which crucially implies that the electric fields are strongly dissipative and guarantees the stability of wave patterns.

Note that this new decomposition for F_2 is quite universal and will play an important role in the stability analysis towards wave patterns to the bipolar VPB system (1.1). Then as the applications of this new decomposition, the stability of viscous shock waves and the rarefaction wave are proved for the 1D bipolar VPB system as the first step. Note that for the stability of superposition of two viscous shock waves in the first and third characteristic fields, there are no zero macroscopic mass conditions for the initial perturbations by introducing suitable shifts on the two viscous shock waves, the linear diffusion wave in the second characteristic field and the coupled diffusion waves, a more motivated by LIU [19], SZEPESSY AND XIN [33], HUANG AND MATSUMURA [12] and WANG AND WANG [35]. Roughly speaking, it is first proved that the linear superposition of two Boltzmann shock profiles is nonlinearly stable time-asymptotically to the 1D bipolar VPB system up to some suitable shifts without imposing the zero macroscopic conditions on the initial perturbations. Moreover, we proved the nonlinear stability of the rarefaction wave solution to the Riemann problem of inviscid Euler system time-asymptotically to the 1D bipolar VPB system. The precise statements of the stability of viscous shock waves and rarefaction waves can be referred to Theorems 3.1 and 4.1, respectively. Future works will be done on the stability of other wave patterns and their linear superpositions.

There have been important works on the existence and behavior of solutions to the VPB system. The global existence of the renormalized solution for large initial data was proved in Mischler [29]. The first global existence result on a classical solution in torus when the initial data is near a global Maxwellian was established in Guo [10]. The global existence of the classical solution in $\mathbf{R}^3$ was given in [39,41]. The case with a general stationary background density function was studied in [6], and the perturbation of vacuums was investigated in [5,7]. Recently, Li et al. [17,18] analyze the spectrum of the linearized VPB system (unipolar and bipolar) and obtain the optimal decay rate of solutions to the nonlinear system near global Maxwellian. See also the works on the stability of global Maxwellian and the optimal time decay rate in [34,36,38], and on boundary value problems of the stationary VPB system [1]. Recently, Duan and Liu [4] proved the stability of the rarefaction wave to a unipolar VPB system, which can be viewed as an approximation of bipolar VPB system (1.1) when the electron density is very rarefied and reaches a local equilibrium state with small electron mass compared with the ion. However, there is not any analysis made concerned with the stability of viscous shock waves to the bipolar VPB system (1.1), as far as we know.

It should be also mentioned that deep investigation has been achieved on the asymptotic stability of wave patterns for viscous conservation laws, which are extremely helpful for understanding the wave phenomenon of the kinetic equations. The time-asymptotic stability of the viscous shock profile started from Goodman [8] for the uniformly viscous conservation laws and Matsumura and Nishihara [27] for the compressible Navier–Stokes equations independently by the anti-derivative methods. Note that in both of the above results the zero mass conditions are imposed on the initial perturbation. Then Liu [19] and Szepessy and Xin [33] removed the zero mass condition by introducing the linear and nonlinear diffusion waves and the coupled diffusion waves in the transverse characteristic field for the uniformly viscous conservation laws and Liu and Zeng [25] proved the physical viscosity case. Zumbrun [44] proved the stability of large-amplitude shock waves of compressible Navier–Stokes equations by the Evans function approach. Then the stability of rarefaction waves for the compressible Navier–Stokes was proved by Matsumura and Nishihara [28] and Nishihara et al. [30]. The stability of the viscous contact wave for the uniformly viscous conservation laws was proved by Liu and Xin [20] and Xin [37] with the zero mass condition. Then for the compressible Navier–Stokes equations with physical viscosities, the stability of the viscous contact wave was proved by Huang et al. [13] with the zero mass condition, and by Huang et al. [14] without the zero mass condition. For composite waves, Huang and Matsumura [12] first studied the asymptotic stability of two viscous shock waves under general initial perturbation without zero mass conditions on initial perturbations for the full Navier–Stokes system and Huang et al. [11] justified the stability of a combination wave of a viscous contact wave and rarefaction waves.

The rest of the paper is arranged as follows. In Sect. 2 we present the classical micro–macro decomposition and introduce a new micro–macro decomposition for the bipolar VPB system (1.1). Then the main results on the stability of viscous shock waves and the rarefaction wave are stated and proved in Sects. 3 and 4, respectively. Finally, "Appendix A and B" are devoted to a priori estimates for the stability of viscous shock waves and the rarefaction wave, respectively.

2. Micro–Macro Decompositions

We reformulate the bipolar VPB system (1.1) and give a new micro–macro decomposition around the local Maxwellian in order to study the nonlinear stability of basic wave patterns to the system (1.1). Set

$$F_1 = \frac{F_A + F_B}{2}, \quad F_2 = \frac{F_A - F_B}{2},$$

then the system (1.1) is changed into

$$\begin{cases} F_{1t} + v \cdot \nabla_x F_1 + \nabla_x \Pi \cdot \nabla_v F_2 = 2Q(F_1, F_1), \\ F_{2t} + v \cdot \nabla_x F_2 + \nabla_x \Pi \cdot \nabla_v F_1 = 2Q(F_1, F_2), \\ \Delta \Pi = 2 \displaystyle\int F_2 \, \mathrm{d}v. \end{cases} \tag{2.1}$$

We present the micro–macro decompositions around the local Maxwellian to the bipolar VPB system (2.1). The equation $(2.1)_1$ can be viewed as the Boltzmann equation with additional electric potential force; we make use of the micro–macro decomposition as introduced by LIU AND YU [21] and LIU ET AL. [23]. In fact, for any solution $F_1(t, x, v)$ to equation $(2.1)_1$, there are five macroscopic (fluid) quantities: the mass density $\rho(t, x)$, the momentum $m(t, x) = \rho u(t, x)$, and the total energy $E(t, x) = \rho\big(e + \frac{1}{2}|u|^2\big)(t, x)$ defined by

$$\begin{cases} \rho(t, x) = \displaystyle\int_{\mathbf{R}^3} \xi_0(v) F_1(t, x, v) \, \mathrm{d}v, \\ \rho u_i(t, x) = \displaystyle\int_{\mathbf{R}^3} \xi_i(v) F_1(t, x, v) \, \mathrm{d}v, \quad i = 1, 2, 3, \\ \rho \left(e + \dfrac{|u|^2}{2}\right)(t, x) = \displaystyle\int_{\mathbf{R}^3} \xi_4(v) F_1(t, x, v) \, \mathrm{d}v, \end{cases} \tag{2.2}$$

where $\xi_i(v)$ $(i = 0, 1, 2, 3, 4)$ are the collision invariants given by

$$\xi_0(v) = 1, \quad \xi_i(v) = v_i \ (i = 1, 2, 3), \quad \xi_4(v) = \frac{1}{2}|v|^2, \tag{2.3}$$

and which satisfy

$$\int_{\mathbf{R}^3} \xi_i(v) Q(g_1, g_2) \, \mathrm{d}v = 0, \quad \text{for} \quad i = 0, 1, 2, 3, 4.$$

Define the local Maxwellian $\mathbf{M}$ associated to the solution $F_1(t, x, v)$ to equation (2.1)$_1$ in terms of the fluid quantities by

$$\mathbf{M} := \mathbf{M}_{[\rho,u,\theta]}(t, x, v) = \frac{\rho(t, x)}{\sqrt{(2\pi R\theta(t, x))^3}} e^{-\frac{|v-u(t,x)|^2}{2R\theta(t,x)}}, \tag{2.4}$$

where $\theta(t, x)$ is the temperature which is related to the internal energy $e(t, x)$ by $e = \frac{3}{2}R\theta$ with $R > 0$ the gas constant, and $u(t, x) = \big(u_1(t, x), u_2(t, x), u_3(t, x)\big)^t$ is the fluid velocity. Then, the collision operator of $Q(f, f)$ can be linearized to be $\mathbf{L_M}$ with respect to the local Maxwellian $\mathbf{M}$ by

$$\mathbf{L_M} g = 2Q(\mathbf{M}, g) + 2Q(g, \mathbf{M}). \tag{2.5}$$

The null space $\mathfrak{N}_1$ of $\mathbf{L_M}$ is spanned by $\xi_i(v)$ $(i = 0, 1, 2, 3, 4)$.

Define an inner product $\langle g_1, g_2\rangle_{\widetilde{\mathbf{M}}}$ for $g_i \in L(\mathbf{R}^3_v)$ with respect to the given local Maxwellian $\widetilde{\mathbf{M}}$ as

$$\langle g_1, g_2\rangle_{\widetilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\widetilde{\mathbf{M}}} g_1(v) g_2(v)\, \mathrm{d}v. \tag{2.6}$$

For simplicity, if $\widetilde{\mathbf{M}}$ is the local Maxwellian $\mathbf{M}$ in (2.4), we shall use the notation $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{\mathbf{M}}$. Furthermore, there exists a positive constant $\widetilde{\sigma}_1 > 0$ such that it holds for any function $g(v) \in \mathfrak{N}_1^{\perp}$ (cf. [3,9]) that

$$\langle g, \mathbf{L_M} g\rangle \leqq -\widetilde{\sigma}_1 \langle \nu(|v|) g, g\rangle, \tag{2.7}$$

where $\nu(|v|) \sim (1 + |v|)$ is the collision frequency for the hard sphere collision.

With respect to the inner product $\langle\cdot,\cdot\rangle$, the following pairwise orthogonal basis span the macroscopic space $\mathfrak{N}_1$:

$$\begin{cases} \chi_0(v) \equiv \dfrac{1}{\sqrt{\rho}}\mathbf{M}, \quad \chi_i(v) \equiv \dfrac{v_i - u_i}{\sqrt{R\theta\rho}}\mathbf{M} \quad \text{for} \quad i = 1, 2, 3, \\ \chi_4(v) \equiv \dfrac{1}{\sqrt{6\rho}}\left(\dfrac{|v-u|^2}{R\theta} - 3\right)\mathbf{M}, \quad \langle\chi_i, \chi_j\rangle = \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4. \end{cases} \tag{2.8}$$

In terms of the above orthogonal basis, the macroscopic projection $\mathbf{P}_0$ from $L^2(\mathbf{R}^3_v)$ to $\mathfrak{N}_1$ and the microscopic projection $\mathbf{P}_1$ from $L^2(\mathbf{R}^3_v)$ to $\mathfrak{N}_1^{\perp}$ can be defined as

$$\mathbf{P}_0 g = \sum_{j=0}^{4} \langle g, \chi_j\rangle \chi_j, \quad \mathbf{P}_1 g = g - \mathbf{P}_0 g.$$

A function $g(v)$ is said to be microscopic or non-fluid if it holds

$$\int g(v)\xi_i(v)\, \mathrm{d}v = 0, \quad i = 0, 1, 2, 3, 4,$$

where $\xi_i(v)$ are the collision invariants defined in (2.3).

Based on the above preparation, the solution $F_1(t, x, v)$ to equation (2.1)$_1$ can be decomposed into the macroscopic (fluid) part, that is, the local Maxwellian

$\mathbf{M} = \mathbf{M}(t, x, v)$ defined in (2.4), and the microscopic (non-fluid) part, that is $\mathbf{G} = \mathbf{G}(t, x, v)$:

$$F_1(t, x, v) = \mathbf{M}(t, x, v) + \mathbf{G}(t, x, v), \quad \mathbf{P}_0 F_1 = \mathbf{M}, \quad \mathbf{P}_1 F_1 = \mathbf{G},$$

and the equation $(2.1)_1$ becomes

$$(\mathbf{M} + \mathbf{G})_t + v \cdot \nabla_x (\mathbf{M} + \mathbf{G}) + \nabla_x \Pi \cdot \nabla_v F_2 = \mathbf{L_M G} + 2Q(\mathbf{G}, \mathbf{G}). \tag{2.9}$$

Taking the inner product of the equation (2.9) and the collision invariants $\xi_i(v)$ $(i = 0, 1, 2, 3, 4)$ with respect to v over $\mathbf{R}^3$, one has the following system for the fluid variables (ρ, u, θ):

$$\begin{cases} \rho_t + \mathrm{div}_x(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p - n_2 \nabla_x \Pi = -\displaystyle\int v \otimes v \cdot \nabla_x \mathbf{G}\, \mathrm{d}v, \\ \left[\rho \left(e + \dfrac{|u|^2}{2}\right)\right]_t + \mathrm{div}_x \left[\rho u \left(e + \dfrac{|u|^2}{2}\right) + pu\right] - \nabla_x \Pi \cdot \displaystyle\int v F_2\, \mathrm{d}v \\ \quad = -\int \frac{1}{2}|v|^2 v \cdot \nabla_x \mathbf{G}\, \mathrm{d}v, \end{cases} \tag{2.10}$$

where

$$n_2 = n_2(x, t) = \int F_2(x, t, v)\, \mathrm{d}v. \tag{2.11}$$

It must be noted that the above fluid-type system (2.10) is not self-contained and the equation for the microscopic component $\mathbf{G}$ is needed, which can be derived by applying the projection operator $\mathbf{P}_1$ into equation (2.10):

$$\mathbf{G}_t + \mathbf{P}_1(v \cdot \nabla_x \mathbf{M}) + \mathbf{P}_1(v \cdot \nabla_x \mathbf{G}) + \mathbf{P}_1(\nabla_x \Pi \cdot \nabla_v F_2) = \mathbf{L_M G} + 2Q(\mathbf{G}, \mathbf{G}). \tag{2.12}$$

Recall that the linearized collision operator $\mathbf{L_M}$ defined by (2.5) is dissipative on $\mathfrak{N}_1^{\perp}$, and its inverse $\mathbf{L_M^{-1}}$ is a bounded operator on $\mathfrak{N}_1^{\perp}$. Thus, it follows from (2.12) that

$$\mathbf{G} = \mathbf{L_M^{-1}}[\mathbf{P}_1(v \cdot \nabla_x \mathbf{M})] + \Gamma, \tag{2.13}$$

with

$$\Gamma = \mathbf{L_M^{-1}}[\mathbf{G}_t + \mathbf{P}_1(v \cdot \nabla_x \mathbf{G}) + \mathbf{P}_1(\nabla_x \Pi \cdot \nabla_v F_2) - 2Q(\mathbf{G}, \mathbf{G})]. \tag{2.14}$$

Substituting (2.13) into (2.10), we finally obtain the compressible Navier–Stokes-type equations for the macroscopic fluid quantities (ρ, u, θ):

$$\begin{cases} \rho_t + \mathrm{div}_x(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p - n_2 \nabla_x \Pi \\ \qquad = -\displaystyle\int v \otimes v \cdot \nabla_x \left(\mathbf{L_M^{-1}}[\mathbf{P}_1(v \cdot \nabla_x \mathbf{M})]\right) \mathrm{d}v - \int v \otimes v \cdot \nabla_x \Gamma\, \mathrm{d}v, \\ \left[\rho \left(\theta + \dfrac{|u|^2}{2}\right)\right]_t + \mathrm{div}_x \left[\rho u \left(\theta + \dfrac{|u|^2}{2}\right) + pu\right] - \nabla_x \Pi \cdot \displaystyle\int v F_2\, \mathrm{d}v \\ \qquad = -\displaystyle\int \frac{1}{2}|v|^2 v \cdot \nabla_x \left(\mathbf{L_M^{-1}}[\mathbf{P}_1(v \cdot \nabla_x \mathbf{M})]\right) \mathrm{d}v - \int \frac{1}{2}|v|^2 v \cdot \nabla_x \Gamma\, \mathrm{d}v. \end{cases} \tag{2.15}$$

A direct computation gives rise to

$$-\int v_i v_j \cdot \nabla_{x_j} \left(\mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(v \cdot \nabla_x \mathbf{M})] \right) \mathrm{d}v$$
$$= \sum_{j=1}^{3} \left[\mu(\theta) \left(u_{ix_j} + u_{jx_i} - \frac{2}{3} \delta_{ij} \mathrm{div}_x u \right) \right]_{x_j}$$

and

$$-\int \frac{1}{2} |v|^2 v \cdot \nabla_x \left(\mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(v \cdot \nabla_x \mathbf{M})] \right) \mathrm{d}v$$
$$= \sum_{j=1}^{3} (\kappa(\theta)\theta_{x_j})_{x_j} + \sum_{i,j=1}^{3} \left\{ \mu(\theta) u_i \left(u_{ix_j} + u_{jx_i} - \frac{2}{3} \delta_{ij} \mathrm{div}_x u \right) \right\}_{x_j},$$

where the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\kappa(\theta) > 0$ are smooth functions of the temperature θ. Here, we renormalize the gas constant R to be $\frac{2}{3}$ so that $e = \theta$ and $p = \frac{2}{3}\rho\theta$.

Now we decompose F_2 in the equation $(2.1)_2$, which is one of main contributions of the present paper. Roughly speaking, the macroscopic part of F_2 satisfies the diffusive equation with damping term and the microscopic couplings, which ensure the strong dissipation of the electric forces and further guarantee the stability of wave patterns. More precisely, the equation $(2.1)_2$ is a system of Vlasov–Poisson–Boltzmann type, which in virtue of the decomposition $F_1 = \mathbf{M} + \mathbf{G}$ becomes

$$F_{2t} + v \cdot \nabla_x F_2 + \nabla_x \Pi \cdot \nabla_v F_1 = \mathbf{N_M} F_2 + 2Q(F_2, \mathbf{G}), \tag{2.16}$$

with the linearized operator $\mathbf{N_M}$ defined by

$$\mathbf{N_M} h = 2Q(h, \mathbf{M}).$$

The null space $\mathfrak{N}_2$ of $\mathbf{N_M}$ is spanned by the single macroscopic variable

$$\chi_0(v) = \frac{\mathbf{M}}{\sqrt{\rho}},$$

which is totally different from the linearized operator $\mathbf{L_M}$ due to the quite different collision structures, whose null space $\mathfrak{N}_1$ is spanned by five macroscopic variables $\chi_j(v)$ $(j = 0, 1, 2, 3, 4)$; one can refer Sotirov and Yu [32] for the delicate analysis of the structure of the linearized operator $\mathbf{N_M}$ for the gas mixture without the electric effects. Furthermore, there exists a positive constant $\widetilde{\sigma}_2 > 0$ such that it holds for any function $g(v) \in \mathfrak{N}_2^{\perp}$ (cf. [32]) that

$$\langle g, \mathbf{N_M} g \rangle \leq -\widetilde{\sigma}_2 \langle \nu(|v|) g, g \rangle,$$

where $\nu(|v|) \sim (1 + |v|)$ is the collision frequency for the hard sphere collision. Consequently, the linearized collision operator $\mathbf{N_M}$ is dissipative on $\mathfrak{N}_2^{\perp}$, and its inverse $\mathbf{N_M}^{-1}$ is a bounded operator on $\mathfrak{N}_2^{\perp}$.

Then we introduce a new micro–macro decomposition around the local Maxellian $\mathbf{M}(x, t, v)$ associated with F_1 as follows:

$$\mathbf{P}_d g = \langle g, \mathbf{M}\rangle \frac{\mathbf{M}}{\rho}, \quad \mathbf{P}_c g = g - \mathbf{P}_d g, \quad \forall g.$$

Then the solution $F_2(t, x, v)$ to the VPB equation $(2.1)_2$ can be decomposed into

$$F_2(t, x, v) = \frac{\mathbf{M}}{\rho} n_2 + \mathbf{P}_c F_2, \quad \mathbf{P}_d F_2 = \frac{\mathbf{M}}{\rho} n_2, \quad \mathbf{P}_c F_2 = F_2 - \frac{\mathbf{M}}{\rho} n_2. \tag{2.17}$$

Taking the inner product of the equation (2.16) and the collision invariants $\xi_0(v) = 1$ with respect to v over $\mathbf{R}^3$, one has the following conservation law:

$$n_{2t} + \operatorname{div}_x \left(\int v F_2 \, \mathrm{d}v \right) = 0. \tag{2.18}$$

Substituting the micro–macro decomposition of F_2 in (2.17) into the above conservation law yields

$$n_{2t} + \operatorname{div}_x (u n_2) + \operatorname{div}_x \left(\int v \mathbf{P}_c F_2 \, \mathrm{d}v \right) = 0. \tag{2.19}$$

Applying the projection operator $\mathbf{P}_c$ to the equation (2.16), one has the non-fluid part equation of F_2:

$$\begin{aligned} &\partial_t (\mathbf{P}_c F_2) - \mathbf{N_M}(\mathbf{P}_c F_2) + \mathbf{P}_c (v \cdot \nabla_x F_2) \\ &\quad + \mathbf{P}_c (\nabla_x \Pi \cdot \nabla_v F_1) + \left(\frac{\mathbf{M}}{\rho} \right)_t n_2 = 2Q(F_2, \mathbf{G}), \end{aligned} \tag{2.20}$$

where we have used the facts that

$$\mathbf{P}_c (\partial_t F_2) = \partial_t (\mathbf{P}_c F_2) + \left(\frac{\mathbf{M}}{\rho} \right)_t n_2, \quad \mathbf{N_M}(F_2) = \mathbf{N_M}(\mathbf{P}_c F_2).$$

By (2.20) and continuity of the inverse operator $\mathbf{N}_{\mathbf{M}}^{-1}$ on $\mathfrak{N}_2^{\perp}$, the $\mathbf{P}_c F_2$ can be expressed by

$$\begin{aligned} \mathbf{P}_c F_2 = \mathbf{N}_{\mathbf{M}}^{-1} \Big[&\partial_t (\mathbf{P}_c F_2) + \mathbf{P}_c (v \cdot \nabla_x F_2) + \mathbf{P}_c (\nabla_x \Pi \cdot \nabla_v F_1) + \left(\frac{\mathbf{M}}{\rho} \right)_t \\ &n_2 - 2Q(F_2, \mathbf{G}) \Big], \end{aligned} \tag{2.21}$$

and it follows that

$$\begin{aligned} \operatorname{div}_x \left(\int v \mathbf{P}_c F_2 \, \mathrm{d}v \right) = \operatorname{div}_x \Bigg(\int v \mathbf{N}_{\mathbf{M}}^{-1} \Big[&\partial_t (\mathbf{P}_c F_2) + \mathbf{P}_c (v \cdot \nabla_x F_2) \\ &+ \mathbf{P}_c (\nabla_x \Pi \cdot \nabla_v F_1) + \left(\frac{\mathbf{M}}{\rho} \right)_t n_2 - 2Q(F_2, \mathbf{G}) \Big] \, \mathrm{d}v \Bigg). \end{aligned} \tag{2.22}$$

The second and third terms on the right hand side of (2.22) can be further transformed as follows: by the decomposition $F_2 = \frac{\mathbf{M}}{\rho}n_2 + \mathbf{P}_c F_2$, it holds that

$$\begin{aligned}
&\mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x F_2) \right] \right) = \mathrm{div}_x \left(\nabla_{x_j} \left(\frac{n_2}{\rho} \right) \int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v_j \mathbf{M})) \right] \right) \\
&\quad + \mathrm{div}_x \left(\frac{n_2}{\rho} \int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x \mathbf{M})) \right] \right) + \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x (\mathbf{P}_c F_2)) \right] \right) \\
&= -\mathrm{div}_x \left(\kappa_1(\theta) \nabla_x \left(\frac{n_2}{\rho} \right) \right) + \mathrm{div}_x \left(\frac{n_2}{\rho} \int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x \mathbf{M})) \right] \right) \\
&\quad + \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x (\mathbf{P}_c F_2)) \right] \right),
\end{aligned} \tag{2.23}$$

where we have used the fact that

$$\begin{aligned}
\int v_i \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v_j \mathbf{M}) \right] &= \int v_i \mathbf{N}_{\mathbf{M}}^{-1} \left[(v_j - u_j)\mathbf{M})) \right] \\
&= \int (v_i - u_i) \mathbf{N}_{\mathbf{M}}^{-1} \left[(v_j - u_j)\mathbf{M})) \right] = -\kappa_1(\theta)\delta_{ij},
\end{aligned}$$

with $i, j = 1, 2, 3$ and $\kappa_1(\theta) > 0$ being a smooth function of the temperature θ. On the other hand, by the decomposition $F_1 = \mathbf{M} + \mathbf{G}$, it holds that

$$\begin{aligned}
&\mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(\nabla_x \Pi \cdot \nabla_v F_1) \right] \right) = \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\nabla_x \Pi \cdot \nabla_v F_1 \right] \right) \\
&= -\mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\nabla_x \Pi \cdot \frac{v-u}{R\theta} \mathbf{M}) \right] \right) + \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\nabla_x \Pi \cdot \nabla_v \mathbf{G} \right] \right) \\
&= \mathrm{div}_x \left(\frac{\kappa_1(\theta)}{R\theta} \nabla_x \Pi \right) + \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\nabla_x \Pi \cdot \nabla_v \mathbf{G} \right] \right).
\end{aligned} \tag{2.24}$$

Substituting (2.22) into (2.19) and making use of (2.23) and (2.24), we can derive the governing equation on the fluid-part of F_2:

$$\begin{aligned}
&n_{2t} + \mathrm{div}_x(u n_2) + \mathrm{div}_x \left(\frac{\kappa_1(\theta)}{R\theta} \nabla \Pi \right) - \mathrm{div}_x \left(\kappa_1(\theta) \nabla \left(\frac{n_2}{\rho} \right) \right) \\
&= -\mathrm{div}_x \left(\frac{n_2}{\rho} \int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x \mathbf{M})) \right] \right) - \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\mathbf{P}_c(v \cdot \nabla_x (\mathbf{P}_c F_2)) \right] \right) \\
&\quad - \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\nabla_x \Pi \cdot \nabla_v \mathbf{G} \right] \right) \\
&\quad - \mathrm{div}_x \left(\int v \mathbf{N}_{\mathbf{M}}^{-1} \left[\partial_t (\mathbf{P}_c F_2) + \left(\frac{\mathbf{M}}{\rho} \right)_t n_2 - 2Q(F_2, \mathbf{G}) \right] \mathrm{d}v \right).
\end{aligned} \tag{2.25}$$

Note that the equation (2.25) is a diffusive equation with the damping term and higher order source terms, which is new and observed first time. The equation (2.25) is one of the main contributions of the present paper such that we can handle the electric potential force term in order to prove the stability of basic wave patterns to the bipolar VPB system (1.1).

To conclude, the VPB system (2.1) can be decomposed to consist of the fluid-dynamical system (2.15) and non-fluid type equation (2.12) for the distribution function F_1, as well as the fluid-dynamical equation (2.25) and the non-fluid type equation (2.20) for F_2 coupled with the Poisson equation

$$\Delta\Pi = 2n_2.$$

Note here that the micro–macro decomposition for F_1 is similar to the one for the Boltzmann equation in [21,23] but with some additional electric potential force terms, and the new micro–macro decomposition for F_2 is made in such a way that it is possible to get the dissipative property of the electric field and further to study the stability toward wave patterns for the bipolar VPB system (2.1). It should be remarked that this decomposition is quite universal and gives a unified framework for the stability analysis towards the wave patterns. As an application of the new decomposition, in the following sections, we prove the nonlinear stability of viscous shock waves and rarefaction waves to the 1D bipolar VPB system as the first step. Furthermore, it can be expected that we will prove the nonlinear stability of the other elementary wave patterns, such as contact discontinuity, boundary layers and so on, to the system (2.1), but this is left in the future works.

Now, for later use, we list some lemmas on the estimates and dissipative properties of collision operators in the weighted L^2 space. The following lemmas are based on the celebrated H-theorem (the first lemma is from [21]):

Lemma 2.1. *There exists a positive constant C such that*

$$\begin{aligned}&\int \frac{\nu(|v|)^{-1}Q(f,g)^2}{\widetilde{\mathbf{M}}}\,\mathrm{d}v\\ &\leqq C\left\{\int \frac{\nu(|v|)f^2}{\widetilde{\mathbf{M}}}\,\mathrm{d}v\cdot\int\frac{g^2}{\widetilde{\mathbf{M}}}\,\mathrm{d}v+\int\frac{f^2}{\widetilde{\mathbf{M}}}\,\mathrm{d}v\cdot\int\frac{\nu(|v|)g^2}{\widetilde{\mathbf{M}}}\,\mathrm{d}v\right\},\end{aligned}$$

where $\widetilde{\mathbf{M}}$ can be any Maxwellian so that the above integrals are well-defined.

Based on Lemma 2.1, the following two lemmas are taken from [24]; their proofs are straightforward by using Cauchy inequality:

Lemma 2.2. *If $\theta/2 < \theta_* < \theta$, then there exist two positive constants $\widetilde{\sigma} = \widetilde{\sigma}(\rho,u,\theta;$
$\rho_*,u_*,\theta_*)$ and $\eta_0 = \eta_0(\rho,u,\theta;\rho_*,u_*,\theta_*)$ such that if $|\rho-\rho_*|+|u-u_*|+|\theta-\theta_*|<\eta_0$, we have for $g_i(v)\in\mathfrak{N}_i^{\perp}$ $(i=1,2)$,*

$$-\int\frac{g_1\mathbf{L}_{\mathbf{M}}g_1}{\mathbf{M}_*}\,\mathrm{d}v \geqq \widetilde{\sigma}\int\frac{\nu(|v|)g_1^2}{\mathbf{M}_*}\,\mathrm{d}v,\quad -\int\frac{g_2\mathbf{N}_{\mathbf{M}}g_2}{\mathbf{M}_*}\,\mathrm{d}v\geqq\widetilde{\sigma}\int\frac{\nu(|v|)g_2^2}{\mathbf{M}_*}\,\mathrm{d}v.$$

Lemma 2.3. *Under the assumptions in Lemma 2.2, we have, for each $g_i(v)\in\mathfrak{N}_i^{\perp}$ $(i=1,2)$,*

$$\begin{aligned}&\int\frac{\nu(|v|)}{\mathbf{M}_*}|\mathbf{L}_{\mathbf{M}}^{-1}g_1|^2\,\mathrm{d}v\leqq\widetilde{\sigma}^{-2}\int\frac{\nu(|v|)^{-1}g_1^2}{\mathbf{M}_*}\,\mathrm{d}v,\quad\text{and}\\ &\int\frac{\nu(|v|)}{\mathbf{M}_*}|\mathbf{N}_{\mathbf{M}}^{-1}g_2|^2\,\mathrm{d}v\leqq\widetilde{\sigma}^{-2}\int\frac{\nu(|v|)^{-1}g_2^2}{\mathbf{M}_*}\,\mathrm{d}v.\end{aligned}$$

Remark 2.1. In Lemmas 2.2 and 2.3, η_0 may not be sufficiently small positive constant. However, in the proof of Theorem 4.1 in the following sections, the smallness of η_0 is crucially used to close the a priori assumptions (3.47) and (4.20).

3. Stability of Boltzmann Shock Profiles for the Bipolar VPB System

In this section, we want to use the micro–macro decomposition introduced in the previous section to prove the nonlinear stability of Boltzmann shock profiles to the 1D bipolar Vlasov–Poisson–Boltzmann system (1.1)

$$\begin{cases} F_{At} + v_1 \partial_x F_A + \partial_x \Pi \partial_{v_1} F_A = Q(F_A, F_A + F_B), \\ F_{Bt} + v_1 \partial_x F_B - \partial_x \Pi \partial_{v_1} F_B = Q(F_B, F_A + F_B), \\ \partial_{xx} \Pi = \displaystyle\int (F_A - F_B)\, \mathrm{d}v, \end{cases} \tag{3.1}$$

with $x \in \mathbf{R}^1$, $t \in \mathbf{R}^+$, $v = (v_1, v_2, v_3)^t \in \mathbf{R}^3$ and the initial values and the far-field states given by

$$\begin{aligned} &F_A(t=0, x, v) = F_{A0}(x, v) \to \mathbf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}(v), \quad \text{as } x \to \pm\infty, \\ &F_B(t=0, x, v) = F_{B0}(x, v) \to \mathbf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}(v), \quad \text{as } x \to \pm\infty, \\ &\Pi_x \to 0, \quad \text{as } x \to \pm\infty, \end{aligned} \tag{3.2}$$

where $u_\pm = (u_{1\pm}, 0, 0)^t$ and $\rho_\pm > 0, u_{1\pm}, \theta_\pm > 0$ are prescribed constant states such that the two states $(\rho_\pm, u_\pm, \theta_\pm)$ are connected by the superposition of 1-shock and 3-shock wave solutions to the corresponding 1D Euler system

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2, 3, \\ \left[\rho\left(e + \dfrac{|u|^2}{2}\right)\right]_t + \left[\rho u_1 \left(e + \dfrac{|u|^2}{2}\right) + p u_1\right]_x = 0. \end{cases} \tag{3.3}$$

As in the previous section, set

$$F_1 = \frac{F_A + F_B}{2}, \quad F_2 = \frac{F_A - F_B}{2}.$$

Then one has the system

$$\begin{cases} F_{1t} + v_1 \partial_x F_1 + \partial_x \Pi \partial_{v_1} F_2 = 2Q(F_1, F_1), \\ F_{2t} + v_1 \partial_x F_2 + \partial_x \Pi \partial_{v_1} F_1 = 2Q(F_2, F_1), \\ \partial_{xx} \Pi = 2 \displaystyle\int F_2\, \mathrm{d}v = 2n_2, \end{cases} \tag{3.4}$$

with the initial values (F_{10}, F_{20}) satisfying

$$\begin{aligned} &F_1(t=0, x, v) = F_{10}(x, v) \to \mathbf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}(v), \quad \text{as } x \to \pm\infty, \\ &F_2(t=0, x, v) = F_{20}(x, v) \to 0, \quad \text{as } x \to \pm\infty, \\ &\Pi_x \to 0, \quad \text{as } x \to \pm\infty. \end{aligned} \tag{3.5}$$

Remark that the equation $(3.4)_1$ is just the Boltzmann equation coupled with the Vlasov–Poisson electric potential terms. By the decomposition for $F_1 = \mathbf{M} + \mathbf{G}$, one can derive from (2.15) and (2.12) the following fluid-part system

$$
\begin{cases}
\rho_t + (\rho u_1)_x = 0, \\
(\rho u_1)_t + (\rho u_1^2 + p)_x - \left(\dfrac{\Pi_x^2}{4}\right)_x = \dfrac{4}{3}(\mu(\theta)u_{1x})_x - \displaystyle\int v_1^2 \Gamma_x \, dv, \\
(\rho u_i)_t + (\rho u_1 u_i)_x = (\mu(\theta)u_{ix})_x - \displaystyle\int v_1 v_i \Gamma_x \, dv, \quad i = 2, 3, \\
\left[\rho\left(\theta + \dfrac{|u|^2}{2}\right) + \dfrac{\Pi_x^2}{4}\right]_t + \left[\rho u_1\left(\theta + \dfrac{|u|^2}{2}\right) + p u_1\right]_x = (\kappa(\theta)\theta_x)_x + \dfrac{4}{3}(\mu(\theta)u_1 u_{1x})_x \\
\qquad + \displaystyle\sum_{i=2}^{3}(\mu(\theta)u_i u_{ix})_x - \int \dfrac{1}{2} v_1 |v|^2 \Gamma_x \, dv,
\end{cases}
\tag{3.6}
$$

and the non-fluid equation for F_1:

$$
\mathbf{G}_t + \mathbf{P}_1(v_1 \mathbf{M}_x) + \mathbf{P}_1(v_1 \mathbf{G}_x) + \mathbf{P}_1(\Pi_x \partial_{v_1} F_2) = \mathbf{L_M G} + 2Q(\mathbf{G}, \mathbf{G}), \tag{3.7}
$$

where $\mathbf{G}$ can be expressed explicitly by (2.13) and (2.14). Note that the fluid-system (3.6) is the compressible Navier–Stokes type system strongly coupled with microscopic terms and electric field terms. By the new decomposition to F_2,

$$
F_2 = \frac{n_2}{\rho}\mathbf{M} + \mathbf{P}_c F_2,
$$

from the equations (2.24) and (2.20), the macroscopic part n_2 satisfies the following equation:

$$
n_{2t} + \left(\int v_1 F_2 \, dv\right)_x = 0, \tag{3.8}
$$

or equivalently,

$$
\begin{aligned}
& n_{2t} + (u_1 n_2)_x + \left(\frac{\kappa_1(\theta)}{R\theta}\Pi_x\right)_x - \left(\kappa_1(\theta)\left(\frac{n_2}{\rho}\right)_x\right)_x \\
&\quad = -\left(\frac{n_2}{\rho}\int v_1 \mathbf{N_M}^{-1}\left[\mathbf{P}_c(v_1 \mathbf{M}_x)\right] dv\right)_x - \left(\int v_1 \mathbf{N_M}^{-1}\left[\mathbf{P}_c(v_1(\mathbf{P}_c F_2)_x)\right] dv\right)_x \\
&\qquad - \left(\int v_1 \mathbf{N_M}^{-1}\left[\Pi_x \mathbf{G}_{v_1}\right] dv\right)_x \\
&\qquad - \left(\int v_1 \mathbf{N_M}^{-1}\left[\partial_t(\mathbf{P}_c F_2) + \left(\frac{\mathbf{M}}{\rho}\right)_t n_2 - 2Q(F_2, \mathbf{G})\right] dv\right)_x,
\end{aligned}
\tag{3.9}
$$

and the microscopic part $\mathbf{P}_d F_2$ satisfies the equation

$$
\partial_t(\mathbf{P}_c F_2) - \mathbf{N_M}(\mathbf{P}_c F_2) + \mathbf{P}_c(v_1 F_{2x}) + \mathbf{P}_c(\Pi_x \partial_{v_1} F_1) + \left(\frac{\mathbf{M}}{\rho}\right)_t n_2 = 2Q(F_2, \mathbf{G}). \tag{3.10}
$$

Then the stability analysis towards the Boltzmann shock profiles for the bipolar VPB system (2.1) will be carried out for the equivalent system (3.6)–(3.10).

