

DISSIPATIVE QUASI-GEOSTROPHIC EQUATIONS IN CRITICAL SOBOLEV SPACES: SMOOTHING EFFECT AND GLOBAL WELL-POSEDNESS

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ABSTRACT. We study the critical and super-critical dissipative quasi-geostrophic equations in \mathbb{R}^2 or \mathbb{T}^2 . An optimal local smoothing effect of solutions with arbitrary initial data in $H^{2-\gamma}$ is proved. As a main application, we establish the global well-posedness for the critical 2D quasi-geostrophic equations with periodic H^1 data. Some decay in time estimates are also provided.

1. INTRODUCTION

We are interested in the initial value problem of two dimensional dissipative quasi-geostrophic equations

$$\begin{cases} \theta_t + u \cdot \nabla \theta + (-\Delta)^{\gamma/2} \theta = 0 & \text{on } \mathbb{R}^2 \times (0, \infty), \\ \theta(0, x) = \theta_0(x) & x \in \mathbb{R}^2, \end{cases} \quad (1.1)$$

where $\gamma \in (0, 2]$ is a fixed parameter and the velocity $u = (u_1, u_2)$ is divergence free and is determined by the Riesz transforms of the potential temperature θ :

$$u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) = (-\partial_{x_2} (-\Delta)^{-1/2} \theta, \partial_{x_1} (-\Delta)^{-1/2} \theta).$$

Equation (1.1) is an important model in geophysical fluid dynamics. It is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency. Mathematically, the equation has also been considered to be a 2D model of the 3D incompressible Navier-Stokes equations. It is therefore an interesting model for investigating existence issues on genuine 3D Navier-Stokes equations. Recently, this equation has been studied by many authors, see [4, 5, 6, 16, 17, 24, 25, 26] and references therein.

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The global existence of a weak solution to (1.1) follows from Resnick [22]. The cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$ are called sub-critical, critical and super-critical respectively. The sub-critical case is well understood. Wu established in [24] the global existence of a unique regular solution to (1.1) with initial data θ_0 in L^p for $p > 2/(\gamma - 1)$. With initial data in the scaling invariant space $L^{2/(\gamma-1)}$, the proof of the global well-posedness can be found, for example, in recent [2], where the asymptotic behavior of the solutions is also studied. By using a Fourier splitting method, Constantin and Wu [6] showed the global existence of a regular solution on the torus with periodic boundary conditions and also a sharp L^2 decay estimate for weak solutions with data in $L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Furthermore, very recently in [12] the author and Li estimated the higher order derivatives of the solution and proved that it is actually spatial analytic.

However, the cases of critical and super-critical quasi-geostrophic equations still have quite a few unsolved problems. In the critical case, Constantin, Córdoba and Wu [5] gave a construction of global regular solutions for the initial data in H^1 under a smallness assumption of L^∞ norm of the data. Moreover, they showed that the solutions are spatial analytic for sufficiently large t . In Chae and Lee [3], the global existence and uniqueness were obtained for small initial data in the critical Besov space $B_{2,1}^{2-\gamma}$. In [16], Ju improved Chae and Lee's result by showing that (1.1) is globally well-posed for small data in H^s if $s \geq 2 - \gamma$, and locally well-posed for large data if $s > 2 - \gamma$.

Very recently, there are two important papers [15] and [1]. In [15] the global well-posedness for the critical quasi-geostrophic equations with periodic C^∞ data was established by Kiselev, Nazarov and Volberg by proving certain non-local maximum principle. In [1] Caffarelli and Vasseur constructed a global regular solution for the critical quasi-geostrophic equations with L^2 initial data. To the best of our knowledge, the uniqueness of such weak solution is still open.

For results with minimal regularity assumptions, in recent [19], Miura improved the result in [16] and proved the local in time existence of a unique regular solution for large initial data in the critical Sobolev space $H^{2-\gamma}$. A similar result was also obtained independently in Ju [18] by using a different approach. For other results about the critical and super-critical cases, we also refer the readers to [3, 7, 16, 17, 25, 26, 27].

Next we shall describe the main results of the present paper.

Our first result (Theorem 2.2 and 2.5) is concerning the *optimal* local smoothing effect of solutions. It says, roughly speaking, that the smoothing effect of the equations in spaces is the same for the

corresponding linear equations. We remark that in the critical or super-critical cases, one has higher derivative in the flow term $u \cdot \nabla \theta$ than in the dissipation term $(-\Delta)^{\gamma/2} \theta$. A general understanding is that the former term tends to make the smoothness of θ worse, while the latter term tends to make it better. We show that for small $\gamma \in (0, 1]$, the dissipation is still strong enough to balance the nonlinear term. This result implies, in particular, that the solution is infinitely differentiable. For the critical quasi-geostrophic equation, although we have H^1 local well-posedness, to get global existence the authors of [15] have to assume that the initial data is smooth. In this connection, we note that the chief purpose of the current article is to fill in this gap.

As a main application of Theorem 2.2, in the second result (Theorem 2.7), we obtain the global well-posedness of the critical quasi-geostrophic equation with period H^1 data. We remark that the problem of H^1 global well-posedness of the critical quasi-geostrophic equation has been open for years.¹ Moreover, we prove an exponential decay estimate of the solution and all its derivatives, and show that the solution is spatial analytic for large t . Although some results here are based on the main result of [19], the proof of which uses the contraction argument, this article is not a simple extension of [19]. The contraction argument is not sufficient to establish the infinite differentiability of the solution, since the time of existence of the solution in $H^\beta, \beta \geq 1$ may be dependent on β . Instead, a suitable arrangement of the nonlinear term enable us to use a bootstrap argument to get the infinite differentiability as well as an exponential decay estimate.

In a forthcoming article, we are going to generalize these results to more general Besov spaces. Although the main idea is similar, more complicated arguments and estimates are involved.

The remaining part of the article is organized as follows: our main theorems (Theorem 2.2, 2.4, 2.5 and 2.7) are stated in the next section. We define some notation which we shall use later and recall some basic estimates in Section 3. The proof of a commutator estimate (Lemma 3.5) is deferred to Section 7. These estimate enable us to prove Theorem 2.2 and 2.4 in Section 4 by adapting an idea, which has been used in [10, 12, 23, 14]. Section 5 and 6 are devoted to the proofs of a Theorem 2.5 and 2.7.

