

# A characterization of probability measure with finite moment and an application to the Boltzmann equation

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## Abstract

We characterize probability measure with finite moment of any order in terms of the symmetric difference operators of their Fourier transforms. By using our new characterization, we prove the continuity  $f(t, v) \in C((0, \infty); L^1_{2k-2+\alpha})$ , where  $f(t, v)$  stands for the density of unique measure-valued solution  $(F_t)_{t \geq 0}$  of the Cauchy problem for the homogeneous non-cutoff Boltzmann equation, with Maxwellian molecules, corresponding to a probability measure initial datum  $F_0$  satisfying

$$\int |v|^{2k-2+\alpha} dF_0(v) < \infty, \quad 0 \leq \alpha < 2, \quad k = 2, 3, 4, \dots,$$

provided that  $F_0$  is not a single Dirac mass.

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## 1. Introduction

The spatially homogeneous Boltzmann equation states

$$\partial_t f(t, v) = Q(f, f)(t, v), \quad (1.1)$$

where  $f(t, v)$  is the density distribution of particles with velocity  $v \in \mathbb{R}^3$  at time  $t$ . The most interesting and important part of this equation is the collision

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operator given on the right hand side that captures the change rates of the density distribution through elastic binary collision:

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*,$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma$$

for each  $\sigma \in \mathbb{S}^2$ , which follows from the conservation of momentum and energy,

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

Motivated by some physical models, we assume that the non-negative cross section  $B$  takes the form

$$B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where

$$\begin{aligned} \Phi(|z|) &= \Phi_\gamma(|z|) = |z|^\gamma \text{ for some } \gamma > -3 \text{ and} \\ b(\cos \theta)\theta^{2+\nu} &\rightarrow K \text{ as } \theta \rightarrow 0+ \text{ for some } 0 < \nu < 2 \text{ and } K > 0. \end{aligned} \quad (1.2)$$

Throughout this paper, we will only consider the case when

$$\gamma = 0, \quad 0 < \nu < 2,$$

which is called the Maxwellian molecule type cross section, because the analysis relies on a simpler form of the equation after taking Fourier transform in  $v$  by the Bobylev formula. As usual, we may restrict  $\theta \in [0, \pi/2]$  by considering *symmetrized* cross section

$$[b(\cos \theta) + b(\cos(\pi - \theta))]\mathbf{1}_{0 \leq \theta \leq \pi/2}$$

(see [16]). In what follows, we make a slightly general assumption on the cross section that

$$\sin^{\alpha_0}(\theta/2)b(\cos \theta) \sin \theta \in L^1((0, \pi/2]) \text{ for some } \alpha_0 \in (0, 2), \quad (1.3)$$

which is fulfilled for  $b$  with (1.2) if  $\nu < \alpha_0$ .

For  $k \in \mathbb{N}^+$ ,  $\alpha \in [0, 2)$ ,  $k + \alpha > 1$ , we denote by  $P_{2k-2+\alpha}(\mathbb{R}^d)$  the set of all probability measure  $F$  on  $\mathbb{R}^d$ ,  $d \geq 1$ , such that

$$\int_{\mathbb{R}^d} |v|^{2k-2+\alpha} dF(v) < \infty$$

and

$$\int_{\mathbb{R}^d} v_j dF(v) = 0, \quad j = 1, \dots, d, \quad (1.4)$$

when  $2k - 2 + \alpha \geq 1$ . In this paper, we consider the Cauchy problem of (1.1) with initial datum

$$f(0, v) = F_0 \in P_{2k-2+\alpha}. \quad (1.5)$$

Let  $\mathcal{F}$  denote the Fourier transform operator defined as

$$\mathcal{F}(F)(\xi) = \hat{F}(\xi) = \int_{\mathbb{R}^d} e^{-iv \cdot \xi} dF(v)$$

for each  $F \in P_0(\mathbb{R}^d)$ , called the characteristic function of  $F$ , and  $\mathcal{K} = \mathcal{F}(P_0(\mathbb{R}^d))$ . Inspired by a series of works by Toscani and his co-authors [5, 7, 14], Cannone-Karch introduced the space  $\mathcal{K}^\alpha$  for  $\alpha > 0$  defined as follows:

$$\mathcal{K}^\alpha = \{\varphi \in \mathcal{K}; \|\varphi - 1\|_\alpha < \infty\}, \quad (1.6)$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi \in \mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}. \quad (1.7)$$

The space  $\mathcal{K}^\alpha$  endowed with the distance

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi \in \mathbb{R}^d} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha} \quad (1.8)$$

is a complete metric space (see Proposition 3.10 of [4]). Moreover,  $\mathcal{K}^\alpha = \{1\}$  for all  $\alpha > 2$  and the following embeddings (Lemma 3.12 of [4]) hold

$$\{1\} \subset \mathcal{K}^\alpha \subset \mathcal{K}^\beta \subset \mathcal{K} \quad \text{for all } 2 \geq \alpha \geq \beta > 0.$$

With this classification on characteristic functions, the global existence of solution in  $\mathcal{K}^\alpha$  was studied in [4] (see also [9]) with the assumption (1.3). The space  $\mathcal{K}^\alpha$  arises in connection with the Fourier image of  $P_\alpha(\mathbb{R}^d)$  and it is proved  $\mathcal{F}(P_\alpha(\mathbb{R}^d)) \subset \mathcal{K}^\alpha$ . However, the inclusion is proper. Indeed, it is shown (see Remark 3.16 of [4]) that the function  $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$ , with  $\alpha \in (0, 2)$ , belongs to  $\mathcal{K}^\alpha$  but  $p_\alpha(v) = \mathcal{F}^{-1}(\varphi_\alpha)(v)$ , the density of  $\alpha$ -stable symmetric Lévy process, is not contained in  $P_\alpha(\mathbb{R}^d)$ . Hence, the solution obtained in the function space  $\mathcal{K}^\alpha$  does not represent the moment properties in physics even when it is assumed initially.

