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# A STUDY ON THE BOUNDARY LAYER FOR THE PLANAR MAGNETOHYDRODYNAMICS SYSTEM*

Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

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**Abstract** The paper aims to estimate the thickness of the boundary layer for the planar MHD system with vanishing shear viscosity $\mu$. Under some conditions on the initial and boundary data, we show that the thickness is of the order $\sqrt{\mu}|\ln\mu|$. Note that this estimate holds also for the Navier-Stokes system so that it extends the previous works even without the magnetic effect.

**Key words** magnetohydrodynamics system; boundary layer; vanishing shear viscosity

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## 1 Introduction

Consider the planar magnetohydrodynamics system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \Big(\rho u^2 + p + \dfrac{1}{2}|\mathbf{b}|^2\Big)_x = \lambda u_{xx}, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x = \mu \mathbf{w}_{xx}, \\ \mathbf{b}_t + (u\mathbf{b} - \mathbf{w})_x = \nu \mathbf{b}_{xx}, \\ (\rho e)_t + (\rho e u)_x - (\kappa\theta_x)_x + p u_x = \lambda u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2 =: \mathcal{Q}, \end{cases} \tag{1.1}$$

where $\rho$ denotes the density of the flow, $\theta$ the temperature, $u \in \mathbb{R}$ the longitudinal velocity, $\mathbf{w} \in \mathbb{R}^2$ the transversal velocity, $\mathbf{b} \in \mathbb{R}^2$ the transversal magnetic field, $p = p(\rho,\theta)$ the pressure and $e = e(\rho,\theta) > 0$ the internal energy, respectively. Moreover, $\kappa = \kappa(\rho,\theta)$ is the heat

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conductivity coefficient, and the positive constants $\lambda$, $\mu$ and $\nu$ represent the bulk viscosity, shear viscosity and the magnetic diffusivity coefficients, respectively. In this paper, we consider the perfect gas with the equation of state given by

$$p = R\rho\theta, \qquad e = C_v\theta, \tag{1.2}$$

with physical constants $R > 0$ and $C_v > 0$. Without loss of generality, set $C_v = 1$. Motivated by some physical models, such as the Boltzmann collision operator, assume $\kappa$ depends only on $\theta$ as

$$\kappa = \kappa(\theta) = \theta^q, \quad q > 0. \tag{1.3}$$

In this paper, we consider the system (1.1) in a bounded domain $Q_T = \Omega \times (0, T)$ with $\Omega = (0, 1)$ under the following initial and boundary conditions:

$$\begin{cases} (\rho, u, \theta, \mathbf{w}, \mathbf{b})(x, 0) = (\rho_0, u_0, \theta_0, \mathbf{w}_0, \mathbf{b}_0)(x), \\ (u, \mathbf{b}, \theta_x)|_{x=0,1} = \mathbf{0}, \\ \mathbf{w}(0, t) = \mathbf{w}_1(t), \quad \mathbf{w}(1, t) = \mathbf{w}_2(t), \end{cases} \tag{1.4}$$

and try to understand the thickness of the boundary layer when the shear viscosity $\mu$ vanishes.

For this, let us first review some of the related works. First of all, the MHD system has been extensively studied because of its physical importance and mathematical difficulties, cf. [1, 2, 4, 11, 14, 15, 22–24] and the references therein.

Without the boundary effect, Vol'pert and Hudjaev [24] proved the existence and uniqueness of local solutions to this system. Some results on the system with small initial data were obtained in [10, 17, 19, 20]. When the heat conductivity coefficient $\kappa$ is of the order of $\theta^q$ for some $q > 0$, some global existence solutions to the system (1.1) with large initial data were studied in [2, 3, 21] for $q \geq 2$, in [5] for $q \geq 1$ and in [6] for $q > 0$. On the other hand, for the case $q = 0$, the problem on the global existence of smooth solution to (1.1)–(1.4) with large initial data remains unsolved even though the corresponding problem for the Navier-Stokes equations was solved in [13] long time ago.

For problem in a bounded domain, the presence of boundary layer is a fundamental problem in fluid dynamics that can be traced back to the seminal work by Prandtl in 1904. For this, some results on the vanishing shear viscosity for the Navier-Stokes equations can be found in [7–9, 12, 19, 25] and the references therein. With the effect of magnetic field, the vanishing shear viscosity for the planar flow was studied in [5, 6] under the following condition on $\kappa$:

$$C^{-1}(1 + \theta^q) \leq \kappa \equiv \kappa(\theta) \leq C(1 + \theta^q)\,(q > 0), \text{ or, } \kappa \equiv \kappa(\rho) \geq C/\rho, \tag{1.5}$$

that avoids the degeneracy of $\theta$ around zero.

Without the magnetic effect, Frid and Shelukhin in [8] investigated the boundary layer of the compressible isentropic Navier-Stokes equations with cylindrical symmetry, and estimated the thickness of boundary layer (cf. Definition 1.1 below) in the order of $O(\mu^\alpha)(0 < \alpha < \frac{1}{2})$. For the non-isentropic Navier-Stokes equations, by imposing the following assumption on $\kappa$:

$$C^{-1}(1 + \theta^q) \leq \kappa(\rho, \theta) \leq C(1 + \theta^q), \quad |\kappa_\rho(\rho, \theta)| \leq C(1 + \theta^q) \quad (q > 1), \tag{1.6}$$

Jiang and Zhang in [12] obtained the same thickness estimate. In a recent paper [18], we improved these results to remove the constraint on the heat conductivity coefficient.

The purpose of this paper is to estimate the thickness of the boundary layer for the vanishing shear viscosity with the magnetic effect under some physical condition on the heat conductivity coefficient.

As in [8], the thickness of a boundary layer is defined as follows.

**Definition 1.1** A function $\delta(\mu)$ is called the thickness function of the boundary layer, denoted by BL-thickness, of the problem (1.1)–(1.4) with vanishing shear viscosity $\mu$ if $\delta(\mu) \to 0$ as $\mu \to 0$, and

$$\lim_{\mu\to 0} \|(\rho-\overline{\rho}, u-\overline{u}, \mathbf{w}-\overline{\mathbf{w}}, \mathbf{b}-\overline{\mathbf{b}}, \theta-\overline{\theta})\|_{L^\infty(0,T;L^\infty(\Omega_{\delta(\mu)}))} = 0,$$
$$\inf_{\mu\to 0}\lim \|(\rho-\overline{\rho}, u-\overline{u}, \mathbf{w}-\overline{\mathbf{w}}, \mathbf{b}-\overline{\mathbf{b}}, \theta-\overline{\theta})\|_{L^\infty(0,T;L^\infty(\Omega))} > 0,$$

where $\Omega_\delta = (\delta, 1-\delta)$ for $\delta \in (0, 1/2)$, and $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$ and $(\overline{\rho}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}, \overline{\theta})$ are the solutions to the problem (1.1)–(1.4) and the problem (1.1)–(1.4) with $\mu = 0$ respectively.

To obtain a lower bound of $\delta(\mu)$, as in [8, 12], we only consider the system (1.1) with the following initial boundary conditions:

$$\rho_0 \equiv \text{constant} > 0, \quad u_0 \equiv 0, \quad \mathbf{v}_0 \equiv \mathbf{0}, \quad \mathbf{b}_0 \equiv \mathbf{0}, \quad \theta_0 \equiv \text{constant} > 0, \tag{1.7}$$

$$\mathbf{w}_i(t) \in \big(C^1[0,T]\big)^2, \quad \mathbf{w}_i(0) = 0, \quad i = 1, 2. \tag{1.8}$$

Note that $(\overline{\rho}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}, \overline{\theta}) \equiv (\rho_0, 0, \mathbf{0}, \mathbf{0}, \theta_0)$ is a solution for the problem (1.1)–(1.4) with $\mu = 0$.

The main result of the paper is stated as follows.

**Theorem 1.2** Let the initial and boundary functions satisfy (1.7) and (1.8). Then

(1) the problem (1.1)–(1.4) admits a unique solution;

(2) the following $L^2$ estimates hold

$$\begin{aligned}\sup_{0<t<T}\int_\Omega \big[(\rho-\rho_0)^2 + u^2 + (\theta-\theta_0)^2 + u_x^2 + \rho_x^2 + |\mathbf{b}|^2 + |\mathbf{w}|^2\big]\mathrm{d}x \\ + \iint_{Q_T} \big(u_t^2 + \theta_x^2 + u_{xx}^2\big)\mathrm{d}x\mathrm{d}t \le C\sqrt{\mu}, \\ \sup_{0<t<T}\int_\Omega (|\mathbf{b}_x|^2 + \theta_x^2)\mathrm{d}x + \iint_{Q_T} |\mathbf{b}_t|^2 \mathrm{d}x\mathrm{d}t \le C\mu^{1/4};\end{aligned} \tag{1.9}$$

(3) any function $\delta(\mu)$ satisfying $\delta(\mu) \to 0$ and $\sqrt{\mu}/\delta(\mu) \to 0$ as $\mu \to 0$ is a BL-thickness for the problem (1.1)–(1.4) when the boundary value $(\mathbf{w}_1, \mathbf{w}_2)$ is not identically zero. In this case,

$$\int_\delta^{1-\delta} |\mathbf{w}_x|\mathrm{d}x \le C\Big(\mu^{1/8} + \frac{\sqrt{\mu}}{\delta}\Big), \quad \forall \delta \in (0, 1/2). \tag{1.10}$$

Here and in the following, we use $C$ to denote a positive generic constant independent of $\mu$.

Note that the existence of solution stated in Theorem 1.2 was proved in [6]. Thus, we only need to show (1.9) and (1.10) for the estimation on the BL-thickness, as in [8]. However, the arguments used in [8, 12] for the Navier-Stokes equations can not be applied directly to the system (1.1) because of the strong interaction between the fluid variables with the magnetic field.

For example, without the magnetic field, the following pointwise estimate on $u$:

$$u \le C\xi_\delta(x), \quad x \in [0, \delta]; \quad u \ge -C\xi_\delta(x), \quad x \in [1-\delta, 1], \tag{1.11}$$

with

$$\xi_\delta(x)=\begin{cases}x, & 0\le x\le\delta,\\ \delta, & \delta<x<1-\delta,\\ 1-x, & 1-\delta\le x\le 1\end{cases}$$

can be obtained by either the maximal principle as in [8] or by obtaining a uniform bound on $u_x$ as in [12, 18]. However, it seems very difficult to achieve this estimate in the appearance of the magnetic field $\mathbf{b}$.

