Global-in-time Stability of Hartmann Layer in Two Space Dimension

Feng Xie

School of Mathematical Sciences, and LSC-MOE,

Shanghai Jiao Tong University, Shanghai 200240, P.R.China

Tong Yang

Department of Mathematics, City University of Hong Kong,

Tat Chee Avenue, Kowloon, Hong Kong;
and School of Mathematical Sciences

Shanghai Jiao Tong University, Shanghai 200240, P.R.China

Abstract: In this paper, we prove global existence of solutions with Gevrey regularity to the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime derived by formal multi-scale expansion in [9]. The analysis shows that the combined effect of the magentic diffusivity and transveral magnetic field to the boundary leads to a linear damping on the tangential velocity field in the Prandtl regime near the boundary. Precisely, such damping term on tangential velocity yields global in time energy estimate in Gevrey norms in the tangential space variable for the perturbation of the classical Hartmann profile.

2000 Mathematical Subject Classification: 76N20, 35Q35, 76W05, 35M33.

Keywords: MHD boundary layer, Prandtl-Hartmann mixed regime, global existence, Gevrey regularity.

1 Introduction

The following mixed Prandtl and Hartmann boundary layer equations from the classical incompressible MHD system derived in [9] for flat boundary in two space dimension when the physical parameters such as Reynolds number, magnetic Reynols number and the Hartmann number satisfy some constraits in the high Reynolds numbers limit.

$$
\begin{cases}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 = \partial_y b_1 + \partial_y^2 u_1, \\
\partial_y u_1 + \partial_y^2 b_1 = 0, \\
\partial_x u_1 + \partial_y u_2 = 0,
\end{cases} \tag{1.1}
$$

with $x \in \mathbb{R}, y \in \mathbb{R}_+$. (u_1, u_2) denotes the velocity boundary layer functions and b_1 is the magnetic field boundary layer.

[1]*E-mail address:* tzxief@sjtu.edu.cn (F. Xie), matyang@cityu.edu.hk(T. Yang)

For this system, there is a well known solution called Hartmann layer. In this paper, we will study the global-in-time stability of this Hartmann layer when the solution is in the Gevrey function space. For this, we consider the system (1.1) with initial data denoted by

$$u_1(t=0,x,y)=u_{10}(x,y), \tag{1.2}$$

and the no-slip boundary condition

$$u_1|_{y=0}=0, \quad u_2|_{y=0}=0. \tag{1.3}$$

The far-field conditions are taken as the uniform constant states. Consequently, there is no pressure term appearing in equation of $(1.1)_1$.

$$\lim_{y\to+\infty} u_1=\bar{u}, \qquad \lim_{y\to+\infty} b_1=\bar{b}. \tag{1.4}$$

Integration the equation of $(1.1)_2$ over $[y,+\infty]$ yields that

$$-u_1(t,x,y)+\bar{u}=\partial_y b_1. \tag{1.5}$$

Thus, the equations (1.1) can be reduced into the following form.

$$\begin{cases} \partial_t u_1+u_1\partial_x u_1+u_2\partial_y u_1=-u_1(t,x,y)+\bar{u}+\partial_y^2 u_1, \\ \partial_x u_1+\partial_y u_2=0. \end{cases} \tag{1.6}$$

It is not difficult to find that

$$u_1=(1-e^{-y})\bar{u}, \qquad u_2=0, \tag{1.7}$$

is a special steady solution to (1.6), which is called the classical Hartmann profiles.
Without loss of generality, we assume $\bar{u}=1$ here and in sequel.
By setting

$$u_1=(1-e^{-y})+u, \qquad u_2=v, \tag{1.8}$$

we obtain

$$\begin{cases} \partial_t u+(1-e^{-y}+u)\partial_x u+v\partial_y(-e^{-y}+u)=-u+\partial_y^2 u, \\ \partial_x u+\partial_y v=0, \end{cases} \tag{1.9}$$

and initial and boundary conditions of u are given by

$$u_0(x,y)=u_{10}(x,y)-(1-e^{-y}), \tag{1.10}$$

and

$$\begin{cases} u|_{y=0}=0, \\ v|_{y=0}=0. \end{cases} \tag{1.11}$$

Our main results can be stated as follows.

Theorem 1.1 *There exists a small constant $\delta_0 > 0$. Let the initial data $u_{10}(x, y)$ be a small perturbation around the Hartmann profile $\bar{u}(1 - e^{-y})$ with a Gevrey regularity radius $\tau_0 > 0$. Precisely speaking, the initial data (1.2) satisfies*

$$\|\partial_y u_{10} + u_{10} - \bar{u}\|_{X^{r,\beta}_{\tau_0,\alpha}} \leq \delta_0, \tag{1.12}$$

with $0 < \alpha < \sqrt{2}/2$ and $1 \leq \beta < \min\{(2r+1)/3, 2r-1\}, r > 1$. Then there exists a unique global solution (u_1, u_2) to the mixed Prandtl and Hartmann boundary layer equations (1.1)-(1.4), which is Gevrey regular for x variable, and belongs to a weighted H^1 space with respect to y variable.

