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A class of self-similar solutions to a singular and degenerate diffusion equation

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Abstract

In this paper we study self-similar solutions to the singular and degenerate diffusion equation

$$u_t = (|(p(u))_x|^{\lambda-2}(p(u))_x)_x, \quad -\infty < x < +\infty,\ t > 0,$$

where $1 < \lambda < 2$. The existence and uniqueness for the solutions are established. In addition, the asymptotic behavior is investigated.

Keywords: Singular and degenerate diffusion equation; Existence; Uniqueness and asymptotic behavior

1. Introduction

Consider the singular and degenerate diffusion equation

$$u_t = (|(p(u))_x|^{\lambda-2}(p(u))_x)_x, \quad -\infty < x < +\infty,\ t > 0, \tag{1.1}$$

where $1 < \lambda < 2$, $p(s) \in C^1([0, +\infty))$, $p'(s) > 0$ for $s > 0$ and $p'(s)$ is a monotone increasing function in $(0, +\infty)$. The equation may have singularity or degeneracy. In fact, the equation is singular at the points where $p(u)_x = 0$, while degenerate at the points where $u = 0$ if $p'(0) = 0$. This type of equation has a wide range of applications in physics and engineering sciences, see [9,15,17,18,20] and references therein.

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During the past decades, equations of the form (1.1) have been paid much attention by many mathematicians. In particular, the investigation for the important case $\lambda = 2$ has been the subject of intensive study, see [1,3–8,14,16,19,21,22]. When $\lambda \neq 2$ and $p(s) = s^m$ $(m \geqslant 1)$, (1.1) is called the non-Newtonian polytropic filtration equation which has also been studied extensively, see [12] for the existence, uniqueness and regularity of solutions in one-dimensional case, and [11] and references therein for multi-dimensional case, see also [2,10,13].

Another way to view this equation and the degeneracy comes from the nonlinear damping problem for Euler equations with vacuum. Consider the following system for isentropic flow:

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2 + P(\rho))_x = -\kappa(\rho u)^\alpha,$$

where ρ, u and $P(\rho)$ are density, velocity and pressure, respectively, while $\kappa > 0$ is the damping coefficient, and $\alpha > 0$ is the power in the nonlinear damping term. It is known that usually when time t tends to infinity, the convection term in the second equation, i.e., $(\rho u)_t + (\rho u^2)_x$, decays faster than the other terms. This makes it possible to combine the above two equations to have the nonlinear porous media equation in the form of

$$\rho_t = \frac{1}{\kappa}((P(\rho)_x)^{1/\alpha})_x.$$

If the pressure function satisfies $P'(0) = 0$, then it is clear that the above equation is degenerate at $\rho = 0$, i.e., at vacuum states. We believe that the study in this paper will be useful to the study of the large time behavior of solutions to the above system.

In this paper, we will consider Eq. (1.1) with general function $p(u)$ under some assumption to obtain some existence and regularity results. Moreover, the necessary and sufficient condition on the solutions with compact support is given with detailed behavior of the solutions at the interface. For this, we consider the self-similar solutions of the form

$$u(x,t) = w(\xi), \quad \xi = x(t+1)^{-1/\lambda}.$$

A direct calculation shows that $w = w(\xi)$ satisfies the following equation:

$$-\frac{1}{\lambda}\,\xi w' = (|(p(w))'|^{\lambda-2}(p(w))')'. \tag{1.2}$$

We will investigate the infinite two-point boundary value problem of (1.2) with

$$w(-\infty) = w_-, \quad w(+\infty) = w_+, \tag{1.3}$$

where $w_\pm \geqslant 0$.

Since Eq. (1.2) is singular at the points where $(p(w))' = 0$ and may be degenerate at the points where $u = 0$, the classical solution may not exist. For this, the solution to (1.2) and (1.3) is defined as follows.

Definition 1. A non-negative function $w(\xi) \in C(-\infty, +\infty)$ is a solution of Eq. (1.2), if $p(w) \in C^1(-\infty, +\infty)$, w and $(p(w))'$ are absolutely continuous in $(-\infty, +\infty)$ so that

(1.2) holds almost everywhere. Moreover, if

$$\lim_{\xi\to-\infty} w(\xi) = w_-, \qquad \lim_{\xi\to+\infty} w(\xi) = w_+,$$

$w(\xi)$ is called a solution of the infinite two-point boundary value problem (1.2) and (1.3).

If $w(\xi)$ is a solution of the problem (1.2) and (1.3), then $\tilde{w}(\xi) = w(-\xi)$ is also a solution of Eq. (1.2) with

$$w(-\infty) = w_+, \quad w(+\infty) = w_-.$$

Therefore, we can assume $0 \leqslant w_- \leqslant w_+$ without loss of generality.

The main results of this paper are the following:

Theorem 1. *There exists a unique solution of the infinite two-point boundary value problem* (1.2) *and* (1.3).

Theorem 2. *Let $w(\xi)$ be the solution of the infinite two-point boundary value problem* (1.2) *and* (1.3) *with $0 < w_- < w_+$. Then $w(\xi)$ is strictly increasing and satisfies*

$$\lim_{\xi\to-\infty} |\xi|^{\lambda/(2-\lambda)}(w(\xi) - w_-) = \frac{2-\lambda}{\lambda p'(w_-)}\left(\frac{2\lambda(\lambda-1)p'(w_-)}{2-\lambda}\right)^{1/(2-\lambda)},$$

$$\lim_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(w_+ - w(\xi)) = \frac{2-\lambda}{\lambda p'(w_+)}\left(\frac{2\lambda(\lambda-1)p'(w_+)}{2-\lambda}\right)^{1/(2-\lambda)}.$$

Theorem 3. *Let $w(\xi)$ be the solution of the infinite two-point boundary value problem* (1.2) *and* (1.3) *with $w_- = 0$ and $w_+ > 0$. Set*

$$\hat{\xi} = \inf\{\xi \in (-\infty, +\infty) : w(\xi) > 0\}.$$

For $\hat{\xi}$, we have the following two cases:

(i) *If $\int_0^1 p'(s)s^{-1/(\lambda-1)}\,\mathrm{d}s < +\infty$, then $-\infty < \hat{\xi} < 0$ and the solution satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\int_0^{w(\xi)} p'(s)s^{-1/(\lambda-1)}\,\mathrm{d}s\right)\bigg|_{\xi=\hat{\xi}^+} = \left(\frac{-\hat{\xi}}{\lambda}\right)^{1/(\lambda-1)} > 0.$$

Moreover, $w(\xi)$ is strictly increasing on $[\hat{\xi}, +\infty)$, while $w(\xi) \equiv 0$ on $(-\infty, \hat{\xi}]$.

(ii) *If $\int_0^1 p'(s)s^{-1/(\lambda-1)}\,\mathrm{d}s = +\infty$, then $\hat{\xi} = -\infty$ and the solution satisfies*

$$\lim_{\xi\to-\infty} |\xi| w(\xi) = 0, \qquad \lim_{\xi\to-\infty} |\xi|^{\lambda/(2-\lambda)} p(w(\xi)) = 0.$$

And $w(\xi)$ is strictly increasing in $(-\infty, +\infty)$.

Furthermore, for both cases, we have

$$\lim_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(w_+ - w(\xi)) = \frac{2-\lambda}{\lambda p'(w_+)}\left(\frac{2\lambda(\lambda-1)p'(w_+)}{2-\lambda}\right)^{1/(2-\lambda)}.$$

Remark 1. For $p(s) = s^m$ $(m \geqslant 1)$, Theorem 3 yields the following properties on the solutions:

(i) If $m > 1/(\lambda - 1)$, then $-\infty < \hat{\xi} < 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(w^{m-1/(\lambda-1)}(\xi))|_{\xi=\hat{\xi}^+} = \frac{1}{m}\left(m - \frac{1}{\lambda-1}\right)\left(\frac{-\hat{\xi}}{\lambda}\right)^{1/(\lambda-1)} > 0.$$

(ii) If $1 \leqslant m \leqslant 1/(\lambda - 1)$, then $\hat{\xi} = -\infty$.