3.1. Existence of Boltzmann Shock Profiles

In this subsection, we state the existence and list the properties of Boltzmann shock profile. Consider the viscous shock profile to the 1D slab symmetric Boltzmann equation

$$F_{1t} + v_1 F_{1x} = 2Q(F_1, F_1). \tag{3.11}$$

We recall the Riemann problem for the compressible Euler equation (3.3) with the Riemann initial data

$$(\rho, m, E)(x, 0) = \begin{cases} (\rho_-, m_-, E_-), & x < 0, \\ (\rho_+, m_+, E_+), & x > 0, \end{cases} \tag{3.12}$$

where $m_\pm = \rho_\pm u_\pm$, $E_\pm = \rho_\pm(\theta_\pm + \frac{|u_\pm|^2}{2})$ and $u_\pm = (u_{1pm}, 0, 0)^t$. It is well-known that the system (3.3) has three eigenvalues: $\lambda_1 = u_1 - \frac{\sqrt{10\theta}}{3}$, $\lambda_2 = u_1$, $\lambda_3 = u_1 + \frac{\sqrt{10\theta}}{3}$ where the second characteristic field is linear degenerate and the other two are genuinely nonlinear. In the present section, we focus our attention on the situation where the Riemann solution of (3.3), (3.12) consists of two shock waves (and three constant states), that is, there exists an intermediate state $(\rho_\#, m_\# = \rho_\# u_\#, E_\# = \rho_\#(\theta_\# + \frac{|u_\#|^2}{2}))$ with $u_\# = (u_{1\#}, 0, 0)^t$ such that (ρ_-, m_-, E_-) connects with $(\rho_\#, m_\#, E_\#)$ by the 1-shock wave with the shock speed s_1 and $(\rho_\#, m_\#, E_\#)$ connects with (ρ_+, m_+, E_+) by the 3-shock wave with the shock speed s_3. Here the shock speeds s_1 and s_3 are constants determined by R–H conditions (3.16) and (3.15) and the entropy conditions (3.17). By the standard arguments (for example [31]) for each (ρ_-, m_-, E_-), we can see that our situation takes place provided (ρ_+, m_+, E_+) is located on a curved surface in a neighborhood of (ρ_-, m_-, E_-). In what follows, the neighborhood of (ρ_-, m_-, E_-) is denoted by Ω_-. To describe the strengths of the shock waves for later use, we set

$$\begin{aligned} \delta^{S_1} &= |\rho_\# - \rho_-| + |m_\# - m_-| + |E_\# - E_-|, \\ \delta^{S_3} &= |\rho_\# - \rho_+| + |m_\# - m_+| + |E_\# - E_+| \end{aligned}$$

and $\delta = \min\{\delta^{S_1}, \delta^{S_3}\}$. When we choose $|(\rho_+ - \rho_-, m_+ - m_-, E_+ - E_-)|$ small in our situation, for the fixed (ρ_-, m_-, E_-), we note that it holds that

$$\delta^{S_1} + \delta^{S_3} \le C|(\rho_+ - \rho_-, m_+ - m_-, E_+ - E_-)|,$$

where C is a positive constant depending only on (ρ_-, m_-, E_-). Then, if it also holds that

$$\delta^{S_1} + \delta^{S_3} \le C\delta, \quad \text{as } \delta^{S_1} + \delta^{S_3} \to 0 \tag{3.13}$$

for a positive constant C, we call the strengths of the shock waves are "small with same order". In what follows, we always assume (3.13).

Then we recall the i-shock profile $F_1^{S_i}(x - s_i t, v)$ $(i = 1, 3)$ of the Boltzmann equation (3.11) in Eulerian coordinates with its existence and properties given in the papers by Caflisch and Nicolaenko [2] and Liu and Yu [21,22]. Note that the compressibility of the Boltzmann shock profile is first proved in [22], which is crucial for the stability analysis towards the Boltzmann shock profile. First of all, the i-shock profile $F_1^{S_i}(x-s_it,v)$ satisfies

$$\begin{cases} -s_i\left(F_1^{S_i}\right)'+v_1\left(F_1^{S_i}\right)'=2Q\left(F_1^{S_i},F_1^{S_i}\right), \quad i=1,3, \\ F_1^{S_1}(-\infty,v)=\mathbf{M}_{[\rho_-,u_-,\theta_-]}(v), \quad F_1^{S_3}(-\infty,v)=\mathbf{M}_{[\rho_\#,u_\#,\theta_\#]}(v), \\ F_1^{S_1}(+\infty,v)=\mathbf{M}_{[\rho_\#,u_\#,\theta_\#]}(v), \quad F_1^{S_3}(+\infty,v)=\mathbf{M}_{[\rho_+,u_+,\theta_+]}(v), \end{cases} \tag{3.14}$$

where $'=\frac{\mathrm{d}}{\mathrm{d}\vartheta_i}$, $\vartheta_i=x-s_it$, $u_\pm=(u_{1\pm},0,0)^t$, $u_\#=(u_{1\#},0,0)^t$ and $(\rho_\pm,u_\pm,\theta_\pm)$, $(\rho_\#,u_\#,\theta_\#)$ satisfy Rankine-Hugoniot conditions

$$\begin{cases} -s_1(\rho_\#-\rho_-)+(\rho_\#u_{1\#}-\rho_-u_{1-})=0, \\ -s_1(\rho_\#u_{1\#}-\rho_-u_{1-})+(\rho_\#u_{1\#}^2+p_\#-\rho_-u_{1-}^2-p_-)=0, \\ -s_1(\rho_\#E_\#-\rho_-E_-)+(\rho_\#u_{1\#}E_\#+p_\#u_{1\#}-\rho_-u_{1-}E_--p_-u_{1-})=0, \end{cases} \tag{3.15}$$

$$\begin{cases} -s_3(\rho_+-\rho_\#)+(\rho_+u_{1+}-\rho_\#u_{1\#})=0, \\ -s_3(\rho_+u_{1+}-\rho_\#u_{1\#})+(\rho_+u_{1+}^2+p_+-\rho_\#u_{1\#}^2-p_\#)=0, \\ -s_3(\rho_+E_+-\rho_\#E_\#)+(\rho_+u_{1+}E_++p_+u_{1+}-\rho_\#u_{1\#}E_\#-p_\#u_{1\#})=0, \end{cases} \tag{3.16}$$

and Lax entropy conditions

$$\lambda_{1\#}<s_1<\lambda_{1-}, \quad \lambda_{3+}<s_3<\lambda_{3\#}, \tag{3.17}$$

with s_i being i-shock wave speed and $\lambda_i=u_1+(-1)^{\frac{i+1}{2}}\frac{\sqrt{10\theta}}{3}$ $(i=1,3)$ being the i-th characteristic eigenvalue of the Euler equations in the Eulerian coordinate and $\lambda_{1-}=u_{1-}-\frac{\sqrt{10\theta_-}}{3}$, $\lambda_{i\#}=u_{1\#}+(-1)^{\frac{i+1}{2}}\frac{\sqrt{10\theta_\#}}{3}$ $(i=1,3)$ and $\lambda_{3+}=u_{1+}+\frac{\sqrt{10\theta_+}}{3}$. By the micro–macro decomposition to the Boltzmann equation (3.11) around the local Maxwellian $\mathbf{M}^{S_i}$ $(i=1,3)$ (cf. [21,23]), it holds that

$$F_1^{S_i}(x-s_it,v)=\mathbf{M}^{S_i}(x-s_it,v)+\mathbf{G}^{S_i}(x-s_it,v),$$

where

$$\begin{aligned} \mathbf{M}^{S_i}(x-s_it,v)&=\mathbf{M}_{[\rho^{S_i},u^{S_i},\theta^{S_i}]}(x-s_it,v) \\ &=\frac{\rho^{S_i}(x-s_it)}{\sqrt{(2\pi R\theta^{S_i}(x-s_it))^3}}e^{-\frac{|v-u^{S_i}(x-s_it)|^2}{2R\theta^{S_i}(x-s_it)}}, \end{aligned}$$

with

$$\begin{pmatrix} \rho^{S_i} \\ \rho^{S_i}u_j^{S_i} \\ \rho^{S_i}(\theta^{S_i}+\frac{|u^{S_i}|^2}{2}) \end{pmatrix}=\int_{\mathbf{R}^3}\begin{pmatrix} 1 \\ v_j \\ \frac{|v|^2}{2} \end{pmatrix}F_1^{S_i}(x-s_it,v)\,\mathrm{d}v, \quad j=1,2,3.$$

With respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbf{M}^{S_i}}$ defined in (2.6), we can now define the macroscopic projection $\mathbf{P}_0^{S_i}$ and microscopic projection $\mathbf{P}_1^{S_i}$ by

$$\mathbf{P}_0^{S_i} g = \sum_{j=0}^{4} \left\langle g, \chi_j^{S_i} \right\rangle_{\mathbf{M}^{S_i}} \chi_j^{S_i}, \quad \mathbf{P}_1^{S_i} g = g - \mathbf{P}_0^{S_i} g,$$

where $\chi_j^{S_i}$ $(j = 0, 1, 2, 3, 4)$ are the corresponding pairwise orthogonal base defined in (2.8) by replacing $(\rho, u, \theta, \mathbf{M})$ by $(\rho^{S_i}, u^{S_i}, \theta^{S_i}, \mathbf{M}^{S_i})$. Under the above micro–macro decomposition, the solution $F_1^{S_i} = F_1^{S_i}(x - s_i t, v)$ satisfies

$$\mathbf{P}_0^{S_i} F_1^{S_i} = \mathbf{M}^{S_i}, \quad \mathbf{P}_1^{S_i} F^{S_i} = \mathbf{G}^{S_i},$$

and the Boltzmann equation (3.11) becomes

$$(\mathbf{M}^{S_i}+\mathbf{G}^{S_i})_t+v_1(\mathbf{M}^{S_i}+\mathbf{G}^{S_i})_x = 2\Big[Q(\mathbf{M}^{S_i}, \mathbf{G}^{S_i})+Q(\mathbf{G}^{S_i}, \mathbf{M}^{S_i})\Big]+2Q(\mathbf{G}^{S_i}, \mathbf{G}^{S_i}).$$

Correspondingly, we have the following fluid-type system for the fluid components of shock profile:

$$\begin{cases} \rho_t^{S_i} + \left(\rho^{S_i} u_1^{S_i}\right)_x = 0, \\ \left(\rho^{S_i} u_1^{S_i}\right)_t + [\rho^{S_i} \left(u_1^{S_i}\right)^2 + p^{S_i}]_x = \frac{4}{3}\left(\mu(\theta^{S_i}) u_{1x}^{S_i}\right)_x - \int v_1^2 \Gamma_x^{S_i}\, dv, \\ \left(\rho^{S_i} u_j^{S_i}\right)_t + \left(\rho^{S_i} u_1^{S_i} u_j^{S_i}\right)_x = \left(\mu(\theta^{S_i}) u_{jx}^{S_i}\right)_x - \int v_1 v_j \Gamma_x^{S_i}\, dv, \quad j = 2, 3, \\ \left[\rho^{S_i}\left(\theta^{S_i} + \frac{|u^{S_i}|^2}{2}\right)\right]_t + \left[\rho^{S_i} u_1^{S_i}\left(\theta^{S_i} + \frac{|u^{S_i}|^2}{2}\right) + p^{S_i} u_1^{S_i}\right]_x = \left(\kappa(\theta^{S_i})\theta_x^{S_i}\right)_x \\ \quad +\frac{4}{3}\left(\mu(\theta^{S_i}) u_1^{S_i} u_{1x}^{S_i}\right)_x + \sum_{j=2}^{3}\left(\mu(\theta^{S_i}) u_j^{S_i} u_{jx}^{S_i}\right)_x - \int \frac{1}{2} v_1 |v|^2 \Gamma_x^{S_i}\, dv. \end{cases} \tag{3.18}$$

In fact, from the invariance of the equation (3.14) by changing v_j with $-v_j$ for $j = 2, 3$ and the fact that $u_{j\pm} = 0$, we have $u_j^{S_i} = \int v_1 v_j \Gamma_x^{S_i}\, dv \equiv 0$ for $j = 2, 3$. And the equation for the non-fluid component $\mathbf{G}^{S_i}$ $(i = 1, 3)$ is

$$\mathbf{G}_t^{S_i} + \mathbf{P}_1^{S_i}\left(v_1 \mathbf{M}_x^{S_i}\right) + \mathbf{P}_1^{S_i}\left(v_1 \mathbf{G}_x^{S_i}\right) = \mathbf{L}_{\mathbf{M}^{S_i}} \mathbf{G}^{S_i} + 2Q\left(\mathbf{G}^{S_i}, \mathbf{G}^{S_i}\right). \tag{3.19}$$

Here $\mathbf{L}_{\mathbf{M}^{S_i}}$ is the linearized collision operator of $Q(F_1^{S_i}, F_1^{S_i})$ with respect to the local Maxwellian $\mathbf{M}^{S_i}$:

$$\mathbf{L}_{\mathbf{M}^{S_i}} g = 2[Q(\mathbf{M}^{S_i}, g) + Q(g, \mathbf{M}^{S_i})].$$

Thus

$$\mathbf{G}^{S_i} = \mathbf{L}_{\mathbf{M}^{S_i}}^{-1}\left[\mathbf{P}_1^{S_i}\left(v_1 \mathbf{M}_x^{S_i}\right)\right] + \Gamma_x^{S_i},$$

$$\Gamma_x^{S_i} = \mathbf{L}_{\mathbf{M}^{S_i}}^{-1}\left[\left(\mathbf{G}_t^{S_i} + \mathbf{P}_1^{S_i}\left(v_1 \mathbf{G}_x^{S_i}\right)\right) - 2Q\left(\mathbf{G}^{S_i}, \mathbf{G}^{S_i}\right)\right]. \tag{3.20}$$

Now we recall the properties of the shock profile $F_1^{S_i}(x - s_i t, v)$ $(i = 1, 3)$ that are given or can be induced by LIU AND YU [22] in Theorem 6.8.

Lemma 3.1. [22] *For $i = 1, 3$, if the shock wave strength δ^{S_i} is small enough, then the Boltzmann equation (3.11) admits a i-shock profile solution $F_1^{S_i}(x - s_i t, v)$ uniquely up to a shift satisfying the following properties:*

(1) *The shock profile $F_1^{S_i}(x - s_i t, v)$ converges to its far fields exponentially fast with an exponent proportional to the magnitude of the shock wave strength, that is*

$$\begin{cases} \left|\left(\rho^{S_1} - \rho_-, u_1^{S_1} - u_{1-}, \theta^{S_1} - \theta_-\right)\right| \leqq C\delta^{S_1} e^{-c\delta^{S_1}|\vartheta_1|}, & \text{as } \vartheta_1 < 0, \\ \left|\left(\rho^{S_1} - \rho_\#, u_1^{S_1} - u_{1\#}, \theta^{S_1} - \theta_\#\right)\right| \leqq C\delta^{S_1} e^{-c\delta^{S_1}|\vartheta_1|}, & \text{as } \vartheta_1 > 0, \\ \left|\left(\rho^{S_3} - \rho_+, u_1^{S_3} - u_{1+}, \theta^{S_3} - \theta_+\right)\right| \leqq C\delta^{S_3} e^{-c\delta^{S_3}|\vartheta_3|}, & \text{as } \vartheta_3 > 0, \\ \left|\left(\rho^{S_3} - \rho_\#, u_1^{S_3} - u_{1\#}, \theta^{S_3} - \theta_\#\right)\right| \leqq C\delta^{S_3} e^{-c\delta^{S_3}|\vartheta_3|}, & \text{as } \vartheta_3 < 0, \\ \left(\int \frac{\nu(|v|)|\mathbf{G}^{S_i}|^2}{\mathbf{M}_0}\, dv\right)^{\frac{1}{2}} \leqq C(\delta^{S_i})^2 e^{-c\delta^{S_i}|\vartheta_i|}, \quad i = 1, 3, \end{cases}$$

with δ^{S_i} being the i-shock strength and $\mathbf{M}_0$ being the global Maxwellian which is close to the shock profile with its precise definition given in Theorem 6.8, [22].

(2) *Compressibility of i-shock profile:*

$$(\lambda_i^{S_i})_{\vartheta_i} < 0, \quad \lambda_i^{S_i} = u_1^{S_i} + (-1)^{\frac{i+1}{2}} \frac{\sqrt{10\theta^{S_i}}}{3}.$$

(3) *The following properties hold:*

$$\rho_{\vartheta_i}^{S_i} \sim u_{1\vartheta_i}^{S_i} \sim \theta_{\vartheta_i}^{S_i} \sim (\lambda_i^{S_i})_{\vartheta_i} \sim \left(\int \frac{\nu(|v|)|\mathbf{G}^{S_i}|^2}{\mathbf{M}_0}\, dv\right)^{\frac{1}{2}},$$

where $A \sim B$ denotes the equivalence of the quantities A and B, and

$$\begin{cases} u_j^{S_i} \equiv 0, \quad \int v_1 v_j \Gamma_{\vartheta_i}^{S_i}\, dv \equiv 0, \quad j = 2, 3, \\ \left|\partial_{\vartheta_i}^k \left(\rho^{S_i}, u_1^{S_i}, \theta^{S_i}\right)\right| \leqq C(\delta^{S_i})^{k-1}\left|\left(\rho_{\vartheta_i}^{S_i}, u_{1\vartheta_i}^{S_i}, \theta_{\vartheta_i}^{S_i}\right)\right|, \quad k \geq 2, \\ \left(\int \frac{\nu(|v|)|\partial_{\vartheta_i}^k \mathbf{G}^{S_i}|^2}{\mathbf{M}_0}\, dv\right)^{\frac{1}{2}} \leqq C(\delta^{S_i})^k \left(\int \frac{\nu(|v|)|\mathbf{G}^{S_i}|^2}{\mathbf{M}_0}\, dv\right)^{\frac{1}{2}}, \quad k \geq 1, \\ \left|\int v_1 \xi_j(v) \Gamma_{\vartheta_i}^{S_i}\, dv\right| \leqq C\delta^{S_i}\left|u_{1\vartheta_i}^{S_i}\right|, \quad j = 1, 2, 3, 4, \end{cases}$$

where $\xi_j(v)$ $(j = 1, 2, 3, 4)$ are the collision invariants defined in (2.3).

Let $U(x,t) = (\rho, m, E + \frac{\Pi_x^2}{4})^t$ with $m = \rho u$ and $E = \rho(\theta + \frac{u^2}{2})$ being the solution of the system (3.6) and $\bar{U}(x,t) = (\bar{\rho}, \bar{m}, \bar{E})^t$ being the linear superposition of 1-shock and 3-shock profile with

$$\begin{cases} \bar{\rho} = \rho^{S_1}(x - s_1 t) + \rho^{S_3}(x - s_3 t) - \rho_{\#}, \quad \bar{E} = E^{S_1}(x - s_1 t) + E^{S_3}(x - s_3 t) - E_{\#}, \\ \bar{m}_1 = m_1^{S_1}(x - s_1 t) + m_1^{S_3}(x - s_3 t) - m_{1\#}, \quad \bar{m}_i = 0, \quad i = 2, 3, \end{cases} \tag{3.21}$$

where $(\cdot)^t$ denotes the transpose of the vector $(\cdot)$. Note that in order to keep the conservative form of the system (3.6), an additional term $\frac{\Pi_x^2}{4}$, which means the electric potential energy should be put on the total energy $E + \frac{\Pi_x^2}{4}$, which is quite different from the classic Boltzmann equation.

Since the present paper is concerning with general initial perturbation, that is, the integral $\int_{-\infty}^{\infty} (U - \bar{U})(x, 0)\, \mathrm{d}x$ may not be zero, to use the anti-derivative technique, we need to find an ansatz $\tilde{U}$ such that $\int_{-\infty}^{\infty} (U(x,t) - \tilde{U}(x,t))\, \mathrm{d}x = 0$, meanwhile, $\tilde{U}$ and $\bar{U}$ are time-asymptotically equivalent, that is, $|\tilde{U} - \bar{U}|$ tends to zero as $t \to \infty$. As observed by Liu [19], the general perturbations will not only produce the shift on the viscous shock wave itself, but also the linear or nonlinear diffusion waves in the transverse fields. Let

$$A(\rho, m_1, E) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{m_1^2}{\rho^2} + \frac{m^2}{3\rho^2} & \frac{4m_1}{3\rho} & \frac{2}{3} \\ -\frac{5m_1 E}{3\rho^2} + \frac{2m^2 m_1}{3\rho^3} & \frac{5E}{3\rho} - \frac{2m_1^2}{3\rho^2} - \frac{m^2}{3\rho^2} & \frac{5m_1}{3\rho} \end{pmatrix}$$

be the Jacobi matrix of the flux $(m_1, \frac{2}{3}E + \frac{m_1^2}{\rho} - \frac{m^2}{3\rho}, \frac{5m_1 E}{3\rho} - \frac{m_1 m^2}{3\rho^2})^t$ in the Euler system (3.3) with respect to (ρ, m_1, E). Then the second right eigenvector of the matrix $A(\rho, m_1, E)$ at the intermediate state $(\rho_{\#}, m_{1\#}, E_{\#})$ is

$$r_2 = \left(1, u_{1\#}, \frac{u_{1\#}^2}{2}\right)^t.$$

Furthermore, a direct computation yields that the three vectors $r_1 = (\rho_{\#} - \rho_-, m_{\#} - m_-, E_{\#} - E_-)^t$, r_2 above and $r_3 = (\rho_+ - \rho_{\#}, m_+ - m_{\#}, E_+ - E_{\#})^t$ are linearly independent in $\mathbf{R}^3$ if $\delta^{S_1} + \delta^{S_3}$ is suitably small. So if the initial mass $\int (U(x,0) - \bar{U}(x,0))\, \mathrm{d}x$ is not zero, we can distribute the initial mass along the three independent directions r_1, r_2 and r_3 as in Liu [19], that is,

$$\int (U(x,0) - \bar{U}(x,0))\, \mathrm{d}x = \sum_{i=1}^{3} \alpha_i r_i, \tag{3.22}$$

where α_i $(i = 1, 2, 3)$ are constants uniquely determined by the initial data. The excessive mass $\alpha_1 r_1$ in the first characteristic field can be eliminated by the translated 1-viscous shock wave with a shift α_1, that is, $\rho^{S_1}(x - s_1 t + \alpha_1)$. Similarly, we can eliminate $\alpha_3 r_3$ by replacing $\rho^{S_3}(x - s_3 t)$ by $\rho^{S_3}(x - s_3 t + \alpha_3)$, so the remaining problem is how to remove the excessive mass in the second characteristic field, that is, $\alpha_2 r_2$. Motivated by HUANG AND MATSUMURA [12], the desired ansatz $\tilde{U} = (\tilde{\rho}, \tilde{m}, \tilde{E})^t$ is constructed in the following form:

$$\begin{aligned} \tilde{\rho} &= \rho^{S_1}(x - s_1 t + \alpha_1) + \rho^{S_3}(x - s_3 t + \alpha_3) - \rho_\# + \Theta(x,t), \quad \tilde{m}_i = 0, \quad i = 2, 3, \\ \tilde{m}_1 &= m_1^{S_1}(x - s_1 t + \alpha_1) + m_1^{S_3}(x - s_3 t + \alpha_3) - m_{1\#} + u_{1\#}\Theta(x,t) - a\Theta_x(x,t), \\ \tilde{E} &= E^{S_1}(x - s_1 t + \alpha_1) + E^{S_3}(x - s_3 t + \alpha_3) - E_\# + \frac{1}{2}u_{1\#}^2\Theta(x,t) - a u_{1\#}\Theta_x(x,t), \end{aligned} \tag{3.23}$$

where α_i $(i = 1, 3)$ are the shifts of the i-viscous shock wave and Θ is the linear diffusion wave

$$\Theta(x,t) = \frac{\alpha_2}{\sqrt{4\pi a(1+t)}} e^{-\frac{(x - u_{1\#}t)^2}{4a(1+t)}}, \quad a = \frac{3\kappa(\theta_\#)}{5\rho_\#} > 0 \tag{3.24}$$

satisfying the heat equation

$$\Theta_t + u_{1\#}\Theta_x = a\Theta_{xx}, \quad \Theta|_{t=-1} = \alpha_2 \delta(x), \quad \int_{-\infty}^{\infty} \Theta(x,t)\,\mathrm{d}x = \alpha_2, \tag{3.25}$$

and the terms $-a\Theta_x$ and $-au_{1\#}\Theta_x$ are the coupled diffusion waves as first introduced by SZEPESSY AND XIN [33] which does not carry the initial mass, but get rid of some bad error terms decaying not enough with respect to the time t. Then it follows from (3.22) that

$$\begin{aligned} \int_{-\infty}^{\infty} (U(x,0) - \tilde{U}(x,0))\,\mathrm{d}x &= \int_{-\infty}^{\infty} (U(x,0) - \bar{U}(x,0))\,\mathrm{d}x \\ &\quad + \int_{-\infty}^{\infty} (\bar{U}(x,0) - \tilde{U}(x,0))\,\mathrm{d}x = 0, \end{aligned} \tag{3.26}$$

where we have used the fact that $\int_{-\infty}^{\infty} \Theta\,\mathrm{d}x = \alpha_2$. Thus $\tilde{U}(x,t)$ is the desired ansatz.

By Lemma 3.1, for suitably small δ and α_2, we have the wave interaction estimates between two viscous shock waves:

$$\begin{aligned} |\rho^{S_1} - \rho_\#| \cdot |\rho^{S_3} - \rho_\#| &\le C\delta^{S_1}\delta^{S_3}\left(e^{-c\delta^{S_1}(|x|+t) + c\delta^{S_1}|\alpha_1|} + e^{-c\delta^{S_3}(|x|+t) + c\delta^{S_3}|\alpha_3|}\right) \\ &\leqq C\delta^2 e^{-c\delta(|x|+t)}, \end{aligned} \tag{3.27}$$

and between the i-viscous shock waves $(i = 1, 3)$ and the diffusion waves:

$$\begin{aligned} |\rho^{S_i} - \rho_\#| \cdot |\Theta| &\le C|\alpha_2|\delta^{\frac{3}{2}} e^{-c\delta(|x|+t)} \\ &\quad + C\frac{|\alpha_2|}{(1+t)^{\frac{3}{2}}} e^{-\frac{c(x - u_{1\#}t)^2}{1+t}} + C(\delta + |\alpha_2|)e^{-c(|x|+t)}, \end{aligned} \tag{3.28}$$

where we used the fact that $\delta^{S_1}\alpha_1$ and $\delta^{S_3}\alpha_3$ are uniformly bounded by (3.22) for small δ^{S_1} and δ^{S_3} as long as the initial perturbation stays bounded. Set

$$Q = \Bigg\{ q(t,x) : |q| \le C\Big(\delta^2 + |\alpha_2|\delta^{\frac{3}{2}}\Big)e^{-c\delta(|x|+t)} \\ + C\frac{|\alpha_2|}{(1+t)^{\frac{3}{2}}}e^{-\frac{c(x-u_{1\#}t)^2}{1+t}} + C(\delta + |\alpha_2|)e^{-c(|x|+t)} \Bigg\}. \tag{3.29}$$

From now on, we denote q as a function belonging to the set Q if without confusions. Applying (3.23), we can calculate directly that

$$\tilde{u}_1 \sim u_1^{S_1} + u_1^{S_3} - u_{1\#} - \frac{a}{\rho_\#}\Theta_x, \quad \text{and}$$
$$\frac{\tilde{m}_1^2}{\tilde{\rho}} \sim \frac{(m_1^{S_1})^2}{\rho^{S_1}} + \frac{(m_1^{S_3})^2}{\rho^{S_3}} - \frac{m_{1\#}^2}{\rho_\#} + u_{1\#}^2\Theta - 2u_{1\#}a\Theta_x,$$

where $A \sim B$ means that $A = B + q$ with $q \in Q$. Similarly, we can calculate $\tilde{p}$, $\tilde{\theta}$, $\tilde{u}_{1x}$, $\mu(\tilde{\theta})\tilde{u}_{1x}$, $\frac{\tilde{m}_1\tilde{E}}{\tilde{\rho}}$, $\frac{\tilde{m}_1\tilde{p}}{\tilde{\rho}}$, $\tilde{\theta}_x$, $\kappa(\tilde{\theta})\tilde{\theta}_x$ and $\mu(\tilde{\theta})\tilde{u}_1\tilde{u}_{1x}$. Henceforth, we can check that the ansatz $\tilde{U} = (\tilde{\rho}, \tilde{m}, \tilde{E})^t$ defined in (3.23) well approximates solution of the VPB system as

$$\begin{cases} \tilde{\rho}_t + \tilde{m}_{1x} = 0, \\ \tilde{m}_{1t} + \left(\dfrac{\tilde{m}_1^2}{\tilde{\rho}} + \tilde{p}\right)_x = \dfrac{4}{3}(\mu(\tilde{\theta})\tilde{u}_{1x})_x - \displaystyle\int v_1^2\left(\Gamma_x^{S_1} + \Gamma_x^{S_3}\right) \mathrm{d}v + Q_{1x}, \\ \tilde{E}_t + \left(\dfrac{\tilde{m}_1\tilde{E}}{\tilde{\rho}} + \dfrac{\tilde{p}\tilde{m}_1}{\tilde{\rho}}\right)_x = (\kappa(\tilde{\theta})\tilde{\theta}_x)_x + \dfrac{4}{3}(\mu(\tilde{\theta})\tilde{u}_1\tilde{u}_{1x})_x \\ \quad -\int v_1\frac{|v|^2}{2}\left(\Gamma_x^{S_1} + \Gamma_x^{S_3}\right) \mathrm{d}v + Q_{2x}, \end{cases} \tag{3.30}$$

where

$$Q_1 = \left(\frac{\tilde{m}_1^2}{\tilde{\rho}} - \frac{(m_1^{S_1})^2}{\rho^{S_1}} - \frac{(m_1^{S_3})^2}{\rho^{S_3}} + \frac{m_{1\#}^2}{\rho_\#}\right) + \left(\tilde{p} - p^{S_1} - p^{S_3} + p_\#\right) \\ -\frac{4}{3}\left(\mu(\tilde{\theta})\tilde{u}_{1x} - \mu(\theta^{S_1})u_{1x}^{S_1} - \mu(\theta^{S_3})u_{1x}^{S_3}\right) + 2u_{1\#}a\Theta_x - u_{1\#}^2\Theta - a^2\Theta_{xx}, \tag{3.31}$$

and

$$Q_2 = \left(\frac{\tilde{m}_1\tilde{E}}{\tilde{\rho}} - \frac{m_1^{S_1}E^{S_1}}{\rho^{S_1}} - \frac{m_1^{S_3}E^{S_3}}{\rho^{S_3}} + \frac{m_{1\#}E_\#}{\rho_\#}\right) \\ + \left(\frac{\tilde{m}_1\tilde{p}}{\tilde{\rho}} - \frac{m_1^{S_1}p^{S_1}}{\rho^{S_1}} - \frac{m_1^{S_3}p^{S_3}}{\rho^{S_3}} + \frac{m_{1\#}p_\#}{\rho_\#}\right) \\ - \left(\kappa(\tilde{\theta})\tilde{\theta}_x - \kappa(\theta^{S_1})\theta_x^{S_1} - \kappa(\theta^{S_3})\theta_x^{S_3}\right)$$

$$
\begin{aligned}
&-\frac{4}{3}\left(\mu(\tilde{\theta})\tilde{u}_1\tilde{u}_{1x}-\mu(\theta^{S_1})u^{S_1}u_{1x}^{S_1}-\mu(\theta^{S_3})u^{S_3}u_{1x}^{S_3}\right)\\
&+\frac{3}{2}u_{1\#}^2 a\Theta_x-\frac{1}{2}u_{1\#}^3\Theta-a^2u_{1\#}\Theta_{xx}.
\end{aligned}
\tag{3.32}
$$

Obviously, it holds that $Q_1, Q_2 \sim 0$, that is, $Q_1, Q_2 \in Q$. Under the above preparation, we are now at the stage to state our main result. We first fix any (ρ_-, m_-, E_-), and assume that $(\rho_+, m_+, E_+) \in \Omega_-$ and the Riemann solution of (3.3), (3.12) consists of two shock waves. Then the macrpscopic composite wave $(\tilde{\rho}, \tilde{m} = \tilde{\rho}\tilde{u}, \tilde{E} = \tilde{\rho}(\tilde{\theta} + \frac{|\tilde{u}|^2}{2}))(x, t)$ defined in (3.23) is well defined. Denote the perturbation around the ansatz by

$$
\begin{aligned}
(\Phi_x, \Psi_x, W_x)(t,x) &= (\phi, \psi, \omega)(t,x) = \left(\rho-\tilde{\rho}, m-\tilde{m}, E+\frac{\Pi_x^2}{4}-\tilde{E}\right)(t,x),\\
\widetilde{\mathbf{G}}(t,x,v) &= \mathbf{G}(t,x,v)-\mathbf{G}^{S_1}(x-s_1t+\alpha_1, v)-\mathbf{G}^{S_3}(x-s_3t+\alpha_3, v),\\
\widetilde{F}_1(t,x,v) &= F_1(t,x,v)-F^S_{\alpha_1,\alpha_3}(t,x,v),
\end{aligned}
\tag{3.33}
$$

with

$$
F^S_{\alpha_1,\alpha_3}(t,x,v) = F_1^{S_1}(x-s_1t+\alpha_1, v)+F_1^{S_3}(x-s_3t+\alpha_3, v)-\mathbf{M}_\# \tag{3.34}
$$

being the superposition of shifted 1-shock profile and 3-shock profile to the 1D Boltzmann equation (3.11) and $\mathbf{M}_\# = \frac{\rho_\#}{\sqrt{(2\pi R\theta_\#)^3}} e^{-\frac{|v-u_\#|^2}{2R\theta_\#}}$ is the intermediate equi-llbrium state.

3.2. Main Result

Denote

$$
\begin{aligned}
\mathcal{E}(t) = \sup_{s\in[0,t]}\Bigg\{&\|(\Phi,\Psi,W)(\cdot,s)\|^2_{H_x^2}\\
&+\sum_{0\leqq|\beta|\leqq 2}\int\int\frac{|\partial^\beta(\widetilde{\mathbf{G}}, \mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x+\|(\Pi_x, n_2, n_{2x})(\cdot,s)\|^2\\
&+\sum_{|\alpha|=1, 0\leqq|\beta|\leqq 1}\int\int\frac{|\partial^\alpha\partial^\beta(\widetilde{\mathbf{G}}, \mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\\
&+\sum_{|\alpha|=2}\int\int\frac{|\partial^\alpha(\widetilde{F}_1, F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\Bigg\},
\end{aligned}
\tag{3.35}
$$

with $\mathbf{M}_*$ being the global Maxellian chosen in Theorem 3.1 and $\partial^\alpha = \partial^\alpha_{x,t}, \partial^\beta = \partial^\beta_v$. Now we can state our main result as follows:

Theorem 3.1. *There exist positive constants* δ_0 *and* ε_0 *and a global Maxellian* $\mathbf{M}_* = \mathbf{M}_{[\rho_*,u_*,\theta_*]}$, *such that if the shock wave strength and the diffusion wave strength* α_2 *satisfy* $|(\rho_+ - \rho_-, m_+ - m_-, E_+ - E_-)| + |\alpha_2| \leqq \delta_0$ *and* (3.13) *and the initial data satisfies that*

$$\mathcal{E}(0) \leqq \varepsilon_0,$$

then the Cauchy problem of the bipolar VPB system (2.1) *admits a unique global solution* $(F_1, F_2)(t, x, v)$ *satisfying*

$$\mathcal{E}(t) \leqq C\left(\mathcal{E}(0) + \delta_0^{\frac{1}{2}}\right)$$

for the uniform-in-time positive constant C *and the time-asymptotic behaviors*

$$\left\|\left(\widetilde{F}_1(t,x,v), F_2(t,x,v)\right)\right\|_{L_x^\infty L_v^2\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)} + \|(\Pi_x, n_2)(t,x)\|_{L_x^\infty} \to 0, \quad \text{as } t \to \infty.$$

Consequently, it holds that

$$\left\|\left(F_A - F^S_{\alpha_1,\alpha_3}, F_B - F^S_{\alpha_1,\alpha_3}\right)\right\|_{L_x^\infty L_v^2\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)} + \|(\Pi_x, n_2)\|_{L_x^\infty} \to 0, \quad \text{as } t \to \infty.$$

Here and in the sequel, $f(v) \in L_v^2(\frac{1}{\sqrt{\mathbf{M}_*}})$ *means that* $\frac{f(v)}{\sqrt{\mathbf{M}_*}} \in L_v^2(\mathbf{R}^3)$.

Remark 3.1. Theorem 3.1 implies that the linear superposition of two Boltzmann shock profiles in the first and third characteristic fields is nonlinearly stable time-asymptotically to the 1D bipolar VPB system (2.1) up to some suitable shifts. Note that there is no zero macroscopic mass conditions on the initial perturbations.