Remark 1.1 *We will introduce the Gevrey regularity function spaces $X^{r,\beta}_{\tau_0,\alpha}$ and others in details in Subsection 2. It is necessary to explain the condition $1 \leq \beta < \min\{(2r+1)/3, 2r-1\}, r > 1$ a little more. That means that the regime of Gevrey regularity index β depends on the choice of parameter $r > 1$. Precisely speaking, the parameter r is more large, the value interval of index β is more large. In particular, if one takes $r > 1$ and $\beta = 1$, $X^{r,1}_{\tau_0,\alpha}$ is exactly the weighted analytic function space. It shows that Theorem 1.12 also holds true for the weighted analytic function spaces. That is, the well-posedness of (1.1)-(1.4) in the analytic setting is a direct corollary of Theorem 1.12. In addition, the Gevrey regularity radius τ_0 and the small parameter δ_0 should satisfy the condition (3.19). When the initial Gevrey regularity radius τ_0 is fixed, then the condition (3.19) is guaranteed by choosing suitably small δ_0.*

Remark 1.2 *It is not necessary to require the initial data u_{10} is the small perturbation around the given Hartmann profile $\bar{u}(1 - e^{-y})$. It can be taken any small Gevrey regularity initial data with suitable small far-field state $\bar{u}$. The main results in this paper still hold true.*

As is well-known that the Prandtl boundary layer is the leading order characteristic boundary layer for the incompressible Navier-Stokes equations with non-slip boundary condition in high Reynolds number case, which were first proposed to describe the structure of viscous high Reynolds number flows by Prandtl [25] in 1904. From that time on, such an important fluid dynamics model attracts much attention from mathematicians. Under the monotone assumption on the tangential velocity in the normal direction, Oleinik firstly obtained the local existence of classical solutions in the two spatial dimension by using the classical Crocco transformation, cf. [23]. This result together with some other extensions in this direction are presented in Oleinik-Samokhin's classical book [24]. Recently, this well-posedness result was re-proved by using an energy method in the framework of weighted Sobolev spaces in [1] and [21] independently. It is noted that all the above well-posedness results are achieved in the local time interval. By imposing an additional favorable pressure condition, a global in time weak solution was given by Xin and Zhang in [27].

Very recently, the lower bound of life-span for small analytic solution to the classical Prandtl equations with small perturbation analytic initial data was given by Zhang ang Zhang in [28]. Precisely speaking, when the strength of background shear flow $(u_s(t, y), 0)$ is a size of $\varepsilon^{5/3}$, and the initial perturbation around such a shear flow is a size of ε, then the classical Prandtl system admits a unique analytic solution with the life-span being greater than $\varepsilon^{-4/3}$. Furthermore, when the initial data is chosen to a small perturbation around some given Guassian function, the almost global existence for the Prandtl boundary layer equations is obtained by Ignatova and

Vicol in [13]. Where the cancellation methods developed in [21] and the monotonicity conditions of background solution are essentially used in [13].

When the monotonicity condition is violated, seperation of the boundary layer is well expected and observed. For this, E-Engquist constructed a finite time blowup solution to the Prandtl equations in [5]. And this kind of blowup result is extended for the case of van Dommelen and Shen singularity in [15], where the outer Euler flows are periodic functions of sin and cos functions. In addition, when the background shear flow $u_s(t, y)$ has a non-degenerate critical point, that is, there exists a curve of $y = y(t)$ such that $\partial_y u(t, y(t)) = 0$ and $\partial_y^2 u(t, y(t)) \neq 0$. And the existence of such a curve is guaranteed by imposing the same conditions on initial data of $u_(t, y)$. Then some interesting ill-posedness (or instability) phenomena of solutions to both the linear and nonlinear Prandtl equations around the shear flow are studied, cf. [6, 7, 11]. All these results show that the monotone assumption on the tangential velocity plays a key role for well-posedness except in the frameworks of analytic functions and Gevrey classes. For the data is analytic in terms of the space variables x, y, Sammartino and Caflisch [26, 3] established the local well-posedness result of the Prandtl system. Later, the analyticity requirement about the normal variable y was removed by Lombardo, Cannone and Sammartino in [20]. The main argument used in [20, 26] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem. For Gevrey regularity solutions to Prandtl equations, one can refer to [8, 16] and references cited therein.

A natural question is that whether the global existence of smooth (or strong) solution can be achieved for the classical Prandtl equations in either analytic setting or Gevrey regularity function spaces. To our knowledge, this problem is still open up to now, even for the monotone data. It is believed that suitable magnetic field can stabilize the boundary layer in some physical regime [22, 2, 4, 10]. One can also refer to some very recent results in [9, 17, 18, 19] for the derivation of MHD boundary layer equations, stability analysis of magnetic field on the boundary layer from the mathematical point of view. Upon this consideration, we will establish the global existence of solutions to a mixed Prandtl and Hartmann MHD boundary layer equations in this paper. These results, together with those in [9, 17, 18], indeed show that the magnetic field has stability effect on boundary layers. One of the key point of this paper is that the effect of magnetic field boundary layer function behaves like a damping term in the classical Prandtl equations. By using the good properties of such a damping term, we can obtain the exponential decay of Gevrey regularity norms for solutions with respect to the time variable. Then the global Gevrey regularity solutions to the mixed Prandtl and Hartmann boundary layer equations near the classical Hartmann profile are proved. The proof depends on Gevrey class energy estimates with respect to tangential variable t.

Finally, the rest of the paper is organized as follows. We will reformulate the problem in a convenient form, introduce some weighted Gevrey regularity function spaces, and then give some preliminaries in Section 2. The uniform estimates of the solutions are established in Section 3, Based on these uniform estimates of solutions, the global existence and unique solutions to (1.1) are proved in Section 4.

In this paper the constant C, $\bar{C}$ and C_0 are generic, which may change line by line..