One of the key observations in this paper is that since

$$\begin{aligned}&|u(x,t)|\le x\|u_x\|_{L^\infty(\Omega)}\le\xi_\delta(x)\int_\Omega|u_{xx}|\mathrm{d}x,\forall x\in[0,\delta],\\ &|u(x,t)|\le(1-x)\|u_x\|_{L^\infty(\Omega)}\le\xi_\delta(x)\int_\Omega|u_{xx}|\mathrm{d}x,\forall x\in[1-\delta,1],\end{aligned}\tag{1.12}$$

the estimate (1.11) can be replaced by (1.12) if $u_{xx}$ is uniformly bounded in $L^1(Q_T)$. Indeed, for $q>1$, we can show by using an argument similar to [8, 12] that $u_{xx}$ is uniformly bounded in $L^2(Q_T)$. For the more difficult case when $q\in(0,1]$, we prove by using the $L^p$-estimate of a linear parabolic equation that

$$\|u_{xx}\|_{L^{4/3}(Q_T)}\le C.\tag{1.13}$$

On the other hand, for the estimation on the magnetic field, we will show that

$$\sup_{0<t<T}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x+\iint_{Q_T}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t\le C\mu^{1/4}.\tag{1.14}$$

With (1.13) and (1.14), one can prove (1.10).

The rest of the paper will be arranged as follows. Theorem 1.2 will be proved in the next section. Precisely, in Section 2.1, we give some basic estimates, in particular, the estimate (1.13). In Section 2.2, we establish the $L^2$-estimates on $u$, $\theta$ and $\mathbf{w}$ in terms of the shear viscosity for proving (1.14). The proof of (1.10) will be given in Section 2.3.

## 2 Proof of Theorem 1.2

We divide the proof of the Theorem 1.2 into the following three subsections.

### 2.1 A Priori Estimates

In this subsection, we first give some basic a priori estimates. First of all, it follows from the equations (1.1) that

$$\begin{aligned}&\mathcal{E}_t+\Big[u\Big(\mathcal{E}+p+\frac12|\mathbf{b}|^2\Big)-\mathbf{w}\cdot\mathbf{b}\Big]_x=\big(\lambda uu_x+\mu\mathbf{w}\cdot\mathbf{w}_x+\nu\mathbf{b}\cdot\mathbf{b}_x+\kappa\theta_x\big)_x,\\ &(\rho\mathcal{S})_t+(\rho u\mathcal{S})_x-\Big(\frac{\kappa\theta_x}{\theta}\Big)_x=\frac{\lambda u_x^2+\mu|\mathbf{w}_x|^2+\nu|\mathbf{b}_x|^2}{\theta}+\frac{\kappa\theta_x^2}{\theta^2},\end{aligned}\tag{2.1}$$

where

$$\mathcal{E}=\rho\Big(e+\frac12(u^2+|\mathbf{w}|^2)\Big)+\frac12|\mathbf{b}|^2,\quad \mathcal{S}=\ln\theta-R\ln\rho.$$

Some basic estimates are given in

**Lemma 2.1** Under the conditions in Theorem 1.2, it holds that

$$\begin{aligned}&\int_\Omega \rho(x,t)\mathrm{d}x = \int_\Omega \rho_0(x)\mathrm{d}x, \quad \forall t\in(0,T),\\ &\sup_{0<t<T}\int_\Omega \big[\rho(\theta+u^2+|\mathbf{w}|^2)+|\mathbf{b}|^2\big]\mathrm{d}x \le C,\\ &\iint_{Q_T}\Big(\frac{\lambda u_x^2+\mu|\mathbf{w}_x|^2+\nu|\mathbf{b}_x|^2}{\theta}+\frac{\kappa\theta_x^2}{\theta^2}\Big)\mathrm{d}x\mathrm{d}t\le C.\end{aligned} \tag{2.2}$$

**Proof** Integrating $(2.1)_1$ over $Q_t=\Omega\times(0,t)$ with $t\in(0,T)$ yields

$$\int_\Omega \mathcal{E}\mathrm{d}x=\int_\Omega \mathcal{E}|_{t=0}\mathrm{d}x+\mu\int_0^t \mathbf{w}\cdot\mathbf{w}_x\,|_{x=0}^{x=1}\,\mathrm{d}s. \tag{2.3}$$

Let $d=0$ or 1. We first integrate $(1.1)_3$ from $x=d$ to $x$, and then integrate again over $\Omega$ to obtain

$$\mu\mathbf{w}_x(d,t)=\mu\big(\mathbf{w}_2-\mathbf{w}_1\big)-\int_\Omega(\rho u\mathbf{w}-\mathbf{b})\mathrm{d}x-\frac{\partial}{\partial t}\int_\Omega\int_d^x\rho\mathbf{w}\mathrm{d}y\mathrm{d}x. \tag{2.4}$$

Taking the inner product of (2.4) with $\mathbf{w}(d,t)$ and integrating over $(0,t)$, we have

$$\begin{aligned}\mu\int_0^t\mathbf{w}\cdot\mathbf{w}_x|_{x=d}\mathrm{d}s=&\mu\int_0^t\big(\mathbf{w}_2-\mathbf{w}_1\big)\cdot\mathbf{w}(d,s)\mathrm{d}s-\int_0^t\mathbf{w}(d,s)\cdot\Big(\int_\Omega(\rho u\mathbf{w}-\mathbf{b})\mathrm{d}x\Big)\mathrm{d}s\\ &-\mathbf{w}(d,t)\cdot\Big(\int_\Omega\int_d^x\rho\mathbf{w}\mathrm{d}y\mathrm{d}x\Big)+\int_0^t\mathbf{w}_t(d,t)\cdot\Big(\int_\Omega\int_d^x\rho\mathbf{w}\mathrm{d}y\mathrm{d}x\Big)\mathrm{d}s.\end{aligned}$$

Hence, $(2.2)_1$ gives

$$\begin{aligned}\Big|\mu\int_0^t\mathbf{w}\cdot\mathbf{w}_x|_{x=d}\mathrm{d}s\Big|\le& C+C\int_\Omega\rho|\mathbf{w}|\mathrm{d}x+C\iint_{Q_t}(\rho u|\mathbf{w}|+|\mathbf{b}|+\rho|\mathbf{w}|)\mathrm{d}x\mathrm{d}s\\ \le& C+\frac12\int_\Omega\rho|\mathbf{w}|^2\mathrm{d}x+C\iint_{Q_t}\mathcal{E}\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Substituting it into (2.3) yields

$$\int_\Omega\mathcal{E}\mathrm{d}x\le C+C\iint_{Q_t}\mathcal{E}\mathrm{d}x\mathrm{d}s,$$

that implies $(2.2)_2$ by Gronwall inequality.

Finally, $(2.2)_3$ can be proved by integrating $(2.1)_2$ and using $(2.2)_2$. Then the proof of the lemma is completed. □

Some other estimates given in the following lemma can be found [6, 8] except the lower bound of the temperature $\theta$.

**Lemma 2.2** Let $q>0$. Under the conditions of Theorem 1.2, we have

$$\begin{aligned}&C^{-1}\le\rho\le C,\quad \theta\ge C,\\ &\iint_{Q_T}\Big(\frac{u_x^2+\mu|\mathbf{w}_x|^2+|\mathbf{b}_x|^2}{\theta^\alpha}+\frac{\kappa\theta_x^2}{\theta^{1+\alpha}}\Big)\mathrm{d}x\mathrm{d}t\le C,\quad\forall\alpha\in(0,\min\{1,q\}),\\ &\int_0^T\|\theta\|_{L^\infty}^{q+1-\alpha}\mathrm{d}t\le C,\quad\forall\alpha\in(0,\min\{1,q\}),\\ &\iint_{Q_T}\big(u_x^2+\mu|\mathbf{w}_x|^2+|\mathbf{b}_x|^2\big)\mathrm{d}x\mathrm{d}t\le C,\\ &\int_0^T\|\mathbf{b}\|_{L^\infty}^2\mathrm{d}t\le C;\quad\iint_{Q_T}|\theta_x|^\beta\mathrm{d}x\mathrm{d}t\le C,\quad\forall\beta\in(1,3/2).\end{aligned} \tag{2.5}$$

**Proof** We only need to prove the lower bound on $\theta$ because the other estimates can be found in [6, 8].

Based on the fact that $\rho$ is bounded, it follows from $(1.1)_5$ that

$$\theta_t + u\theta_x - \frac{1}{\rho}(\kappa\theta_x)_x \geq \frac{\lambda}{\rho}\Big(u_x^2 - \frac{p}{\lambda}u_x\Big) = \frac{\lambda}{\rho}\left(u_x - \frac{p}{2\lambda}\right)^2 - \frac{R^2}{4\lambda}\rho\theta^2.$$

Hence,

$$\theta_t + u\theta_x - \frac{1}{\rho}(\kappa\theta_x)_x + K\theta^2 \geq 0,$$

where $K$ is a positive constant independent of $\mu$. Let $\underline{\theta} = \frac{\min_{\overline{\Omega}}\theta_0}{Ct+1}$ with $C = K\min_{\overline{\Omega}}\theta_0$, and $z = \theta - \underline{\theta}$. We have

$$z_x|_{x=0,1} = 0, \qquad z|_{t=0} \geq 0,$$

and

$$\begin{aligned}
&z_t + uz_x - \frac{1}{\rho}(\kappa z_x)_x + K(\theta + \underline{\theta})z \\
&= \theta_t + C\frac{\min_{\overline{\Omega}}\theta_0}{(Ct+1)^2} + u\theta_x - \frac{1}{\rho}(\kappa\theta_x)_x + K\theta^2 - K\left(\frac{\min_{\overline{\Omega}}\theta_0}{Ct+1}\right)^2 \\
&\geq C\frac{\min_{\overline{\Omega}}\theta_0}{(Ct+1)^2} - K\left(\frac{\min_{\overline{\Omega}}\theta_0}{Ct+1}\right)^2 = 0.
\end{aligned}$$

Then, by the comparison theorem, $z \geq 0$ on $\overline{Q}_T$. The proof is completed. □

For the magnetic field, we have the following

**Lemma 2.3** Under the conditions of Theorem 1.2, we have

$$\sup_{0<t<T}\int_\Omega |\mathbf{b}|^4\mathrm{d}x + \iint_{Q_T}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}t \leq C.$$