Remark 2. Consider the Euler equations for polytropic gas. When $P(\rho) = \sigma^2\rho^\gamma$ with $\gamma \geqslant 1$ being the adiabatic constant and σ being a constant, the explanation on the vacuum behavior from Theorem 3 can be stated as follows. When $\gamma > \alpha$, the gas can connect to vacuum in finite distance at any time with the physical vacuum boundary condition being $(\rho^{\gamma-\alpha})_x \neq 0$ and bounded at the vacuum interface. This is consistent with the work done on the Euler equations with linear damping when $\alpha = 1$. On the other hand, when $1 \leqslant \gamma \leqslant \alpha$, the gas canonically does not connect to vacuum in finite distance for any time, but at infinity.

Theorem 4. *There exists one solution to the Cauchy problem of Eq.* (1.1) *with the initial value*

$$u(x, 0) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \tag{1.4}$$

where $u_\pm \geqslant 0$.

2. Preliminaries

Let $v = p(w)$. Then Eq. (1.2) is transformed to

$$-\frac{1}{\lambda}\xi(q(v))' = (|v'|^{\lambda-2}v')', \tag{2.1}$$

i.e.

$$v'' = -\frac{1}{\lambda(\lambda-1)}\xi q'(v)|v'|^{2-\lambda}v', \tag{2.2}$$

where $q(s) = p^{-1}(s)$ is the inverse function of p. And the boundary value (1.3) is transformed to

$$v(-\infty) = v_-, \quad v(+\infty) = v_+, \tag{2.3}$$

where $v_- = p(w_-)$ and $v_+ = p(w_+)$. It is easy to see that $q(s) \in C(R(p)) \cap C^1(R_+(p))$, $q'(s) > 0$ for $s \in R_+(p)$ and $q'(s)$ is a monotone decreasing function in $R_+(p)$, where

$$R(p) = \{p(s) : s \in [0, +\infty)\}, \quad R_+(p) = \{p(s) : s \in (0, +\infty)\}.$$

According to Definition 1, we see that

Definition 2. A function $v(\xi) \in C(I)$ is called a solution of Eq. (2.2) in an interval I, if $v(I) \subset R(p)$, $v \in C^1(I)$, $q(v)$ and v' are absolutely continuous in I so that (2.2) holds almost everywhere. Additionally, if $I = (-\infty, +\infty)$ and

$$\lim_{\xi \to -\infty} v(\xi) = v_-, \qquad \lim_{\xi \to +\infty} v(\xi) = v_+,$$

we say $v(\xi)$ is a solution of the infinite two-point boundary value problem (2.2) and (2.3).

From Eq. (2.2) and Definition 2, if v is a solution of (2.2) in an interval I, then v is a classical solution of (2.2) in $\{\xi \in I : v(\xi) \in R_+(p)\}$. Noticing $2 - \lambda > 0$, it is evident that for any solution v of Eq. (2.2), if $v(\xi_0) \in R_+(p)$ and $v'(\xi_0) = 0$ for some $\xi_0 \in (-\infty, +\infty)$, then $v' \equiv 0$. That is,

Proposition 2.1. *Assume I is an interval, $v(\xi)$ $(\xi \in I)$ is a solution of Eq.* (2.2) *and $v(I) \subset R_+(p)$.*

(i) *If there exists $\xi_0 \in I$ such that $v'(\xi_0) > 0$, then*

$$v'(\xi) > 0, \quad \forall \xi \in I.$$

(ii) *If there exists $\xi_0 \in I$ such that $v'(\xi_0) = 0$, then*

$$v'(\xi) = 0, \quad \forall \xi \in I.$$

(iii) *If there exists $\xi_0 \in I$ such that $v'(\xi_0) < 0$, then*

$$v'(\xi) < 0, \quad \forall \xi \in I.$$

Lemma 2.1. *Assume I is an interval, $v(\xi)$ $(\xi \in I)$ is a strictly increasing solution of Eq.* (2.2), *$0 \in I$, $v(I) \subset R_+(p)$, $b \in R_+(p)$ and*

$$v(\xi) \leqslant b, \quad \forall \xi \in I.$$

Then

$$v'(\xi) \leqslant ({v'}^{\lambda-2}(0) + C\xi^2)^{1/(\lambda-2)}, \quad \forall \xi \in I,$$

where

$$C = \frac{(2-\lambda)q'(b)}{2\lambda(\lambda-1)} > 0.$$

Proof. Since v is strictly increasing, (2.2) can be rewritten as

$$v'' = -\frac{1}{\lambda(\lambda-1)} \xi q'(v) {v'}^{3-\lambda},$$

i.e.

$$(v'^{\lambda-2})' = \frac{2-\lambda}{\lambda(\lambda-1)}\xi q'(v).$$

Thus

$$v'^{\lambda-2}(\xi) = v'^{\lambda-2}(0) + \frac{2-\lambda}{\lambda(\lambda-1)}\int_0^{\xi} sq'(v(s))\,\mathrm{d}s, \quad \forall \xi \in I.$$

Since $q'(s) > 0$ for $s \in R_+(p)$ and $q'(s)$ is a monotone decreasing function in $R_+(p)$, we have

$$0 < q'(b) \leqslant q'(v(s)), \quad \forall s \in I.$$

Noticing that $1 < \lambda < 2$ and $v'(0) > 0$, we see that

$$0 < v'^{\lambda-2}(0) + \frac{(2-\lambda)q'(b)}{2\lambda(\lambda-1)}\xi^2 \leqslant v'^{\lambda-2}(\xi), \quad \forall \xi \in I.$$

Thus

$$v'(\xi) \leqslant \left(v'^{\lambda-2}(0) + \frac{(2-\lambda)q'(b)}{2\lambda(\lambda-1)}\xi^2\right)^{1/(\lambda-2)}, \quad \forall \xi \in I.$$

The proof is complete. □

Lemma 2.2. *Let $v_1(\xi)$ and $v_2(\xi)$ be two strictly increasing solutions of Eq.* (2.2) *in an interval I and $v_1(I), v_2(I) \subset R_+(p)$.*

(i) *If there exists $\xi_0 \in [0, +\infty) \cap I$ such that*

$$v_1(\xi_0) \geqslant v_2(\xi_0), \quad v_1'(\xi_0) > v_2'(\xi_0),$$

then

$$v_1'(\xi) > v_2'(\xi), \quad \forall \xi \in [\xi_0, +\infty) \cap I.$$

(ii) *If there exists $\xi_0 \in (-\infty, 0) \cap I$ such that*

$$v_1(\xi_0) = v_2(\xi_0), \quad v_1'(\xi_0) > v_2'(\xi_0),$$

then

$$v_1'(\xi) > v_2'(\xi), \quad \forall \xi \in [\xi_0, +\infty) \cap I.$$

Proof. (i) Assume the conclusion was invalid. Let

$$\xi^* = \inf\{\xi \in [\xi_0, +\infty) \cap I : v_1'(\xi) \leqslant v_2'(\xi)\}.$$

Then $0 \leqslant \xi_0 < \xi^*$, $v_1'(\xi^*) = v_2'(\xi^*)$ and $v_1'(\xi) > v_2'(\xi)$ for $\xi_0 < \xi < \xi^*$. Since $v_1(\xi_0) \geqslant v_2(\xi_0)$, $v_1(\xi^*) > v_2(\xi^*)$. Eq. (2.2) implies $v_1''(\xi^*) > v_2''(\xi^*)$, which contradicts that $v_1'(\xi^*) = v_2'(\xi^*)$ and $v_1'(\xi) > v_2'(\xi)$ for $\xi_0 < \xi < \xi^*$.

