

WELL-POSEDNESS IN GEVREY FUNCTION SPACE FOR THE THREE-DIMENSIONAL PRANDTL EQUATIONS

WEI-XI LI AND TONG YANG

ABSTRACT. In the paper, we study the three-dimensional Prandtl equations without any monotonicity condition on the velocity field. We prove that when one tangential component of the velocity field has a single curve of non-degenerate critical points with respect to the normal variable, the system is locally well-posed in the Gevrey function space with Gevrey index in $]1, 2]$. The proof is based on some new observation of cancellation mechanism in the three space dimensional system in addition to those in the two-dimensional setting obtained in [1, 7, 19, 22].

1. INTRODUCTION AND MAIN RESULTS

The inviscid limit for the incompressible Navier-Stokes equations is one of the most fundamental problems in fluid mechanics. The justification remains a challenging problem from the mathematical point of view in particular with physical boundary conditions because of the appearance of boundary layer. Under the no-slip boundary condition, the behavior of the boundary layer can be described by the Prandtl system introduced by Prandtl [25] to study the behavior of the incompressible flow near a rigid wall at high Reynolds number. Formally, the asymptotic limit of the Navier-Stokes equations is represented by the Euler equations away from boundary and by the Prandtl equations within the boundary layer. A mathematically rigorous justification of the vanishing viscosity limit basically remains unsolved up to now even though it has been achieved in some special settings, see for example [3, 8, 9, 10, 12, 20, 26] and the references therein for the recent progress.

The mathematical study on the boundary layer has a very long history, however, so far the theory is only well developed in various function spaces for the 2D Prandtl system. In fact, the 2D Prandtl system can be reduced to a scalar nonlinear and nonlocal degenerate parabolic equation that has loss of derivative in tangential variable. The degeneracy in the viscosity dissipation coupled with the loss of derivative in the nonlocal term is the main difficulty in the well-posedness theories. To overcome the degeneracy, it is natural, in the spirit of abstract Cauchy-Kovalevskaya theorem, to perform estimates within the category of the analytic function space, and in this context the well-posedness was obtained in [26] (see the earlier work [2], and also [15, 23] for further generalization), even with the justification of the vanishing viscosity limit and in 3D. If the initial data have only finite order of regularity, the well-posedness in Sobolev space was first obtained by Oleinik (cf. [24]) under the monotonicity assumption in the normal direction, where the Crocco transformation was used to overcome the loss of the derivative. Recently, an approach based on energy method was developed for the well-posedness in Sobolev space under Oleinik's monotonicity assumption, see [1, 22] where the key observation is some kind of cancellation property in the convection term due to the monotonicity. Furthermore, the

2010 *Mathematics Subject Classification.* 35Q30, 35Q31.

Key words and phrases. 3D Prandtl boundary layer, non-degenerate critical points, Gevrey class.

well-posedness results in Gevrey space were achieved in the recent works [4, 7, 19] for the initial data without analyticity or monotonicity, where some further cancellation properties were observed near the non-degenerate critical points. Different from the analytic context, the Gevrey space with index $\sigma > 1$ contains compactly supported functions. We also refer to [18] for the smoothing effect in Gevrey space under the monotonicity assumption. In the aforementioned works, only local-in-time well-posedness results are obtained. On the other hand, the global weak solution was established by [27] with additional assumption on pressure while the existence of global strong solutions still remains unsolved, although there are several works (see [13, 28, 29] for example) about the lifespan of solutions in different settings.

On the other hand, in general boundary separation happens that implies the Prandtl equations can no longer be a suitable model for describing the behavior of the flow near the boundary. In mathematics, this is related to the fact that the Prandtl equations without the analyticity or monotonicity are in general ill-posed, cf. [5, 6, 11] and the references therein.

Compared to the 2D case, much less is known about the three-dimensional Prandtl equations. As for the well-posedness theories, only partial results have been obtained in some specific settings, cf. [26] in the analytic context and [16] under some constraint on flow structure. In addition to the difficulties for 2D, another major difficulty in 3D arises from the secondary flow. As it will be seen in the later analysis, the cancellation properties observed in 2D case are not enough to overcome the difficulties in the analysis. In addition, we need to use some new cancellations in the 3D setting. We will explain this further in Subsection 2.2 about the new ideas and approach to be used.

In this paper, we will study the well-posedness of 3D Prandtl system without the analyticity or monotonicity. For this, let us first mention the paper on the ill-posedness [17] which shows that even for a perturbation of shear flow, without the structural condition, the linearized Prandtl equations are ill-posed. In fact, the ill-posedness estimate on the solution operator implies that the optimal Gevrey index for the well-posedness without any structural condition is 2. Hence, the result of this paper about the well-posedness in Gevrey function space with index in $]1, 2]$ complements the ill-posedness estimate in [17]. Without loss of generality, we will consider the system in a periodic domain in tangential direction, that is, in $\Omega = \mathbb{T}^2 \times \mathbb{R}_+$.

Denote by (u, v) the tangential component and by w the vertical component of the velocity field, then the 3D Prandtl system in Ω reads

$$\begin{cases} \partial_t u + (u\partial_x + v\partial_y + w\partial_z)u - \partial_z^2 u + \partial_x p = 0, & t > 0, (x, y, z) \in \Omega, \\ \partial_t v + (u\partial_x + v\partial_y + w\partial_z)v - \partial_z^2 v + \partial_y p = 0, & t > 0, (x, y, z) \in \Omega, \\ \partial_x u + \partial_y v + \partial_z w = 0, & t > 0, (x, y, z) \in \Omega, \\ u|_{z=0} = v|_{z=0} = w|_{z=0} = 0, & \lim_{z \rightarrow +\infty} (u, v) = (U(t, x, y), V(t, x, y)), \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, & (x, y, z) \in \Omega, \end{cases} \quad (1)$$

where $(U(t, x, y), V(t, x, y))$ and $p(t, x, y)$ are the boundary traces of the tangential velocity field and pressure of the outer flow, satisfying Bernoulli's law

$$\begin{cases} \partial_t U + U\partial_x U + V\partial_y U + \partial_x p = 0, \\ \partial_t V + U\partial_x V + V\partial_y V + \partial_y p = 0. \end{cases}$$

Note p, U, V are given functions determined by the Euler flow, and (1) is a degenerate parabolic system losing one order derivative in the tangential variable. We refer to [21, 24, 25] for the background and mathematical presentation of this fundamental system.

We will consider the Prandtl system (1) under the assumption that one component of the initial tangential velocity component, for example u_0 admits a single curve of non-degenerate critical points. The precise assumption on the initial data is given as follows.

Assumption 1.1. Let $\delta > 2$ be a given number. Suppose there exists a single curve $z = \gamma_0(x, y)$ in Ω with $0 < \sup_{\mathbb{T}^2} \gamma_0 < +\infty$ and several constants $C_0 > 0$, $0 < c_0 < 1$ and $0 < \epsilon_0 < \frac{1}{4} \sup_{\mathbb{T}^2} \gamma_0$, such that for any $(x, y) \in \mathbb{T}^2$ the initial datum $u_0 \in C^6(\Omega)$ satisfies the following properties:

$$\begin{cases} \partial_z u_0(x, y, z) = 0 \text{ iff } z = \gamma_0(x, y), \text{ and } \partial_z^2 u_0(x, y, \gamma_0(x, y)) \neq 0; \\ c_0 \langle z \rangle^{-\delta} \leq \partial_z u_0(x, y, z) \leq c_0^{-1} \langle z \rangle^{-\delta} \text{ if } |z - \gamma_0(x, y)| \geq \epsilon_0; \\ \sum_{2 \leq j \leq 6} |\partial_z^j u_0(x, y, z)| \leq C_0 \langle z \rangle^{-\delta-1} \text{ for any } z \geq 0, \end{cases}$$

where $\langle z \rangle = (1 + |z|^2)^{1/2}$.

Next we introduce the Gevrey function space in the tangential variables x, y . In this paper, we use ℓ, κ to denote two fixed constants satisfying

$$\kappa \geq 1, \quad \ell > 3/2 \text{ and } \ell + 1/2 < \delta \leq \ell + 1, \quad (2)$$

where $\delta > 2$ is given in Assumption 1.1.

Definition 1.2 (Gevrey space in tangential variables x and y). Let U, V be the data given in (1). With each pair (ρ, σ) , $\rho > 0$ and $\sigma \geq 1$, a Banach space $X_{\rho, \sigma}$ consists of all smooth vector functions (u, v) such that $\|(u, v)\|_{\rho, \sigma} < +\infty$, where the Gevrey norm $\|\cdot\|_{\rho, \sigma}$ is defined below. For each multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, denote $\partial^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}$ and

$$\psi = \partial_z(u - U) = \partial_z u, \quad \eta = \partial_z(v - V) = \partial_z v.$$

Then the Gevrey norm is defined by

$$\begin{aligned} \|(u, v)\|_{\rho, \sigma} &= \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-6}}{[(|\alpha| - 7)!]^\sigma} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha(u - U)\|_{L^2} + \|\langle z \rangle^\kappa \partial^\alpha(v - V)\|_{L^2} \right) \\ &+ \sup_{|\alpha| \leq 6} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha(u - U)\|_{L^2} + \|\langle z \rangle^\kappa \partial^\alpha(v - V)\|_{L^2} \right) \\ &+ \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-6}}{[(|\alpha| - 7)!]^\sigma} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L^2} + \sup_{|\alpha| \leq 6} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L^2} \\ &+ \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-5}}{[(|\alpha| - 6)!]^\sigma} |\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} + \sup_{|\alpha| \leq 6} \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} \\ &+ \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \geq 7}} \frac{\rho^{|\alpha|+j-6}}{[(|\alpha| + j - 7)!]^\sigma} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L^2} + \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \leq 6}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L^2} \\ &+ \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \geq 7}} \frac{\rho^{|\alpha|+j-5}}{[(|\alpha| + j - 6)!]^\sigma} |\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L^2} + \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \leq 6}} \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L^2}, \end{aligned} \quad (3)$$

where and throughout the paper we use L^2 instead of $L^2(\Omega)$ without confusion. Moreover, we define another Gevrey space $Y_{\rho,\sigma}$ consist of smooth functions $F(x, y)$ such that $\|F\|_{\rho,\sigma} < +\infty$, where

$$\|F\|_{\rho,\sigma} = \sup_{|\alpha| \geq 0} \frac{\rho^{|\alpha|}}{(|\alpha|!)^\sigma} \|\partial^\alpha F\|_{L^2(\mathbb{T}^2)}.$$

Remark 1.3. Note the factors in front of the L^2 -norms of $\partial^\alpha \psi$ and $\partial^\alpha \eta$ are anisotropic, likewise, for the mixed derivatives in the last two lines of (3).

We will look for the solutions to (1) in the Gevrey function space $X_{\rho,\sigma}$. For this, the initial data (u_0, v_0) satisfy the following compatibility conditions

$$\begin{cases} (u_0, v_0)|_{z=0} = (0, 0), \quad \lim_{z \rightarrow +\infty} (u_0, v_0) = (U, V) \text{ and } (\partial_z \psi_0, \partial_z \eta_0)|_{z=0} = (\partial_x p, \partial_y p), \\ \partial_z^3 \psi_0|_{z=0} = \psi_0 (\partial_x \psi_0 - \partial_y \eta_0)|_{z=0} + 2\eta_0 \partial_y \psi_0|_{z=0} + \partial_t \partial_x p, \\ \partial_z^3 \eta_0|_{z=0} = \eta_0 (\partial_y \eta_0 - \partial_x \psi_0)|_{z=0} + 2\psi_0 \partial_x \eta_0|_{z=0} + \partial_t \partial_y p, \end{cases} \quad (4)$$

where $\psi_0 = \partial_z u_0$ and $\eta_0 = \partial_z v_0$.

The main result of this paper can be stated as follows.

Theorem 1.4. *Let $1 < \sigma \leq 2$, under the compatibility conditions (4), suppose $U, V, p \in Y_{2\rho_0,\sigma}$ and $(u_0, v_0) \in X_{2\rho_0,\sigma}$ for some $\rho_0 > 0$. Moreover, suppose u_0 satisfies Assumption 1.1. Then the Prandtl system (1) admits a unique solution $(u, v) \in L^\infty([0, T]; X_{\rho,\sigma})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.*

Remark 1.5. Up to a coordinate transformation, the assertion in the above theorem still holds if we impose the existence of non-degenerate points on the tangential component of the initial velocity field in an arbitrary but fixed direction instead of u_0 .

Remark 1.6. The assumption that $U, V, p \in Y_{2\rho_0,\sigma}$ is closely related to the up-to-boundary Gevrey regularity for the Euler equations, see for example [14] and the references therein.

Remark 1.7. It remains unsolved about the well-posedness in the category of Sobolev space with finite regularity that has been well developed in 2D. And a more important open problem is the mathematical justification on the approximation of the solution to Navier-Stokes equation with no-slip boundary condition by the those of Euler and Prandtl equations. We expect the present work may shed some light on the vanishing viscosity limit for Navier-Stokes equations in 3D setting.

To simplify the notations, we will mainly focus only on the constant outer flow when $(U, V) \equiv (0, 0)$, and the argument can be extended, without essential difficulty to general functions U and V as given in the proof of Theorem 1.4 at the end of the paper.

For the constant outer flow, the Prandtl system (1) can be written as

$$\begin{cases} \partial_t u + (u \partial_x + v \partial_y + w \partial_z) u - \partial_z^2 u = 0, & t > 0, (x, y, z) \in \Omega, \\ \partial_t v + (u \partial_x + v \partial_y + w \partial_z) v - \partial_z^2 v = 0, & t > 0, (x, y, z) \in \Omega, \\ (u, v)|_{z=0} = (0, 0), \quad \lim_{z \rightarrow +\infty} (u, v) = (0, 0), \\ (u, v)|_{t=0} = (u_0, v_0), \quad (x, y, z) \in \Omega, \end{cases} \quad (5)$$

with

$$w(t, x, y, z) = - \int_0^z \partial_x u(t, x, y, \tilde{z}) d\tilde{z} - \int_0^z \partial_y v(t, x, y, \tilde{z}) d\tilde{z}.$$

The existence and uniqueness of solutions to system (5) can be stated as follows.

Theorem 1.8. *Let $1 < \sigma \leq 2$. Suppose the initial datum (u_0, v_0) belongs to $X_{2\rho_0, \sigma}$ for some $\rho_0 > 0$, and satisfies the compatibility condition (4) with constant pressure. Then the system (5) admits a unique solution $(u, v) \in L^\infty([0, T]; X_{\rho, \sigma})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.*

We will focus on proving Theorem 1.8 and show at the end of the last section about how to extend the argument to the case with general outer flow when $U, V \in Y_{2\rho_0, \sigma}$. Furthermore, we will present in detail the proof of Theorem 1.8 for $\sigma \in [3/2, 2]$. Note that the constraint $\sigma \geq 3/2$ is not essential and indeed it is just a technical assumption for clear presentation. And we will explain at the end of Section 8 about how to modify the proof for the case when $1 < \sigma < 3/2$.

The rest of the paper is organized as follows. In Section 2, we first introduce the notations used in the paper, and then explain the main difficulties and the new ideas. We will prove in Sections 3-8 the a priori estimate given in Section 2. The proof of the well-posedness for the Prandtl system is given in the last section.

2. NOTATIONS AND METHODOLOGY

This section and Sections 3-8 are to derive a priori estimate for the Prandtl system (5), which is crucial for proving Theorem 1.8. In Subsection 2.1, we introduce some auxiliary functions to be estimated. In Subsection 2.2, we explain the difficulties for the well-posedness of 3D Prandtl system and then present the ideas to overcome them. The a priori estimate is given in Subsection 2.3 with its proof given in Sections 3-8.

Let δ, ℓ and κ be some given numbers satisfying (2), and let (u, v) be a solution to the Prandtl system (5) in $[0, T] \times \Omega$, satisfying the properties stated as follows. There exists a single curve $z = \gamma(t, x, y)$ of non-degenerate critical points, that is, for $(t, x, y) \in [0, T] \times \mathbb{T}^2$,

$$\partial_z u(t, x, y, z) = 0 \text{ iff } z = \gamma(t, x, y, z), \text{ and } |\partial_z^2 u(t, x, y, \gamma(t, x, y))| > 0.$$

To simply the argument we may assume without loss of generality that

$$\gamma \equiv 1,$$

that is 1 is the only non-degenerate critical point of u , and the general case can be derived quite similarly with slight modification. Moreover, there exist three constants $0 < c, \epsilon < 1/4$ and $C > 0$ such that for any $t \in [0, T]$ and for any $(x, y) \in \mathbb{T}^2$ we have

$$\begin{cases} \partial_z u(t, x, y, z) \equiv 0 \text{ iff } z = 1, \text{ and } |\partial_z^2 u(t, x, y, 1)| > 0; \\ |\partial_z^2 u(t, x, y, z)| \geq c, \text{ if } |z - 1| \leq 2\epsilon; \\ c \langle z \rangle^{-\delta} \leq |\partial_z u(t, x, y, z)| \leq c^{-1} \langle z \rangle^{-\delta}, \text{ if } |z - 1| \geq \epsilon; \\ \sum_{j=1}^6 |\partial_z^j u(t, x, y, z)| \leq c^{-1} \langle z \rangle^{-\delta-1} \text{ for } z \geq 0, \end{cases} \quad (6)$$

and

$$\begin{aligned} & \sum_{|\alpha| \leq 3} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha u\|_{L^\infty} + \|\langle z \rangle^\kappa \partial^\alpha v\|_{L^\infty} + \|\partial^\alpha w\|_{L^\infty} \right) \\ & + \sum_{|\alpha| \leq 4} \|\langle z \rangle^\ell \partial^\alpha \partial_z u\|_{L_{x,y}^\infty(L_z^2)} + \sum_{\substack{|\alpha|+j \leq 4 \\ j \geq 1}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \partial_z u\|_{L_{x,y}^\infty(L_z^2)} \\ & + \sum_{|\alpha|+j \leq 4} \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \partial_z v\|_{L_{x,y}^\infty(L_z^2)} \leq C, \end{aligned} \quad (7)$$

where $L_{x,y}^\infty(L_z^2) = L^\infty(\mathbb{T}^2; L^2(\mathbb{R}_+))$ stands for the classical Sobolev space, so does the Sobolev space $L_{x,y}^2(L_z^\infty)$.