**Proof** Taking the inner product of $(1.1)_4$ with $4|\mathbf{b}|^2\mathbf{b}$ and integrating over $Q_t$ give

$$\begin{aligned}
&\int_\Omega |\mathbf{b}|^4\mathrm{d}x + 4\nu\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + 8\nu\iint_{Q_t}|\mathbf{b}\cdot\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s \\
&= \int_\Omega |\mathbf{b}_0|^4\mathrm{d}x + 4\iint_{Q_t}\mathbf{w}_x\cdot(|\mathbf{b}|^2\mathbf{b})\mathrm{d}x\mathrm{d}s - 4\iint_{Q_t}(u\mathbf{b})_x\cdot(|\mathbf{b}|^2\mathbf{b})\mathrm{d}x\mathrm{d}s. \qquad (2.6)
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\iint_{Q_t}\mathbf{w}_x\cdot(\mathbf{b}|\mathbf{b}|^2)\mathrm{d}x\mathrm{d}s &= -\iint_{Q_t}\mathbf{w}\cdot(\mathbf{b}_x|\mathbf{b}|^2)\mathrm{d}x\mathrm{d}s - 2\iint_{Q_t}(\mathbf{w}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{b}_x)\mathrm{d}x\mathrm{d}s \\
&\leq 3\iint_{Q_t}|\mathbf{w}||\mathbf{b}|^2|\mathbf{b}_x|\mathrm{d}x\mathrm{d}s \\
&\leq \frac{\nu}{4}\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}|\mathbf{w}|^2|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s \\
&\leq \frac{\nu}{4}\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C\int_0^t\|\mathbf{b}\|_{L^\infty}^2\int_\Omega|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s \\
&\leq \frac{\nu}{4}\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C, \qquad (2.7)
\end{aligned}$$

where we have used $(2.2)_2$ and $(2.5)_4$, and

$$-\iint_{Q_t}(u\mathbf{b})_x\cdot|\mathbf{b}|^2\mathbf{b}\mathrm{d}x\mathrm{d}s = 3\iint_{Q_t}u(\mathbf{b}_x\cdot\mathbf{b})|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s$$

$$\le \frac{\nu}{4}\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}u^2|\mathbf{b}|^4\mathrm{d}x\mathrm{d}s$$
$$\le \frac{\nu}{4}\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C\int_0^t\|u^2\|_{L^\infty(\Omega)}\int_\Omega|\mathbf{b}|^4\mathrm{d}x\mathrm{d}s. \quad (2.8)$$

Plugging (2.7) and (2.8) into (2.6) yields

$$\int_\Omega|\mathbf{b}|^4\mathrm{d}x + \iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s \le C + C\int_0^t\|u^2\|_{L^\infty(\Omega)}\int_\Omega|\mathbf{b}|^4\mathrm{d}x\mathrm{d}s.$$

By Gronwall inequality and noticing $\|u^2\|_{L^1(0,T;L^\infty(\Omega))} \le \|u_x\|^2_{L^2(Q_T)} \le C$ by $(2.5)_4$, we complete the proof of the lemma. □

The following lemma gives further estimates on the density function.

**Lemma 2.4** Under the condition of Theorem 1.2, we have

$$\begin{aligned}&\sup_{0<t<T}\int_\Omega\rho_x^2\mathrm{d}x + \iint_{Q_T}\left(\rho_t^2+\theta\rho_x^2\right)\mathrm{d}x\mathrm{d}t \le C,\\ &|\rho(x,s)-\rho(y,t)| \le C\Big(|x-y|+|s-t|^{1/2}\Big)^{1/2}.\end{aligned} \quad (2.9)$$

**Proof** Set $\eta = 1/\rho$. It follows from the equation $(1.1)_1$ that

$$u_x = \rho(\eta_t + u\eta_x).$$

Substituting it into $(1.1)_2$ yields

$$[\rho(u-\lambda\eta_x)]_t + [\rho u(u-\lambda\eta_x)]_x = R\rho^2(\theta\eta_x - \eta\theta_x) - \mathbf{b}\cdot\mathbf{b}_x.$$

Multiplying it by $(u-\lambda\eta_x)$ and integrating over $Q_t$ give

$$\begin{aligned}&\frac{1}{2}\int_\Omega\rho(u-\lambda\eta_x)^2\mathrm{d}x + R\lambda\iint_{Q_t}\theta\rho^2\eta_x^2\mathrm{d}x\mathrm{d}s\\ =&\frac{1}{2}\int_\Omega\rho_0(u_0+\lambda\rho_0^{-2}\rho_{0x})^2\mathrm{d}x + R\iint_{Q_t}\rho^2\theta u\eta_x\mathrm{d}x\mathrm{d}s\\ &-R\iint_{Q_t}\rho^2\eta\theta_x(u-\lambda\eta_x)\mathrm{d}x\mathrm{d}s - \iint_{Q_t}\mathbf{b}\cdot\mathbf{b}_x(u-\lambda\eta_x)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}R\iint_{Q_t}\rho^2\theta u\eta_x\mathrm{d}x\mathrm{d}s &\le \frac{R\lambda}{2}\iint_{Q_t}\theta\rho^2\eta_x^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}\theta u^2\mathrm{d}x\mathrm{d}s\\ &\le \frac{R\lambda}{2}\iint_{Q_t}\theta\rho^2\eta_x^2\mathrm{d}x\mathrm{d}s + C\int_0^t\|u^2\|_{L^\infty}\int_\Omega\theta\mathrm{d}x\mathrm{d}s\\ &\le C + \frac{R\lambda}{2}\iint_{Q_t}\theta\rho^2\eta_x^2\mathrm{d}x\mathrm{d}s.\end{aligned}$$

By Lemma 2.3 and $(2.5)_2$, we then obtain

$$\begin{aligned}&-R\iint_{Q_t}\rho^2\eta\theta_x(u-\lambda\eta_x)\mathrm{d}x\mathrm{d}s - \iint_{Q_t}\mathbf{b}\cdot\mathbf{b}_x(u-\lambda\eta_x)\mathrm{d}x\mathrm{d}s\\ \le&\, C + C\iint_{Q_t}\theta^{1+\alpha-q}\rho(u-\lambda\eta_x)^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}\frac{\theta^q\theta_x^2}{\theta^{1+\alpha}}\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}\rho(u-\lambda\eta_x)^2\mathrm{d}x\mathrm{d}s\\ \le&\, C + C\int_0^t\left(1+\|\theta^{1+\alpha-q}\|_{L^\infty(\Omega)}\right)\int_\Omega\rho(u-\lambda\eta_x)^2\mathrm{d}x\mathrm{d}s, \quad \alpha\in(0,\min\{1,q\}).\end{aligned}$$

Combining the above estimates yields

$$\int_\Omega \rho(u-\lambda\eta_x)^2\mathrm{d}x+\iint_{Q_t}\theta\rho^2\eta_x^2\mathrm{d}x\mathrm{d}s\le C+C\int_0^t\|\theta^{1+\alpha-q}\|_{L^\infty(\Omega)}\int_\Omega\rho(u-\lambda\eta_x)^2\mathrm{d}x\mathrm{d}s.$$

Hence, Gronwall inequality gives

$$\sup_{0<t<T}\int_\Omega \rho_x^2\mathrm{d}x+\iint_{Q_T}\theta\rho_x^2\mathrm{d}x\mathrm{d}t\le C. \tag{2.10}$$

With this estimate, it follows from the equation $(1.1)_1$ and (2.5) that

$$\|\rho_t\|^2_{L^2(Q_T)}\le C\|(u,u_x,\rho_x)\|^2_{L^2(Q_T)}\le C,$$

and this gives $(2.9)_1$.

To prove the second estimate in the lemma, let $\beta(x)=\rho(x,s)-\rho(x,t)$ for $s,t\in[0,T]$ with $s\ne t$. Then for any $x\in[0,1]$ and $\delta\in(0,\frac12]$, there exist some $y\in[0,1]$ and $\xi$ between $x$ and $y$ such that $\delta=|y-x|$ and $\beta(\xi)=\frac{1}{x-y}\int_y^x\beta(z)\mathrm{d}z$. Since

$$\beta(x)=\frac{1}{x-y}\int_y^x\beta(z)\mathrm{d}z+\int_\xi^x\beta'(z)\mathrm{d}z,$$

by (2.10), we have

$$\begin{aligned}|\beta(x)|&\le\frac1\delta\left|\int_y^x\beta(z)\mathrm{d}z\right|+\left|\int_\xi^x\beta'(z)\mathrm{d}z\right|\\&\le\frac1\delta\left|\int_y^x\int_s^t\rho_\tau\mathrm{d}\tau\mathrm{d}z\right|+\left|\int_\xi^x\Big(\rho_z(z,t)-\rho_z(z,s)\Big)\mathrm{d}z\right|\\&\le\frac1\delta\left(\iint_{Q_T}\rho_\tau^2\mathrm{d}\tau\mathrm{d}z\right)^{1/2}|x-y|^{1/2}|s-t|^{1/2}\\&\quad+\left(\int_0^1(|\rho_z(z,s)|^2+|\rho_z(z,t)|^2)\mathrm{d}z\right)^{1/2}|x-\xi|^{1/2}\\&\le C\delta^{-1/2}|s-t|^{1/2}+C\delta^{1/2}.\end{aligned}$$

Taking $\delta=|s-t|^{1/2}$ with $s\ne t$ gives

$$|\rho(x,s)-\rho(x,t)|\le C|s-t|^{1/4}.$$

On the other hand, (2.10) implies that

$$|\rho(x,t)-\rho(y,t)|\le C|x-y|^{1/2}.$$

Thus, $(2.9)_2$ holds and the proof of the lemma is completed. □

The following lemma gives a $L^p$ estimate on the second order derivative of the velocity by using the parabolic equation properties.

**Lemma 2.5** Under the conditions of Theorem 1.2, we have

$$\iint_{Q_T}|u_{xx}|^{4/3}\mathrm{d}x\mathrm{d}t\le C. \tag{2.11}$$

**Proof** Rewrite the equation of $u$ as

$$u_t-\frac{\lambda}{\rho}u_{xx}=-uu_x-\theta_x-\frac{R}{\rho}\rho_x\theta-\frac1\rho\mathbf{b}\cdot\mathbf{b}_x=:f.$$

By Lemmas 2.2 and 2.3, it is clear that the second term and the fourth term of $f$ are uniform bounded in $L^{4/3}(Q_T)$.