¹After the paper was finished, the author and Dapeng Du realized that by adapting a method in [15] with suitable modifications, the results here can be used to establish the H^1 global well-posedness of the critical quasi-geostrophic equation in the whole space. We present this in a subsequent paper [11].

2. MAIN THEOREMS

Define $G(t, x) = G_\gamma(t, x)$ by its Fourier transform $\widehat{G}_\gamma(t, \xi) = e^{-t|\xi|^\gamma}$ for $t > 0$. Then $G_\gamma(t, x)$ is the fundamental solution of the linear operator $\partial_t + (-\Delta)^{\gamma/2}$. It also has the scaling property

$$G_\gamma(t, x) = t^{-\frac{2}{\gamma}} G_\gamma(1, xt^{-\frac{1}{\gamma}}).$$

It is well-known that (1.1) can be rewritten into an integral equation

$$\theta(t, \cdot) = G(t, \cdot) * \theta_0 - \int_0^t G(t-s, \cdot) * (u \cdot \nabla \theta)(s, \cdot) ds.$$

Since u is divergence free, integration by parts yields

$$\theta(t, \cdot) = G(t, \cdot) * \theta_0 - \int_0^t \nabla G(t-s, \cdot) * (u\theta)(s, \cdot) ds.$$

In the sub-critical case, after obtaining suitable linear and bilinear estimates in certain Banach spaces, one can use the classical Kato's contraction method [13] to prove the local existence results. However, due to the weak dissipations, this method seems not applicable in the usual way for the critical and super-critical cases. In particular, it is difficult to find a suitable Banach space X so that the bilinear term is continuous from $X \times X$ to X .

The following theorem is recently proved in Miura [19] by using a variation of the Kato's method combined with a commutator estimate associated with the Littlewood-Paley operator in the Sobolev space (see also recent Ju [18] for a different approach).

Proposition 2.1. *Let $\gamma \in (0, 1]$ and $\theta_0 \in H^{2-\gamma}$. Then there exists $T > 0$ such that the initial value problem for (1.1) has a unique solution*

$$\theta(t, x) \in C([0, T]; H^{2-\gamma}) \cap L^2(0, T; H^{2-\gamma/2}). \quad (2.1)$$

The solution θ satisfies

$$\sup_{0 < t < T} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} < \infty, \quad (2.2)$$

for any $\beta \in [0, \gamma)$ and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} = 0, \quad (2.3)$$

for any $\beta \in (0, \gamma)$. Furthermore, there exists $\varepsilon_0 > 0$ such that if $\|\theta_0\|_{\dot{H}^{2-\gamma}} < \varepsilon_0$, then we can take $T = \infty$.

By adapting the idea which were used in [10, 23, 12, 14, 20], we are able to get the optimal local smoothing effect of the solution. Next we state our main results.

Theorem 2.2. *Let $\gamma \in (0, 1]$ and $\theta_0 \in H^{2-\gamma}$. Then the solution θ in Proposition 2.1 satisfies*

$$\sup_{0 < t < T} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} < \infty, \quad (2.4)$$

for any $\beta \geq 0$ and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} = 0, \quad (2.5)$$

for any $\beta > 0$.

Remark 2.3. If we assume $\theta_0 \in H^{2-\gamma}$, the Sobolev embedding theorem, the boundedness of Riesz transforms on $L^p, 1 < p < \infty$ and Theorem 2.2 together with the L^p maximum principle imply that the solution θ and u are smooth in x in $(0, T) \times \mathbb{R}^2$. Then from the equation (1.1) itself, we see that they are also smooth in t in that region. Consequently, the mild solution θ is in fact a classical solution of (1.1).

The proof of Theorem 2.2 also yields an optimal decay in time estimate of higher order Sobolev norms in case of small initial data.

Theorem 2.4. *Under the assumptions of Theorem 2.2, there exists $\varepsilon_0 > 0$ such that if $\|\theta_0\|_{\dot{H}^{2-\gamma}} < \varepsilon_0$, then*

i) the initial value problem for (1.1) has a unique global regular solution $\theta(t, x)$ in

$$C_b([0, \infty); H^{2-\gamma}) \cap L^2((0, \infty); H^{2-\gamma/2}).$$

ii) for any $\beta \geq 0$, the solution θ satisfies

$$\sup_{t > 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} < \infty. \quad (2.6)$$

Without much more work, a modification of the proof of Theorem 2.2 gives the integrability of the solution, along with its derivatives, in time variable (See, e.g. [10]).

Theorem 2.5. *Let $\gamma \in (0, 1]$ and $\theta_0 \in H^{2-\gamma}$. Then the solution θ in Proposition 2.1 satisfies*

$$\|t^{\beta_1/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}}\|_{L_t^{\gamma/\beta_2}(0, T)} < \infty, \quad (2.7)$$

for any $\beta = \beta_1 + \beta_2$ with $\beta_1 \geq 0$ and $\beta_2 \in [0, \gamma/2]$.

Remark 2.6. As in Theorem 2.4, from the proof below we can clearly see that if the $\dot{H}^{2-\gamma}$ norm of the initial data is sufficiently small (but independent of β_1 or β_2), then one may take $T = \infty$ in Theorem 2.5.

We can also consider the 2D quasi-geostrophic equations on the torus with periodic boundary condition:

$$\begin{cases} \theta_t + u \cdot \nabla \theta + (-\Delta)^{\gamma/2} \theta = 0 & \text{on } \mathbb{T}^2 \times (0, \infty), \\ \theta(0, x) = \theta_0(x) & x \in \mathbb{T}^2, \end{cases} \quad (2.8)$$

where $\mathbb{T}^2 = [0, 1]^2$ and $\theta_0 \in \dot{H}^{2-\gamma}(\mathbb{T}^2)$. As usual, the zero-average condition is assumed:

$$\int_{\mathbb{T}^2} \theta_0(x) dx = 0.$$

Then by the Poincaré inequality, we have $\theta_0 \in H^{2-\gamma}(\mathbb{T}^2)$. The proofs of Proposition 2.1 and Theorem 2.2, 2.4, 2.5 can be easily modified to get the corresponding results for (2.8). Also owing to a well-known fact $\int_{\mathbb{T}^2} \theta(t, \cdot) dx = 0$ and Poincaré's inequality, the homogeneous Sobolev norms in these estimates can be replaced by the corresponding inhomogeneous norms. We leave the details to interested readers.

For the critical quasi-geostrophic equations on the torus, we have the following global existence result and *exponential* decay estimate.