Let $0 \leq \tau_1, \tau_2 \leq 1$ be two $C^\infty(\mathbb{R})$ smooth functions such that

$$\tau_1 \equiv 1 \text{ on } \{|z-1| > 3\epsilon/2\}, \quad \tau_1 \equiv 0 \text{ on } \{|z-1| \leq \epsilon\}, \quad (8)$$

and

$$\tau_2 \equiv 1 \text{ on } \{|z-1| \leq 3\epsilon/2\}, \quad \text{supp } \tau_2 \subset \{|z-1| \leq 2\epsilon\}. \quad (9)$$

Observe

$$\tau_1 + \tau_2 \geq 1, \quad \tau_1' = \tau_1' \tau_2, \quad \tau_2' = \tau_2' \tau_1, \quad \text{and } (1 - \tau_2) = (1 - \tau_2) \tau_1, \quad (10)$$

because $\tau_2 \equiv 1$ on $\text{supp } \tau_1'$, $\tau_1 \equiv 1$ on $\text{supp } \tau_2'$, and $\tau_1 \equiv 1$ on $\text{supp}(1 - \tau_2)$. Here and throughout the paper, f' and f'' stand for the first and the second order derivatives of f .

2.1. Notations. From now on, we write ∂^α instead of $\partial_x^{\alpha_1} \partial_y^{\alpha_2}$ for $\alpha \in \mathbb{Z}_+^2$. Let (u, v) solve the system (5). Define ψ and η by

$$\psi = \partial_z u \text{ and } \eta = \partial_z v. \quad (11)$$

Applying ∂_z to the equations for u and v in (5), we obtain the equations solved by ψ and η , that is,

$$\begin{cases} \partial_t \psi + (u \partial_x + v \partial_y + w \partial_z) \psi - \partial_z^2 \psi = g, \\ \partial_t \eta + (u \partial_x + v \partial_y + w \partial_z) \eta - \partial_z^2 \eta = h, \end{cases} \quad (12)$$

where g, h are given by

$$\begin{cases} g = (\partial_y v) \psi - (\partial_y u) \eta, \\ h = (\partial_x u) \eta - (\partial_x v) \psi. \end{cases} \quad (13)$$

Moreover, denote

$$\xi = \partial_z \psi = \partial_z^2 u, \quad \zeta = \partial_z \eta = \partial_z^2 v, \quad (14)$$

and it follows from (12) that

$$\begin{cases} \partial_t \xi + (u \partial_x + v \partial_y + w \partial_z) \xi - \partial_z^2 \xi = \sum_{j=1}^3 \theta_j, \\ \partial_t \zeta + (u \partial_x + v \partial_y + w \partial_z) \zeta - \partial_z^2 \zeta = \sum_{j=1}^3 \mu_j, \end{cases} \quad (15)$$

where

$$\begin{cases} \theta_1 = 2\xi \partial_y v - 2\eta \partial_y \psi, & \theta_2 = \psi \partial_y \eta - \zeta \partial_y u, & \theta_3 = \xi \partial_x u - \psi \partial_x \psi, \\ \mu_1 = 2\zeta \partial_x u - 2\psi \partial_x \eta, & \mu_2 = \eta \partial_x \psi - \xi \partial_x v, & \mu_3 = \zeta \partial_y v - \eta \partial_y \eta. \end{cases} \quad (16)$$

For each multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, we define g_α and h_α by

$$g_\alpha = \partial^\alpha g = \partial_x^{\alpha_1} \partial_y^{\alpha_2} g, \quad \text{and} \quad h_\alpha = \partial^\alpha h = \partial_x^{\alpha_1} \partial_y^{\alpha_2} h. \quad (17)$$

Similarly, define

$$\vec{\theta}_\alpha = (\partial^\alpha \theta_1, \dots, \partial^\alpha \theta_3), \quad \vec{\theta}_\mu = (\partial^\alpha \mu_1, \dots, \partial^\alpha \mu_3) \quad (18)$$

And for each $m \geq 1$ define

$$\begin{cases} \Gamma_m = \psi \partial_x^m v - \eta \partial_x^m u, & \tilde{\Gamma}_m = (\partial_y^m v) \psi - (\partial_y^m u) \eta, \\ H_m = \xi \partial_x^m v - \eta \partial_x^m \psi, & \tilde{H}_m = \xi \partial_y^m v - \eta \partial_y^m \psi, \\ G_m = \xi \partial_x^m u - \psi \partial_x^m \psi, & \tilde{G}_m = \xi \partial_y^m u - \psi \partial_y^m \psi. \end{cases} \quad (19)$$

Let τ_1, τ_2 be given by (8) and (9). Motivated by [1, 22], when $\psi \neq 0$ in $\text{supp } \tau_1$ then we can define f_m with $m \geq 1$, by

$$f_m = \tau_1 \partial_x^m \psi - \tau_1 \frac{\xi}{\psi} \partial_x^m u = \tau_1 \psi \partial_z \left(\frac{\partial_x^m u}{\psi} \right), \quad (20)$$

recalling ψ, ξ are defined in (11) and (14). Likewise, define \tilde{f}_m by

$$\tilde{f}_m = \tau_1 \partial_y^m \psi - \tau_1 \frac{\xi}{\psi} \partial_y^m u = \tau_1 \psi \partial_z \left(\frac{\partial_y^m u}{\psi} \right).$$

On the other hand, when $\xi \neq 0$ in $\text{supp } \tau_2$ then we define

$$q_m = \tau_2 \partial_x^m \xi - \tau_2 \frac{\partial_z \xi}{\xi} \partial_x^m \psi = \tau_2 \xi \partial_z \left(\frac{\partial_x^m \psi}{\xi} \right), \quad (21)$$

and similarly for \tilde{q}_m . It follows from (5) that

$$(\partial_z \psi, \partial_z \eta)|_{z=0} = (\partial_z f_m, \Gamma_m)|_{z=0} = (\partial_z \tilde{f}_m, \tilde{\Gamma}_m)|_{z=0} = (g_\alpha, h_\alpha)|_{z=0} = (0, 0).$$

Note the definitions of f_m and \tilde{f}_m are motivated by [1, 22], and meanwhile the type of auxiliary functions q_m, \tilde{q}_m are used by [19]. As in [19] if we can control the term $\tau_2 \partial_x^m \psi$ then the estimates on $\partial_x^m u$ and $\partial_x^m \psi$ can be derived from the weighted L^2 -norm of f_m and q_m by Hardy inequality and Poincaré inequality, likewise, for $\partial_y^m u$ and $\partial_y^m \psi$. As a result, we have the upper bounds on $\partial^\alpha u$ and $\partial^\alpha \psi$, by the inequality

$$\forall \alpha \in \mathbb{Z}_+^2, \forall F \in H^\infty, \quad \|\partial^\alpha F\|_{L^2(\mathbb{T}^2)}^2 \leq \|\partial_x^{|\alpha|} F\|_{L^2(\mathbb{T}^2)}^2 + \|\partial_y^{|\alpha|} F\|_{L^2(\mathbb{T}^2)}^2. \quad (22)$$

In order to obtain the upper bound of $\partial^\alpha \eta$, we will apply the energy method to the equation (12) for η . It remains to estimate $\partial^\alpha v$ with $|\alpha| = m$, and this can be deduced from the estimation on the auxiliary functions Γ_m and $\tilde{\Gamma}_m$ on the monotonicity part where $\psi \neq 0$, and from H_m, \tilde{H}_m on the concave part where $\xi \neq 0$. Finally, we need to estimate g_α and h_α since they appear in the equations for f_m and $\partial^\alpha \eta$ with the degeneracy in tangential variables. We will explain further in the next subsection about the motivation for introducing the above auxiliary functions.

2.2. Difficulties and methodologies. In this subsection, we will explain the main difficulties and the new ideas introduced in this paper.

When applying ∂^α to the equations in (5), we lose derivatives in the tangential variables x and y in the terms

$$(\partial^\alpha w) \psi, \quad (\partial^\alpha w) \eta. \quad (23)$$

Similar to 2D case, part of these terms can be handled under the the existence of single curve of non-degenerate critical points, by some kind of cancellation properties, cf. [7, 19]. In the 3D setting considered in this paper, for the monotonicity part that $|\psi| > 0$, we can apply the same cancellation as in 2D case, to the equations of $\partial^\alpha u$ and $\partial^\alpha \psi$. This indeed eliminates the first term in (23), but meanwhile a new term

$$g_\alpha = \partial^\alpha g$$

appears when applying ∂^α to the equation (12) for ψ . Note that g also has the loss of one order derivative in y variable. This prevents us to investigate the well-posedness in Sobolev space with finite order of regularity. When performing estimates in the concave (or convex) part where $|\xi| > 0$, the situation is quite different from the one in [7, 19]. Precisely, in addition to the term

$$\left(\tau_2' \int_0^z \partial_x^{m+1} u \, d\tilde{z}, \tau_2 \partial_x^{m+1} u \right)_{L^2(\Omega)}$$

that can be handled by some kind of crucial representation of $\partial_x^m u$ introduced by [7], we are faced two new terms caused by the appearance of the secondary component v and the cutoff function τ_2 :

$$\left(\tau_2' \int_0^z \partial_x^m \partial_y v \, dz, \tau_2 \partial_x^{m+1} u \right)_{L^2(\Omega)} \quad \text{and} \quad (\tau_2 \partial_x^m \partial_y v, \tau_2 \partial_x^{m+1} u)_{L^2(\Omega)}, \quad (24)$$

where the degeneracy occurs in the nonlocal term caused by v .

Estimates on $\partial^\alpha u$ and $\partial^\alpha \psi$. If we have the desired upper bounds for the terms in (24) then we can make use of the same cancellation as in 2D case [19] to obtain a new equation for the auxilliary function f_m, q_m defined by (20) and (21). Even though this can not avoid the degeneracy in the tangential variables because of $g_\alpha, \vec{\theta}_\alpha$. Our observation is that this kind of cancellation transfers the degeneracy coming from the first non-local term in (23) to a local term g_α . To work with g_α , we have the advantage to use another kind of cancellation. Precisely, multiplying the first equation in (12) by $\partial_y v$, and the second equation by $\partial_y u$, and then their subtraction yields the equation for g . Based on this, we can apply our approach used in 2D [19] to perform energy estimate for g_α and similarly for $\vec{\theta}_\alpha$, and then for f_m , in the context of Gevrey function space rather than the analytic function space, if we ignore at this moment the degeneracy caused by the second term in (23). Similar argument applies also to h_α and \tilde{f}_m, \tilde{q}_m . As a result, we can obtain the estimates as desired for $\partial^\alpha u$ and $\partial^\alpha \psi$ by Hardy inequality and Poincaré inequality.

Estimates on $\partial^\alpha v$ and $\partial^\alpha \eta$. Now we turn to the second tangential component of velocity field. Firstly, for η , note that if the order of its derivatives in the energy are one order less than the ones of u, v and ψ , then we do not have derivative loss for this term. This is why in the definition of the Gevrey norm, there is an anisotropic term for η (see Remark 1.3).

To handle v and its derivatives, we will not apply the energy estimation directly because it involves the second term in (23). Instead, we observe that the estimate on $\partial_x^m v$ can be derived through $h_{m-1,0} = \partial_x^{m-1} h$ defined in (17) in the monotonicity part where $|\psi| > 0$, and through $\partial_x^{m-1} \mu_2$ defined in (16) in the concave part of $|\xi| > 0$. On the other hand, it is not easy to estimate the bounds of the lower order derivatives $\partial_x^j v, j < m$, in the expression of $h_{m-1,0}$. This is why we introduce the auxilliary functions Γ_m and H_m , which contain only the leading term in the representations of $h_{m-1,0}$ and $\partial_x^{m-1} \mu_2$. It is then clear that we can get the upper bound of $\partial_x^m v$ from Γ_m in the monotonicity part and from H_m in concave part. Furthermore, Γ_m can be handled by a new cancellation as shown in the equation for Γ_m , where the terms in (23) do not appear due to cancellation. Similar argument applies to $H_m, \tilde{\Gamma}_m$ and \tilde{H}_m . And this leads to the desired estimate on $\partial^\alpha v$.

Estimates on the terms in (24). We will use the similar idea as above for treatment of $\partial_x^m v$, together with the crucial representation of $\partial_x^m u$ introduced by [7]; see Subsections 6.1 and 6.2 for detailed computation.

2.3. A priori estimate. We first introduce the Gevrey function space and the equipped norm.

Definition 2.1. Denote $\vec{a} = (u, v)$ with (u, v) satisfying the Prandtl system (5) and the conditions (6)-(7). Let $X_{\rho,\sigma}$ be the Gevrey function space given in Definition 1.2, equipped

with the norm $\|\cdot\|_{\rho,\sigma}$ defined by (3). Set

$$\begin{aligned}
|\vec{a}|_{\rho,\sigma} &= \|\vec{a}\|_{\rho,\sigma} + \sup_{m \geq 7} \frac{\rho^{m-6}}{[(m-7)!]^\sigma} \left(\|\langle z \rangle^\ell f_m\|_{L^2} + \|q_m\|_{L^2} + \|\tau_2 \partial_x^m \xi\|_{L^2} \right) \\
&+ \sup_{m \geq 7} \frac{\rho^{m-6}}{[(m-7)!]^\sigma} \left(\|\langle z \rangle^\ell \tilde{f}_m\|_{L^2} + \|\tilde{q}_m\|_{L^2} + \|\tau_2 \partial_y^m \xi\|_{L^2} \right) \\
&+ \sup_{m \geq 7} \frac{\rho^{m-6}}{[(m-7)!]^\sigma} \left(\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + \|G_m\|_{L^2} + \|H_m\|_{L^2} \right) \\
&+ \sup_{m \geq 7} \frac{\rho^{m-6}}{[(m-7)!]^\sigma} \left(\|\langle z \rangle^{\kappa+\delta} \tilde{\Gamma}_m\|_{L^2} + \|\tilde{G}_m\|_{L^2} + \|\tilde{H}_m\|_{L^2} \right) \\
&+ \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-5}}{[(|\alpha|-6)!]^\sigma} |\alpha| \left(\|\langle z \rangle^{\kappa+\delta} g_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} h_\alpha\|_{L^2} + \|\vec{\theta}_\alpha\|_{L^2} + \|\vec{\mu}_\alpha\|_{L^2} \right) \\
&+ \sup_{1 \leq m \leq 6} \left(\|\langle z \rangle^\ell f_m\|_{L^2} + \|q_m\|_{L^2} + \|\tau_2 \partial_x^m \xi\|_{L^2} + \|\langle z \rangle^\ell \tilde{f}_m\|_{L^2} + \|\tilde{q}_m\|_{L^2} + \|\tau_2 \partial_y^m \xi\|_{L^2} \right) \\
&+ \sup_{1 \leq m \leq 6} \left(\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + \|G_m\|_{L^2} + \|H_m\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} \tilde{\Gamma}_m\|_{L^2} + \|\tilde{G}_m\|_{L^2} + \|\tilde{H}_m\|_{L^2} \right) \\
&+ \sup_{0 \leq |\alpha| \leq 6} \left(\|\langle z \rangle^{\kappa+\delta} g_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} h_\alpha\|_{L^2} + \|\vec{\theta}_\alpha\|_{L^2} + \|\vec{\mu}_\alpha\|_{L^2} \right),
\end{aligned}$$

where the functions $\tau_j, g_\alpha, h_\alpha, f_m$, etc., are defined in Subsection 2.1. Similarly, we define $|\vec{a}_0|_{\rho,\sigma}$ for $\vec{a}_0 = (u_0, v_0)$, the initial datum in (5). The L^2 -norm of a vector $\vec{p} = (p_1, p_2, p_3)$ is defined by

$$\|\vec{p}\|_{L^2}^2 = \sum_{1 \leq j \leq 3} \|p_j\|_{L^2}^2.$$

Remark 2.2. It is clear that $|\vec{a}|_{\rho,\sigma} \leq |\vec{a}|_{\tilde{\rho},\sigma}$ for any $\rho \leq \tilde{\rho}$. Moreover, direct calculation gives

$$\|\vec{a}\|_{\rho,\sigma} \leq |\vec{a}|_{\rho,\sigma} \leq C_{\rho,\rho^*} (\|\vec{a}\|_{\rho^*,\sigma} + \|\vec{a}\|_{\rho^*,\sigma}^2)$$

for any $\rho < \rho^*$, with C_{ρ,ρ^*} being a constant depending only on the difference $\rho^* - \rho$.

The main result of this part can be stated as follows.

Theorem 2.3 (A priori estimate in Gevrey space). *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) such that the properties listed in (6)-(7) hold. Then there exists a constant $C_* > 1$, such that the estimate*

$$|\vec{a}(t)|_{\rho,\sigma}^2 \leq C_* |\vec{a}_0|_{\rho,\sigma}^2 + C_* \int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + C_* \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \quad (25)$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$ and any $t \in [0, T]$. Here, the constant C_* depends only on the constants in (6)-(7) as well as the Sobolev embedding constants.