By Lemma 2.1, we have

$$u^2 \le 2\int_\Omega |uu_x|\mathrm{d}x \le 2\Big(\int_\Omega u^2\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega u_x^2\mathrm{d}x\Big)^{1/2} \le C\Big(\int_\Omega u_x^2\mathrm{d}x\Big)^{1/2},$$

that implies

$$\int_0^T \|u\|_{L^\infty}^4\mathrm{d}t \le C\iint_{Q_T} u_x^2\mathrm{d}x\mathrm{d}t \le C.$$

Thus,

$$\begin{aligned}\iint_{Q_T} |uu_x|^{3/2}\mathrm{d}x\mathrm{d}t &\le 2\iint_{Q_T} u^6\mathrm{d}x\mathrm{d}t + 2\iint_{Q_T} u_x^2\mathrm{d}x\mathrm{d}t\\ &\le C + C\int_0^T \|u\|_{L^\infty}^4 \int_\Omega u^2\mathrm{d}x\mathrm{d}t\\ &\le C + C\int_0^T \|u\|_{L^\infty}^4\mathrm{d}t \le C.\end{aligned}$$

By $(2.9)_1$, we then obtain

$$\begin{aligned}\iint_{Q_T} |\rho_x\theta|^{4/3}\mathrm{d}x\mathrm{d}t &= \iint_{Q_T} (|\rho_x|^2\theta)^{2/3}\theta^{2/3}\mathrm{d}x\mathrm{d}t\\ &\le C\iint_{Q_T} \rho_x^2\theta\mathrm{d}x\mathrm{d}t + C\iint_{Q_T}\theta^2\mathrm{d}x\mathrm{d}t\\ &\le C + C\int_0^T \|\theta\|_{L^\infty}\int_\Omega \theta\mathrm{d}x\mathrm{d}t \le C.\end{aligned}$$

Combining the above estimates yields $\|f\|_{L^{4/3}(Q_T)} \le C$. Therefore, from the $L^p$-estimate of linear parabolic equations (cf. [16, Corollary 7.16]) and by noticing $(2.9)_2$, the estimate given in the lemma is proved. □

### 2.2 Estimates in Terms of Shear Viscosity

In this subsection, we will give some detailed estimates on the solution in terms of the shear viscosity for the estimation on the boundary layer thickness. The next lemma gives estimates on the transversal magnetic and velocity fields.

**Lemma 2.6** Under the conditions of Theorem 1.2, we have

$$\begin{aligned}&\sup_{0<t<T}\int_\Omega \big(|\mathbf{b}|^2 + |\mathbf{w}|^2\big)\mathrm{d}x + \iint_{Q_T} |\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}t \le C\sqrt{\mu},\\ &\sqrt{\mu}\sup_{0<t<T}\int_\Omega |\mathbf{w}_x|^2\mathrm{d}x + \mu^{3/2}\iint_{Q_T} |\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}t \le C.\end{aligned} \tag{2.12}$$

In particular,

$$\sqrt{\mu}\int_0^T \|\mathbf{w}_x\|_{L^\infty}\mathrm{d}t + \mu\int_0^T \|\mathbf{w}_x\|_{L^\infty}^2\mathrm{d}t \le C. \tag{2.13}$$

**Proof** We first prove the following estimate

$$\mu\int_\Omega |\mathbf{w}_x|^2\mathrm{d}x + \mu^2\iint_{Q_t} |\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s \le C\sqrt{\mu} + C\iint_{Q_t} |\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s. \tag{2.14}$$

Rewrite $(1.1)_3$ in the form of

$$\mathbf{w}_t - \frac{\mu}{\rho}\mathbf{w}_{xx} = \frac{1}{\rho}\mathbf{b}_x - u\mathbf{w}_x. \tag{2.15}$$

Taking the inner product of (2.15) with $-\mu\mathbf{w}_{xx}$ and integrating over $Q_t$ give

$$\begin{aligned}
&\mu\int_\Omega |\mathbf{w}_x|^2\mathrm{d}x + \mu^2\iint_{Q_t}\frac{1}{\rho}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s\\
=& -\mu\iint_{Q_t}\frac{1}{\rho}\mathbf{b}_x\cdot\mathbf{w}_{xx}\mathrm{d}x\mathrm{d}s + \mu\iint_{Q_t} u\mathbf{w}_x\cdot\mathbf{w}_{xx}\mathrm{d}x\mathrm{d}s + \mu\int_0^t \mathbf{w}_t\cdot\mathbf{w}_x\Big|_{x=0}^{x=1}\mathrm{d}s\\
=& -\mu\iint_{Q_t}\frac{1}{\rho}\mathbf{b}_x\cdot\mathbf{w}_{xx}\mathrm{d}x\mathrm{d}s - \frac{\mu}{2}\iint_{Q_t} u_x|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s + \mu\int_0^t \mathbf{w}_t\cdot\mathbf{w}_x\Big|_{x=0}^{x=1}\mathrm{d}s\\
\le& \frac{\mu^2}{4}\iint_{Q_t}\frac{1}{\rho}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\\
&+C\int_0^t\|u_x\|_{L^\infty}\Big(\mu\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)\mathrm{d}s + \mu\int_0^t \mathbf{w}_t\cdot\mathbf{w}_x\Big|_{x=0}^{x=1}\mathrm{d}s.
\end{aligned}\tag{2.16}$$

Since

$$\begin{aligned}
|\mathbf{w}_x|^2 &\le \Big|\frac{\mathbf{w}(b,t)-\mathbf{w}(a,t)}{b-a}\Big|^2 + 2\int_\Omega|\mathbf{w}_x||\mathbf{w}_{xx}|\mathrm{d}x\\
&\le C + C\Big(\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega|\mathbf{w}_{xx}|^2\mathrm{d}x\Big)^{1/2},
\end{aligned}$$

that is,

$$|\mathbf{w}_x| \le C + C\Big(\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)^{1/4}\Big(\int_\Omega|\mathbf{w}_{xx}|^2\mathrm{d}x\Big)^{1/4},$$

we have

$$\begin{aligned}
\mu^{3/2}\int_0^t\|\mathbf{w}_x\|_\infty^2\mathrm{d}s &\le C\mu^{3/2} + C\int_0^t\Big(\mu\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)^{1/2}\Big(\mu^2\int_\Omega|\mathbf{w}_{xx}|^2\mathrm{d}x\Big)^{1/2}\mathrm{d}s\\
&\le C\mu^{3/2} + C\mu\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s + C\mu^2\iint_{Q_t}\frac{1}{\rho}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s,\\
\mu\int_0^t\|\mathbf{w}_x\|_\infty\mathrm{d}s &\le C\mu + C\int_0^t\mu^{1/4}\Big(\mu\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)^{1/4}\Big(\mu^2\int_\Omega|\mathbf{w}_{xx}|^2\mathrm{d}x\Big)^{1/4}\mathrm{d}s\\
&\le C\sqrt{\mu} + \frac{C\mu}{\epsilon}\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s + \epsilon\mu^2\iint_{Q_t}\frac{1}{\rho}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{2.17}$$

By taking $\epsilon > 0$ small enough, we obtain

$$\begin{aligned}
\mu\int_0^t \mathbf{w}_t\cdot\mathbf{w}_x\Big|_{x=0}^{x=1}\mathrm{d}s &\le C\mu\int_0^t\|\mathbf{w}_x\|_\infty\mathrm{d}s\\
&\le C\sqrt{\mu} + C\mu\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s + \frac{\mu^2}{4}\iint_{Q_t}\frac{1}{\rho}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{2.18}$$

Inserting (2.18) into (2.16) gives

$$\begin{aligned}
&\mu\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x + \mu^2\iint_{Q_t}\frac{1}{\rho}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}s\\
\le& C\sqrt{\mu} + C\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C\int_0^t\big(1+\|u_x\|_{L^\infty}\big)\Big(\mu\int_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\Big)\mathrm{d}s.
\end{aligned}$$

By Gronwall inequality and $\iint_{Q_T}\|u_x\|_{L^\infty}\mathrm{d}x\mathrm{d}t \le \iint_{Q_T}|u_{xx}|\mathrm{d}x\mathrm{d}t \le C$ because of (2.11), we obtain (2.14).

Next, we will show that

$$\int_{\Omega}|\mathbf{b}|^2\mathrm{d}x+\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\le C\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s. \tag{2.19}$$

Taking the inner product of $(1.1)_4$ with $\mathbf{b}$ and integrating over $Q_t$ give

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}|\mathbf{b}|^2\mathrm{d}x+\nu\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\\
=&\iint_{Q_t}(\mathbf{w}-u\mathbf{b})_x\cdot\mathbf{b}\mathrm{d}x\mathrm{d}s=-\iint_{Q_t}(\mathbf{w}-u\mathbf{b})\cdot\mathbf{b}_x\mathrm{d}x\mathrm{d}s\\
\le&\frac{\nu}{2}\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}u^2|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s\\
\le&\frac{\nu}{2}\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s+C\int_0^t\|u^2\|_{L^\infty(\Omega)}\int_{\Omega}|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

that gives (2.19) by Gronwall inequality.

Now we turn to show

$$\int_{\Omega}|\mathbf{w}|^2\mathrm{d}x\le C\sqrt{\mu}. \tag{2.20}$$

In fact, taking the inner product of $(1.1)_3$ with $\mathbf{w}$ and integrating over $Q_t$ yield

$$\frac{1}{2}\int_{\Omega}\rho|\mathbf{w}|^2\mathrm{d}x+\mu\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s=\iint_{Q_t}\mathbf{b}_x\cdot\mathbf{w}\mathrm{d}x\mathrm{d}s+\mu\int_0^t\mathbf{w}_x\cdot\mathbf{w}\Big|_{x=0}^{x=1}\mathrm{d}s. \tag{2.21}$$

By (2.19), we obtain

$$\iint_{Q_t}\mathbf{b}_x\cdot\mathbf{w}\mathrm{d}x\mathrm{d}s\le 2\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s+2\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s\le C\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s.$$

From (2.14), (2.19) and (2.17) with $\epsilon=1$, it follows that

$$\mu\int_0^t\mathbf{w}_x\cdot\mathbf{w}\Big|_{x=0}^{x=1}\mathrm{d}s\le C\mu\int_0^t\|\mathbf{w}_x\|_{L^\infty(\Omega)}\mathrm{d}s\le C\sqrt{\mu}+C\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s.$$

Thus, we have from (2.21) that

$$\int_{\Omega}|\mathbf{w}|^2\mathrm{d}x+\mu\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s\le C\sqrt{\mu}+C\iint_{Q_t}|\mathbf{w}|^2\mathrm{d}x\mathrm{d}s,$$

which implies (2.20) by Gronwall inequality. Then (2.12) follows.