2 Preliminary

To establish the global existence of solution to (1.9), we shall reformulate the problem in a convenient form. To this end, the vorticity of the velocity $\omega = \partial_y u$ are taken into consideration.

$$\partial_t \omega + (1 - e^{-y} + u)\partial_x \omega - v e^{-y} + v \partial_y \omega = -\omega + \partial_y^2 \omega. \tag{2.1}$$

Introduce

$$g = \omega + u \tag{2.2}$$

then one has

$$\partial_t g + (1 - e^{-y} + u)\partial_x g + v \partial_y g = -g + \partial_y^2 g. \tag{2.3}$$

The relationship between the new unknown function of g and u can be written as follows.

$$u = e^{-y} \int_0^y e^z g(t, x, z) dz. \tag{2.4}$$

It is easy to obtain the initial data of g in term of (2.2).

$$g(0, x, y) = \partial_y u_{10}(x, y) + u_{10}(x, y) - 1. \tag{2.5}$$

And the boundary condition of g is derived as follows.
It implies from (1.9) and (1.11) that

$$(\partial_y \omega - u)|_{y=0} = 0,$$

then by the definition of g in (2.2), one has

$$(\partial_y g - g)|_{y=0} = 0. \tag{2.6}$$

In the remainder of this paper, we shall establish the global existence of solutions to (2.3)-(2.6). Then, by the definition of u in (2.4), we indeed show the global existence of solutions to (1.9)-(1.11).
Below, we introduce the weighted Gevrey regularity function spaces used in this paper. Denote Gevrey class weights M_m given by

$$M_m = \frac{(m+1)^r}{(m!)^\beta}$$

for some $r > 1$ and $\beta \geq 1$. In particular, when $\beta = 1$, it corresponds to the analytic function class. Let

$$X_m = \|e^{\alpha y} \partial_x^m g\|_{L^2} \tau^m M_m, \tag{2.7}$$
$$Z_m = \|e^{\alpha y} \partial_y \partial_x^m g\|_{L^2} \tau^m M_m, \tag{2.8}$$
$$Y_m = \|e^{\alpha y} \partial_x^m g\|_{L^2} \tau^{m-1/2} m^{1/2} M_m, \tag{2.9}$$

$$D_m = \|e^{\alpha y}\partial_x^m g\|_{L_y^\infty L_x^2}\tau^m M_m, \tag{2.10}$$

and we define

$$\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} = \sum_{m\geq 0} X_m^2, \tag{2.11}$$

$$\|g\|^2_{Z^{r,\beta}_{\tau,\alpha}} = \sum_{m\geq 0} Z_m^2, \tag{2.12}$$

$$\|g\|^2_{Y^{r,\beta}_{\tau,\alpha}} = \sum_{m\geq 0} Y_m^2, \tag{2.13}$$

$$\|g\|^2_{D^{r,\beta}_{\tau,\alpha}} = \sum_{m\geq 0} D_m^2. \tag{2.14}$$

Here τ denotes the Gevrey radius, which depends on the time variable t with initial data $\tau_0 > 0$. We write $g \in X^{r,\beta}_{\tau,\alpha}$ if $\|g\|_{X^{r,\beta}_{\tau,\alpha}} < \infty$, and the definitions of $Z^{r,\beta}_{\tau,\alpha}, Y^{r,\beta}_{\tau,\alpha}$ and $D^{r,\beta}_{\tau,\alpha}$ are similar.

Remark 2.1 *Once the global uniform estimates of g is established, that is, $\|g\|_{X^{r,\beta}_{\tau,\alpha}} < \infty$, then it follows from (2.4) that $\|u\|_{X^{r,\beta}_{\tau,\alpha'}} < \infty$ and $\|u\|_{Z^{r,\beta}_{\tau,\alpha'}} < \infty$ with $0 \leq \alpha' < \alpha$. Moreover, $\|u\|_{D^{r,\beta}_{\tau,\alpha}} < \infty$.*

3 Uniform *A Priori* Energy Estimates

In this section, we shall derive the uniform estimates of solutions to (2.3)-(2.6) in Gevrey regularity norms through energy estimate methods.
Applying the operator ∂_x^m on the equation (2.3), multiplying the resulting equation by $e^{2\alpha y}\partial_x^m g$, and integrating it over $\mathbb{R}^2_+$ lead to

$$\int_{\mathbb{R}^2_+} \partial_x^m(\partial_t g + (1-e^{-y}+u)\partial_x g + v\partial_y g + g - \partial_y^2 g)e^{2\alpha y}\partial_x^m g dxdy = 0 \tag{3.1}$$

We weill estimate each term below. It is obvious to obtain

$$\int_{\mathbb{R}^2_+} \partial_x^m \partial_t g e^{2\alpha y}\partial_x^m g dxdy = \frac{1}{2}\frac{d}{dt}\|e^{\alpha y}\partial_x^m g\|^2_{L^2} \tag{3.2}$$

and

$$\int_{\mathbb{R}^2_+} \partial_x^m g e^{2\alpha y}\partial_x^m g dxdy = \|e^{\alpha y}\partial_x^m g\|^2_{L^2}. \tag{3.3}$$

Integration by parts leads to

$$\begin{aligned}
&-\int_{\mathbb{R}^2_+} \partial_y^2\partial_x^m g e^{2\alpha y}\partial_x^m g dxdy \\
&= \int_{\mathbb{R}} \partial_y\partial_x^m g(t,x,0)\partial_x^m g(t,x,0)dx + \|e^{\alpha y}\partial_y\partial_x^m g\|^2_{L^2} + 2\alpha\int_{\mathbb{R}^2_+} \partial_y\partial_x^m g e^{2\alpha y}\partial_x^m g dxdy
\end{aligned}$$

$$
\begin{aligned}
=&\|\partial_x^m g(t,x,0)\|_{L^2_x}^2+\|e^{\alpha y}\partial_y\partial_x^m g\|_{L^2}^2-\alpha\|\partial_x^m g(t,x,0)\|_{L^2_x}^2-2\alpha^2\|e^{\alpha y}\partial_x^m g\|_{L^2}^2\\
=&(1-\alpha)\|\partial_x^m g(t,x,0)\|_{L^2_x}^2+\|e^{\alpha y}\partial_y\partial_x^m g\|_{L^2}^2-2\alpha^2\|e^{\alpha y}\partial_x^m g\|_{L^2}^2, \qquad (3.4)
\end{aligned}
$$