(ii) From (i), we need to only prove that

$$v_1'(\xi) > v_2'(\xi), \quad \forall \xi \in [\xi_0, 0] \cap I.$$

Assume the conclusion was invalid. Let

$$\xi_* = \inf\{x \in [\xi_0, 0] \cap I : v_1'(\xi) \leqslant v_2'(\xi)\}.$$

Then $\xi_0 < \xi_* \leqslant 0$, $v_1'(\xi_*) = v_2'(\xi_*)$ and $v_1'(\xi) > v_2'(\xi)$ for $\xi_0 < \xi < \xi_*$. Since $v_1(\xi_0) = v_2(\xi_0)$, $v_1(\xi) > v_2(\xi)$ for $\xi_0 < \xi \leqslant \xi_*$. Integrating (2.1) from ξ_0 to ξ_*, we get

$$v'^{\lambda-1}\Big|_{\xi_0}^{\xi_*} = -\frac{1}{\lambda}\int_{\xi_0}^{\xi_*} \xi(q(v))'\,\mathrm{d}\xi = \frac{1}{\lambda}\int_{\xi_0}^{\xi_*} q(v)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v)\Big|_{\xi_0}^{\xi_*}.$$

Therefore,

$$v_2'^{\lambda-1}(\xi_0) - v_1'^{\lambda-1}(\xi_0) = \frac{1}{\lambda}\int_{\xi_0}^{\xi_*} (q(v_1) - q(v_2))\,\mathrm{d}\xi - \frac{1}{\lambda}\xi_*(q(v_1(\xi_*)) - q(v_2(\xi_*))).$$

Since $\xi_* \leqslant 0$, $q(v_1(\xi_*)) - q(v_2(\xi_*)) > 0$, $\int_{\xi_0}^{\xi_*}(q(v_1) - q(v_2))\,\mathrm{d}\xi > 0$, we get that

$$v_2'^{\lambda-1}(\xi_0) > v_1'^{\lambda-1}(\xi_0),$$

which contradicts $v_1'(\xi_0) > v_2'(\xi_0)$. The proof is complete. □

3. The case without degeneracy

In this section, we consider the case without degeneracy, i.e. $v_\pm \in R_+(p)$. By Proposition 2.1, problems (2.2) and (2.3) have a unique solution for $v_- = v_+$. Therefore, we just need to investigate the case $v_- < v_+$.

Lemma 3.1. *Assume* $\xi_0 \in (-\infty, +\infty)$ *and* $a \in R_+(p)$. *For sufficiently small* $\delta > 0$, *Eq.* (2.2) *with the initial value*

$$\begin{cases} v(\xi_0) = a, \\ v'(\xi_0) = \delta \end{cases} \tag{3.1}$$

can be extended to $+\infty$.

Proof. We first prove the conclusion for $\xi_0 = 0$. Since $p \in C^1(0, +\infty)$, $R_+(p)$ is an open interval. Thus, there exists $b \in R_+(p)$ such that $b > a$. Choose $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$\int_0^{+\infty} (\delta^{\lambda-2} + C\xi^2)^{1/(\lambda-2)}\,\mathrm{d}\xi < b - a,$$

where

$$C=\frac{(2-\lambda)q'(b)}{2\lambda(\lambda-1)}>0.$$

By Lemma 2.1, it is easy to see for any $0<\delta<\delta_0$, the solution of (2.2) and (3.1) with $\xi_0=0$ can be extended to $+\infty$.

If $\xi_0>0$, we can obtain the conclusion directly by the result of $\xi_0=0$ and Lemma 2.2.

If $\xi_0<0$, by the result of $\xi_0=0$ and the continuity of solution to initial, we need only prove that the solution v of (2.2) and (3.1) can be extended to 0 and satisfies $v(0)\in R_+(p)$ for sufficiently small $\delta>0$. By Proposition 2.1, $v'>0$. Therefore, Eq. (2.2) can be transformed to

$$({v'}^{\lambda-2})'=\frac{2-\lambda}{\lambda(\lambda-1)}\xi q'(v).$$

Integrating from ξ_0 to ξ, we get

$${v'}^{\lambda-2}(\xi)=\delta^{\lambda-2}+\frac{2-\lambda}{\lambda(\lambda-1)}\int_{\xi_0}^{\xi} sq'(v(s))\,\mathrm{d}s,$$

which implies v can be extended to 0 and satisfies $v'(0)\in R_+(p)$ when $\delta>0$ is sufficiently small. The proof is complete. □

Lemma 3.2. *Assume $\xi_1<\xi_2$, $a<b$ and $a,b\in R_+(p)$. There exists a solution of Eq.* (2.2) *with the boundary value*

$$v(\xi_1)=a,\quad v(\xi_2)=b. \tag{3.2}$$

Proof. We denote $v(\xi;\delta)$ the solution of Eq. (2.2) with the initial value

$$\begin{cases} v(\xi_1)=a,\\ v'(\xi_1)=\delta,\end{cases}$$

where $\delta>0$. From Lemma 3.1, we know that $v(\xi;\delta)$ is existent on $[\xi_1,+\infty)$ for sufficiently small $\delta>0$. By the continuity of solution to initial,

$$\lim_{\delta\to 0^+} v(\xi_2;\delta)=a.$$

Thus the set

$$E=\{\delta>0: v(\xi;\delta) \text{ is existent on } [\xi_1,\xi_2] \text{ and } v(\xi_2;\delta)\leqslant b\}$$

is not empty. We declare that E is bounded additionally.

(i) The case $\xi_1\geqslant 0$. Eq. (2.2) can be transformed to

$$({v'}^{\lambda-2})'=\frac{2-\lambda}{\lambda(\lambda-1)}\xi q'(v).$$

Integrating from ξ_1 to ξ, we get

$$v'^{\lambda-2}(\xi) = \delta^{\lambda-2} + \frac{2-\lambda}{\lambda(\lambda-1)} \int_{\xi_1}^{\xi} s q'(v(s))\,\mathrm{d}s.$$

Therefore, for sufficiently large $\delta > 0$, if $v(\xi;\delta)$ is existent on $[\xi_1, \xi_2]$, then $v(\xi_2;\delta) > b$. This implies E is bounded.

(ii) The case $\xi_1 < 0$. Since $v''(\xi;\delta) > 0$ for $\xi < 0$ and $v(\xi;\delta)$ is strictly increasing,

$$\left[\frac{b-a}{\min\{\xi_2, 0\} - \xi_1}, +\infty\right) \cap E = \emptyset,$$

which implies E is bounded too.

Let

$$\delta_0 = \sup\, E.$$

Then $0 < \delta_0 < +\infty$. By the continuity of solution to initial, $\delta_0 \in E$. Assume $v(\xi_2;\delta_0) < b$. By Lemma 2.2 and the continuity of solution to initial, $v(\xi;\delta_0+\varepsilon)$ is existent on $[\xi_1, \xi_2]$ and $v(\xi_2;\delta_0+\varepsilon) < b$ for sufficiently small $\varepsilon > 0$, which contradicts the definition of δ_0. Hence $v(\xi_2;\delta_0) = b$. This implies that $v(\xi;\delta_0)$ is a solution of the problem (2.2) and (3.2). The proof is complete. $\square$

Now we can study existence, uniqueness and asymptotic behavior of solutions of problems (2.2) and (2.3) for the case without degeneracy.

Proposition 3.1. *Assume $v_- < v_+$. There exists at least one solution of problems* (2.2) *and* (2.3).