In summary, we obtain (2.13) and the proof of the lemma is completed. □

The following estimate about the relation between the fluid variables and the magnetic field will also be useful.

**Lemma 2.7** Under the conditions of Theorem 1.2, we have

$$\begin{aligned}
&\sup_{0<t<T}\int_{\Omega}\left[u_x^2+\rho_x^2+(\rho-\rho_0)^2+(\theta-\theta_0)^2\right]\mathrm{d}x+\iint_{Q_T}\left(u_t^2+u_{xx}^2+\theta_x^2\right)\mathrm{d}x\mathrm{d}t\\
\le& C\sqrt{\mu}+\iint_{Q_T}|\mathbf{b}\cdot\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}t+C\iint_{Q_T}|\mathbf{b}_x|^2|\theta-\theta_0|\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{2.22}$$

**Proof** Rewrite the equation $(1.1)_3$ into

$$\sqrt{\rho}u_t-\frac{\lambda}{\sqrt{\rho}}u_{xx}=-\sqrt{\rho}uu_x-R\sqrt{\rho}\theta_x-R\frac{\rho_x}{\sqrt{\rho}}\theta-\frac{1}{\sqrt{\rho}}\mathbf{b}\cdot\mathbf{b}_x.$$

Taking the square on both sides and integrating over $Q_t$, we obtain

$$\begin{aligned}
&\frac{\lambda}{2}\int_\Omega u_x^2\mathrm{d}x+\iint_{Q_t}\rho u_t^2\mathrm{d}x\mathrm{d}t+\lambda^2\iint_{Q_t}\frac{1}{\rho}u_{xx}^2\mathrm{d}x\mathrm{d}s\\
&\le C\iint_{Q_t}u^2u_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\theta_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\rho_x^2\theta^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\\
&\le C\int_0^t\|u^2\|_{L^\infty(\Omega)}\int_\Omega u_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\theta_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\rho_x^2\theta^2\mathrm{d}x\mathrm{d}s\\
&\quad+C\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s. \qquad (2.23)
\end{aligned}$$

Note that

$$\begin{aligned}
\iint_{Q_t}\rho_x^2\theta^2\mathrm{d}x\mathrm{d}s&\le 2\iint_{Q_t}\rho_x^2[(\theta-\theta_0)^2+\theta_0^2]\mathrm{d}x\mathrm{d}s\\
&\le C\int_0^t\|\theta-\theta_0\|_{L^\infty}^2\int_\Omega\rho_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\rho_x^2\mathrm{d}x\mathrm{d}s\\
&\le C\int_0^t\|\theta-\theta_0\|_{L^\infty}^2\mathrm{d}s+C\iint_{Q_t}\rho_x^2\mathrm{d}x\mathrm{d}s\\
&\le C\iint_{Q_t}(\theta-\theta_0)^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\theta_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\rho_x^2\mathrm{d}x\mathrm{d}s. \qquad (2.24)
\end{aligned}$$

On the other hand, we can derive from the equation $(1.1)_1$ that

$$\Big(\frac{\rho_x^2}{2}\Big)_t+\Big(u\rho_{xx}+2\rho_xu_x+\rho u_{xx}\Big)\rho_x=0,$$

and integrating over $Q_t$ yields

$$\begin{aligned}
\frac{1}{2}\int_\Omega\rho_x^2\mathrm{d}x&=-\iint_{Q_t}\Big(u\rho_{xx}\rho_x+2\rho_x^2u_x+\rho\rho_xu_{xx}\Big)\mathrm{d}x\mathrm{d}s\\
&=-\frac{3}{2}\iint_{Q_t}\rho_x^2u_x\mathrm{d}x\mathrm{d}s+\iint_{Q_t}\rho\rho_xu_{xx}\mathrm{d}x\mathrm{d}s\\
&\le C\int_0^t\|u_x\|_{L^\infty}\int_\Omega\rho_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\rho_x^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}u_{xx}^2\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

By Gronwall inequality, we have

$$\int_\Omega\rho_x^2\mathrm{d}x\le C\iint_{Q_t}u_{xx}^2\mathrm{d}x\mathrm{d}s.$$

Substituting it into (2.24) yields

$$\iint_{Q_t}\rho_x^2\theta^2\mathrm{d}x\mathrm{d}s\le C\iint_{Q_t}(\theta-\theta_0)^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\theta_x^2\mathrm{d}x\mathrm{d}s+C\int_0^t\Big(\iint_{Q^s}u_{xx}^2\mathrm{d}xd\tau\Big)\mathrm{d}s.$$

Then, we can derive from (2.23) that

$$\begin{aligned}
&\int_\Omega u_x^2\mathrm{d}x+\iint_{Q_t}u_t^2\mathrm{d}x\mathrm{d}t+\iint_{Q_t}u_{xx}^2\mathrm{d}x\mathrm{d}s\\
&\le C\iint_{Q_t}|\mathbf{b}|^2|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}(\theta-\theta_0)^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}\theta_x^2\mathrm{d}x\mathrm{d}s\\
&\quad+C\int_0^t\Big(1+\|u^2\|_{L^\infty(\Omega)}\Big)\Big(\int_\Omega u_x^2\mathrm{d}x+\iint_{Q_s}|u_{xx}|^2\mathrm{d}xd\tau\Big)\mathrm{d}s,
\end{aligned}$$

which gives

$$\int_\Omega u_x^2 \mathrm{d}x + \iint_{Q_t} u_t^2 \mathrm{d}x\mathrm{d}t + \iint_{Q_t} u_{xx}^2 \mathrm{d}x\mathrm{d}s$$
$$\le C\iint_{Q_t} |\mathbf{b}|^2|\mathbf{b}_x|^2 \mathrm{d}x\mathrm{d}s + C\iint_{Q_t} (\theta-\theta_0)^2 \mathrm{d}x\mathrm{d}s + C\iint_{Q_t} \theta_x^2 \mathrm{d}x\mathrm{d}s. \tag{2.25}$$

Multiplying $(1.1)_5$ by $(\theta-\theta_0)$ and integrating over $Q_t$, we obtain

$$\frac{1}{2}\int_\Omega \rho|\theta-\theta_0|^2 \mathrm{d}x + \iint_{Q_t} \kappa\theta_x^2 \mathrm{d}x\mathrm{d}s$$
$$= -\iint_{Q_t} p u_x(\theta-\theta_0)\mathrm{d}x\mathrm{d}s + \iint_{Q_t} (\theta-\theta_0)\mathcal{Q}\mathrm{d}x\mathrm{d}s. \tag{2.26}$$

Thus,

$$-\iint_{Q_t} p u_x(\theta-\theta_0)\mathrm{d}x\mathrm{d}s$$
$$= -R\iint_{Q_t} \rho u_x(\theta-\theta_0)^2 \mathrm{d}x\mathrm{d}s - R\theta_0\iint_{Q_t} \rho(\theta-\theta_0)u_x \mathrm{d}x\mathrm{d}s$$
$$\le C\int_0^t \|u_x\|_{L^\infty}\int_\Omega (\theta-\theta_0)^2 \mathrm{d}x\mathrm{d}s + C\iint_{Q_t} u_x^2 \mathrm{d}x\mathrm{d}s + C\iint_{Q_t} (\theta-\theta_0)^2 \mathrm{d}x\mathrm{d}s. \tag{2.27}$$

By $(2.5)_4$ and $(2.12)_2$, we have

$$\iint_{Q_t} (\theta-\theta_0)\mathcal{Q}\mathrm{d}x\mathrm{d}s \le C\sqrt{\mu} + C\int_0^t \|\theta-\theta_0\|_{L^\infty}\int_\Omega u_x^2 \mathrm{d}x\mathrm{d}s + C\iint_{Q_t} |\mathbf{b}_x|^2|\theta-\theta_0|\mathrm{d}x\mathrm{d}s. \tag{2.28}$$

Combining (2.27) and (2.28) with (2.26) yields

$$\int_\Omega |\theta-\theta_0|^2 \mathrm{d}x + \iint_{Q_t} \theta_x^2 \mathrm{d}x\mathrm{d}s$$
$$\le C\sqrt{\mu} + C\iint_{Q_t} |\mathbf{b}_x|^2|\theta-\theta_0|\mathrm{d}x\mathrm{d}s + C\int_0^t (1+\|\theta-\theta_0\|_{L^\infty})\int_\Omega u_x^2 \mathrm{d}x\mathrm{d}s$$
$$+C\int_0^t (1+\|u_x\|_{L^\infty})\int_\Omega |\theta-\theta_0|^2 \mathrm{d}x\mathrm{d}s.$$

By (2.25), Gronwall inequality yields

$$\int_\Omega |\theta-\theta_0|^2 \mathrm{d}x + \iint_{Q_t} \theta_x^2 \mathrm{d}x\mathrm{d}s \le C\sqrt{\mu} + C\iint_{Q_t} |\mathbf{b}_x|^2|\theta-\theta_0|\mathrm{d}x\mathrm{d}s + C\iint_{Q_t} |\mathbf{b}|^2|\mathbf{b}_x|^2 \mathrm{d}x\mathrm{d}s,$$

and it completes the proof of the lemma. □

**Lemma 2.8** Under the conditions of Theorem 1.2, we have

$$\begin{aligned}
&\sqrt{\mu}\iint_{Q_T} |\mathbf{b}_{xx}|^2 \mathrm{d}x\mathrm{d}t \le C,\\
&\int_0^T \|\mathbf{b}_x\|_{L^\infty}^2 \mathrm{d}t \le C,\\
&\sup_{0<t<T}\int_\Omega \big(u^2+u_x^2+(\theta-\theta_0)^2+\rho_x^2\big)\mathrm{d}x + \iint_{Q_T} \big(u_t^2+\theta_x^2+u_{xx}^2\big)\mathrm{d}x\mathrm{d}t \le C\sqrt{\mu}.
\end{aligned} \tag{2.29}$$