In order to capture the precise moment condition in the Fourier space, a precise classification on characteristic functions was introduced in [12] (see also [11]). Let

$$\widetilde{\mathcal{M}}^\alpha = \{\varphi \in \mathcal{K}; \|\operatorname{Re} \varphi - 1\|_{\mathcal{M}^\alpha} + \|\varphi - 1\|_\alpha < \infty\}, \quad \alpha \in (0, 2), \quad (1.9)$$

where

$$\|\varphi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^{d+\alpha}} d\xi, \quad (1.10)$$

and  $\operatorname{Re} \varphi$  stands for the real part of  $\varphi(\xi)$ . Clearly,  $\widetilde{\mathcal{M}}^\alpha \subset \mathcal{K}^\alpha$ . Moreover, it was shown in [12] that for any  $\alpha \in (0, 2)$ ,  $\widetilde{\mathcal{M}}^\alpha = \mathcal{F}(P_\alpha)$ .

For  $\varphi, \tilde{\varphi} \in \widetilde{\mathcal{M}}^\alpha$ , put

$$\|\varphi - \tilde{\varphi}\|_{\widetilde{\mathcal{M}}^\alpha} = \int_{\mathbb{R}^d} \frac{|\operatorname{Re} \varphi(\xi) - \operatorname{Re} \tilde{\varphi}(\xi)|}{|\xi|^{d+\alpha}} d\xi,$$

and we introduce the distance in  $\widetilde{\mathcal{M}}^\alpha$  as

$$\operatorname{dis}_{\alpha, \beta, \epsilon}(\varphi, \tilde{\varphi}) = \|\varphi - \tilde{\varphi}\|_{\widetilde{\mathcal{M}}^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta + \|\varphi - \tilde{\varphi}\|_\beta^\epsilon, \quad (1.11)$$

where  $0 < \beta < \alpha < 2, 0 < \epsilon < 1$ . Then  $(\widetilde{\mathcal{M}}^\alpha, \operatorname{dis}_{\alpha, \beta, \epsilon})$  is a complete metric space. In [12], the well-posedness of the Cauchy problem (1.1)-(1.5) is established in  $\widetilde{\mathcal{M}}^\alpha$ .

In this paper, we are devoted to characterizing the Fourier images of spaces  $P_{2k-2+\alpha}(\mathbb{R}^d)$ . In [6], Cho characterized the Fourier images of probability measure having finite absolute moment, without vanishing momentum condition, in terms of the forward difference operator and its iterates. As a modification of his results, we introduce a new classification of characteristic functions defined in terms of the symmetric central difference operator and its iterates as follows:

For any  $k \in \mathbb{N}^+, \alpha \in [0, 2), k + \alpha > 1$ , set

$$\mathcal{M}_k^\alpha = \left\{ \varphi \in \mathcal{K}; \int \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi < \infty \right\}, \quad (1.12)$$

where

$$\Delta^1 \varphi(\xi) = \frac{2 - \varphi(\xi) - \varphi(-\xi)}{4} = \frac{1 - \operatorname{Re} \varphi(\xi)}{2} = \int \sin^2 \frac{v \cdot \xi}{2} dF(v),$$

$$\begin{aligned} \Delta^2 \varphi(\xi) &= \frac{6 - 4\varphi(\xi) - 4\varphi(-\xi) + \varphi(2\xi) + \varphi(-2\xi)}{16} \\ &= \frac{3 - 4 \operatorname{Re} \varphi(\xi) + \operatorname{Re} \varphi(2\xi)}{8} = \int \sin^4 \frac{v \cdot \xi}{2} dF(v) \end{aligned}$$

and generally for  $k \in \mathbb{N}^+$ ,

$$\begin{aligned} \Delta^k \varphi(\xi) &= \frac{1}{2} \sum_{j=0}^k c_{k,j} (\varphi(j\xi) + \varphi(-j\xi)) \\ &= \sum_{j=0}^k c_{k,j} \operatorname{Re} \varphi(j\xi) = \int \sin^{2k} \frac{v \cdot \xi}{2} dF(v). \end{aligned}$$

Here,  $c_{k,j}$  are the coefficients of the expansion

$$\sin^{2k} \frac{x}{2} = \sum_{j=0}^k c_{k,j} \cos(jx) \quad \text{for all } x \in \mathbb{R}, \quad (1.13)$$

and an inductive calculation gives

$$c_{k,0} = 2^{-2k} \binom{2k}{k}, \quad c_{k,j} = (-1)^j 2^{-2k+1} \binom{2k}{k+j}, \quad j = 1, \dots, k.$$

For  $\varphi, \tilde{\varphi} \in \mathcal{M}_k^\alpha$ , put

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{M}_k^\alpha} = \int_{\mathbb{R}^d} \frac{|\Delta^k \varphi(\xi) - \Delta^k \tilde{\varphi}(\xi)|}{|\xi|^{d+2k-2+\alpha}} d\xi,$$

and introduce the distance

$$dis_{k,\alpha,\beta}(\varphi, \tilde{\varphi}) = \|\varphi - \tilde{\varphi}\|_{\mathcal{M}_k^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta, \quad (1.14)$$

where  $0 < \beta < \alpha$  if  $k = 1$  and  $0 < \beta < 2$  if  $k \geq 2$ .