where in the second equality, we used the boundary condition (2.6).
Below, we will estimate the two mixed nonlinear terms in (3.1). First,

$$
\begin{aligned}
&\int_{\mathbb{R}^2_+}\partial_x^m((1-e^{-y}+u)\partial_x g)e^{2\alpha y}\partial_x^m g dxdy\\
=&\sum_{j=0}^m\begin{pmatrix} m\\ j\end{pmatrix}\int_{\mathbb{R}^2_+}\partial_x^{m-j}u\partial_x^{j+1}ge^{2\alpha y}\partial_x^m g dxdy\\
\triangleq& R_1,
\end{aligned}
$$

and

$$
\begin{aligned}
|R_1|\le&\sum_{j=0}^{[m/2]}\begin{pmatrix} m\\ j\end{pmatrix}\|\partial_x^{m-j}u\|_{L^2_xL^\infty_y}\|e^{\alpha y}\partial_x^{j+1}g\|_{L^\infty_xL^2_y}\|e^{\alpha y}\partial_x^m g\|_{L^2}\\
&+\sum_{j=[m/2]+1}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\|\partial_x^{m-j}u\|_{L^\infty_{xy}}\|e^{\alpha y}\partial_x^{j+1}g\|_{L^2}\|e^{\alpha y}\partial_x^m g\|_{L^2}.
\end{aligned}
$$

For $0\le j\le [m/2]$, by (2.4), one has

$$
\begin{aligned}
&\|\partial_x^{m-j}u\|_{L^2_xL^\infty_y}=\|\partial_x^{m-j}\int_0^y e^{-(y-z)}g(t,x,z)dz\|_{L^2_xL^\infty_y}\\
\le&\|e^{\alpha y}\partial_x^{m-j}\int_0^y e^{-(y-z)}g(t,x,z)e^{\alpha z}e^{-\alpha z}dz\|_{L^2_xL^\infty_y}\\
=&\|\int_0^y e^{-(1-\alpha)(y-z)}\partial_x^{m-j}g(t,x,z)e^{\alpha z}dz\|_{L^2_xL^\infty_y}\\
\le&C\|e^{\alpha y}\partial_x^{m-j}g\|_{L^2},
\end{aligned}
$$

provided that $\alpha<1$.
Using the Agmon inequality gives that

$$
\|e^{\alpha y}\partial_x^{j+1}g\|_{L^\infty_xL^2_y}\le C\|e^{\alpha y}\partial_x^{j+1}g\|_{L^2}^{1/2}\|e^{\alpha y}\partial_x^{j+2}g\|_{L^2}^{1/2}.
$$

For $[m/2]+1\le j\le m$,

$$
\begin{aligned}
&\|\partial_x^{m-j}u\|_{L^\infty_{xy}}\\
=&\|\partial_x^{m-j}\int_0^y e^{-(y-z)}g(t,x,z)dz\|_{L^\infty_{xy}}\\
\le&\|e^{\alpha y}\partial_x^{m-j}\int_0^y e^{-(y-z)}g(t,x,z)e^{\alpha z}e^{-\alpha z}dz\|_{L^\infty_{xy}}\\
=&\|\int_0^y e^{-(1-\alpha)(y-z)}\partial_x^{m-j}g(t,x,z)e^{\alpha z}dz\|_{L^\infty_{xy}}
\end{aligned}
$$

$$\le C\|e^{\alpha y}\partial_x^{m-j}g\|_{L^2_yL^\infty_x},$$
$$\le C\|e^{\alpha y}\partial_x^{m-j}g\|_{L^2}^{1/2}\|e^{\alpha y}\partial_x^{m-j+1}g\|_{L^2}^{1/2}.$$

Consequently

$$\begin{aligned}
&\sum_{m\ge 0}|R_1|\tau^{2m}M_m^2\\
\le&\frac{C}{(\tau(t))^{1/2}}\left\{\sum_{m\ge 0}\sum_{j=0}^{[m/2]}X_{m-j}Y_{j+1}^{1/2}Y_{j+2}^{1/2}Y_m\frac{\binom{m}{j}M_m}{M_{m-j}M_{j+1}^{1/2}M_{j+2}^{1/2}(j+1)^{1/4}(j+2)^{1/4}m^{1/2}}\right. \quad (3.5)\\
&\left.+\sum_{m\ge 0}\sum_{j=[m/2]+1}^{m}X_{m-j}^{1/2}X_{m-j+1}^{1/2}Y_{j+1}Y_m\frac{\binom{m}{j}M_m}{M_{m-j}^{1/2}M_{m-j+1}^{1/2}M_{j+1}(j+1)^{1/2}m^{1/2}}\right\}.
\end{aligned}$$

And the second nonlinear term is estimated as follows.

$$\begin{aligned}
&\int_{\mathbb{R}^2_+}\partial_x^m(v\partial_y g)e^{2\alpha y}\partial_x^m g dxdy\\
=&\sum_{j=0}^{m}\binom{m}{j}\int_{\mathbb{R}^2_+}\partial_x^{m-j}v\partial_x^j\partial_y g e^{2\alpha y}\partial_x^m g dxdy\\
\triangleq&R_2
\end{aligned}$$

and

$$\begin{aligned}
|R_2|\le&\sum_{j=0}^{[m/2]}\binom{m}{j}\|\partial_x^{m-j}v\|_{L^2_xL^\infty_y}\|e^{\alpha y}\partial_x^j\partial_y g\|_{L^\infty_xL^2_y}\|e^{\alpha y}\partial_x^m g\|_{L^2}\\
&+\sum_{j=[m/2]+1}^{m}\binom{m}{j}\|\partial_x^{m-j}v\|_{L^\infty_{xy}}\|e^{\alpha y}\partial_x^j\partial_y g\|_{L^2}\|e^{\alpha y}\partial_x^m g\|_{L^2}.
\end{aligned}$$