**Proof** Taking the inner product of $(1.1)_4$ with $\mathbf{b}_{xx}$ and integrating over $Q_t$ give

$$\frac{1}{2}\int_\Omega |\mathbf{b}_x|^2 \mathrm{d}x + \nu\iint_{Q_t} |\mathbf{b}_{xx}|^2 \mathrm{d}x\mathrm{d}s = \iint_{Q_t} (u\mathbf{b}-\mathbf{w})_x\cdot\mathbf{b}_{xx}\mathrm{d}x\mathrm{d}s$$

$$
\begin{aligned}
&= \iint_{Q_t} u_x \mathbf{b}\cdot\mathbf{b}_{xx}\mathrm{d}x\mathrm{d}s - \frac{1}{2}\iint_{Q_t} u_x|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s - \iint_{Q_t} \mathbf{w}_x\cdot\mathbf{b}_{xx}\mathrm{d}x\mathrm{d}s \\
&\le \frac{\nu}{2}\iint_{Q_t}|\mathbf{b}_{xx}|^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s \\
&\quad + C\iint_{Q_t}|u_x|^2|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s + C\int_0^t \|u_x\|_{L^\infty}\int_\Omega |\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s. \qquad (2.30)
\end{aligned}
$$

From (2.22) and Lemma 2.3, we have

$$
\int_\Omega u_x^2\mathrm{d}x \le C + C\iint_{Q_t}|\mathbf{b}_x|^2\theta\mathrm{d}x\mathrm{d}s.
$$

Hence, this together with $(2.5)_6$ imply

$$
\begin{aligned}
\iint_{Q_t}|u_x|^2|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s &\le \int_0^t \|\mathbf{b}\|_{L^\infty}^2\int_\Omega |u_x|^2\mathrm{d}x\mathrm{d}s \\
&\le C\int_0^t \|\mathbf{b}\|_{L^\infty}^2\mathrm{d}s + C\int_0^t \|\mathbf{b}\|_{L^\infty}^2 \iint_{Q_s}|\mathbf{b}_x|^2\theta\mathrm{d}x\mathrm{d}\tau\mathrm{d}s \\
&\le C + C\iint_{Q_t}|\mathbf{b}_x|^2\theta\mathrm{d}x\mathrm{d}s \\
&\le C + C\int_0^t \|\theta\|_{L^\infty}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Substituting it into (2.30) yields

$$
\begin{aligned}
&\int_\Omega |\mathbf{b}_x|^2\mathrm{d}x + \iint_{Q_t}|\mathbf{b}_{xx}|^2\mathrm{d}x\mathrm{d}s \\
&\le C\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s + C\int_0^t \big(\|u_x\|_{L^\infty} + \|\theta\|_{L^\infty}\big)\int_\Omega |\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s,
\end{aligned}
$$

so that

$$
\int_\Omega |\mathbf{b}_x|^2\mathrm{d}x + \iint_{Q_t}|\mathbf{b}_{xx}|^2\mathrm{d}x\mathrm{d}s \le C\iint_{Q_t}|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}s.
$$

This together with (2.12) give $(2.29)_1$.

By $(2.12)_1$ and $(2.29)_1$, we have

$$
\begin{aligned}
\int_0^t \|\mathbf{b}_x\|_{L^\infty}^2\mathrm{d}s &\le C\iint_{Q_t}|\mathbf{b}_x||\mathbf{b}_{xx}|\mathrm{d}x\mathrm{d}s \\
&\le C\Big(\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\iint_{Q_t}|\mathbf{b}_{xx}|^2\mathrm{d}x\mathrm{d}s\Big)^{1/2} \\
&\le C\Big(\sqrt{\mu}\iint_{Q_t}|\mathbf{b}_{xx}|^2\mathrm{d}x\mathrm{d}s\Big)^{1/2} \le C,
\end{aligned}
$$

so that $(2.29)_2$ holds.

By $(2.12)_1$ and $(2.29)_2$, we obtain

$$
\begin{aligned}
&\iint_{Q_t}|\mathbf{b}\cdot\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + \iint_{Q_t}|\mathbf{b}_x|^2|\theta-\theta_0|\mathrm{d}x\mathrm{d}s \\
&\le \int_0^t \|\mathbf{b}_x\|_{L^\infty}^2\int_\Omega|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s + C\iint_{Q_t}|\mathbf{b}_x|^2(\theta-\theta_0)^2\mathrm{d}x\mathrm{d}s \\
&\le C\sqrt{\mu} + C\int_0^t \|\mathbf{b}_x\|_{L^\infty}^2\int_\Omega(\theta-\theta_0)^2\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

By (2.22) we have $(2.29)_3$ by Gronwall inequality. And then it completes the proof of the lemma. □

The following lemma gives estimates on the derivatives of the transversal magnetic and velocity fields.

**Lemma 2.9** Under the conditions of Theorem 1.2, we have

$$\begin{aligned}&\iint_{Q_T}|\mathbf{w}_t|^2\mathrm{d}x\mathrm{d}t\le C,\\&\sup_{0<t<T}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x+\iint_{Q_T}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t\le C\mu^{1/4}.\end{aligned}\tag{2.31}$$

**Proof** From (2.15), (2.29) and (2.12), it follows that

$$\begin{aligned}\iint_{Q_T}|\mathbf{w}_t|^2\mathrm{d}x\mathrm{d}t&=\iint_{Q_T}\Big|\frac{\mu}{\rho}\mathbf{w}_{xx}+\frac{1}{\rho}\mathbf{b}_x-u\mathbf{w}_x\Big|^2\mathrm{d}x\mathrm{d}t\\&\le C\mu^2\iint_{Q_T}|\mathbf{w}_{xx}|^2\mathrm{d}x\mathrm{d}t+C\iint_{Q_T}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}t+C\iint_{Q_T}u^2|\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}t\\&\le C\sqrt{\mu}+C\frac{1}{\sqrt{\mu}}\int_0^T\|u^2\|_{L^\infty(\Omega)}\Big(\sqrt{\mu}\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)\mathrm{d}t\\&\le C\sqrt{\mu}+C\frac{1}{\sqrt{\mu}}\int_0^T\|u^2\|_{L^\infty(\Omega)}\mathrm{d}t\\&\le C\sqrt{\mu}+C\frac{1}{\sqrt{\mu}}\iint_{Q_T}u_x^2\mathrm{d}x\mathrm{d}t\le C,\end{aligned}$$

which implies $(2.31)_1$.

Taking the inner product of $(1.1)_4$ with $\mathbf{b}_t$ and integrating over $Q_t$ yield

$$\frac{\nu}{2}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x+\iint_{Q_t}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t=\iint_{Q_t}\big(\mathbf{w}_x-(u\mathbf{b})_x\big)\cdot\mathbf{b}_t\mathrm{d}x\mathrm{d}t.\tag{2.32}$$

Since

$$\begin{aligned}\iint_{Q_t}\mathbf{w}_x\cdot\mathbf{b}_t\mathrm{d}x\mathrm{d}t&=\int_\Omega\mathbf{w}_x\cdot\mathbf{b}\mathrm{d}x-\iint_{Q_t}(\mathbf{w}_t)_x\cdot\mathbf{b}\mathrm{d}x\mathrm{d}t\\&=-\int_\Omega\mathbf{w}\cdot\mathbf{b}_x\mathrm{d}x+\iint_{Q_t}\mathbf{w}_t\cdot\mathbf{b}_x\mathrm{d}x\mathrm{d}t,\end{aligned}$$

by $(2.12)_1$, we have

$$\begin{aligned}\iint_{Q_t}\mathbf{w}_x\cdot\mathbf{b}_t\mathrm{d}x\mathrm{d}t&\le C\sqrt{\mu}+\frac{\nu}{4}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x+\Big(\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\Big)^{1/2}\Big(\iint_{Q_t}|\mathbf{w}_t|^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}\\&\le C\mu^{1/4}+\frac{\nu}{4}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x.\end{aligned}\tag{2.33}$$

By using $(2.12)_1$, $\|u\|_{L^\infty(Q_T)}\le C$ and $\int_0^T\|u_x\|^2_{L^\infty}\mathrm{d}t\le\iint_{Q_T}|u_{xx}|^2\mathrm{d}x\mathrm{d}t\le C$ because of $(2.29)_3$, we have

$$\begin{aligned}-\iint_{Q_t}(u\mathbf{b})_x\cdot\mathbf{b}_t\mathrm{d}x\mathrm{d}t&\le\frac{1}{2}\iint_{Q_t}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\iint_{Q_t}|(u\mathbf{b})_x|^2\mathrm{d}x\mathrm{d}s\\&\le\frac{1}{2}\iint_{Q_t}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t+C\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s+C\iint_{Q_t}|\mathbf{b}u_x|^2\mathrm{d}x\mathrm{d}s\\&\le C\sqrt{\mu}+\frac{1}{2}\iint_{Q_t}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t+C\int_0^t\|u_x\|^2_{L^\infty(\Omega)}\int_\Omega|\mathbf{b}|^2\mathrm{d}x\mathrm{d}s\end{aligned}$$

$$\le C\sqrt{\mu}+\frac{1}{2}\iint_{Q_t}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t+C\sqrt{\mu}\int_0^t\|u_x\|^2_{L^\infty(\Omega)}\mathrm{d}s$$
$$\le C\sqrt{\mu}+\frac{1}{2}\iint_{Q_t}|\mathbf{b}_t|^2\mathrm{d}x\mathrm{d}t. \tag{2.34}$$

Substituting (2.33) and (2.34) into (2.32) completes the proof of the lemma. □

The following lemma is about further estimate on the temperature.