Proof. We denote $v_0(\xi)$ the solution of Eq. (2.2) with the boundary value

$$v(0) = v_-, \quad v(1) = v_+,$$

and $v_n(\xi)$ the solution of Eq. (2.2) with the boundary value

$$v(-n) = v_-, \quad v(n) = v_+,$$

where n is a positive integer. By Lemma 3.2, v_0 and v_n are existent. We declare that $v_n'(0) \leqslant v_0'(0)$ for each n. Otherwise, for some $n \geqslant 1$,

$$v_n'(0) > v_0'(0).$$

Since $v_n(0) > a = v_0(0)$, Lemma 2.2 implies that

$$v_n'(\xi) > v_0'(\xi), \quad \forall \xi \in [0,1]$$

which contradicts that $v_n(0) > a = v_0(0)$ and $v_n(1) \leqslant b = v_0(1)$. By Lemma 2.1,

$$v_n'(\xi) \leqslant ({v_n'}^{\lambda-2}(0) + C\xi^2)^{1/(\lambda-2)} \leqslant ({v_0'}^{\lambda-2}(0) + C\xi^2)^{1/(\lambda-2)}, \quad \forall \xi \in [-n, +n], \tag{3.3}$$

where $C > 0$ is a constant independent of n. This implies $\{v_n\}$, $\{v'_n\}$ and $\{v''_n\}$ are all uniformly bounded. Therefore, $\{v_n\}$ and $\{v'_n\}$ are both uniformly bounded and equi-continuous on any compact subset of $(-\infty, +\infty)$. By Arzela–Ascoli's theorem, there exists a subsequence, denoted by $\{v_{n_k}\}$, and a function $v \in C^1(-\infty, +\infty)$ such that $\{v_{n_k}\}$ and $\{v'_{n_k}\}$ converge uniformly to v and v' on any compact subset of $(-\infty, +\infty)$, respectively. Since v_{n_k} satisfy Eq. (2.2), $v \in C^2(-\infty, +\infty)$ and satisfies Eq. (2.2). From (3.3) and the boundary value of v_n, it is easy to see that

$$\lim_{\xi\to-\infty} v(\xi) = v_-, \qquad \lim_{\xi\to+\infty} v(\xi) = v_+.$$

Thus, v is a solution of problems (2.2) and (2.3). The proof is complete. □

Proposition 3.2. *Assume $v_- < v_+$. Problems* (2.2) *and* (2.3) *admit at most one solution.*

Proof. We argue by contradiction. Assume v_1 and v_2 are two solutions of problems (2.2) and (2.3), and there exists $\xi_0 \in (-\infty, +\infty)$ such that

$$v_1(\xi_0) > v_2(\xi_0).$$

For $\xi_1 < \xi_0 < \xi_2$, integrating (2.1) from ξ_1 to ξ_2, we get

$$v_i'^{\lambda-1}\Big|_{\xi_1}^{\xi_2} = -\frac{1}{\lambda}\int_{\xi_1}^{\xi_2} \xi(q(v_i))'\,\mathrm{d}\xi = -\frac{1}{\lambda}\xi q(v_i)\Big|_{\xi_1}^{\xi_2} + \frac{1}{\lambda}\int_{\xi_1}^{\xi_2} q(v_i)\,\mathrm{d}\xi \quad (i = 1, 2). \tag{3.4}$$

Since v_i $(i = 1, 2)$ are solutions of problems (2.2) and (2.3),

$$\begin{aligned}\xi_1 q(v_i(\xi_1)) &= \xi_1 q(v_i(-\infty)) + \xi_1 \int_{-\infty}^{\xi_1} q'(v_i)v_i'\,\mathrm{d}\xi \\ &= \xi_1 q(v_-) + \xi_1 \int_{-\infty}^{\xi_1} q'(v_i)v_i'\,\mathrm{d}\xi \quad (i = 1, 2)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\xi_2 q(v_i(\xi_2)) &= \xi_2 q(v_i(+\infty)) - \xi_2 \int_{\xi_2}^{+\infty} q'(v_i)v_i'\,\mathrm{d}\xi \\ &= \xi_2 q(v_+) - \xi_2 \int_{\xi_2}^{+\infty} q'(v_i)v_i'\,\mathrm{d}\xi \quad (i = 1, 2).\end{aligned} \tag{3.6}$$

We distinguish four cases to complete the proof.

(i) If

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (-\infty, +\infty).$$

From (3.4)–(3.6), we can get

$$v_1'^{\lambda-1}\Big|_{\xi_1}^{\xi_2} - v_2'^{\lambda-1}\Big|_{\xi_1}^{\xi_2} = \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_1)v_1'\,\mathrm{d}\xi + \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_1)v_1'\,\mathrm{d}\xi - \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_2)v_2'\,\mathrm{d}\xi - \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_2)v_2'\,\mathrm{d}\xi + \frac{1}{\lambda}\int_{\xi_1}^{\xi_2}(q(v_1)-q(v_2))\,\mathrm{d}\xi.$$

Since $v_1(\xi) > v_2(\xi)$ in $(-\infty, +\infty)$ and $q'(s) > 0$ for $s \in R_+(p)$, for $\xi_1 \leqslant \xi_0 - 1$ and $\xi_2 \geqslant \xi_0 + 1$, we have

$$\begin{aligned} & v_1'^{\lambda-1}\Big|_{\xi_1}^{\xi_2} - v_2'^{\lambda-1}\Big|_{\xi_1}^{\xi_2} - \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_1)v_1'\,\mathrm{d}\xi - \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_1)v_1'\,\mathrm{d}\xi \\ & + \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_2)v_2'\,\mathrm{d}\xi + \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_2)v_2'\,\mathrm{d}\xi \\ & = \frac{1}{\lambda}\int_{\xi_1}^{\xi_2}(q(v_1)-q(v_2))\,\mathrm{d}\xi \\ & \geqslant \frac{1}{\lambda}\int_{\xi_0-1}^{\xi_0+1}(q(v_1)-q(v_2))\,\mathrm{d}\xi. \end{aligned}$$

Letting $\xi_1 \to -\infty$ and $\xi_2 \to +\infty$, and noticing that each term in the left converges to 0 from Lemma 2.1, we see that

$$0 \geqslant \frac{1}{\lambda}\int_{\xi_0-1}^{\xi_0+1}(q(v_1)-q(v_2))\,\mathrm{d}\xi,$$

which contradicts that $v_1(\xi) > v_2(\xi)$ in $(-\infty, +\infty)$ and $q'(s) > 0$ for $s \in R_+(p)$.

(ii) If

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (-\infty, \xi_0),$$

but

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (\xi_0, +\infty)$$

is invalid. Let

$$\xi^* = \inf\{\xi > \xi_0 : v_1(\xi) \leqslant v_2(\xi)\}.$$

Then $\xi^* > \xi_0$, $v_1(\xi^*) = v_2(\xi^*)$ and $v_1(\xi) > v_2(\xi)$ in $(-\infty, \xi^*)$. We first prove that $v_1'(\xi^*) = v_2'(\xi^*)$. On the one hand, if $v_1'(\xi^*) > v_2'(\xi^*)$, then Lemma 2.2 implies that

$$v_1'(\xi) > v_2'(\xi), \quad \forall \xi \geqslant \xi^*,$$

which contradicts that $v_1(\xi^*) = v_2(\xi^*)$ and $v_1(+\infty) = v_2(+\infty) = v_+$. On the other hand, if $v_1'(\xi^*) < v_2'(\xi^*)$, then Lemma 2.2 implies that

$$v_1'(\xi) < v_2'(\xi), \quad \forall \xi \geqslant \xi^*,$$

which contradicts that $v_1(\xi^*) = v_2(\xi^*)$ and $v_1(+\infty) = v_2(+\infty) = v_+$ too. Hence $v_1'(\xi^*) = v_2'(\xi^*)$.