With the above preparation, we will give the proof of the main theorem in the following section. For this, we will first reformulate the problem. It follows from (3.6), (3.30) and (3.33) that (Φ, Ψ, W) solves

$$a\begin{cases} \Phi_t + \Psi_{1x} = 0, \\ \Psi_{1t} + \left(\dfrac{m_1^2}{\rho} + p - \dfrac{\tilde{m}_1^2}{\tilde{\rho}} - \tilde{p}\right) - \dfrac{\Pi_x^2}{4} = \dfrac{4}{3}(\mu(\theta)u_{1x} - \mu(\tilde{\theta})\tilde{u}_{1x}) \\ \quad -\displaystyle\int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v - Q_1, \\ \Psi_{it} + \left(\dfrac{m_1 m_i}{\rho} - \dfrac{\tilde{m}_1\tilde{m}_i}{\tilde{\rho}}\right) = (\mu(\theta)u_{ix} - \mu(\tilde{\theta})\tilde{u}_{ix}) \\ \quad -\displaystyle\int v_1 v_i(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v, \quad i = 2, 3, \\ W_t + \left(\dfrac{Em_1}{\rho} + \dfrac{pm_1}{\rho} - \dfrac{\tilde{E}\tilde{m}_1}{\tilde{\rho}} - \dfrac{\tilde{p}\tilde{m}_1}{\tilde{\rho}}\right) = (\kappa(\theta)\theta_x - \kappa(\tilde{\theta})\tilde{\theta}_x) \\ \quad +\dfrac{4}{3}(\mu(\theta)u_1u_{1x} - \mu(\tilde{\theta})\tilde{u}_1\tilde{u}_{1x}) + \displaystyle\sum_{i=2}^{3}\mu(\theta)u_iu_{ix} \\ \quad -\frac{1}{2}\displaystyle\int v_1|v|^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v - Q_2. \end{cases} \tag{3.36}$$

To capture the viscous effect of velocity and temperature, set

$$\Psi = \tilde{\rho}\widetilde{\Psi} + \tilde{u}\Phi, \tag{3.37}$$

$$W = \tilde{\rho}\widetilde{W} + \tilde{u}\cdot\Psi + \left(\tilde{\theta} - \frac{|\tilde{u}|^2}{2}\right)\Phi = \tilde{\rho}\widetilde{W} + \tilde{\rho}\tilde{u}\cdot\widetilde{\Psi} + \left(\tilde{\theta} + \frac{|\tilde{u}|^2}{2}\right)\Phi. \tag{3.38}$$

We also denote

$$(\tilde{\psi}, \tilde{\omega})(t,x) = (\widetilde{\Psi}_x, \widetilde{W}_x)(t,x). \tag{3.39}$$

Then we have the following linearized system for $(\Phi, \widetilde{\Psi}, \widetilde{W})$:

$$\begin{cases} \Phi_t + \tilde{\rho}\widetilde{\Psi}_{1x} + \tilde{u}_1\Phi_x + \tilde{\rho}_x\widetilde{\Psi}_1 + \tilde{u}_{1x}\Phi = 0, \\ \tilde{\rho}\widetilde{\Psi}_{1t} + \tilde{\rho}\tilde{u}_1\widetilde{\Psi}_{1x} - \frac{1}{3}\tilde{\rho}\tilde{u}_{1x}\widetilde{\Psi}_1 + \frac{2}{3}\tilde{\rho}_x\widetilde{W} + \frac{2}{3}\tilde{\rho}\widetilde{W}_x + \frac{2}{3}\tilde{\theta}\Phi_x - \frac{2\tilde{\theta}\tilde{\rho}_x}{3\tilde{\rho}}\Phi \\ \qquad = \frac{4}{3}\mu(\tilde{\theta})\widetilde{\Psi}_{1xx} - \int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v + J_1 + N_1 - Q_1, \\ \tilde{\rho}\widetilde{\Psi}_{it} + \tilde{\rho}\tilde{u}_1\widetilde{\Psi}_{ix} - \tilde{\rho}\tilde{u}_{1x}\widetilde{\Psi}_i = \mu(\tilde{\theta})\widetilde{\Psi}_{ixx} \\ \quad - \int v_1 v_i(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v + J_i + N_i, \quad i = 2,3, \\ \tilde{\rho}\widetilde{W}_t + \tilde{\rho}\tilde{u}_1\widetilde{W}_x - \tilde{\rho}\tilde{u}_{1x}\widetilde{W} + \frac{2}{3}\tilde{\rho}\tilde{\theta}\widetilde{\Psi}_{1x} - \frac{2}{3}\tilde{\rho}\tilde{\theta}_x\widetilde{\Psi}_1 = \kappa(\tilde{\theta})\widetilde{W}_{xx} \\ \quad -\frac{1}{2}\int v_1|v|^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v + \tilde{u}_1\int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v \\ \quad + J_4 + N_4 - (Q_2 - \tilde{u}_1 Q_1), \end{cases} \tag{3.40}$$

where

$$\begin{aligned} J_1 = {} & \left[\left(\int v_1^2(\Gamma^{S_1} + \Gamma^{S_3})\,\mathrm{d}v - Q_1\right)_x - \frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\tilde{u}_{1x}\right]\frac{\Phi}{\tilde{\rho}} + \frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{\rho}}\tilde{u}_{1x}\Phi_x \\ & + \frac{4}{3}\left(\tilde{u}_{1x}^2 - \frac{\tilde{\rho}_x\tilde{\theta}_x}{\tilde{\rho}}\right)\mu'(\tilde{\theta})\widetilde{\Psi}_1 + \frac{4}{3}\left(\frac{\mu(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\widetilde{\Psi}_1\right)_x \\ & + \frac{4}{3}\frac{\mu'(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\tilde{u}_{1x}\widetilde{W} + \frac{4}{3}\mu'(\tilde{\theta})\tilde{u}_{1x}\widetilde{W}_x, \\ J_i = {} & \left(\frac{\mu(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\widetilde{\Psi}_i\right)_x - \frac{\mu'(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\tilde{\theta}_x\widetilde{\Psi}_i, \quad i = 2,3, \\ J_4 = {} & \left(\int v_1\frac{|v|^2}{2}(\Gamma_x^{S_1} + \Gamma_x^{S_3})\,\mathrm{d}v - \tilde{u}_1\int v_1^2(\Gamma_x^{S_1} + \Gamma_x^{S_3})\,\mathrm{d}v - Q_{2x} + \tilde{u}_1 Q_{1x} \right. \\ & \left. - \frac{\kappa(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\tilde{\theta}_x\right)\frac{\Phi}{\tilde{\rho}} \\ & + \frac{\kappa(\tilde{\theta})}{\tilde{\rho}}\tilde{\theta}_x\Phi_x + \left(\int v_1^2(\Gamma_x^{S_1} + \Gamma_x^{S_3})\,\mathrm{d}v - Q_{1x} + \frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{\rho}}\tilde{u}_{1x}\tilde{\rho}_x\right)\widetilde{\Psi}_1 \\ & + \frac{8}{3}\mu(\tilde{\theta})\tilde{u}_{1x}\widetilde{\Psi}_{1x} \\ & + \left[\left(\kappa(\tilde{\theta}) - \frac{4}{3}\mu(\tilde{\theta})\right)\tilde{u}_{1x}\widetilde{\Psi}_1\right]_x + \left(\frac{\kappa(\tilde{\theta})}{\tilde{\rho}}\tilde{\rho}_x\widetilde{W}\right)_x + \kappa'(\tilde{\theta})\tilde{\theta}_x\widetilde{W}_x, \end{aligned} \tag{3.41}$$

and the nonlinear terms

$$N_i = O(1)|(\Phi_x, \Psi_x, W_x, \Pi_x, \Phi_{xx}, \Psi_{ixx})|^2, \quad i = 1, 2, 3, \tag{3.42}$$
$$N_4 = O(1)|(\Phi_x, \Psi_x, W_x, \Pi_x, \Phi_{xx}, \Psi_{xx}, W_{xx})|^2. \tag{3.43}$$

From (3.7), (3.19) and (3.33), we can derive the equation for the non-fluid component $\widetilde{\mathbf{G}}(t, x, v)$ as

$$\begin{aligned}\widetilde{\mathbf{G}}_t - \mathbf{L_M}\widetilde{\mathbf{G}} = & -\mathbf{P}_1(v_1\widetilde{\mathbf{G}}_x) - \mathbf{P}_1(\Pi_x\partial_{v_1}F_2) \\ & +2Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}) + 2[Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_1} + \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_1} + \mathbf{G}^{S_3}, \widetilde{\mathbf{G}})] \\ & +2[Q(\mathbf{G}^{S_1}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_1})] \\ & -\left[\mathbf{P}_1(v_1\mathbf{M}_x) - \mathbf{P}_1^{S_1}\left(v_1\mathbf{M}_x^{S_1}\right) - \mathbf{P}_1^{S_3}\left(v_1\mathbf{M}_x^{S_3}\right)\right] + \sum_{i=1,3} R_i,\end{aligned} \tag{3.44}$$

with R_i given by

$$R_i = (\mathbf{L_M} - \mathbf{L}_{\mathbf{M}^{S_i}})\mathbf{G}^{S_i} - \left[\mathbf{P}_1\left(v_1\mathbf{G}_x^{S_i}\right) - \mathbf{P}_1^{S_i}\left(v_1\mathbf{G}_x^{S_i}\right)\right], \quad i = 1, 3. \tag{3.45}$$

From (2.1) and (3.14), we have the equation for $\widetilde{F}_1$ defined in (3.33)

$$\begin{aligned}&\widetilde{F}_{1t} + v_1\widetilde{F}_{1x} + \Pi_x\partial_{v_1}F_2 = \mathbf{L_M}\widetilde{\mathbf{G}} + 2Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}) + (\mathbf{L_M} - \mathbf{L}_{\mathbf{M}^{S_1}})(\mathbf{G}^{S_1}) \\ &\quad + (\mathbf{L_M} - \mathbf{L}_{\mathbf{M}^{S_3}})(\mathbf{G}^{S_3}) + 2[Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_1} + \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_1} + \mathbf{G}^{S_3}, \widetilde{\mathbf{G}})] \\ &\quad +2[Q(\mathbf{G}^{S_1}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_1})].\end{aligned} \tag{3.46}$$

Consider the reformulated system (3.40), (3.44), (3.46), (3.8), (3.9) and (3.10). Since the local existence of solution to the VPB system can be proved similarly, as in [10], to prove the global existence on the time interval $[0, T]$, we only need to close the following a priori assumption by the continuity argument:

$$\mathcal{E}(T) \leqq \chi_T^2, \tag{3.47}$$

where $\mathcal{E}(T)$ is defined in (3.35). Here and in the sequel χ_T is a small positive constant depending on the initial data and wave strengths. Note that even χ_T is denoted by the subscription T, which means that the a priori assumption (3.47) is imposed on the time interval $[0, T]$, however, χ_T is to be chosen to be independent of the time T.

3.3. The Proof of Main Result

By the continuum argument for the local solution to the system (2.1) or equivalently the system (3.6)–(3.10), to prove Theorem 3.1, it is sufficient to close the a priori assumption (3.47) and verify the time-asymptotic behaviors of the solution. We start from the lower order estimates in the following Proposition:

Proposition 3.1. *For each (ρ_-, m_-, E_-), there exists a positive constant C such that, if $\delta + |\alpha_2| \leqq \delta_0$ for a suitably small positive constant δ_0, then it holds that*

$$
\begin{aligned}
&\|(\Phi, \Psi, W, \Phi_x, \Pi_x, n_2)(\cdot, t)\|^2 + \int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, t)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_0^t \|(\Pi_x, \Pi_{x\tau}, n_2)\|^2\,\mathrm{d}\tau \\
&\quad + \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\Phi, \Psi, W, \widetilde{\Psi}, \widetilde{W}, n_2)\|^2\,\mathrm{d}\tau \\
&\quad + \int_0^t \|\sqrt{|u_{1x}^{S_1}| + |u_{1x}^{S_3}| + |\Theta_x|}(\Phi, \widetilde{\Psi}_1, \widetilde{W})\|^2\,\mathrm{d}\tau \\
&\quad + \int_0^t \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&\leqq C(\chi_T + \delta_0) \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\phi, \psi, \omega)\|^2\,\mathrm{d}\tau \\
&\quad + C\int_0^t \|(\phi_x, \tilde{\psi}_x, \tilde{\omega}_x)\|^2\,\mathrm{d}\tau + C(\chi_T \\
&\quad + \delta_0)\int_0^t \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&\quad + C \sum_{|\alpha'|=1} \int_0^t \int\int \frac{\nu(|v|)|\partial^{\alpha'}(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau + C\left(\mathcal{E}(0)^2 + \delta_0^{\frac{1}{2}}\right).
\end{aligned}
$$

The proof of Proposition 3.1 will be given in "Appendix A". Then we perform the higher order estimates. Firstly, we apply ∂_x to the system (3.40) to get

$$
\begin{cases}
\phi_t + \tilde{\rho}\tilde{\psi}_{1x} + \tilde{u}_1\phi_x + \tilde{\rho}_x\tilde{\psi}_1 + \tilde{u}_{1x}\phi = -L_0, \\
\tilde{\rho}\tilde{\psi}_{1t} + \tilde{\rho}\tilde{u}_1\tilde{\psi}_{1x} - \frac{1}{3}\tilde{\rho}\tilde{u}_{1x}\tilde{\psi}_1 + \frac{2}{3}\tilde{\rho}_x\tilde{\omega} + \frac{2}{3}\tilde{\rho}\tilde{\omega}_x + \frac{2}{3}\tilde{\theta}\phi_x - \frac{2\tilde{\theta}\tilde{\rho}_x}{3\tilde{\rho}}\phi \\
\quad = \left(\frac{4}{3}\mu(\tilde{\theta})\tilde{\psi}_{1x}\right)_x - \int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})_x\,\mathrm{d}v + (J_1 + N_1 - Q_1)_x - L_1, \\
\tilde{\rho}\tilde{\psi}_{it} + \tilde{\rho}\tilde{u}_1\tilde{\psi}_{ix} - \tilde{\rho}\tilde{u}_{1x}\tilde{\psi}_i = (\mu(\tilde{\theta})\tilde{\psi}_{ix})_x \\
\quad - \int v_1 v_i(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})_x\,\mathrm{d}v + (J_i + N_i)_x - L_i, \quad i = 2, 3, \\
\tilde{\rho}\tilde{\omega}_t + \tilde{\rho}\tilde{u}_1\tilde{\omega}_x - \tilde{\rho}\tilde{u}_{1x}\tilde{\omega} + \frac{2}{3}\tilde{\rho}\tilde{\theta}\tilde{\psi}_{1x} - \frac{2}{3}\tilde{\rho}\tilde{\theta}_x\tilde{\psi}_1 = (\kappa(\tilde{\theta})\tilde{\omega}_x)_x \\
\quad -\frac{1}{2}\int v_1|v|^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})_x\,\mathrm{d}v \\
\quad + \left(\tilde{u}_1 \int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v\right)_x + \left(J_4 + N_4 - (Q_2 - \tilde{u}_1 Q_1)\right)_x - L_4,
\end{cases}
\tag{3.48}
$$

where

$$
\begin{aligned}
L_0 &= \tilde{\rho}_x\tilde{\psi}_1 + \tilde{u}_{1x}\phi + \tilde{\rho}_{xx}\widetilde{\Psi}_1 + \tilde{u}_{1xx}\Phi, \\
L_1 &= \tilde{\rho}_x\widetilde{\Psi}_{1t} + (\tilde{\rho}\tilde{u}_1)_x\tilde{\psi}_1 - \frac{1}{3}(\tilde{\rho}\tilde{u}_{1x})_x\widetilde{\Psi}_1 + \frac{2}{3}\tilde{\rho}_{xx}\widetilde{W} + \frac{2}{3}\tilde{\rho}_x\tilde{\omega}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3}\tilde{\theta}_x\phi-\frac{2}{3}\left(\frac{\tilde{\theta}\tilde{\rho}_x}{\tilde{\rho}}\right)_x\Phi,\\
L_i&=\tilde{\rho}_x\widetilde{\Psi}_{it}+(\tilde{\rho}\tilde{u}_1)_x\tilde{\psi}_i-(\tilde{\rho}\tilde{u}_{1x})_x\widetilde{\Psi}_i,\quad i=2,3,\\
L_4&=\tilde{\rho}_x\widetilde{W}_t+(\tilde{\rho}\tilde{u}_1)_x\tilde{\omega}-(\tilde{\rho}\tilde{u}_{1x})_x\widetilde{W}+\frac{2}{3}(\tilde{\rho}\tilde{\theta})_x\tilde{\psi}_1-\frac{2}{3}(\tilde{\rho}\tilde{\theta}_x)_x\widetilde{\Psi}_1.
\end{aligned}
$$

To derive the estimate on the more higher order derivatives, we apply ∂_x to the system (3.48) to obtain

$$
\begin{cases}
\phi_{xt}+\tilde{\rho}\tilde{\psi}_{1xx}+\tilde{u}_1\phi_{xx}+\tilde{\rho}_x\tilde{\psi}_{1x}+\tilde{u}_{1x}\phi_x=-\widetilde{L}_0,\\
\tilde{\rho}\tilde{\psi}_{1xt}+\tilde{\rho}\tilde{u}_1\tilde{\psi}_{1xx}-\frac{1}{3}\tilde{\rho}\tilde{u}_{1x}\tilde{\psi}_{1x}+\frac{2}{3}\tilde{\rho}_x\tilde{\omega}_x+\frac{2}{3}\tilde{\rho}\tilde{\omega}_{xx}+\frac{2}{3}\tilde{\theta}\phi_{xx}-\frac{2\tilde{\theta}\tilde{\rho}_x}{3\tilde{\rho}}\phi_x\\
\quad=\left(\frac{4}{3}\mu(\tilde{\theta})\tilde{\psi}_{1x}\right)_{xx}-\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})_{xx}\,\mathrm{d}v+(J_1+N_1-Q_1)_{xx}-\widetilde{L}_1,\\
\tilde{\rho}\tilde{\psi}_{ixt}+\tilde{\rho}\tilde{u}_1\tilde{\psi}_{ixx}-\tilde{\rho}\tilde{u}_{1x}\tilde{\psi}_{ix}=(\mu(\tilde{\theta})\tilde{\psi}_{ix})_{xx}-\int v_1v_i(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})_{xx}\,\mathrm{d}v\\
\quad+(J_i+N_i)_{xx}-\widetilde{L}_i,\quad i=2,3,\\
\tilde{\rho}\tilde{\omega}_{xt}+\tilde{\rho}\tilde{u}_1\tilde{\omega}_{xx}-\tilde{\rho}\tilde{u}_{1x}\tilde{\omega}_x+\frac{2}{3}\tilde{\rho}\tilde{\theta}\tilde{\psi}_{1xx}-\frac{2}{3}\tilde{\rho}\tilde{\theta}_x\tilde{\psi}_{1x}=(\kappa(\tilde{\theta})\tilde{\omega}_x)_{xx}\\
\quad-\frac{1}{2}\int v_1|v|^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})_{xx}\,\mathrm{d}v+\left(\tilde{u}_1\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v\right)_{xx}\\
\quad+(J_4+N_4-(Q_2-\tilde{u}_1Q_1))_{xx}-\widetilde{L}_4,
\end{cases}
\tag{3.49}
$$

where

$$
\begin{aligned}
\widetilde{L}_0&=\tilde{\rho}_x\tilde{\psi}_{1x}+\tilde{u}_{1x}\phi_x+\tilde{\rho}_{xx}\tilde{\psi}_1+\tilde{u}_{1xx}\phi+L_{0x},\\
\widetilde{L}_1&=\tilde{\rho}_x\tilde{\psi}_{1t}+(\tilde{\rho}\tilde{u}_1)_x\tilde{\psi}_{1x}-\frac{1}{3}(\tilde{\rho}\tilde{u}_{1x})_x\tilde{\psi}_1\\
&\quad+\frac{2}{3}\tilde{\rho}_{xx}\tilde{\omega}+\frac{2}{3}\tilde{\rho}_x\tilde{\omega}_x+\frac{2}{3}\tilde{\theta}_x\phi_x-\frac{2}{3}\left(\frac{\tilde{\theta}\tilde{\rho}_x}{\tilde{\rho}}\right)_x\phi+L_{1x},\\
\widetilde{L}_i&=\tilde{\rho}_x\tilde{\psi}_{it}+(\tilde{\rho}\tilde{u}_1)_x\tilde{\psi}_{ix}-(\tilde{\rho}\tilde{u}_{1x})_x\tilde{\psi}_i+L_{ix},\quad i=2,3,\\
\widetilde{L}_4&=\tilde{\rho}_x\tilde{\omega}_t+(\tilde{\rho}\tilde{u}_1)_x\tilde{\omega}_x-(\tilde{\rho}\tilde{u}_{1x})_x\tilde{\omega}+\frac{2}{3}(\tilde{\rho}\tilde{\theta})_x\tilde{\psi}_{1x}-\frac{2}{3}(\tilde{\rho}\tilde{\theta}_x)_x\tilde{\psi}_1+L_{4x}.
\end{aligned}
$$

By using the above two systems and the equations for n_2 and the non-fluid component $\mathbf{P}_cF_2$, we can establish the following proposition for the higher order energy estimates:

Proposition 3.2. *Under the assumptions of Proposition 3.1, it holds that*

$$
\begin{aligned}
\sup_{0\leqq t<\infty}\Bigg[&\|(\phi,\psi,\omega)(\cdot,t)\|_{H^1}^2+\|(\Pi_x,n_2,n_{2x})(\cdot,t)\|^2\\
&+\sum_{0\leqq|\beta|\leqq 2}\int\int\frac{|\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}
$$

$$+\sum_{|\alpha'|=1,0\leqq|\beta'|\leqq1}\int\int\frac{|\partial^{\alpha'}\partial^{\beta'}(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v$$

$$+\sum_{|\alpha|=2}\int\int\frac{|\partial^{\alpha}(\widetilde{F}_1,F_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\Bigg]$$

$$+\sum_{1\leqq|\alpha|\leqq2}\int_0^\infty\|\partial^\alpha(\phi,\psi,\omega,n_2)\|^2\,\mathrm{d}\tau+\int_0^\infty\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau$$

$$+\sum_{1\leqq|\alpha|\leqq2}\int_0^\infty\int\int\frac{\nu(|v|)|\partial^{\alpha}(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$

$$+\sum_{0\leqq|\beta|\leqq2}\int_0^\infty\int\int\frac{\nu(|v|)|\partial^{\beta}(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$

$$+\sum_{|\alpha'|=1,|\beta'|=1}\int_0^\infty\int\int\frac{\nu(|v|)|\partial^{\alpha'}\partial^{\beta'}(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$

$$\leqq C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right).$$

The proof of Proposition 3.2 will be given in the "Appendix A".

With Propositions 3.1 and 3.2, we can close the a priori assumption (3.47) and one has

$$\int_0^{+\infty}\int\int\frac{|(\widetilde{F}_1,F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\leqq\int_0^{+\infty}\int\int\frac{\left|\left(\mathbf{M}-\mathbf{M}^{S_1}-\mathbf{M}^{S_3}+\mathbf{M}_\#,\frac{n_2}{\rho}\mathbf{M}\right)_x\right|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+\int_0^{+\infty}\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\leqq C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right).$$

From the Vlasov–Poisson–Boltzmann system, we can obtain

$$\int_0^{+\infty}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int\int\frac{|(\widetilde{F}_1,F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right|\mathrm{d}\tau\leqq C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right).$$

Therefore, one has

$$\int_0^{+\infty}\left(\int\int\frac{|(\widetilde{F}_1,F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x+\left|\frac{\mathrm{d}}{\mathrm{d}t}\int\int\frac{|(\widetilde{F}_1,F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right|\right)\mathrm{d}\tau<\infty,$$

which implies that

$$\lim_{t\to+\infty}\int\int\frac{|(\widetilde{F}_1,F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x=0.$$

By Sobolev inequality,

$$\left\| \int \frac{|(\widetilde{F}_1, F_2)|^2}{\mathbf{M}_*} \, dv \right\|_{L_x^\infty}^2 \leqq C \int\int \frac{|(\widetilde{F}_1, F_2)|^2}{\mathbf{M}_*} \, dv \, dx \int\int \frac{|(\widetilde{F}_1, F_2)_x|^2}{\mathbf{M}_*} \, dv \, dx,$$

and we can prove the asymptotic behavior of the solutions and complete the proof of Theorem 3.1.

4. Stability of Rarefaction Wave to the Bipolar VPB System

In this section, we employ the micro–macro decomposition introduced in Sect. 2 to prove the nonlinear stability of planar rarefaction wave to the Cauchy problem of the bipolar VPB system (3.1)–(3.2) in spatial one-dimension by the two states $(\rho_\pm, u_\pm, \theta_\pm)$ with $u_\pm = (u_{1\pm}, 0, 0)^t$ and $\rho_\pm > 0, u_{1\pm}, \theta_\pm > 0$ being connected by the rarefaction wave solution to the Riemann problem of the corresponding 1D Euler system (3.3) with the Riemann initial data

$$(\rho_0^r, u_0^r, \theta_0^r)(x) = \begin{cases} (\rho_+, u_+, \theta_+), \ x > 0, \\ (\rho_-, u_-, \theta_-), \ x < 0. \end{cases} \tag{4.1}$$

Correspondingly, the initial values to the transformed system (3.4) satisfy

$$\begin{cases} F_1(t = 0, x, v) = F_{10}(x, v) \to \mathbf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}(v), \quad \text{as } x \to \pm\infty, \\ F_2(t = 0, x, v) = F_{20}(x, v) \to 0, \quad \text{as } x \to \pm\infty, \\ \Pi_x \to 0, \quad \text{as } x \to \pm\infty. \end{cases} \tag{4.2}$$

4.1. Rarefaction Wave and Main Result

First we describe the rarefaction wave solution to the Euler system (3.3), (4.1) with the state equation

$$p = \frac{2}{3}\rho\theta = k\rho^{\frac{5}{3}} \exp(S), \quad k = \frac{1}{2\pi e}.$$

It is straight to calculate that the Euler system (3.3) for (ρ, u_1, θ) has three distinct eigenvalues

$$\lambda_i(\rho, u_1, S) = u_1 + (-1)^{\frac{i+1}{2}} \sqrt{p_\rho(\rho, S)}, \quad i = 1, 3, \quad \lambda_2(\rho, u_1, S) = u_1,$$

with corresponding right eigenvectors

$$r_i(\rho, u_1, S) = \left((-1)^{\frac{i+1}{2}} \rho, \sqrt{p_\rho(\rho, S)}, 0 \right)^t, \\ i = 1, 3, \ r_2(\rho, u_1, S) = (p_S, 0, -p_\rho)^t,$$

such that

$$r_i(\rho, u_1, S) \cdot \nabla_{(\rho, u_1, S)} \lambda_i(\rho, u_1, S) \neq 0, \quad i = 1, 3, \quad \text{and} \\ r_2(\rho, u_1, S) \cdot \nabla_{(\rho, u_1, S)} \lambda_2(\rho, u_1, S) \equiv 0.$$

Thus the two i-Riemann invariants $\Sigma_i^{(j)} (i = 1, 3,\ j = 1, 2)$ can be defined by (cf. [31])

$$\Sigma_i^{(1)} = u_1 + (-1)^{\frac{i-1}{2}} \int^{\rho} \frac{\sqrt{p_z(z, S)}}{z} \,\mathrm{d}z, \quad \Sigma_i^{(2)} = S, \tag{4.3}$$

such that

$$\nabla_{(\rho,u_1,S)} \Sigma_i^{(j)}(\rho, u_1, S) \cdot r_i(\rho, u_1, S) \equiv 0, \quad i = 1, 3, \quad j = 1, 2.$$

Given the right state $(\rho_+, u_{1+}, \theta_+)$ with $\rho_+ > 0, \theta_+ > 0$, the i-Rarefaction wave curve $(i = 1, 3)$ in the phase space (ρ, u_1, θ) with $\rho > 0$ and $\theta > 0$ can be defined by (cf. [16]):

$$\begin{aligned} R_i(\rho_+, u_{1+}, \theta_+) : = \Big\{ (\rho, u_1, \theta) \Big| \lambda_{ix}(\rho, u_1, S) > 0,\ & \Sigma_i^{(j)}(\rho, u_1, S) \\ = \Sigma_i^{(j)}(\rho_+, u_{1+}, S_+), \quad & j = 1, 2 \Big\}. \end{aligned} \tag{4.4}$$

Without loss of generality, we consider stability of 3-rarefaction wave to the Euler system (3.3), (4.1) in the present paper and the stability of 1-rarefaction wave can be done similarly. The 3-rarefaction wave to the Euler system (3.3), (4.1) can be expressed explicitly by the Riemann solution to the inviscid Burgers equation:

$$\begin{cases} w_t + w w_x = 0, \\ w(x, 0) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases} \tag{4.5}$$

If $w_- < w_+$, then the Riemann problem (4.5) admits a rarefaction wave solution $w^r(x, t) = w^r(\frac{x}{t})$ given by

$$w^r\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} \leqq w_-, \\ \frac{x}{t}, & w_- \leqq \frac{x}{t} \leqq w_+, \\ w_+, & \frac{x}{t} \geqq w_+. \end{cases} \tag{4.6}$$

Then the 3-rarefaction wave solution $(\rho^r, u^r, \theta^r)(\frac{x}{t})$ to the compressible Euler equations (3.3), (4.1) can be defined explicitly by

$$\begin{cases} w_\pm = \lambda_3(\rho_\pm, u_{1\pm}, \theta_\pm), \quad w^r\left(\frac{x}{t}\right) = \lambda_3(\rho^r, u_1^r, \theta^r)\left(\frac{x}{t}\right), \\ \Sigma_3^{(j)}(\rho^r, u_1^r, \theta^r)\left(\frac{x}{t}\right) = \Sigma_3^{(j)}(\rho_\pm, u_{1\pm}, \theta_\pm), \quad j = 1, 2, \quad u_2^r = u_3^r = 0, \end{cases} \tag{4.7}$$

where $\Sigma_3^{(j)}$ $(j = 1, 2)$ are the 3-Riemann invariants defined in (4.3).

By the previous micro–macro decomposition for $F_1 = \mathbf{M} + \mathbf{G}$, one has the fluid-part system (3.6) and the non-fluid equation (3.7) for F_1. By the new micro–macro decomposition to

$$F_2 = \frac{n_2}{\rho}\mathbf{M} + \mathbf{P}_c F_2,$$

the macroscopic part n_2 satisfy the nonlinear diffusion equation (3.8), or equivalently (3.9), and the microscopic part $\mathbf{P}_c F_2$ satisfy the equation (3.10).

The analysis of nonlinear stability towards the rarefaction wave to the bipolar VPB system (3.4) will be carried out for the equivalent system (3.6)–(3.10), which can be roughly viewed as the strong couplings of the system of viscous conservation laws with the microscopic terms and the electric field terms.

Denote the energy functional $\mathcal{E}(t)$ by

$$\begin{aligned}\mathcal{E}(t) = \sup_{\tau\in[0,t]} \Bigg\{ & \|(\rho-\rho^r, u-u^r, \theta-\theta^r)\|^2_{H^1(\mathbf{R})} + \|(\Pi_x, n_2, n_{2x})\|^2 \\ & + \sum_{0\leqq|\beta|\leqq 2} \int_{\mathbf{R}}\int \frac{|\partial^\beta(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\, \mathrm{d}v\,\mathrm{d}x \\ & + \sum_{|\alpha|=1, 0\leqq|\beta|\leqq 1} \int\int \frac{|\partial^\alpha\partial^\beta(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\, \mathrm{d}v\,\mathrm{d}x \\ & + \sum_{|\alpha|=2} \int\int \frac{|\partial^\alpha(F_1, F_2)|^2}{\mathbf{M}_*}\, \mathrm{d}v\,\mathrm{d}x \Bigg\},\end{aligned} \tag{4.8}$$

where and in the sequel $\partial^\alpha = \partial^\alpha_{x,t}$, $\partial^\beta = \partial^\beta_v$. Note that the global Maxellian $\mathbf{M}_*$ in (4.8) is determined in Theorem 4.1. Then the main result for the stability of rarefaction wave to bipolar VPB system (3.4) can be stated as follows:

Theorem 4.1. *Assume that Riemann solution of Euler equation* (3.3), (4.1) *consists of one 3-rarefaction wave given by* (4.7). *Then, there exist positive constants* δ_0 *and* ε_0 *and a global Maxellian* $\mathbf{M}_* = \mathbf{M}_{[\rho_*, u_*, \theta_*]}$ *with* $\rho_* > 0, \theta_* > 0$, *such that if the wave strength* $\delta = |(\rho_+ - \rho_-, u_+ - u_-, \theta_+ - \theta_-)| \leqq \delta_0$ *and the initial data satisfies*

$$\mathcal{E}(0) \leqq \varepsilon_0, \tag{4.9}$$

then Cauchy problem of the bipolar VPB system (3.4)–(4.2) *admits a unique global strong solution* (F_1, F_2, Φ) *satisfying*

$$\mathcal{E}(t) \leqq C\left(\mathcal{E}(0) + \delta_0^{\frac{1}{8}}\right), \tag{4.10}$$

with the positive constant C *independent of* $t \geqq 0$, *and the time-asymptotic behaviors*

$$\left\| \left(F_1(t,x,v) - \mathbf{M}_{[\rho^r, u^r, \theta^r]}(t,x,v),\ F_2(t,x,v) \right) \right\|_{L^\infty_x L^2_v\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)} \to 0, \quad \text{as } t\to\infty,$$

$$\left\| (\rho,u,\theta)(t,x) - (\rho^r, u^r, \theta^r)\left(\frac{x}{t}\right) \right\|_{L^\infty_x} + \|(\Pi_x, n_2)(t,x)\|_{L^\infty_x} \to 0, \quad \text{as } t\to\infty.$$

Here $f(v) \in L^2_v(\frac{1}{\sqrt{\mathbf{M}_*}})$ *means that* $\frac{f(v)}{\sqrt{\mathbf{M}_*}} \in L^2_v(\mathbf{R}^3)$.

Remark 4.1. From Theorem 4.1 it follows that there exists a unique global strong solution (F_A, F_B, Π_x) to the original bipolar VPB system (3.1)–(3.2) which satisfies

$$\left\| \left(F_A(t,x,v) - \mathbf{M}_{[\rho^r,u^r,\theta^r]}(t,x,v), F_B(t,x,v) - \mathbf{M}_{[\rho^r,u^r,\theta^r]}(t,x,v) \right) \right\|_{L_x^\infty L_v^2\left(\frac{1}{\sqrt{\mathbf{M}_*}}\right)}$$

$$+ \|(\Pi_x, n_2)(t,x)\|_{L_x^\infty} \to 0,$$

as $t \to \infty$, which implies the time-asymptotic stability of rarefaction wave to the inviscid Euler system for the 1D bipolar VPB system (2.1). Note that the time-asymptotic stability of rarefaction wave in Theorem 4.1 is independent of the approximation for the rarefaction wave fan in the following Sect. 4.2:

Remark 4.2. The initial values Π_{0x} is defined through the Poisson equation $\Pi_{0xx} = 2n_{20}$, while $n_{20} = \displaystyle\int F_{20}(x,v)\,\mathrm{d}v$.

4.2. Approximate Rarefaction Wave

We first construct an approximate smooth rarefaction wave to the 3-rarefaction wave defined in (4.7). Motivated by [28], the approximate rarefaction wave can be constructed by the Burgers equation

$$\begin{cases} \bar{w}_t + \bar{w}\bar{w}_x = 0, \\ \bar{w}(0,x) = \bar{w}_0(x) = \dfrac{w_+ + w_-}{2} + \dfrac{w_+ - w_-}{2}\tanh x, \end{cases} \tag{4.11}$$

where the hyperbolic tangent function $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Note that the solution $\bar{w}(t,x)$ of the problem (4.11) can be given explicitly by

$$\bar{w}(t,x) = \bar{w}_0(x_0(t,x)), \quad x = x_0(t,x) + \bar{w}_0(x_0(t,x))t, \tag{4.12}$$

and $\bar{w}(t,x)$ has the following properties:

Lemma 4.1. [28] *The problem* (4.11) *has a unique smooth global solution* $\bar{w}(x,t)$ *such that:*

(1) $w_- < \bar{w}(x,t) < w_+,\ \partial_x \bar{w}(x,t) > 0,$ *for* $x \in \mathbf{R},\ t \geqq 0$*;*
(2) *The following estimates hold for all* $t > 0$ *and* $p \in [1,\infty]$ *we italics*

$$\begin{aligned} &\|\bar{w}(\cdot,t) - w^r(\tfrac{\cdot}{t})\|_{L^p} \leqq C(w_+ - w_-), \\ &\|\partial_x \bar{w}(\cdot,t)\|_{L^p} \leqq C\min\{(w_+ - w_-), (w_+ - w_-)^{1/p} t^{-1+1/p}\}, \\ &\|\partial_x^2 \bar{w}(\cdot,t)\|_{L^p} \leqq C\min\{(w_+ - w_-), t^{-1}\}, \\ &\left|\frac{\partial^2 \bar{w}(x,t)}{\partial x^2}\right| \leqq C\frac{\partial \bar{w}(x,t)}{\partial x}; \end{aligned}$$

(3) *The approximate rarefaction wave* $\bar{w}(x,t)$ *and the original rarefaction wave* $w^r(\frac{x}{t})$ *are time-asymptotically equivalent, that is,*

$$\lim_{t \to +\infty} \sup_{x \in \mathbf{R}} |\bar{w}(x,t) - w^r\left(\frac{x}{t}\right)| = 0.$$

Correspondingly, the approximate rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})(x,t)$ to the 3-rarefaction wave $(\rho^r, u^r, \theta^r)(\frac{x}{t})$ in (4.7) to compressible Euler equations (3.3), (4.1) can be defined by

$$\begin{cases} w_\pm = \lambda_3(\rho_\pm, u_{1\pm}, \theta_\pm), \quad \bar{w}(x, 1+t) = \lambda_3(\bar{\rho}, \bar{u}_1, \bar{\theta})(x,t), \\ \Sigma_3^{(j)}(\bar{\rho}, \bar{u}_1, \bar{\theta})(x,t) = \Sigma_3^{(j)}(\rho_\pm, u_{1\pm}, \theta_\pm), \quad j = 1, 2, \quad \bar{u}_2 = \bar{u}_3 = 0, \end{cases} \tag{4.13}$$

where $\bar{w}(x,t)$ is the solution of Burger's equation (4.11) defined in (4.12). Then the approximate rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})(x,t)$ satisfies the Euler system

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u}_1)_x = 0, \\ (\bar{\rho}\bar{u}_1)_t + (\bar{\rho}\bar{u}_1^2 + \bar{p})_x = 0, \\ (\bar{\rho}\bar{u}_i)_t + (\bar{\rho}\bar{u}_1\bar{u}_i)_x = 0, \quad i = 2, 3, \\ (\bar{\rho}\bar{\theta})_t + (\bar{\rho}\bar{u}_1\bar{\theta})_x + \bar{p}\bar{u}_{1x} = 0, \end{cases} \tag{4.14}$$

and the following properties, which can be proven by similar arguments as used in [28] and is omitted for brevity:

Lemma 4.2. *The approximate 3-rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})$ defined in (4.13) satisfies the following properties we italics:*

(i) $\bar{u}_{1x}(x,t) > 0$; *for* $x \in \mathbf{R}$, $t \geqq 0$.