For $0\le j\le[m/2]$,

$$\begin{aligned}
\|\partial_x^{m-j}v\|_{L^2_xL^\infty_y}=&\|\int_0^y(e^{-z}\int_0^z e^s\partial_x^{m-j+1}g(t,x,s)ds)dz\|_{L^2_xL^\infty_y}\\
=&\|\int_0^y e^{-\alpha z}(\int_0^z e^{(1-\alpha)(s-z)}\partial_x^{m-j+1}g(t,x,s)e^{\alpha s}ds)dz\|_{L^2_xL^\infty_y}\\
\le&C\|e^{\alpha y}\partial_x^{m-j+1}g\|_{L^2}
\end{aligned}$$

here the fact that $0<\alpha<1$ is also used. And

$$\|e^{\alpha y}\partial_x^j\partial_y g\|_{L^\infty_xL^2_y}\le C\|e^{\alpha y}\partial_x^j\partial_y g\|_{L^2}^{1/2}\|e^{\alpha y}\partial_x^{j+1}\partial_y g\|_{L^2}^{1/2}.$$

For $[m/2]+1 \le j \le m$,

$$\begin{aligned}\|\partial_x^{m-j} v\|_{L^\infty} =& \|\int_0^y (e^{-z}\int_0^z e^s \partial_x^{m-j+1} g(t,x,s)ds)dz\|_{L^\infty}\\ =& \|\int_0^y e^{-\alpha z}(\int_0^z e^{(1-\alpha)(s-z)}\partial_x^{m-j+1} g(t,x,s)e^{\alpha s}ds)dz\|_{L^\infty}\\ \le& C\|e^{\alpha y}\partial_x^{m-j+1} g\|_{L^2}^{1/2}\|e^{\alpha y}\partial_x^{m-j+2} g\|_{L^2}^{1/2}.\end{aligned}$$

Consequently

$$\begin{aligned}&\sum_{m\ge0}|R_2|\tau^{2m}M_m^2\\ \le& \frac{C}{(\tau(t))^{1/2}}\left\{\sum_{m\ge0}\sum_{j=0}^{[m/2]} Y_{m-j+1}Z_j^{1/2}Z_{j+1}^{1/2}Y_m \frac{\binom{m}{j}M_m}{M_{m-j+1}M_j^{1/2}M_{j+1}^{1/2}(m-j+1)^{1/2}m^{1/2}}\right. \qquad (3.6)\\ &\left.+\sum_{m\ge0}\sum_{j=[m/2]+1}^{m} Y_{m-j+1}^{1/2}Y_{m-j+2}^{1/2}Z_jY_m \frac{\binom{m}{j}M_m}{M_{m-j+1}^{1/2}M_{m-j+2}^{1/2}M_j(m-j+1)^{1/4}(m-j+2)^{1/4}m^{1/2}}\right\}.\end{aligned}$$

Combining the above estimates and using the definitions of $X_{\tau,\alpha}, Z_{\tau,\alpha}$ and $Y_{\tau,\alpha}$ yield that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|g\|_{X_{\tau,\alpha}}^2 - \dot{\tau}\|g\|_{Y_{\tau,\alpha}}^2 + \|g\|_{Z_{\tau,\alpha}}^2 + (1-2\alpha^2)\|g\|_{X_{\tau,\alpha}}^2 + (1-\alpha)\|g(t,x,0)\|_{X_{\tau,\alpha}}^2\\ \le& \sum_{m\ge0}|R_1|\tau^{2m}M_m^2 + \sum_{m\ge0}|R_2|\tau^{2m}M_m^2 \qquad (3.7)\end{aligned}$$

Since

$$\begin{aligned}&\frac{\binom{m}{j}M_m}{M_{m-j}M_{j+1}^{1/2}M_{j+2}^{1/2}(j+1)^{1/4}(j+2)^{1/4}m^{1/2}}\\ =& \frac{((m-j)!)^{\beta-1}(j!)^{\beta-1}(m+1)^r(j+1)^\beta(j+2)^{\beta/2}}{(m!)^{\beta-1}(m-j+1)^r(j+2)^{r/2}(j+3)^{r/2}(j+1)^{1/4}(j+2)^{1/4}m^{1/2}}\\ \le& C(1+j)^{3\beta/2-r-1} \qquad (3.8)\end{aligned}$$

for all $0 \le j \le [m/2]$. In the last inequality, we used that $(p!q!)^{\beta-1} \le ((p+q)!)^{\beta-1}$. Then,

$$\begin{aligned}&\frac{\binom{m}{j}M_m}{M_{m-j}^{1/2}M_{m-j+1}^{1/2}M_{j+1}(j+1)^{1/2}m^{1/2}}\\ =& \frac{((m-j)!)^{\beta-1}((j+1)!)^{\beta-1}(m+1)^r(m-j+1)^{\beta/2}(j+1)}{(m!)^{\beta-1}(m-j+1)^{r/2}(m-j+2)^{r/2}(j+2)^r(j+1)^{1/2}m^{1/2}}\end{aligned}$$

$$\leq C(m-j+1)^{\beta/2-r} \tag{3.9}$$

In the last inequality, we used that $((p+1)!q!)^{\beta-1} \leq C((p+q)!)^{\beta-1}$.
Consequently

$$\begin{aligned}
\sum_{m\geq 0}|R_1|\tau^{2m}M_m^2 \leq & \frac{C}{(\tau(t))^{1/2}}\left\{\sum_{m\geq 0}\sum_{j=0}^{[m/2]} X_{m-j}Y_{j+1}^{1/2}Y_{j+2}^{1/2}Y_m(1+j)^{3\beta/2-r-1}\right.\\
& \left. +\sum_{m\geq 0}\sum_{j=[m/2]+1}^{m} X_{m-j}^{1/2}X_{m-j+1}^{1/2}Y_{j+1}Y_m(m-j+1)^{\beta/2-r}\right\}\\
\leq & \frac{C}{(\tau(t))^{1/2}}\|g\|_{X^{r,\beta}_{\tau,\alpha}}\|g\|^2_{Y^{r,\beta}_{\tau,\alpha}} \qquad (3.10)
\end{aligned}$$