**Theorem 1.1.** *Let  $k \in \mathbb{N}^+, \alpha \in [0, 2), k + \alpha > 1$ , then the space  $\mathcal{M}_k^\alpha$  is a complete metric space endowed with the distance (1.14). Moreover, we have*

$$\mathcal{M}_k^\alpha = \mathcal{F}(P_{2k-2+\alpha}(\mathbb{R}^d)), \quad (1.15)$$

and the condition that  $\lim_{n \rightarrow \infty} dis_{k,\alpha,\beta}(\varphi_n, \varphi) = 0$  with  $\varphi_n, \varphi \in \mathcal{M}_k^\alpha$  implies

$$\lim_{n \rightarrow \infty} \int \psi(v) dF_n(v) = \int \psi(v) dF(v) \quad (1.16)$$

for any  $\psi \in C(\mathbb{R}^d)$  satisfying the growth condition  $|\psi(v)| \lesssim \langle v \rangle^{2k-2+\alpha}$ , where  $F_n = \mathcal{F}^{-1}(\varphi_n), F = \mathcal{F}^{-1}(\varphi) \in P_{2k-2+\alpha}(\mathbb{R}^d)$ .

**Remark 1.2.** (1) *In the case  $k = 1, 0 < \alpha < 2$ , the newly defined space  $\mathcal{M}_k^\alpha$  coincides with  $\widetilde{\mathcal{M}}^\alpha$ . Since the solution in  $\widetilde{\mathcal{M}}^\alpha$  has been well studied in [11, 12], we will mainly focus on the case  $k \geq 2, 0 \leq \alpha < 2$  in this paper.*

(2) *For a probability measure  $F$  on the real line, it has been shown in [8] that, if  $\int |x|^{2k-2+\alpha} dF(x) < \infty$ , then there exists  $C_{k,\alpha} > 0$ , which depends only on  $k, \alpha$ , such that*

$$\int_0^\infty \frac{1}{t^{1+\alpha}} \left\{ 1 - \operatorname{Re} \varphi(t) + \sum_{j=1}^{k-1} \frac{t^{2j} \varphi^{(2j)}(0)}{(2j)!} \right\} dt = C_{k,\alpha} \int_{-\infty}^\infty |x|^{2k-2+\alpha} dF(x),$$

where  $\varphi = \mathcal{F}(F)$ . However, this characterization is different from the one given in (1.12).

Thanks to this new characterization of  $P_{2k-2+\alpha}$  by its exact Fourier image  $\mathcal{M}_k^\alpha$ , we can obtain the following theorems, that improves the continuity results of solutions established in [12].

**Theorem 1.3.** *Assume that  $b$  satisfies (1.3). Let  $k \in \mathbb{N}$ ,  $k \geq 2$ ,  $\alpha \in [0, 2)$ . If the initial datum  $F_0 \in P_{2k-2+\alpha}(\mathbb{R}^3)$ , then there exists a unique measure valued solution  $F_t \in C([0, \infty), P_{2k-2+\alpha}(\mathbb{R}^3))$  to the Cauchy problem (1.1)-(1.5) which preserves the energy and momentum for all time  $t > 0$ , that is,*

$$\int |v|^2 dF_t = \int |v|^2 dF_0, \quad \int v_j dF_t = 0 \quad \text{for } j = 1, 2, 3. \quad (1.17)$$

Furthermore, if  $b$  satisfies (1.2) and if  $F_0$  is not a single Dirac mass, then  $F_t$  admits the density distribution function  $f(t, v)$ ,  $dF_t(v) = f(t, v)dv$ , satisfying

$$f \in C((0, \infty); L_{2k-2+\alpha}^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)).$$

The proof of the above theorem is given in the Fourier space. In fact, by letting  $\varphi(t, \xi) = \int_{\mathbb{R}^3} e^{-iv \cdot \xi} dF_t(v)$  and  $\varphi_0 = \mathcal{F}(F_0)$ , it follows from the Bony formula [2, 3] that the Cauchy problem (1.1)-(1.5) is reduced to

$$\begin{cases} \partial_t \varphi(t, \xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left( \varphi(t, \xi^+) \varphi(t, \xi^-) - \varphi(t, \xi) \varphi(t, 0) \right) d\sigma, \\ \varphi(0, \xi) = \varphi_0(\xi), \quad \text{where } \xi^\pm = \frac{\xi}{2} \pm \frac{|\xi|}{2} \sigma. \end{cases} \quad (1.18)$$