Theorem 2.7. *Let $\gamma = 1$ and $\theta_0 \in \dot{H}^1(\mathbb{T}^2)$. Then the initial value problem for (2.8) has a unique global smooth solution θ in*

$$C_b([0, \infty); H^1(\mathbb{T}^2)) \cap L^2((0, \infty); H^{3/2}(\mathbb{T}^2)). \quad (2.9)$$

For some $T_0 > 0$, $\theta(t, \cdot)$ is spatial analytic for any $t \geq T_0$. Furthermore, the solution and all its derivatives decay exponentially as t goes to infinity. More precisely, we have

$$\sup_{t>0} e^{t/4} t^\beta \|\theta(t, \cdot)\|_{H^{2-\gamma+\beta}} < \infty, \quad (2.10)$$

for any $\beta \geq 0$.

3. NOTATION AND SOME PRELIMINARY ESTIMATES

First we recall the Littlewood-Paley decomposition. For any integer j , define Δ_j to be the Littlewood-Paley projection operator with $\Delta_j v = \phi_j * v$, where

$$\begin{aligned} \hat{\phi}_j(\xi) &= \hat{\phi}(2^{-j}\xi), \quad \hat{\phi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \quad \hat{\phi} \geq 0, \\ \text{supp } \hat{\phi} &\subset \{\xi \in \mathbb{R}^2 \mid 1/2 \leq |\xi| \leq 2\}, \quad \sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \text{ for } \xi \neq 0. \end{aligned}$$

Modulo a polynomials, formally we have the Littlewood-Paley decomposition

$$v(\cdot, t) = \sum_{j \in \mathbb{Z}} \Delta_j v(\cdot, t).$$

For any $p \in (1, \infty)$ and $s \geq 0$, as usual we denote $\dot{W}^{s,p}$ and $W^{s,p}$ to be the homogeneous and inhomogeneous Sobolev spaces with norms

$$\|v\|_{\dot{W}^{s,p}} := \left\| \left(\sum_{k \in \mathbb{Z}} |2^{ks} \Delta_k v|^2 \right)^{1/2} \right\|_{L^p} \sim \|\Lambda^s v\|_{L^p},$$

$$\|v\|_{W^{s,p}} := \|v\|_{\dot{W}^{s,p}} + \|v\|_{L^p},$$

with implicit constants depending on p and s . When $p = 2$, we use \dot{H}^s and H^s instead of $\dot{W}^{s,p}$ and $W^{s,p}$.

Denote $\Lambda = (-\Delta)^{1/2}$. The following Bernstein's inequality is well-known.

Lemma 3.1. *Let $p \in [1, \infty]$ and $s \in \mathbb{R}$. Then for any $j \in \mathbb{Z}$, we have*

$$\lambda 2^{js} \|\Delta_j v\|_{L^p} \leq \|\Lambda^s \Delta_j v\|_{L^p} \leq \lambda' 2^{js} \|\Delta_j v\|_{L^p} \quad (3.1)$$

with some constants λ and λ' depending only on p and s . Moreover, for $1 \leq p \leq q \leq \infty$, there exists a positive constant C such that

$$\|\Delta_j v\|_{L^q} \leq C 2^{(2/p-2/q)j} \|\Delta_j v\|_{L^p}. \quad (3.2)$$

We shall use the next two standard linear estimates, the proofs of which can be found, for example, in [19].

Lemma 3.2. *For any $\gamma > 0$ and any function $v \in L^2$, we have*

$$e^{-2^{\gamma j+1} \lambda' t} \|\Delta_j v\|_{L^2} \leq \|G(t, \cdot) * \Delta_j v\|_{L^2} \leq e^{-2^{\gamma j+1} \lambda t} \|\Delta_j v\|_{L^2}, \quad (3.3)$$

where λ and λ' are some positive constants depending only on γ .

Lemma 3.3. *For any $\gamma > 0$ and $s \geq 0$, there exists a positive constant C depending only on s and γ such that for any $v \in L^2$, we have*

$$\sup_{t \in (0, \infty)} t^{s/\gamma} \|G(t, \cdot) * v\|_{\dot{H}_x^s} \leq C \|v\|_{L^2}, \quad (3.4)$$

$$\lim_{t \rightarrow 0} t^{s/\gamma} \|G(t, \cdot) * v\|_{\dot{H}_x^s} = 0, \quad \forall s > 0. \quad (3.5)$$

Moreover, for $s \in [0, \gamma/2]$ we have

$$\|G(t, \cdot) * v\|_{L_t^{\gamma/s} \dot{H}_x^s} \leq C \|v\|_{L^2} \quad (3.6)$$

As an easy consequence of Lemma 3.3, we have:

Lemma 3.4. *For any $\beta = \beta_1 + \beta_2$ with $\beta_1 \geq 0$ and $\beta_2 \in [0, \gamma/2]$, it holds that*

$$\|t^{\beta_1/\gamma} \|G(t, \cdot) * v\|_{\dot{H}^\beta} \|_{L_t^{\gamma/\beta_2}} < C \|v\|_{L^2}, \quad (3.7)$$

where C is a positive constant depending only on β_1, β_2 and γ .

Proof. By the semi-group property of the kernel $G(t, \cdot)$, (3.4) and (3.6), we get

$$\begin{aligned} & \|t^{\beta_1/\gamma} \|G(t, \cdot) * v\|_{\dot{H}^\beta} \|_{L_t^{\gamma/\beta_2}} \\ &= \|t^{\beta_1/\gamma} \|G(t/2, \cdot) * G(t/2, \cdot) * v\|_{\dot{H}^\beta} \|_{L_t^{\gamma/\beta_2}} \\ &\leq C \|G(t/2, \cdot) * v\|_{L_t^{\gamma/\beta_2} \dot{H}_x^{\beta_2}} \leq C \|v\|_{L^2}. \end{aligned}$$

The lemma is proved. \square

The next lemma is a commutator estimate, which is a key estimate in our proof. The proof of the lemma essentially follows that of Proposition 2 [19]. We defer it to Section 7.