Remark 2.4. When $1 < \sigma < 3/2$, we can obtain a similar a priori estimate as (25), cf. Theorem 8.1 in Section 8.

In view of Remark 2.2, each term in (25) is well-defined. We will proceed to derive the upper bound of $|\vec{a}|_{\rho,\sigma}$ in the next Sections 3-8.

Before proving Theorem 2.3, we first list some basic inequalities to be used.

Lemma 2.5. *With the notations in Subsection 2.1, the following inequalities hold.*

(i) *For any integer $k \geq 1$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq 1$, we have*

$$k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho}} k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho} - \rho}. \quad (26)$$

(ii) *For any suitable function F ,*

$$\|F\|_{L^\infty(\Omega)} \leq \sqrt{2} \left(\|F\|_{L^2_{x,y}(L^\infty_z)} + \|\partial_x F\|_{L^2_{x,y}(L^\infty_z)} + \|\partial_y F\|_{L^2_{x,y}(L^\infty_z)} + \|\partial_x \partial_y F\|_{L^2_{x,y}(L^\infty_z)} \right), \quad (27)$$

and

$$\begin{aligned} \|F\|_{L^\infty(\Omega)} \leq & 2 \left(\|F\|_{L^2} + \|\partial_x F\|_{L^2} + \|\partial_y F\|_{L^2} + \|\partial_z F\|_{L^2} \right) \\ & + 2 \left(\|\partial_x \partial_y F\|_{L^2} + \|\partial_x \partial_z F\|_{L^2} + \|\partial_y \partial_z F\|_{L^2} + \|\partial_x \partial_y \partial_z F\|_{L^2} \right). \end{aligned} \quad (28)$$

(iii) *For any $0 < r \leq 1$ and any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ with $|\alpha| = m$, we have*

$$\begin{aligned} & \|\langle z \rangle^{\ell-1} \partial^\alpha u\|_{L^2} + \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L^2} + \|\tau_2 \partial^\alpha \xi\|_{L^2} + \|\langle z \rangle^\ell f_m\|_{L^2} + \|\langle z \rangle^\kappa \partial^\alpha v\|_{L^2} \\ & + \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + \|H_m\|_{L^2} + \|G_m\|_{L^2} \\ \leq & \begin{cases} \frac{[(m-7)!]^\sigma}{r^{(m-6)}} |\vec{a}|_{r,\sigma}, & \text{if } m \geq 7, \\ |\vec{a}|_{r,\sigma}, & \text{if } m \leq 6, \end{cases} \end{aligned} \quad (29)$$

and

$$\|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L^2} \leq \begin{cases} \frac{[(|\alpha|+j-7)!]^\sigma}{r^{(|\alpha|+j-6)}} |\vec{a}|_{r,\sigma}, & \text{if } |\alpha|+j \geq 7 \text{ and } 1 \leq j \leq 4, \\ |\vec{a}|_{r,\sigma}, & \text{if } |\alpha|+j \leq 6 \text{ and } 1 \leq j \leq 4. \end{cases} \quad (30)$$

(iv) *For any $0 < r \leq 1$ and any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, we have*

$$\begin{aligned} & \|\partial^\alpha w\|_{L^2_{x,y}(L^\infty_z)} + |\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} \\ & + |\alpha| \left(\|\langle z \rangle^{\kappa+\delta} g_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} h_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} + \|\vec{\theta}_\alpha\|_{L^2} + \|\vec{\mu}_\alpha\|_{L^2} \right) \\ \leq & \begin{cases} \frac{[(|\alpha|-6)!]^\sigma}{r^{(|\alpha|-5)}} |\vec{a}|_{r,\sigma}, & \text{if } |\alpha| \geq 7, \\ |\vec{a}|_{r,\sigma}, & \text{if } |\alpha| \leq 6, \end{cases} \end{aligned} \quad (31)$$

and

$$|\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L^2} \leq \begin{cases} \frac{[(|\alpha|+j-6)!]^\sigma}{r^{(|\alpha|+j-5)}} |\vec{a}|_{r,\sigma}, & \text{if } |\alpha|+j \geq 7 \text{ and } 1 \leq j \leq 4, \\ |\vec{a}|_{r,\sigma}, & \text{if } |\alpha|+j \leq 6 \text{ and } 1 \leq j \leq 4. \end{cases} \quad (32)$$

(v) *Let $\sigma \geq 1$ and let $m \geq 7$. Then for any $0 < r \leq 1$ we have*

$$\|\partial_x^{m-1} \xi\|_{L^2} \leq \|\partial_z f_{m-1}\|_{L^2} + C m^{-\sigma} \frac{[(m-7)!]^\sigma}{r^{m-7}} |\vec{a}|_{r,\sigma}. \quad (33)$$

Proof. We refer to [19, Lemma 3.2] for the proof of (i) and (v), and the proof of (ii) follows from the standard Sobolev inequalities (see for example [19, Lemma A.1]). The other inequalities are direct consequences of the definition of $|\vec{a}|_{r,\sigma}$. \square

3. ESTIMATES ON f_m, \tilde{f}_m, q_m AND \tilde{q}_m

This section is for deriving the upper bounds of the weighted L^2 -norms of f_m and q_m , defined in (20) and (21). And \tilde{f}_m and \tilde{q}_m can be handled similarly. To estimate f_m , we will use the cancellation introduced in [1, 22] for 2D Prandtl equations. The estimation on q_m relies on another kind of cancellation used in [19].

To simplify the notation, we use from now on the capital letter C to denote some generic constant that may vary from line to line, and it depends only on the constants in (6)-(7) as well as the Sobolev embedding constants, in particular, it is independent of the order of derivatives denoted by m .

The proposition below is the main result in this section.

Proposition 3.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (6)-(7). Then for any $m \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left(\|\langle z \rangle^\ell f_m(t)\|_{L^2}^2 + \|q_m(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \tilde{f}_m(t)\|_{L^2}^2 + \|\tilde{q}_m(t)\|_{L^2}^2 \right) \\ & + \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \int_0^t \left(\|\langle z \rangle^\ell \partial_z f_m(s)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_z \tilde{f}_m(s)\|_{L^2}^2 \right) ds \\ & \leq C |\tilde{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\tilde{a}(s)|_{\rho, \sigma}^2 + |\tilde{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\tilde{a}(s)|_{\rho, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We will only estimate f_m and q_m because \tilde{f}_m and \tilde{q}_m can be estimated similarly. To prove the above proposition, we first derive the equations solved by f_m , $m \geq 1$, as

$$\partial_t f_m + (u \partial_x + v \partial_y + w \partial_z) f_m - \partial_z^2 f_m = \tau_1 \partial_x^m g + \mathcal{J}_m, \quad (34)$$

where g is defined by (13), and

$$\begin{aligned} \mathcal{J}_m &= \tau_1 \chi \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] + \tau_1 \chi \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \psi \\ & - \tau_1 \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi] - \tau_1 \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \\ & - \tau_1 \left[\frac{\partial_z g - \eta \partial_y \psi - \chi g}{\psi} - \partial_x \psi + \chi \partial_x u + \chi \partial_y v + 2\chi \partial_z \chi \right] \partial_x^m u + 2\tau_1 (\partial_z \chi) \partial_x^m \psi \\ & + (w \tau_1' - \tau_1'') \left(\partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m u \right) - 2\tau_1' \partial_z \left(\partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m u \right) \end{aligned}$$

with $\chi \stackrel{\text{def}}{=} \partial_z \psi / \psi$. Here and throughout the paper, $\binom{m}{j}$ denotes the binomial coefficient.

We just give a sketch for obtaining (34)-(39). The derivation of (34) relies on the cancellation property observed in [1, 22]. Applying ∂_x^m to the equations (5) and (12) for u and ψ , it follows from Leibniz formula that

$$\begin{aligned} & \partial_t \partial_x^m u + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m u - \partial_z^2 \partial_x^m u + (\partial_x^m w) \psi \\ & = - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \psi \end{aligned} \quad (35)$$

and

$$\begin{aligned} & \partial_t \partial_x^m \psi + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m \psi - \partial_z^2 \partial_x^m \psi + (\partial_x^m w) \partial_z \psi \\ &= \partial_x^m g - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_z \partial_x^{m-j} \psi. \end{aligned} \quad (36)$$

Multiplying the first equation above by $\tau_1 \partial_z \psi / \psi$ and the second equation by τ_1 and then subtracting one by another, we obtain the equation for f_m .

The following three lemmas are for the estimates on the terms involving $\partial_x^m g$ and \mathcal{J}_m appearing on the right sides of (34).

Lemma 3.2. *Let $\sigma \leq 2$. Then for any $m \geq 7$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\left(\langle z \rangle^\ell \tau_1 \partial_x^m g, \langle z \rangle^\ell f_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho}, \sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

Proof. Note that $\partial_x^m g = g_\alpha$ with $\alpha = (m, 0) \in \mathbb{Z}_+^2$, and that $\ell \leq \kappa + \delta$ in view of (2). Then we use (31) and (29) to get

$$\begin{aligned} \left(\langle z \rangle^\ell \tau_1 \partial_x^m g, \langle z \rangle^\ell f_m \right)_{L^2} &\leq m^{-1} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |\vec{a}|_{\tilde{\rho}, \sigma} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-6}} |\vec{a}|_{\tilde{\rho}, \sigma} \\ &\leq \tilde{\rho}^{-1} m^{\sigma-1} \frac{\rho^{2(m-6)}}{\tilde{\rho}^{2(m-6)}} \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\tilde{\rho}, \sigma}^2. \end{aligned}$$

Moreover, it follows from the fact $\sigma \leq 2$ and (26) that

$$\tilde{\rho}^{-1} m^{\sigma-1} \frac{\rho^{2(m-6)}}{\tilde{\rho}^{2(m-6)}} \leq \tilde{\rho}^{-1} m \frac{\rho^{m-6}}{\tilde{\rho}^{m-6}} \leq \frac{C}{\tilde{\rho} - \rho}.$$

Then combining these estimates completes the proof. \square

Lemma 3.3. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\left(\langle z \rangle^\ell \mathcal{J}_m, \langle z \rangle^\ell f_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho, \sigma}^2 + |\vec{a}|_{\rho, \sigma}^3) + \frac{C[(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho}, \sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

Proof. We divide the term into

$$\left(\langle z \rangle^\ell \mathcal{J}_m, \langle z \rangle^\ell f_m \right)_{L^2} = A_1 + A_2 + A_3 + A_4$$

with

$$\begin{aligned}
A_1 &= \sum_{j=1}^m \binom{m}{j} \left(\tau_1 \chi \langle z \rangle^\ell [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u], \langle z \rangle^\ell f_m \right)_{L^2} \\
&\quad - \sum_{j=1}^m \binom{m}{j} \left(\tau_1 \langle z \rangle^\ell [(\partial_x^j u) \partial_x \partial_x^{m-j} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi], \langle z \rangle^\ell f_m \right)_{L^2}, \\
A_2 &= \sum_{j=1}^{m-1} \binom{m}{j} \left(\tau_1 \chi \langle z \rangle^\ell (\partial_x^j w) \partial_x^{m-j} \psi, \langle z \rangle^\ell f_m \right)_{L^2} \\
&\quad - \sum_{j=1}^{m-1} \binom{m}{j} \left(\tau_1 \langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi, \langle z \rangle^\ell f_m \right)_{L^2}, \\
A_3 &= 2 \left(\langle z \rangle^\ell \tau_1 (\partial_z \chi) \partial_x^m \psi, \langle z \rangle^\ell f_m \right)_{L^2} \\
&\quad - \left(\langle z \rangle^\ell \tau_1 \left[\frac{\partial_z g - \eta \partial_y \psi - \chi g}{\psi} - \partial_x \psi + \chi \partial_x u + \chi \partial_y v + 2\chi \partial_z \chi \right] \partial_x^m u, \langle z \rangle^\ell f_m \right)_{L^2}
\end{aligned}$$

and

$$A_4 = \left(\langle z \rangle^\ell \left[(w \tau_1' - \tau_1'') (\partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m u) - 2\tau_1' \partial_z (\partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m u) \right], \langle z \rangle^\ell f_m \right)_{L^2}.$$

We now derive the upper bounds for $A_j, 1 \leq j \leq 4$, term by term.

Upper bound for A_3 and A_4 . In view of the conditions (6)-(7) and the fact that $|\chi| \leq C \langle z \rangle^{-1}$ on $\text{supp } \tau_1$, by observing (2), we have

$$\| \tau_1 \partial_z \chi \|_{L^\infty} + \| \langle z \rangle \tau_1 \left[\frac{-\eta \partial_y \psi}{\psi} - \partial_x \psi + \chi \partial_x u + \chi \partial_y v + 2\chi \partial_z \chi \right] \|_{L^\infty} \leq C.$$

And using (2) and the representation (13) of g gives

$$\| \langle z \rangle \tau_1 \frac{\partial_z g - \chi g}{\psi} \|_{L^\infty} \leq C.$$

Thus,

$$A_3 \leq C \left(\| \langle z \rangle^{\ell-1} \partial_x^m u \|_{L^2} + \| \langle z \rangle^\ell \partial_x^m \psi \|_{L^2} \right) \| \langle z \rangle^\ell f_m \|_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho, \sigma}^2,$$

the last inequality following from (29). Similarly, we have by observing that $\tau_1' = \tau_1' \tau_2$ due to (10) and that $\langle z \rangle$ is bounded on $\text{supp } \tau_1'$ or $\text{supp } \tau_1''$,

$$A_4 \leq \| f_m \|_{L^2} \left(\| \partial_x^m u \|_{L^2} + \| \partial_x^m \psi \|_{L^2} + \| \tau_2 \partial_x^m \xi \|_{L^2} \right) \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho, \sigma}^2,$$

where in the last inequality we have used (29). Thus, we conclude

$$A_3 + A_4 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho, \sigma}^2.$$

Upper bound for A_2 . We will show that A_2 satisfies

$$A_2 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho, \sigma}^3 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}. \quad (37)$$

For this, firstly, we have

$$\begin{aligned} A_2 &\leq \| \langle z \rangle^\ell f_m \|_{L^2} \sum_{1 \leq j \leq m-1} \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \|_{L^2} \\ &\quad + \| \langle z \rangle^\ell f_m \|_{L^2} \sum_{1 \leq j \leq m-1} \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j w) \partial_x^{m-j} \psi \|_{L^2} \\ &\stackrel{\text{def}}{=} A_{2,1} + A_{2,2}. \end{aligned}$$

We will only estimate $A_{2,1}$ because $A_{2,2}$ can be handled similarly. By (7), (30) and (31), we have

$$\begin{aligned} &\left[\sum_{1 \leq j \leq 2} + \sum_{m-2 \leq j \leq m-1} \right] \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \|_{L^2} \\ &\leq C m \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-6}} |\tilde{a}|_{\tilde{\rho}, \sigma} \leq \frac{C[(m-7)!]^\sigma}{\rho^{m-6}} \frac{|\tilde{a}|_{\tilde{\rho}, \sigma}}{\tilde{\rho} - \rho}, \end{aligned} \quad (38)$$

where in the last inequality we have used (26). Next, we estimate the remaining terms in the summation. Note that

$$\sum_{3 \leq j \leq m-3} \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \|_{L^2} \leq T_1 + T_2$$

with

$$\begin{aligned} T_1 &= \sum_{j=3}^{[(m-3)/2]} \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \|_{L^2}, \\ T_2 &= \sum_{j=[(m-3)/2]+1}^{m-3} \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \|_{L^2}, \end{aligned}$$

where as standard, $[p]$ denotes the largest integer less than or equal to p . By using Sobolev inequality (27) and (29)-(31), we have

$$\begin{aligned} T_1 &\leq \sum_{j=3}^{[(m-3)/2]} \binom{m}{j} \| (\partial_x^j w) \|_{L^\infty} \| \langle z \rangle^\ell \partial_z \partial_x^{m-j} \psi \|_{L^2} \\ &\leq C \frac{m!}{3!(m-3)!} |\tilde{a}|_{\rho, \sigma} \frac{[(m-9)!]^\sigma}{\rho^{m-8}} |\tilde{a}|_{\rho, \sigma} \\ &\quad + C \sum_{j=4}^{[(m-3)/2]} \frac{m!}{j!(m-j)!} \frac{[(j-4)!]^\sigma}{\rho^{j-3}} |\tilde{a}|_{\rho, \sigma} \frac{[(m-j-6)!]^\sigma}{\rho^{m-j-5}} |\tilde{a}|_{\rho, \sigma} \\ &\leq \frac{C |\tilde{a}|_{\rho, \sigma}^2}{\rho^{m-8}} m^3 [(m-9)!]^\sigma + \frac{C |\tilde{a}|_{\rho, \sigma}^2}{\rho^{m-8}} \sum_{j=4}^{[(m-3)/2]} \frac{m! [(j-4)!]^{\sigma-1} [(m-j-6)!]^{\sigma-1}}{j^4 (m-j)^6} \\ &\leq \frac{C |\tilde{a}|_{\rho, \sigma}^2}{\rho^{m-6}} [(m-7)!]^\sigma (m^3 m^{-2\sigma}) + \frac{C |\tilde{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \sum_{j=4}^{[(m-3)/2]} \frac{(m-7)! m^7}{j^4 m^6} [(m-10)!]^{\sigma-1} \\ &\leq \frac{C [(m-7)!]^\sigma |\tilde{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \left(m^3 m^{-2\sigma} + \sum_{j=4}^{[(m-3)/2]} \frac{m}{j^4 m^{3(\sigma-1)}} \right) \\ &\leq \frac{C [(m-7)!]^\sigma}{\rho^{m-6}} |\tilde{a}|_{\rho, \sigma}^2, \end{aligned}$$

where in the last inequality we have used the fact that $\sigma \geq 3/2$. Similarly,

$$\begin{aligned}
T_2 &\leq \sum_{j=[(m-3)/2]+1}^{m-3} \binom{m}{j} \|(\partial_x^j w)\|_{L_{x,y}^2(L_z^\infty)} \|\langle z \rangle^\ell \partial_z \partial_x^{m-j} \psi\|_{L_{x,y}^\infty(L_z^2)} \\
&\leq \sum_{j=[(m-3)/2]+1}^{m-4} \frac{m!}{j!(m-j)!} \frac{[(j-6)!]^\sigma}{\rho^{j-5}} |\vec{a}|_{\rho,\sigma} \frac{[(m-j-4)!]^\sigma}{\rho^{m-j-3}} |\vec{a}|_{\rho,\sigma} \\
&\quad + \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-8}} m^3 [(m-9)!]^\sigma \\
&\leq \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-8}} \sum_{j=[(m-3)/2]+1}^{m-4} \frac{m! [(j-6)!]^{\sigma-1} [(m-j-4)!]^{\sigma-1}}{j^6 (m-j)^4} + \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-8}} m^3 [(m-9)!]^\sigma \\
&\leq \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-6}} \sum_{j=[(m-3)/2]+1}^{m-4} \frac{(m-7)! m^7}{m^6 (m-j)^4} [(m-10)!]^{\sigma-1} + \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-6}} m^3 [(m-9)!]^\sigma \\
&\leq \frac{C [(m-7)!]^\sigma |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-6}} \left(m^3 m^{-2\sigma} + \sum_{j=[(m-3)/2]+1}^{m-4} \frac{m}{(m-j)^4 m^{3(\sigma-1)}} \right) \\
&\leq \frac{C [(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho,\sigma}^2.
\end{aligned}$$

Combining these inequalities yields

$$\sum_{3 \leq j \leq m-3} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \leq \frac{C [(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho,\sigma}^2,$$

which along with (38) gives

$$\sum_{1 \leq j \leq m-1} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \leq \frac{C [(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho,\sigma}^2 + \frac{C [(m-7)!]^\sigma |\vec{a}|_{\tilde{\rho},\sigma}}{\tilde{\rho} - \rho},$$

and thus,

$$A_{2,1} \leq \frac{C [(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho,\sigma}^3 + \frac{C [(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho},\sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

The upper bound for $A_{2,2}$ is similar and thus (37) follows.