(ii) *The following estimates hold for all $t > 0$ and $q \in [1, \infty]$:*

$$\begin{aligned} &\|(\bar{\rho}, \bar{u}_1, \bar{\theta})(\cdot, t) - (\rho^r, u^r, \theta^r)\left(\frac{\cdot}{t}\right)\|_{L^q} \leqq C\delta, \\ &\|(\bar{\rho}, \bar{u}_1, \bar{\theta})_x(\cdot, t)\|_{L^q} \leqq C_q\, \delta^{1/q}(1+t)^{-1+1/q}, \\ &\|(\bar{\rho}, \bar{u}_1, \bar{\theta})_{xx}(\cdot, t)\|_{L^q} \leqq C_q \min\{\delta^{-1}, (1+t)^{-1}\}; \end{aligned}$$

(iii) *Time-asymptotically, the approximation rarefaction wave and the inviscid rarefaction wave are equivalent, that is,*

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left|(\bar{\rho}, \bar{u}, \bar{\theta})(x,t) - (\rho^r, u^r, \theta^r)\left(\frac{x}{t}\right)\right| = 0.$$

Denote the perturbation around the approximate rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})(x,t)$ by

$$(\phi, \psi, \omega)(t,x) = (\rho - \bar{\rho}, u - \bar{u}, \theta - \bar{\theta})(x,t) \tag{4.15}$$

where $(\rho, u, \theta)(x,t)$ is the fluid-dynamical quantities related to the solution $F_1(t, x, v)$ of the VPB equation $(3.4)_1$. By (3.6) and (4.14), we obtain the system for the perturbation (ϕ, ψ, ω) in (4.15) as follows:

$$\begin{cases} \phi_t + \bar{\rho}\psi_{1x} + \bar{u}_1\phi_x + \bar{\rho}_x\psi_1 + \bar{u}_{1x}\phi = -(\phi\psi_1)_x, \\ \psi_{1t} + \bar{u}_1\psi_{1x} + \bar{u}_{1x}\psi_1 + \frac{2}{3}\omega_x \\ \quad + \frac{2\bar{\theta}}{3\bar{\rho}}\phi_x + \frac{2}{3}\rho_x\left(\frac{\theta}{\rho} - \frac{\bar{\theta}}{\bar{\rho}}\right) - \Pi_x\frac{n_2}{\rho} + \psi_1\psi_{1x} = \frac{4}{3\rho}(\mu(\theta)u_{1x})_x - \frac{1}{\rho}\int v_1^2\Gamma_x\, dv, \\ \psi_{it} + \bar{u}_1\psi_{ix} + \psi_1\psi_{ix} = \frac{1}{\rho}(\mu(\theta)u_{ix})_x - \frac{1}{\rho}\int v_1 v_i\Gamma_x\, dv, \, i = 2, 3, \\ \omega_t + \bar{u}_1\omega_x + \theta_x\psi_1 + \frac{2}{3}u_{1x}\omega + \frac{2}{3}\bar{\theta}\psi_{1x} + \frac{\Pi_x}{\rho}\left(\int v_1 F_2\, dv - u_1 n_2\right) = \frac{1}{\rho}(\kappa(\theta)\theta_x)_x \\ \quad + \frac{4}{3\rho}\mu(\theta)u_{1x}^2 + \frac{1}{\rho}\sum_{i=2}^{3}\mu(\theta)u_{ix}^2 - \frac{1}{\rho}\int v_1\frac{|v|^2}{2}\Gamma_x\, dv + \frac{1}{\rho}\sum_{i=1}^{3}u_i\int v_1 v_i\Gamma_x\, dv. \end{cases} \tag{4.16}$$

Set the correction function $\bar{\mathbf{G}}$ as

$$\bar{\mathbf{G}} = \frac{3}{2\theta}\mathbf{L}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_1\Big(v_1\Big(\frac{|v-u|^2}{2\theta}\bar{\theta}_x + v_1\bar{u}_{1x}\Big)\Big)\mathbf{M}\Big], \tag{4.17}$$

and let $\widetilde{\mathbf{G}}$ be the rest part related to the microscopic function $\mathbf{G}$

$$\widetilde{\mathbf{G}} = \mathbf{G} - \bar{\mathbf{G}}, \tag{4.18}$$

which satisfies

$$\begin{aligned}\widetilde{\mathbf{G}}_t - \mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}} = &-\frac{3}{2\theta}\Big[\mathbf{P}_1\Big(v_1\Big(\frac{|v-u|^2}{2\theta}\omega_x + v\cdot\psi_x\Big)\Big)\mathbf{M}\Big] \\ &- \mathbf{P}_1(v_1\mathbf{G}_x) - \mathbf{P}_1(\Pi_x\partial_{v_1}F_2) + 2Q(\mathbf{G},\mathbf{G}) - \bar{\mathbf{G}}_t. \end{aligned}\tag{4.19}$$

Notice that in (4.18), $\bar{\mathbf{G}}$ is subtracted from $\mathbf{G}$ when carrying out the lower order energy estimates because $\int |(\bar{u}_x, \bar{\theta}_x)|^2\,\mathrm{d}x \sim (1+t)^{-1}$ is not integrable with respect to the time t.

4.3. The Proof of Main Result

Consider the reformulated system (4.16), (4.19) and (3.8), (3.9) and (3.10). Since the local existence of solution to the VPB system can be proved similarly as in [10], to prove the global existence on the time interval $[0, T]$ with $T > 0$ be any positive time, we only need to close the following a-priori estimates:

$$\begin{aligned}\mathcal{N}(T) = \sup_{0\leqq t\leqq T}\Big\{&\|(\phi,\psi,\omega)(t,\cdot)\|_{H^1}^2 + \|(\Pi_x, n_2, n_{2x})\|^2 \\ &+ \sum_{0\leqq|\beta|\leqq 2}\int\int\frac{|\partial^\beta(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x \\ &+ \sum_{|\alpha'|=1, 0\leqq|\beta'|\leqq 1}\int\int\frac{|\partial^{\alpha'}\partial^{\beta'}(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x \\ &+ \sum_{|\alpha|=2}\int\int\frac{|\partial^\alpha(F_1, F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\Big\} \leqq \chi_T^2;\end{aligned}\tag{4.20}$$

in the sequel χ_T is a small positive constant only depending on the initial data and wave strengths and independent of the time T. Notice that the difference of energy functionals $\mathcal{E}(T)$ in (4.8) and $\mathcal{N}(T)$ in (4.20) lie in the perturbations around the original inviscid rarefaction wave and the approximate rarefaction wave. By Lemma 4.2, it can be seen easily that

$$\mathcal{N}(T) \leqq C(\mathcal{E}(T) + \delta) \quad \text{and} \quad \mathcal{E}(T) \leqq C(\mathcal{N}(T) + \delta),$$

with some uniform positive constant C. Under the a priori assumption (4.20), we can prove that

$$
\begin{aligned}
&\sup_{0\leqq t<\infty}\Bigg[\|(\phi,\psi,\omega)(\cdot,t)\|_{H^1}^2+\|(\Pi_x,n_2,n_{2x})(\cdot,t)\|^2\\
&\quad+\sum_{0\leqq|\beta|\leqq 2}\int\int\frac{|\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad+\sum_{|\alpha'|=1,0\leqq|\beta'|\leqq 1}\int\int\frac{|\partial^{\alpha'}\partial^{\beta'}(\mathbf{G},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad+\sum_{|\alpha|=2}\int\int\frac{|\partial^\alpha(F_1,F_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\Bigg]\\
&\quad+\int_0^\infty\|\sqrt{\bar u_{1x}}(\phi,\psi_1,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+\sum_{1\leqq|\alpha|\leqq 2}\int_0^\infty\|\partial^\alpha(\phi,\psi,\omega,n_2)\|^2\,\mathrm{d}\tau+\int_0^\infty\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau\\
&\quad+\sum_{1\leqq|\alpha|\leqq 2}\int_0^\infty\int\int\frac{\nu(|v|)|\partial^\alpha(\mathbf{G},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+\sum_{0\leqq|\beta|\leqq 2}\int_0^\infty\int\int\frac{\nu(|v|)|\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+\sum_{|\alpha'|=1,|\beta'|=1}\int_0^\infty\int\int\frac{\nu(|v|)|\partial^{\alpha'}\partial^{\beta'}(\mathbf{G},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\leqq C(\mathcal{N}(0)^2+\delta^{\frac18})\leqq C\left(\mathcal{E}(0)^2+\delta_0^{\frac18}\right).
\end{aligned}
\tag{4.21}
$$

The detailed proof of the a priori estimates (4.21) will be given in "Appendix B".

Therefore, from (4.21), we can close the a-priori assumption by choosing suitably small positive constants ε_0, δ_0 and one has

$$
\begin{aligned}
&\int_0^{+\infty}\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq\int_0^{+\infty}\int\int\frac{|\left(\mathbf{M}_x-(\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]})_x,(\frac{n_2}{\rho}\mathbf{M})_x\right)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^{+\infty}\int\int\frac{|(\mathbf{G}_x,(\mathbf{P}_cF_2)_x)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C\int_0^{+\infty}\|(\phi,\psi,\omega,n_2)_x\|^2\,\mathrm{d}\tau+C\delta_0\int_0^{+\infty}\|(\phi,\psi,\omega)\|^2\mathrm{d}\tau
\end{aligned}
$$

$$+\int_0^{+\infty}\int\int\frac{|(\mathbf{G}_x,(\mathbf{P}_cF_2)_x)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C\delta_0^2\leqq C\Big(\mathcal{N}(0)^2+\delta_0^{\frac{1}{8}}\Big)$$
$$\leqq C\Big(\mathcal{E}(0)^2+\delta_0^{\frac{1}{8}}\Big).$$

From the Vlasov–Poisson–Boltzmann system (3.4), we can obtain

$$\int_0^{+\infty}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right|\mathrm{d}\tau\leqq C\Big(\mathcal{E}(0)^2+\delta_0^{\frac{1}{8}}\Big).$$

Therefore, one has

$$\int_0^{+\infty}\Big(\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x$$
$$+\left|\frac{\mathrm{d}}{\mathrm{d}t}\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right|\Big)\mathrm{d}\tau<\infty,$$

which implies that

$$\lim_{t\to+\infty}\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x=0.$$

By Sobolev inequality

$$\|\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\|_{L_x^\infty}^2$$
$$\leqq C\Big(\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\Big)$$
$$\cdot\Big(\int\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\Big)$$

we can prove that

$$\lim_{t\to+\infty}\sup_x\int\frac{|(F_1-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]},F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v=0. \tag{4.22}$$

Similarly, one can prove that

$$\lim_{t\to+\infty}\|(\Pi_x,n_2)\|=0.$$

By Lemma 4.2, it holds that

$$\lim_{t\to+\infty}\sup_x\int\frac{|\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}-\mathbf{M}_{[\rho^r,u^r,\theta^r]}|^2}{\mathbf{M}_*}\,\mathrm{d}v=0. \tag{4.23}$$

Thus the time-asymptotic convergence of the solutions (F_1, F_2) to the rarefaction wave $\mathbf{M}_{[\rho^r,u^r,\theta^r]}$ can be derived directly from (4.22) and (4.23). By (4.21) and Lemma 4.2, it holds that

$$\mathcal{E}(t) \leqq C(\mathcal{N}(t)+\delta) \leqq C\left(\mathcal{E}(0)+\delta_0^{\frac{1}{8}}\right), \quad \forall t \in [0,+\infty),$$

which proves (4.10). Thus the proof of Theorem 4.1 is complete.

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Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: A Priori Estimates for Stability of Boltzmann Shock Profiles

By (3.6) and (3.18), one has

$$\begin{cases}\phi_t+\psi_{1x}=0,\\ \psi_{1t}+(\rho u_1^2-\tilde{\rho}\tilde{u}_1^2+p-\tilde{p})_x-\left(\dfrac{\Pi_x^2}{4}\right)_x=-\dfrac{4}{3}\left(\mu(\tilde{\theta})\tilde{u}_{1x}-\mu(\theta^{S_1})u_{1x}^{S_1}-\mu(\theta^{S_3})u_{1x}^{S_3}\right)_x\\ \qquad\qquad -\displaystyle\int v_1^2\widetilde{\mathbf{G}}_x\,\mathrm{d}v-Q_{1x},\\ \psi_{it}+(\rho u_1u_i)_x=-\displaystyle\int v_1v_i\widetilde{\mathbf{G}}_x\,\mathrm{d}v,\quad i=2,3,\\ \omega_t+(\rho u_1\theta-\tilde{\rho}\tilde{u}_1\tilde{\theta}+\rho u_1\dfrac{|u|^2}{2}-\tilde{\rho}\tilde{u}_1\dfrac{|\tilde{u}|^2}{2}+pu_1-\tilde{p}\tilde{u}_1)_x\\ \quad=-\left(\kappa(\tilde{\theta})\tilde{\theta}_x-\kappa(\theta^{S_1})\theta_x^{S_1}-\kappa(\theta^{S_3})\theta_x^{S_3}\right)_x\\ \qquad-\dfrac{4}{3}\left(\mu(\tilde{\theta})\tilde{u}_1\tilde{u}_{1x}-\mu(\theta^{S_1})u_1^{S_1}u_{1x}^{S_1}-\mu(\theta^{S_3})u_1^{S_3}u_{1x}^{S_3}\right)_x-\dfrac{1}{2}\displaystyle\int v_1|v|^2\widetilde{\mathbf{G}}_x\,\mathrm{d}v-Q_{2x}.\end{cases} \tag{A.1}$$

In fact, by the a priori assumption (3.47), one also has from the system (A.1) that

$$\|(\widetilde{\Psi},\widetilde{W})\|_{H^2}^2,\ \|(\Phi,\Psi,W,\widetilde{\Psi},\widetilde{W})\|_{L_x^\infty}^2,\ \|(\phi,\psi,\omega,\Pi_x,n_2)\|_{L_x^\infty}^2\leqq C(\chi_T+\delta_0)^2, \tag{A.2}$$

and

$$\|(\phi_t,\psi_t,\omega_t)\|^2\leqq C(\chi_T+\delta_0)^2,$$

hence, one has

$$\begin{aligned}\|(\rho_t,u_t,\theta_t)\|^2 &\leqq C\|(\rho_t,m_t,E_t)\|^2\\ &\leqq C\|(\phi_t,\psi_t,\omega_t,\Pi_x,\Pi_{xt})\|^2+C\|(\tilde\rho_t,\tilde m_t,\tilde E_t)\|^2\leqq C(\chi_T+\delta_0)^2.\end{aligned}$$

For $|\alpha|=2$, it follows from (2.2) and (2.11) that

$$\begin{aligned}&\|\partial^\alpha\left(\rho,\rho u,\rho\left(\theta+\frac{|u|^2}{2}\right),n_2\right)\|^2\leqq C\int\int\frac{|\partial^\alpha(F_1,F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\\ &\leqq C\int\int\frac{|\partial^\alpha(\widetilde F_1,F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x+C\int\int\frac{|\partial^\alpha(F^S_{\alpha_1,\alpha_3})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\leqq C(\chi_T+\delta_0)^2,\end{aligned}\tag{A.3}$$

and

$$\begin{aligned}\|\partial^\alpha(\rho,u,\theta)\|^2 &\leqq C\|\partial^\alpha\left(\rho,\rho u,\rho\left(\theta+\frac{|u|^2}{2}\right)\right)\|^2\\ &\quad+C\sum_{|\alpha'|=1}\int|\partial^{\alpha'}\left(\rho,\rho u,\rho\left(\theta+\frac{|u|^2}{2}\right)\right)|^4\,\mathrm{d}x\\ &\leqq C(\chi_T+\delta_0)^2.\end{aligned}\tag{A.4}$$

Therefore, for $|\alpha|=2$, we have

$$\|\partial^\alpha(\phi,\psi,\omega,n_2)\|^2\leqq C(\chi_T+\delta_0)^2.\tag{A.5}$$

By (3.8) and a priori assumption (3.47), it holds that

$$\begin{aligned}\|n_{2t}\|^2 &\leqq C\left[\|n_{2x}\|^2+\int|n_2|^2|(\rho_x,u_x,\theta_x)|^2\,\mathrm{d}x+\int\int\frac{|(\mathbf{P}_cF_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right]\\ &\leqq C(\chi_T+\delta_0)^2.\end{aligned}\tag{A.6}$$

By (3.47), (A.5) and (A.6), for $|\alpha'|=1$, it holds that

$$\|\partial^{\alpha'}(\phi,\psi,\omega,n_2)\|^2_{L^\infty}\leqq C(\chi_T+\delta_0)^2.$$

By (3.8), one has

$$\frac12\Pi_{xt}+\int v_1F_2\,\mathrm{d}v=0,$$

that is,

$$\frac12\Pi_{xt}+u_1n_2+\int v_1\mathbf{P}_cF_2\,\mathrm{d}v=0.\tag{A.7}$$

Therefore, one has

$$\|\Pi_{xt}\|^2\leqq C\left[\|n_2\|^2+\int\int\frac{|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right]\leqq C(\chi_T+\delta_0)^2.\tag{A.8}$$

By (A.7), it holds that

$$\frac12\Pi_{xtt}+(u_1n_2)_t+\int v_1(\mathbf{P}_cF_2)_t\,\mathrm{d}v=0.\tag{A.9}$$

Thus, one has

$$\|\Pi_{xtt}\|^2 \leqq C\left[\|n_{2t}\|^2+\int |n_2|^2|u_t|^2\,\mathrm{d}x+\int\int \frac{|(\mathbf{P}_c F_2)_t|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right] \leqq C(\chi_T+\delta_0)^2. \tag{A.10}$$

By (A.8), (A.6), (A.10) and (A.5), it holds that

$$\|(\Pi_{xt},\Pi_{xtt})\|_{L^\infty}^2 \leqq C\|(\Pi_{xt},\Pi_{xtt})\|\|(n_{2t},n_{2tt})\| \leqq C(\chi_T+\delta_0)^2.$$

Moreover, it holds that

$$\begin{aligned}\left\|\int \frac{|(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\right\|_{L_x^\infty} &\leqq C\left(\int\int \frac{|(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right)^{\frac12}\\ &\quad\cdot\left(\int\int \frac{|(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right)^{\frac12}\\ &\leqq C(\chi_T+\delta_0)^2.\end{aligned}$$

Furthermore, for $|\alpha'|=1$, it holds that

$$\begin{aligned}\left\|\int \frac{|\partial^{\alpha'}(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\right\|_{L_x^\infty} &\leqq C\left(\int\int \frac{|\partial^{\alpha'}(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right)^{\frac12}\\ &\quad\cdot\left(\int\int \frac{|\partial^{\alpha'}(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right)^{\frac12}\\ &\leqq C\chi_T(\chi_T+\delta_0) \leq C(\chi_T+\delta_0)^2.\end{aligned} \tag{A.11}$$

Finally, by noticing the facts that $F_1=\mathbf{M}+\mathbf{G}$ and $F_2=\frac{n_2}{\rho}\mathbf{M}+\mathbf{P}_c F_2$, it holds that

$$\begin{aligned}\int\int \frac{|\partial^\alpha(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x &\leq C\int\int \frac{|\partial^\alpha(\widetilde{F}_1,F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\\ &+C\int\int \frac{|\partial^\alpha(\mathbf{M}-\mathbf{M}^{S_1}-\mathbf{M}^{S_3})|^2+|\partial^\alpha\left(\frac{n_2}{\rho}\mathbf{M}\right)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x \leqq C(\chi_T+\delta_0)^2.\end{aligned}$$

Appendix A.1: Proof of Proposition 3.1: Lower Order Estimates

The proof of the lower order estimates in Proposition 3.1 involves five steps.
Step 1 Estimation on $\|(\Phi,\widetilde{\Psi},\widetilde{W})(t,\cdot)\|^2$.
Multiplying $(3.40)_1$ by $\dfrac{2\Phi}{3\tilde{\rho}}$, $(3.40)_2$ by $\dfrac{\widetilde{\Psi}_1}{\tilde{\theta}}$, $(3.40)_3$ by $\widetilde{\Psi}_i$, $(3.40)_4$ by $\dfrac{\widetilde{W}}{\tilde{\theta}^2}$, respectively, and collecting the resulted equations together, we can get

$$
\begin{aligned}
&\left(\frac{\Phi^2}{3\tilde\rho}+\frac{\tilde\rho\widetilde\Psi_1^2}{2\tilde\theta}+\sum_{i=2}^3\frac{\tilde\rho\widetilde\Psi_i^2}{2}+\frac{\tilde\rho\widetilde W^2}{2\tilde\theta^2}\right)_t+(\cdots)_x-\tilde\rho\tilde u_{1x}\left(\frac{\widetilde\Psi_1^2}{3\tilde\theta}+\sum_{i=2}^3\widetilde\Psi_i^2+\frac{\widetilde W^2}{\tilde\theta^2}\right)\\
&\quad+\frac{4\mu(\tilde\theta)}{3\tilde\theta}\widetilde\Psi_{1x}^2+\sum_{i=2}^3\mu(\tilde\theta)\widetilde\Psi_{ix}^2+\frac{\kappa(\tilde\theta)}{\tilde\theta^2}\widetilde W_x^2\\
&=-\left(\frac{4\mu(\tilde\theta)}{3\tilde\theta}\right)_x\widetilde\Psi_1\widetilde\Psi_{1x}-\sum_{i=2}^3(\mu(\tilde\theta))_x\widetilde\Psi_i\widetilde\Psi_{ix}-\left(\frac{\kappa(\tilde\theta)}{\tilde\theta^2}\right)_x\widetilde W\widetilde W_x\\
&\quad-\left(\frac{\widetilde\Psi_1^2}{2\tilde\theta^2}+\frac{\widetilde W^2}{\tilde\theta^3}\right)(\tilde\rho\tilde\theta_t+\tilde\rho\tilde u_1\tilde\theta_x)+K_1+\frac{\widetilde\Psi_1}{\tilde\theta}(J_1+N_1-Q_1)\\
&\quad+\sum_{i=2}^3\widetilde\Psi_i(J_i+N_i)+\frac{\widetilde W}{\tilde\theta^2}(J_4+N_4-Q_2+\tilde u_1Q_1)
\end{aligned}\tag{A.1.1}
$$

with

$$
\begin{aligned}
K_1=&-\frac{\widetilde\Psi_1}{\tilde\theta}\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v-\sum_{i=2}^3\widetilde\Psi_i\int v_1v_i(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v\\
&-\frac{\widetilde W}{\tilde\theta^2}\left[\int v_1\frac{|v|^2}{2}(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v-\tilde u_1\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v\right].
\end{aligned}
$$

Here and in the sequel the notation $(\cdots)_x$ represents the term in the conservative form so that it vanishes after integration. By (3.30), one has

$$
\begin{aligned}
\tilde\rho\tilde\theta_t+\tilde\rho\tilde u_1\tilde\theta_x=&-\frac{2}{3}\tilde\rho\tilde\theta\tilde u_{1x}+\frac{4}{3}\mu(\tilde\theta)\tilde u_{1x}^2+(\kappa(\tilde\theta)\tilde\theta_x)_x-\int v_1\frac{|v|^2}{2}\left(\Gamma_x^{S_1}+\Gamma_x^{S_3}\right)\mathrm{d}v\\
&+\tilde u_1\int v_1^2\left(\Gamma_x^{S_1}+\Gamma_x^{S_3}\right)\mathrm{d}v+Q_{2x}-\tilde u_1Q_{1x}.
\end{aligned}\tag{A.1.2}
$$

Substituting (A.1.2) into (A.1.1), carring out similar estimates as to [35] and choosing δ_0 and χ_T suitably small yields that

$$
\begin{aligned}
&\|(\Phi,\widetilde\Psi,\widetilde W)(t,\cdot)\|^2+\int_0^t\left[\left\|\sqrt{|u_{1x}^{S_1}|+|u_{1x}^{S_3}|}(\widetilde\Psi,\widetilde W)\right\|^2+\|(\widetilde\Psi_x,\widetilde W_x)\|^2\right]\mathrm{d}\tau\\
&\leqq C\int\int\frac{|\widetilde{\mathbf G}|^2}{\mathbf M_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x+C(\chi_T+\delta_0)\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+\sigma\int_0^t\|(\widetilde\Psi_\tau,\widetilde W_\tau)\|^2\,\mathrm{d}\tau+C\delta_0\int_0^t\|\sqrt{|u_{1x}^{S_1}|+|u_{1x}^{S_3}|}\Phi\|^2\,\mathrm{d}\tau\\
&\quad+C_\sigma\int_0^t\int\int\frac{\nu(|v|)}{\mathbf M_*}|\widetilde{\mathbf G}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf P_cF_2)|^2}{\mathbf M_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau+C\left(\mathcal E(0)^2+\delta_0^{\frac12}\right),
\end{aligned}\tag{A.1.3}
$$

where and in the sequel section the global Maxellian $\mathbf{M}_* = \mathbf{M}_{[\rho_*,u_*,\theta_*]}$ is chosen such that

$$\rho_* > 0, \quad \frac{1}{2}\theta(t,x) < \theta_* < \theta(t,x), \tag{A.1.4}$$

and

$$|\rho(x,t)-\rho_*| + |u(x,t)-u_*| + |\theta(x,t)-\theta_*| < \eta_0, \tag{A.1.5}$$

with η_0 being the small positive constant in Lemma 2.2. In fact, if the wave strength $\delta + |\alpha_2|$ is suitably small, then it is holds that

$$\frac{1}{2}\sup_{x,t}\tilde{\theta}(x,t) < \inf_{x,t}\tilde{\theta}(x,t).$$

Therefore, we can choose the global Mawellian $\mathbf{M}_* = \mathbf{M}_{[\rho_*,u_*,\theta_*]}$ satisfying (A.1.4) and (A.1.5) provided that the solution $(\rho,u,\theta)(x,t)$ is near the ansatz $(\tilde{\rho},\tilde{u},\tilde{\theta})(x,t)$ as in a priori assumption (3.47).

Step 2 Estimation on $\int_0^t\int(|u_{1x}^{S_1}| + |u_{1x}^{S_3}|)\Phi^2\,\mathrm{d}x\,\mathrm{d}\tau$.

In order to estimate $\int_0^t\int(|u_{1x}^{S_1}|+|u_{1x}^{S_3}|)\Phi^2\,\mathrm{d}x\,\mathrm{d}\tau$, we borrow the ideas of the vertical estimates in Goodman [8] to carry out the following characteristic weight estimates. First, we diagonalize the system (3.40). Let $V = (\Phi, \widetilde{\Psi}_1, \widetilde{W})^t$, then

$$V_t + A_1V_x + A_2V = A_3V_{xx} + A_4, \tag{A.1.6}$$

where

$$A_1 = \begin{pmatrix} \tilde{u}_1 & \tilde{\rho} & 0 \\ \frac{2\tilde{\theta}}{3\tilde{\rho}} & \tilde{u}_1 & \frac{2}{3} \\ 0 & \frac{2}{3}\tilde{\theta} & \tilde{u}_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \tilde{u}_{1x} & \tilde{\rho}_x & 0 \\ -\frac{2\tilde{\theta}\tilde{\rho}_x}{3\tilde{\rho}^2} & -\frac{\tilde{u}_{1x}}{3} & \frac{2\tilde{\rho}_x}{3\tilde{\rho}} \\ 0 & -\frac{2}{3}\tilde{\theta}_x & -\tilde{u}_{1x} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4\mu(\tilde{\theta})}{3\tilde{\rho}} & 0 \\ 0 & 0 & \frac{\kappa(\tilde{\theta})}{\tilde{\rho}} \end{pmatrix},$$

and

$$A_4 = \frac{1}{\tilde{\rho}}\begin{pmatrix} 0 \\ -\int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v + J_1 + N_1 - Q_1 \\ \int\left(\tilde{u}_1v_1^2 - \frac{v_1|v|^2}{2}\right)(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})\,\mathrm{d}v + J_4 + N_4 - (Q_2 - \tilde{u}_1Q_1) \end{pmatrix}.$$

We can compute three eigenvalues of the matrix A_1 in the system (A.1.6)

$$\tilde{\lambda}_1 = \tilde{u}_1 - \frac{\sqrt{10\tilde{\theta}}}{3}, \quad \tilde{\lambda}_2 = \tilde{u}_1, \quad \tilde{\lambda}_3 = \tilde{u}_1 + \frac{\sqrt{10\tilde{\theta}}}{3},$$

with corresponding left and right eigenvectors given by

$$l_1 = \left(\tilde{\theta}, -\frac{\sqrt{10\tilde{\theta}}}{2}\tilde{\rho}, \tilde{\rho}\right), \quad l_2 = \left(\tilde{\theta}, 0, -\frac{3}{2}\tilde{\rho}\right), \quad l_3 = \left(\tilde{\theta}, \frac{\sqrt{10\tilde{\theta}}}{2}\tilde{\rho}, \tilde{\rho}\right),$$

and

$$r_1 = \frac{3}{10\tilde{\rho}\tilde{\theta}}\left(\tilde{\rho}, -\frac{\sqrt{10\tilde{\theta}}}{3}, \frac{2}{3}\tilde{\theta}\right)^t, \quad r_2 = \frac{2}{5\tilde{\rho}\tilde{\theta}}(\tilde{\rho}, 0, -\tilde{\theta})^t,$$
$$r_3 = \frac{3}{10\tilde{\rho}\tilde{\theta}}\left(\tilde{\rho}, \frac{\sqrt{10\tilde{\theta}}}{3}, \frac{2}{3}\tilde{\theta}\right)^t,$$

respectively. If we denote that the matrix composed with the left and right eigenvalues by $L = (l_1, l_2, l_3)^t$, $R = (r_1, r_2, r_3)$, then we have

$$LR = Id., \quad LA_1R = \Lambda := \mathrm{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3),$$

with $Id.$ being the 3×3 identity matrix. Denote that $Z = LV$ with $Z = (Z_1, Z_2, Z_3)^t$, then we have $V = RZ$. Multiplying the system (A.1.6) by the matrix L on the left, we can obtain the diagonalized system for Z

$$Z_t + \Lambda Z_x - LA_3RZ_{xx} = -L(R_t + A_1R_x - A_3R_{xx})Z - LA_2RZ + 2LA_3R_xZ_x + LA_4. \tag{A.1.7}$$

Introduce the weight function

$$\alpha(t,x) = \frac{\rho^{S_1}(t,x)}{\rho_{\#}}, \quad \beta(t,x) = \frac{\rho^{S_3}(t,x)}{\rho_{\#}}.$$

From the properties of the shock profile to the Boltzmann equation, we have

$$\lambda_{ix}^{S_i} < 0, \quad \text{and} \quad \rho_x^{S_i} < 0 \quad (i = 1, 3).$$

Thus it holds that

$$\alpha, \beta < 1 \quad \text{and} \quad |\alpha - 1|, \quad |\beta - 1| \leqq C\delta \ll 1, \quad \text{if}\, \delta \ll 1.$$

Taking inner product by multiplying $\bar{Z} := (Z_1, \alpha^N Z_2, \alpha^N Z_3)^t$ with N being a large positive constant to be determined and noting that for $i = 2, 3$,

$$\alpha_t + \tilde{\lambda}_i\alpha_x = -s_1\alpha_x + \tilde{\lambda}_i\alpha_x = \left(\lambda_i^{S_1} - s_1\right)\alpha_x + \left(\tilde{\lambda}_i - \lambda_i^{S_1}\right)\alpha_x,$$

we have

$$\begin{aligned}
&\left[\frac{Z_1^2 + \alpha^N(Z_2^2 + Z_3^2)}{2}\right]_t + (\cdots)_x - \lambda_{1x}^{S_1}\frac{Z_1^2}{2} - \alpha^N\sum_{i=2}^3 \frac{\lambda_{ix}^{S_3} Z_i^2}{2} \\
&\quad - N\alpha^{N-1}\alpha_x\sum_{i=2}^3\frac{(\lambda_i^{S_1} - s_1)Z_i^2}{2} - \bar{Z}\cdot LA_3RZ_{xx} \\
&= -\bar{Z}\cdot L(R_t + A_1R_x - A_3R_{xx})Z - \bar{Z}\cdot LA_2RZ \\
&\quad + 2\bar{Z}\cdot LA_3R_xZ_x + \bar{Z}\cdot LA_4 + (\tilde{\lambda}_{1x} - \lambda_{1x}^{S_1})\frac{Z_1^2}{2} \\
&\quad + \alpha^N\sum_{i=2}^3\frac{(\tilde{\lambda}_{ix} - \lambda_{ix}^{S_3})Z_i^2}{2} + N\alpha^{N-1}\alpha_x\sum_{i=2}^3\frac{(\tilde{\lambda}_i - \lambda_i^{S_1})Z_i^2}{2}.
\end{aligned}$$

Note that

$$
\begin{aligned}
-\bar{Z}\cdot LA_3RZ_{xx} &= -(\bar{Z}\cdot LA_3RZ_x)_x + \bar{Z}_x\cdot LA_3RZ_x + \bar{Z}\cdot(LA_3R)_xZ_x \\
&= -(\bar{Z}\cdot LA_3RZ_x)_x + Z_x\cdot LA_3RZ_x + (\bar{Z}-Z)_x\cdot LA_3RZ_x \\
&\quad + \bar{Z}\cdot(LA_3R)_xZ_x.
\end{aligned}
$$

We can directly compute that the matrix LA_3R is non-negatively definite, and so

$$
Z_x\cdot LA_3RZ_x \geqq 0.
$$

On the other hand, it holds that

$$
\begin{aligned}
(\bar{Z}-Z)_x &= (0, (\alpha^N-1)Z_2, (\alpha^N-1)Z_3)_x{}^t \\
&= (\alpha^N-1)(0, Z_{2x}, Z_{3x})^t + N\alpha^{N-1}\alpha_x(0, Z_2, Z_3)^t.
\end{aligned}
$$

By the Lax entropy condition to 1-shock, we have

$$
\lambda_2^{S_1} - s_1 > \lambda_2^{S_1} - \lambda_{1-} = u_1^{S_1} - \left(u_{1-} - \frac{\sqrt{10\theta_-}}{3}\right) \geqq \frac{\sqrt{10\theta_-}}{3} - C\delta > \frac{\sqrt{10\theta_-}}{6}
$$

and

$$
\lambda_3^{S_1} - s_1 > \lambda_2^{S_1} - \lambda_{1-} > \frac{\sqrt{10\theta_-}}{6} \quad \text{if } \delta \ll 1.
$$