**Lemma 2.10** Under the conditions of Theorem 1.2, we have

$$\sup_{0<t<T}\int_\Omega\theta^q\theta_x^2\mathrm{d}x+\iint_{Q_T}\theta^{2q}\theta_t^2\mathrm{d}x\mathrm{d}t\le C\mu^{1/4}.$$

**Proof** Rewrite the equation $(1.1)_5$ as

$$\rho\theta_t-(\kappa\theta_x)_x=\mathcal{Q}-\rho u\theta_x-R\rho\theta u_x=:f. \tag{2.35}$$

We first estimate $\|f\|_{L^2(Q_T)}$. Obviously,

$$\iint_{Q_T}f^2\mathrm{d}x\mathrm{d}t\le C\iint_{Q_T}(u_x^2\theta^2+u^2\theta_x^2+u_x^4+\mu^2|\mathbf{w}_x|^4+|\mathbf{b}_x|^4)\mathrm{d}x\mathrm{d}t.$$

By Lemma 2.8 and noticing $\|u\|_{L^\infty(Q_T)}\le C$, we have

$$\iint_{Q_T}\big(u_x^4+u^2\theta_x^2+u_x^2\theta^2\big)\mathrm{d}x\mathrm{d}t$$
$$\le C\sqrt{\mu}+C\iint_{Q_T}\big(u_x^4+u_x^2(\theta-\theta_0)^2+\theta_0^2u_x^2\big)\mathrm{d}x\mathrm{d}t$$
$$\le C\sqrt{\mu}+\int_0^T\big(\|u_x\|^2_{L^\infty}+\|\theta-\theta_0\|^2_{L^\infty}\big)\int_\Omega u_x^2\mathrm{d}x\mathrm{d}t$$
$$\le C\sqrt{\mu}+C\int_0^T\big(\|u_x\|^2_{L^\infty}+\|\theta-\theta_0\|^2_{L^\infty}\big)\mathrm{d}t\le C\sqrt{\mu}.$$

By $(2.12)_2$ and (2.13), we also have

$$\mu^2\iint_{Q_T}|\mathbf{w}_x|^4\mathrm{d}x\mathrm{d}t\le\mu^{3/2}\int_0^T\|\mathbf{w}_x\|^2_{L^\infty}\Big(\sqrt{\mu}\int_\Omega|\mathbf{w}_x|^2\mathrm{d}x\Big)\mathrm{d}t$$
$$\le\mu^{3/2}\int_0^T\|\mathbf{w}_x\|^2_{L^\infty}\mathrm{d}t\le C\sqrt{\mu}.$$

From (2.31) and (2.29), it follows that

$$\iint_{Q_T}|\mathbf{b}_x|^4\mathrm{d}x\mathrm{d}t\le\int_0^T\|\mathbf{b}_x\|^2_{L^\infty}\int_\Omega|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}t\le C\mu^{1/4}\int_0^T\|\mathbf{b}_x\|^2_{L^\infty}\mathrm{d}t\le C\mu^{1/4}.$$

Combining the above estimates gives

$$\|f\|^2_{L^2(Q_T)}\le C\mu^{1/4}. \tag{2.36}$$

Multiplying (2.35) by $\theta^q\theta_t$ and integrating by parts give

$$\iint_{Q_t}\rho\theta^q\theta_t^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_\Omega\theta^{2q}\theta_x^2\mathrm{d}x=\iint_{Q_t}f\theta^q\theta_t\mathrm{d}x\mathrm{d}t,$$

so that (2.36) implies

$$\iint_{Q_t}\rho\theta^q\theta_t^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_\Omega\theta^{2q}\theta_x^2\mathrm{d}x\le C\mu^{1/4}\|\theta\|^q_{L^\infty(Q_t)}+\frac{1}{2}\iint_{Q_t}\rho\theta^q\theta_t^2\mathrm{d}x\mathrm{d}t.$$

Thus,

$$\iint_{Q_t} \theta^q \theta_t^2 \mathrm{d}x\mathrm{d}t + \int_\Omega \theta^{2q}\theta_x^2 \mathrm{d}x \le C\mu^{1/4}\|\theta\|^q_{L^\infty(Q_t)}. \tag{2.37}$$

On the other hand, we have

$$\theta^{1+q} \le \Big(\int_\Omega \theta \mathrm{d}x\Big)^{1+q} + (1+q)\int_\Omega \theta^q|\theta_x|\mathrm{d}x \le C + C\int_\Omega \theta^{2q}\theta_x^2\mathrm{d}x. \tag{2.38}$$

Combining (2.37) and (2.38) yields

$$\|\theta\|^{1+q}_{L^\infty(Q_T)} \le C(1+\|\theta\|^q_{L^\infty(Q_T)}),$$

which implies $\|\theta\|_{L^\infty(Q_T)} \le C$. This together with (2.37) complete the proof of the lemma. □

### 2.3 Variation Estimate on Transversal Velocity

As shown in [8], the thickness of the boundary layer can be estimated once we obtain (1.10). Hence, in this subsection, we will prove (1.10) based on the estimates obtained in the previous subsections.

Let $\mathbf{y} = \mathbf{w}_x$. Differentiating (2.15) with respect to $x$ gives

$$\mathbf{y}_t = \mu\Big(\frac{\mathbf{y}_x}{\rho}\Big)_x - (u\mathbf{y})_x + \Big(\frac{\mathbf{b}_x}{\rho}\Big)_x. \tag{2.39}$$

Denote $\Phi_\epsilon(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ for $\epsilon \in (0,1)$ and $\xi_\delta : [0,1] \to [0,1]$ for $\delta \in (0, \frac{1}{2})$ by

$$\Phi_\epsilon(\xi) = \sqrt{\epsilon^2 + |\xi|^2}, \quad \forall \xi \in \mathbb{R}^2; \qquad \xi_\delta(x) = \begin{cases} x, & 0 \le x \le \delta, \\ \delta, & \delta < x < 1-\delta, \\ 1-x, & 1-\delta \le x \le 1. \end{cases}$$

Observe that $\xi_\delta$ has the properties

$$0 \le \xi_\delta(x) \le \delta, \quad \forall x \in [0,1]. \tag{2.40}$$

Note that $\Phi_\epsilon$ satisfies

$$\begin{cases} |\xi| \le |\Phi_\epsilon(\xi)| \le |\xi| + \epsilon, & \forall \xi \in \mathbb{R}^2, \\ |\nabla_\xi \Phi_\epsilon(\xi)| \le 1, & \forall \xi \in \mathbb{R}^2, \\ 0 \le \xi \cdot \nabla_\xi \Phi_\epsilon(\xi) \le \Phi_\epsilon(\xi), & \forall \xi \in \mathbb{R}^2, \\ \eta D_\xi^2 \Phi_\epsilon(\xi)\eta^\top \ge 0, & \forall \xi, \eta \in \mathbb{R}^2, \\ \lim\limits_{\epsilon \to 0^+} \Phi_\epsilon(\xi) = |\xi|, & \forall \xi \in \mathbb{R}^2, \end{cases} \tag{2.41}$$

where $\xi^\top$ stands for the transpose of the vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, and $D_\xi^2 g$ is the Hessian matrix of the function $g : \mathbb{R}^2 \to \mathbb{R}$:

$$D_\xi^2 g(\xi) = \begin{pmatrix} g_{\xi_1\xi_1} & g_{\xi_1\xi_2} \\ g_{\xi_2\xi_1} & g_{\xi_2\xi_2} \end{pmatrix}.$$

Taking the inner product of (2.39) with $\big(\nabla_\xi \Phi_\epsilon(\mathbf{y})\big)\xi_\delta(x)$ and integrating over $Q_t$, we obtain

$$\begin{aligned} &\int_\Omega \Phi_\epsilon(\mathbf{y})\xi_\delta \mathrm{d}x - \epsilon\int_\Omega \xi_\delta \mathrm{d}x \\ = &-\mu\iint_{Q_t} \frac{1}{\rho}\mathbf{y}_x D_\xi^2 \Phi_\epsilon(\mathbf{y})(\mathbf{y}_x)^\top \xi_\delta \mathrm{d}x\mathrm{d}s - \mu\iint_{Q_t} \frac{\mathbf{y}_x \cdot \nabla_\xi \Phi_\epsilon(\mathbf{y})}{\rho}\xi_\delta' \mathrm{d}x\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&-\iint_{Q_t} u_x\mathbf{y}\cdot\nabla_\xi\Phi_\epsilon(\mathbf{y})\xi_\delta \mathrm{d}x\mathrm{d}s-\iint_{Q_t} u\mathbf{y}_x\cdot\nabla_\xi\Phi_\epsilon(\mathbf{y})\xi_\delta \mathrm{d}x\mathrm{d}s\\
&+\iint_{Q_t}\frac{\mathbf{b}_{xx}\cdot\nabla_\xi\Phi_\epsilon(\mathbf{y})}{\rho}\xi_\delta \mathrm{d}x\mathrm{d}s-\iint_{Q_t}\frac{\mathbf{b}_{x}\cdot\nabla_\xi\Phi_\epsilon(\mathbf{y})}{\rho^2}\rho_x\xi_\delta \mathrm{d}x\mathrm{d}s\\
=:&\sum_{n=1}^{6} I_n. \qquad (2.42)
\end{aligned}
$$

From $(2.41)_4$, it follows

$$I_1\le 0.$$

For $I_2$, by using $(2.41)_1$, (2.9) and (2.13), we have

$$
\begin{aligned}
I_2&=\mu\int_0^t\frac{\Phi_\epsilon(\mathbf{y})}{\rho}\Big|_{x=1-\delta}^{x=1}\mathrm{d}s-\mu\int_0^t\frac{\Phi_\epsilon(\mathbf{y})}{\rho}\Big|_{x=0}^{x=\delta}\mathrm{d}s-\mu\iint_{Q_t}\frac{\Phi_\epsilon(\mathbf{y})\rho_x}{\rho^2}\xi_\delta'\mathrm{d}x\mathrm{d}s\\
&\le\mu\int_0^t\left(\frac{\Phi_\epsilon(\mathbf{y})}{\rho}\Big|_{x=1}+\frac{\Phi_\epsilon(\mathbf{y})}{\rho}\Big|_{x=0}\right)\mathrm{d}s-\mu\iint_{Q_t}\frac{\Phi_\epsilon(\mathbf{y})\rho_x}{\rho^2}\xi_\delta'\mathrm{d}x\mathrm{d}s\\
&\le C\mu\int_0^t\big(\|\mathbf{y}\|_{L^\infty(\Omega)}+\epsilon\big)\mathrm{d}s+C\mu\iint_{Q_t}(|\mathbf{y}|+1)|\rho_x|\mathrm{d}x\mathrm{d}s\\
&\le C\sqrt{\mu}+C\mu\Big(\iint_{Q_t}(|\mathbf{y}|+1)^2\mathrm{d}x\mathrm{d}s\Big)^{1/2}\Big(\iint_{Q_t}\rho_x^2\mathrm{d}x\mathrm{d}s\Big)^{1/2}\le C\sqrt{\mu}.
\end{aligned}
$$

Note that

$$I_4=\iint_{Q_t}u_x\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s+\iint_{Q_t}u\Phi_\epsilon(\mathbf{y})\xi_\delta'\mathrm{d}x\mathrm{d}s. \qquad (2.43)$$