Upper bound for A_1 . Just following the argument used above for A_2 with slight modification, we can obtain

$$A_1 \leq \frac{C [(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho,\sigma}^3 + \frac{C [(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho},\sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

This completes the proof of Lemma 3.3. \square

Proof of Proposition 3.1. We first estimate f_m . Multiplying both sides of equation (34) by $\langle z \rangle^{2\ell} f_m$ and then integrating over Ω , by integration by parts and observing $\partial_z f_m|_{z=0} = 0$,

we have, for any $\varepsilon > 0$,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\langle z \rangle^\ell f_m\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_z f_m\|_{L^2}^2 \\
&= \left(\langle z \rangle^\ell (\partial_x^m g + \mathcal{J}_m), \langle z \rangle^\ell f_m \right)_{L^2} + \left(w(\partial_z \langle z \rangle^\ell) f_m, \langle z \rangle^\ell f_m \right)_{L^2} + \frac{1}{2} \left((\partial_z^2 \langle z \rangle^{2\ell}) f_m, f_m \right)_{L^2} \\
&\leq \left(\langle z \rangle^\ell \partial_x^m g, \langle z \rangle^\ell f_m \right)_{L^2} + \left(\langle z \rangle^\ell \mathcal{J}_m, \langle z \rangle^\ell f_m \right)_{L^2} + C \|\langle z \rangle^\ell f_m\|_{L^2}^2 \\
&\leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\bar{a}|_{\rho,\sigma}^2 + |\bar{a}|_{\rho,\sigma}^3 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\bar{a}|_{\bar{\rho},\sigma}^2}{\bar{\rho} - \rho},
\end{aligned}$$

where the last line follows from Lemmas 3.2-3.3 as well as (29). Then integrating over $[0, t]$, we obtain the upper bound as desired for the term $\|\langle z \rangle^\ell f_m\|_{L^2}$. The estimate on $\|q_m\|_{L^2}$ is similar. In fact, we multiply equation (36) by $\tau_2(\partial_z \xi)/\xi$, and apply $\tau_2 \partial_x^m$ to the first equation in (15), and finally subtracting one by another. This gives the equation solved by q_m :

$$\begin{aligned}
& (\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2) q_m = \tau_2 \sum_{j=1}^3 \partial_x^m \theta_j - b\tau_2 \partial_x^m g \\
& - \tau_2 \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x^{m-j+1} \xi + (\partial_x^j v) \partial_y \partial_x^{m-j} \xi] - \tau_2 \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_z \partial_x^{m-j} \xi \\
& + b\tau_2 \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x^{m-j+1} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi] + b\tau_2 \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \xi \\
& + 2(\partial_z b) \partial_z (\tau_2 \partial_x^m \psi) - \tau_2' b w \partial_x^m \psi + b\tau_2'' \partial_x^m \psi + 2b\tau_2' \partial_x^m \xi + \tau_2' v \partial_x^m \xi - \tau_2'' \partial_x^m \xi - 2\tau_2' \partial_x^m \partial_z \xi + \mathcal{P}_m,
\end{aligned}$$

where $b = \frac{\partial_z \xi}{\xi}$ and

$$\begin{aligned}
\mathcal{P}_m &= \frac{2(\psi \partial_x \xi - (\partial_x u) \partial_z \xi)}{\xi} \tau_2 \partial_x^m \psi - \frac{(\psi \partial_x \psi - (\partial_x u) \xi) \partial_z \xi}{\xi^2} \tau_2 \partial_x^m \psi \\
&\quad - \frac{2(\partial_z^2 \xi) \partial_z \xi}{\xi^2} \tau_2 \partial_x^m \psi + \frac{2(\partial_z \xi)^2 \partial_z \xi}{\xi^3} \tau_2 \partial_x^m \psi.
\end{aligned}$$

Then by repeating the argument for treating f_m , we can obtain the desired upper bound for $\|q_m\|_{L^2}$; see also [19, Subsection 5.2] for the detailed computation. The estimate on \tilde{f}_m and \tilde{q}_m can be obtained similarly. Then we have completed the proof of Proposition 3.1. \square

4. ESTIMATES ON Γ_m, H_m, G_m AND $\tilde{\Gamma}_m, \tilde{H}_m, \tilde{G}_m$

This section is for deriving the upper bounds of the L^2 -norms of Γ_m, H_m and G_m , defined in (19). And $\tilde{\Gamma}_m, \tilde{H}_m, \tilde{G}_m$ can be handled similarly. The main tool here is another kind of cancellation without monotonicity or concave condition.

The proposition below is the main result in this section.

Proposition 4.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (6)-(7). Then for any*

$m \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left(\|\langle z \rangle^{\kappa+\delta} \Gamma_m(t)\|_{L^2}^2 + \|H_m(t)\|_{L^2}^2 + \|G_m(t)\|_{L^2}^2 \right) \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

and similarly for the upper bound of $\tilde{\Gamma}_m, \tilde{H}_m$ and \tilde{G}_m .

We first handle Γ_m , and to do so we apply another kind of cancellation by virtue of the equations solved by u and v . Precisely, applying ∂_x^m to the equation (5) for v gives

$$\begin{aligned} & \partial_t \partial_x^m v + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m v - \partial_z^2 \partial_x^m v + (\partial_x^m w) \eta \\ & = - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} v + (\partial_x^j v) \partial_y \partial_x^{m-j} v] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \eta. \end{aligned}$$

We multiply the above equation by ψ and multiply the equation (35) by η , and then subtract one by another to have

$$\partial_t \Gamma_m + (u \partial_x + v \partial_y + w \partial_z) \Gamma_m - \partial_z^2 \Gamma_m = \mathcal{L}_m, \quad (39)$$

where

$$\begin{aligned} \mathcal{L}_m & = \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] \eta + \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) (\partial_x^{m-j} \psi) \eta \\ & \quad - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} v + (\partial_x^j v) \partial_y \partial_x^{m-j} v] \psi - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) (\partial_x^{m-j} \eta) \psi \\ & \quad + g \partial_x^m v - 2 (\partial_z \psi) \partial_x^m \eta - \left(h \partial_x^m u - 2 (\partial_z \eta) \partial_x^m \psi \right) \end{aligned}$$

with g, h defined by (13). Note the terms in the last line above come from the commutators between the functions ψ, η and the differential operators, where we have used the equations (12) for ψ and η .

Lemma 4.2. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\left(\langle z \rangle^{\kappa+\delta} \mathcal{L}_m, \langle z \rangle^{\kappa+\delta} \Gamma_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho, \sigma}^2 + |\vec{a}|_{\rho, \sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}.$$

Proof. The proof is similar to those for Lemmas 3.2-3.3. For instance, following the argument in Lemma 3.2, it yields

$$-2 \left(\langle z \rangle^{\kappa+\delta} (\partial_z \psi) \partial_x^m \eta, \langle z \rangle^{\kappa+\delta} \Gamma_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}.$$

Meanwhile the other terms involving in \mathcal{L}_m can be handled as in Lemma 3.3. We omit the detail here for brevity. \square

Proof of Proposition 4.1. We multiply both sides of equation (39) by $\langle z \rangle^{2(\kappa+\delta)} \Gamma_m$ and then integrating over Ω , by integration by parts and observing $\Gamma_m|_{z=0} = 0$, we have,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2}^2 + \|\langle z \rangle^{\kappa+\delta} \partial_z \Gamma_m\|_{L^2}^2 \right) \\ &= \left(\langle z \rangle^{\kappa+\delta} \mathcal{L}_m, \langle z \rangle^\ell \Gamma_m \right)_{L^2} + \left(w(\partial_z \langle z \rangle^{\kappa+\delta}) \Gamma_m, \langle z \rangle^{\kappa+\delta} \Gamma_m \right)_{L^2} + \frac{1}{2} \left((\partial_z^2 \langle z \rangle^{2(\kappa+\delta)}) \Gamma_m, \Gamma_m \right)_{L^2} \\ &\leq \left(\langle z \rangle^{\kappa+\delta} \mathcal{L}_m, \langle z \rangle^\ell \Gamma_m \right)_{L^2} + C \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2}^2 \\ &\leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^3 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}, \end{aligned}$$

where the last line follows from Lemmas 4.2 as well as (29). Then integrating over $[0, t]$, we obtain the upper bound as desired for the term $\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2}$.

The other terms $\|H_m\|_{L^2}$ and $\|G_m\|_{L^2}$ can be treated in the same way. In fact, we apply similar kind of cancellation by virtue of the equations solved by v and ψ to get the desired upper bound for $\|H_m\|_{L^2}$, and meanwhile by virtue of the equations for u and ψ to obtain the estimate on $\|G_m\|_{L^2}$. Since the argument is quite similar as that for handling $\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2}$ we omit it for brevity. The estimate on $\tilde{\Gamma}_m, \tilde{H}_m$ and \tilde{G}_m can be obtained in the same way. Then we have completed the proof of Proposition 4.1. \square

5. ESTIMATES ON g_α, h_α AND $\vec{\theta}_\alpha, \vec{\mu}_\alpha$

In this section, we estimate g_α, h_α and $\vec{\theta}_\alpha, \vec{\mu}_\alpha$, which are defined by (17) and (18), and the main result can be stated as follows.

Proposition 5.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (6)-(7). Then for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} \left(|\alpha|^2 \|\langle z \rangle^{\kappa+\delta} g_\alpha(t)\|_{L^2}^2 + |\alpha|^2 \|\langle z \rangle^{\kappa+\delta} h_\alpha(t)\|_{L^2}^2 \right) \\ &+ \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} \left(|\alpha|^2 \|\vec{\theta}_\alpha(t)\|_{L^2}^2 + |\alpha|^2 \|\vec{\mu}_\alpha(t)\|_{L^2}^2 \right) \\ &\leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We will only estimate g_α because the treatment of $h_\alpha, \vec{\theta}_\alpha$ and $\vec{\mu}_\alpha$ is similar. By (22), it suffices to estimate g_m and \tilde{g}_m with

$$g_m \stackrel{\text{def}}{=} \partial_x^m g, \quad \tilde{g}_m \stackrel{\text{def}}{=} \partial_y^m g. \quad (40)$$

For the same reason, we handle only g_m . And we begin with the equation solved by g_m . Let $m \geq 1$ and let g_m be given by (40). Then

$$\partial_t g_m + (u \partial_x + v \partial_y + w \partial_z) g_m - \partial_z^2 g_m = \mathcal{K}_m, \quad (41)$$

where

$$\begin{aligned} \mathcal{K}_m &= - \sum_{1 \leq j \leq m} \binom{m}{j} [(\partial_x^j u) \partial_x g_{m-j} + (\partial_x^j v) \partial_y g_{m-j} + (\partial_x^j w) \partial_z g_{m-j}] \\ &+ 2 \partial_x^m [(\partial_y \psi) \partial_z \eta] - 2 \partial_x^m [(\partial_y \eta) \partial_z \psi]. \end{aligned}$$

To see this, multiplying the first equation in (12) by $\partial_y v$ and the second equation by $\partial_y u$, and then subtracting one by another, we get the equation solved by g , i.e.,

$$\partial_t g + (u\partial_x + v\partial_y + w\partial_z)g - \partial_z^2 g = 2(\partial_y \psi)\partial_z \eta - 2(\partial_y \eta)\partial_z \psi. \quad (42)$$

Taking ∂_x^m on both sides and then using Leibniz formula, we obtain the equation (41).

The following three lemmas are for the estimates on the terms in \mathcal{K}_m .

Lemma 5.2. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, and any $\varepsilon > 0$, we have*

$$\begin{aligned} & m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \psi)\partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \| \langle z \rangle^{\kappa+\delta} \partial_z g_m \|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^3 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right), \end{aligned} \quad (43)$$

where C_ε is a constant depending on ε .

Proof. We use Leibniz formula to get

$$\begin{aligned} & m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \psi)\partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & = m^2 \left(\sum_{0 \leq j \leq [m/2]} + \sum_{[m/2]+1 \leq j \leq m} \right) \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}. \end{aligned}$$

Furthermore, using integration by parts for the first term on the right side gives

$$m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \psi)\partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \leq B_1 + B_2 + B_3 + B_4$$

with

$$\begin{aligned} B_1 &= m^2 \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \| \langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_x^{m-j} \eta \|_{L^2} \| \langle z \rangle^{\kappa+\delta} \partial_z g_m \|_{L^2}, \\ B_2 &= m^2 \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \| \langle z \rangle^{\kappa+\delta} (\partial_y \partial_z \partial_x^j \psi) \partial_x^{m-j} \eta \|_{L^2} \| \langle z \rangle^{\kappa+\delta} g_m \|_{L^2}, \\ B_3 &= m^2 \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \| \langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_x^{m-j} \eta \|_{L^2} \| \langle z \rangle^{\kappa+\delta} g_m \|_{L^2}, \\ B_4 &= m^2 \sum_{[m/2]+1 \leq j \leq m} \binom{m}{j} \| \langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_z \partial_x^{m-j} \eta \|_{L^2} \| \langle z \rangle^{\kappa+\delta} g_m \|_{L^2}. \end{aligned}$$

By (29)-(31) and the Sobolev inequality (28), we have

$$\begin{aligned} & m \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \| \langle z \rangle^{\kappa+\delta} (\partial_y \partial_z \partial_x^j \psi) \partial_x^{m-j} \eta \|_{L^2} \\ & \leq C m \sum_{j=2}^{[m/2]} \frac{m!}{j!(m-j)!} \frac{[(j-2)!]^\sigma}{\rho^{j-1}} |\bar{a}|_{\rho,\sigma} \frac{[(m-j-6)!]^\sigma}{\rho^{m-j-5}(m-j)} |\bar{a}|_{\rho,\sigma} + \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\bar{a}|_{\rho,\sigma}^2 \\ & \leq \frac{C |\bar{a}|_{\rho,\sigma}^2}{\rho^{m-5}} \sum_{j=2}^{[m/2]} \frac{m! [(j-2)!]^{\sigma-1} [(m-j-6)!]^{\sigma-1} m}{j^2 (m-j)^7} + \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\bar{a}|_{\rho,\sigma}^2 \\ & \leq \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\bar{a}|_{\rho,\sigma}^2. \end{aligned}$$

Similarly,

$$m \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_x^{m-j} \eta\|_{L^2} \leq \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho,\sigma}^2.$$

Thus, we combine the above estimates to have for any $\varepsilon > 0$,

$$B_1 + B_2 + B_3 \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 \right).$$

Applying a similar argument as for (37), we have

$$\begin{aligned} & m \left(\sum_{[m/2]+1 \leq j \leq m-2} + \sum_{m-1 \leq j \leq m} \right) \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_z \partial_x^{m-j} \eta\|_{L^2} \\ & \leq \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}}{\tilde{\rho} - \rho}. \end{aligned}$$

Thus,

$$B_4 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\vec{a}|_{\rho,\sigma}^3 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

Combining the estimates on B_1 - B_4 , we obtain the desired inequality and complete the proof. \square

Lemma 5.3. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, and any $\varepsilon > 0$, we have*

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} [(\partial_x^j u) \partial_x g_{m-j} + (\partial_x^j v) \partial_y g_{m-j} + (\partial_x^j w) \partial_z g_{m-j}], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right), \end{aligned}$$

where C_ε is a constant depending on ε .