Therefore, it holds that

$$
\begin{aligned}
|(\bar{Z}-Z)_x\cdot LA_3RZ_x| &\leqq |(\alpha^N-1)(0, Z_{2x}, Z_{3x})^t\cdot LA_3RZ_x| \\
&\quad + |N\alpha^{N-1}\alpha_x(0, Z_2, Z_3)^t\cdot LA_3RZ_x| \\
&\leqq C\delta|Z_x|^2 + N\alpha^{N-1}|\alpha_x|\sum_{i=2}^{3}|Z_i||Z_x| \leqq \frac{N\alpha^{N-1}|\alpha_x|}{4}\sum_{i=2}^{3}\frac{(\lambda_i^{S_1}-s_1)Z_i^2}{2} \\
&\quad + C\sqrt{\delta_0}|Z_x|^2,
\end{aligned}
$$

if we choose $N = \frac{1}{\sqrt{\delta_0}}$ with $\delta_0 \ll 1$. Then one has

$$
\begin{aligned}
|\bar{Z}\cdot(LA_3R)_xZ_x| &\leqq C\left(\left|\lambda_{1x}^{S_1}\right| + \left|\lambda_{1x}^{S_3}\right| + |\Theta_x| + q\right)|Z||Z_x| \\
&\leqq C\sqrt{\delta_0}|Z_x|^2 + C\sqrt{\delta_0}\left(\left|\lambda_{1x}^{S_1}\right| + \left|\lambda_{1x}^{S_3}\right|\right)|Z|^2 + q|Z|^2,
\end{aligned}
\tag{A.1.8}
$$

where $q \in Q$ defined in (3.29). Similarly to [35], we can get

$$
\begin{aligned}
&\|Z(t,\cdot)\|^2 + \int_0^t\int\left[\left|\lambda_{1x}^{S_1}\right|Z_1^2 + \sum_{i=2}^{3}\left|\lambda_{ix}^{S_3}\right|Z_i^2 + N|\alpha_x|\sum_{i=2}^{3}Z_i^2\right]\mathrm{d}x\,\mathrm{d}\tau \\
&\leq C\int\int\frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + C\sqrt{\delta_0}\int_0^t\|(Z_\tau, Z_x)\|^2\,\mathrm{d}\tau \\
&\quad + C(\delta_0+\chi_T)\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau \\
&\quad + C\int_0^t\int\left|\lambda_{3x}^{S_3}\right|\left(Z_1^2+Z_2^2\right)\mathrm{d}x\,\mathrm{d}\tau + C\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C\chi_T\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau\\
&+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau+C\int_0^t\|\sqrt{|\Theta_x|}Z\|^2\,\mathrm{d}\tau\\
&+C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right). \qquad\qquad (A.1.9)
\end{aligned}
$$

Taking the inner product by multiplying $\widetilde{Z}:=(\beta^{-N}Z_1,\beta^{-N}Z_2,Z_3)^t$ with $N=\frac{1}{\sqrt{\delta_0}}$ as before, similarly to (A.1.9), we can get

$$
\begin{aligned}
&\|Z(t,\cdot)\|^2+\int_0^t\int\left[\left|\lambda_{3x}^{S_3}\right|Z_3^2+\sum_{i=1}^{2}\left|\lambda_{ix}^{S_1}\right|Z_i^2+N|\beta_x|\sum_{i=2}^{3}Z_i^2\right]\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C\int\int\frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x+C\sqrt{\delta_0}\int_0^t\|(Z_\tau,Z_x)\|^2\,\mathrm{d}\tau\\
&\quad+C(\delta_0+\chi_T)\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+C\int_0^t\int\left|\lambda_{1x}^{S_1}\right|(Z_2^2+Z_3^2)\,\mathrm{d}x\,\mathrm{d}\tau+C\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\chi_T\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau+C\int_0^t\|\sqrt{|\Theta_x|}Z\|^2\,\mathrm{d}\tau\\
&\quad+C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right)\\
&\quad+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau. \qquad\qquad (A.1.10)
\end{aligned}
$$

Combining (A.1.9) and (A.1.10) and choosing δ_0 sufficiently small, it holds that

$$
\begin{aligned}
&\|Z(t,\cdot)\|^2+\int_0^t\left\|\sqrt{\left|\lambda_{1x}^{S_1}\right|+\left|\lambda_{3x}^{S_3}\right|}Z\right\|^2\mathrm{d}\tau\leq C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right)\\
&\quad+C\int\int\frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad+C\int_0^t\left\|\sqrt{|\Theta_x|}Z\right\|^2\mathrm{d}\tau+C\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C(\delta_0+\chi_T)\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+C\chi_T\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau+C\sqrt{\delta_0}\int_0^t\|(Z_\tau,Z_x)\|^2\,\mathrm{d}\tau
\end{aligned}
$$

$$+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau. \tag{A.1.11}$$

Step 3 Estimation on $\int_0^t\|\sqrt{|\Theta_x|}Z\|^2\,\mathrm{d}\tau$.
In this step, we estimate $\int_0^t\|\sqrt{|\Theta_x|}Z\|^2\,\mathrm{d}\tau$ on the right hand side of (A.1.11). Note that the linear diffusion wave $\Theta(t,x)$ in (3.24) and the coupled diffusion wave $\Theta_x(t,x)$ propagate along the constant speed $u_{1\#}$. Therefore, somehow we can view these diffusion waves as the viscous contact wave in the linearly degenerate field as constructed in [13,14]. In fact, the viscous contact waves is exactly constructed through the self-similar solution to the diffusion equation. Thus we can borrow some ideas of the weighted estimates for the viscous contact wave in [11] to estimate the term $\int_0^t\|\sqrt{|\Theta_x|}Z\|^2\,\mathrm{d}\tau$. Similar to the viscous contact wave in the second characteristic field, the diffusion waves here have the extra dissipation on the first and third transverse characteristic fields. By the delicate weighted estimates, we can get such estimates as

$$|\alpha_2|\int_0^t\int\left[h\left(Z_1^2+Z_3^2\right)+h^2Z_2^2\right]\mathrm{d}x\,\mathrm{d}\tau, \tag{A.1.12}$$

with $h\sim(1+t)^{-\frac{1}{2}}e^{-\frac{c(x-u_{1\#}t)^2}{1+t}}$ defined in (A.1.15). Note that (A.1.12) means that the diffusion wave in the second characteristic field has some extra dissipative effects on the first and third transverse characteristic fields compared with the second diffusion wave field. By the inequality

$$\begin{aligned}|\Theta_x||Z|^2&\leqq C|\alpha_2|(1+t)^{-1}e^{-\frac{(x-u_{1\#}t)^2}{8a(1+t)}}\left(Z_1^2+Z_2^2+Z_3^2\right)\\&\leqq C|\alpha_2|\left[h\left(Z_1^2+Z_3^2\right)+h^2Z_2^2\right]\end{aligned} \tag{A.1.13}$$

we can get the desired estimate for $\int_0^t\|\sqrt{|\Theta_x|}Z\|^2\,\mathrm{d}\tau$ from (A.1.12). In what follows, we focus on the proof of (A.1.12).
We first set the matrices $(c_{ij})_{n\times n}$ and $(b_{ij})_{n\times n}$ as

$$LA_3R:=(c_{ij})_{n\times n},\quad L(R_t+A_1R_x-A_3R_{xx}+A_2R):=(b_{ij})_{n\times n}.$$

Then from (A.1.7), one has the equation for Z_1:

$$Z_{1t}+\tilde{\lambda}_1Z_{1x}=\sum_{j=1}^3c_{1j}Z_{jxx}-\sum_{j=1}^3b_{1j}Z_j+(2LA_3R_xZ_x+LA_4)_1. \tag{A.1.14}$$

Here $(\cdot)_i$ $(i=1,2,3)$ denotes the i-th component of the vector $(\cdot)$. Set

$$h=\frac{1}{\sqrt{16\pi a(1+t)}}\exp\left(-\frac{(x-u_{1\#}t)^2}{16a(1+t)}\right)\quad\text{and}\quad\eta_1=\exp\left(\int_{-\infty}^xh(y,t)\,\mathrm{d}y\right), \tag{A.1.15}$$

with $a = \frac{3\kappa(\theta_\#)}{5\rho_\#} > 0$. Then h satisfies

$$h_t + u_{1\#}h_x = ah_{xx}.$$

Obviously, it holds that $1 \leqq \eta_1 \leqq e$ and

$$\eta_{1t} = \eta_1 \int_{-\infty}^{x} h_t(y,t)\,\mathrm{d}y = \eta_1(ah_x - u_{1\#}h), \quad \eta_{1x} = \eta_1 h.$$

Multiplying (A.1.14) by $\eta_1 Z_1$, we can get

$$\left(\eta_1 \frac{Z_1^2}{2}\right)_t - (\eta_{1t} + \tilde{\lambda}_1 \eta_{1x})\frac{Z_1^2}{2} - \tilde{\lambda}_{1x}\eta_1 \frac{Z_1^2}{2} = -\sum_{j=1}^{3} Z_{jx}(c_{1j}\eta_1 Z_1)_x$$
$$+ \left[-\sum_{j=1}^{3} b_{1j}Z_j + (2LA_3R_xZ_x + LA_4)_1\right]\eta_1 Z_1 + (\cdots)_x. \quad \text{(A.1.16)}$$

Integrating (A.1.16) with respect to x, t and similar to [35], one has

$$\int Z_1^2\,\mathrm{d}x + \int_0^t \int hZ_1^2\,\mathrm{d}x\,\mathrm{d}\tau \le C\int\int \frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$
$$+ C\int_0^t \left[\left\|\sqrt{\left|\lambda_{1x}^{S_1}\right| + \left|\lambda_{3x}^{S_3}\right|}Z\right\|^2 + \|Z_x\|^2\right]\mathrm{d}\tau$$
$$+ \sigma\int_0^t \|Z_{1\tau}\|^2\,\mathrm{d}\tau + C_\sigma \int_0^t\int\int \frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ C(\delta_0 + \chi_T)\int_0^t \|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau$$
$$+ C\chi_T \sum_{|\alpha'|=1}\int_0^t \|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau$$
$$+ C\int_0^t\int |\Theta_x|\left(Z_2^2 + Z_3^2\right)\mathrm{d}x\,\mathrm{d}\tau + C\left(\mathcal{E}(0)^2 + \delta_0^{\frac{1}{2}}\right)$$
$$+ C\chi_T\int_0^t\int\int \frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$
$$+ C\,(\chi_T + \delta_0)\int_0^t \|(\Pi_x, n_2, \Pi_{x\tau})\|^2\,\mathrm{d}\tau, \quad \text{(A.1.17)}$$

with sufficiently small positive constant $\sigma > 0$ and positive constant C_σ. Similarly, one can derive that

$$\int Z_3^2\,\mathrm{d}x + \int_0^t\int hZ_3^2\,\mathrm{d}x\,\mathrm{d}\tau \le C\left(\mathcal{E}(0)^2 + \delta_0^{\frac{1}{2}}\right)$$
$$+ C\int\int \frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + \sigma\int_0^t \|Z_{3\tau}\|^2\,\mathrm{d}\tau$$

$$
\begin{aligned}
&+C\int_0^t\left[\left\|\sqrt{\left|\lambda_{1x}^{S_1}\right|+\left|\lambda_{3x}^{S_3}\right|}Z\right\|^2+\|Z_x\|^2\right]\mathrm{d}\tau+C(\delta_0+\chi_T)\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&+C_\sigma\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&+C\chi_T\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau+C\int_0^t\int|\Theta_x|\left(Z_1^2+Z_2^2\right)\mathrm{d}x\,\mathrm{d}\tau\\
&+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau. \qquad\qquad (A.1.18)
\end{aligned}
$$

Then we estimate $\int_0^t\int|\Theta_x|Z_2^2\,\mathrm{d}x\,\mathrm{d}\tau$. Set

$$
\eta_2(x,t)=\int_{-\infty}^x h(y,t)\mathrm{d}y;
$$

it is easy to check that

$$
\eta_{2t}=ah_x-u_{1\#}h,\quad \|n\|_{L^\infty}=1.
$$

The following lemma is from Huang et al. [11] for the stability of viscous contact wave, which plays an important role for weighted characteristic estimate to $\int_0^t\int|\Theta_x|Z_2^2\,\mathrm{d}x\,\mathrm{d}\tau$:

Lemma 4.3. *For* $0<\tau\le\infty$*, suppose that* $Z(t,x)$ *satisfies*

$$
Z\in L^\infty(0,t;L^2(\mathbf{R})),\quad Z_x\in L^2(0,t;L^2(\mathbf{R})),\quad Z_t\in L^2(0,t;H^{-1}(\mathbf{R})).
$$

Then the following estimate holds for any $\tau\in(0,t]$,

$$
\begin{aligned}
\int_0^t\int h^2Z^2\,\mathrm{d}x\,\mathrm{d}\tau\le{}&\frac{1}{a}\int Z^2(x,0)\,\mathrm{d}x+4\int_0^t\|Z_x\|^2\,\mathrm{d}\tau\\
&+\frac{2}{a}\left(\int_0^t<Z_\tau,Zn^2>\mathrm{d}\tau-u_{1\#}\int_0^t\int Z^2nh\,\mathrm{d}x\,\mathrm{d}\tau\right).
\end{aligned}
$$

From (A.1.7), we have

$$
Z_{2t}+\tilde{\lambda}_2Z_{2x}=\sum_{j=1}^3c_{2j}Z_{jxx}-\sum_{j=1}^3b_{2j}Z_j+(2LA_3R_xZ_x+LA_4)_2,
$$

and we can get that

$$
\begin{aligned}
&\int_0^t < Z_{2\tau}, Z_2 n^2 > \mathrm{d}\tau - u_{1\#}\int_0^t \int Z_2^2 nh \,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^t \int \left\{ \left(\sum_{j=1}^{3} c_{2j} Z_{jxx} - \sum_{j=1}^{3} b_{2j} Z_j + (2LA_3 R_x Z_x + LA_4)_2 \right) Z_2 n^2 \right. \\
&\quad \left. - (\tilde{\lambda}_2 Z_{2x} Z_2 n^2 + u_{1\#} Z_2^2 nh) \right\} \mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Taking $Z = Z_2$ in Lemma 4.3 and using the above equation, one can get

$$
\begin{aligned}
&\int_0^t \int h^2 Z_2^2 \,\mathrm{d}x\,\mathrm{d}\tau \le C_\sigma \int\int \frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + C\int_0^t \left[\left\| \sqrt{\left|\lambda_{1x}^{S_1}\right| + \left|\lambda_{3x}^{S_3}\right|} Z \right\|^2 + \|Z_x\|^2 \right] \mathrm{d}\tau \\
&\quad + \sigma \left[\|Z_2\|^2 + \int_0^t \|Z_{2\tau}\|^2 \,\mathrm{d}\tau \right] + C_\sigma \int_0^t \int\int \frac{\nu(|v|)}{\mathbf{M}_*} |\widetilde{\mathbf{G}}|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C\chi_T \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\phi,\psi,\omega)\|^2 \,\mathrm{d}\tau + C\int_0^t \int |\Theta_x| \left(Z_1^2 + Z_3^2 \right) \mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C\left(\mathcal{E}(0)^2 + \delta_0^{\frac{1}{2}} \right) + C\chi_T \int_0^t \int\int \frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&\quad + C(\delta_0 + \chi_T) \int_0^t \|(\phi,\psi,\omega,\Pi_x,n_2,\Pi_{x\tau})\|^2 \,\mathrm{d}\tau. \qquad \text{(A.1.19)}
\end{aligned}
$$

Combining (A.1.17), (A.1.18) and (A.1.19), noting that (A.1.11) and (A.1.13) and choosing δ_0 and σ sufficiently small, one has

$$
\begin{aligned}
&\|Z(t,\cdot)\|^2 + \int_0^t \left\| \sqrt{\left|\lambda_{1x}^{S_1}\right| + \left|\lambda_{3x}^{S_3}\right| + |\Theta_x|}\, Z \right\|^2 \mathrm{d}\tau \\
&\le C\int\int \frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + C\left(\mathcal{E}(0)^2 + \delta_0^{\frac{1}{2}} \right) \\
&\quad + C(\sigma + \sqrt{\delta_0}) \int_0^t \|(Z_\tau, Z_x)\|^2 \,\mathrm{d}\tau + C\chi_T \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\phi,\psi,\omega)\|^2 \,\mathrm{d}\tau \\
&\quad + C_\sigma \int_0^t \int\int \frac{\nu(|v|)}{\mathbf{M}_*} |\widetilde{\mathbf{G}}|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C(\delta_0 + \chi_T) \int_0^t \|(\phi,\psi,\omega,\Pi_x,n_2,\Pi_{x\tau})\|^2 \,\mathrm{d}\tau \\
&\quad + C\chi_T \int_0^t \int\int \frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau. \qquad \text{(A.1.20)}
\end{aligned}
$$

Noting that Z is a linear vector function of Φ, $\widetilde{\Psi}$, $\widetilde{W}$ and combining (A.1.3) and (A.1.20), we can get

$$
\begin{aligned}
&\|(\Phi,\widetilde{\Psi},\widetilde{W})(t,\cdot)\|^2 \\
&\quad+\int_0^t\left[\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}\,(\Phi,\widetilde{\Psi},\widetilde{W})\right\|^2+\|(\widetilde{\Psi}_x,\widetilde{W}_x)\|^2\right]\mathrm{d}\tau \\
&\leqq C\int\int\frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad+C(\sigma+\sqrt{\delta_0})\int_0^t\|(\Phi,\widetilde{\Psi},\widetilde{W})_\tau\|^2\,\mathrm{d}\tau \\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau \\
&\quad+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau \\
&\quad+C_\sigma\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+\;C(\sigma+\chi_T+\sqrt{\delta_0})\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau \\
&\quad+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau+C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right).
\end{aligned}
$$

Step 4 Estimation on $\|\Phi_x(t,\cdot)\|^2$.
Note that the dissipation term does not contain the term $\|\Phi_x\|^2$. From $(3.40)_2$, we have

$$
\begin{aligned}
&\frac{4}{3}\frac{\mu(\tilde\theta)}{\tilde\rho}\Phi_{xt}+\tilde\rho\widetilde{\Psi}_{1t}+\tilde\rho\tilde u_1\widetilde{\Psi}_{1x}+\frac{2}{3}\tilde\theta\Phi_x=-\frac{4\mu(\tilde\theta)}{3\tilde\rho}(2\tilde\rho_x\widetilde{\Psi}_{1x}+\tilde\rho_{xx}\widetilde{\Psi}_1)-\frac{4\mu(\tilde\theta)}{3\tilde\rho}(\tilde u_1\Phi)_{xx} \\
&+\frac{1}{3}\tilde\rho\tilde u_{1x}\widetilde{\Psi}_1-\frac{2}{3}\tilde\rho_x\widetilde{W}-\frac{2}{3}\tilde\rho\widetilde{W}_x+\frac{2\tilde\theta\tilde\rho_x}{3\tilde\rho}\Phi-\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v+J_1+N_1-Q_1.
\end{aligned}
\tag{A.1.21}
$$

Multiplying (A.1.21) by Φ_x yields that

$$
\begin{aligned}
&\left(\frac{2\mu(\tilde\theta)}{3\tilde\rho}\Phi_x^2+\tilde\rho\Phi_x\widetilde{\Psi}_1\right)_t+\frac{2\tilde\theta}{3}\Phi_x^2=\left(\frac{2\mu(\tilde\theta)}{3\tilde\rho}\right)_t\Phi_x^2+(\tilde\rho\widetilde{\Psi}_1)_x^2+(\tilde\rho\widetilde{\Psi}_1)_x\tilde u_{1x}\Phi \\
&\quad+\Big[-\frac{4\mu(\tilde\theta)}{3\tilde\rho}(2\tilde\rho_x\widetilde{\Psi}_{1x}+\tilde\rho_{xx}\widetilde{\Psi}_1)-\frac{4\mu(\tilde\theta)}{3\tilde\rho}(\tilde u_1\Phi)_{xx}-\frac{2}{3}\tilde\rho\tilde u_{1x}\widetilde{\Psi}_1-\frac{2}{3}\tilde\rho_x\widetilde{W}-\frac{2}{3}\tilde\rho\widetilde{W}_x \\
&\quad+\frac{2\tilde\theta\tilde\rho_x}{3\tilde\rho}\Phi-\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v+J_1+N_1-Q_1\Big]\Phi_x+(\cdots)_x.
\end{aligned}
\tag{A.1.22}
$$

By using Lemmas 2.1, 2.2 and 2.3, one has

$$
\begin{aligned}
&\int_0^t\int\left|\int v_1^2(\Gamma-\Gamma^{S_1}-\Gamma^{S_3})\,\mathrm{d}v\right|^2\mathrm{d}x\,\mathrm{d}\tau\leqq C\delta_0^2+C\delta_0\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau \\
&\quad+C(\delta_0+\chi_T)\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^{\alpha'}\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

$$+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau. \tag{A.1.23}$$

Integrating (A.1.22) with respect to x and t, which together with Hölder's inequality and (A.1.23) gives that

$$\|\Phi_x(t,\cdot)\|^2+\int_0^t\|\Phi_x\|^2\,\mathrm{d}\tau\leqq C\|\widetilde{\Psi}_1(t,\cdot)\|^2+C\int_0^t\|(\widetilde{\Psi}_x,\widetilde{W}_x)\|^2\,\mathrm{d}\tau$$
$$+C\chi_T\int_0^t\|(\phi_x,\psi_{1x})\|^2\,\mathrm{d}\tau$$
$$+C\delta_0\int_0^t\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}(\Phi,\widetilde{\Psi}_1,\widetilde{W})\right\|^2\,\mathrm{d}\tau$$
$$+C\chi_T\int_0^t\int\int\frac{\nu(|v|)|\partial_{v_1}(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\int_0^t\|(\psi,\omega,\Pi_x,n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau$$
$$+C(\delta_0+\chi_T)\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right)$$
$$+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^{\alpha'}\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

Step 5 Estimation on the non-fluid component.

Next we do the microscopic estimates for the Vlasov–Poisson–Boltzmann system. Multiplying the equation (3.44) and the equation (3.10) by $\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}$ and $\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}$, respectively, one has

$$\left(\frac{|\widetilde{\mathbf{G}}|^2}{2\mathbf{M}_*}\right)_t-\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}}=\Bigg\{-\mathbf{P}_1(v_1\widetilde{\mathbf{G}}_x)-\mathbf{P}_1(\Pi_x\partial_{v_1}F_2)+2[Q(\mathbf{G}^{S_1},\mathbf{G}^{S_3})$$
$$+Q(\mathbf{G}^{S_3},\mathbf{G}^{S_1})]+2[Q(\widetilde{\mathbf{G}},\mathbf{G}^{S_1}+\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_1}+\mathbf{G}^{S_3},\widetilde{\mathbf{G}})]$$
$$-\left[\mathbf{P}_1(v_1\mathbf{M}_x)-\mathbf{P}_1^{S_1}\left(v_1\mathbf{M}_x^{S_1}\right)-\mathbf{P}_1^{S_3}\left(v_1\mathbf{M}_x^{S_3}\right)\right]$$
$$+2Q(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}})+\sum_{j=1,3}R_j\Bigg\}\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}. \tag{A.1.24}$$

and

$$\left(\frac{|\mathbf{P}_c F_2|^2}{2\mathbf{M}_*}\right)_t - \frac{\mathbf{P}_c F_2}{\mathbf{M}_*}\mathbf{N}_{\mathbf{M}}(\mathbf{P_c} F_2)$$
$$= \left[- v_1 \partial_x F_2 - \left(\frac{n_2}{\rho}\mathbf{M}\right)_t - \mathbf{P}_c(\Pi_x \partial_{v_1} F_1) + 2Q(F_2, \mathbf{G})\right]\frac{\mathbf{P}_c F_2}{\mathbf{M}_*}, \tag{A.1.25}$$

where in the equation (A.1.25), one has used the fact that

$$-\mathbf{P}_c(v_1\partial_x F_2) - \left(\frac{\mathbf{M}}{\rho}\right)_t n_2 = -v_1\partial_x F_2 + \mathbf{P}_d(v_1\partial_x F_2) - \left(\frac{\mathbf{M}}{\rho}\right)_t n_2$$
$$= -v_1\partial_x F_2 + \frac{\mathbf{M}}{\rho}\left(\int v_1 F_2 \, \mathrm{d}v\right)_x - \left(\frac{\mathbf{M}}{\rho}\right)_t n_2$$
$$= -v_1\partial_x F_2 - \frac{\mathbf{M}}{\rho} n_{2t} - \left(\frac{\mathbf{M}}{\rho}\right)_t n_2 = -v_1\partial_x F_2 - \left(\frac{n_2}{\rho}\mathbf{M}\right)_t.$$

It can be computed that

$$\mathbf{P}_1(\Pi_x\partial_{v_1} F_2) = \Pi_x \mathbf{P}_1\left[\partial_{v_1}\left(\frac{n_2}{\rho}\mathbf{M} + \mathbf{P}_c F_2\right)\right] = \Pi_x\mathbf{P}_1\left[\partial_{v_1}\left(\mathbf{P}_c F_2\right)\right]$$
$$= \Pi_x\left(\mathbf{P}_c F_2\right)_{v_1} - \Pi_x \sum_{j=0}^{4}\int \left(\mathbf{P}_c F_2\right)_{v_1}\frac{\chi_j}{\mathbf{M}}\,\mathrm{d}v\chi_j = \Pi_x\left(\mathbf{P}_c F_2\right)_{v_1}$$
$$+ \Pi_x \sum_{j=0}^{4}\int (\mathbf{P}_c F_2)\left(\frac{\chi_j}{\mathbf{M}}\right)_{v_1}\mathrm{d}v\chi_j, \tag{A.1.26}$$

and

$$\mathbf{P}_c(\Pi_x\partial_{v_1} F_1) = \Pi_x\partial_{v_1} F_1 - \Pi_x\mathbf{P}_d\left(\partial_{v_1} F_1\right) = \Pi_x\partial_{v_1} F_1$$
$$= \Pi_x \mathbf{M}_{v_1} + \Pi_x \widetilde{\mathbf{G}}_{v_1} + \Pi_x\left(\mathbf{G}^{S_1}_{v_1} + \mathbf{G}^{S_3}_{v_1}\right). \tag{A.1.27}$$

Substituting (A.1.26) and (A.1.27) into (A.1.24) and (A.1.25), respectively, then summing the resulting equations together and noting the cancelation

$$\Pi_x\left(\mathbf{P}_c F_2\right)_{v_1}\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*} + \Pi_x\widetilde{\mathbf{G}}_{v_1}\frac{\mathbf{P}_c F_2}{\mathbf{M}_*} = \left(\Pi_x\frac{\mathbf{P}_c F_2\cdot\widetilde{\mathbf{G}}}{\mathbf{M}_*}\right)_{v_1} + \Pi_x\frac{\mathbf{P}_c F_2\cdot\widetilde{\mathbf{G}}}{\mathbf{M}_*^2}\left(\mathbf{M}_*\right)_{v_1},$$

it holds that

$$\left(\frac{|\widetilde{\mathbf{G}}|^2 + |\mathbf{P}_c F_2|^2}{2\mathbf{M}_*}\right)_t - \frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}} - \frac{\mathbf{P}_c F_2}{\mathbf{M}_*}\mathbf{N}_{\mathbf{M}}(\mathbf{P_c} F_2) + (\cdots)_{v_1}$$
$$= \Big\{ - \mathbf{P}_1(v_1\widetilde{\mathbf{G}}_x) + 2Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}) + 2[Q(\mathbf{G}^{S_1}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_1})]$$
$$+ 2[Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_1} + \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_1} + \mathbf{G}^{S_3}, \widetilde{\mathbf{G}})] + \sum_{j=1,3} R_j$$

$$
\begin{aligned}
&-\Big[\mathbf{P}_1(v_1\mathbf{M}_x)-\mathbf{P}_1^{S_1}\left(v_1\mathbf{M}_x^{S_1}\right)-\mathbf{P}_1^{S_3}\left(v_1\mathbf{M}_x^{S_3}\right)\Big]\\
&-\Pi_x\sum_{j=0}^{4}\int(\mathbf{P}_cF_2)\left(\frac{\chi_j}{\mathbf{M}}\right)_{v_1}\mathrm{d}v\chi_j\bigg\}\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}\\
&+\bigg[-v_1\partial_xF_2-\left(\frac{n_2}{\rho}\mathbf{M}\right)_t+2Q(F_2,\mathbf{G})-\Pi_x\mathbf{M}_{v_1}-\Pi_x\left(\mathbf{G}_{v_1}^{S_1}+\mathbf{G}_{v_1}^{S_3}\right)\bigg]\\
&\times\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}+\Pi_x\frac{\mathbf{P}_cF_2\cdot\widetilde{\mathbf{G}}}{\mathbf{M}_*^2}\left(\mathbf{M}_*\right)_{v_1}.
\end{aligned}
$$

Integrating the above equation with respect to x, t, v yields that

$$
\begin{aligned}
&\int\int\frac{|\widetilde{\mathbf{G}}|^2+|\mathbf{P}_cF_2|^2}{2\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v-\int\int\frac{|\widetilde{\mathbf{G}}|^2+|\mathbf{P}_cF_2|^2}{2\mathbf{M}_*}(x,v,0)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad-\int_0^t\int\int\left[\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}}+\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}\mathbf{N}_{\mathbf{M}}(\mathbf{P_c}F_2)\right]\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&=-\int_0^t\int\int\bigg\{\bigg[-\mathbf{P}_1(v_1\widetilde{\mathbf{G}}_x)+2Q(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}})\\
&\quad+2[Q(\mathbf{G}^{S_1},\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_3},\mathbf{G}^{S_1})]\\
&\quad+2[Q(\widetilde{\mathbf{G}},\mathbf{G}^{S_1}+\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_1}+\mathbf{G}^{S_3},\widetilde{\mathbf{G}})]+\sum_{j=1,3}R_j\\
&\quad-\Big[\mathbf{P}_1(v_1\mathbf{M}_x)-\mathbf{P}_1^{S_1}(v_1\mathbf{M}_x^{S_1})-\mathbf{P}_1^{S_3}(v_1\mathbf{M}_x^{S_3})\Big]\\
&\quad-\Pi_x\sum_{j=0}^{4}\int(\mathbf{P}_cF_2)\left(\frac{\chi_j}{\mathbf{M}}\right)_{v_1}\mathrm{d}v\chi_j\bigg]\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}+\bigg[-v_1\partial_xF_2\\
&\quad-\left(\frac{n_2}{\rho}\mathbf{M}\right)_\tau+2Q(F_2,\mathbf{G})-\Pi_x\mathbf{M}_{v_1}-\Pi_x(\mathbf{G}_{v_1}^{S_1}+\mathbf{G}_{v_1}^{S_3})\bigg]\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}\\
&\quad+\Pi_x\frac{\mathbf{P}_cF_2\cdot\widetilde{\mathbf{G}}}{\mathbf{M}_*^2}\left(\mathbf{M}_*\right)_{v_1}\bigg\}\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&:=\sum_{i=1}^{13}Y_i.
\end{aligned}
\tag{A.1.28}
$$

Now we calculate the right hand side of (A.1.28). The estimations of Y_i ($i = 1, 2, \cdots 7, 10, 12, 13$) are standard and will be skipped for brevity. Noting that

$$
\begin{aligned}
-v_1\partial_xF_2\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}&=-v_1\partial_x\left(\frac{n_2}{\rho}\mathbf{M}+\mathbf{P}_cF_2\right)\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}\\
&=-v_1\left(\frac{n_2}{\rho}\mathbf{M}\right)_x\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}-\left(v_1\frac{|\mathbf{P}_cF_2|^2}{2\mathbf{M}_*}\right)_x
\end{aligned}
$$

$$
\begin{aligned}
&= -v_1 \frac{n_{2x}\mathbf{M}}{\rho}\frac{\mathbf{P}_c F_2}{\mathbf{M}_*} - n_2 v_1 \left(\frac{\mathbf{M}}{\rho}\right)_x \frac{\mathbf{P}_c F_2}{\mathbf{M}_*} + (\cdots)_x \\
&= -\frac{n_{2x}}{\rho} v_1 \left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2 - n_2 v_1 \left(\frac{\mathbf{M}}{\rho}\right)_x \frac{\mathbf{P}_c F_2}{\mathbf{M}_*} - \frac{n_{2x}}{\rho} v_1 \mathbf{P}_c F_2 + (\cdots)_x,
\end{aligned}
$$

it follows from (A.7) that

$$
\begin{aligned}
Y_8 &= -\int_0^t \int\int v_1 \partial_x F_2 \frac{\mathbf{P}_c F_2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&= \int_0^t \int\int \Big[-\frac{n_{2x}}{\rho} v_1 \left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2 - n_2 v_1 \left(\frac{\mathbf{M}}{\rho}\right)_x \frac{\mathbf{P}_c F_2}{\mathbf{M}_*} \\
&\quad - \frac{n_{2x}}{\rho} v_1 \mathbf{P}_c F_2 \Big]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^t \int\int \Big[-\frac{n_{2x}}{\rho} v_1 \left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2 - n_2 v_1 \left(\frac{\mathbf{M}}{\rho}\right)_x \frac{\mathbf{P}_c F_2}{\mathbf{M}_*} \Big]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t \int \frac{n_{2x}}{\rho}\left(\frac{1}{2}\Pi_{x\tau} + u_1 n_2\right)\mathrm{d}x\,\mathrm{d}\tau. \qquad \text{(A.1.29)}
\end{aligned}
$$

By integration by parts, we can compute that

$$
\begin{aligned}
&\int_0^t \int \frac{n_{2x}}{\rho}\left(\frac{1}{2}\Pi_{x\tau} + u_1 n_2\right)\mathrm{d}x\,\mathrm{d}\tau \\
&= -\int_0^t \int \left[\frac{n_2}{\rho} n_{2\tau} + \frac{1}{2} n_2 \Pi_{x\tau}\left(\frac{1}{\rho}\right)_x + \left(\frac{u_1}{2\rho}\right)_x n_2^2\right]\mathrm{d}x\,\mathrm{d}\tau \\
&= -\int \frac{n_2^2}{2\rho}(x,t)\,\mathrm{d}x + \int \frac{n_{20}^2}{2\rho_0}\,\mathrm{d}x \\
&\quad - \int_0^t \int \left[\left(\frac{1}{2\rho}\right)_\tau n_2^2 + \frac{1}{2} n_2 \Pi_{x\tau}\left(\frac{1}{\rho}\right)_x + \left(\frac{u_1}{2\rho}\right)_x n_2^2\right]\mathrm{d}x\,\mathrm{d}\tau. \qquad \text{(A.1.30)}
\end{aligned}
$$

Substituting (A.1.30) into (A.1.29) and estimating the other terms yields that

$$
\begin{aligned}
Y_8 + \int \frac{n_2^2}{2\rho}(x,t)\,\mathrm{d}x &\leqq C\|n_{20}\|^2 + C(\chi_T + \delta_0) \\
&\quad \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0)\int_0^t \|(n_2, \Pi_{x\tau})\|^2\,\mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0 + \eta_0)\int_0^t \left[\|n_{2x}\|^2 + \int\int \frac{\nu(|v|)|\mathbf{P}_c F_2|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right]\mathrm{d}\tau.
\end{aligned}
$$

Similarly, it holds that

$$
\begin{aligned}
|Y_9| &= |\int_0^t \int\int \left[\frac{n_{2\tau}}{\rho}\frac{\mathbf{M}}{\mathbf{M}_*}\mathbf{P}_c F_2 + n_2\left(\frac{\mathbf{M}}{\rho}\right)_\tau \frac{\mathbf{P}_c F_2}{\mathbf{M}_*}\right] \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau| \\
&= |\int_0^t \int\int \left[\frac{n_{2\tau}}{\rho}\left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2 + n_2\left(\frac{\mathbf{M}}{\rho}\right)_t \frac{\mathbf{P}_c F_2}{\mathbf{M}_*}\right] \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau| \\
&\leqq C(\chi_T + \delta_0 + \eta_0)\int_0^t \left[\|n_{2\tau}\|^2 + \int\int \frac{\nu(|v|)|\mathbf{P}_c F_2|^2}{\mathbf{M}_*}\, \mathrm{d}x\, \mathrm{d}v\right] \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0)\int_0^t \|(\Pi_{x\tau}, n_2)\|^2\, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0)\sum_{|\alpha'|=1}\int_0^t \|\partial^{\alpha'}(\phi, \psi, \omega)\|^2\, \mathrm{d}\tau.
\end{aligned}
$$

Again it follows from (A.7) that

$$
\begin{aligned}
Y_{11} &= \int_0^t \int\int \Pi_x \mathbf{M}\frac{v_1 - u_1}{R\theta}\frac{\mathbf{P}_c F_2}{\mathbf{M}_*}\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau \\
&= \int_0^t \int\int \Pi_x \frac{v_1 - u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau \\
&\quad + \int_0^t \int\int \Pi_x \frac{v_1 - u_1}{R\theta}\mathbf{P}_c F_2\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau \\
&= \int_0^t \int\int \Pi_x \frac{v_1 - u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau \\
&\quad + \int_0^t \int \frac{\Pi_x}{R\theta}\left(\int v_1 \mathbf{P}_c F_2\, \mathrm{d}v\right) \mathrm{d}x\, \mathrm{d}\tau \\
&= \int_0^t \int\int \Pi_x \frac{v_1 - u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*} - 1\right)\mathbf{P}_c F_2\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}\tau \\
&\quad - \int_0^t \int \frac{\Pi_x}{R\theta}\left(\frac{1}{2}\Pi_{xt} + u_1 n_2\right) \mathrm{d}x\, \mathrm{d}\tau \\
&\leqq -\int \frac{\Pi_x^2}{4R\theta}\, \mathrm{d}x + C\|\Pi_{x0}\|^2 + C(\chi_T + \delta_0 + \eta_0) \\
&\quad \int_0^t \left[\|\Pi_x\|^2 + \int\int \frac{\nu(|v|)|\mathbf{P}_c F_2|^2}{\mathbf{M}_*}\, \mathrm{d}x\, \mathrm{d}v\right] \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0)\int_0^t \|(\Pi_x, \Pi_{x\tau}, n_2, \omega_\tau, \psi_{1x}, \omega_x)\|^2\, \mathrm{d}\tau,
\end{aligned}
$$

where in the last inequality one has used the fact that

$$
\begin{aligned}
-\int_0^t \int \frac{\Pi_x}{R\theta}\left(\frac{1}{2}\Pi_{x\tau} + u_1 n_2\right) \mathrm{d}x\, \mathrm{d}\tau &= -\int \frac{\Pi_x^2}{4R\theta}\, \mathrm{d}x + \int \frac{\Pi_{x0}^2}{4R\theta_0}\, \mathrm{d}x \\
&\quad + \int_0^t \int \left[\left(\frac{1}{4R\theta}\right)_\tau + \left(\frac{u_1}{4R\theta}\right)_x\right]\Pi_x^2\, \mathrm{d}x\, \mathrm{d}\tau
\end{aligned}
$$

$$\leqq -\int \frac{\Pi_x^2}{4R\theta}\,\mathrm{d}x + C\|\Pi_{0x}\|^2 + C(\chi_T+\delta_0)$$
$$\int_0^t \|(\Pi_x, \Pi_{x\tau}, n_2, \omega_\tau, \psi_{1x}, \omega_x)\|^2\,\mathrm{d}\tau.$$