By Theorem 1.1, to prove Theorem 1.3 it suffices to show

**Theorem 1.4.** *Assume that  $b$  satisfies (1.3). Let  $k \in \mathbb{N}$ ,  $k \geq 2$ ,  $\alpha \in [0, 2)$ ,  $\beta \in (\alpha_0, 2)$ . If the initial datum  $\varphi_0 = \mathcal{F}(F_0) \in \mathcal{M}_k^\alpha$ , with  $F_0$  satisfying (1.4), then there exists a unique classical solution  $\varphi(t, \xi) \in C([0, \infty), \mathcal{M}_k^\alpha)$  to the Cauchy problem (1.18). Moreover, for any  $T > 0$  and  $0 \leq s < t \leq T$ , we have*

$$\|\varphi(t, \cdot) - \varphi(s, \cdot)\|_\beta \leq |t - s| \cdot e^{\lambda_\beta T} \|1 - \varphi_0\|_\beta, \quad (1.19)$$

$$\|\varphi(t, \cdot) - \varphi(s, \cdot)\|_{\mathcal{M}_k^\alpha} \lesssim |t - s| \cdot \sup_{\tau \in [0, T]} \int |v|^{2k-2+\alpha} dF_\tau, \quad (1.20)$$

where  $\lambda_\beta > 0$  is a constant defined as

$$\lambda_\beta = \int_0^{\pi/2} b(\cos \theta) \left( \sin^\beta \frac{\theta}{2} + \cos^\beta \frac{\theta}{2} - 1 \right) \sin \theta d\theta.$$

**Remark 1.5.** (1.19) and (1.20) imply that

$$dis_{k, \alpha, \beta}(\varphi(t, \xi), \varphi(s, \xi)) \leq C_{T, \varphi_0} |t - s|, \quad (1.21)$$

where  $C_{T, \varphi_0} > 0$  only depends on  $T$  and the initial data.

## 2. Characterization of $P_{2k+\alpha}(\mathbb{R}^3)$

**Proposition 2.1.** *Let  $k \in \mathbb{N}$ ,  $\alpha \in [0, 2)$ ,  $k + \alpha > 1$  and let  $\mathcal{M}_k^\alpha$  be a subspace of  $\mathcal{K} = \mathcal{F}(P_0(\mathbb{R}^d))$  defined by (1.9). Then we have the formula (1.15). Furthermore, for  $M \in [1, \infty]$ , if we put*

$$c_{\alpha, d, M, k} = \int_{\{|\zeta| \leq M\}} \frac{\sin^{2k}(\mathbf{e}_1 \cdot \zeta/2)}{|\zeta|^{d+2k-2+\alpha}} d\zeta > 0, \quad (2.1)$$

and if  $F = \mathcal{F}^{-1}(\varphi)$  for  $\varphi \in \mathcal{M}_k^\alpha$ , then for any  $R > 0$  we have

$$\int_{\{|v| \geq R\}} |v|^{2k-2+\alpha} dF(v) \leq \frac{1}{c_{\alpha,d,1,k}} \int_{\{|\xi| \leq 1/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi. \quad (2.2)$$

Moreover,

$$\int_{\mathbb{R}^d} |v|^{2k-2+\alpha} dF(v) \leq \frac{1}{c_{\alpha,d,\infty,k}} \int_{\mathbb{R}^d} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi. \quad (2.3)$$

*Proof.* Note

$$\int_{\{|\xi| \leq M/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi = \int_{\mathbb{R}_+^d} \left( \int_{\{|\xi| \leq M/R\}} \frac{\sin^{2k}(v \cdot \xi/2)}{|\xi|^{d+2k-2+\alpha}} d\xi \right) dF(v).$$

By the change of variable  $|v|\xi = \zeta$  and by using the invariance of the rotation, we have

$$\begin{aligned} \int_{\{|\xi| \leq M/R\}} \frac{\sin^{2k}(v \cdot \xi/2)}{|\xi|^{d+2k-2+\alpha}} d\xi &= |v|^{2k-2+\alpha} \int_{\{|\zeta| \leq M|v|/R\}} \frac{\sin^{2k}(\mathbf{e}_1 \cdot \zeta/2)}{|\zeta|^{d+2k-2+\alpha}} d\zeta \\ &\geq |v|^{2k-2+\alpha} \mathbf{1}_{\{|v| \geq R\}} c_{\alpha,d,M,k}, \end{aligned}$$

which yields (2.2), with the choice of  $M = 1$ . By letting  $M \rightarrow \infty$  and  $R \rightarrow 0$ , we obtain (2.3). The formula (1.15) is now obvious since

$$\lim_{M \rightarrow \infty} \int_{\{|\xi| \leq M\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \leq c_{\alpha,d,\infty,k} \int |v|^{2k-2+\alpha} dF(v). \quad (2.4)$$

□

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Suppose that  $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{M}_k^\alpha$  satisfies

$$dis_{k,\alpha,\beta}(\varphi_n, \varphi_m) \rightarrow 0 \quad (n, m \rightarrow \infty).$$

Since it follows from Proposition 3.10 of [4] that  $\mathcal{K}^\beta$  is a complete metric space, we have the limit (pointwise convergence)

$$\varphi(\xi) = \lim_{n \rightarrow \infty} \varphi_n(\xi) \in \mathcal{K}^\beta \subset \mathcal{K}.$$

For any fixed  $R > 1$  we have

$$\int_{\{R^{-1} \leq |\xi| \leq R\}} \frac{|\Delta^k \varphi_n(\xi)|}{|\xi|^{d+2k-2+\alpha}} d\xi \leq \sup_n \|\varphi_n\|_{\mathcal{M}_k^\alpha} < \infty.$$

Taking the limit with respect to  $n$  and letting  $R \rightarrow \infty$ , we have  $\varphi \in \mathcal{M}_k^\alpha$ . Now it is easy to see that  $dis_{k,\alpha,\beta}(\varphi_n, \varphi) \rightarrow 0$ .