Proof. Applying the argument used in the proof of Lemma 3.3, we have

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} [(\partial_x^j u) \partial_x g_{m-j} + (\partial_x^j v) \partial_y g_{m-j}], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

Moreover, we write

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & = -2m^2 \left[\sum_{1 \leq j \leq m-2} + \sum_{m-1 \leq j \leq m} \right] \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}. \end{aligned}$$

Direct calculation shows that, by (6)-(7) and the representation (13) of g ,

$$-2m^2 \sum_{m-1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

Furthermore, following the approach used to estimate B_1 - B_3 in Lemma 5.2 as well as the argument in Lemma 3.3, we can obtain

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m-2} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^3 + |\bar{a}|_{\rho,\sigma}^4 \right). \end{aligned}$$

Thus, combining these estimates yields

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^3 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

Then the desired upper bound estimate follows, and it completes the proof. \square

Lemma 5.4. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, and any $\varepsilon > 0$, we have*

$$\begin{aligned} & -2m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \eta) \partial_z \psi], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^2 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right), \end{aligned} \quad (44)$$

where C_ε is a constant depending on ε .

Proof. By using (7) and (29)-(32), direct calculation yields

$$\begin{aligned} & -2m^2 \left(\sum_{0 \leq j \leq 1} + \sum_{m-2 \leq j \leq m-1} \right) \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \eta) \partial_z \partial_x^{m-j} \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \frac{C [(m-6)!]^{2\sigma} |\bar{a}|_{\rho,\sigma}^2}{\rho^{2(m-5)} \tilde{\rho} - \rho}. \end{aligned}$$

Moreover, following the argument used in Lemma 5.2, we have

$$-2m^2 \sum_{2 \leq j \leq m-3} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \eta) \partial_z \partial_x^{m-j} \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\bar{a}|_{\rho,\sigma}^3.$$

Then it remains to show that, for any $\varepsilon > 0$,

$$\begin{aligned} & -2m^2 \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^m \eta) \partial_z \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^2 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned} \quad (45)$$

For this, by the representation of g , applying $\partial_z \partial_x^m$ to the first equation in (13) gives

$$\begin{aligned} \partial_z g_m &= (\partial_y \partial_x^m \eta) \psi + \sum_{0 \leq j \leq m-1} \binom{m}{j} (\partial_y \partial_x^j \eta) \partial_x^{m-j} \psi \\ & \quad + \partial_x^m [(\partial_y v) \partial_z \psi] - \partial_x^m [(\partial_y \psi) \eta] - \partial_x^m [(\partial_y u) \partial_z \eta]. \end{aligned}$$

Thus

$$-2m^2 \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^m \eta) \partial_z \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} = \sum_{1 \leq j \leq 5} S_j,$$

where S_j are given by, using the notation $\chi = \partial_z \psi / \psi$,

$$\begin{aligned} S_1 &= -2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_z g_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_2 &= 2m^2 \sum_{0 \leq j \leq m-1} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} \chi (\partial_y \partial_x^j \eta) \partial_x^{m-j} \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_3 &= 2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_x^m [(\partial_y v) \partial_z \psi], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_4 &= -2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_x^m [(\partial_y \psi) \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_5 &= -2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_x^m [(\partial_y u) \partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}. \end{aligned}$$

The estimation on S_3 - S_5 is similar to that for (43), in fact, we have

$$S_3 + S_4 + S_5 \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^3 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right).$$

And following the argument in Lemma 3.3, it gives

$$S_2 \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^3 + \frac{|\bar{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right).$$

Finally,

$$S_1 = m^2 \left([\partial_z (\langle z \rangle^{2(\kappa+\delta)} \chi)] g_m, g_m \right)_{L^2} \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\bar{a}|_{\rho,\sigma}^2.$$

Combining the upper bounds for S_1 - S_5 gives (45), and then (44) follows. The proof is completed. \square

We are now ready to give

Proof of Proposition 5.1. Observe $g_m|_{z=0} = 0$. We multiply both sides of the equation (41) by $m^2 \langle z \rangle^{2(\kappa+\delta)} g_m$, then integrate over Ω , to have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 \\ &= m^2 \left(\langle z \rangle^{\kappa+\delta} \mathcal{K}_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} + m^2 \left(w(\partial_z \langle z \rangle^{\kappa+\delta}) g_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \quad + \frac{1}{2} m^2 \left((\partial_z^2 \langle z \rangle^{2(\kappa+\delta)}) g_m, g_m \right)_{L^2} \\ & \leq m^2 \left(\langle z \rangle^{\kappa+\delta} \mathcal{K}_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} + C m^2 \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}^2. \end{aligned}$$

Furthermore, by Lemmas 5.2-5.4 and (31) we see the terms in the last line are bounded from above by

$$\varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^2 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right)$$

for any $\varepsilon > 0$. Then choosing ε small enough gives

$$\frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho,\sigma}^2 + |\bar{a}|_{\rho,\sigma}^4 + \frac{|\bar{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right),$$

and thus, integrating over $[0, t]$ yields

$$\begin{aligned} & \frac{\rho^{2(m-5)}}{[(m-6)!]^{2\sigma}} m^2 \|\langle z \rangle^{\kappa+\delta} g_m(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

A similar argument applies to the estimation on the upper bound for \tilde{g}_m . As a result, it follows from (22) that

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+\delta} g_\alpha(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We obtain the desired upper bound for g_α . The other terms $h_\alpha, \vec{\theta}_\alpha$ and $\vec{\mu}_\alpha$ can be handled similarly. In fact, when applying a similar kind of cancellation as for g_α , we see h, θ_j and μ_j satisfy equations quite similar to (42), where $\partial_x w$ or $\partial_y w$ is not involved in the source term due to the cancellation. This enables us to repeat the argument for treating g_α , to obtain the desired estimate for $h_\alpha, \vec{\theta}_\alpha$ and $\vec{\mu}_\alpha$. We omit the details for brevity. Then we have completed the proof of Proposition 5.1. \square

6. UPPER BOUND OF $\tau_2 \partial_x^m \psi$ AND $\tau_2 \partial_x^m \xi$

This section is devoted to handling $\tau_2 \partial_x^m \xi$, which relies on the estimate on $\tau_2 \partial_x^m \psi$. The main result can be stated as follows.

Proposition 6.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (6)-(7). Then for any $m \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left(\|\tau_2 \partial_x^m \psi(t)\|_{L^2}^2 + \|\tau_2 \partial_x^m \xi(t)\|_{L^2}^2 \right) \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

recalling $\xi = \partial_z \psi$.

We will follow the same cancellation method used in [7] to prove the above proposition. Note $|\xi| > 0$ on $\text{supp } \tau_2$. We may assume $-\xi > 0$ on $\text{supp } \tau_2$ without loss of generality. In view of (36) we see $(-\xi)^{-1/2} \tau_2 \partial_x^m \psi$ solves the equation

$$(\partial_t + u \partial_x + v \partial_y + w \partial_z - \partial_z^2) (-\xi)^{-1/2} \tau_2 \partial_x^m \psi - \tau_2 (\partial_x^m w) (-\xi)^{1/2} = \Xi \quad (46)$$

with

$$\begin{aligned} \Xi &= (-\xi)^{-1/2} w \tau_2' \partial_x^m \psi - (-\xi)^{-1/2} \tau_2'' \partial_x^m \psi - 2(-\xi)^{-1/2} \tau_2' \partial_x^m \xi \\ & \quad + \left[(\partial_t + u \partial_x + v \partial_y + w \partial_z - \partial_z^2) (-\xi)^{-1/2} \right] \tau_2 \partial_x^m \psi - 2[\partial_z (-\xi)^{-1/2}] \partial_z (\tau_2 \partial_x^m \psi) \\ & \quad + (-\xi)^{-\frac{1}{2}} \tau_2 \left[\partial_x^m g - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x^{m-j+1} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \xi \right]. \end{aligned}$$

Following the argument for proving Lemma 3.2 and Lemma 3.3, we can compute that

$$\left| \left(\Xi, (-\xi)^{-1/2} \tau_2 \partial_x^m \psi \right)_{L^2} \right| \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

As a result, multiplying both side of the equation (46) by the factor $(-\xi)^{-1/2} \tau_2 \partial_x^m \psi$ and then taking integration over Ω , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(-\xi)^{-1/2} \tau_2 \partial_x^m \psi\|_{L^2}^2 \\ & \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} + |(\tau_2 \partial_x^m w, \tau_2 \partial_x^m \psi)_{L^2}|. \end{aligned}$$

As for the last term on the right side we use integration by parts to get

$$\begin{aligned} & |(\tau_2 \partial_x^m w, \tau_2 \partial_x^m \psi)_{L^2}| \\ & \leq 2 |(\tau_2' \partial_x^m w, \tau_2 \partial_x^m u)_{L^2}| + |(\tau_2 \partial_x^{m+1} u, \tau_2 \partial_x^m u)_{L^2}| + |(\tau_2 \partial_x^m \partial_y v, \tau_2 \partial_x^m u)_{L^2}| \\ & = 2 |(\tau_2' \partial_x^m w, \tau_2 \partial_x^m u)_{L^2}| + |(\tau_2 \partial_x^m \partial_y v, \tau_2 \partial_x^m u)_{L^2}|. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(-\xi)^{-1/2} \tau_2 \partial_x^m \psi\|_{L^2}^2 & \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \\ & \quad + 2 |(\tau_2' \partial_x^m w, \tau_2 \partial_x^m u)_{L^2}| + |(\tau_2 \partial_x^m \partial_y v, \tau_2 \partial_x^m u)_{L^2}|. \end{aligned} \quad (47)$$

The following two lemmas are devoted to estimating the last two terms on the right side of (47).

Lemma 6.2. *We have*

$$\begin{aligned} & |(\tau_2' \partial_x^m w, \tau_2 \partial_x^m u)_{L^2}| \\ & \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

Lemma 6.3. *We have*

$$\begin{aligned} & |(\tau_2 \partial_x^m \partial_y v, \tau_2 \partial_x^m u)_{L^2}| \\ & \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

The proof of Lemmas 6.2 and 6.3 are postponed to the following Subsections 6.1 and 6.2. Now we continue to the proof of Proposition 6.1. Integrating both sides of (47) over $[0, t]$ and using the estimates in Lemmas 6.2 and 6.3, it follows that

$$\begin{aligned} \|(-\xi)^{-1/2} \tau_2 \partial_x^m \psi(t)\|_{L^2}^2 & \leq \frac{2m^{2\sigma}}{\rho^2} \int_0^t \|\partial_z f_{m-1}(s)\|_{L^2}^2 ds \\ & \quad + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}_0|_{\rho,\sigma}^2 + \int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds) \\ & \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}_0|_{\rho,\sigma}^2 + \int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds), \end{aligned}$$

where in the last inequality we have used Proposition 3.1. Observe $(-\xi)^{-1/2}$ has a strictly positive lower bound on $\text{supp } \tau_2$. Then we obtain the desired upper bound for $\tau_2 \partial_x^m \psi$:

$$\|\tau_2 \partial_x^m \psi(t)\|_{L^2}^2 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\bar{a}_0|_{\rho,\sigma}^2 + \int_0^t \left(|\bar{a}(s)|_{\rho,\sigma}^2 + |\bar{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).$$

It remains to handle $\tau_2 \partial_x^m \xi$ and we use (21) and Proposition 3.1 to compute

$$\begin{aligned} \|\tau_2 \partial_x^m \xi(t)\|_{L^2}^2 &\leq \|q_m(t)\|_{L^2}^2 + C \|\tau_2 \partial_x^m \psi(t)\|_{L^2}^2 \\ &\leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\bar{a}_0|_{\rho,\sigma}^2 + \int_0^t \left(|\bar{a}(s)|_{\rho,\sigma}^2 + |\bar{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Combining the above two inequalities we prove Proposition 6.1.

So in order to complete the proof of Proposition 6.1, it remains to prove Lemmas 6.2 and 6.3 giving in the following two subsections.

6.1. Proof of Lemma 6.2. To do so we first recall the crucial representations of $\partial_x^m u$ in terms of φ_m (see [7, Lemma 3]), with φ_m defined by

$$\varphi_m = \left(\vartheta \psi + 1 - \vartheta \right) \left(\partial_x^m \psi - \frac{\xi}{\psi} \partial_x^m u \right) = \left(\vartheta \psi + 1 - \vartheta \right) \psi \partial_z \left(\frac{\partial_x^m u}{\psi} \right), \quad (48)$$

where $m \geq 1$ and $\vartheta(z) \in C_0^\infty(\mathbb{R})$ is a given function such that $\vartheta \equiv 1$ in $[0, 2]$. Then $\partial_x^m u$ can be represented as (see [7, Lemma 3])

$$\partial_x^m u(t, x, y, z) = \alpha_m(t, x, y, z) + \psi(t, x, y, z) \beta_m(t, x, y) \mathbf{1}_{\{z > 1\}}, \quad (49)$$

where $\mathbf{1}_A$ stands for the characteristic function on a set A , and α_m, β_m are defined as follows:

$$\alpha_m(t, x, y, z) = \begin{cases} \psi(t, x, y, z) \int_0^z \frac{\varphi_m}{(\vartheta \psi + 1 - \vartheta) \psi} dz, & \text{if } z < 1, \\ \psi(t, x, y, z) \int_2^z \frac{\varphi_m}{(\vartheta \psi + 1 - \vartheta) \psi} dz, & \text{if } z > 1, \end{cases}$$

recalling 1 is the only non-degenerate critical point of u , and

$$\beta_m(t, x, y) = \frac{\partial_x^m u(t, x, y, 2)}{\psi(t, x, y, 2)}.$$

Observe

$$\forall |z - 1| \geq \epsilon \text{ with } 0 \leq z \leq 2, \quad \frac{1}{|(\vartheta \psi + 1 - \vartheta) \psi|} \leq C,$$

and thus

$$\sup_{0 \leq z \leq 1 - \epsilon} \|\alpha_m(\cdot, z)\|_{L^2(\mathbb{T}^2)} + \sup_{1 + \epsilon \leq z \leq 2} \|\alpha_m(\cdot, z)\|_{L^2(\mathbb{T}^2)} \leq C \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0, 2])}. \quad (50)$$

This implies

$$\|\alpha_m\|_{L^2(\mathbb{T}^2 \times \text{supp } \tau_2)} \leq C \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0, 2])}. \quad (51)$$

The following estimates on β_m are obtained by [7, Lemma 6]:

$$\begin{cases} \|\beta_m\|_{L^2(\mathbb{T}^2)} \leq C \|\partial_x^m \psi\|_{L^2(\mathbb{T}^2 \times \mathbb{R}_+)} \\ \|\partial_y \beta_m\|_{L^2(\mathbb{T}^2)} \leq C \|\partial_x^m \partial_y \psi\|_{L^2(\mathbb{T}^2 \times \mathbb{R}_+)} + C \|\partial_x^m \psi\|_{L^2(\mathbb{T}^2 \times \mathbb{R}_+)}. \end{cases} \quad (52)$$

Lemma 6.4. *Let φ_m, G_m and θ_3 be defined respectively by (48), (19) and (16). Then*

$$\begin{aligned} \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0,2])} &= \|G_m\|_{L^2} \\ &\leq \|\partial_x^{m-1}\theta_3\|_{L^2} + C\|\partial_z f_{m-1}\| + Cm^{1-\sigma} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-7}} |\bar{a}|_{\tilde{\rho},\sigma} + Cm^{2-2\sigma} \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |\bar{a}|_{\rho,\sigma}^2. \end{aligned}$$

Sketch of the proof. Observe $\varphi_m = -G_m$ on $\mathbb{T}^2 \times [0, 2]$ and

$$G_m = \partial_x^{m-1}\theta_3 - \sum_{j=1}^{m-1} \binom{m-1}{j} \left[(\partial_x^j \xi) \partial_x^{m-j} u - (\partial_x^j \psi) \partial_x^{m-j} \psi \right].$$

Furthermore, direct computation shows

$$\begin{aligned} \sum_{j=1}^{m-1} \binom{m-1}{j} \|(\partial_x^j \psi) \partial_x^{m-j} \psi\|_{L^2} &\leq m \|(\partial_x \psi) \partial_x^{m-1} \psi\|_{L^2} + \sum_{j=2}^{m-1} \binom{m-1}{j} \|(\partial_x^j \psi) \partial_x^{m-j} \psi\|_{L^2} \\ &\leq Cm^{1-\sigma} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-7}} |\bar{a}|_{\tilde{\rho},\sigma} + Cm^{2-2\sigma} \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |\bar{a}|_{\rho,\sigma}^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{j=1}^{m-1} \binom{m-1}{j} \|(\partial_x^j \xi) \partial_x^{m-j} u\|_{L^2} &\leq \|(\partial_x^{m-1} \xi) \partial_x u\|_{L^2} + m \|(\partial_x^{m-2} \xi) \partial_x^2 u\|_{L^2} + \sum_{j=1}^{m-3} \binom{m-1}{j} \|(\partial_x^j \xi) \partial_x^{m-j} u\|_{L^2} \\ &\leq C\|\partial_z f_{m-1}\|_{L^2} + Cm^{1-\sigma} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-7}} |\bar{a}|_{\tilde{\rho},\sigma} + Cm^{2-2\sigma} \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |\bar{a}|_{\rho,\sigma}^2, \end{aligned}$$

the last inequality following from (30) and (33) as well as (29). Combining the above inequalities gives the desired estimate and we have completed the proof of Lemma 6.4. \square

Lemma 6.5. *Let $z_0 \in [0, 1]$ be a given number and let $p_0 \in L^\infty(\Omega)$. Then we have*

$$\left\| \int_{z_0}^z p_0 \alpha_m dz \right\|_{L^2(\mathbb{T}^2 \times [1+\frac{3}{2}\epsilon, 2])} \leq C \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0,2])},$$

where the constant C depends on $\|p_0\|_{L^\infty}$.