In the last inequality, we used the following discrete Young inequality.

$$\|\zeta\cdot(\eta*\xi)\|_{l^1} \leq C\|\zeta\|_{l^2}\|\eta\|_{l^1}\|\xi\|_{l^2}$$

with $\zeta_k = Y_k, \eta_k = Y_k^{1/2}Y_{k+1}^{1/2}k^{3\beta/2-r-1}, \xi_k = X_k$ for the first term of the RHS in (3.10), and then the Hölder inequality is also used

$$\|\eta\|_{l^1} \leq C\|g\|_{Y^{r,\beta}_{\tau,\alpha}}$$

provided that $3\beta/2 - r - 1 < -1/2$. That is, $\beta < (2r+1)/3$. And $\zeta_k = Y_k, \eta_k = X_k^{1/2}X_{k+1}^{1/2}(k+1)^{\beta/2-r}, \xi_k = Y_{k+1}$ for the second term of the RHS in (3.10), then

$$\|\eta\|_{l^1} \leq C\|g\|_{X^{r,\beta}_{\tau,\alpha}}.$$

provided that $\beta/2 - r < -1/2$. That is, $\beta < 2r-1$. In conclusion, (3.10) holds true, provided that

$$\beta < \min\{(2r+1)/3, 2r-1\}.$$

Similarly,

$$\begin{aligned}
\sum_{m\geq 0}|R_2|\tau^{2m}M_m^2 \leq & \frac{C}{(\tau(t))^{1/2}}\left\{\sum_{m\geq 0}\sum_{j=0}^{[m/2]} Y_{m-j+1}Z_j^{1/2}Z_{j+1}^{1/2}Y_m(1+j)^{\beta/2-r}\right.\\
& \left. +\sum_{m\geq 0}\sum_{j=[m/2]+1}^{m} Y_{m-j+1}^{1/2}Y_{m-j+2}^{1/2}Z_jY_m(m-j+1)^{3\beta/2-r-1}\right\}\\
\leq & \frac{C}{(\tau(t))^{1/2}}\|g\|_{Z^{r,\beta}_{\tau,\alpha}}\|g\|^2_{Y^{r,\beta}_{\tau,\alpha}} \qquad (3.11)
\end{aligned}$$

provided that

$$\beta < \min\{(2r+1)/3, 2r-1\}.$$

It follows, from (3.7), (3.10) and (3.11), that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + \|g\|^2_{Z^{r,\beta}_{\tau,\alpha}} + (1-2\alpha^2)\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + (1-\alpha)\|g(\cdot,0)\|^2_{X^{r,\beta}_{\tau,\alpha}} \\
\leq &(\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g\|_{X^{r,\beta}_{\tau,\alpha}} + \|g\|_{Z^{r,\beta}_{\tau,\alpha}}))\|g\|^2_{Y^{r,\beta}_{\tau,\alpha}}
\end{aligned}
\tag{3.12}
$$

Set

$$
\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g\|_{X^{r,\beta}_{\tau,\alpha}} + \|g\|_{Z^{r,\beta}_{\tau,\alpha}}) = 0 \tag{3.13}
$$

Then we have

$$
\frac{1}{2}\frac{d}{dt}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + \|g\|^2_{Z^{r,\beta}_{\tau,\alpha}} + (1-2\alpha^2)\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + (1-\alpha)\|g(\cdot,0)\|^2_{X^{r,\beta}_{\tau,\alpha}} \leq 0 \tag{3.14}
$$

When

$$
0 < \alpha < \sqrt{2}/2,
$$

the following differential inequality holds

$$
\frac{1}{2}\frac{d}{dt}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + \|g\|^2_{Z^{r,\beta}_{\tau,\alpha}} + (1-2\alpha^2)\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} \leq 0. \tag{3.15}
$$

with $(1-2\alpha^2) > 0$.
It follows that

$$
e^{2(1-2\alpha^2)t}\frac{d}{dt}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + 2e^{2(1-2\alpha^2)t}\|g\|^2_{Z^{r,\beta}_{\tau,\alpha}} + 2(1-2\alpha^2)e^{2(1-2\alpha^2)t}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}}, \leq 0 \tag{3.16}
$$

which implies

$$
e^{2(1-2\alpha^2)t}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + \int_0^t 2e^{2(1-2\alpha^2)s}\|g(s)\|^2_{Z^{r,\beta}_{\tau,\alpha}} ds \leq \|g(0)\|^2_{X^{r,\beta}_{\tau,\alpha}} \tag{3.17}
$$

From the ordinary differential equation (4.1), one has

$$
\begin{aligned}
&\tau(t)^{3/2} - \tau_0^{3/2} \\
=& -C\int_0^t (\|g(s)\|_{X^{r,\beta}_{\tau,\alpha}} + \|g(s)\|_{Z^{r,\beta}_{\tau,\alpha}})ds \\
\geq& -C\int_0^t \|g(0)\|_{X^{r,\beta}_{\tau_0,\alpha}} e^{-(1-2\alpha^2)s} ds - C\int_0^t \|g(s)\|_{Z^{r,\beta}_{\tau,\alpha}} e^{(1-2\alpha^2)s} e^{-(1-2\alpha^2)s} ds \\
\geq& -C_1\|g(0)\|_{X^{r,\beta}_{\tau_0,\alpha}}.
\end{aligned}
\tag{3.18}
$$