Choosing $\xi_2 = \xi^*$ in (3.4), by (3.5) we get

$$-{v_1'}^{\lambda-1}(\xi_1) + {v_2'}^{\lambda-1}(\xi_1) = \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_1)v_1'\,\mathrm{d}\xi - \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_2)v_2'\,\mathrm{d}\xi + \frac{1}{\lambda}\int_{\xi_1}^{\xi^*}(q(v_1) - q(v_2))\,\mathrm{d}\xi.$$

Since $v_1(\xi) > v_2(\xi)$ in $(-\infty, \xi^*)$ and $q'(s) > 0$ for $s \in R_+(p)$, we have

$$\begin{aligned}
&-{v_1'}^{\lambda-1}(\xi_1) + {v_2'}^{\lambda-1}(\xi_1) - \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_1)v_1'\,\mathrm{d}\xi + \frac{\xi_1}{\lambda}\int_{-\infty}^{\xi_1} q'(v_2)v_2'\,\mathrm{d}\xi \\
&= \frac{1}{\lambda}\int_{\xi_1}^{\xi^*}(q(v_1) - q(v_2))\,\mathrm{d}\xi \\
&\geqslant \frac{1}{\lambda}\int_{\xi_0}^{\xi^*}(q(v_1) - q(v_2))\,\mathrm{d}\xi.
\end{aligned}$$

Letting $\xi_1 \to -\infty$, we see that

$$0 \geqslant \frac{1}{\lambda}\int_{\xi_0}^{\xi^*}(q(v_1) - q(v_2))\,\mathrm{d}\xi,$$

which contradicts that $v_1(\xi) > v_2(\xi)$ in $(-\infty, \xi^*)$ and $q'(s) > 0$ for $s \in R_+(p)$.

(iii) If

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (\xi_0, +\infty),$$

but

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (-\infty, \xi_0)$$

is invalid. Let

$$\xi_* = \sup\{\xi < \xi_0 : v_1(\xi) \leqslant v_2(\xi)\}.$$

Then $\xi_* < \xi_0$, $v_1(\xi_*) = v_2(\xi_*)$ and $v_1(\xi) > v_2(\xi)$ in $(\xi_*, +\infty)$. Similar to (ii), we can prove that $v_1'(\xi_*) = v_2'(\xi_*)$ by Lemma 2.2.

Choosing $\xi_1 = \xi_*$ in (3.4), by (3.6) we get

$${v_1'}^{\lambda-1}(\xi_2) - {v_2'}^{\lambda-1}(\xi_2) = \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_1)v_1'\,\mathrm{d}\xi - \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_2)v_2'\,\mathrm{d}\xi + \frac{1}{\lambda}\int_{\xi_*}^{\xi_2}(q(v_1) - q(v_2))\,\mathrm{d}\xi.$$

Since $v_1(\xi) > v_2(\xi)$ in $(\xi_*, +\infty)$ and $q'(s) > 0$ for $s \in R_+(p)$, we have

$$\begin{aligned} &{v_1'}^{\lambda-1}(\xi_2) - {v_2'}^{\lambda-1}(\xi_2) - \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_1)v_1'\,\mathrm{d}\xi + \frac{\xi_2}{\lambda}\int_{\xi_2}^{+\infty} q'(v_2)v_2'\,\mathrm{d}\xi \\ &\quad = \frac{1}{\lambda}\int_{\xi_*}^{\xi_2} (q(v_1) - q(v_2))\,\mathrm{d}\xi \\ &\quad \geqslant \frac{1}{\lambda}\int_{\xi_*}^{\xi_0} (q(v_1) - q(v_2))\,\mathrm{d}\xi. \end{aligned}$$

Letting $\xi_2 \to +\infty$, we see that

$$0 \geqslant \frac{1}{\lambda}\int_{\xi_*}^{\xi_0} (q(v_1) - q(v_2))\,\mathrm{d}\xi,$$

which contradicts that $v_1(\xi) > v_2(\xi)$ in $(\xi_*, +\infty)$ and $q'(s) > 0$ for $s \in R_+(p)$.

(iv) If

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (-\infty, \xi_0)$$

and

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in (\xi_0, +\infty)$$

are both invalid. Let

$$\xi_* = \sup\{\xi < \xi_0 : v_1(\xi) \leqslant v_2(\xi)\}, \quad \xi^* = \inf\{\xi > \xi_0 : v_1(\xi) \leqslant v_2(\xi)\}.$$

Then $\xi_* < \xi_0 < \xi^*$, $v_1(\xi_*) = v_2(\xi_*)$, $v_1(\xi^*) = v_2(\xi^*)$ and $v_1(\xi) > v_2(\xi)$ in (ξ_*, ξ^*). Similar to (ii), we can prove that $v_1'(\xi_*) = v_2'(\xi_*)$ and $v_1'(\xi^*) = v_2'(\xi^*)$ by Lemma 2.2.

Choosing $\xi_1 = \xi_*$ and $\xi_2 = \xi^*$ in (3.4), we get

$$0 = \frac{1}{\lambda}\int_{\xi_*}^{\xi^*} (q(v_1) - q(v_2))\,\mathrm{d}\xi,$$

which contradicts that $v_1(\xi) > v_2(\xi)$ in (ξ_*, ξ^*) and $q'(s) > 0$ for $s \in R_+(p)$. The proof is complete. □

Proposition 3.3. *Assume $v_- < v_+$ and v is the solution of problems* (2.2) *and* (2.3). *Then v is strictly increasing and satisfies*

$$\lim_{\xi\to-\infty} |\xi|^{\lambda/(2-\lambda)}(v(\xi) - v_-) = \frac{2-\lambda}{\lambda}\left(\frac{2\lambda(\lambda-1)}{(2-\lambda)q'(v_-)}\right)^{1/(2-\lambda)}, \tag{3.7}$$

$$\lim_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(v_+ - v(\xi)) = \frac{2-\lambda}{\lambda}\left(\frac{2\lambda(\lambda-1)}{(2-\lambda)q'(v_+)}\right)^{1/(2-\lambda)}. \tag{3.8}$$

Proof. The monotone property of v can be obtained directly from Proposition 2.1. The proof of (3.7) and (3.8) is similar. Here we only prove (3.8). Since $q \in C^1(R_+(p))$ and

q' is monotone decreasing, for any $\varepsilon > 0$, there exists a constant $a \in (v_-, v_+)$ such that $q'(v_+) \leqslant q'(a) \leqslant q'(v_+) + \varepsilon$. Choose $\xi_0 > 0$ such that $v(\xi_0) \geqslant a$. Since $v' > 0$, Eq. (2.2) can be transformed to

$$({v'}^{\lambda-2})' = \frac{2-\lambda}{\lambda(\lambda-1)} \xi q'(v).$$

Integrating from ξ_0 to ξ, we get

$${v'}^{\lambda-2}(\xi) = {v'}^{\lambda-2}(\xi_0) + \frac{2-\lambda}{\lambda(\lambda-1)} \int_{\xi_0}^{\xi} s q'(v(s))\, \mathrm{d}s, \quad \forall \xi > \xi_0.$$

Since $v' > 0$,

$$a \leqslant v(\xi_0) < v(\xi) < v_+, \quad \forall \xi > \xi_0.$$

Thus

$$0 < q'(v_+) \leqslant q'(v(\xi)) \leqslant q'(a) \leqslant q'(v_+) + \varepsilon, \quad \forall \xi > \xi_0.$$

Therefore,

$$(C_1 + \theta_1 \xi^2)^{1/(\lambda-2)} \leqslant v'(\xi) \leqslant (C_2 + \theta_2 \xi^2)^{1/(\lambda-2)}, \quad \forall \xi > \xi_0,$$

where

$$\theta_1 = \frac{(2-\lambda)(q'(v_+) + \varepsilon)}{2\lambda(\lambda-1)}, \quad C_1 = {v'}^{\lambda-2}(\xi_0) - \theta_1 \xi_0^2,$$

$$\theta_2 = \frac{(2-\lambda) q'(v_+)}{2\lambda(\lambda-1)}, \quad C_2 = {v'}^{\lambda-2}(\xi_0) - \theta_2 \xi_0^2.$$