Proof. We will follow the proof of [7, Lemma 6] with slight modification. Since $p_0 \in L^\infty(\Omega)$ then for any $z \in [1 + \frac{3}{2}\epsilon, 2]$,

$$\begin{aligned} \left| \int_{z_0}^z p_0 \alpha_m d\tilde{z} \right| &\leq C \int_0^z |\alpha_m| d\tilde{z} \leq C \int_0^1 |\psi| \left(\int_0^{\tilde{z}} \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| dz_* \right) d\tilde{z} \\ &\quad + C \int_1^z |\psi| \left(\int_{\tilde{z}}^2 \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| dz_* \right) d\tilde{z}. \end{aligned}$$

Observe 1 is the only critical point of u so that ψ doesn't change sign in the interval $[0, 1]$, saying $\psi \geq 0$. Then

$$\begin{aligned} \int_0^1 |\psi| \left(\int_0^{\tilde{z}} \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| dz_* \right) d\tilde{z} &= \int_0^1 \psi \left(\int_0^{\tilde{z}} \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| dz_* \right) d\tilde{z} \\ &= - \int_0^1 u \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| d\tilde{z} + u(\cdot, 1) \int_0^1 \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| dz_* \\ &\leq \int_0^1 \left| \frac{u - u(\cdot, 1)}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \cdot |\varphi_m| d\tilde{z}. \end{aligned}$$

Similarly

$$\begin{aligned} \int_1^z |\psi| \left(\int_{\tilde{z}}^2 \left| \frac{\varphi_m}{(\vartheta\psi + 1 - \vartheta)\psi} \right| dz_* \right) d\tilde{z} \\ \leq \int_1^z \left| \frac{u - u(\cdot, 1)}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \cdot |\varphi_m| d\tilde{z} + |u(\cdot, z) - u(\cdot, 1)| \int_z^2 \left| \frac{1}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \cdot |\varphi_m| d\tilde{z}. \end{aligned}$$

Thus combining the above inequalities gives, for any $z \in [1 + \frac{3}{2}\epsilon, 2]$,

$$\begin{aligned} \left| \int_{b_0}^z p_0 \alpha_m d\tilde{z} \right| &\leq C \int_0^z \left| \frac{u - u(\cdot, 1)}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \cdot |\varphi_m| d\tilde{z} \\ &\quad + C |u(\cdot, z) - u(\cdot, 1)| \int_z^2 \left| \frac{1}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \cdot |\varphi_m| d\tilde{z} \leq C \|\varphi_m\|_{L^2([0,2])}, \end{aligned}$$

where the last inequality holds because

$$\left| \frac{u - u(\cdot, 1)}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \leq C \text{ for any } z \in [0, 2],$$

and

$$\left| \frac{1}{(\vartheta\psi + 1 - \vartheta)\psi} \right| \leq C \text{ for any } z \in [1 + \frac{3}{2}\epsilon, 2].$$

As a result, the assertion follows. We complete the proof of Lemma 6.5. \square

The rest of this subsection is devoted to

Proof of Lemma 6.2. Note that $|(\tau_2' \partial_x^m w, \tau_2 \partial_x^m u)_{L^2}|$ is bounded from above by the sum of the following two terms:

$$\left| \left(\tau_2' \int_0^z \partial_x^{m+1} u(\cdot, \tilde{z}) d\tilde{z}, \tau_2 \partial_x^m u \right)_{L^2} \right|, \quad \left| \left(\tau_2' \int_0^z \partial_x^m \partial_y v(\cdot, \tilde{z}) d\tilde{z}, \tau_2 \partial_x^m u \right)_{L^2} \right|.$$

We only need to handle the second term, since the first one has been estimated in [7] (see the treatment of E_j in [7]). For this, we claim

$$\begin{aligned} &\left| \left(\tau_2' \int_0^z \partial_x^m \partial_y v(\cdot, \tilde{z}) d\tilde{z}, \tau_2 \partial_x^m u \right)_{L^2} \right| \\ &\leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\bar{a}|_{\rho, \sigma}^2 + |\bar{a}|_{\rho, \sigma}^4 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\bar{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}. \end{aligned} \quad (53)$$

Observe $\text{supp } \tau'_2 \subset \Omega_1 \cup \Omega_2$ with $\Omega_1 = \mathbb{T}^2 \times [1 + \frac{3}{2}\epsilon, 1 + 2\epsilon]$ and $\Omega_2 = \mathbb{T}^2 \times [1 - 2\epsilon, 1 - \frac{3}{2}\epsilon]$, and thus the desired estimate (53) follows if we can prove that

$$\begin{aligned} & \sum_{j=1}^2 \left| \left(\tau'_2 \int_0^z \partial_x^m \partial_y v(\cdot, \tilde{z}) d\tilde{z}, \tau_2 \partial_x^m u \right)_{L^2(\Omega_j)} \right| \\ & \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\bar{a}|_{\rho, \sigma}^2 + |\bar{a}|_{\rho, \sigma}^4 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\bar{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}. \end{aligned} \quad (54)$$

In the following argument we will focus on deriving the upper bound for the integration over Ω_1 , and the estimate for Ω_2 can be handled in the same way with simpler argument. To do so we write, for $1 + \frac{3}{2}\epsilon \leq z \leq 1 + 2\epsilon$,

$$\begin{aligned} \int_0^z \partial_x^m v(\cdot, \tilde{z}) d\tilde{z} &= \int_0^{1-2\epsilon} \partial_x^m v(\cdot, \tilde{z}) d\tilde{z} + \int_{1-2\epsilon}^z \partial_x^m v(\cdot, \tilde{z}) d\tilde{z} \\ &= \int_0^{1-2\epsilon} \left[\frac{\Gamma_m}{\psi} + \frac{\eta}{\psi} \partial_x^m u \right] d\tilde{z} + \int_{1-2\epsilon}^z \left[\frac{H_m}{\xi} + \frac{\eta}{\xi} \partial_x^m \psi \right] d\tilde{z}, \end{aligned}$$

where the last line holds because of (6) and (19). Moreover, for the last term above,

$$\int_{1-2\epsilon}^z \frac{\eta}{\xi} \partial_x^m \psi d\tilde{z} = - \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z} + \frac{\eta}{\xi} \partial_x^m u - \left(\frac{\eta}{\xi} \partial_x^m u \right) \Big|_{z=1-2\epsilon}.$$

Then

$$\begin{aligned} \left| \left(\tau'_2 \int_0^z \partial_x^m \partial_y v(\cdot, \tilde{z}) d\tilde{z}, \tau_2 \partial_x^m u \right)_{L^2(\Omega_1)} \right| &= \left| \left(\tau'_2 \int_0^z \partial_x^m v(\cdot, \tilde{z}) d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ &\leq \sum_{j=1}^3 R_j \end{aligned}$$

with

$$\begin{aligned} R_1 &= \left| \left(\tau'_2 \int_0^{1-2\epsilon} \frac{\Gamma_m}{\psi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| + \left| \left(\tau'_2 \int_{1-2\epsilon}^z \frac{H_m}{\xi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right|, \\ R_2 &= \left| \left(\tau'_2 \int_0^{1-2\epsilon} \frac{\eta}{\psi} \partial_x^m u d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ &\quad + \left| \left(\tau'_2 \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right|, \\ R_3 &= \left| \left(\tau'_2 \frac{\eta}{\xi} \partial_x^m u, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| + \left| \left(\left(\frac{\eta}{\xi} \partial_x^m u \right) \Big|_{z=1-2\epsilon}, \tau'_2, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right|. \end{aligned}$$

Estimate on R_1 . To estimate R_1 , we use the relationship

$$\Gamma_m = -\partial_x^{m-1} h + \sum_{j=0}^{m-2} \frac{(m-1)!}{j!(m-1-j)!} [(\partial_x^{j+1} u) \partial_x^{m-1-j} \eta - (\partial_x^{j+1} v) \partial_x^{m-1-j} \psi]$$

due to the definition (13) and (19) of h and Γ_m . This yields

$$\begin{aligned} & \left| \left(\tau_2' \int_0^{1-2\epsilon} \frac{\Gamma_m}{\psi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \leq C \|\partial_x^{m-1} h\|_{L^2} \|\partial_x^m \partial_y u\|_{L^2} \\ & + C \sum_{j=0}^{m-2} \binom{m-1}{j} \left| \left(\tau_2' \int_0^{1-2\epsilon} \frac{(\partial_x^{j+1} v) \partial_x^{m-1-j} \psi}{\psi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ & + C \sum_{j=0}^{m-2} \binom{m-1}{j} \left| \left(\tau_2' \int_0^{1-2\epsilon} \frac{(\partial_x^{j+1} u) \partial_x^{m-1-j} \eta}{\psi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right|. \end{aligned}$$

As for the first and second terms on the right side, we use (29)-(31) and follow the argument for proving Lemmas 3.2 and (3.3), to conclude with direct computation that they are bounded from above by

$$\frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

By using additionally (31), we can see the last term is also controlled by the above upper bound. Then

$$\begin{aligned} & \left| \left(\tau_2' \int_0^{1-2\epsilon} \frac{\Gamma_m}{\psi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ & \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

Similarly, we have the upper bound of

$$\left| \left(\tau_2' \int_{1-2\epsilon}^z \frac{H_m}{\xi} d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right|.$$

Thus

$$R_1 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \quad (55)$$

Estimate on R_2 . We only need to handle the second term in the representation of R_2 , since the upper bound for the first one can be derived in the same way. Using the representation (49), we have

$$\begin{aligned} & \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ & \leq \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \partial_y \alpha_m \right)_{L^2(\Omega_1)} \right| \\ & \quad + \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 (\beta_m \partial_y \psi + \psi \partial_y \beta_m) \right)_{L^2(\Omega_1)} \right| \\ & \leq C \|\beta_m\|_{L^2(\mathbb{T}^2)} \|\partial_x^m u\|_{L^2} + C \|\alpha_m\|_{L^2(\mathbb{T}^2 \times \text{supp } \tau_2')} (\|\partial_x^m u\|_{L^2} + \|\partial_x^m \partial_y u\|_{L^2}) \\ & \quad + \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \psi \partial_y \beta_m \right)_{L^2(\Omega_1)} \right| \\ & \leq C \|\beta_m\|_{L^2(\mathbb{T}^2)} \|\partial_x^m u\|_{L^2} + C \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0,2])} (\|\partial_x^m u\|_{L^2} + \|\partial_x^m \partial_y u\|_{L^2}) \\ & \quad + \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \psi \partial_y \beta_m \right)_{L^2(\Omega_1)} \right|, \end{aligned}$$

where in the last inequality we have used (51). As for the last term in the above inequality, we use (49) again to get

$$\begin{aligned}
& \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \psi \partial_y \beta_m \right)_{L^2(\Omega_1)} \right| \\
& \leq \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \alpha_m d\tilde{z}, \tau_2 \psi \partial_y \beta_m \right)_{L^2(\Omega_1)} \right| \\
& \quad + \left| \left(\tau_2' \beta_m \int_1^z \left(\partial_z \frac{\eta}{\xi} \right) \psi d\tilde{z}, \tau_2 \psi \partial_y \beta_m \right)_{L^2(\Omega_1)} \right| \\
& \leq C \|\partial_y \beta_m\|_{L^2(\mathbb{T}^2)} \left\| \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \alpha_m d\tilde{z} \right\|_{L^2(\mathbb{T}^2 \times [1+\frac{3}{2}\epsilon, 2])} + C \|\beta_m\|_{L^2(\mathbb{T}^2)}^2 \\
& \leq C \|\partial_y \beta_m\|_{L^2(\mathbb{T}^2)} \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0, 2])} + C \|\beta_m\|_{L^2(\mathbb{T}^2)}^2,
\end{aligned}$$

where in the second inequality we have used the fact that

$$\left| \left(\tau_2' \beta_m \int_1^z \left(\partial_z \frac{\eta}{\xi} \right) \psi d\tilde{z}, \tau_2 \psi \partial_y \beta_m \right)_{L^2(\Omega_1)} \right| = \left| \frac{1}{2} \left(\partial_y \left[\tau_2' \tau_2 \psi \int_1^z \left(\partial_z \frac{\eta}{\xi} \right) \psi d\tilde{z} \right], \beta_m^2 \right)_{L^2(\Omega_1)} \right|$$

and the last inequality follows from Lemma 6.5. Thus combining the above inequalities and observing the estimate (52), we obtain

$$\begin{aligned}
& \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \leq C \|\beta_m\|_{L^2(\mathbb{T}^2)}^2 + C \|\beta_m\|_{L^2(\mathbb{T}^2)} \|\partial_x^m u\|_{L^2} \\
& \quad + C \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0, 2])} \left(\|\partial_y \beta_m\|_{L^2(\mathbb{T}^2)} + \|\partial_x^m u\|_{L^2} + \|\partial_x^m \partial_y u\|_{L^2} \right) \\
& \quad \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\tilde{a}|_{\tilde{\rho}, \sigma}^2 + \frac{C[(m-6)!]^\sigma}{\tilde{\rho}^{(m-5)}} |\tilde{a}|_{\tilde{\rho}, \sigma} \|G_m\|_{L^2}, \quad (56)
\end{aligned}$$

where in the last inequality we have used the fact that $\varphi_m = -G_m$ on $\mathbb{T}^2 \times [0, 2]$ with G_m defined in (19). As for the last term in the above inequality, we use Lemma 6.4 as well as (31) to obtain

$$\begin{aligned}
\|G_m\|_{L^2} & \leq m^{-1} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{(m-6)}} |\tilde{a}|_{\tilde{\rho}, \sigma} + C \|\partial_z f_{m-1}\| \\
& \quad + C m^{1-\sigma} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-7}} |\tilde{a}|_{\tilde{\rho}, \sigma} + C m^{2-2\sigma} \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |\tilde{a}|_{\rho, \sigma}^2,
\end{aligned}$$

and thus, recalling $3/2 \leq \sigma \leq 2$ and $0 < \rho < \tilde{\rho} \leq 1$,

$$\begin{aligned}
& \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{(m-5)}} |\tilde{a}|_{\tilde{\rho}, \sigma} \|G_m\|_{L^2} \leq C \frac{m + m^{\sigma-1}}{\tilde{\rho}} \frac{[(m-7)!]^{2\sigma}}{\tilde{\rho}^{2(m-6)}} |\tilde{a}|_{\tilde{\rho}, \sigma}^2 \\
& \quad + C \frac{m^{2-\sigma} \rho}{\tilde{\rho}} \frac{[(m-7)!]^{2\sigma}}{\tilde{\rho}^{m-6} \rho^{m-6}} |\tilde{a}|_{\tilde{\rho}, \sigma} |\tilde{a}|_{\rho, \sigma}^2 + C \frac{\rho^2}{m^{2\sigma}} \frac{[(m-6)!]^{2\sigma}}{\tilde{\rho}^{2(m-5)}} |\tilde{a}|_{\tilde{\rho}, \sigma}^2 + \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 \\
& \leq C \frac{m}{\tilde{\rho}} \frac{[(m-7)!]^{2\sigma}}{\tilde{\rho}^{2(m-6)}} |\tilde{a}|_{\tilde{\rho}, \sigma}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\tilde{a}|_{\rho, \sigma}^4 + \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 \\
& \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\tilde{a}|_{\rho, \sigma}^4 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}. \quad (57)
\end{aligned}$$

As a result, combining the above inequalities (56) and (57) yields

$$\begin{aligned} & \left| \left(\tau_2' \int_{1-2\epsilon}^z \left(\partial_z \frac{\eta}{\xi} \right) \partial_x^m u d\tilde{z}, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ & \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^4 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

Similar estimate holds for the other term in the representation of R_2 . Thus

$$R_2 \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^4 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \quad (58)$$

Estimate on R_3 . It is clear that, using integration by parts,

$$\left| \left(\tau_2' \frac{\eta}{\xi} \partial_x^m u, \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \leq C \|\partial_x^m u\|_{L^2}^2 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\tilde{a}|_{\rho,\sigma}^2.$$

Using again the representation (49) of $\partial_x^m u$, we have

$$\begin{aligned} & \left| \left(\left(\frac{\eta}{\xi} \partial_x^m u \right) \Big|_{z=1-2\epsilon}, \tau_2', \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \leq \left| \left(\left(\frac{\eta}{\xi} \alpha_m \right) \Big|_{z=1-2\epsilon}, \tau_2', \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ & \leq C \|\alpha_m(\cdot, 1-2\epsilon)\|_{L^2(\mathbb{T}^2)} \|\partial_x^m \partial_y u\|_{L^2} \leq C \|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0,2])} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{(m-5)}} |\tilde{a}|_{\tilde{\rho},\sigma}, \end{aligned}$$

where the last inequality follows from (50) as well as (29). This, along with the fact that $\|\varphi_m\|_{L^2(\mathbb{T}^2 \times [0,2])} = \|G_m\|_{L^2}$ and the estimate (57), yields

$$\begin{aligned} & \left| \left(\left(\frac{\eta}{\xi} \partial_x^m u \right) \Big|_{z=1-2\epsilon}, \tau_2', \tau_2 \partial_x^m \partial_y u \right)_{L^2(\Omega_1)} \right| \\ & \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\tilde{a}|_{\rho,\sigma}^4 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

and thus

$$R_3 \leq \frac{m^{2\sigma}}{\rho^2} \|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\tilde{a}|_{\rho,\sigma}^2 + |\tilde{a}|_{\rho,\sigma}^4 \right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