Substituting the above estimations into (A.1.28), then choosing χ_T, δ, η_0 sufficiently small imply that

$$\begin{aligned}
&\int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v + \|(\Pi_x, n_2)(\cdot,t)\|^2 \\
&\quad + \int_0^t\int\int \frac{\nu(|v|)\left|\left(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2\right)\right|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqq C\int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x,v,0)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + C\|(\Pi_{x0}, n_{20})\|^2 + C\int_0^t \|(\phi_x, \psi_x, \omega_x)\|^2\,\mathrm{d}\tau + C\delta_0^{\frac{1}{2}} \\
&\quad + C(\chi_T+\delta_0)\int_0^t \left[\|(\Pi_{x\tau}, n_2)\|^2 + \sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\omega)\|^2\right]\mathrm{d}\tau \\
&\quad + C(\chi_T+\delta_0+\eta_0)\int_0^t \|(\Pi_x, n_{2x}, n_{2\tau})\|^2\,\mathrm{d}\tau \\
&\quad + C\sum_{|\alpha'|=1}\int_0^t\int\int \frac{\nu(|v|)|\partial^{\alpha'}\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{A.1.31}$$

Step 6 Estimation of electric field terms.
Now we estimate the Poisson term in the electric fields, which is one of the key ingredients of the present paper. Multiplying the equation (3.9) by $-\Pi$ yields that

$$\begin{aligned}
&\left(\frac{\Pi_x^2}{2}\right)_t + \frac{3\kappa_1(\theta)}{2\theta}\Pi_x^2 + \frac{2\kappa_1(\theta)}{\rho}n_2^2 - \frac{1}{4}u_{1x}\Pi_x^2 \\
&= (\cdots)_x - \Pi_x\frac{n_2}{\rho}\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1\mathbf{M}_x)\Big]\mathrm{d}v \\
&\quad - \Pi_x\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1(\mathbf{P}_cF_2)_x)\Big]\mathrm{d}v - \Pi_x^2\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{G}_{v_1}\Big]\mathrm{d}v \\
&\quad - \Pi_x\left(\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\left[\partial_t(\mathbf{P}_cF_2) + \left(\frac{\mathbf{M}}{\rho}\right)_t n_2 - 2Q(F_2,\mathbf{G})\right]\mathrm{d}v\right).
\end{aligned}$$

Integrating the above equation with respect to x, t implies that

$$\|\Pi_x\|^2(t) - \|\Pi_x\|^2(0) + \int_0^t \|(\Pi_x, n_2)\|^2\,\mathrm{d}\tau \leqq C(\chi_T+\delta)$$
$$\int_0^t \|(\Pi_x, n_2, \psi_{1x}, \omega_x)\|^2\,\mathrm{d}\tau$$

$$+ C\int_0^t \int |\Pi_x \frac{n_2}{\rho} \int v_1 \mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1\mathbf{M}_x))\Big] \mathrm{d}v| \,\mathrm{d}x \,\mathrm{d}\tau$$

$$+ C\int_0^t \int |\Pi_x \int v_1 \mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1(\mathbf{P}_c F_2)_x)\Big] \mathrm{d}v| \,\mathrm{d}x \,\mathrm{d}\tau$$

$$+ C\int_0^t \int |\Pi_x^2 \int v_1 \mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{G}_{v_1}\Big] \mathrm{d}v| \,\mathrm{d}x \,\mathrm{d}\tau$$

$$+ C\int_0^t \int |\Pi_x \int v_1 \mathbf{N}_{\mathbf{M}}^{-1}\Big[\partial_\tau(\mathbf{P}_c F_2) + \Big(\frac{\mathbf{M}}{\rho}\Big)_\tau n_2\Big] \mathrm{d}v| \,\mathrm{d}x \,\mathrm{d}\tau$$

$$+ C\int_0^t \int |\Pi_x \int v_1 \mathbf{N}_{\mathbf{M}}^{-1} Q(F_2, \mathbf{G}) \,\mathrm{d}v| \,\mathrm{d}x \,\mathrm{d}\tau$$

$$:= C(\chi_T + \delta_0)\int_0^t \|(\Pi_x, n_2, \psi_{1x}, \omega_x)\|^2 \,\mathrm{d}\tau + \sum_{i=1}^5 T_i. \tag{A.1.32}$$

Now we estimate T_i $(i = 1, 2, 3, 4, 5)$ in (A.1.32) one by one. First, it holds that

$$\begin{aligned}
T_1 &\leqq C\int_0^t \int |n_2||\Pi_x|\Big(\int \frac{\nu(|v|)|\mathbf{N}_{\mathbf{M}}^{-1}[\mathbf{P}_c(v_1\mathbf{M}_x)]|^2}{\mathbf{M}_*} \mathrm{d}v\Big)^{\frac{1}{2}} \\
&\quad \Big(\int \nu(|v|)^{-1} v_1^2 \mathbf{M}_* \,\mathrm{d}v\Big)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}\tau \\
&\leqq C\int_0^t \int |n_2||\Pi_x|\Big(\int \frac{\nu(|v|)^{-1}|\mathbf{P}_c(v_1\mathbf{M}_x)|^2}{\mathbf{M}_*} \mathrm{d}v\Big)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}\tau \\
&\leqq C\int_0^t \int |n_2||\Pi_x|\Big(\int \frac{\nu(|v|)^{-1}|v_1\mathbf{M}_x - \frac{\mathbf{M}}{\rho}(\rho u_1)_x|^2}{\mathbf{M}_*} \mathrm{d}v\Big)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}\tau \\
&\leqq C\int_0^t \int |n_2||\Pi_x||(\rho_x, u_x, \theta_x)| \,\mathrm{d}x \,\mathrm{d}\tau \leqq C(\chi_T + \delta_0) \\
&\quad \int_0^t \Big[\|(n_2, \Pi_x)\|^2 + \|(\phi_x, \psi_x, \omega_x)\|^2\Big] \mathrm{d}x \,\mathrm{d}\tau,
\end{aligned} \tag{A.1.33}$$

and

$$\begin{aligned}
T_2 &\leqq C\int_0^t \int |\Pi_x|\Big(\int \frac{\nu(|v|)|\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1(\mathbf{P}_c F_2)_x)\Big]|^2}{\mathbf{M}_{[\rho_*, u_*, 2\theta_*]}} \mathrm{d}v\Big)^{\frac{1}{2}} \\
&\quad \Big(\int \nu(|v|)^{-1} v_1^2 \mathbf{M}_{[\rho_*, u_*, 2\theta_*]} \,\mathrm{d}v\Big)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}\tau \\
&\leqq C\int_0^t \int |\Pi_x|\Big(\int \frac{\nu(|v|)^{-1}|\mathbf{P}_c(v_1(\mathbf{P}_c F_2)_x)|^2}{\mathbf{M}_{[\rho_*, u_*, 2\theta_*]}} \mathrm{d}v\Big)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}\tau
\end{aligned}$$

$$\leqq C\int_0^t\int|\Pi_x|\Big(\int\frac{\nu(|v|)^{-1}|(\mathbf{P}_cF_2)_x)|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqq \frac{1}{8}\int_0^t\|\Pi_x\|^2\,\mathrm{d}\tau+C\int_0^t\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_x)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.$$

Then one has

$$\begin{aligned}
T_3 &\leqq C\int_0^t\int|\Pi_x|^2\Big(\int\frac{\nu(|v|)^{-1}|\mathbf{G}_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C(\chi_T+\delta_0)\int_0^t\Big[\|\Pi_x\|^2+\int\int\frac{\nu(|v|)^{-1}|\widetilde{\mathbf{G}}_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\Big]\,\mathrm{d}\tau,\\
T_4 &\leqq C\int_0^t\int|\Pi_x|\Big(\int\frac{\nu(|v|)^{-1}|\partial_\tau(\mathbf{P}_cF_2)+(\frac{\mathbf{M}}{\rho})_\tau\ n_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C(\delta_0+\chi_T)\int_0^t\Big[\|(\Pi_x,\Pi_{x\tau},n_2)\|^2+\|(\phi_\tau,\psi_\tau,\omega_\tau)\|^2\Big]\,\mathrm{d}\tau\\
&\quad+\frac{1}{8}\int\|\Pi_x\|^2\,\mathrm{d}\tau+C\int_0^t\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_t|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned}$$

and

$$\begin{aligned}
T_5 &\leqq C\int_0^t\int|\Pi_x|\Big(\int\frac{\nu(|v|)^{-1}|Q(F_2,\mathbf{G})|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C\int_0^t\int|\Pi_x||n_2|\Big(\int\frac{\nu(|v|)|\mathbf{G}|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\int_0^t\int|\Pi_x|\Big(\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\Big(\int\frac{|\mathbf{G}|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\int_0^t\int|\Pi_x|\Big(\int\frac{|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\Big(\int\frac{\nu(|v|)|\mathbf{G}|^2}{\mathbf{M}_*}\,\mathrm{d}v\Big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C(\chi_T+\delta_0)\Big[\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau+\int_0^t\int\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^t\int\int\frac{\nu(|v|)|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\Big].
\end{aligned}\tag{A.1.34}$$

Substituting (A.1.33)–(A.1.34) into (A.1.32) and choosing χ_T,δ sufficiently small yields that

$$\|\Pi_x\|^2(t)+\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau\leqq C\|\Pi_x\|^2(0)+C(\chi_T+\delta_0)\int_0^t\|\Pi_{x\tau}\|^2\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau$$

$$+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\alpha'}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+C(\chi_T+\delta_0)\Big[\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+\int_0^t\int\int\frac{\nu(|v|)|\widetilde{\mathbf{G}}_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\Big] \tag{A.1.35}$$

Multiplying the equation (3.9) by n_2 yields that

$$\Big(\frac{n_2^2}{2}\Big)_t+\frac{3\kappa_1(\theta)}{2\theta}n_2^2+\Big(\frac{3\kappa_1(\theta)}{2\theta}\Big)_x\Pi_xn_2$$
$$-\Big(\kappa_1(\theta)\Big(\frac{n_2}{\rho}\Big)_xn_2\Big)_x+\kappa_1(\theta)\Big(\frac{n_2}{\rho}\Big)_xn_{2x}+(u_1n_2^2)_x$$
$$-u_1n_2n_{2x}=(\cdots)_x+n_{2x}\frac{n_2}{\rho}\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1\mathbf{M}_x))\Big]\mathrm{d}v$$
$$+n_{2x}\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1(\mathbf{P}_cF_2)_x)\Big]\mathrm{d}v$$
$$+n_{2x}\Pi_x\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{G}_{v_1}\Big]\mathrm{d}v$$
$$+n_{2x}\Big(\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\partial_t(\mathbf{P}_cF_2)+\Big(\frac{\mathbf{M}}{\rho}\Big)_tn_2-2Q(F_2,\mathbf{G})\Big]\mathrm{d}v\Big).$$

Integrating the above equation with respect to x, t, one has

$$\|n_2\|^2(t)+\int_0^t\|(n_{2x},n_2)\|^2\,\mathrm{d}\tau\leqq C\|n_{20}\|^2$$
$$+C(\chi_T+\delta_0)\int_0^t\|\Pi_x\|^2\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau$$
$$+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\alpha'}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\Big[\int_0^t\int\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}}_{v_1})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\Big]. \tag{A.1.36}$$

By (A.9), it holds that

$$\int_0^t\|\Pi_{x\tau}\|^2\,\mathrm{d}\tau\leqq C\int_0^t\Big[\|n_2\|^2+\int\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\Big]\mathrm{d}\tau. \tag{A.1.37}$$

By the equation (3.8), one has

$$\begin{aligned}
\int_0^t \|n_{2\tau}\|^2 \,\mathrm{d}\tau &= \int_0^t \|\int v_1 F_{2x}\,\mathrm{d}v\|^2 \,\mathrm{d}\tau \\
&= \int_0^t \|\int v_1 \left(\frac{\mathbf{M}}{\rho} n_2 + \mathbf{P}_c F_2\right)_x \mathrm{d}v\|^2 \,\mathrm{d}\tau \\
&\leqq C \int_0^t \left[\|n_{2x}\|^2 + \int\int \frac{\nu(|v|)|(\mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right]\mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0)\int_0^t \|(n_2, \phi_x, \psi_x, \omega_x)\|^2 \,\mathrm{d}\tau. \qquad \text{(A.1.38)}
\end{aligned}$$

On the other hand, from (3.37), (3.38) and (3.39), it holds that

$$\begin{aligned}
\int_0^t \|(\phi, \psi, \omega)\|^2 \,\mathrm{d}\tau &\leqq C \int_0^t \|(\phi, \tilde{\psi}, \tilde{\omega})\|^2 \,\mathrm{d}\tau \\
&\quad + C\delta_0 \int_0^t \left\|\sqrt{\left|u_{1x}^{S_1}\right| + \left|u_{1x}^{S_3}\right| + |\Theta_x|}(\Phi, \widetilde{\Psi}, \widetilde{W})\right\|^2 \,\mathrm{d}\tau \\
&\quad + \int_0^t \int q|(\Phi, \widetilde{\Psi}, \widetilde{W})|^2 \,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|(\phi_x, \psi_x, \omega_x)\|^2 \,\mathrm{d}\tau &\leqq C \int_0^t \|(\phi_x, \tilde{\psi}_x, \tilde{\omega}_x)\|^2 \,\mathrm{d}\tau + C\delta_0 \int_0^t \|(\phi, \tilde{\psi}, \tilde{\omega})\|^2 \,\mathrm{d}\tau \\
&\quad + C\delta_0 \int_0^t \left\|\sqrt{\left|u_{1x}^{S_1}\right| + \left|u_{1x}^{S_3}\right| + |\Theta_x|}(\Phi, \widetilde{\Psi}, \widetilde{W})\right\|^2 \,\mathrm{d}\tau \\
&\quad + \int_0^t \int q|(\Phi, \widetilde{\Psi}, \widetilde{W})|^2 \,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

On the other hand, from the fluid-type system (3.40), we can get

$$\begin{aligned}
\int_0^t \|(\Phi_\tau, \widetilde{\Psi}_\tau, \widetilde{W}_\tau)\|^2 \,\mathrm{d}\tau &\leqq C \int_0^t \|(\Phi_{xx}, \widetilde{\Psi}_{xx}, \widetilde{W}_{xx}, \Phi_x, \widetilde{\Psi}_x, \widetilde{W}_x)\|^2 \,\mathrm{d}\tau \\
&\quad + C\delta_0 \int_0^t \left\|\sqrt{\left|u_{1x}^{S_1}\right| + \left|u_{1x}^{S_3}\right| + |\Theta_x|}(\Phi, \widetilde{\Psi}, \widetilde{W})\right\|^2 \,\mathrm{d}\tau \\
&\quad + C(\delta_0 + \chi_T)\int_0^t \int\int \frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C(\delta_0 + \chi_T)\int_0^t \|\Pi_x\|^2 \,\mathrm{d}\tau \\
&\quad + C \sum_{|\alpha'|=1} \int_0^t \int\int \frac{\nu(|v|)}{\mathbf{M}_*}|\partial^{\alpha'}\widetilde{\mathbf{G}}|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau + C\delta_0^2.
\end{aligned}$$

In summary, collecting all the above lower order estimates and choosing suitably small χ_T, δ and η_0, we complete the proof of Proposition 3.1.

Appendix A.2: Proof of Proposition 3.2: Higher Order Estimates

Step 1 Estimation on $\|(\phi, \tilde{\psi}, \tilde{\omega})\|^2$.

Similar to the lower order estimates (A.1.1), we multiply $(4.16)_1$ by $\frac{2\phi}{3\tilde{\rho}}$, $(4.16)_2$ by $\frac{\tilde{\psi}_1}{\tilde{\theta}}$, $(4.16)_3$ by $\tilde{\psi}_i$, $(4.16)_4$ by $\frac{\tilde{\omega}}{\tilde{\theta}^2}$ respectively and adding them together to get

$$\begin{aligned}
&\|(\phi, \tilde{\psi}, \tilde{\omega})(t, \cdot)\|^2 + \int_0^t \|(\tilde{\psi}_x, \tilde{\omega}_x)\|^2 \, \mathrm{d}\tau \leqq C\left(\mathcal{E}(0)^2 + \delta_0^{\frac{1}{2}}\right) \\
&\quad + \int_0^t \int q|(\Phi, \widetilde{\Psi}, \widetilde{W})|^2 \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + C\delta_0 \int_0^t \left\| \sqrt{\left|u_{1x}^{S_1}\right| + \left|u_{1x}^{S_3}\right| + |\Theta_x|}(\Phi, \widetilde{\Psi}, \widetilde{W}) \right\|^2 \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \int_0^t \|(\phi_x, \phi_{xx}, \tilde{\psi}_{xx}, \tilde{\omega}_{xx})\|^2 \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \int_0^t \|(\phi, \tilde{\psi}, \tilde{\omega}, \Pi_x, n_2)\|^2 \, \mathrm{d}\tau \\
&\quad + C \sum_{|\alpha'|=1} \int_0^t \int \int \frac{\nu(|v|)}{\mathbf{M}_*} |\partial^{\alpha'} \widetilde{\mathbf{G}}|^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + C\delta_0 \int_0^t \|(\widetilde{\Psi}_\tau, \widetilde{W}_\tau)\|^2 \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \int_0^t \int \int \frac{\nu(|v|)|(\partial_{v_1}(\mathbf{P}_c F_2), \widetilde{\mathbf{G}})|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned}$$

Step 2 Estimation on $\|\phi_x(t, \cdot)\|^2$.

To estimate the term $\int_0^t \|\phi_x\|^2 \, \mathrm{d}\tau$, we rewrite the equation $(4.16)_2$ as

$$\begin{aligned}
&\frac{4\mu(\tilde{\theta})}{3\tilde{\rho}} \phi_{xt} + \tilde{\rho}\tilde{\psi}_{1t} + \tilde{\rho}\tilde{u}_1\tilde{\psi}_{1x} + \frac{2}{3}\tilde{\theta}\phi_x \\
&= -\frac{4\mu(\tilde{\theta})}{3\tilde{\rho}} \left(2\tilde{\rho}_x\tilde{\psi}_{1x} + \tilde{\rho}_{xx}\tilde{\psi}_1 + (\tilde{u}_1\phi)_{xx} + L_{0x}\right) + \frac{4}{3}\mu'(\theta)\tilde{\theta}_x\tilde{\psi}_{1x} \\
&\quad + \frac{1}{3}\tilde{\rho}\tilde{u}_{1x}\tilde{\psi}_1 - \frac{2}{3}\tilde{\rho}_x\tilde{\omega} - \frac{2}{3}\tilde{\rho}\tilde{\omega}_x + \frac{2\tilde{\theta}\tilde{\rho}_x}{3\tilde{\rho}}\phi \\
&\quad - \int v_1^2(\Gamma - \Gamma^{S_1} - \Gamma^{S_3})_x \, \mathrm{d}v + (J_1 + N_1 - Q_1)_x - L_1. \qquad \text{(A.2.1)}
\end{aligned}$$

Multiplying the equation (A.2.1) by ϕ_x and integrating the resulting equation with respect to t, x, we get

$$
\begin{aligned}
&\|\phi_x(t,\cdot)\|^2+\int_0^t\|\phi_x\|^2\,\mathrm{d}\tau \leqq C\|\tilde{\psi}_1(t,\cdot)\|^2+C\int_0^t\|(\tilde{\psi}_{1x},\tilde{\omega}_x)\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\phi_{xx},\tilde{\psi}_{xx},\tilde{\omega}_{xx})\|^2\,\mathrm{d}\tau+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau\\
&\quad+C\delta_0\int_0^t\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}(\Phi,\widetilde{\Psi},\widetilde{W})\right\|^2\mathrm{d}\tau+C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right)\\
&\quad+C\delta_0\int_0^t\|(\phi,\tilde{\psi},\tilde{\omega})\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|(\partial^{\alpha'}\widetilde{\mathbf{G}},\widetilde{\mathbf{G}},\partial_{v_1}(\mathbf{P}_cF_2),\partial_{v_1x}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu^{-1}(|v|)}{\mathbf{M}_*}|\partial^\alpha\mathbf{G}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Now we turn to the time-derivative terms. To estimate $\|(\phi_t,\psi_t,\omega_t)\|^2$, we need to use the system (A.1). By multiplying $(A.1)_1$ by ϕ_t, $(B.1)_2$ by ψ_{1t}, $(A.1)_3$ by ψ_{it} $(i=2,3)$ and $(A.1)_4$ by ω_t respectively, and adding them together, after integrating with respect to t and x, we have

$$
\begin{aligned}
&\int_0^t\|(\phi_\tau,\psi_\tau,\omega_\tau)(\tau,\cdot)\|^2\,\mathrm{d}\tau\\
&\quad\leqq C\delta_0\int_0^t\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}(\Phi,\widetilde{\Psi},\widetilde{W})\right\|^2\mathrm{d}\tau\\
&\qquad+C\int_0^t\|(\phi_x,\tilde{\psi}_x,\tilde{\omega}_x)\|^2\,\mathrm{d}\tau\\
&\qquad+C\delta_0\int_0^t\|(\phi,\tilde{\psi},\tilde{\omega})\|^2\,\mathrm{d}\tau+C\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\widetilde{\mathbf{G}}_x|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau+C\delta_0.
\end{aligned}
$$

Step 3 Estimation on $\|(\phi_x,\tilde{\psi}_x,\tilde{\omega}_x)(t,\cdot)\|^2$.

Multiplying $(3.49)_1$ by $\dfrac{2\phi_x}{3\tilde{\rho}}$, $(3.49)_2$ by $\dfrac{\tilde{\psi}_{1x}}{\tilde{\theta}}$, $(3.49)_3$ by $\tilde{\psi}_{ix}$ and $(3.49)_4$ by $\dfrac{\tilde{\omega}_x}{\tilde{\theta}^2}$, adding them together, and integrating the resulting equation with respect to t, x, we have

$$
\begin{aligned}
&\|(\phi_x,\tilde{\psi}_x,\tilde{\omega}_x)(t,\cdot)\|^2+\int_0^t\|(\tilde{\psi}_{xx},\tilde{\omega}_{xx})\|^2\,\mathrm{d}\tau\leqq C\left(\mathcal{E}(0)^2+\delta_0\right)\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\phi,\tilde{\psi},\tilde{\omega})\|_{H^1}^2\,\mathrm{d}\tau
\end{aligned}
$$

$$+C\delta_0\int_0^t\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}(\Phi,\widetilde{\Psi},\widetilde{W})\right\|^2\,\mathrm{d}\tau$$
$$+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|(\partial^{\alpha'}\widetilde{\mathbf{G}},\widetilde{\mathbf{G}},\partial_{v_1}(\mathbf{P}_cF_2),\partial_{v_1x}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau. \tag{A.2.2}$$

To get the estimation of $\|\phi_{xx}\|^2$, we apply ∂_x to $(A.1)_2$, we get

$$\psi_{1xt}+(\rho u_1^2-\tilde{\rho}\tilde{u}_1^2+p-\tilde{p})_{xx}-\left(\frac{\Pi_x^2}{4}\right)_{xx}$$
$$=-\frac{4}{3}\left[\mu(\tilde{\theta})\tilde{u}_{1x}-\mu(\theta^{S_1})u_{1x}^{S_1}-\mu(\theta^{S_3})u_{1x}^{S_3}\right]_{xx}$$
$$-\int v_1^2\widetilde{\mathbf{G}}_{xx}\,\mathrm{d}v-Q_{1xx}. \tag{A.2.3}$$

Multiplying (A.2.3) by ϕ_{xx} and integrating the resulting equations, we obtain

$$\int\psi_{1x}\phi_{xx}(t,x)\,\mathrm{d}x+\int_0^t\|\phi_{xx}\|^2\,\mathrm{d}\tau\leqq C\|\psi_{1xx}\|^2$$
$$+C(\chi_T+\delta_0)\int_0^t\|(\phi,\tilde{\psi},\tilde{\omega})\|_{H^1}^2\,\mathrm{d}\tau+C\left(\mathcal{E}(0)^2+\delta_0^{\frac{1}{2}}\right)$$
$$+C\int_0^t\|(\tilde{\psi}_{xx},\tilde{\omega}_{xx})\|^2\,\mathrm{d}\tau+C\delta_0\int_0^t\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}(\Phi,\widetilde{\Psi},\widetilde{W})\right\|^2\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\int_0^t\|(n_2,n_{2x})\|^2\,\mathrm{d}\tau+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau. \tag{A.2.4}$$

To estimate $\|(\phi_{xt},\psi_{xt},\omega_{xt})\|^2$ and $\|(\phi_{tt},\psi_{tt},\omega_{tt})\|^2$, we use the system (A.1) again. Applying ∂_x first, and multiplying the four equations of (A.1) by ϕ_{xt}, ψ_{1xt}, ψ_{ixt} $(i=2,3)$, ω_{xt} respectively, then adding them together and integrating with respect to t and x, we have

$$\int_0^t\|(\phi_{x\tau},\psi_{x\tau},\omega_{x\tau})\|^2\,\mathrm{d}\tau\leqq C\int_0^t\|(\phi_{xx},\tilde{\psi}_{xx},\tilde{\omega}_{xx})\|^2\,\mathrm{d}\tau$$
$$+C(\chi_T+\delta_0)\int_0^t\|(\phi,\tilde{\psi},\tilde{\omega})\|_{H^1}^2\,\mathrm{d}\tau$$
$$+C\delta_0\int_0^t\left\|\sqrt{\left|u_{1x}^{S_1}\right|+\left|u_{1x}^{S_3}\right|+|\Theta_x|}(\Phi,\widetilde{\Psi},\widetilde{W})\right\|^2\,\mathrm{d}\tau$$

$$
\begin{aligned}
&+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\int_0^t\|(\Pi_{x\tau},n_{2\tau})\|^2\,\mathrm{d}\tau+C\delta_0^{\frac12}.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
\int_0^t\|(\phi_{\tau\tau},\psi_{\tau\tau},\omega_{\tau\tau})\|^2\,\mathrm{d}\tau &\leqq C\int_0^t\|(\phi_{x\tau},\psi_{x\tau},\omega_{x\tau})\|^2\,\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\int_0^t\|(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C\delta_0^{\frac12}. \qquad \text{(A.2.5)}
\end{aligned}
$$

A suitable linear combination of (A.2.2), (A.2.4)–(A.2.5) gives

$$
\begin{aligned}
&\|(\phi_x,\tilde{\psi}_x,\tilde{\omega}_x)(t,\cdot)\|^2+\int_0^t\Big[\|(\phi_{xx},\tilde{\psi}_{xx},\tilde{\omega}_{xx})\|^2+\sum_{|\alpha'|=1}\|\partial^{\alpha'}(\phi_\tau,\psi_\tau,\omega_\tau)\|^2\Big]\mathrm{d}\tau\\
&\leq C\|(\phi_{xx},\psi_{1xx})\|^2+C(\chi_T+\delta_0)\int_0^t\Big[\|(\phi,\tilde{\psi},\tilde{\omega})\|_{H^1}^2+\|(\phi,\psi,\omega)_\tau\|^2\Big]\mathrm{d}\tau\\
&+C\delta_0\int_0^t\Big\|\sqrt{\big|u_{1x}^{S_1}\big|+\big|u_{1x}^{S_3}\big|+|\Theta_x|}(\Phi,\widetilde{\Psi},\widetilde{W})\Big\|^2\,\mathrm{d}\tau\\
&+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha\widetilde{\mathbf{G}}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|(\partial^{\alpha'}\widetilde{\mathbf{G}},\widetilde{\mathbf{G}},\partial_{v_1}(\mathbf{P}_cF_2),\partial_{v_1x}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\int_0^t\int q|(\Phi,\widetilde{\Psi},\widetilde{W})|^2\,\mathrm{d}x\,\mathrm{d}\tau+C(\chi_T+\delta_0)\\
&\int_0^t\|(\Pi_x,\Pi_{x\tau},n_2,n_{2x},n_{2\tau})\|^2\,\mathrm{d}\tau+C\Big(\mathcal{E}(0)^2+\delta_0^{\frac12}\Big). \qquad \text{(A.2.6)}
\end{aligned}
$$

Step 4 Estimation on the non-fluid component.
To close the above estimate, we need to estimate the derivatives on the non-fluid component $\partial^\alpha\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)(1\leqq|\alpha|\leqq 2, 0\leqq|\beta|\leqq 2)$. First, applying ∂_x to (3.7) and (3.10), respectively, we have

$$
\begin{aligned}
\widetilde{\mathbf{G}}_{xt}-(\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}})_x=\Big\{&-\mathbf{P}_1(v_1\widetilde{\mathbf{G}}_x)-\mathbf{P}_1(\Pi_x F_{2v_1})\\
&+2[Q(\widetilde{\mathbf{G}},\mathbf{G}^{S_1}+\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_1}+\mathbf{G}^{S_3},\widetilde{\mathbf{G}})]\\
&+2Q(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}})+2[Q(\mathbf{G}^{S_1},\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_3},\mathbf{G}^{S_1})]\\
&-\Big[\mathbf{P}_1(v_1\mathbf{M}_x)-\mathbf{P}_1^{S_1}\left(v_1\mathbf{M}_x^{S_1}\right)\\
&-\mathbf{P}_1^{S_3}\left(v_1\mathbf{M}_x^{S_3}\right)\Big]+\sum_{j=1,3}R_j\Big\}_x.
\end{aligned}
\tag{A.2.7}
$$

and

$$
\begin{aligned}
&(\mathbf{P}_c F_2)_{xt}+\Big[\mathbf{P}_c(v_1F_{2x})\Big]_x+\Big[\mathbf{P}_c(\Pi_x\partial_{v_1}F_1)\Big]_x+\Big[\Big(\frac{\mathbf{M}}{\rho}\Big)_t n_2\Big]_x\\
&\quad=\mathbf{N}_{\mathbf{M}}(\mathbf{P}_cF_2)_x+2Q(\mathbf{P}_cF_2,\mathbf{M}_x)+2Q(F_{2x},\mathbf{G})+2Q(F_2,\mathbf{G}_x).
\end{aligned}
\tag{A.2.8}
$$

Multiplying (A.2.7) and (A.2.8) by $\frac{\widetilde{\mathbf{G}}_x}{\mathbf{M}_*}$ and $\frac{(\mathbf{P}_cF_2)_x}{\mathbf{M}_*}$, respectively, and integrating with respect to x, v and t, and also using Lemmas 2.1, 2.2 and 2.3, we obtain

$$
\begin{aligned}
&\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_x|^2}{2\mathbf{M}_*}(x,v,t)\,\mathrm{d}v\,\mathrm{d}x+\|n_2(\cdot,t)\|^2\\
&\quad+\frac{\widetilde{\sigma}}{2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_x|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_x|^2}{2\mathbf{M}_*}(x,v,0)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad+C\|n_{20}\|^2\\
&\quad+C\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_{xx}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+C\int_0^t\|(\phi_{xx},\psi_{xx},\omega_{xx},n_{2x},n_{2xx})\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\phi_{x\tau},\psi_{x\tau},\omega_{x\tau})\|^2\,\mathrm{d}\tau+C\delta_0\\
&\quad+C(\chi_T+\delta_0+\eta_0)\int_0^t\|n_2\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\sum_{0\leqq|\beta'|\leqq1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\beta'}(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{A.2.9}
$$

Note that in the above inequality, we have used the fact that

$$
\begin{aligned}
&-\int_0^t\int\int \Pi_{xx}\mathbf{M}_{v_1}\frac{(\mathbf{P}_cF_2)_x}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&=2\int_0^t\int n_2\mathbf{M}\frac{v_1-u_1}{R\theta}\frac{(\mathbf{P}_cF_2)_x}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}\tau\\
&=2\int_0^t\int\int n_2\frac{v_1-u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*}-1\right)(\mathbf{P}_cF_2)_x\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+2\int_0^t\int\int n_2\frac{v_1-u_1}{R\theta}(\mathbf{P}_cF_2)_x\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&=2\int_0^t\int\int n_2\frac{v_1-u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*}-1\right)(\mathbf{P}_cF_2)_x\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+2\int_0^t\int\frac{n_2}{R\theta}\left(\int v_1\mathbf{P}_cF_2\,\mathrm{d}v\right)_x\mathrm{d}x\,\mathrm{d}\tau\\
&=2\int_0^t\int\int n_2\frac{v_1-u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*}-1\right)(\mathbf{P}_cF_2)_x\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad-2\int_0^t\int\frac{n_2}{R\theta}\left[n_{2\tau}+(u_1n_2)_x\right]\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq-\int\frac{n_2^2}{R\theta}(x,t)\,\mathrm{d}x+C\|n_{20}\|^2+C(\chi_T+\delta_0+\eta_0)\\
&\quad\int_0^t\left[\|n_2\|^2+\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_x|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right]\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\omega_\tau,\psi_{1x},\omega_x)\|^2\,\mathrm{d}\tau.
\end{aligned}
$$

Similarly, one has

$$
\begin{aligned}
&-\int_0^t\int\int \Pi_{xt}\mathbf{M}_{v_1}\frac{(\mathbf{P}_cF_2)_\tau}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&=\int_0^t\int \Pi_{xt}\mathbf{M}\frac{v_1-u_1}{R\theta}\frac{(\mathbf{P}_cF_2)_x}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}\tau\\
&=\int_0^t\int\int \Pi_{x\tau}\frac{v_1-u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*}-1\right)(\mathbf{P}_cF_2)_\tau\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+\int_0^t\int\frac{\Pi_{x\tau}}{R\theta}\left(\int v_1\mathbf{P}_cF_2\,\mathrm{d}v\right)_\tau\mathrm{d}x\,\mathrm{d}\tau\\
&=\int_0^t\int\int \Pi_{x\tau}\frac{v_1-u_1}{R\theta}\left(\frac{\mathbf{M}}{\mathbf{M}_*}-1\right)(\mathbf{P}_cF_2)_x\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad-\int_0^t\int\frac{\Pi_{x\tau}}{R\theta}\left[\frac{1}{2}\Pi_{x\tau\tau}+(u_1n_2)_\tau\right]\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq-\int\frac{\Pi_{x\tau}^2}{4R\theta}(x,t)\,\mathrm{d}x+C\|\Pi_{0xt}\|^2+C(\chi_T+\delta_0+\eta_0)
\end{aligned}
$$

$$\int_0^t \left[\|\Pi_{x\tau}\|^2 + \int\int \frac{\nu(|v|)|(\mathbf{P}_c F_2)_\tau|^2}{\mathbf{M}_*} \, \mathrm{d}x \, \mathrm{d}v \right] \mathrm{d}\tau$$
$$+ C(\chi_T + \delta_0) \int_0^t \|(\omega_\tau, \psi_{1x}, \omega_x)\|^2 \, \mathrm{d}\tau,$$

and further that

$$\begin{aligned}
&\int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_t|^2}{2\mathbf{M}_*}(x, v, t) \, \mathrm{d}v \, \mathrm{d}x + \|\Pi_{xt}(\cdot, t)\|^2 \\
&\quad + \frac{\widetilde{\sigma}}{2} \int_0^t \int\int \frac{\nu(|v|)}{\mathbf{M}_*} |(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_\tau|^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau \\
&\leqq C \int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_t|^2}{2\mathbf{M}_*}(x, v, 0) \, \mathrm{d}v \, \mathrm{d}x \\
&\quad + C\|\Pi_{0xt}\|^2 + C \int_0^t \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_{x\tau}|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \int_0^t \|(\phi_x, \varphi_{\tau\tau}, \psi_{\tau\tau}, \omega_{\tau\tau})\|^2 \, \mathrm{d}\tau \\
&\quad + C\delta_0 + C \int_0^t \|(\phi_{x\tau}, \psi_{x\tau}, \omega_{x\tau}, n_{2\tau}, n_{2x\tau})\|^2 \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\phi, \psi, \omega)\|^2 \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0 + \eta_0) \int_0^t \|\Pi_{x\tau}\|^2 \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \sum_{0 \leqq |\beta'| \leqq 1} \int_0^t \int\int \frac{\nu(|v|)\left|\left(\partial^{\beta'}(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2), (\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_x\right)\right|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned} \tag{A.2.10}$$

The combination of (A.2.9) and (A.2.10), and choosing χ_T, δ suitably small yields that

$$\begin{aligned}
&\sum_{|\alpha|=1} \int\int \frac{|\partial^\alpha(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, t) \, \mathrm{d}v \, \mathrm{d}x \\
&\quad + \sum_{|\alpha|=1} \int_0^t \int\int \frac{\nu(|v|)|\partial^\alpha(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + \|(n_2, \Pi_{xt})(\cdot, t)\|^2 \leqq C \int |\phi_{xx}\psi_{1x}(x, t)| \, \mathrm{d}x \\
&\quad + C \sum_{|\alpha|=2} \int_0^t \int\int \frac{\nu(|v|)|\partial^\alpha(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \sqrt{\delta}_0) \int_0^t \left[\|\sqrt{\bar{u}_{1x}}(\phi, \psi_1, \omega)\|^2 + \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \omega)\|^2 + \|\Pi_x\|^2 \right] \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+C(\chi_T+\delta_0)\sum_{0\leqq|\beta'|\leqq 1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\beta'}(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&+C(\mathcal{E}(0)^2+\delta_0)+\ C(\chi_T+\delta_0)\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_{vx}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&+C(\chi_T+\sqrt{\delta}_0+\eta_0)\int_0^t\|(n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau.
\end{aligned}
$$