Lemma 3.5. *Assume $m \geq 0$, $1 \leq s < 2$, $t < 1$ satisfying $m + t + s > 0$. Then there exists positive constant $C = C(s, t)$ such that*

$$\|[f, \Delta_j]g\|_{\dot{H}^m} \leq C 2^{-(s+t-1)j} c_j (\|f\|_{\dot{H}^{m+s}} \|g\|_{\dot{H}^t} + \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^{m+t}})$$

for any $j \in \mathbb{Z}$, $f \in H^{m+t}$ and $g \in H^{m+s}$ with $\|c_j\|_{l^2} \leq 1$. Here,

$$[f, \Delta_j]g = f \Delta_j g - \Delta_j (fg).$$

Remark 3.6. Define $\tilde{\Delta}_j = \sum_{|k-j| \leq 1} \Delta_j$. It is clear from the proofs later that we only need a weaker estimate

$$\begin{aligned} & \|\tilde{\Delta}_j [f, \Delta_j]g\|_{\dot{H}^m} \\ & \leq C 2^{-(s+t-1)j} c_j (\|f\|_{\dot{H}^{m+s}} \|g\|_{\dot{H}^t} + \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^{m+t}}). \end{aligned} \quad (3.8)$$

To get this estimate, the condition in Lemma 3.5 can be relaxed to $s < 2$, $t < 1$ and $m + t + s > 0$.

Finally, we shall also make use of the following lemma, which follows simply from Plancherel's equality and localization property of Littlewood-Paley projections. However, it is important in our proofs.

Lemma 3.7. *For any $j \in \mathbb{Z}$ and $u, v \in L^2$, we have*

$$\int_{\mathbb{R}^2} u \Delta_j v \, dx = \int_{\mathbb{R}^2} (\tilde{\Delta}_j u) (\Delta_j v) \, dx. \quad (3.9)$$

4. LOCAL SMOOTHING EFFECT I

Firstly, we give a general remark on our proofs. Recall that equation (1.1) can be rewritten as

$$\theta(t, \cdot) = G(t, \cdot) * \theta_0 - \int_0^t G(t-s, \cdot) * (u \cdot \nabla \theta)(s, \cdot) \, ds. \quad (4.1)$$

For the linear part, the estimate follows straightforwardly from Lemma 3.3 and 3.4. As usual, it is more difficult to get a good estimate of the

nonlinear term, especially in the critical and super-critical case. Notice that the kernel $G(t-s, \cdot)$ becomes singular as $s \rightarrow t$, and the initial data θ_0 is rough and only in $H^{2-\gamma}$. To deal with the nonlinear term, the idea is to divide the integral into two parts. For small s , we use the smoothness of the kernel $G(t-s, \cdot)$. For large s we should make use of the smoothness of $\theta(s)$ and $u(s)$. This technique has been used in [10, 23], and extensively in recent [12, 14, 20]. Although the formulation (4.1) does not appear explicitly in the proof below, we are still able to exploit this idea. Moreover, thanks to the flexibility of Lemma 3.5, the proof of the local smoothing effect is considerably simpler comparing to those in [10, 23, 12, 14, 20].

However, since the estimates such as Bernstein's inequality and fractional Leibniz's rule are quite rough, at present we are not able to get any analyticity rate estimate as in [23, 12, 20]. On the other hand, it would be very interesting to find out whether the mild solution of the critical quasi-geostrophic equation with arbitrary H^1 initial data is spatially analytic. We note here that in the super-critical case even the solutions to the corresponding linear equations are not spatially analytic. So one should not expect that for the nonlinear equations.

Proof of Theorem 2.2: Let θ be the solution in Proposition 2.1. Denote $\theta_j = \Delta_j \theta$ and recall $\Lambda = (-\Delta)^{1/2}$. For each $j \in \mathbb{Z}$, we apply the operator Δ_j to the both sides of (1.1) and get

$$\partial_t \theta_j + \Delta_j(u \cdot \nabla \theta) + \Lambda^\gamma \theta_j = 0.$$

Thus,

$$\partial_t \theta_j + u \cdot \nabla \theta_j + \Lambda^\gamma \theta_j = [u, \Delta_j] \nabla \theta. \quad (4.2)$$

After multiplying both sides of (4.2) by θ_j , integrating in x and noticing that u is divergence free, we obtain by using (3.1), Lemma 3.7 and Hölder's inequality that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_j\|_{L^2}^2 + \lambda 2^{\gamma j} \|\theta_j\|_{L^2}^2 &\leq \int_{\mathbb{R}^2} ([u, \Delta_j] \nabla \theta) \theta_j \, dx \\ &= \int_{\mathbb{R}^2} (\tilde{\Delta}_j [u, \Delta_j] \nabla \theta) \theta_j \, dx \\ &\leq \|\tilde{\Delta}_j [u, \Delta_j] \nabla \theta\|_{L^2} \|\theta_j\|_{L^2}. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \|\theta_j\|_{L^2} + \lambda 2^{\gamma j} \|\theta_j\|_{L^2} \leq 2 \|\tilde{\Delta}_j [u, \Delta_j] \nabla \theta\|_{L^2}. \quad (4.3)$$

Gronwall's inequality together with (4.3) yields

$$\begin{aligned} & \|\theta_j(t, \cdot)\|_{L^2} \\ & \leq e^{-2^{\gamma j} \lambda t} \|\theta_j(0)\|_{L^2} + 2 \int_0^t e^{-2^{\gamma j} \lambda(t-s)} \|\tilde{\Delta}_j[u, \Delta_j] \nabla \theta(s, \cdot)\|_{L^2} ds. \end{aligned} \quad (4.4)$$

We prove the theorem by an induction on β . Proposition (2.1) gives (2.4) and (2.5) for $\beta \in (0, \gamma)$. Now assume $\beta_0 \geq \gamma$, and (2.4) and (2.5) are true for any $\beta \in (0, \beta_0 - \gamma/6]$. Let's consider the case when $\beta = \beta_0$. We multiply the both sides of (4.4) by $2^{(2-\gamma+\beta_0)j}$, use (3.1), and split the integral in to two parts,

$$\|\theta_j(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta_0}} \leq 2^{(2-\gamma+\beta_0)j} e^{-2^{\gamma j} \lambda t} \|\theta_j(0, \cdot)\|_{L^2} + I_1 + I_2, \quad (4.5)$$

where

$$\begin{aligned} I_1 &= 2 \int_0^{t/2} 2^{(2-\gamma+\beta_0)j} e^{-2^{\gamma j} \lambda(t-s)} \|\tilde{\Delta}_j[u, \Delta_j] \nabla \theta(s, \cdot)\|_{L^2} ds, \\ I_2 &= 2 \int_{t/2}^t 2^{(2-\gamma+\beta_0)j} e^{-2^{\gamma j} \lambda(t-s)} \|\tilde{\Delta}_j[u, \Delta_j] \nabla \theta(s, \cdot)\|_{L^2} ds. \end{aligned}$$

We estimate I_1 and I_2 differently. In I_1 , we absorb (most part of) the factor $2^{(2-\gamma+\beta_0)j}$ to the 'kernel' $e^{-2^{\gamma j} \lambda(t-s)}$. While in I_2 , we absorb (most part of) that factor to the commutator term $\|\tilde{\Delta}_j[u, \Delta_j] \nabla \theta(s, \cdot)\|_{L^2}$ and use the localization property of $\tilde{\Delta}_j$ in the frequency space.