Hence,

$$
\begin{aligned}
&|u(x,t)|\le x\|u_x\|_{L^\infty(\Omega)}\le x\int_\Omega|u_{xx}|\mathrm{d}x\le\xi_\delta(x)\int_\Omega|u_{xx}|\mathrm{d}x,\forall x\in[0,\delta],\\
&|u(x,t)|\le (1-x)\|u_x\|_{L^\infty(\Omega)}\le (1-x)\int_\Omega|u_{xx}|\mathrm{d}x\le\xi_\delta(x)\int_\Omega|u_{xx}|\mathrm{d}x,\forall x\in[1-\delta,1],
\end{aligned}
$$

so that

$$
\begin{aligned}
\iint_{Q_t}u\Phi_\epsilon(\mathbf{y})\xi_\delta'\mathrm{d}x\mathrm{d}s&=\int_0^t\int_0^\delta u\Phi_\epsilon(\mathbf{y})\mathrm{d}x\mathrm{d}s-\int_0^t\int_{1-\delta}^1 u\Phi_\epsilon(\mathbf{y})\mathrm{d}x\mathrm{d}s\\
&\le\int_0^t\Big(\int_\Omega|u_{xx}|\mathrm{d}x\Big)\int_0^\delta\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s\\
&\quad+\int_0^t\Big(\int_\Omega|u_{xx}|\mathrm{d}x\Big)\int_{1-\delta}^1\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s\\
&\le\int_0^t\Big(\int_\Omega|u_{xx}|\mathrm{d}x\Big)\int_0^1\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s. \qquad (2.44)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\iint_{Q_t}u_x\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s&\le C\int_0^t\|u_x\|_{L^\infty(\Omega)}\int_\Omega\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s\\
&\le C\int_0^t\Big(\int_\Omega|u_{xx}|\mathrm{d}x\Big)\int_\Omega\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s. \qquad (2.45)
\end{aligned}
$$

Plugging (2.44) and (2.45) into (2.43) yields

$$I_4\le C\int_0^t\Big(\int_\Omega|u_{xx}|\mathrm{d}x\Big)\int_\Omega\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s.$$

By $(2.41)_3$, we have

$$\begin{aligned} I_3 &\le C\int_0^t \|u_x\|_{L^\infty(\Omega)}\int_\Omega \Phi_\epsilon(\mathbf{y})\xi_\delta \mathrm{d}x\mathrm{d}s \\ &\le C\int_0^t \Big(\int_\Omega |u_{xx}|\mathrm{d}x\Big)\int_\Omega \Phi_\epsilon(\mathbf{y})\xi_\delta \mathrm{d}x\mathrm{d}s. \end{aligned}$$

From the equation $(1.1)_4$, by using $(2.41)_3$, $(2.41)_2$, (2.31) and (2.40), we obtain

$$\begin{aligned} I_5 &= \frac{1}{\nu}\iint_{Q_t} \frac{\big[\mathbf{b}_t + (u\mathbf{b})_x - \mathbf{y}\big]\cdot\nabla_\xi\Phi_\epsilon(\mathbf{y})}{\rho}\xi_\delta\mathrm{d}x\mathrm{d}t \\ &\le \frac{1}{\nu}\iint_{Q_t} \frac{\big[\mathbf{b}_t + (u\mathbf{b})_x\big]\cdot\nabla_\xi\Phi_\epsilon(\mathbf{y})}{\rho}\xi_\delta\mathrm{d}x\mathrm{d}t \\ &\le C\delta\iint_{Q_t}\big[|\mathbf{b}_t| + |(u\mathbf{b})_x|\big]\mathrm{d}x\mathrm{d}t \\ &\le C\delta\iint_{Q_t}\big[|\mathbf{b}_t| + |u\mathbf{b}_x| + |u_x\mathbf{b}|\big]\mathrm{d}x\mathrm{d}t \le C\delta\mu^{1/8}. \end{aligned}$$

Therefore, by $(2.29)_3$, (2.31) and (2.40), we have

$$\begin{aligned} I_6 &\le C\delta\iint_{Q_t}|\mathbf{b}_x||\rho_x|\mathrm{d}x\mathrm{d}s \\ &\le C\delta\Big(\iint_{Q_t}|\mathbf{b}_x|^2\mathrm{d}x\mathrm{d}s\Big)^{1/2}\Big(\iint_{Q_t}|\rho_x|^2\mathrm{d}x\mathrm{d}s\Big)^{1/2} \le C\delta\mu^{1/2}. \end{aligned}$$

Combining the above estimates with (2.42) yields

$$\int_\Omega \Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x \le \delta\epsilon + C\sqrt{\mu} + C\delta\mu^{1/8} + C\int_0^t\Big(\int_\Omega|u_{xx}|\mathrm{d}x\Big)\int_\Omega\Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x\mathrm{d}s.$$

Hence, by (2.11) and the Gronwall inequality, we have

$$\int_\Omega \Phi_\epsilon(\mathbf{y})\xi_\delta\mathrm{d}x \le C(\delta\epsilon + \sqrt{\mu} + \delta\mu^{1/8}).$$

Since the constant is independent of $\epsilon$, taking $\epsilon \to 0$ yields

$$\int_\Omega |\mathbf{y}|\xi_\delta\mathrm{d}x \le C\sqrt{\mu} + C\delta\mu^{1/8}.$$

Therefore, this gives (1.10) by the definition of $\xi_\delta$.

Finally, for the boundary layer respect to the transversal velocity, we have from (1.10) and $(2.12)_1$ that

$$\begin{aligned} \|\mathbf{w}\|_{L^\infty(\delta,1-\delta)} &\le C\left(\int_\delta^{1-\delta}|\mathbf{w}|\mathrm{d}x + \int_\delta^{1-\delta}|\mathbf{w}_x|\mathrm{d}x\right) \\ &\le C\Big(\mu^{1/4} + \mu^{1/8} + \frac{\sqrt{\mu}}{\delta}\Big), \quad \forall\delta\in(0,1/4), \end{aligned}$$

which implies that for any function $\delta(\mu)$ satisfying $\sqrt{\mu}/\delta(\mu)\to 0$ as $\mu\to 0$, we have

$$\lim_{\mu\to 0}\|\mathbf{w}\|_{L^\infty(0,T;L^\infty(\delta(\mu),1-\delta(\mu)))} = 0.$$

Thus, the proof of Theorem 1.2 is completed.

## References


[1] Balescu R. Transport Processes in Plasmas I: Classical Transport Theory. New York: North-Holland, 1988

[2] Chen G, Wang D. Global solutions of nonlinear magnetohydrodynamics with large initial data. J Differential Equations, 2002, **182**: 344–376

[3] Chen G, Wang D. Existence and continuous dependence of large solutions for the magnetohydrodynamic equations. Z Angew Math Phys, 2003, **54**: 608–632

[4] Clemmow P, Dougherty J. Electrodynamics of Particles and Plasmas. New York: Addison-Wesley, 1990

[5] Fan J, Jiang S, Nakamura G. Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun Math Phys, 2007, **270**: 691–708

[6] Fan J, Huang S, Li F. Global strong solutions to the planar compressible magnetohydrodynamic equations with large initial data and vaccum. arXiv: 1206.3624v3[math.AP]19 Jul 2012

[7] Fan J, Jiang S. Zero shear viscosity limit of the Navier-Stokes equations of compressible isentropic fluids with cylindrical symmetry. Rend Sem Mat Univ Politec Torino, 2007, **65**: 35–52

[8] Frid H, Shelukhin V V. Boundary layer for the Navier-Stokes equations of compressible fluids. Commun Math Phys, 1999, **208**: 309–330

[9] Frid H, Shelukhin V V. Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry. SIAM J Math Anal, 2000, **31**: 1144–1156

[10] Hoff D, Tsyganov E. Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z Angew Math Phys, 2005, **56**: 791–804

[11] Hu X, Wang D. Global existence and large-time behavior of solutions to the threedimensional equations of compressible magnetohydrodynamic flows. Arch Ration Mech Anal, 2010, **197**: 203–238

[12] Jiang S, Zhang J W. Boundary layers for the Navier-Stokes equations of compressible heat-conducting flows with cylindrical symmetry. SIAM J Math Anal, 2009, **41**: 237–268

[13] Kazhikhov A, Shelukhin V V. Unique global solutions in time of initial boundary value problems for one-dimensional equations of a viscous gas. J Appl Math Mech, 1977, **41**: 273–283

[14] Kawashima S, Okada M. Smooth global solutions for the one-dimensional equations in magnetohydrodynamics. Proc Japan Acad Ser A, Math Sci, 1982, **58** 384–387

[15] Laudau L, Lifshitz E. Electrodynamics of Continuous Media. 2nd ed. New York: Pergamon, 1984

[16] Lieberman G M. Second Order Parabolic Diffrential Equations. Singapore: World Scientific, 1996

[17] Liu T, Zeng Y. Large-time behavior of solutions for general quasilinear hyperbolicparabolic systems of conservation laws. Mem Amer Math Soc, 1997, **125**(599)

[18] Qin X, Yang T, Yao Z, Zhou W. Vanishing shear viscosity and boundary layer for the Navier-Stokes equations with cylindrical symmetry. Arch Rational Mech Anal, 2015, **216**(3): 1049–1086

[19] Shelukhin V V. The limit of zero shear viscosity for compressible fluids. Arch Rational Mech Anal, 1998, **143**: 357–374

[20] Tsyganov E, Hoff D. Systems of partial differential equations of mixed hyperbolic-parabolic type. J Differential Equations, 2004, **204**: 163–201

[21] Wang D. Large solutions to the initial boundary value problem for planar magnetohydrodynamics. SIAM J Appl Math, 2003, **23**: 1424–1441

[22] Woods L. Principles of Magnetoplasma Dynamics. New York: Oxford University Press, 1987

[23] Wu C. Formation, structure, and stability of MHD intermediate shocks. J Geophys Res, 1990, **95**: 8149–8175

[24] Vol'pert A, Hudjaev S. On the Cauchy problem for composite systems of nonlinear differential equations. Math USSR-Sb, 1972, **16**: 517–544

[25] Yao L, Zhang T, Zhu C J. Boundary layers for compressible Navier-Stokes equations with density dependent viscosity and cylindrical symmetry. Ann I H Poincaré, 2011, **28**: 677–709