Hence for any $\xi > \xi_0$,

$$\int_{\xi}^{\infty} (C_1 + \theta_1 s^2)^{1/(\lambda-2)}\, \mathrm{d}s \leqslant v_+ - v(\xi) \leqslant \int_{\xi}^{\infty} (C_2 + \theta_2 s^2)^{1/(\lambda-2)}\, \mathrm{d}s.$$

By L'Hospital's Rule, we see that

$$\begin{aligned}
&\lim_{\xi \to +\infty} \xi^{\lambda/(2-\lambda)} \int_{\xi}^{\infty} (C_i + \theta_i s^2)^{1/(\lambda-2)}\, \mathrm{d}s \\
&\quad = \frac{2-\lambda}{\lambda} \lim_{\xi \to +\infty} \xi^{2/(2-\lambda)} (C_i + \theta_i \xi^2)^{1/(\lambda-2)} \\
&\quad = \frac{2-\lambda}{\lambda} \theta_i^{1/(\lambda-2)} \quad (i = 1, 2).
\end{aligned}$$

Therefore,

$$\lim_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)} \int_{\xi}^{\infty} (C_1+\theta_1 s^2)^{1/(\lambda-2)}\,\mathrm{d}s$$
$$\leqslant \underline{\lim}_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(v_+ - v(\xi))$$
$$\leqslant \overline{\lim}_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(v_+ - v(\xi))$$
$$\leqslant \lim_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)} \int_{\xi}^{\infty} (C_2+\theta_2 s^2)^{1/(\lambda-2)}\,\mathrm{d}s,$$

i.e.

$$\frac{2-\lambda}{\lambda}\left(\frac{(2-\lambda)(q'(v_+)+\varepsilon)}{2\lambda(\lambda-1)}\right)^{1/(\lambda-2)}$$
$$\leqslant \underline{\lim}_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(v_+ - v(\xi))$$
$$\leqslant \overline{\lim}_{\xi\to+\infty} \xi^{\lambda/(2-\lambda)}(v_+ - v(\xi))$$
$$\leqslant \frac{2-\lambda}{\lambda}\left(\frac{(2-\lambda)q'(v_+)}{2\lambda(\lambda-1)}\right)^{1/(\lambda-2)}.$$

Owing to the arbitrariness of $\varepsilon > 0$, we get (3.8). The proof is complete. □

Since problems (1.2) and (1.3) are equivalent to problems (2.2) and (2.3), we get Theorems 1 and 2 with $0 < w_- < w_+$ from Proposition 3.1–3.3 directly.

4. The case with possible degeneracy

In this section, we consider the case with degeneracy, i.e. $v_\pm \in R(p)$ and $v_- = p(0)$. By Proposition 2.1, problems (2.2) and (2.3) have a unique solution for $v_- = v_+ = p(0)$. Therefore, we need only investigate the case $v_- = p(0)$ and $v_+ \in R_+(p)$. Without generality, we assume $v_- = p(0) = 0$ and $v_+ = p(w_+) = 1$, and investigate the infinite two-point boundary value problem of (2.2) with

$$v(-\infty) = 0, \quad v(+\infty) = 1. \tag{4.1}$$

Proposition 4.1. *There exists at least one solution of problems* (2.2) *and* (4.1).

Proof. We denote $v_n(\xi)$ the solution of Eq. (2.2) with the boundary value

$$v(-\infty) = \frac{1}{n}, \quad v(+\infty) = 1,$$

where n is a positive integer. By Proposition 3.1, Lemma 2.2 and the proof of Proposition 3.2, v_n are existent and

$$0 < v_{n+1}(\xi) < v_n(\xi) < 1, \quad \forall \xi \in (-\infty, +\infty) \quad (n = 1, 2, \ldots).$$

Let

$$v(\xi)=\lim_{n\to\infty} v_n(\xi), \quad \forall \xi\in(-\infty,+\infty).$$

Since $0<v_n<1$ and v_n are strictly increasing in $(-\infty,+\infty)$, $0\leqslant v<1$ and v is monotone increasing in $(-\infty,+\infty)$.

Now we show that v is a solution of problems (2.2) and (4.1). Since $v_n'>0$ in $(-\infty,+\infty)$ and $v_n''>0$ in $(-\infty,0)$,

$$0<v_n'(-1)<1 \quad (n=1,2,\ldots).$$

For any $\xi_1<\xi_2$, integrating Eq. (2.1) with $v=v_n$ from ξ_1 to ξ_2, we get

$$v_n'^{\,\lambda-1}\Big|_{\xi_1}^{\xi_2}=-\frac{1}{\lambda}\int_{\xi_1}^{\xi_2}\xi(q(v_n))'\,\mathrm{d}\xi=\frac{1}{\lambda}\int_{\xi_1}^{\xi_2}q(v_n)\,\mathrm{d}\xi-\frac{1}{\lambda}\xi q(v_n)\Big|_{\xi_1}^{\xi_2}. \tag{4.2}$$

Choosing $\xi_1=-1$ and $\xi_2=0$, we get

$$v_n'^{\,\lambda-1}(0)=v_n'^{\,\lambda-1}(-1)+\frac{1}{\lambda}\int_{-1}^{0}q(v_n)\,\mathrm{d}\xi-\frac{1}{\lambda}q(v_n(-1))\leqslant 1+\frac{q(1)}{\lambda}.$$

By Lemma 2.1,

$$0<v_n'(\xi)\leqslant C(1+\xi^2)^{1/(\lambda-2)}, \quad \forall \xi\in(-\infty,+\infty), \tag{4.3}$$

where $C>0$ is a constant independent of n. This implies that v_n are uniformly bounded and equi-continuous in $(-\infty,+\infty)$. Therefore, $v\in C(-\infty,+\infty)$ and

$$v(-\infty)=0, \quad v(+\infty)=1.$$

(i) Assume $v(\xi_0)>0$ for some $\xi_0\in(-\infty,+\infty)$. Then

$$v_n(\xi)>v(\xi)\geqslant v(\xi_0)>0, \quad \forall \xi\geqslant\xi_0 \ \ (n=1,2,\ldots). \tag{4.4}$$

For any $\xi_2>\xi_1\geqslant\xi_0$, from (4.2) we get

$$\begin{aligned}|v_n'^{\,\lambda-1}(\xi_2)-v_n'^{\,\lambda-1}(\xi_1)|&=\frac{1}{\lambda}\left|\int_{\xi_1}^{\xi_2}\xi(q(v_n))'\,\mathrm{d}\xi\right|\\&\leqslant\frac{\max\{|\xi_1|,|\xi_2|\}}{\lambda}\int_{\xi_1}^{\xi_2}|(q(v_n))'|\,\mathrm{d}\xi\\&=\frac{\max\{|\xi_1|,|\xi_2|\}}{\lambda}\int_{\xi_1}^{\xi_2}(q(v_n))'\,\mathrm{d}\xi\\&\leqslant\frac{\max\{|\xi_1|,|\xi_2|\}}{\lambda}(q(v_n(\xi_2))-q(v_n(\xi_1)))\\&\leqslant\frac{\max\{|\xi_1|,|\xi_2|\}}{\lambda}q'(v(\xi_0))v'(0)(\xi_2-\xi_1).\end{aligned}$$

This estimate and (4.3) imply that v_n' are uniformly bounded and equi-continuous on any compact subset of $[\xi_0, +\infty)$. Therefore, $v \in C^1([\xi_0, +\infty))$. Noticing (4.4), we see that $v \in C^2([\xi_0, +\infty))$ and satisfies Eq. (2.2) on $[\xi_0, +\infty)$ since v_n satisfy Eq. (2.2) in $(-\infty, +\infty)$.