If one choose the initial perturbation data is small, such that

$$
\frac{\tau_0}{2} > C_1^{2/3}\|g(0)\|^{2/3}_{X^{r,\beta}_{\tau_0,\alpha}}, \tag{3.19}
$$

then (3.18) implies that

$$\tau(t) > \tau_0/2 \tag{3.20}$$

for all $t \geq 0$.
Consequently,

Proposition 3.1 *Under the same conditions stated in Theorem 1.1, suppose the g is the Gevrey regularity solution to (2.3)-(2.6), then the following estimates hold true.*

$$e^{2(1-2\alpha^2)t}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + \int_0^t 2e^{2(1-2\alpha^2)s}\|g(s)\|^2_{Z^{r,\beta}_{\tau,\alpha}} ds \leq \|g(0)\|^2_{X^{r,\beta}_{\tau_0,\alpha}} \tag{3.21}$$

with $\tau(t) > \tau_0/2$ for all $t \geq 0$.

4 The Proof of Theorem 1.1

Once the uniform estimates of solutions to (2.3)-(2.6) are achieved in Proposition 3.1, and the local existence of solutions in Gevrey regularity class can be done by the similar arguments as in [8, 16], then the global existence of solutions to (2.3)-(2.6) is proved. Moreover, by the relationship of (2.4), one can show the global existence of u to the initial-boundary value problem (1.9)-(1.11). In addition, according to Remark 2.1 and Proposition 3.1, it follows that $\|u\|_{X^{r,\beta}_{\tau,\alpha'}} + \|\partial_y u\|_{X^{r,\beta}_{\tau,\alpha'}} < \infty$ with $0 \leq \alpha' < \alpha$ and $\tau > \tau_0/2$. As a consequence, We finish the proof of global existence part in Theorem 1.1.

Below we will prove the uniqueness part in Theorem 1.1. It suffices to show the uniqueness of solutions to (2.3)-(2.6).

Assume there are two solutions g_1 and g_2 to (2.3)-(2.6) with the same initial data g_0, which satisfies $\|g_0\|_{X^{r,\beta}_{\tau_0,\alpha}} \leq \delta_0$. And the radii of Gevrey regularity of g_1 and g_2 are $\tau_1(t)$ and $\tau_2(t)$,respectively.
Define $\tau(t)$ by

$$\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g_1\|_{X^{r,\beta}_{\tau_1,\alpha}} + \|g_1\|_{Z^{r,\beta}_{\tau_1,\alpha}}) = 0 \tag{4.1}$$

with initial data

$$\tau(0) = \frac{\tau_0}{4}. \tag{4.2}$$

In view of the estimate in Section 3, there exists a time interval $[0, T]$, such that

$$\frac{\tau_0}{8} \leq \tau(t) \leq \frac{\tau_0}{4} \leq \frac{\min\{\tau_1(t), \tau_2(t)\}}{2} \tag{4.3}$$

for all $t \in [0, T]$.
We consider the equations of difference of solutions by setting $\bar{g} = g_1 - g_2$, one has

$$\partial_t \bar{g} + (1 - e^{-y} + u_1)\partial_x \bar{g} + (v_1 - v_2)\partial_y g_1 = -\bar{g} + \partial_y^2 \bar{g} + R, \tag{4.4}$$

with the source terms R being defined by

$$R = -(u_1 - u_2)\partial_x g_2 - v_2 \partial_y \bar{g}. \tag{4.5}$$

The initial dat and the boundary conditions are thus given by

$$\bar{g}(t = 0, x, y) = 0, \tag{4.6}$$

and

$$(\partial_y \bar{g} - \bar{g})|_{y=0} = 0. \tag{4.7}$$

By the similar arguments as those in Section 3, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + \|g\|^2_{Z^{r,\beta}_{\tau,\alpha}} + (1-2\alpha^2)\|g\|^2_{X^{r,\beta}_{\tau,\alpha}} + (1-\alpha)\|g(\cdot,0)\|^2_{X^{r,\beta}_{\tau,\alpha}} \\
\le&(\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g_1\|_{X^{r,\beta}_{\tau,\alpha}} + \|g_1\|_{Z^{r,\beta}_{\tau,\alpha}}))\|\bar{g}\|^2_{Y^{r,\beta}_{\tau,\alpha}} + \frac{C_0}{(\tau(t))^{1/2}}\|g_2\|_{Y^{r,\beta}_{\tau,\alpha}}(\|\bar{g}\|^2_{X^{r,\beta}_{\tau,\alpha}} + \|\bar{g}\|^2_{Z^{r,\beta}_{\tau,\alpha}})
\end{aligned} \tag{4.8}$$

From (4.1), we have

$$\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g_1\|_{X^{r,\beta}_{\tau,\alpha}} + \|g_1\|_{Z^{r,\beta}_{\tau,\alpha}}) \le 0, \tag{4.9}$$

due to the facts that $\tau(t) \le \tau_1(t)$ and the $X^{r,\beta}_{\tau,\alpha}$ and $Z^{r,\beta}_{\tau,\alpha}$ norms are increasing with respect to τ. Moreover,

$$\begin{aligned}
\frac{C_0}{(\tau(t))^{1/2}}\|g_2\|_{Y^{r,\beta}_{\tau,\alpha}} \le& \frac{C}{\tau}\|g_2\|_{X^{r,\beta}_{2\tau,\alpha}} \le \frac{C}{\tau}\|g_2\|_{X^{r,\beta}_{\tau_2,\alpha}} \\
\le& \frac{2C}{\tau_0}\|g(0)\|_{X^{r,\beta}_{\tau_0,\alpha}} e^{-2(1-2\alpha^2)t} \\
\le& \bar{C}\delta_0^{1/3}.
\end{aligned} \tag{4.10}$$

In the second inequality, $2\tau \le \tau_2$. In the third inequality, (3.19) and (3.21) are used. Since δ_0 is suitably small, it follows, from (4.8)-(4.10), that

$$\frac{d}{dt}\|\bar{g}\|^2_{X^{r,\beta}_{\tau,\alpha}} \le C\|\bar{g}\|^2_{X^{r,\beta}_{\tau,\alpha}} \tag{4.11}$$

It implies uniqueness of solutions to (2.3) due to $\bar{g} = 0$ in whole time interval $[0, T]$.