Suppose that, for  $F_n, F \in P_\alpha(\mathbb{R}^d)$ , we have

$$\varphi_n = \mathcal{F}(F_n), \varphi = \mathcal{F}(F) \in \mathcal{M}_k^\alpha, \text{ and } \lim_{n \rightarrow \infty} \text{dis}_{k,\alpha,\beta}(\varphi_n, \varphi) = 0.$$

Note that for  $R > 1$

$$\int_{\{|\xi| \leq 1/R\}} \frac{\Delta^k \varphi_n(\xi)}{|\xi|^{d+\alpha}} d\xi \leq \int_{\{|\xi| \leq 1/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+\alpha}} d\xi + \|\varphi_n - \varphi\|_{\mathcal{M}_k^\alpha}.$$

It follows from (2.2) that for any  $\varepsilon > 0$  there exist  $R > 1$  and  $N \in \mathbb{N}$  such that

$$\int_{\{|v| \geq R\}} |v|^{2k-2+\alpha} dF_n(v) + \int_{\{|v| \geq R\}} |v|^{2k-2+\alpha} dF(v) < \varepsilon \text{ if } n \geq N.$$

This shows (1.16) because  $\varphi_n \rightarrow \varphi$  in  $\mathcal{S}'(\mathbb{R}^d)$  and so  $F_n \rightarrow F$  in  $\mathcal{S}'(\mathbb{R}^d)$ . This completes the proof of the theorem.  $\square$

### 3. Proof of Theorem 1.4

The main purpose of this section concerns with the continuity of solutions in the new classification of characteristic functions. We only need to prove Theorem 1.4 because Theorem 1.3 follows by using Theorem 1.1.

*Proof.* Since  $k \geq 2$ , we have  $\varphi_0 \in \mathcal{M}_k^\alpha \subset \mathcal{K}^2 \subset \widetilde{\mathcal{M}}^{\beta_1}$ , for some  $\beta_1 \in (\beta, 2)$ . By Theorem 1.8 of [12], we can obtain a unique classical solution  $\varphi(t, \xi) \in C([0, \infty), \widetilde{\mathcal{M}}^{\beta_1})$  corresponding to the initial data  $\varphi_0$ . Since  $F_0 \in P_{2k-2+\alpha}(\mathbb{R}^3)$ , by the Corollary 1.7 of [12] and (2.4), it is easy to see that, for any  $t \geq 0$ ,  $\varphi(t, \xi) \in \mathcal{M}_k^\alpha$ . More precisely, there exists a constant  $C_T > 0$  such that

$$\sup_{s \leq \tau \leq t} \|\varphi(\tau)\|_{\mathcal{M}_k^\alpha} \leq C_T \text{ for all } s, t \in [0, T]. \quad (3.5)$$

To complete the proof of Theorem 1.4, it remains to prove (1.19) and (1.20). The first estimate (1.19) is a direct consequence of the formula

$$\varphi(t, \xi) - \varphi(s, \xi) = \int_s^t \int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(\varphi(\tau, \xi^+) \varphi(\tau, \xi^-) - \varphi(\tau, \xi)\right) d\sigma d\tau,$$

and Lemma 2.2 of [9]. The second one (1.20) follows from the following calculation which is a variant of Lemma 2.2 of [9] and Lemma 3.4 of [11] (see also the proof of (1.23) in [12]).



For any  $k \geq 2, \alpha \in [0, 2)$ , we have

$$\begin{aligned}
& \int \frac{|\Delta^k \varphi(\xi, t) - \Delta^k \varphi(\xi, s)|}{|\xi|^{3+2k-2+\alpha}} d\xi \\
&= \int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}} \left| \sum_{j=1}^k c_{k,j} (\operatorname{Re} \varphi(j\xi, t) - \operatorname{Re} \varphi(j\xi, s)) \right| \\
&= \int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}} \left| \sum_{j=1}^k c_{k,j} \operatorname{Re} \int_s^t \int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \right. \\
&\quad \left. \cdot (\varphi(j\xi^+, \tau) \varphi(j\xi^-, \tau) - \varphi(j\xi, \tau)) d\sigma d\tau \right|. \tag{3.6}
\end{aligned}$$

As in [9], we put  $\zeta = (\xi^+ \cdot \frac{\xi}{|\xi|}) \frac{\xi}{|\xi|}$  and consider  $\tilde{\xi}^+ = \zeta - (\xi^+ - \zeta)$ , which is symmetric to  $\xi^+$  on  $\mathbb{S}^2$ , see Figure 1. Then, we split (3.6) into three parts:

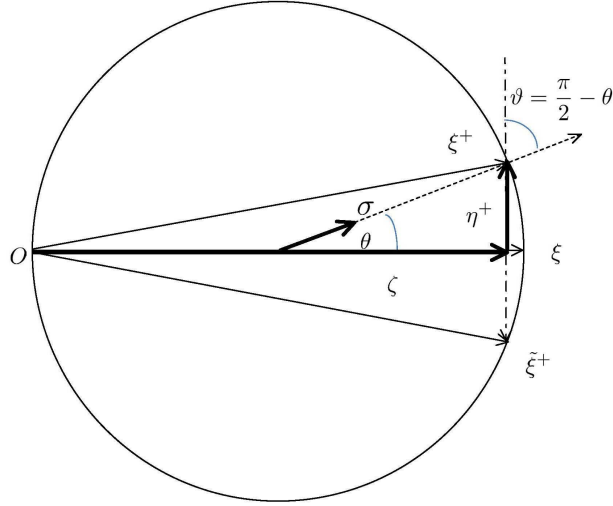


Figure 1:  $\theta = \langle \frac{\xi}{|\xi|}, \sigma \rangle$ ,  $\vartheta = \langle \frac{\eta^+}{|\eta^+|}, \sigma \rangle$ ,  $\eta^+ = \xi^+ - \zeta$ .

$$I_{j,1} = \frac{1}{2} \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) (\varphi(j\xi^+) + \varphi(j\tilde{\xi}^+) - 2\varphi(j\zeta)) d\sigma, \tag{3.7}$$

$$I_{j,2} = \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) (\varphi(j\zeta) - \varphi(j\xi)) d\sigma, \tag{3.8}$$

$$I_{j,3} = \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \varphi(j\xi^+) (\varphi(j\xi^-) - 1) d\sigma. \tag{3.9}$$

Summing over  $j$ , we have

$$\begin{aligned} \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,1} &= \frac{1}{2} \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \int \left( \sin^{2k} \frac{\xi^+ \cdot v}{2} + \sin^{2k} \frac{\tilde{\xi}^+ \cdot v}{2} \right. \\ &\quad \left. - 2 \sin^{2k} \frac{\zeta \cdot v}{2} \right) dF_\tau d\sigma, \\ \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,2} &= \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \int \left( \sin^{2k} \frac{\zeta \cdot v}{2} - \sin^{2k} \frac{\xi \cdot v}{2} \right) dF_\tau d\sigma. \end{aligned}$$

By a proper transformation to the variable  $\xi$ , that is,  $\eta = |v|\xi$ , and then use  $\xi$  again to replace  $\eta$ , we have

$$\begin{aligned} &\int \frac{1}{|\xi|^{3+2k-2+\alpha}} \left| \int_s^t \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,1} d\tau \right| d\xi \\ &\leq \frac{1}{2} \int_s^t \int_{\mathbb{S}^2} \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left| \sin^{2k} \frac{\xi^+ \cdot \mathbf{e}_1}{2} + \sin^{2k} \frac{\tilde{\xi}^+ \cdot \mathbf{e}_1}{2} - 2 \sin^{2k} \frac{\zeta \cdot \mathbf{e}_1}{2} \right| d\sigma d\xi \\ &\quad \cdot \int |v|^{2k-2+\alpha} dF_\tau d\tau. \end{aligned}$$

If we put  $A = \zeta \cdot \mathbf{e}_1/2$  and  $B = \eta^+ \cdot \mathbf{e}_1/2$ , ( $\eta^+ = \xi^+ - \zeta$ ), then

$$\begin{aligned} &\sin^{2k} \frac{\xi^+ \cdot \mathbf{e}_1}{2} + \sin^{2k} \frac{\tilde{\xi}^+ \cdot \mathbf{e}_1}{2} - 2 \sin^{2k} \frac{\zeta \cdot \mathbf{e}_1}{2} \\ &= (\sin A \cos B + \sin B \cos A)^{2k} + (\sin A \cos B - \sin B \cos A)^{2k} - 2 \sin^{2k} A \\ &= 2 \sin^{2k} A \left( (\cos^2 B)^k - 1 \right) \\ &\quad + 2 \sum_{j=1}^k \binom{2k}{2j} \sin^{2j} B \sin^{2k-2j} A \cos^{2k-2j} B \cos^{2j} A. \end{aligned}$$

Since

$$\sin^2 \frac{\eta^+ \cdot \mathbf{e}_1}{2} \lesssim |\xi|^2 \sin^2 \frac{\theta}{2},$$

we have

$$\begin{aligned} &\left| \sin^{2k} \frac{\xi^+ \cdot \mathbf{e}_1}{2} + \sin^{2k} \frac{\tilde{\xi}^+ \cdot \mathbf{e}_1}{2} - 2 \sin^{2k} \frac{\zeta \cdot \mathbf{e}_1}{2} \right| \\ &\lesssim \min \left\{ |\xi|^2 \sin^2 \frac{\theta}{2}, 1 \right\} \cdot \mathbf{1}_{\{|\xi| \geq 1\}} + |\xi|^{2k} \sin^2 \frac{\theta}{2} \cdot \mathbf{1}_{\{|\xi| < 1\}}. \end{aligned}$$