Now we combine the above inequality and the estimates (55) and (58) on R_1 and R_2 to obtain the desired upper bound for the integration over Ω_1 on the left side of (54). The estimation for the integration over Ω_2 is similar. Then the estimate (53) follows. We then complete the proof of Lemma 6.2. \square

6.2. Proof of Lemma 6.3. Observe $|\xi| > 0$ on $\text{supp } \tau_2$. Thus we use the representation (16) and (19) of θ_1 and G_m , to write

$$\begin{aligned} \left| (\tau_2 \partial_x^m \partial_y v, \tau_2 \partial_x^m u)_{L^2} \right| & \leq \frac{1}{2} \left| (\xi^{-1} \tau_2 \partial_x^m \theta_1, \tau_2 \partial_x^m u)_{L^2} \right| + \left| (\xi^{-1} \tau_2 \eta \partial_x^m \partial_y \psi, \tau_2 \partial_x^m u)_{L^2} \right| \\ & \quad + \sum_{j=1}^m \binom{m}{j} \left| (\xi^{-1} \tau_2 (\partial_x^j \xi) \partial_x^{m-j} \partial_y v, \tau_2 \partial_x^m u)_{L^2} \right| \\ & \quad + \sum_{j=1}^m \binom{m}{j} \left| (\xi^{-1} \tau_2 (\partial_x^j \eta) \partial_x^{m-j} \partial_y \psi, \tau_2 \partial_x^m u)_{L^2} \right|. \end{aligned}$$

Then Lemma 6.3 holds if we can show that

$$\left| (\xi^{-1} \tau_2 \partial_x^m \theta_1, \tau_2 \partial_x^m u)_{L^2} \right| \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\tilde{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}, \quad (59)$$

$$\begin{aligned}
& |(\xi^{-1}\tau_2\eta\partial_x^m\partial_y\psi, \tau_2\partial_x^m u)_{L^2}| \\
& \leq \frac{m^{2\sigma}}{\rho^2}\|\partial_z f_{m-1}\|_{L^2}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}\left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4\right) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}\frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
& \sum_{j=1}^m \binom{m}{j} \left[|(\xi^{-1}\tau_2(\partial_x^j\xi)\partial_x^{m-j}\partial_y v, \tau_2\partial_x^m u)_{L^2}| + |(\xi^{-1}\tau_2(\partial_x^j\eta)\partial_x^{m-j}\partial_y\psi, \tau_2\partial_x^m u)_{L^2}| \right] \\
& \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}|\vec{a}|_{\rho,\sigma}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}|\vec{a}|_{\rho,\sigma}^2.
\end{aligned} \tag{61}$$

Proof of (59). This is just a direct consequence of (29) and (31), using the same argument in the proof of Lemma 3.2. \square

Proof of (60). By virtue of the representation (19) of G_m , we have, using (29),

$$\begin{aligned}
& |(\xi^{-1}\tau_2\eta\partial_x^m\partial_y\psi, \tau_2\partial_x^m u)_{L^2}| \\
& \leq |(\xi^{-1}\tau_2\eta\partial_x^m\partial_y\psi, \xi^{-1}\tau_2 G_m)_{L^2}| + |(\xi^{-1}\tau_2\eta\partial_x^m\partial_y\psi, \xi^{-1}\tau_2\psi\partial_x^m\psi)_{L^2}| \\
& \leq C\|\partial_x^m\partial_y\psi\|_{L^2}\|G_m\|_{L^2} + C\|\partial_x^m\psi\|_{L^2}^2 \\
& \leq \frac{C[(m-6)!]^\sigma}{\tilde{\rho}^{(m-5)}}|\vec{a}|_{\tilde{\rho},\sigma}\|G_m\|_{L^2} + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}|\vec{a}|_{\rho,\sigma}^2,
\end{aligned}$$

which together with (57) give (60). \square

Proof of (61). Following the argument in the proof of Lemma 3.3 and using (29), we have

$$\begin{aligned}
& \sum_{j=1}^m \binom{m}{j} |(\xi^{-1}\tau_2(\partial_x^j\xi)\partial_x^{m-j}\partial_y v, \tau_2\partial_x^m u)_{L^2}| \leq \sum_{j=1}^m \binom{m}{j} \|(\tau_2\partial_x^j\xi)\partial_x^{m-j}\partial_y v\|_{L^2}\|\partial_x^m u\|_{L^2} \\
& \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}|\vec{a}|_{\rho,\sigma}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}\frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}.
\end{aligned}$$

Similarly, using (31) and (29) gives

$$\begin{aligned}
& \sum_{j=1}^m \binom{m}{j} |(\xi^{-1}\tau_2(\partial_x^j\eta)\partial_x^{m-j}\partial_y\psi, \tau_2\partial_x^m u)_{L^2}| \\
& \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}|\vec{a}|_{\rho,\sigma}^2 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}}\frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}.
\end{aligned}$$

Combining these inequalities gives (61). \square

7. ESTIMATE ON $\|\vec{a}\|_{\rho,\sigma}$

This section is about the estimate on $\|\vec{a}\|_{\rho,\sigma}$ with $\vec{a} = (u, v)$. We will handle the tangential derivatives and mixed derivatives in Subsection 7.1 and Subsection 7.2 respectively.

7.1. Upper bound for the tangential derivatives. For the derivatives in x, y variables, we have the following

Proposition 7.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (6)-(7). Then for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-6)}}{[(|\alpha|-7)!]^{2\sigma}} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial^\alpha \psi(t)\|_{L^2}^2 + \|\langle z \rangle^\kappa \partial^\alpha v(t)\|_{L^2}^2 \right) \\ & + \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The proof of this proposition is based on the following two lemmas.

Lemma 7.2 (Estimates on $\partial^\alpha u, \partial^\alpha \psi$ and $\partial^\alpha v$). *Under the same assumption as in Proposition 7.1, for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-6)}}{[(|\alpha|-7)!]^{2\sigma}} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial^\alpha \psi(t)\|_{L^2}^2 + \|\langle z \rangle^\kappa \partial^\alpha v(t)\|_{L^2}^2 \right) \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. The estimates on $\partial^\alpha u$ and $\partial^\alpha \psi$ can be obtained in the same way as [19, Proposition 6.1]. In fact, it follows from the Hardy type inequality that

$$\begin{aligned} & \|\tau_1 \langle z \rangle^{\ell-1} \partial_x^m u\|_{L^2} + \|\tau_1 \langle z \rangle^\ell \partial_x^m \psi\|_{L^2} \\ & \leq C \|\langle z \rangle^\ell f_m\|_{L^2} + C \left\| \tau_1' \partial_x^m \psi - \tau_1' \frac{\partial_z \xi}{\psi} \partial_x^m u \right\|_{L^2} + C \|\tau_2 \partial_x^m \psi\|_{L^2}; \end{aligned}$$

see [19, Lemma 6.2] for the detailed proof of the above inequality. Moreover, just repeating the argument for estimating $\|\langle z \rangle^\ell f_m\|_{L^2}$ (see Proposition 3.1), we have

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left\| \tau_1' \partial_x^m \psi - \tau_1' \frac{\partial_z \xi}{\psi} \partial_x^m u \right\|_{L^2} \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

This together with Propositions 3.1 and 6.1 give

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left(\|\tau_1 \langle z \rangle^{\ell-1} \partial_x^m u\|_{L^2} + \|\tau_1 \langle z \rangle^\ell \partial_x^m \psi\|_{L^2} \right) \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

and thus, using Proposition 6.1 and the fact that $1 \leq \tau_1 + \tau_2$, we have

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left(\|\tau_1 \langle z \rangle^{\ell-1} \partial_x^m u(t)\|_{L^2} + \|\langle z \rangle^\ell \partial_x^m \psi(t)\|_{L^2} \right) \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \quad (62) \end{aligned}$$

On the other hand, it follows from Poincaré inequality as well as the relationship in (10) that

$$\begin{aligned} \|\tau_2 \langle z \rangle^{\ell-1} \partial_x^m u(t)\|_{L^2} &\leq C \|\tau_2 \partial_x^m u(t)\|_{L^2} \leq C (\|\tau_2' \partial_x^m u(t)\|_{L^2} + \|\tau_2 \partial_x^m \psi(t)\|_{L^2}) \\ &\leq C (\|\tau_1 \partial_x^m u(t)\|_{L^2} + \|\partial_x^m \psi(t)\|_{L^2}) \\ &\leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\vec{a}_0|_{\rho,\sigma}^2 + \int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where in the last inequality we have used (62). We then combine the above two inequalities and observe the fact that $1 \leq \tau_1 + \tau_2$, to obtain

$$\begin{aligned} &\frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} (\|\langle z \rangle^{\ell-1} \partial_x^m u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_x^m \psi(t)\|_{L^2}^2) \\ &\leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned} \quad (63)$$

Similar estimates hold for the weighted L^2 -norms of $\partial_y^m u$ and $\partial_y^m \psi$. Then we use (22) with $|\alpha| = m$ for the desired estimates on $\partial^\alpha u$ and $\partial^\alpha \psi$.

It remains to handle $\partial^\alpha v$. And again it suffices to consider $\partial_x^m v$ and $\partial_y^m v$ with $m = |\alpha|$, due to (22). In view of (19), we have

$$\partial_x^m v = \begin{cases} (\Gamma_m + \eta \partial_x^m u) / \psi, & \text{if } z \in \text{supp } \tau_1, \\ (H_m + \eta \partial_x^m \psi) / \xi, & \text{if } z \in \text{supp } \tau_2, \end{cases}$$

and thus, recalling $\psi \sim \langle z \rangle^{-\delta}$ on $\text{supp } \tau_1$,

$$\begin{aligned} \|\langle z \rangle^\kappa \partial_x^m v\|_{L^2} &\leq \|\tau_1 \langle z \rangle^\kappa \partial_x^m v\|_{L^2} + C \|\tau_2 \partial_x^m v\|_{L^2} \\ &\leq C \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + C \|\langle z \rangle^{\kappa+\delta} \eta \partial_x^m u\|_{L^2} + C \|H_m\|_{L^2} + C \|\partial_x^m \psi\|_{L^2} \\ &\leq C \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + C \|\langle z \rangle^{\ell-1} \partial_x^m u\|_{L^2} + C \|H_m\|_{L^2} + C \|\partial_x^m \psi\|_{L^2}, \end{aligned}$$

where in the last inequality we have used (2) and (7). This together with Proposition 3.1 and (63), yield the upper bound for $\partial_x^m v$, i.e.,

$$\begin{aligned} &\frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \|\langle z \rangle^\kappa \partial_x^m v(t)\|_{L^2}^2 \\ &\leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The same argument applies to $\partial_y^m v$. Thus, the desired bound for $\partial^\alpha v$ follows and the proof is completed. \square

Lemma 7.3 (Estimate on $\partial^\alpha \eta$). *Under the same assumption as in Proposition 7.1, for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\begin{aligned} &\frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta(t)\|_{L^2}^2 \\ &\leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. As before, it suffices to consider only $\partial_x^m \eta$ and $\partial_y^m \eta$. We apply ∂_x^m to the equation for η in (12) to obtain

$$\begin{aligned} & \partial_t \partial_x^m \eta + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m \eta - \partial_z^2 \partial_x^m \eta \\ &= \partial_x^m h - \sum_{1 \leq j \leq m} \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} \eta + (\partial_x^j v) \partial_y \partial_x^{m-j} \eta + (\partial_x^j w) \partial_z \partial_x^{m-j} \eta]. \end{aligned}$$

Multiplying both sides above by $m^2 \langle z \rangle^{2(\kappa+2)} \partial_x^m \eta$, and then integrating over Ω , we have by noting $\partial_z \eta|_{z=0} = 0$,

$$\frac{1}{2} \frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+2} \partial_x^m \eta\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+2} \partial_z \partial_x^m \eta\|_{L^2}^2 = P_1 + P_2 + P_3$$

with

$$\begin{aligned} P_1 &= m^2 \left(\langle z \rangle^{\kappa+2} \partial_x^m h, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} + m^2 \left(w (\partial_z \langle z \rangle^{\kappa+2}) \partial_x^m \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} \\ &\quad + \frac{1}{2} m^2 \left((\partial_z^2 \langle z \rangle^{2(\kappa+2)}) \partial_x^m \eta, \partial_x^m \eta \right)_{L^2}, \\ P_2 &= -m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+2} [(\partial_x^j u) \partial_x \partial_x^{m-j} \eta + (\partial_x^j v) \partial_y \partial_x^{m-j} \eta], \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2}, \\ P_3 &= -m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+2} (\partial_x^j w) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2}. \end{aligned}$$

Furthermore, using (31)-(32) we obtain by noting $\kappa + 2 \leq \kappa + \delta$ due to (2) and $\partial_x^m h = h_\gamma$ with $\gamma = (m, 0)$,

$$P_1 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\bar{a}|_{\rho, \sigma}^2.$$

We now estimate P_3 . By (31), (32) and (7), we have

$$\begin{aligned} & -m^2 \left(\sum_{1 \leq j \leq 2} + \sum_{m-1 \leq j \leq m} \right) \binom{m}{j} \left(\langle z \rangle^{\kappa+2} (\partial_x^j w) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \frac{|\bar{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

Moreover, for the terms in the middle of the summation, it follows from the argument used in Lemma 3.3 that

$$-m^2 \sum_{3 \leq j \leq m-2} \binom{m}{j} \left(\langle z \rangle^{\kappa+2} (\partial_x^j w) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\bar{a}|_{\rho, \sigma}^3.$$

Thus, we obtain

$$P_3 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho, \sigma}^3 + \frac{|\bar{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Similarly, we have

$$P_2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho, \sigma}^3 + \frac{|\bar{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Combining these estimates gives

$$\frac{1}{2} \frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+2} \partial_x^m \eta\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+2} \partial_z \partial_x^m \eta\|_{L^2}^2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\bar{a}|_{\rho, \sigma}^2 + |\bar{a}|_{\rho, \sigma}^3 + \frac{|\bar{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Integrating over $[0, t]$ yields

$$\begin{aligned} & \frac{\rho^{2(m-5)}}{[(m-6)!]^{2\sigma}} m^2 \|\langle z \rangle^{\kappa+2} \partial_x^m \eta(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The upper bound of $\|\langle z \rangle^{\kappa+2} \partial_y^m \eta(t)\|_{L^2}$ can be obtained similarly. Thus, the estimate on $\partial^\alpha \eta$ follows and this completes the proof. \square

As an immediate consequence of Lemmas 7.2-7.3, we have Proposition 7.1.

7.2. Upper bound for the mixed derivatives. Now we estimate the mixed derivatives that is stated in

Proposition 7.4. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (6)-(7). Then we have, for any pair $(\alpha, j) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ with $1 \leq j \leq 4$ and $|\alpha| + j \geq 7$, any $t \in [0, T]$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$,*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|+j-6)}}{[(|\alpha|+j-7)!]^{2\sigma}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi(t)\|_{L^2}^2 + \frac{\rho^{2(|\alpha|+j-5)}}{[(|\alpha|+j-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Sketch of the proof. Based on the above discussion, we will only sketch the proof for brevity. First of all, we need to consider here the boundary conditions since the normal derivatives are involved when we use integration by parts. The situation is exactly the same as in 2D, where we need to use the equation (12) and the boundary conditions

$$\partial_z \psi|_{z=0} = \partial_z \eta|_{z=0} = 0,$$

to reduce the order of normal derivatives in the boundary terms. Precisely, we use the equation for ψ in (12) to obtain

$$\partial_z^3 \psi|_{z=0} = \partial_z (\partial_t \psi + (u \partial_x + v \partial_y + w \partial_z) \psi - g)|_{z=0} = \psi (\partial_x \psi - \partial_y \eta)|_{z=0} + 2\eta \partial_y \psi|_{z=0},$$

and moreover, direct computation yields

$$\partial_z^5 \psi|_{z=0} = -(\partial_z^2 \psi) (\partial_x \psi + \partial_y \eta)|_{z=0} + 4\psi \partial_x \partial_z^2 \psi|_{z=0} + 4\eta \partial_y \partial_z^2 \psi|_{z=0}.$$

Similar equalities hold for $\partial_z^3 \eta|_{z=0}$ and $\partial_z^5 \eta|_{z=0}$. Then, we can follow the argument in [19] to deduce the energy estimates on $\partial^\alpha \partial_z^j \psi$ and $\partial^\alpha \partial_z^j \eta$, with only difference coming from the appearance of two additional terms

$$\left(\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j g, \langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi \right)_{L^2} \text{ and } |\alpha|^2 \left(\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j h, \langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta \right)_{L^2}. \quad (64)$$

However, there will be no additional difficulty to control the above two terms because we can use (13) to write, by noting $1 \leq j \leq 4$,

$$\partial^\alpha \partial_z^j g = \partial^\alpha \partial_z^{j-1} \left[(\partial_y \eta) \psi + (\partial_y v) \partial_z \psi - (\partial_y \psi) \eta - (\partial_x u) \partial_z \eta \right].$$

Then following the argument in the proof of Lemma 3.3 yields that

$$\|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^{j-1} g\|_{L^2} \leq \frac{C[(|\alpha|+j-7)!]^\sigma}{\rho^{|\alpha|+j-6}} |\vec{a}|_{\rho, \sigma}^2 + \frac{C[(|\alpha|+j-7)!]^\sigma}{\rho^{|\alpha|+j-6}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}}{\tilde{\rho} - \rho}.$$

Hence, we have

$$\left(\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j g, \langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi \right)_{L^2} \leq \frac{C[(|\alpha| + j - 7)!]^{2\sigma}}{\rho^{2(|\alpha| + j - 6)}} \left(|\tilde{a}|_{\rho, \sigma}^3 + \frac{|\tilde{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Similarly, we can show that the second term in (64) is bounded from above by

$$\frac{C[(|\alpha| + j - 6)!]^{2\sigma}}{\rho^{2(|\alpha| + j - 5)}} |\tilde{a}|_{\rho, \sigma}^3.$$

And the other terms can be estimated in the same way as [19], so that we omit the detail for brevity, cf. Subsection 6.2 in [19]. Then the estimate given in Proposition 7.4 follows. \square

8. PROOF OF A PRIORI ESTIMATE

In this section, we will prove the a priori estimate, in the two cases of $\sigma \in [3/2, 2]$ and $1 < \sigma < 3/2$ separately. For this, we will first prove Theorem 2.3 for the case when $\sigma \in [3/2, 2]$.