Estimate of I_1 : In Lemma 3.5 we take $m = 0$, $s = 2 - \gamma$, $t = 1 - 3\gamma/4$, $f = u$, $g = \nabla \theta$, and get

$$\begin{aligned} I_1 &\leq C c_j \int_0^{t/2} 2^{(3\gamma/4+\beta_0)j} e^{-2^{\gamma j} \lambda(t-s)} \|u(s, \cdot)\|_{\dot{H}^{2-\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}} ds \\ &\leq C c_j \int_0^{t/2} (t-s)^{-3/4-\beta_0/\gamma} \|\theta(s, \cdot)\|_{\dot{H}^{2-\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}} ds \\ &\leq C c_j t^{-\beta_0/\gamma} \sup_{s \in (0, t)} \|\theta(s, \cdot)\|_{\dot{H}^{2-\gamma}} \sup_{s \in (0, t)} (s^{1/4} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}}) \\ &\quad \cdot \int_0^{t/2} (t-s)^{-3/4} s^{-1/4} ds \\ &\leq C c_j t^{-\beta_0/\gamma} \sup_{s \in (0, t)} \|\theta(s, \cdot)\|_{\dot{H}^{2-\gamma}} \sup_{s \in (0, t)} (s^{1/4} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}}), \end{aligned}$$

where in the second inequality we use the boundedness of Riesz transforms in L^2 .

Estimate of I_2 : By the Bernstein's inequality, it holds that

$$2^{kj} \|\tilde{\Delta}_j[u, \Delta_j] \nabla \theta(s)\|_{L^2} \leq C \|\tilde{\Delta}_j[u, \Delta_j] \nabla \theta(s)\|_{\dot{H}^k}.$$

Recall that here we assume $\beta_0 \geq \gamma$. In Lemma 3.5 we take $m = \beta_0 - \gamma/2 \geq 0$, $s = 2 - 2\gamma/3$, $t = 1 - 2\gamma/3$, $f = u$ and $g = \nabla\theta$, and get

$$\begin{aligned}
I_2 &\leq C \int_{t/2}^t 2^{(2-\gamma/2)j} e^{-2^{\gamma j} \lambda(t-s)} \|\tilde{\Delta}_j[u, \Delta_j] \nabla\theta(s, \cdot)\|_{\dot{H}^{\beta_0-\gamma/2}} ds \\
&\leq C c_j \int_{t/2}^t 2^{5\gamma j/6} e^{-2^{\gamma j} \lambda(t-s)} (\|u(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}} \|\theta(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}} \\
&\quad + \|u(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}}) ds \\
&\leq C c_j \int_{t/2}^t (t-s)^{-5/6} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}} \|\theta(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}} ds \\
&\leq C c_j \sup_{s \in (0,t)} (s^{\beta_0/\gamma-1/6} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}}) \sup_{s \in (0,t)} (s^{1/3} \|\theta(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}}) \\
&\quad \cdot t^{-\beta_0/\gamma} \int_{t/2}^t (t-s)^{-5/6} s^{-1/6} ds \\
&\leq C c_j t^{-\beta_0/\gamma} \sup_{s \in (0,t)} (s^{\beta_0/\gamma-1/6} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}}) \\
&\quad \cdot \sup_{s \in (0,t)} (s^{1/3} \|\theta(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}}),
\end{aligned}$$

where in the second inequality we again use the boundedness of Riesz transforms.

Now we take the l_2 norm of both sides of (4.5) in $j \in \{-N, -N+1, \dots, N-1, N\}$ for some positive integer N and then multiply both sides by $t^{\beta_0/\gamma}$. Owing to (3.1) and Lemma 3.2, 3.3, it holds that

$$\begin{aligned}
&t^{\beta_0/\gamma} \left(\sum_{j=-N}^N (\|\theta_j\|_{\dot{H}_x^{2-\gamma+\beta_0}}^2) \right)^{1/2} \\
&\leq C \sup_{s \in (0, C_1 t)} (s^{\beta_0/\gamma} \|G(s, \cdot) * \theta_0\|_{\dot{H}^{2-\gamma+\beta_0}}) \\
&\quad + C \sup_{s \in (0,t)} \|\theta(s, \cdot)\|_{\dot{H}^{2-\gamma}} \sup_{s \in (0,t)} (s^{1/4} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}}) \\
&\quad + C \sup_{s \in (0,t)} (s^{\frac{\beta_0}{\gamma}-\frac{1}{6}} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-\frac{7\gamma}{6}}}) \sup_{s \in (0,t)} (s^{1/3} \|\theta(s, \cdot)\|_{\dot{H}^{2-\frac{2\gamma}{3}}}), \quad (4.6)
\end{aligned}$$

where C and C_1 are positive constants independent of t . In the above inequality, the first term on the right-hand side is bounded with respect to t and goes to zero as $t \rightarrow 0$ due to Lemma 3.3. The second and the third term is bounded for $t \in (0, T)$ and go to zero as $t \rightarrow 0$ by the inductive assumption. Letting $N \rightarrow +\infty$ in (4.6) yields (2.4) and (2.5) for $\beta = \beta_0$. Theorem 2.2 is then proved.

Proof of Theorem 2.4: The proofs of the first part of the theorem and the second part for $\beta \in [0, \gamma)$ can be found in [19]. We only need to show the second part for $\beta \geq \gamma$. However, this follows immediately from the induction argument in the proof of Theorem 2.2 and (4.6). This completes the proof of Theorem 2.4.