For  $|\xi| > 1$ ,

$$\begin{aligned}
& \int_{|\xi|>1} \frac{1}{|\xi|^{3+2k-2+\alpha}} \left| \int_s^t \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,1} d\tau \right| d\xi \\
& \lesssim \int_{|\xi|>1} \frac{1}{|\xi|^{3+2k-2+\alpha}} \int_0^{\pi/2} b(\theta) \min\{|\xi|^2 \sin^2(\theta/2), 1\} \sin \theta d\theta d\xi \\
& \lesssim \int_{|\xi|>1} \frac{|\xi|^{\alpha_0}}{|\xi|^{3+2k-2+\alpha}} d\xi < \infty.
\end{aligned}$$

The case  $|\xi| < 1$  is easier. Hence, we proved

$$\begin{aligned}
& \int \frac{1}{|\xi|^{3+2k-2+\alpha}} \left| \int_s^t \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,1} d\tau \right| d\xi \\
& \lesssim |t-s| \cdot \sup_{\tau \in [0, T]} \int |v|^{2k-2+\alpha} dF_\tau. \tag{3.10}
\end{aligned}$$

The second term  $I_{j,2}$  is similar. Now let's consider the last term  $I_{j,3}$ . Recall that the solution conserves the momentum, i.e.

$$\int v dF_t = 0 \text{ for all } t \geq 0. \tag{3.11}$$

We have

$$\begin{aligned}
\operatorname{Re} I_{j,3} &= \int b \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \operatorname{Re} \left( \varphi(j\xi^+) (\varphi(j\xi^-) - 1) \right) d\sigma \\
&= \int b \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \iint \operatorname{Re} \left( e^{-i(j\xi^+ \cdot v)} (e^{-i(j\xi^- \cdot w)} - 1) \right) dF_\tau(v) dF_\tau(w) \right) d\sigma \\
&= \int b \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \iint \{ \cos(j(\xi^+ \cdot v + \xi^- \cdot w)) \right. \\
&\quad \left. - \cos(j\xi^+ \cdot v) \} dF_\tau(v) dF_\tau(w) \right) d\sigma.
\end{aligned}$$

It follows from (1.13) that

$$\begin{aligned}
\sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,3} &= \int b \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \iint \left( \sin^{2k} \frac{\xi^+ \cdot v + \xi^- \cdot w}{2} \right. \right. \\
&\quad \left. \left. - \sin^{2k} \frac{\xi^+ \cdot v}{2} \right) dF_\tau(v) dF_\tau(w) \right) d\sigma.
\end{aligned}$$

Denoting  $x = \frac{\xi^+ \cdot v}{2}, y = \frac{\xi^- \cdot w}{2}$ , we see

$$\begin{aligned}
& \sin^{2k} \frac{\xi^+ \cdot v + \xi^- \cdot w}{2} - \sin^{2k} \frac{\xi^+ \cdot v}{2} \\
&= \left( \sin x \cos y + \sin y \cos x \right)^{2k} - \sin^{2k} x \\
&= 2k \sin y \sin^{2k-1} x \cos x \\
&\quad + 2k \sin y (\cos^{2k-1} y - 1) \sin^{2k-1} x \cos x \\
&\quad + \sum_{\ell=2}^{2k} \binom{2k}{\ell} \sin^\ell y \cos^{2k-\ell} y \sin^{2k-\ell} x \cos^\ell x \\
&\quad + \sin^{2k} x (\cos^{2k} y - 1) \\
&= J_{3,1} + J_{3,2} + J_{3,3} + J_{3,4}.
\end{aligned}$$

Since the solution conserves the momentum, we obtain

$$\begin{aligned}
& \left| \iint J_{3,1} dF_\tau(v) dF_\tau(w) \right| \\
&= 2k \left| \int \sin^{2k-1} \frac{\xi^+ \cdot v}{2} \cos \frac{\xi^+ \cdot v}{2} dF_\tau(v) \int \left( \sin \frac{\xi^- \cdot w}{2} - \frac{\xi^- \cdot w}{2} \right) dF_\tau(w) \right|.
\end{aligned}$$

Since  $|z - \sin z| \lesssim |z| \min\{|z|, 1\}$ , by the change of the variable  $|v|\xi \rightarrow \zeta$ , we have

$$\begin{aligned}
& \int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}} \int b \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left| \iint J_{3,1} dF_\tau(v) dF_\tau(w) \right| d\sigma \\
&\lesssim \int \frac{\min\{|\zeta|^{2k-1}, 1\} d\zeta}{|\zeta|^{3+2k-2+\alpha}} \int b(\cos \theta) \theta^{\max\{\alpha_0, 1\}} \sin \theta d\theta \\
&\quad \times \int |v|^{2k-2+\alpha} \left( \int \left( \frac{|\zeta||w|}{|v|} \right)^{\max\{\alpha_0, 1\}} dF_\tau(w) \right) dF_\tau(v) \\
&\lesssim \int \langle v \rangle^{2k-2+\alpha} dF_\tau(v) \int \langle w \rangle^2 dF_\tau(w).
\end{aligned}$$

We have the same upper bound for the integrals corresponding to  $J_{3,2}$  and  $J_{3,4}$ . Furthermore, if one use another change of variable  $|w|\xi \rightarrow \zeta$  for terms with