(ii) Assume $v(\xi_0) = 0$ for some $\xi_0 \in (-\infty, +\infty)$. Then $v(\xi) \equiv 0$ on $(-\infty, \xi_0]$ by Proposition 2.1. Thus $v \in C^2((-\infty, \xi_0])$ and satisfies Eq. (2.2) on $(-\infty, \xi_0]$.

(iii) Assume there is $\xi_0 \in (-\infty, +\infty)$ such that $v(\xi_0) = 0$ and $v(\xi) > 0$ for all $\xi > \xi_0$. Choosing $\xi_1 = \xi_0$ in (4.2), we see that for any $\xi_2 > \xi_0$,

$$v_n'^{\,\lambda-1}(\xi_2) - v_n'^{\,\lambda-1}(\xi_0) = \frac{1}{\lambda}\int_{\xi_0}^{\xi_2} q(v_n)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi_2 q(v_n(\xi_2)) + \frac{1}{\lambda}\xi_0 q(v_n(\xi_0)).$$

Letting $n \to \infty$, we get

$$v'^{\lambda-1}(\xi_2) = \frac{1}{\lambda}\int_{\xi_0}^{\xi_2} q(v)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi_2 q(v(\xi_2)), \quad \forall \xi_2 > \xi_0.$$

Letting $\xi_2 \to \xi_0^+$, we see that $\lim_{\xi \to \xi_0^+} v'(\xi) = 0$.

From the above discussion, we see that $v \in C^1(-\infty, +\infty)$ and v is a solution of problems (2.2) and (4.1). The proof is complete. □

Lemma 4.1. *Let v be a solution of problems* (2.2) *and* (4.1) *and*

$$\hat{\xi} = \inf\{\xi \in (-\infty, +\infty) : v(\xi) > 0\}.$$

Then v is monotone increasing in $(-\infty, +\infty)$ and $\hat{\xi} < 0$. In addition, for $\hat{\xi}$ and $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s$ we have the following two cases:

(i) *If $\hat{\xi} > -\infty$, then $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s < +\infty$.*
(ii) *If $\hat{\xi} = -\infty$, then $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$.*

Proof. From (2.2) and Proposition 2.1, v is monotone increasing and $v'(\xi) \leqslant v'(0)$ for all $\xi \in (-\infty, +\infty)$. Hence $\hat{\xi} < 0$.

(i) The case $\hat{\xi} > -\infty$. For any $\hat{\xi} < \xi < 0$, integrating (2.1) from $\hat{\xi}$ to ξ, we get

$$v'^{\lambda-1}\Big|_{\hat{\xi}}^{\xi} = -\frac{1}{\lambda}\int_{\hat{\xi}}^{\xi} s(q(v(s)))'\,\mathrm{d}s.$$

Thus

$$\begin{aligned} v'^{\lambda-1}(\xi) &= -\frac{1}{\lambda}\int_{\hat{\xi}}^{\xi} s(q(v(s)))'\,\mathrm{d}s \\ &\leqslant \frac{|\hat{\xi}|}{\lambda}\int_{\hat{\xi}}^{\xi} |(q(v(s)))'|\,\mathrm{d}s = \frac{|\hat{\xi}|}{\lambda}\int_{\hat{\xi}}^{\xi}(q(v(s)))'\,\mathrm{d}s = \frac{|\hat{\xi}|}{\lambda} q(v(\xi)). \end{aligned}$$

Therefore,

$$q^{-1/(\lambda-1)}(v(\xi))v'(\xi) \leqslant \left(\frac{|\hat{\xi}|}{\lambda}\right)^{1/(\lambda-1)}, \quad \forall \xi \in (\hat{\xi}, 0).$$

Integrating from $\hat{\xi}$ to 0, we get

$$\int_0^{v(0)} q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = \int_{\hat{\xi}}^0 q^{-1/(\lambda-1)}(v(\xi))v'(\xi)\,\mathrm{d}\xi \leqslant \left(\frac{|\hat{\xi}|}{\lambda}\right)^{1/(\lambda-1)} |\hat{\xi}|.$$

Therefore, $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s < +\infty$.

(ii) The case $\hat{\xi} = -\infty$. For any $\xi_1 < \xi < 0$, integrating (2.1) from ξ_1 to ξ, we get

$$v'^{\lambda-1}\Big|_{\xi_1}^{\xi} = -\frac{1}{\lambda}\int_{\xi_1}^{\xi} s(q(v(s)))'\,\mathrm{d}s.$$

Letting $\xi_1 \to -\infty$, we get

$$\begin{aligned} v'^{\lambda-1}(\xi) &= -\frac{1}{\lambda}\int_{-\infty}^{\xi} s(q(v(s)))'\,\mathrm{d}s \\ &= -\frac{1}{\lambda}\int_{-\infty}^{\xi} s|(q(v(s)))'|\,\mathrm{d}s \\ &\geqslant \frac{|\xi|}{\lambda}\int_{-\infty}^{\xi} (q(v(s)))'\,\mathrm{d}s = \frac{|\xi|}{\lambda} q(v(\xi)). \end{aligned} \tag{4.5}$$

Therefore,

$$q^{-1/(\lambda-1)}(v(\xi))v'(\xi) \geqslant \left(\frac{|\xi|}{\lambda}\right)^{1/(\lambda-1)}, \quad \forall \xi < 0.$$

Integrating from $-\infty$ to 0, we get

$$\begin{aligned} \int_0^{v(0)} q^{-1/(\lambda-1)}(s)\,\mathrm{d}s &= \int_{-\infty}^0 q^{-1/(\lambda-1)}(v(\xi))v'(\xi)\,\mathrm{d}\xi \\ &\geqslant \int_{-\infty}^0 \left(\frac{|\xi|}{\lambda}\right)^{1/(\lambda-1)} \mathrm{d}\xi = +\infty. \end{aligned}$$

Therefore, $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$. The proof is complete. □

Proposition 4.2. *Problems* (2.2) *and* (4.1) *admit at most one solution.*

Proof. Assume v_1 and v_2 are two different solutions of problems (2.2) and (4.1). If $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$, then $v_1(\xi), v_2(\xi) \in R_+(p)$ for all $\xi \in (-\infty, +\infty)$ by Lemma 4.1. Letting $\xi \to -\infty$ in (4.5), by Lemma 2.1, we get

$$\lim_{\xi \to -\infty} |\xi| q(v(\xi)) = 0,$$

substituting which for (3.5), we can show that $v_1 \equiv v_2$ is similar to the proof of Proposition 3.2.

Now we assume $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s < +\infty$. By Lemma 4.1, there exists $\hat{\xi}_i \in (-\infty, 0)$ such that $v_i(\xi) > 0$ in $(\hat{\xi}_i, +\infty)$ while $v_i \equiv 0$ on $(-\infty, \hat{\xi}_i]$ for $i = 1, 2$. If $\hat{\xi}_1 = \hat{\xi}_2$, the same argument as the proof of Proposition 3.2 (iii) can show that $v_1 \equiv v_2$. Therefore, we just need to consider the case $\hat{\xi}_1 \neq \hat{\xi}_2$ and assume $\hat{\xi}_1 < \hat{\xi}_2$ without generality. We declare that

$$v_1(\xi) > v_2(\xi), \quad \forall \xi \in [\hat{\xi}_2, 0]. \tag{4.6}$$

Otherwise, let

$$\xi^* = \inf\{\xi \in [\hat{\xi}_2, 0] : v_1(\xi) \leqslant v_2(\xi)\}.$$

Then $\hat{\xi}_2 < \xi^* \leqslant 0$, $v_1(\xi^*) = v_2(\xi^*)$ and $v_1(\xi) > v_2(\xi)$ for all $\xi \in [\hat{\xi}_2, \xi^*)$. Integrating Eq. (2.1) with $v = v_i$ from $\hat{\xi}_i$ to ξ^*, we get

$$v_i'^{\lambda-1}\Big|_{\hat{\xi}_i}^{\xi^*} = -\frac{1}{\lambda}\int_{\hat{\xi}_i}^{\xi^*} \xi(q(v_i))'\,\mathrm{d}\xi = \frac{1}{\lambda}\int_{\hat{\xi}_i}^{\xi^*} q(v_i)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v_i)\Big|_{\hat{\xi}_i}^{\xi^*} \quad (i = 1, 2).$$