5. LOCAL SMOOTHING EFFECT II

This section is devoted to the proof of Theorem 2.5. First we consider the case when $\beta \in [0, \gamma/2]$. As

$$\theta \in L^\infty([0, T]; H^{2-\gamma}) \cap L^2(0, T; H^{2-\gamma/2}),$$

for $\beta_1 = 0$ and $\beta_2 = \beta \in [0, \gamma/2]$, by Hölder's inequality and the interpolation estimate, we obtain

$$\theta \in L^{\gamma/\beta}([0, T]; \dot{H}^{2-\gamma+\beta}). \quad (5.1)$$

This together with (2.4) concludes Theorem 2.5 in its full generality when $\beta \in [0, \gamma/2]$.

Next we assume $\beta_0 > \gamma/2$ and proceed by an induction on β . Suppose (2.7) has been proved for $\beta \in [0, \beta_0 - \gamma/6]$. Let's consider the case when $\beta = \beta_0$ and assume $\beta_0 = \beta_1 + \beta_2$ for some $\beta_1 > 0$ and $\beta_2 \in [0, \gamma/2]$. Note that the estimates of both I_1 and I_2 still holds true if we only assume $\beta_0 > \gamma/2$. Because of Theorem 2.2, we already know that $\theta(t, \cdot) \in \dot{H}^{2-\gamma+\beta}$ for any $t > 0$. Taking the l_2 norm of both sides of (4.5) in $j \in \mathbb{Z}$ and then multiply both sides by $t^{\beta_1/\gamma}$ instead of $t^{\beta_0/\gamma}$ in the previous section, we obtain

$$t^{\beta_1/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta_0}} \leq C t^{\beta_0/\gamma} \|G(C_1 t, \cdot) * \theta_0\|_{\dot{H}^{2-\gamma+\beta_0}} + C I_3 + C I_4, \quad (5.2)$$

where

$$I_3 = \int_0^{t/2} t^{\beta_1/\gamma} (t-s)^{-3/4-\beta_0/\gamma} \|u(s, \cdot)\|_{\dot{H}^{2-\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}} ds,$$

$$I_4 = \int_{t/2}^t t^{\beta_1/\gamma} (t-s)^{-5/6} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}} \|\theta(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}} ds.$$

We then show that all the three terms on the right-hand side of (5.2) are in $L^{\gamma/\beta_2}(0, T)$.

Due to Lemma 3.4, the first term is indeed in $L^{\gamma/\beta_2}(0, \infty)$. For I_3 , we compute

$$I_3 \leq C \int_0^{t/2} (t-s)^{-3/4-\beta_2/\gamma} \|\theta(s, \cdot)\|_{\dot{H}^{2-\gamma}} \|\theta(s, \cdot)\|_{\dot{H}^{2-3\gamma/4}} ds.$$

Owing to Proposition 2.1, we have

$$\|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma}} \|\theta(t, \cdot)\|_{\dot{H}^{2-3\gamma/4}} \in L^4((0, T)).$$

This together with the fractional integration yields

$$I_3 \in L^{\gamma/\beta_2}(0, T).$$

Finally, I_4 is less than

$$C \int_{t/2}^t (t-s)^{-5/6} (s^{\beta_1/\gamma-1/6} \|\theta(s, \cdot)\|_{\dot{H}^{2+\beta_0-7\gamma/6}}) (s^{1/6} \|\theta(s, \cdot)\|_{\dot{H}^{2-2\gamma/3}}) ds$$

By the inductive assumption, we have,

$$\begin{aligned} t^{1/6} \|\theta(t, \cdot)\|_{\dot{H}^{2-2\gamma/3}} &\in L^6((0, T)), \\ t^{\beta_1/\gamma-1/4} \|\theta(t, \cdot)\|_{\dot{H}^{2+\beta_0-5\gamma/4}} &\in L^{\gamma/\beta_2}((0, T)). \end{aligned}$$

These estimate together with the fractional integration yield $I_4 \in L^{\gamma/\beta_2}(0, T)$. It follows that (2.7) holds for $\beta = \beta_0$. The theorem is proved.

6. GLOBAL WELL-POSEDNESS WHEN $\gamma = 1$

As we discussed in Remark 2.3, the solution θ and u become smooth immediately for $t > 0$. Fix a $t_1 \in (0, T)$. Then we can consider $\theta(t_1)$ as initial data and apply the result of the global existence for smooth initial data in [15]. The boundedness of θ and its derivatives follows from the uniform bound

$$\|\nabla\theta(t, \cdot)\|_{L^\infty} \leq C \|\nabla\theta_0\|_{L^\infty} e^{C\|\theta_0\|_{L^\infty}} \quad (6.1)$$

established in [15] and Theorem 2.2. The solution is in $C([0, \infty); H^1) \cap L^2_{\text{loc}}((0, \infty); H^{3/2})$. The uniqueness then follows in a standard way from the local uniqueness result (see, e.g. [19]). To see the solution is also in (2.9), it suffices to verify the decay estimate (2.10).

Denote $\hat{\theta}(t, j), j \in \mathbb{Z}^2$ to be the Fourier coefficients of $\hat{\theta}(t, \cdot)$. Recall that $\theta(t, 0) \equiv 0$ for any $t \geq 0$. Since θ and u are smooth, Theorem 4.1 of Córdoba and Córdoba [4] yields the following lemma.

Lemma 6.1. *Under the assumptions of Theorem 2.7, there exists a positive constant C depending only on θ_0 so that*

$$\|\theta(t, \cdot)\|_{L^\infty} \leq C/(1+t)$$

for any $t \geq t_1$.

Thus we can choose t large so that $\|\theta(t, \cdot)\|_{L^\infty}$ is as small as we want. This together with a small data result due to Constantin, Córdoba and Wu [5] implies the spatial analyticity of θ for $t \geq T_0$ for some $T_0 \geq t_1$. More precisely, we have

Lemma 6.2. *Under the assumptions of Theorem 2.7, there exists $T_0 \geq t_1$ such that*

$$y(t) := \sum_{j \in \mathbb{Z}^2 \setminus \{(0,0)\}} |\hat{\theta}(t, j)| e^{(t-T_0)|j|/2} \leq 1/2, \quad (6.2)$$

for any $t \geq T_0$.

We claim that (6.2) implies (2.10). Indeed, for $t \in (0, T_0)$ estimate (2.10) is an immediate consequence of Theorem 2.2, (6.1) and Poincaré's inequality. For any $t \geq T_0$, we have

$$\begin{aligned} e^{t/2} t^{2\beta/\gamma} \|\theta(t, \cdot)\|_{H^{2-\gamma+\beta}}^2 &\leq C e^{t/2} t^{2\beta/\gamma} \sum_{j \in \mathbb{Z}^2 \setminus \{(0,0)\}} |\hat{\theta}(t, j)|^2 |j|^{2(2-\gamma+\beta)} \\ &\leq C (y(t))^2 \leq C. \end{aligned}$$

This finishes the proof of Theorem 2.7.