$\ell \geq k$ , then

$$\begin{aligned}
& \int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}} \int b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left| \iint J_{3,3} dF_\tau(v) dF_\tau(w) \right| d\sigma \\
& \lesssim \int \frac{d\zeta}{|\zeta|^{3+2k-2+\alpha}} \int b(\cos \theta) \theta^{\alpha_0} \sin \theta d\theta \\
& \quad \times \iint |v|^{2k-2+\alpha} \sum_{\ell=2}^{k-1} \left( |\zeta|^{2k-\ell} \mathbf{1}_{\{|\zeta| \leq 1\}} + \mathbf{1}_{\{|\zeta| > 1\}} \right) \\
& \quad \times \left( \left( \frac{|w||\zeta|}{|v|} \right)^\ell \mathbf{1}_{\{|\zeta| \leq 1\}} + \left( \frac{|w||\zeta|}{|v|} \right)^{\alpha_0} \mathbf{1}_{\{|\zeta| > 1\}} \right) dF_\tau(v) dF_\tau(w) \\
& + \int \frac{d\zeta}{|\zeta|^{3+2k-2+\alpha}} \int b(\cos \theta) \theta^{\alpha_0} \sin \theta d\theta \\
& \quad \times \iint |w|^{2k-2+\alpha} \sum_{\ell=k}^{2k} \left( \left( \frac{|v||\zeta|}{|w|} \right)^{2k-\ell} \mathbf{1}_{\{|\zeta| \leq 1\}} + \mathbf{1}_{\{|\zeta| > 1\}} \right) \\
& \quad \times \left( |\zeta|^\ell \mathbf{1}_{\{|\zeta| \leq 1\}} + |\zeta|^{\alpha_0} \mathbf{1}_{\{|\zeta| > 1\}} \right) dF_\tau(v) dF_\tau(w) \\
& \lesssim \iint \left( \langle v \rangle^{2k-2+\alpha} \langle w \rangle^2 + \langle w \rangle^{2k-2+\alpha} \langle v \rangle^2 \right) dF_\tau(v) dF_\tau(w).
\end{aligned}$$

Thus, we have similar estimate as (3.10), namely,

$$\int \frac{1}{|\xi|^{3+2k-2+\alpha}} \left| \int_s^t \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,3} d\tau \right| d\xi \lesssim |t-s| \cdot \sup_{\tau \in [0, T]} \int \langle w \rangle^{2k-2+\alpha} dF_\tau(w).$$

□

#### 4. Proof of Theorem 1.3

The existence and uniqueness of the solution to the Cauchy problem (1.1)-(1.5) follow from Theorem 1.4 and the smoothing effect is proved by the Corollary 1.10 of [12]. To finish the proof, it remains to show

$$\mathcal{F}^{-1}(\varphi(t)) = F_t = f(t, v) \in C((0, \infty); L_{2k-2+\alpha}^1(\mathbb{R}^3))$$

if  $b$  has a singularity (1.2).

Indeed, it follows from the smoothing effect that for any  $0 < t_0 < T < \infty$  and for any  $N > 0$ , there exists a constant  $C_{N, t_0, T} > 0$  such that

$$\sup_{t_0 \leq \tau \leq T} \int \langle \xi \rangle^N |\varphi(\tau, \xi)|^2 d\xi \leq C_{N, t_0, T}.$$

Let  $t_1 \in (t_0, T)$  and  $\varepsilon > 0$ . Since  $\varphi(t_1, \xi) \in \mathcal{M}_k^\alpha$ , there exists an  $R = R_\varepsilon > 1$  such that

$$\int_{|\xi| < 1/R} \frac{\Delta^k \varphi(t_1, \xi)}{|\xi|^{3+2k-2+\alpha}} d\xi < c_{\alpha, 3, 1, k} \frac{\varepsilon}{2}$$

and moreover it follows from (1.20) that there exists a  $\delta > 0$  such that if  $|t - t_1| < \delta$  then

$$\int_{|\xi| < 1/R} \frac{\Delta^k \varphi(t, \xi)}{|\xi|^{3+2k-2+\alpha}} d\xi < c_{\alpha, 3, 1, k} \varepsilon.$$

By means of (2.2), we have  $\int_{\{|v| \geq R\}} |v|^{2k+\alpha-2} f(t, v) dv < \varepsilon$  if  $|t - t_1| < \delta$ . Therefore, if  $|t - t_1| < \delta$  then

$$\begin{aligned} & \int \langle v \rangle^{2k-2+\alpha} |f(t, v) - f(t_1, v)| dv \\ & < \langle R \rangle^{2k-2+\alpha} \int_{|v| < R} |f(t, v) - f(t_1, v)| dv + 4\varepsilon. \end{aligned}$$

On the other hand, for any  $M > 1$  we have

$$\begin{aligned} \sup_v |f(t, v) - f(t_1, v)| & \leq \int |\varphi(t, \xi) - \varphi(t_1, \xi)| d\xi \\ & \leq \left( \int_{|\xi| \geq M} \langle \xi \rangle^{-4} d\xi \right)^{1/2} \left( \int \langle \xi \rangle^4 |\varphi(t, \xi) - \varphi(t_1, \xi)|^2 d\xi \right)^{1/2} \\ & \quad + \frac{4\pi M^3}{3} \sup_{|\xi| \leq M} |\varphi(t, \xi) - \varphi(t_1, \xi)|. \end{aligned}$$

We can conclude the continuity since  $\varphi(t, \xi) \in C([0, \infty); \mathcal{K}^2)$ .

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