The case when $\sigma \in [3/2, 2]$. By Propositions 3.1, 5.1, 7.1 and 7.4, we have the upper bound on the terms in $|\tilde{a}|_{\rho, \sigma}$ when the order of derivatives is greater than or equal to 7, that is, these terms are bounded by

$$C |\tilde{a}_0|_{\rho, \sigma}^2 + C \int_0^t \left(|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4 \right) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.$$

For the derivatives with order less than 7, it can be checked straightforwardly that the same upper bound holds. Hence, we have the a priori estimate (25) and then complete the proof of Theorem 2.3.

The case when $1 < \sigma < 3/2$. The proof for this case is similar to the one when $\sigma \in [3/2, 2]$, just replacing the norm $\|\cdot\|_{\rho, \sigma}$ by an equivalent norm. Precisely, if $1 < \sigma < 3/2$, then we can find an integer $N \geq 2$ such that

$$(N + 1)/N \leq \sigma. \quad (65)$$

Define another Gevrey norm $\|\cdot\|_{\rho, \sigma, N}$ by replacing respectively $\sup_{|\alpha| \geq 7}$ and $\sup_{|\alpha| \leq 6}$ in (3) by $\sup_{|\alpha| \geq N+5}$ and $\sup_{|\alpha| \leq N+4}$. Do the same for $\sup_{|\alpha| + j \geq 7}$ and $\sup_{|\alpha| + j \leq 6}$. This new norm $\|\cdot\|_{\rho, \sigma, N}$ is equivalent to $\|\cdot\|_{\rho, \sigma}$ in the sense that

$$\|\cdot\|_{\rho, \sigma} \leq \|\cdot\|_{\rho, \sigma, N} \leq C_N \rho^{2-N} \|\cdot\|_{\rho, \sigma}$$

for $\rho \leq 1$, where C_N is a constant depending only on N . Moreover, following the same argument as above, we replace $|\cdot|_{\rho, \sigma}$ given in Definition 2.1 by $|\cdot|_{\rho, \sigma, N}$, and replace (7) by

$$\begin{aligned} & \sum_{|\alpha| \leq N+1} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha u\|_{L^\infty} + \|\langle z \rangle^\kappa \partial^\alpha v\|_{L^\infty} + \|\partial^\alpha w\|_{L^\infty} \right) \\ & + \sum_{|\alpha| \leq N+2} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L_{x,y}^\infty(L_z^2)} + \sum_{\substack{|\alpha| + j \leq N+2 \\ j \geq 1}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L_{x,y}^\infty(L_z^2)} \\ & + \sum_{|\alpha| + j \leq N+2} \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L_{x,y}^\infty(L_z^2)} \leq \tilde{C}. \end{aligned} \quad (66)$$

And then we have

Theorem 8.1. *Let $1 < \sigma < 3/2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) such that the properties in (6) and (66) hold. Then there exists a constant $C_* > 1$, such that*

$$|\vec{a}(t)|_{\rho, \sigma, N}^2 \leq C_* |\vec{a}_0|_{\rho, \sigma, N}^2 + C_* \int_0^t \left(|\vec{a}(s)|_{\rho, \sigma, N}^2 + |\vec{a}(s)|_{\rho, \sigma, N}^4 \right) ds + C_* \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma, N}^2}{\tilde{\rho} - \rho} ds \quad (67)$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$ and any $t \in [0, T]$.

Sketch of the proof. For brevity, we only give a sketch of the proof. In fact, it is similar to that for Theorem 2.3. Here we will show how to modify the argument used there.

We first recall how the assumption $\sigma \in [3/2, 2]$ is used when proving Theorem 2.3. For $\sigma \geq 3/2$, we use the following type of splitting in the summation

$$\left[\sum_{0 \leq j \leq 1} + \sum_{m-1 \leq j \leq m} \right] \binom{m}{j} \cdots + \sum_{2 \leq j \leq m-2} \binom{m}{j} \cdots \quad (68)$$

By direct computation, we see that the factors m and $m^{2-\sigma}$ appear in the first two summations in (68) respectively. Then we can use (26) and (7) to conclude that the summation

$$\left[\sum_{0 \leq j \leq 1} + \sum_{m-1 \leq j \leq m} \right] \binom{m}{j} \cdots$$

is basically bounded from above by

$$\frac{C[(m-7)!]^\sigma |\vec{a}|_{\tilde{\rho}, \sigma}}{\rho^{m-6} \tilde{\rho} - \rho} \quad \text{or} \quad \frac{C[(m-6)!]^\sigma |\vec{a}|_{\tilde{\rho}, \sigma}}{\rho^{m-5} \tilde{\rho} - \rho}.$$

Meanwhile, for the last summation in (68) we have factors like

$$m^{3-2\sigma}, m^{4-3\sigma}, \dots,$$

which are less than 1 when $\sigma \geq 3/2$. Thus, the last summation in (68) is basically bounded from above by

$$\frac{C[(m-7)!]^\sigma |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \quad \text{or} \quad \frac{C[(m-6)!]^\sigma |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-5}}.$$

Now we turn to the case when $1 < \sigma < 3/2$, and instead of (68), we can use a new splitting

$$\left[\sum_{0 \leq j \leq N-1} + \sum_{m-N+1 \leq j \leq m} \right] \binom{m}{j} \cdots + \sum_{N \leq j \leq m-N} \binom{m}{j} \cdots \quad (69)$$

Then we have factors

$$m, m^{2-\sigma}, m^{3-2\sigma}, m^{4-3\sigma}, m^{N-(N-1)\sigma},$$

appearing in the first two summations. Note that these factors are bounded by m , and then we can use again (26) and (66) to estimate the first two summations. Meanwhile, factors like

$$m^{N+1-N\sigma}, m^{N+2-(N+1)\sigma}, \dots,$$

appear in the last summation in (69), and these factors are less than 1 because of (65). So the situation is similar to the case of $\sigma \in [3/2, 2]$. Then one can apply the argument used in Sections 3-7 to the case when $1 < \sigma < 3/2$, with $|\cdot|_{\rho, \sigma}$ and (7) therein replaced by $|\cdot|_{\rho, \sigma, N}$ and (66). Then the desired a priori estimate (67) follows, and it completes the proof. \square

9. PROOF OF THE MAIN RESULTS

We will prove in this section the main results on the existence and uniqueness for Prandtl system. We first prove Theorem 1.8 that corresponds to the constant outer flow when $(U, V) = (0, 0)$, and then explain how to extend the argument to the general outer flow. Since the proof is similar as in 2D case after we have the a priori estimate, we will only give a sketch.

Sketch of the proof of Theorem 1.8. The proof relies on the a priori estimates given in Theorems 2.3 and 8.1.

In order to obtain the existence of solutions to the Prandtl equations (5), there are two main ingredients, one of which is to investigate the existence of approximate solutions to the regularized Prandtl system

$$\begin{cases} \partial_t u_\varepsilon + (u_\varepsilon \partial_x + v_\varepsilon \partial_y + w_\varepsilon \partial_z) u_\varepsilon - \partial_z^2 u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon - \varepsilon \partial_y^2 u_\varepsilon = 0, \\ \partial_t v_\varepsilon + (u_\varepsilon \partial_x + v_\varepsilon \partial_y + w_\varepsilon \partial_z) v_\varepsilon - \partial_z^2 v_\varepsilon - \varepsilon \partial_x^2 v_\varepsilon - \varepsilon \partial_y^2 v_\varepsilon = 0, \\ (u_\varepsilon, v_\varepsilon)|_{z=0} = (0, 0), \quad \lim_{z \rightarrow +\infty} (u_\varepsilon, v_\varepsilon) = (0, 0), \\ (u_\varepsilon, v_\varepsilon)|_{t=0} = (u_0, v_0), \end{cases} \quad (70)$$

with $w_\varepsilon = -\int_0^z (\partial_x u_\varepsilon + \partial_y v_\varepsilon) d\tilde{z}$. We remark that the regularized equations above share the same compatibility condition (4) as the original system (5). Another ingredient is to derive a uniform estimate with respect to ε for the approximate solutions $(u_\varepsilon, v_\varepsilon)$.

The existence for the parabolic system (70) is standard. Indeed, suppose that $(u_0, v_0) \in X_{2\rho_0, \sigma}$. Then we can construct, following the similar scheme as that in [19, Section 7], a solution $(u_\varepsilon, v_\varepsilon) \in L^\infty([0, T_\varepsilon^*]; X_{3\rho_0/2, \sigma})$ to (70) such that

$$\sup_{t \in [0, T_\varepsilon^*]} \|(u_\varepsilon(t), v_\varepsilon(t))\|_{3\rho_0/2, \sigma} \leq C \|(u_0, v_0)\|_{2\rho_0, \sigma}, \quad (71)$$

where T_ε^* may depend on ε .

It remains to derive a uniform estimate for the approximate solutions $(u_\varepsilon, v_\varepsilon)$, so that we can remove the ε -dependence of the lifespan T_ε^* . For this, we need to verify that $(u_\varepsilon, v_\varepsilon)$ satisfies the condition (6), and the condition (7) if $\sigma \in [3/2, 2]$ and (66) if $1 < \sigma < 3/2$ respectively.

In view of (71) and by Sobolev inequalities and the definition of $\|\cdot\|_{\rho, \sigma}$, we know that u_ε and v_ε satisfy the condition (7) if $\sigma \in [3/2, 2]$, and the condition (66) in the case of $1 < \sigma < 3/2$. In order to verify (6), we suppose u_0 satisfies Assumption 1.1 additionally and show that these properties therein preserve in time. This is clear when we consider a small perturbation around a shear flow. For the general initial data, it was shown in [22] by using the classical maximum principle for the parabolic equation, that such properties listed in Assumption 1.1 also preserve in time. Here, we can adopt the argument in [22, Subsection 5.2], and apply the maximum principle to the first equation in (70). Precisely, applying ∂_z to the first equation in (70) and using the notations $\psi_\varepsilon = \partial_z u_\varepsilon$ and $\eta_\varepsilon = \partial_z v_\varepsilon$, we have

$$\partial_t \psi_\varepsilon + (u_\varepsilon \partial_x + v_\varepsilon \partial_y + w_\varepsilon \partial_z) \psi_\varepsilon - \partial_z^2 \psi_\varepsilon - \varepsilon \partial_x^2 \psi_\varepsilon - \varepsilon \partial_y^2 \psi_\varepsilon = (\partial_y v_\varepsilon) \psi_\varepsilon - (\partial_y u_\varepsilon) \eta_\varepsilon,$$

that is,

$$\begin{aligned} & \left[\partial_t + \left(u_\varepsilon \partial_x + v_\varepsilon \partial_y + (w_\varepsilon + 2\delta(1+z)^{-1}) \partial_z \right) - \partial_z^2 - \varepsilon \partial_x^2 - \varepsilon \partial_y^2 \right] ((1+z)^\delta \psi_\varepsilon) \\ &= \left(\partial_y v_\varepsilon - (\partial_y u_\varepsilon) \eta_\varepsilon / \psi_\varepsilon + \delta w_\varepsilon / (1+z) + \delta(\delta+1)/(1+z)^2 \right) (1+z)^\delta \psi_\varepsilon. \end{aligned}$$

By using (71) and the maximal principle for parabolic equations (see [22, Lemma E.2]), one can show that there exists $c_* > 0$ independent of ε such that for any $(t, x, y, z) \in [0, T_\varepsilon^*] \times \Omega$,

$$\psi_\varepsilon(t, x, y, z) \geq c_*(1+z)^{-\delta}.$$

This gives the lower bound on $\partial_z u_\varepsilon$ given in (6). Similarly, we can derive also the upper bounds on $\partial_z u_\varepsilon$ and its normal derivatives given in (6).

Consequently, following the argument in Sections 3-8, the estimates (25) and (67) also hold, with \bar{a} there replaced by $(u_\varepsilon, v_\varepsilon)$. Finally, by the uniform estimate on $(u_\varepsilon, v_\varepsilon)$, we can repeat the argument in [19, Section 8] to conclude the existence and uniqueness of solutions to the Prandtl system (5). For brevity, we omit the detail here and refer it to [22, Section 5.2] and [19, Section 8] for the comprehensive discussion. Thus, the proof of Theorem 1.8 is completed. \square

Sketch of the proof of Theorem 1.4. Now we consider the general outer flow $U, V, p \in Y_{2\rho_0, \sigma}$. The argument is similar to the case with the constant outer flow discussed above. The main difference comes from the appearance of extra source term and boundary term. Since the extra source terms only involve U, V and p , they are bounded by the Gevrey norms of U, V and p .

In addition, the boundary terms do not create additional difficulty in the estimation. To see this, we consider for instance \hat{f}_m which is defined in the same way as f_m , that is,

$$\hat{f}_m = \partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m (u - U) = \psi \partial_z \left(\frac{\partial_x^m (u - U)}{\psi} \right).$$

Different from the case with constant data U , since $\hat{f}_m \partial_z \hat{f}_m|_{z=0} \neq 0$, then we have boundary terms like

$$\int_{\mathbb{T}^2} \hat{f}_m(x, y, 0) \partial_z \hat{f}_m(x, y, 0) dx dy,$$

when applying integration by parts. Furthermore, note that, denoting $\chi = \partial_z \psi / \psi$,

$$\hat{f}_m(x, y, 0) = \partial_x^m \psi(x, y, 0) + \chi(x, y, 0) \partial_x^m U(x, y),$$

by using the fact that $\partial_z \psi|_{z=0} = \partial_x p$, we have

$$\partial_z \hat{f}_m(x, y, 0) = \partial_x^{m+1} p(x, y) + (\partial_z \chi)(x, y, 0) \partial_x^m U(x, y) - \chi(x, y, 0) \partial_x^m \psi(x, y, 0).$$

Then

$$\left| \int_{\mathbb{T}^2} \hat{f}_m(x, y, 0) \partial_z \hat{f}_m(x, y, 0) dx dy \right| \leq C \|\partial_x^m \psi(\cdot, 0)\|_{L^2(\mathbb{T}^2)}^2 + \text{terms involving } U, V, p.$$

As for the first term on the right side, we have

$$\begin{aligned} \|\partial_x^m \psi(\cdot, 0)\|_{L^2(\mathbb{T}^2)}^2 &\leq \varepsilon \|\partial_z \partial_x^m \psi\|_{L^2(\Omega)}^2 + C_\varepsilon \|\partial_x^m \psi\|_{L^2(\Omega)}^2 \\ &\leq \varepsilon \|\partial_z \hat{f}_m\|_{L^2(\Omega)}^2 + C_\varepsilon \left(\|\partial_x^m \psi\|_{L^2(\Omega)}^2 + \|\partial_x^m (u - U)\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

where in the last inequality we have used the representation of \hat{f}_m . We can choose ε small enough to absorb the first term $\|\partial_z \hat{f}_m\|_{L^2(\Omega)}^2$. Then the quantity

$$\left| \int_{\mathbb{T}^2} \hat{f}_m(x, y, 0) \partial_z \hat{f}_m(x, y, 0) dx dy \right|$$

has a suitable upper bound, just as $\|\partial_x^m \psi\|_{L^2}^2$ and $\|\partial_x^m (u - U)\|_{L^2}^2$. We can apply a similar argument as above to the other boundary terms when estimating $g_\alpha, h_\alpha, \Gamma_m$, etc.

In summary, we can extend the result to general outer flow (U, V) , with (U, V) in a Gevrey space with the same index σ as the initial data. For this, we replace (u, v) by $(u - U, v - V)$ in the estimation, and perform the estimates on $(u - U, v - V)$ following the discussions in Sections 3-8. Since it does not involve any extra essential difficulty, we omit the detail for brevity. \square

Acknowledgements. We would like to thank the reviewer's valuable suggestions for the revision of the paper. The research of the first author was supported by NSFC (11871054, 11771342) and Fok Ying Tung Education Foundation (151001). And the research of the second author was supported by the General Research Fund of Hong Kong, CityU No.11320016 and the NSFC project 11731008.

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(W.-X. Li) SCHOOL OF MATHEMATICS AND STATISTICS, AND COMPUTATIONAL SCIENCE HUBEI KEY LABORATORY, WUHAN UNIVERSITY, 430072 WUHAN, CHINA

E-mail address: wei-xi.li@whu.edu.cn

(T.Yang) DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, HONG KONG, & SCHOOL OF MATHEMATICS AND STATISTICS, CHONGQING UNIVERSITY, CHONGQING, CHINA

E-mail address: matyang@cityu.edu.hk