7. A COMMUTATOR ESTIMATE

This section is devoted to the proof of Lemma 3.5. We follow closely the idea of Proposition 2 in [19] (see also earlier [3, 8, 9] for similar estimates). However, since we also consider higher order Sobolev norms by introducing a parameter m , we give a proof here for the sake of completeness. It is worth noting that from the proof below the condition of Lemma 3.5 can be relaxed.

We start with the definition of Bony's paraproduct operator and some basic estimates for the paraproduct operator (see, e.g. [21]). Define paraproduct operators by

$$T_f g := \sum_{j \in \mathbb{Z}} S_j f \Delta_j g, \quad R(f, g) := \sum_{|i-j| \leq 2} \Delta_i f \Delta_j g,$$

where $S_j f = \sum_{k \leq j-3} \Delta_k f$. Then we have

$$fg = T_f g + T_g f + R(f, g).$$

Lemma 7.1. *i) If $s < 1, t \in \mathbb{R}$, there exists a positive constant C depending only on s and t such that for any $f \in \dot{H}^s(\mathbb{R}^2)$ and $g \in \dot{H}^t(\mathbb{R}^2)$ we have*

$$\|T_f g\|_{\dot{H}^{s+t-1}} \leq C \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^t}. \quad (7.1)$$

ii) If $s+t > 0$, there exists a positive constant C depending only on s and t such that for any $f \in \dot{H}^s(\mathbb{R}^2)$ and $g \in \dot{H}^t(\mathbb{R}^2)$ we have

$$\|R(f, g)\|_{\dot{H}^{s+t-1}} \leq C \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^t}. \quad (7.2)$$

iii) If $s, t < 1$ and $s + t > 0$, there exists a positive constant C depending only on s and t such that for any $f \in \dot{H}^s(\mathbb{R}^2)$ and $g \in \dot{H}^t(\mathbb{R}^2)$ we have

$$\|fg\|_{\dot{H}^{s+t-1}} \leq C\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^t}. \quad (7.3)$$

The following fractional Leibniz's rule is well-known.

Lemma 7.2. *Assume $s \geq 0$ and $p \in (1, \infty)$. Then we have*

$$\|fg\|_{\dot{W}^{s,p}} \leq C\|f\|_{\dot{W}^{s,p_1}}\|g\|_{L^{p_2}} + C\|f\|_{L^{p'_1}}\|g\|_{\dot{W}^{s,p'_2}}$$

if the right-hand side is finite. Here $p_1, p_2, p'_1, p'_2 \in (1, +\infty)$ satisfy

$$1/p = 1/p_1 + 1/p_2 = 1/p'_1 + 1/p'_2.$$

Now we are ready to prove Lemma 3.5. Denote

$$f_j = \Delta_j f, \quad g_j = \Delta_j g, \quad \check{\Delta}_j = \sum_{|k-j| \leq 2} \Delta_k$$

for any $j \in \mathbb{Z}$. In terms of paraproducts, we have

$$\begin{aligned} [f, \Delta_j]g &= -\Delta_j R(f, g) - \Delta_j(T_g f) + [T_f, \Delta_j]g + R(f, g_j) + T_{g_j} f \\ &= -\Delta_j R(f, g) - \Delta_j(T_g f) + \sum_{|k-j| \leq 3} [S_k f, \Delta_j]g_k \\ &\quad + \sum_{|k-j| \leq 2} \check{\Delta}_k f \Delta_k g_j + \sum_{k \geq j+1} S_k g_j f_k \\ &:= I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where in the second equality above we use the localization property of Littlewood-Paley projections in the frequency space. Choose $p_1 \in (2, \infty)$ sufficiently large so that $s + 2/p_1 < 2$. This is possible because $s < 2$. Let $p_2 \in (2, \infty)$ be a number satisfying $1/p_1 + 1/p_2 = 1/2$.

Estimate of I_1 : Because $m + s + t > 0$, by using (7.2) with $m + s$ and t in place of s and t respectively, we get

$$\|I_1\|_{\dot{H}^m} \leq Cc_j 2^{(1-s-t)j} \|f\|_{\dot{H}^{m+s}} \|g\|_{\dot{H}^t},$$

where

$$c_j = 2^{(s+t-1)j} \|\Delta_j R(f, g)\|_{\dot{H}^m} / \|R(f, g)\|_{\dot{H}^{m+s+t-1}}.$$

Estimate of I_2 : Since $t < 1$, (7.1) with $t + m$ in place of t gives

$$\|I_2\|_{\dot{H}^m} \leq \tilde{c}_j 2^{(1-s-t)j} \|f\|_{\dot{H}^{s+m}} \|g\|_{\dot{H}^t},$$

where

$$\tilde{c}_j = 2^{(s+t-1)j} \|\Delta_j T_g f\|_{\dot{H}^m} / \|T_g f\|_{\dot{H}^{m+s+t-1}}$$

Estimate of I_3 : The estimate of I_3 is more delicate. By the mean value theorem, we have

$$\begin{aligned} I_3 &= \sum_{|k-j|\leq 3} \int_{\mathbb{R}^2} \int_0^1 \phi_j(y) y (S_k \nabla f)(x - sy) g_k(x - y) ds dy \\ &= 2^{-j} \sum_{|k-j|\leq 3} \int_{\mathbb{R}^2} \int_0^1 \phi(y) y (S_k \nabla f)(x - 2^{-j} sy) g_k(x - 2^{-j} y) ds dy. \end{aligned}$$

Now due to Minkowski's inequality and Lemma 7.2, we get

$$\|I_3\|_{\dot{H}^m} \leq C 2^{-j} \sum_{|k-j|\leq 3} \left(\|S_k \nabla f\|_{\dot{W}^{m,p_1}} \|g_k\|_{L^{p_2}} + \|S_k \nabla f\|_{L^{p_1}} \|g_k\|_{\dot{W}^{m,p_2}} \right). \quad (7.4)$$

Recall $s + 2/p_1 < 2$. Then by Hölder's inequality,

$$|S_k \Lambda^m \nabla f| \leq C 2^{(2-s-2/p_1)k} \|2^{(s+2/p_1-2)i} \Lambda^m \nabla f_i\|_{l^2}.$$