Applying ∂_{v_j} $(j=1,2,3)$ to the equation (3.44) and noting that (A.1.26), one has

$$
\begin{aligned}
&\widetilde{\mathbf{G}}_{v_jt}-\mathbf{L_M}\widetilde{\mathbf{G}}_{v_j}=-\partial_{v_j}\mathbf{P}_1(v_1\widetilde{\mathbf{G}}_x)-\Pi_x(\mathbf{P}_cF_2)_{v_1v_j}\\
&\quad+\Pi_x\sum_{k=0}^4\int(\mathbf{P}_cF_2)\left(\frac{\chi_k}{\mathbf{M}}\right)_{v_1}\mathrm{d}v(\chi_k)_{v_j}\\
&\quad+(\partial_{v_j}\mathbf{L_M}\widetilde{\mathbf{G}}-\mathbf{L_M}\widetilde{\mathbf{G}}_{v_j})+2\partial_{v_j}Q(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}})\\
&\quad+2\partial_{v_j}[Q(\widetilde{\mathbf{G}},\mathbf{G}^{S_1}+\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_1}+\mathbf{G}^{S_3},\widetilde{\mathbf{G}})]\\
&\quad+2\partial_{v_j}[Q(\mathbf{G}^{S_1},\mathbf{G}^{S_3})+Q(\mathbf{G}^{S_3},\mathbf{G}^{S_1})]\\
&\quad-\left[\mathbf{P}_1(v_1\mathbf{M}_x)-\mathbf{P}_1^{S_1}\left(v_1\mathbf{M}_x^{S_1}\right)-\mathbf{P}_1^{S_3}\left(v_1\mathbf{M}_x^{S_3}\right)\right]_{v_j}+\partial_{v_j}\sum_{k=1,3}R_k. \qquad (A.2.11)
\end{aligned}
$$

Similarly, applying ∂_{v_j} $(j=1,2,3)$ to the equation (3.10) and noting (A.1.27) implies that

$$
\begin{aligned}
&\partial_t(\mathbf{P}_cF_2)_{v_j}-\mathbf{N_M}(\mathbf{P}_cF_2)_{v_j}+\partial_{v_j}\mathbf{P}_c(v_1F_{2x})+\Pi_x\mathbf{M}_{v_1v_j}+\Pi_x\widetilde{\mathbf{G}}_{v_1v_j}\\
&\quad+\Pi_x\left(\mathbf{G}^{S_1}_{v_1v_j}+\mathbf{G}^{S_3}_{v_1v_j}\right)+\left(\frac{\mathbf{M}}{\rho}\right)_{tv_j}n_2=2Q(\mathbf{P}_cF_2,\mathbf{M}_{v_j})+2\partial_{v_j}Q(F_2,\mathbf{G}).
\end{aligned}
\qquad (A.2.12)
$$

Recall the following two facts:

$$
\begin{aligned}
&\partial_{v_j}\mathbf{L_M}g-\mathbf{L_M}(g_{v_j})=2Q(\partial_{v_j}\mathbf{M},g)+2Q(g,\partial_{v_j}\mathbf{M}),\\
&\left(\mathbf{L}_{\mathbf{M}}^{-1}g\right)_{v_j}=\mathbf{L}_{\mathbf{M}}^{-1}(g_{v_j})+\mathbf{L}_{\mathbf{M}}^{-1}\left(\partial_{v_j}\mathbf{L_M}g-\mathbf{L_M}(g_{v_j})\right).
\end{aligned}
$$

Multiplying the equations (A.2.11) and (A.2.12) by $\frac{\widetilde{\mathbf{G}}_{v_j}}{\mathbf{M}_*}$ and $\frac{(\mathbf{P}_cF_2)_{v_j}}{\mathbf{M}_*}$, respectively, and then integrating with respect to x, v, t implies that

$$
\begin{aligned}
&\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_{v_j}|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\\
&\quad+\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_{v_j}|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\leqq C\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)_{v_j}|^2}{\mathbf{M}_*}(x,v,0)\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&+ C(\chi_T + \delta_0) \int_0^t \left[\|n_2\|^2 + \sum_{|\alpha|=1} \|\partial^\alpha(\phi, \psi, \omega)\|^2 \right] \mathrm{d}\tau \\
&+ C \int_0^t \|(\psi_x, \omega_x, \Pi_x, n_{2x})\|^2 \,\mathrm{d}\tau \\
&+ C\delta_0 + C \int_0^t \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2, \widetilde{\mathbf{G}}_x, (\mathbf{P}_c F_2)_x)|^2}{\mathbf{M}_*} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau.
\end{aligned}
\tag{A.2.13}
$$

Applying $\partial_{v_j v_k}$ $(j, k = 1, 2, 3)$ to the equations (3.44) and (2.20) and using the similar analysis as in obtaining (A.2.13), one has for $|\beta| = 2$ that

$$
\begin{aligned}
&\int\int \frac{|\partial^\beta(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, t)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_0^t \int\int \frac{\nu(|v|)|\partial^\beta(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&\leqq C \int\int \frac{|\partial^\beta(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, 0)\,\mathrm{d}x\,\mathrm{d}v + C\int_0^t \|(\psi_x, \omega_x, \Pi_x, n_{2x})\|^2 \,\mathrm{d}\tau \\
&\quad + C \sum_{0\leqq|\beta'|\leqq 1} \int_0^t \int\int \frac{\nu(|v|)|\partial^{\beta'}(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2, \widetilde{\mathbf{G}}_x, (\mathbf{P}_c F_2)_x)|^2}{\mathbf{M}_*} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \sum_{|\alpha|=1} \int_0^t \int |\partial^\alpha(\phi, \psi, \omega)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + C\delta_0.
\end{aligned}
$$

By (A.1.1), (A.1.35) and (A.1.36), it holds that

$$
\begin{aligned}
&\int_0^t \|(\psi_x, \omega_x, \Pi_x, n_{2x})\|^2 \,\mathrm{d}\tau \leqq C\|(\phi, \psi, \omega, \phi_x, \Pi_x, n_2)(\cdot, 0)\|^2 + C\delta_0^{\frac{1}{2}} \\
&\quad + C(\chi_T + \sqrt{\delta}_0) \int_0^t \|(\psi_{1xx}, \phi_{xx})\|^2 \,\mathrm{d}\tau \\
&\quad + C(\chi_T + \delta_0) \int_0^t \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2, \widetilde{\mathbf{G}}_{v_1})|^2}{\mathbf{M}_*} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C \sum_{1\leqq|\alpha|\leqq 2} \int_0^t \int\int \frac{\nu(|v|)|\partial^\alpha(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Similarly, one has for $|\alpha'| = 1$ and $|\beta'| = 1$ that

$$
\begin{aligned}
&\int\int \frac{|\partial^{\alpha'}\partial^{\beta'}(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, t)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_0^t \int\int \frac{\nu(|v|)|\partial^{\alpha'}\partial^{\beta'}(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau \\
&\leqq C \int\int \frac{|\partial^{\alpha'}\partial^{\beta'}(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, 0)\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{1\leqq|\alpha|\leqq 2}\int_0^t\int\int\frac{\nu(|v|)|\partial^\alpha(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&+C\int_0^t\Big[\sum_{|\alpha|=2}\|\partial^\alpha(\varphi,\psi,\omega,n_2)\|^2+\|(n_2,n_{2x},n_{2\tau},\Pi_{x\tau})\|^2\Big]\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&+C(\chi_T+\delta_0)\sum_{0\leqq|\beta|\leqq 1}\int_0^t\int\int\frac{\nu(|v|)|\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau+C\delta_0.
\end{aligned}
$$

By (A.1.36), (A.1.37), (A.1.38) and (A.2.18), it holds that

$$
\begin{aligned}
&\int_0^t\Big[\sum_{|\alpha|=2}\|\partial^\alpha(\varphi,\psi,\omega,n_2)\|^2+\|(n_2,n_{2x},n_{2\tau},\Pi_{x\tau})\|^2\Big]\mathrm{d}\tau\\
&\quad\leqq C\|(\Pi_{x0},\psi_{0x},\omega_{0x},\phi_{0xx},n_{20},n_{20x})\|^2\\
&\qquad+C\int|\varphi_{xx}\psi_{1x}(x,t)|\,\mathrm{d}x+C\sum_{1\leqq|\alpha|\leqq 2}\int_0^t\int\int\frac{\nu(|v|)|\partial^\alpha(\widetilde{\mathbf{G}},\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+C(\chi_T+\sqrt{\delta_0})\int_0^t\Big[\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\omega)\|^2+\|\sqrt{\bar u_{1x}}(\phi,\psi_1,\omega)\|^2+\|\Pi_x\|^2\Big]\mathrm{d}\tau\\
&\qquad+C\int_0^t\int\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C(\chi_T+\sqrt{\delta_0})\\
&\qquad\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}}_{v_1},\widetilde{\mathbf{G}}_{vx})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

In summary, one has

$$
\begin{aligned}
&\int\int\Big[\sum_{|\alpha|=1,0\leqq|\beta|\leqq 1}\frac{|\partial^\alpha\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\\
&\qquad+\sum_{1\leqq|\beta|\leqq 2}\frac{|\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\Big]\mathrm{d}x\,\mathrm{d}v\\
&\qquad+\int_0^t\int\int\Big[\sum_{|\alpha|=1,0\leqq|\beta|\leqq 1}\frac{\nu(|v|)|\partial^\alpha\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\\
&\qquad+\sum_{1\leqq|\beta|\leqq 2}\frac{\nu(|v|)|\partial^\beta(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\Big]\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\leqq\mathcal{E}^2(0)+C\int|\phi_{xx}\psi_{1x}(x,t)|\,\mathrm{d}x+C\delta_0^{\frac{1}{2}}\\
&\qquad+C\sum_{|\alpha|=0,2}\int_0^t\int\int\frac{\nu(|v|)|\partial^\alpha(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau
\end{aligned}
$$

$$+C(\chi_T+\sqrt{\delta_0})\int_0^t\Big[\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\omega)\|^2$$
$$+\|\sqrt{\bar{u}_{1x}}(\phi,\psi_1,\omega)\|^2+\|\Pi_x\|^2+\|(\psi_{1xx},\phi_{xx})\|^2\Big]\mathrm{d}\tau$$
$$+C(\chi_T+\sqrt{\delta_0}+\eta_0)\int_0^t\|(n_2,\Pi_{x\tau})\|^2\,\mathrm{d}\tau.$$

Step 5 Estimation on $\|n_{2x}(t,\cdot)\|^2$.
Now we estimate $\|n_{2x}\|^2$. Applying ∂_x to the equation (3.9), one has

$$\begin{aligned}
&n_{2xt}+(u_1n_2)_{xx}+\left(\frac{\kappa_1(\theta)}{R\theta}\Pi_x\right)_{xx}-\left(\kappa_1(\theta)\left(\frac{n_2}{\rho}\right)_x\right)_{xx}\\
&=-\left(\frac{n_2}{\rho}\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1\mathbf{M}_x))\Big]\mathrm{d}v\right)_{xx}\\
&\quad-\left(\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\mathbf{P}_c(v_1(\mathbf{P}_cF_2)_x)\Big]\mathrm{d}v\right)_{xx}\\
&\quad-\left(\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\Pi_x\mathbf{G}_{v_1}\Big]\mathrm{d}v\right)_{xx}\\
&\quad-\left(\int v_1\mathbf{N}_{\mathbf{M}}^{-1}\Big[\partial_t(\mathbf{P}_cF_2)+\left(\frac{\mathbf{M}}{\rho}\right)_t n_2-2Q(F_2,\mathbf{G})\Big]\mathrm{d}v\right)_{xx}.
\end{aligned}\tag{A.2.14}$$

Multiplying the equation (A.2.14) by n_{2x} and integrating the resulting equation with respect to x,t yields that

$$\begin{aligned}
&\|n_{2x}\|^2(t)+\int_0^t\|(n_{2x},n_{2xx})\|^2\,\mathrm{d}\tau\leqq +C(\chi_T+\delta_0)\\
&\int_0^t\|(\Pi_x,n_2,\psi_x,\phi_x,\omega_x,\psi_{1xx},\phi_{xx})\|^2\,\mathrm{d}\tau+C(\chi_T+\delta_0)\\
&\int_0^t\int\int\frac{\nu(|v|)|\Big((\mathbf{P}_cF_2)_x,(\mathbf{P}_cF_2)_t,\widetilde{\mathbf{G}}_{v_1},\widetilde{\mathbf{G}}_{v_1x},\widetilde{\mathbf{G}},\widetilde{\mathbf{G}}_x\Big)|^2)}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&+C\int_0^t\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_{xx},\mathbf{P}_cF_2)_{xt})|^2)}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&+C(\|n_{2x}\|^2(0)+\delta_0).
\end{aligned}\tag{A.2.15}$$

By the equation (3.8), it holds that

$$\begin{aligned}
\int_0^t\|n_{2x\tau}\|^2\,\mathrm{d}\tau&=\int_0^t\|\int v_1F_{2xx}\,\mathrm{d}v\|^2\,\mathrm{d}\tau\\
&=\int_0^t\|\int v_1\left(\frac{\mathbf{M}}{\rho}n_2+\mathbf{P}_cF_2\right)_{xx}\mathrm{d}v\|^2\,\mathrm{d}\tau\\
&\leqq C\int_0^t\left[\|n_{2xx}\|^2+\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_{xx}|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\right]\mathrm{d}\tau
\end{aligned}$$

$$+C(\chi_T+\delta_0)\int_0^t \|(n_2, n_{2x}, \phi_{xx}, \psi_{xx}, \omega_{xx}, \phi_x, \psi_x, \omega_x)\|^2 \,\mathrm{d}\tau, \tag{A.2.16}$$

and

$$\begin{aligned}\int_0^t \|n_{2\tau\tau}\|^2 \,\mathrm{d}\tau &= \int_0^t \|\int v_1 F_{2x\tau}\,\mathrm{d}v\|^2 \,\mathrm{d}\tau\\
&= \int_0^t \|\int v_1\Big(\frac{\mathbf{M}}{\rho}n_2+\mathbf{P}_c F_2\Big)_{x\tau}\,\mathrm{d}v\|^2\,\mathrm{d}\tau\\
&\leqq C\int_0^t\Big[\|n_{2x\tau}\|^2+\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_{x\tau}|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\Big]\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\Big[\|(n_2,\phi_{x\tau},\psi_{x\tau},\omega_{x\tau})\|^2\\
&\quad+\sum_{|\alpha'|=1}\|\partial'(\phi,\psi,\omega,n_2)\|^2\Big]\mathrm{d}\tau. \end{aligned}\tag{A.2.17}$$

By (A.2.15), (A.2.16) and (A.2.17), one has

$$\begin{aligned}&\|n_{2x}(\cdot,t)\|^2+\int_0^t\Big[\|n_{2x}\|^2+\sum_{|\alpha|=2}\|\partial^\alpha n_2\|^2\Big]\mathrm{d}\tau\\
&\leqq C\|n_{20x}\|^2+C(\chi_T+\delta_0)\sum_{1\leqq|\alpha|\leqq 2}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau\\
&\quad+C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)|\partial^\alpha(\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\Pi_x,n_2,n_{2\tau})\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\int\int\frac{\nu(|v|)|\Big(\mathbf{P}_cF_2)_x,(\mathbf{P}_cF_2)_\tau,\widetilde{\mathbf{G}}_{v_1},\widetilde{\mathbf{G}}_{v_1x},\widetilde{\mathbf{G}}\Big)|^2)}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau.\end{aligned}\tag{A.2.18}$$

By (A.9), it holds that

$$\begin{aligned}\int_0^t\|\Pi_{x\tau\tau}\|^2\,\mathrm{d}\tau &\leqq C\int_0^t\Big[\|n_{2\tau}\|^2+\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_\tau|^2}{\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\Big]\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(n_2,\phi_\tau,\psi_\tau,\omega_\tau)\|^2\,\mathrm{d}\tau.\end{aligned}\tag{A.2.19}$$

Step 6 Highest order estimates.

Finally, we estimate the highest order derivatives, that is, $\int_0^t\int\int\frac{\nu(|v|)|\partial^\alpha\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$ with $|\alpha|=2$ and $\|\phi_{xx}\|^2$ in (A.2.6). To do so, it is sufficient to study $\int\int\frac{|\partial^\alpha\widetilde{F}_1|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x$

($|\alpha| = 2$) in view of (A.3), (A.4) and (A.5). Applying $\partial^\alpha (|\alpha| = 2)$ on the Vlasov–Poisson–Boltzmann equation (2.1), one has

$$
\begin{aligned}
&(\partial^\alpha \widetilde{F}_1)_t + v_1(\partial^\alpha \widetilde{F}_1)_x + \Pi_x \partial_{v_1}(\partial^\alpha F_2) + \partial^\alpha \Pi_x \partial_{v_1} F_2 \\
&\quad + \sum_{|\alpha'|=1, \alpha' \leqq \alpha} C_\alpha^{\alpha'} \partial^{\alpha'} \Pi_x \partial_{v_1} \partial^{\alpha-\alpha'} F_2 \\
&\quad = \partial^\alpha \mathbf{L}_\mathbf{M} \widetilde{\mathbf{G}} + 2\partial^\alpha Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}) + \partial^\alpha \Big\{ (\mathbf{L}_\mathbf{M} - \mathbf{L}_{\mathbf{M}^{S_1}})(\mathbf{G}^{S_1}) + (\mathbf{L}_\mathbf{M} - \mathbf{L}_{\mathbf{M}^{S_3}})(\mathbf{G}^{S_3}) \\
&\qquad + 2[Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_1} + \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_1} + \mathbf{G}^{S_3}, \widetilde{\mathbf{G}})] \\
&\qquad + 2[Q(\mathbf{G}^{S_1}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_1})] \Big\} \qquad \text{(A.2.20)}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\partial^\alpha F_2)_t + v_1(\partial^\alpha F_2)_x + \Pi_x \partial_{v_1}(\partial^\alpha F_2) + \partial^\alpha \Pi_x \partial_{v_1} F_1 \\
&\quad + \sum_{|\alpha'|=1, \alpha' \leqq \alpha} C_\alpha^{\alpha'} \partial^{\alpha'} \Pi_x \partial_{v_1} \partial^{\alpha-\alpha'} F_1 = \partial^\alpha \mathbf{N}_\mathbf{M}(\mathbf{P}_c F_2) + 2\partial^\alpha Q(F_2, \mathbf{G}).
\end{aligned} \qquad \text{(A.2.21)}
$$

Multiplying (A.2.20) and (A.2.21) by $\frac{\partial^\alpha \widetilde{F}_1}{\mathbf{M}_*} = \frac{\partial^\alpha (\mathbf{M} - \mathbf{M}^{S_1} - \mathbf{M}^{S_3})}{\mathbf{M}_*} + \frac{\partial^\alpha \widetilde{\mathbf{G}}}{\mathbf{M}_*}$ and $\frac{\partial^\alpha F_2}{\mathbf{M}_*} = \frac{\partial^\alpha (\frac{n_2}{\rho} \mathbf{M})}{\mathbf{M}_*} + \frac{\partial^\alpha (\mathbf{P}_c F_2)}{\mathbf{M}_*}$, respectively, we obtain

$$
\begin{aligned}
&\left(\frac{|\partial^\alpha \widetilde{F}_1|^2}{2\mathbf{M}_*} \right)_t - \frac{\partial^\alpha \widetilde{\mathbf{G}}}{\mathbf{M}_*} \mathbf{L}_\mathbf{M} \partial^\alpha \widetilde{\mathbf{G}} \\
&\quad = \frac{\partial^\alpha \widetilde{F}_1}{\mathbf{M}_*} \Big\{ \Pi_x \partial_{v_1}(\partial^\alpha F_2) + \partial^\alpha \Pi_x \partial_{v_1} F_2 + \sum_{|\alpha'|=1, \alpha' \leqq \alpha} C_\alpha^{\alpha'} \partial^{\alpha'} \Pi_x \partial_{v_1} \partial^{\alpha-\alpha'} F_2 \\
&\qquad + (\partial^\alpha \mathbf{L}_\mathbf{M} \widetilde{\mathbf{G}} - \mathbf{L}_\mathbf{M} \partial^\alpha \widetilde{\mathbf{G}}) + 2\partial^\alpha Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}) \\
&\qquad + \partial^\alpha \Big[(\mathbf{L}_\mathbf{M} - \mathbf{L}_{\mathbf{M}^{S_1}})(\mathbf{G}^{S_1}) + (\mathbf{L}_\mathbf{M} - \mathbf{L}_{\mathbf{M}^{S_3}})(\mathbf{G}^{S_3}) + 2[Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_1} \\
&\qquad + \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_1} + \mathbf{G}^{S_3}, \widetilde{\mathbf{G}})] + 2[Q(\mathbf{G}^{S_1}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_1})] \Big] \Big\} \\
&\qquad + \frac{\partial^\alpha (\mathbf{M} - \mathbf{M}^{S_1} - \mathbf{M}^{S_3})}{\mathbf{M}_*} \mathbf{L}_\mathbf{M} \partial^\alpha \widetilde{\mathbf{G}} + \left(v_1 \frac{|\partial^\alpha \widetilde{F}_1|^2}{2\mathbf{M}_*} \right)_x,
\end{aligned} \qquad \text{(A.2.22)}
$$

and

$$
\begin{aligned}
&\left(\frac{|\partial^\alpha F_2|^2}{2\mathbf{M}_*} \right)_t + \frac{\partial^\alpha (\mathbf{P}_c F_2)}{\mathbf{M}_*} \mathbf{N}_\mathbf{M} \partial^\alpha (\mathbf{P}_c F_2) \\
&\quad = \Pi_x \frac{\partial^\alpha F_2}{\mathbf{M}_*} \partial_{v_1}(\partial^\alpha \widetilde{F}_1) + \Pi_x \frac{\partial^\alpha F_2}{\mathbf{M}_*} \partial_{v_1}(\partial^\alpha F^S_{\alpha_1, \alpha_3}) \\
&\qquad + \frac{\partial^\alpha F_2}{\mathbf{M}_*} \Big[\partial^\alpha \Pi_x \partial_{v_1} F_1 + \sum_{|\alpha'|=1, \alpha' \leqq \alpha} C_\alpha^{\alpha'} \partial^{\alpha'} \Pi_x \partial_{v_1} \partial^{\alpha-\alpha'} F_1
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{|\alpha'|=1,\alpha'\leqq\alpha} 2C_\alpha^{\alpha'} Q(\partial^{\alpha'}\mathbf{P}_c F_2, \partial^{\alpha-\alpha'}\mathbf{M})\\
&+2Q(\mathbf{P}_c F_2, \partial^\alpha \mathbf{M})\Bigg] + \frac{\partial^\alpha\left(\frac{\mathbf{M}}{\rho}n_2\right)}{\mathbf{M}_*}\mathbf{N}_{\mathbf{M}}\partial^\alpha(\mathbf{P}_c F_2)\\
&+2\partial^\alpha Q(F_2, \mathbf{G})\frac{\partial^\alpha F_2}{\mathbf{M}_*} + \left(v_1\frac{|\partial^\alpha F_2|^2}{2\mathbf{M}_*}\right)_x,
\end{aligned} \tag{A.2.23}
$$

Adding (A.2.22) and (A.2.23) together and noting that

$$
\begin{aligned}
&\Pi_x\frac{\partial^\alpha \widetilde{F}_1}{\mathbf{M}_*}\partial_{v_1}(\partial^\alpha F_2) + \Pi_x\frac{\partial^\alpha F_2}{\mathbf{M}_*}\partial_{v_1}(\partial^\alpha \widetilde{F}_1)\\
&\quad= \left(\Pi_x\frac{\partial^\alpha \widetilde{F}_1\partial^\alpha F_2}{\mathbf{M}_*}\right)_{v_1} + \Pi_x\frac{\partial^\alpha \widetilde{F}_1\partial^\alpha F_2}{(\mathbf{M}_*)^2}(\mathbf{M}_*)_{v_1},
\end{aligned}
$$

and then integrating the resulting equation over x, v, t implies that

$$
\begin{aligned}
&\int\!\!\int \frac{|\partial^\alpha \widetilde{F}_1|^2 + |\partial^\alpha F_2|^2}{2\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v - \int\!\!\int \frac{|\partial^\alpha \widetilde{F}_{10}|^2 + |\partial^\alpha F_{20}|^2}{2\mathbf{M}_*}\,\mathrm{d}x\,\mathrm{d}v\\
&\quad-\int_0^t\int\!\!\int\left[\frac{\partial^\alpha\widetilde{\mathbf{G}}}{\mathbf{M}_*}\mathbf{L}_{\mathbf{M}}\partial^\alpha\widetilde{\mathbf{G}} + \frac{\partial^\alpha(\mathbf{P}_c F_2)}{\mathbf{M}_*}\mathbf{N}_{\mathbf{M}}\partial^\alpha(\mathbf{P}_c F_2)\right]\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau\\
&=\int_0^t\int\!\!\int\Bigg\{\Pi_x\frac{\partial^\alpha \widetilde{F}_1\partial^\alpha F_2}{(\mathbf{M}_*)^2}(\mathbf{M}_*)_{v_1} + \frac{\partial^\alpha\widetilde{F}_1}{\mathbf{M}_*}\partial^\alpha\Pi_x\partial_{v_1}F_2\\
&\quad+\frac{\partial^\alpha\widetilde{F}_1}{\mathbf{M}_*}\sum_{|\alpha'|=1,\alpha'\leqq\alpha} C_\alpha^{\alpha'}\partial^{\alpha'}\Pi_x\partial_{v_1}\partial^{\alpha-\alpha'}F_2\\
&\quad+\frac{\partial^\alpha\widetilde{F}_1}{\mathbf{M}_*}(\partial^\alpha\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}} - \mathbf{L}_{\mathbf{M}}\partial^\alpha\widetilde{\mathbf{G}})\\
&\quad+\frac{\partial^\alpha\widetilde{F}_1}{\mathbf{M}_*}\partial^\alpha\Bigg[(\mathbf{L}_{\mathbf{M}} - \mathbf{L}_{\mathbf{M}^{S_1}})(\mathbf{G}^{S_1}) + (\mathbf{L}_{\mathbf{M}} - \mathbf{L}_{\mathbf{M}^{S_3}})(\mathbf{G}^{S_3})\\
&\quad+2[Q(\widetilde{\mathbf{G}}, \mathbf{G}^{S_1}+\mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_1}+\mathbf{G}^{S_3}, \widetilde{\mathbf{G}})]\\
&\quad+2[Q(\mathbf{G}^{S_1}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_1})]\Bigg] + \frac{\partial^\alpha\widetilde{F}_1}{\mathbf{M}_*}2\partial^\alpha Q(\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}})\\
&\quad+\frac{\partial^\alpha(\mathbf{M}-\mathbf{M}^{S_1}-\mathbf{M}^{S_3})}{\mathbf{M}_*}\mathbf{L}_{\mathbf{M}}\partial^\alpha\widetilde{\mathbf{G}} + \Pi_x\frac{\partial^\alpha F_2}{\mathbf{M}_*}\partial_{v_1}(\partial^\alpha F^S_{\alpha_1,\alpha_3})\\
&\quad+\frac{\partial^\alpha F_2}{\mathbf{M}_*}\sum_{|\alpha'|=1,\alpha'\leqq\alpha} C_\alpha^{\alpha'}\partial^{\alpha'}\Pi_x\partial_{v_1}\partial^{\alpha-\alpha'}F_1\\
&\quad+\frac{\partial^\alpha F_2}{\mathbf{M}_*}\partial^\alpha\Pi_x\partial_{v_1}F_1 + \frac{\partial^\alpha F_2}{\mathbf{M}_*}\sum_{|\alpha'|=1,\alpha'\leqq\alpha} 2C_\alpha^{\alpha'} Q(\partial^{\alpha'}\mathbf{P}_c F_2, \partial^{\alpha-\alpha'}\mathbf{M})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial^\alpha F_2}{\mathbf{M}_*}2Q(\mathbf{P}_c F_2, \partial^\alpha \mathbf{M}) \\
&+\frac{\partial^\alpha\left(\frac{\mathbf{M}}{\rho}n_2\right)}{\mathbf{M}_*}\mathbf{N}_{\mathbf{M}}\partial^\alpha(\mathbf{P}_c F_2)+2\partial^\alpha Q(F_2,\mathbf{G})\frac{\partial^\alpha F_2}{\mathbf{M}_*}\Bigg\}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau.
\end{aligned}
$$

Then we can get

$$
\begin{aligned}
&\sum_{|\alpha|=2}\left[\|\partial^\alpha \Pi_x(\cdot,t)\|^2+\int\int\frac{|\partial^\alpha(\widetilde{F}_1,F_2)|^2}{2\mathbf{M}_*}(x,v,t)\,\mathrm{d}v\,\mathrm{d}x\right] \\
&\quad+\frac{\widetilde{\sigma}}{2}\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\leqq C\sum_{|\alpha|=2}\left[\|\partial^\alpha \Pi_{x0}\|^2+\int\int\frac{|\partial^\alpha(\widetilde{F}_{10},F_{20})|^2}{2\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\right] \\
&\quad+C(\chi_T+\delta_0)\sum_{|\alpha'|=1}\int_0^t\|\partial^{\alpha'}(\phi,\psi,\omega,n_2)\|^2\,\mathrm{d}\tau \\
&\quad+C(\chi_T+\delta_0)\int_0^t\int\int\Bigg[\frac{\nu(|v|)|\left(\widetilde{\mathbf{G}},\mathbf{P}_c F_2\right)|^2}{\mathbf{M}_*} \\
&\quad+\sum_{|\alpha'|=1}\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^{\alpha'}(\widetilde{\mathbf{G}},\mathbf{P}_c F_2)|^2\Bigg]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad+C(\eta_0+\chi_T+\delta_0)\sum_{|\alpha|=2}\int_0^t\|\partial^\alpha(\phi,\psi,\omega,\Pi_x,n_2)\|^2\,\mathrm{d}\tau \\
&\quad+C(\chi_T+\delta_0)\int_0^t\|(\Pi_{x\tau},n_2)\|^2\,\mathrm{d}\tau+C\delta_0.
\end{aligned}
$$

Note that by (A.1.36), (A.1.38) and (A.2.19), for $|\alpha|=2$, it holds that

$$
\begin{aligned}
&\int_0^t\|\partial^\alpha \Pi_x\|^2\,\mathrm{d}\tau\leqq\int_0^t\|(n_{2x},n_{2\tau},\Pi_{x\tau\tau})\|^2\,\mathrm{d}\tau\leqq C\|n_{20}\|^2 \\
&\quad+C(\chi_T+\delta_0)\int_0^t\|\Pi_x\|^2\,\mathrm{d}\tau \\
&\quad+C(\chi_T+\delta_0)\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\alpha'}(\mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

$$+C(\chi_T+\delta_0)\Big[\int_0^t\int\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}}_{v_1})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\Big].$$

Therefore, collecting all the above estimates together, we can get Proposition 3.2.

Appendix B: A Priori Estimates for Stability of Rarefaction Wave

The proof of Theorem 4.1 is shown by the continuum argument for the local solution to the system (3.4) or equivalently the system (3.6)–(3.10), which can be proved in a fashion similar to [10,40]. Therefore, to prove Theorem 4.1, it is sufficient to close the a-priori assumption (4.20) and verify the a priori estimates (4.21) and the time-asymptotic behaviors of the solution. By (3.6) and (4.14), one has

$$\begin{cases}\phi_t+\bar\rho\psi_{1x}+\bar u_1\phi_x+\bar\rho_x\psi_1+\bar u_{1x}\phi=-(\phi\psi_1)_x,\\ \psi_{1t}+\bar u_1\psi_{1x}+\bar u_{1x}\psi_1+\frac{2}{3}\omega_x+\frac{2\bar\theta}{3\bar\rho}\phi_x+\frac{2}{3}\rho_x\left(\frac{\theta}{\rho}-\frac{\bar\theta}{\bar\rho}\right)-\Pi_x\frac{n_2}{\rho}+\psi_1\psi_{1x}\\ \quad=-\frac{1}{\rho}\int v_1^2\mathbf{G}_x\,\mathrm{d}v,\\ \psi_{it}+\bar u_1\psi_{ix}=-\frac{1}{\rho}\int v_1v_i\mathbf{G}_x\,\mathrm{d}v,\quad i=2,3,\\ \omega_t+\bar u_1\omega_x+\theta_x\psi_1+\frac{2}{3}u_{1x}\omega+\frac{2}{3}\bar\theta\psi_{1x}+\frac{\Pi_x}{\rho}\left(\int v_1F_2\,\mathrm{d}v-u_1n_2\right)\\ \qquad=-\frac{1}{\rho}\int v_1\frac{|v|^2}{2}\mathbf{G}_x\,\mathrm{d}v+\frac{1}{\rho}\sum_{i=1}^3u_i\int v_1v_i\mathbf{G}_x\,\mathrm{d}v.\end{cases}\tag{B.1}$$

In fact, by the a-priori assumption (4.20), one also has from the system (B.1) that

$$\|(\phi,\psi,\omega)\|_{L_x^\infty}^2+\|(\phi_t,\psi_t,\omega_t)\|^2\leqq C(\chi_T+\delta_0)^2,$$

hence, one has

$$\|(\rho_t,u_t,\theta_t)\|^2\leqq C\|(\phi_t,\psi_t,\omega_t)\|^2+C\|(\bar\rho_t,\bar u_t,\bar\theta_t)\|^2\leqq C(\chi_T+\delta_0)^2.$$

For $|\alpha|=2$, it follows from (2.2) and (2.11) that

$$\|\partial^\alpha\left(\rho,\rho u,\rho(\theta+\frac{|u|^2}{2}),n_2\right)\|^2\leqq C\int\int\frac{|\partial^\alpha(F_1,F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\leqq C(\chi_T+\delta_0)^2,$$

and

$$\begin{aligned}\|\partial^\alpha(\rho,u,\theta)\|^2&\leqq C\|\partial^\alpha\left(\rho,\rho u,\rho\left(\theta+\frac{|u|^2}{2}\right)\right)\|^2\\&\quad+C\sum_{|\alpha'|=1}\int|\partial^{\alpha'}\left(\rho,\rho u,\rho\left(\theta+\frac{|u|^2}{2}\right)\right)|^4\,\mathrm{d}x\\&\leqq C(\chi_T+\delta_0)^2.\end{aligned}\tag{B.2}$$

Therefore, for $|\alpha| = 2$, we have

$$\|\partial^{\alpha}(\phi, \psi, \omega, n_2)\|^2 \leqq C(\chi_T + \delta_0)^2. \tag{B.3}$$

By (3.8) and a-priori assumption (4.20), it holds that

$$\begin{aligned}\|n_{2t}\|^2 &\leqq C\left[\|n_{2x}\|^2 + \int |n_2|^2|(\rho_x, u_x, \theta_x)|^2 \, \mathrm{d}x + \int\int \frac{|(\mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*} \, \mathrm{d}x \, \mathrm{d}v\right] \\ &\leqq C(\chi_T + \delta_0)^2.\end{aligned} \tag{B.4}$$

By (4.20), (B.3) and (B.4), for $|\alpha'| = 1$, it holds that

$$\|\partial^{\alpha'}(\phi, \psi, \omega, n_2)\|_{L^\infty}^2 \leqq C(\chi_T + \delta_0)^2.$$

By (A.7), one has

$$\|\Pi_{xt}\|^2 \leqq C\left[\|n_2\|^2 + \int\int \frac{|\mathbf{P}_c F_2|^2}{\mathbf{M}_*} \, \mathrm{d}x \, \mathrm{d}v\right] \leqq C(\chi_T + \delta_0)^2. \tag{B.5}$$

By (A.7), it holds that

$$\frac{1}{2}\Pi_{xtt} + \int v_1 F_{2t} \, \mathrm{d}v = 0. \tag{B.6}$$

Thus, one has

$$\begin{aligned}\|\Pi_{xtt}\|^2 &\leqq C\left[\|n_{2t}\|^2 + \int |n_2|^2|(\rho_t, u_t, \theta_t)|^2 \, \mathrm{d}x + \int\int \frac{|(\mathbf{P}_c F_2)_t|^2}{\mathbf{M}_*} \, \mathrm{d}x \, \mathrm{d}v\right] \\ &\leqq C(\chi_T + \delta_0)^2.\end{aligned} \tag{B.7}$$

By (B.5), (B.4), (B.7) and (B.3), it holds that

$$\|(\Pi_{xt}, \Pi_{xtt})\|_{L^\infty}^2 \leqq C\|(\Pi_{xt}, \Pi_{xtt})\|\|(n_{2t}, n_{2tt})\| \leqq C(\chi_T + \delta_0)^2.$$

Moreover, it holds that

$$\begin{aligned}\left\|\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v\right\|_{L_x^\infty} &\leqq C\left(\int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqq C(\chi_T + \delta_0)^2.\end{aligned}$$