So

$$v_1'^{\lambda-1}(\xi^*) - v_2'^{\lambda-1}(\xi^*) = \frac{1}{\lambda}\int_{\hat{\xi}_1}^{\xi^*} q(v_1)\,\mathrm{d}\xi - \frac{1}{\lambda}\int_{\hat{\xi}_2}^{\xi^*} q(v_2)\,\mathrm{d}\xi > 0,$$

i.e. $v_1'(\xi^*) > v_2'(\xi^*)$, which contradict that $v_1(\xi^*) = v_2(\xi^*)$ and $v_1(\xi) > v_2(\xi)$ for all $\xi \in [\hat{\xi}_2, \xi^*)$.

Integrating Eq. (2.1) with $v = v_i$ from $\hat{\xi}_i$ to 0, we get

$$v_i'^{\lambda-1}\Big|_{\hat{\xi}_i}^{0} = -\frac{1}{\lambda}\int_{\hat{\xi}_i}^{0} \xi(q(v_i))'\,\mathrm{d}\xi = \frac{1}{\lambda}\int_{\hat{\xi}_i}^{0} q(v_i)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v_i)\Big|_{\hat{\xi}_i}^{0} \quad (i = 1, 2).$$

Hence

$$v_1'^{\lambda-1}(0) = \frac{1}{\lambda}\int_{\hat{\xi}_1}^{0} q(v_1)\,\mathrm{d}\xi, \quad v_2'^{\lambda-1}(0) = \frac{1}{\lambda}\int_{\hat{\xi}_2}^{0} q(v_2)\,\mathrm{d}\xi.$$

From (4.6), $v_1'(0) > v_2'(0)$. By Lemma 2.2, we get

$$v_1'(\xi) > v_2'(\xi), \quad \forall \xi \geqslant 0,$$

which contradicts that $v_1(+\infty) = v_2(+\infty) = 1$. The proof is complete. □

Proposition 4.3. *Assume v is the solution of problems* (2.2) *and* (4.1).

(i) *If $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s < +\infty$, then there exists $-\infty < \hat{\xi} < 0$ such that $v(\xi) > 0$ is strictly increasing in $(\hat{\xi}, +\infty)$, while $v(\xi) \equiv 0$ on $(-\infty, \hat{\xi}]$. In addition,*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\int_0^{v(\xi)} q^{-1/(\lambda-1)}(s)\,\mathrm{d}s\right)\Bigg|_{\xi=\hat{\xi}^+} = \left(\frac{-\hat{\xi}}{\lambda}\right)^{1/(\lambda-1)} > 0. \tag{4.7}$$

(ii) *If* $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$, *then* $v(\xi) > 0$ *is strictly increasing in* $(-\infty, +\infty)$ *and*

$$\lim_{\xi \to -\infty} |\xi| q(v(\xi)) = 0, \qquad \lim_{\xi \to -\infty} |\xi|^{\lambda/(2-\lambda)} v(\xi) = 0. \tag{4.8}$$

Furthermore, for both cases, we have

$$\lim_{\xi \to +\infty} \xi^{\lambda/(2-\lambda)}(v_+ - v(\xi)) = \frac{2-\lambda}{\lambda}\left(\frac{2\lambda(\lambda-1)}{(2-\lambda)q'(v_+)}\right)^{1/(2-\lambda)}. \tag{4.9}$$

Proof. By Lemma 4.1 and Proposition 2.1, we need only prove (4.7)–(4.9).

We first prove (4.7). For any $\xi > \hat{\xi}$, integrating Eq. (2.1) from $\hat{\xi}$ to ξ, we get

$$v'^{\lambda-1}\Big|_{\hat{\xi}}^{\xi} = -\frac{1}{\lambda}\int_{\hat{\xi}}^{\xi} s(q(v(s)))'\,\mathrm{d}s = \frac{1}{\lambda}\int_{\hat{\xi}}^{\xi} q(v(s))\,\mathrm{d}s - \frac{1}{\lambda} s q(v(s))\Big|_{\hat{\xi}}^{\xi},$$

i.e.

$$v'^{\lambda-1}(\xi) = \frac{1}{\lambda}\int_{\hat{\xi}}^{\xi} q(v(s))\,\mathrm{d}s - \frac{1}{\lambda}\xi q(v(\xi)), \quad \forall \xi > \hat{\xi}.$$

Therefore,

$$\frac{v'(\xi)}{q^{1/(\lambda-1)}(v(\xi))} = \left(\frac{1}{\lambda}\int_{\hat{\xi}}^{\xi} \frac{q(v(s))}{q(v(\xi))}\,\mathrm{d}s - \frac{1}{\lambda}\xi\right)^{1/(\lambda-1)}, \quad \forall \xi > \hat{\xi}.$$

Letting $\xi \to \hat{\xi}^+$ and noticing that $q(v(s)) < q(v(\xi))$ for all $\hat{\xi} < s < \xi$, we get (4.7).

Letting $\xi \to -\infty$ in (4.5), we get the former of (4.8) by Lemma 2.1. Now we show the latter. For any $\xi < 0$, similar to the proof of Lemma 2.1, we can get that

$$v'(s) \leqslant (v'^{\lambda-2}(0) + C(\xi)s^2)^{1/(\lambda-2)} \leqslant C^{1/(\lambda-2)}(\xi)(-s)^{2/(\lambda-2)}, \quad \forall s \leqslant \xi,$$

where $C(\xi) = \frac{(2-\lambda)q'(v(\xi))}{2\lambda(\lambda-1)} > 0$. Therefore,

$$\begin{aligned} v(\xi) = \int_{-\infty}^{\xi} v'(s)\,\mathrm{d}s &\leqslant C^{1/(\lambda-2)}(\xi)\int_{-\infty}^{\xi}(-s)^{2/(\lambda-2)}\,\mathrm{d}s \\ &= \frac{(2-\lambda)}{\lambda} C^{1/(\lambda-2)}(\xi)|\xi|^{-\lambda/(2-\lambda)}, \quad \forall \xi < 0, \end{aligned}$$

i.e.

$$|\xi|^{\lambda/(2-\lambda)} v(\xi) \leqslant \frac{(2-\lambda)}{\lambda} C^{1/(\lambda-2)}(\xi), \quad \forall \xi < 0. \tag{4.10}$$

Since $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$ and $q'(s) \in C(0, +\infty)$ is monotone decreasing, we get $\lim_{s \to 0^+} q'(s) = +\infty$. Hence

$$\lim_{\xi \to -\infty} C(\xi) = +\infty.$$

Letting $\xi \to -\infty$ in (4.10), we get the latter of (4.8).

The proof of (4.9) is the same as (3.8) and the proof of the proposition is complete. □

Since problems (1.2) and (1.3) are equivalent to problems (2.2) and (4.1), we get Theorems 1 and 3 with $0 = w_- < w_+$ from Proposition 4.1–4.3 directly. For Theorem 4, we only notice that

$$u(x,t) = w(xt^{-1/\lambda}), \quad -\infty < x < +\infty, \quad t > 0$$

is a solution of the Cauchy problem (1.1) and (1.4), where w is the solution of problems (1.2) and (1.3) with $w_- = u_-$ and $w_+ = u_+$.

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