Then by the continuous argument, we conclude the proof of uniqueness of solutions for all $t > 0$ in Theorem 1.1.

Acknowledgement: Feng Xie is supported by National Nature Science Foundation of China 11571231, the China Scholarship Council and Shanghai Jiao Tong University SMC(A). Tong Yang is supported by National Nature Science Foundation of China 11771169.

References

[1] Alexander R., Wang Y-G, Xu C-J. and Yang, T., Well posedness of the Prandtl eqauation in Sobolev spaces, J. Amer. Math. Soc., 28 (2015), no. 3, 745-784.

[2] Arkhipov V.N., Influence of a magnetic field on boundary layer stability, Soviet Physics (Doklady) Trans. 4(1960), 1199-1201.

[3] Caflisch R.E. and Sammartino M., Existence and singularities for the Prandtl boundary layer equations, Z. Angew. Math. Mech., 80 (2000), 733-744.

[4] Drazin P.G., Stability of parallel flow in a parallel magnetic field at small magnetic Reynolds, J. Fluid Mech. 8(1960), 130-142.

[5] E W. and Engquist B., Blow up of solutions of the unsteady Prandtl equation, Comm. Pure Appl. Math., 50 (1997), 1287-1293.

[6] Gérard-Varet D. and Dormy E., On the ill-posedness of the Prandtl equation, J. Amer. Math. Soc., 23 (2010), 591-609.

[7] Gerard-Varet D. and Nguyen T., Remarks on the ill-posedness of the Prandtl equation, Asymptot. Anal. 77 (2012), no. 1-2, 71-88.

[8] Gérard-Varet D. and Masmoudi N., Well-posedness for the Prandtl system without analyticity or monotonicity. Ann. Sci. Éc. Norm Supér. (4) 48 (2015), no. 6, 1273-1325.

[9] Gérard-Varet D. and Prestipino M., Formal Derivation and Stability Analysis of Boundary Layer Models in MHD. Z. Angew. Math. Phys. 68 (2017), no. 3, 16 pp.

[10] Gotoh K., Effect of a Magnetic Field upon the Stability of Free Boundary Layers between Two Uniform Streams, J. Phys. Soc.. Japan, 16(1961), 559-570.

[11] Guo Y. and Nguyen T., A note on the Prandtl boundary layers, Comm. Pure Appl. Math. 64 (2011), 1416-1438.

[12] Hörmander L., The analysis of linear partial differential equations III, vol.257. Springer, Berlin, New York, 1983.

[13] Ignatova M. and Vicol V., Almost global existence for the Prandtl boundary layer equations. Arch. Ration. Mech. Anal. 220 (2016), 2, 809-848.

[14] Kukavica I., Vicol V., On the local existence of analytic solutions to the Prandtl boundary layer equations. Commun. Math. Sci. 11 (2013), no. 1, 269-292.

[15] Kukavica I., Vicol V., Wang F., The van Dommelen and Shen singularity in the Prandtl equations. Adv. Math. 307 (2017), 288-311.

[16] Li W.-X., Yang Y., Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points. . arXiv:1609.08430 (2016)

[17] Liu C.-J., Xie F. and Yang T., MHD boundary layers in Sobolev spaces without monotonicity. I. Well-posedness theory. (2016), arXiv 1611.05815.

[18] Liu C.-J., Xie F. and Yang T., MHD boundary layers in Sobolev spaces without monotonicity. II. Convergence theory. (2017), arXiv:1704.00523.

[19] Liu C.-J., Xie F. and Yang T., A note on the Ill-posedness of shear flow for the MHD boundary layers equations. Preprint (2017)

[20] Lombardo M. C., Cannone M. and Sammartino M. , Well-posedness of the boundary layer equations, SIAM J. Math. Anal., 35 (2003), 987-1004.

[21] Masmoudi N. and Wong T.-K., Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, Comm. Pure Appl. Math. 68 (2015), no. 10, 1683-1741.

[22] Rossow V.J., Boundary layer stability diagram for electrically conducting fluids in the presence of a magnetic field, NACA Technical Note 4282, NACA (Washington, 1958).

[23] Oleinik O. A., The Prandtl system of equations in boundary layer theory. Soviet Math Dokl, 4 (1963), 583-586.

[24] Oleinik O. A. and Samokhin V. N., Mathematical Models in Boundary Layers Theory. Chapman and Hall/CRC, 1999.

[25] Prandtl L., Uber flüssigkeits-bewegung bei sehr kleiner reibung. Verhandlungen des III. Internationlen Mathematiker Kongresses, Heidelberg. Teubner, Leipzig, (1904), 484-491.

[26] Sammartino M. and Caflisch R. E., Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, Comm. Math. Phys., 192 (1998), 433-461.

[27] Xin Z-P. and Zhang L., On the global existence of solutions to the Prandtl system, Adv. Math., 181 (2004), 88-133.

[28] Zhang P. and Zhang Z.F., Long time well-posdness of Prandtl system with small and analytic initial data. J. Funct. Anal. 270 (2016), no. 7, 2591-2615.