Furthermore, for $|\alpha'| = 1$, it holds that

$$\begin{aligned}&\left\|\int \frac{|\partial^{\alpha'}(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v\right\|_{L_x^\infty} \leqq C\left(\int\int \frac{|\partial^{\alpha'}(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\cdot \left(\int\int \frac{|\partial^{\alpha'}(\mathbf{G}, \mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x\right)^{\frac{1}{2}} \leqq C\chi_T(\chi_T + \delta_0) \leq C(\chi_T + \delta_0)^2.\end{aligned} \tag{B.8}$$

Finally, by noticing the facts that $F_1 = \mathbf{M} + \mathbf{G}$ and $F_2 = \frac{n_2}{\rho}\mathbf{M} + \mathbf{P}_c F_2$ and (B.2) with $|\alpha| = 2$, it holds that

$$\begin{aligned}\int\int \frac{|\partial^\alpha(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x &\le C\int\int \frac{|\partial^\alpha(F_1, F_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x \\ &\quad + C\int\int \frac{|\partial^\alpha \mathbf{M}|^2 + |\partial^\alpha\left(\frac{n_2}{\rho}\mathbf{M}\right)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x \\ &\leqq C(\chi_T + \delta_0)^2,\end{aligned}$$

where in the last inequality we have used a similar argument as used for (B.8). We start from the lower order estimates. First, the entropy is defined by

$$-\frac{3}{2}\rho S = \int \mathbf{M}\ln \mathbf{M}\,\mathrm{d}v.$$

Multiplying the equation (3.4) by $\ln \mathbf{M}$ and integrating over v, it holds that

$$\begin{aligned}&\left(-\frac{3}{2}\rho S\right)_t + \left(-\frac{3}{2}\rho u_1 S\right)_x + \int \Pi_x \partial_{v_1} F_2 \ln \mathbf{M}\,\mathrm{d}v \\ &+ \left(\int v_1 \mathbf{G}\ln\mathbf{M}\,\mathrm{d}v\right)_x = \int v_1 \mathbf{G}(\ln \mathbf{M})_x\,\mathrm{d}v.\end{aligned}$$

Direct computations yields

$$\begin{aligned}S &= -\frac{2}{3}\ln\rho + \ln\left(\frac{4\pi}{3}\theta\right) + 1, \\ p &= \frac{2}{3}\rho\theta = k\rho^{\frac{5}{3}}\exp(S).\end{aligned}$$

Denote

$$\begin{aligned}\mathbf{X} &= \left(\rho, \rho u_1, \rho u_2, \rho u_3, \rho\left(\theta + \frac{|u|^2}{2}\right)\right)^t, \\ \mathbf{Y} &= \left(\rho u_1, \rho u_1^2 + p, \rho u_1 u_2, \rho u_1 u_3, \rho u_1\left(\theta + \frac{|u|^2}{2}\right) + p u_1\right)^t.\end{aligned}$$

Then the conservation law (3.6) can be rewritten as

$$\mathbf{X}_t + \mathbf{Y}_x = \begin{pmatrix} 0 \\ \frac{4}{3}\mu(\theta)u_{1x} - \int v_1^2 \Gamma\,\mathrm{d}v \\ \mu(\theta)u_{2x} - \int v_1 v_2 \Gamma\,\mathrm{d}v \\ \mu(\theta)u_{3x} - \int v_1 v_3 \Gamma\,\mathrm{d}v \\ \kappa(\theta)\theta_x + \frac{4}{3}\mu(\theta)u_1 u_{1x} + \sum_{i=2}^{3}\mu(\theta)u_i u_{ix} - \int \frac{1}{2}v_1|v|^2\Gamma\,\mathrm{d}v \end{pmatrix}_x.$$

We define an entropy–entropy flux pair (η, q) around a local Maxwellian $\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}$ as

$$\begin{cases} \eta = \bar\theta\left\{-\frac{3}{2}\rho S + \frac{3}{2}\bar\rho\bar S + \frac{3}{2}\nabla_{\mathbf{X}}(\rho S)|_{\mathbf{X}=\bar{\mathbf{X}}}\cdot(\mathbf{X}-\bar{\mathbf{X}})\right\}, \\ q = \bar\theta\left\{-\frac{3}{2}\rho u_1 S + \frac{3}{2}\bar\rho\bar u_1\bar S + \frac{3}{2}\nabla_{\mathbf{X}}(\rho S)|_{\mathbf{X}=\bar{\mathbf{X}}}\cdot(\mathbf{Y}-\bar{\mathbf{Y}})\right\}. \end{cases}$$

Here, we can compute that

$$(\rho S)_\rho = S + \frac{1}{\theta} + \frac{|u|^2}{2\theta} - \frac{5}{3}, \quad (\rho S)_{m_i} = -\frac{u_i}{\theta}, \quad i=1,2,3, \quad (\rho S)_E = \frac{1}{\theta},$$

and

$$\begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \bar\theta\rho S + \rho\left[\left(\bar S - \frac{5}{3}\right)\bar\theta + \frac{|u-\bar u|^2}{2}\right] + \frac{2}{3}\bar\rho\bar\theta\right\} \\ \quad = \rho\bar\theta\Psi\left(\frac{\bar\rho}{\rho}\right) + \frac{3}{2}\rho\bar\theta\Psi\left(\frac{\theta}{\bar\theta}\right) + \frac{3}{4}\rho|u-\bar u|^2, \\ q = u_1\eta + (u_1-\bar u_1)(\rho\theta - \bar\rho\bar\theta). \end{cases}$$

Then, for $\mathbf{X}$ in any closed bounded region in $\sum = \{\mathbf{X} : \rho > 0, \theta > 0\}$, there exists a positive constant C_0 such that

$$C_0^{-1}|\mathbf{X}-\bar{\mathbf{X}}|^2 \leqq \eta \leqq C_0|\mathbf{X}-\bar{\mathbf{X}}|^2.$$

Direct computations yield that

$$\begin{aligned} &\eta_t + q_x + \frac{2\bar\theta\mu(\theta)}{\theta}\psi_{1x}^2 + \frac{3\bar\theta\mu(\theta)}{2\theta}\sum_{i=2}^{3}\psi_{ix}^2 + \frac{3\bar\rho\kappa(\theta)}{2\theta^2}\omega_x^2 \\ &\quad - \left[\nabla_{(\bar\rho,\bar u,\bar S)}\eta\cdot(\bar\rho,\bar u,\bar S)_t + \nabla_{(\bar\rho,\bar u,\bar S)}q\cdot(\bar\rho,\bar u,\bar S)_x\right] \\ &= (\cdots)_x - \frac{3\bar\theta\kappa(\theta)}{2\theta^2}\omega_x\bar\theta_x + \frac{3\kappa(\theta)}{2\theta^2}(\omega_x+\bar\theta_x)\bar\theta_x\omega - \frac{2\bar\theta\mu(\theta)}{\theta}\psi_{1x}\bar u_{1x} \\ &\quad + \frac{2\mu(\theta)}{\theta}(\bar u_{1x}+\psi_{1x})\bar u_{1x}\omega \\ &\quad + \frac{3}{2}\frac{\bar\theta\omega_x - \omega\bar\theta_x}{\theta^2}\int v_1\frac{|v|^2}{2}\Gamma\,\mathrm{d}v - \frac{3}{2}\frac{\bar\theta\omega_x - \omega\bar\theta_x}{\theta^2} \\ &\quad \sum_{i=1}^{3}u_i\int v_1v_i\Gamma\,\mathrm{d}v + \frac{3}{2}\frac{\bar\theta\psi_{1x} - \bar u_{1x}\psi_1}{\theta}\int v_1^2\Gamma\,\mathrm{d}v \\ &\quad + \frac{3}{2}\frac{\bar\theta}{\theta}\sum_{i=2}^{3}u_{ix}\int v_1v_i\Gamma\,\mathrm{d}v + \frac{3}{2}\Pi_x n_2\psi_1 + \frac{3}{2\theta}\Pi_x\omega\int v_1\mathbf{P}_cF_2\,\mathrm{d}v. \end{aligned} \tag{B.9}$$

First, under the a-priori assumption (4.20), one has

$$-\left[\nabla_{(\bar\rho,\bar u,\bar S)}\eta\cdot(\bar\rho,\bar u,\bar S)_t + \nabla_{(\bar\rho,\bar u,\bar S)}q\cdot(\bar\rho,\bar u,\bar S)_x\right]$$

$$
\begin{aligned}
&= \frac{3}{2}\rho\bar{u}_{1x}(u_1-\bar{u}_1)^2 + \frac{2}{3}\rho\bar{\theta}\bar{u}_{1x}\Psi\left(\frac{\bar{\rho}}{\rho}\right) + \rho\bar{\theta}\bar{u}_{1x}\Psi\left(\frac{\theta}{\bar{\theta}}\right)\\
&\quad - \frac{3}{2}\rho\bar{\theta}_x(u_1-\bar{u}_1)\left(\frac{2}{3}\ln\frac{\bar{\rho}}{\rho} - \ln\frac{\theta}{\bar{\theta}}\right)\\
&\geqq C^{-1}\bar{u}_{1x}(\phi^2+\psi_1^2+\omega^2)
\end{aligned}
$$

for some positive constant C and the convex function

$$
\Psi(s) = s - \ln s - 1.
$$

Integrating (B.9) with respect to x, t over $\mathbf{R}^1 \times [0, t]$ yields that

$$
\begin{aligned}
&\int \eta(x,t)\,\mathrm{d}x + \int_0^t \left[\|(\psi_x,\omega_x)\|^2 + \|\sqrt{\bar{u}_{1x}}(\phi,\psi_1,\omega)\|^2\right]\mathrm{d}t\\
&\leqq C\int \eta(x,0)\,\mathrm{d}x + C\left|\int_0^t\int\left[-\frac{3\bar{\theta}\kappa(\theta)}{2\theta^2}\omega_x\bar{\theta}_x + \frac{3\kappa(\theta)}{2\theta^2}(\omega_x+\bar{\theta}_x)\bar{\theta}_x\omega\right]\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad + C\left|\int_0^t\int\left[-\frac{2\bar{\theta}\mu(\theta)}{\theta}\psi_{1x}\bar{u}_{1x} + \frac{2\mu(\theta)}{\theta}(\bar{u}_{1x}+\psi_{1x})\bar{u}_{1x}\omega\right]\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad + C\left|\int_0^t\int \frac{3}{2}\frac{\bar{\theta}\omega_x - \omega\bar{\theta}_x}{\theta^2}\left[\int v_1\frac{|v|^2}{2}\Gamma\,\mathrm{d}v - \sum_{i=1}^{3}u_i\int v_1v_i\Gamma\,\mathrm{d}v\right]\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad + C\left|\int_0^t\int\left[\frac{3}{2}\frac{\bar{\theta}\psi_{1x} - \bar{u}_{1x}\psi_1}{\theta}\int v_1^2\Gamma\,\mathrm{d}v + \frac{3}{2}\frac{\bar{\theta}}{\theta}\sum_{i=2}^{3}u_{ix}\int v_1v_i\Gamma\,\mathrm{d}v\right]\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\quad + C\left|\int_0^t\int\left[\frac{3}{2}\Pi_x n_2\psi_1 + \frac{3}{2\theta}\Pi_x\omega\int v_1\mathbf{P}_c F_2\,\mathrm{d}v\right]\mathrm{d}x\,\mathrm{d}\tau\right|\\
&:= C\int \eta(x,0)\,\mathrm{d}x + \sum_{i=1}^{5} I_i.
\end{aligned}
\tag{B.10}
$$

First, by integration by parts and the Cauchy inequality, it holds that

$$
\begin{aligned}
I_1 &= C\left|\int_0^t\int\left[\frac{3\bar{\theta}\kappa(\theta)}{2\theta^2}\omega\bar{\theta}_{xx} + \left(\frac{3\bar{\theta}\kappa(\theta)}{2\theta^2}\right)_x\omega\bar{\theta}_x + \frac{3\kappa(\theta)}{2\theta^2}(\omega_x+\bar{\theta}_x)\bar{\theta}_x\omega\right]\mathrm{d}x\,\mathrm{d}\tau\right|\\
&\leqq C\int_0^t\int |\omega|\left[|\bar{\theta}_{xx}| + |\bar{\theta}_x|^2 + |\omega_x||\bar{\theta}_x|\right]\mathrm{d}x\,\mathrm{d}\tau\\
&\leqq C\int_0^t \|\omega\|_{L^\infty}\left[\|\bar{\theta}_{xx}\|_{L^1} + \|\bar{\theta}_x\|_{L^2}^2 + \|\omega_x\|_{L^2}\|\bar{\theta}_x\|_{L^2}\right]\mathrm{d}\tau\\
&\leqq C\delta^{\frac{1}{8}}\int_0^t \|\omega_x\|_{L^2}^2\,\mathrm{d}\tau + C\delta^{\frac{1}{8}}\left[\int_0^t \|\sqrt{\eta}\|_{L^2}^2(1+\tau)^{-\frac{7}{6}}\,\mathrm{d}\tau + 1\right].
\end{aligned}
\tag{B.11}
$$

Similarly, one has

$$
I_2 \leqq C\delta^{\frac{1}{8}}\int_0^t \|(\omega_x,\psi_{1x})\|_{L^2}^2\,\mathrm{d}\tau + C\delta^{\frac{1}{8}}\left[\int_0^t \|\sqrt{\eta}\|_{L^2}^2(1+\tau)^{-\frac{7}{6}}\,\mathrm{d}\tau + 1\right]
$$

and

$$I_3 \leqq \sigma \int_0^t \|(\omega_x, \omega\bar{\theta}_x)\|_{L^2}^2 \,\mathrm{d}\tau + C_\sigma \int_0^t \int \left[\left| \int v_1 \frac{|v|^2}{2} \Gamma \,\mathrm{d}v \right|^2 + \sum_{i=1}^3 \left| \int v_1 v_i \Gamma \,\mathrm{d}v \right|^2 \right] \mathrm{d}x \,\mathrm{d}\tau, \quad \text{(B.12)}$$

with some small positive constant $\sigma > 0$ to be determined and the positive constant C_σ depending on σ. Note that by (2.14), it holds that

$$\left| \int v_1 \frac{|v|^2}{2} \Gamma \,\mathrm{d}v \right| = \left| \int v_1 \frac{|v|^2}{2} \mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{G}_t + \mathbf{P}_1(v_1 \mathbf{G}_x) + \mathbf{P}_1(\Pi_x \partial_{v_1} F_2) - 2Q(\mathbf{G}, \mathbf{G})] \,\mathrm{d}v \right| := \sum_{i=1}^4 I_{3i}. \quad \text{(B.13)}$$

Choose the global Maxellian $\mathbf{M}_* = \mathbf{M}_{[\rho_*, u_*, \theta_*]}$ such that

$$\rho_* > 0, \quad \frac{1}{2}\theta(t,x) < \theta_* < \theta(t,x) \quad \text{(B.14)}$$

and

$$|\rho(x,t) - \rho_*| + |u(x,t) - u_*| + |\theta(x,t) - \theta_*| < \eta_0, \quad \text{(B.15)}$$

with η_0 being the small positive constant in Lemma 2.2. Then with such a chosen $\mathbf{M}_*$, it holds that

$$I_{31} = \left| \int v_1 \frac{|v|^2}{2} \mathbf{L}_{\mathbf{M}}^{-1} \mathbf{G}_t \,\mathrm{d}v \right| \leqq \left(\int \frac{\nu(|v|) |\mathbf{L}_{\mathbf{M}}^{-1} \mathbf{G}_t|^2}{\mathbf{M}_*} \,\mathrm{d}v \right)^{\frac{1}{2}} \left(\int \mathbf{M}_* \nu(|v|)^{-1} \left(v_1 \frac{|v|^2}{2} \right)^2 \mathrm{d}v \right)^{\frac{1}{2}} \leqq C \left(\int \frac{\nu(|v|)^{-1} |\mathbf{G}_t|^2}{\mathbf{M}_*} \,\mathrm{d}v \right)^{\frac{1}{2}} \quad \text{(B.16)}$$

and

$$I_{32} = \left| \int v_1 \frac{|v|^2}{2} \mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(v_1 \mathbf{G}_x)] \,\mathrm{d}v \right| \leqq C \left(\int \frac{\nu(|v|) |\mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(v_1 \mathbf{G}_x)]|^2}{\mathbf{M}_{[\rho_*, u_*, 2\theta_*]}} \,\mathrm{d}v \right)^{\frac{1}{2}} \leqq C \left(\int \frac{\nu(|v|)^{-1} |\mathbf{P}_1(v_1 \mathbf{G}_x)|^2}{\mathbf{M}_{[\rho_*, u_*, 2\theta_*]}} \,\mathrm{d}v \right)^{\frac{1}{2}} \leqq C \left(\int \frac{\nu(|v|)^{-1} |\mathbf{G}_x|^2}{\mathbf{M}_*} \,\mathrm{d}v \right)^{\frac{1}{2}}. \quad \text{(B.17)}$$

Furthermore, one has

$$
\begin{aligned}
I_{33} &= \left| \int v_1 \frac{|v|^2}{2} \mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(\Pi_x \partial_{v_1} F_2)] \, \mathrm{d}v \right| \\
&\leqq C|\Pi_x| \Big(\int \frac{\nu(|v|) |\mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(v_1 \mathbf{G}_x)]|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \\
&\leqq C|\Pi_x| \Big(\int \frac{\nu(|v|)^{-1} |\frac{n_2}{\rho} \mathbf{M}_{v_1} + \partial_{v_1}(\mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \\
&\leqq C|\Pi_x||n_2| + C|\Pi_x| \Big(\int \frac{\nu(|v|)^{-1} |\partial_{v_1}(\mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}}
\end{aligned} \tag{B.18}
$$

and

$$
\begin{aligned}
I_{34} &= \left| \int v_1 \frac{|v|^2}{2} \mathbf{L}_{\mathbf{M}}^{-1} [4Q(\mathbf{G}, \mathbf{G})] \, \mathrm{d}v \right| \leqq C \Big(\int \frac{\nu(|v|) |\mathbf{L}_{\mathbf{M}}^{-1} [Q(\mathbf{G}, \mathbf{G})]|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \\
&\leqq C \Big(\int \frac{\nu(|v|)^{-1} |Q(\mathbf{G}, \mathbf{G})|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \leqq C \Big(\int \frac{\nu(|v|) |\mathbf{G}|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \Big(\int \frac{|\mathbf{G}|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \\
&\leqq C \Big(\int \frac{\nu(|v|) |\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \Big(\int \frac{|\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} + C|(\bar{\theta}_x, \bar{u}_{1x})| \Big(\int \frac{\nu(|v|) |\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*} \, \mathrm{d}v \Big)^{\frac{1}{2}} \\
&\quad + C|(\bar{\theta}_x, \bar{u}_{1x})|^2.
\end{aligned} \tag{B.19}
$$

Substituting (B.16)–(B.19) into (B.13) and then into (B.12), one can arrive at

$$
\begin{aligned}
I_3 &\leqq \Big(\sigma + C\delta^{\frac{1}{8}} \Big) \int_0^t \|\omega_x\|_{L^2}^2 \, \mathrm{d}\tau + C\delta^{\frac{1}{8}} \Big[\int_0^t \|\sqrt{\eta}\|_{L^2}^2 (1+\tau)^{-\frac{7}{6}} \, \mathrm{d}\tau + 1 \Big] \\
&\quad + C_\sigma \int_0^t \int \int \frac{\nu(|v|)^{-1} |(\mathbf{G}_t, \mathbf{G}_x)|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau + C\chi_T \int_0^t \|(\Pi_x, n_2)\|^2 \, \mathrm{d}\tau \\
&\quad + C\chi_T \int_0^t \int \int \frac{\nu(|v|) |\mathbf{P}_c F_2|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + C(\chi_T + \delta) \int_0^t \int \int \frac{\nu(|v|)^{-1} |\widetilde{\mathbf{G}}|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned} \tag{B.20}
$$

Similar estimates hold for I_4. Then I_5 can be estimated by

$$
I_5 \leqq C\chi_T \int_0^t \|(\Pi_x, n_2)\|^2 \, \mathrm{d}\tau + C\chi_T \int_0^t \int \int \frac{\nu(|v|) |\mathbf{P}_c F_2|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau. \tag{B.21}
$$

Substituting the estimates for $I_i (i = 1, 2, 3, 4, 5)$ in (B.11)–(B.21) into (B.10) and applying Gronwall inequality yields the first-step lower order estimates

$$
\begin{aligned}
&\|(\phi, \psi, \omega)(\cdot, t)\|^2 + \int_0^t \|(\psi_x, \omega_x)\|^2 \, \mathrm{d}\tau + \int_0^t \|\sqrt{\bar{u}_{1x}} (\phi, \psi_1, \omega)\|^2 \, \mathrm{d}\tau \\
&\leqq C\|(\phi, \psi, \omega)(\cdot, 0)\|^2 \\
&\quad + C\delta^{\frac{1}{8}} + C \int_0^t \int \int \frac{\nu(|v|)^{-1} |(\mathbf{G}_t, \mathbf{G}_x)|^2}{\mathbf{M}_*} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau
\end{aligned}
$$

$$+C(\chi_T+\delta)\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+C\chi_T\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau. \tag{B.22}$$

Next we want to get the estimation of $\|\phi_x\|^2$. By $(4.16)_1$ and $(4.16)_2$, it holds that

$$\begin{aligned}&\frac{4\mu(\theta)}{3\rho\bar\rho}\phi_{xt}+\psi_{1t}+\frac{2}{3}\omega_x+\frac{2\bar\theta}{3\bar\rho}\phi_x+\bar u_1\psi_{1x}+\bar u_{1x}\psi_1\\&\quad+\frac{2}{3}\rho_x\left(\frac{\theta}{\rho}-\frac{\bar\theta}{\bar\rho}\right)-\Pi_x\frac{n_2}{\rho}+\psi_1\psi_{1x}\\&=-\frac{4}{3\rho}\Big(\frac{\mu(\theta)}{\bar\rho}\Big)_x\phi_t-\frac{4}{3\rho}\left[\mu(\theta)\frac{\bar u_1\phi_x+\bar u_{1x}\phi+\bar\rho_x\psi_1+(\phi\psi_1)_x}{\bar\rho}\right]_x\\&\quad+\frac{4}{3\rho}\Big(\mu(\theta)\bar u_{1x}\Big)_x-\frac{1}{\rho}\int v_1^2\Gamma_x\,\mathrm{d}v.\end{aligned}$$

Multiplying the above equation by ϕ_x, one has

$$\begin{aligned}&\left(\frac{2\mu(\theta)}{3\rho\bar\rho}\phi_x^2+\psi_1\phi_x\right)_t+\frac{2\bar\theta}{3\bar\rho}\phi_x^2=\left(\frac{2\mu(\theta)}{3\rho\bar\rho}\right)_t\phi_x^2+\frac{2}{3}\omega_x\phi_x+\bar u_{1x}\phi_x\psi_1\\&+\psi_{1x}\Big[\bar\rho\psi_{1x}+\bar u_{1x}\phi+\bar\rho_x\psi_1+(\phi\psi_1)_x\Big]\\&-\frac{2}{3}\rho_x\phi_x\left(\frac{\theta}{\rho}-\frac{\bar\theta}{\bar\rho}\right)+\Pi_x\phi_x\frac{n_2}{\rho}+\psi_1\psi_{1x}\phi_x-\frac{4}{3\rho}\left(\frac{\mu(\theta)}{\bar\rho}\right)_x\phi_t\phi_x\\&-\frac{4}{3\rho}\left[\mu(\theta)\frac{\bar u_1\phi_x+\bar u_{1x}\phi+\bar\rho_x\psi_1\phi_x+(\phi\psi_1)_x}{\bar\rho}\right]_x\phi_x\\&+\frac{4}{3\rho}\Big(\mu(\theta)\bar u_{1x}\Big)_x\phi_x-\frac{1}{\rho}\int v_1^2\Gamma_x\,\mathrm{d}v\phi_x.\end{aligned} \tag{B.23}$$

Note that

$$\begin{aligned}&\int_0^t\int|\int v_1^2\Gamma_x\,\mathrm{d}v|^2\,\mathrm{d}x\,\mathrm{d}\tau\leqq C\sum_{|\alpha|=2}\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha\mathbf{G}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C\delta^{\frac12}\\&+C(\chi_T+\delta)\int_0^t\Big[\|(\Pi_x,n_2,\phi_x,\psi_x,\omega_x)\|^2\\&+\sum_{|\alpha'|=1}\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\partial^{\alpha'}\mathbf{G})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\\&+\int\int\frac{\nu(|v|)|(\mathbf{P}_cF_2)_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\Big]\,\mathrm{d}\tau\end{aligned} \tag{B.24}$$

and

$$
\begin{aligned}
&|\int_0^t \int -\frac{4}{3\rho}\Big(\mu(\theta)\frac{\bar{u}_1\phi_x}{\bar{\rho}}\Big)_x \phi_x \,\mathrm{d}x\,\mathrm{d}\tau| = |\int_0^t \int \mu(\theta)\frac{\bar{u}_1\phi_x}{\bar{\rho}}\Big(\frac{4\phi_x}{3\rho}\Big)_x \mathrm{d}x\,\mathrm{d}\tau| \\
&= |\int_0^t \int \mu(\theta)\frac{\bar{u}_1\phi_x}{\bar{\rho}}\Big(\frac{4\phi_{xx}}{3\rho} - \frac{4\phi_x\rho_x}{3\rho^2}\Big)\mathrm{d}x\,\mathrm{d}\tau| \\
&= |\int_0^t \int \Big[-\mu(\theta)\frac{\bar{u}_1\phi_x}{\bar{\rho}}\frac{4\phi_x\rho_x}{3\rho^2} - \Big(\frac{4\mu(\theta)\bar{u}_1}{3\rho\bar{\rho}}\Big)_x \frac{\phi_x^2}{2}\Big]\mathrm{d}x\,\mathrm{d}\tau| \\
&\leqq C(\chi_T+\delta)\int_0^t \|\phi_x\|^2\,\mathrm{d}\tau. \qquad\qquad (B.25)
\end{aligned}
$$

Integrating the equation (B.23) with respect to x, t, then using Cauchy inequality and (B.24)-(B.25) and choosing χ_T, δ suitably small, one can obtain

$$
\begin{aligned}
&\|\phi_x(\cdot,t)\|^2 + \int_0^t \|\phi_x\|^2\,\mathrm{d}\tau \leqq C\Big[\|\psi_1(\cdot,t)\|^2 + \|(\phi_{0x},\psi_{10})\|^2 + \delta^{\frac{1}{2}}\Big] \\
&+C\int_0^t \|(\psi_{1x},\omega_x)\|^2\,\mathrm{d}\tau \\
&+C(\sqrt{\delta}+\chi_T)\int_0^t \|(\Pi_x, n_2, \psi_{1xx}, \phi_{xx})\|^2\,\mathrm{d}\tau \\
&+C\sum_{|\alpha|=2}\int_0^t\int\int \frac{\nu(|v|)}{\mathbf{M}_*}|\partial^\alpha \mathbf{G}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&+C(\chi_T+\sqrt{\delta})\int_0^t \|\sqrt{\bar{u}_{1x}}(\psi_1,\phi,\omega)\|^2\,\mathrm{d}\tau \\
&+C(\delta+\chi_T)\int_0^t\int\int \frac{\nu(|v|)|(\mathbf{P}_c F_2)_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&+C(\delta+\chi_T)\sum_{|\alpha'|=1}\int_0^t\int\int \frac{\nu(|v|)}{\mathbf{M}_*}|(\widetilde{\mathbf{G}}, \partial^{\alpha'}\mathbf{G})|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau. \qquad\qquad (B.26)
\end{aligned}
$$

Then we estimate $\|(\phi,\psi,\omega)_t\|^2$. For this, we use the system (B.1). By the equation $(B.1)_2$, one has

$$
\begin{aligned}
\int_0^t \|\psi_{1t}\|^2\,\mathrm{d}\tau \leqq{}& C\int_0^t \|(\phi_x,\psi_x,\omega_x)\|^2\,\mathrm{d}\tau + C(\chi_T+\delta) \\
&\int_0^t \Big[\|\sqrt{\bar{u}_{1x}}(\psi_1,\phi,\omega)\|^2 + \|n_2\|^2\Big]\mathrm{d}\tau \\
&+C\int_0^t\int\int \frac{\nu(|v|)}{\mathbf{M}_*}|\mathbf{G}_x|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Similar estimates hold for ϕ_t, ψ_{2t}, ψ_{3t} and ω_t. Therefore, one can arrive at

$$
\begin{aligned}
\int_0^t \|(\phi_t,\psi_t,\omega_t)\|^2\,\mathrm{d}\tau \leqq{}& C\int_0^t \|(\phi_x,\psi_x,\omega_x)\|^2\,\mathrm{d}\tau \\
&+C(\chi_T+\delta)\int_0^t \Big[\|\sqrt{\bar{u}_{1x}}(\psi_1,\phi,\omega)\|^2 + \|n_2\|^2\Big]\mathrm{d}\tau
\end{aligned}
$$

$$+C\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\mathbf{G}_x|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau+C(\chi_T+\delta)$$
$$\int_0^t\int\int\frac{\nu(|v|)}{\mathbf{M}_*}|\mathbf{P}_cF_2|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau. \tag{B.27}$$

By (B.22), (B.26) and (B.27), it holds that

$$\begin{aligned}
&\|(\phi,\psi,\omega,\phi_x)(\cdot,t)\|^2+\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}\tau+\int_0^t\|\sqrt{\bar{u}_{1x}}(\phi,\psi_1,\omega)\|^2\,\mathrm{d}\tau\\
&\leqq C\|(\phi,\psi,\omega,\phi_x)(\cdot,0)\|^2+C\delta^{\frac{1}{8}}\\
&\quad+C(\chi_T+\delta)\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2,(\mathbf{P}_cF_2)_{v_1})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\sum_{1\leqq|\alpha|\leqq 2}\int_0^t\int\int\frac{\nu(|v|)^{-1}|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\sqrt{\delta})\int_0^t\|(\Pi_x,n_2,\psi_{1xx},\phi_{xx})\|^2\,\mathrm{d}\tau.
\end{aligned}$$

Next we do the microscopic estimates for the Vlasov–Poisson–Boltzmann system. Multiplying the equation (4.19) and the equation (3.10) by $\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}$ and $\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}$, respectively, one has

$$\begin{aligned}
\left(\frac{|\widetilde{\mathbf{G}}|^2}{2\mathbf{M}_*}\right)_t-\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*}\mathbf{L}_\mathbf{M}\widetilde{\mathbf{G}}=\Big\{&-\frac{3}{2\theta}\mathbf{L}_\mathbf{M}^{-1}\Big[\mathbf{P}_1\Big(v_1\Big(\frac{|v-u|^2}{2\theta}\omega_x+v\cdot\psi_x\Big)\Big)\mathbf{M}\Big]\\
&-\mathbf{P}_1(v_1\mathbf{G}_x)-\mathbf{P}_1(\Pi_x\partial_{v_1}F_2)\\
&+2Q(\mathbf{G},\mathbf{G})-\bar{\mathbf{G}}_t\Big\}\frac{\widetilde{\mathbf{G}}}{\mathbf{M}_*},
\end{aligned}\tag{B.28}$$

and

$$\begin{aligned}
\left(\frac{|\mathbf{P}_cF_2|^2}{2\mathbf{M}_*}\right)_t-\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}\mathbf{N}_\mathbf{M}(\mathbf{P_c}F_2)=\Big[&-v_1\partial_xF_2-\Big(\frac{n_2}{\rho}\mathbf{M}\Big)_t\\
&-\mathbf{P}_c(\Pi_x\partial_{v_1}F_1)+2Q(F_2,\mathbf{G})\Big]\frac{\mathbf{P}_cF_2}{\mathbf{M}_*}.
\end{aligned}\tag{B.29}$$

By using methods similar to those for obtaining (A.1.31), one can derive from (B.28) and (B.29) that

$$\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,t)\,\mathrm{d}x\,\mathrm{d}v+\|(\Pi_x,n_2)(\cdot,t)\|^2$$
$$+\int_0^t\int\int\frac{\nu(|v|)\Big|\Big(\widetilde{\mathbf{G}},\mathbf{P}_cF_2\Big)\Big|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau$$

$$
\begin{aligned}
&\leqq C\int\int\frac{|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}(x,v,0)\,\mathrm{d}x\,\mathrm{d}v+C\|(\Pi_{0x},n_{20})\|^2\\
&\quad+C\|(\phi_0,\psi_0,\omega_0)\|^2+C\delta^{\frac{1}{8}}\\
&\quad+C(\chi_T+\delta+\eta_0)\int_0^t\|(\Pi_x,n_{2x},n_{2t})\|^2\,\mathrm{d}\tau+C(\chi_T+\delta)\\
&\quad\int_0^t\Big[\|n_2\|^2+\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\omega)\|^2\Big]\mathrm{d}\tau\\
&\quad+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\alpha'}\mathbf{G}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Similarly to (A.1.35) and (A.1.36), one has

$$
\begin{aligned}
&\|\Pi_x\|^2(t)+\int_0^t\|(\Pi_x,n_2)\|^2\,\mathrm{d}\tau\leqq C\|\Pi_x\|^2(0)\\
&\quad+C(\chi_T+\delta)\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\alpha'}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta)\Big[\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\mathbf{P}_cF_2)|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^t\int\int\frac{\nu(|v|)|\widetilde{\mathbf{G}}_{v_1}|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\|n_2\|^2(t)+\int_0^t\|(n_{2x},n_2,\sqrt{\bar{u}_{1x}}n_2)\|^2\,\mathrm{d}\tau\leqq C\|n_{20}\|^2\\
&\quad+C(\chi_T+\delta)\int_0^t\|\Pi_x\|^2\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta)\sum_{|\alpha|=1}\int_0^t\|\partial^\alpha(\phi,\psi,\omega)\|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\sum_{|\alpha'|=1}\int_0^t\int\int\frac{\nu(|v|)|\partial^{\alpha'}(\mathbf{P}_cF_2))|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C(\chi_T+\delta)\Big[\int_0^t\int\int\frac{\nu(|v|)|\mathbf{P}_cF_2|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^t\int\int\frac{\nu(|v|)|(\widetilde{\mathbf{G}},\widetilde{\mathbf{G}}_{v_1})|^2}{\mathbf{M}_*}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau\Big].
\end{aligned}
$$

By (B.6), it holds that

$$\int_0^t \|\Pi_{xt}\|^2 \, d\tau \leqq C \int_0^t \left[\|n_2\|^2 + \int\int \frac{\nu(|v|)|\mathbf{P}_c F_2|^2}{\mathbf{M}_*} \, dx \, dv \right] d\tau.$$

By the equation (3.8), one has

$$\begin{aligned}\int_0^t \|n_{2t}\|^2 \, d\tau &= \int_0^t \| \int v_1 F_{2x} \, dv\|^2 \, d\tau = \int_0^t \| \int v_1 \left(\frac{\mathbf{M}}{\rho} n_2 + \mathbf{P}_c F_2 \right)_x dv\|^2 \, d\tau \\ &\leqq C \int_0^t \left[\|n_{2x}\|^2 + \int\int \frac{\nu(|v|)|(\mathbf{P}_c F_2)_x|^2}{\mathbf{M}_*} \, dx \, dv \right] d\tau \\ &\quad + C(\chi_T + \delta) \int_0^t \|(n_2, \phi_x, \psi_x, \omega_x)\|^2 \, d\tau.\end{aligned}$$

In summary, collecting all the above lower order estimates and choosing suitably small χ_T, δ and η_0, we arrive at

$$\begin{aligned}&\|(\phi, \psi, \omega, \phi_x, \Pi_x, n_2)(\cdot, t)\|^2 + \int\int \frac{|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*}(x, v, t) \, dx \, dv \\ &\quad + \int_0^t \|\sqrt{\bar{u}_{1x}}(\phi, \psi_1, \omega)\|^2 \, d\tau \\ &\quad + \sum_{|\alpha'|=1} \int_0^t \|\partial^{\alpha'}(\phi, \psi, \omega, n_2)\|^2 \, d\tau \\ &\quad + \int_0^t \|(\Pi_x, \Pi_{xt}, n_2)\|^2 \, d\tau + \int_0^t \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, dx \, dv \, d\tau \\ &\leqq C\|(\phi_0, \psi_0, \omega_0, \phi_{0x}, \Pi_{0x}, n_{20})\|^2 + C\delta^{\frac{1}{8}} \\ &\quad + C \int\int \frac{|(\widetilde{\mathbf{G}}_0, \mathbf{P}_c F_{20})|^2}{\mathbf{M}_*}(x, v) \, dx \, dv \\ &\quad + C(\chi_T + \delta) \int_0^t \left[\|(\psi_{1xx}, \phi_{xx})\|^2 \right. \\ &\quad \left. + \int\int \frac{\nu(|v|)|(\widetilde{\mathbf{G}}, \mathbf{P}_c F_2)_{v_1}|^2}{\mathbf{M}_*} \, dx \, dv \right] d\tau \\ &\quad + C \sum_{|\alpha|=2} \int_0^t \int\int \frac{\nu(|v|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_*} \, dx \, dv \, d\tau \\ &\quad + C \sum_{|\alpha'|=1} \int_0^t \int\int \frac{\nu(|v|)|\partial^{\alpha'}(\mathbf{G}, \mathbf{P}_c F_2)|^2}{\mathbf{M}_*} \, dx \, dv \, d\tau.\end{aligned}$$

The higher order estimates can be done similarly as to "Appendix A" and will be skipped for brevity.

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