Therefore,

$$\begin{aligned} \|S_k \nabla f\|_{\dot{W}^{m,p_1}} &\leq C 2^{(2-s-2/p_1)k} \| \|2^{(s+2/p_1-2)i} \Lambda^m \nabla f_i\|_{l^2} \|_{L^{p_1}} \\ &= C 2^{(2-s-2/p_1)k} \|f\|_{\dot{W}^{m+s+2/p_1-1,p_1}} \\ &\leq C 2^{(2-s-2/p_1)k} \|f\|_{\dot{H}^{m+s}}, \end{aligned}$$

where in the last inequality we use Sobolev embedding theorem. Similarly,

$$\|S_k \nabla f\|_{L^{p_1}} \leq C 2^{(2-s-2/p_1)k} \|f\|_{\dot{H}^s}.$$

These estimates together with (7.4) and Lemma 3.1 yield

$$\begin{aligned} &\|I_3\|_{\dot{H}^m} \\ &\leq C 2^{(1-s-2/p_1)j} \left(\|f\|_{\dot{H}^{m+s}} \sum_{|k-j|\leq 3} \|g_k\|_{L^{p_2}} + \|f\|_{\dot{H}^s} \sum_{|k-j|\leq 3} \|g_k\|_{\dot{W}^{m,p_2}} \right) \\ &\leq C 2^{(1-s-t)j} \left(\|f\|_{\dot{H}^{m+s}} \sum_{|k-j|\leq 3} 2^{(t-2/p_1)k} \|g_k\|_{L^{p_2}} \right. \\ &\quad \left. + \|f\|_{\dot{H}^s} \sum_{|k-j|\leq 3} 2^{(t-2/p_1)k} \|g_k\|_{\dot{W}^{m,p_2}} \right) \\ &\leq C 2^{(1-s-t)j} \left(\|f\|_{\dot{H}^{m+s}} \sum_{|k-j|\leq 3} 2^{tk} \|g_k\|_{L^2} + \|f\|_{\dot{H}^s} \sum_{|k-j|\leq 3} 2^{tk} \|g_k\|_{\dot{H}^m} \right) \\ &\leq C 2^{(1-s-t)j} \bar{c}_j (\|f\|_{\dot{H}^{m+s}} \|g\|_{\dot{H}^t} + \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^{m+t}}), \end{aligned}$$

where

$$\bar{c}_j = \frac{1}{100} \sum_{|k-j| \leq 3} 2^{tk} \|g_k\|_{L^2} / \|g\|_{\dot{H}^t} + \frac{1}{100} \sum_{|k-j| \leq 3} 2^{tk} \|g_k\|_{\dot{H}^m} / \|g\|_{\dot{H}^{m+t}}.$$

It is easily seen that $\|\bar{c}_j\|_{l^2} \leq 1$, which completes the estimate of I_3 .

Estimate of I_4 : Lemma 7.2 yields

$$\begin{aligned} \|I_4\|_{\dot{H}^m} &\leq \sum_{|k-j| \leq 2} \|\check{\Delta}_k f \Delta_k g_j\|_{\dot{H}^m} \\ &\leq C \sum_{|k-j| \leq 2} \left(\|\check{\Delta}_k f\|_{\dot{W}^{m,p_1}} \|\Delta_k g_j\|_{L^{p_2}} + \|\check{\Delta}_k f\|_{L^{p_1}} \|\Delta_k g_j\|_{\dot{W}^{m,p_2}} \right) \\ &\leq C 2^{(1-s-t)j} \sum_{|k-j| \leq 2} \left(2^{k(s+2/p_1-1)} \|\check{\Delta}_k f\|_{\dot{W}^{m,p_1}} 2^{k(t-2/p_1)} \|\Delta_k g_j\|_{L^{p_2}} \right. \\ &\quad \left. + 2^{k(s+2/p_1-1)} \|\check{\Delta}_k f\|_{L^{p_1}} 2^{k(t-2/p_1)} \|\Delta_k g_j\|_{\dot{W}^{m,p_2}} \right) \\ &\leq C 2^{(1-s-t)j} \bar{c}_j (\|f\|_{\dot{H}^{m+s}} \|g\|_{\dot{H}^t} + \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^{m+t}}), \end{aligned}$$

where $\bar{c}_j, j \in \mathbb{Z}$ are the same constants as in the estimate of I_3 .

Estimate of I_5 : By using the boundedness of the operator S_k in $L^p, p \in (1, \infty)$ and Lemma 7.2, we have

$$\begin{aligned} \|I_5\|_{\dot{H}^m} &\leq C \|g_j\|_{\dot{W}^{m,p_2}} \sum_{k \geq j+1} \|f_k\|_{L^{p_1}} + C \|g_j\|_{L^{p_2}} \sum_{k \geq j+1} \|f_k\|_{\dot{W}^{m,p_1}} \\ &:= I_{51} + I_{52}. \end{aligned}$$

By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned} I_{51} &\leq C 2^{(-t+1-2/p_2)j} \|g_j\|_{\dot{H}^{m+t}} \sum_{k \geq j+1} 2^{(1-2/p_1)k} \|f_k\|_{L^2} \\ &\leq C \bar{c}_j 2^{(-t+1-2/p_2)j} \|g\|_{\dot{H}^{m+t}} \left(\sum_{k \geq j+1} 2^{2sk} \|f_k\|_{L^2}^2 \right)^{1/2} 2^{(1-2/p_1-s)j} \\ &\leq C \bar{c}_j 2^{(-t-s+1)j} \|g\|_{\dot{H}^{m+t}} \|f\|_{\dot{H}^s}. \end{aligned}$$

In a similar way,

$$I_{52} \leq C \bar{c}_j 2^{(-t-s+1)j} \|g\|_{\dot{H}^t} \|f\|_{\dot{H}^{m+s}}.$$

Combining all these estimates together finishes the proof of the lemma.

As we mentioned in Remark 3.6, in the proofs of the main theorems we only use the estimate of a frequency localized object $\tilde{\Delta}_j[f, \Delta_j]g$

instead of $[f, \Delta_j]g$ itself. Notice that

$$\tilde{\Delta}_j I_5 = \tilde{\Delta}_j \sum_{j+1 \leq k \leq j+4} S_k g_j f_k.$$

Now due to the finiteness of the number of the sum on k and boundedness of $\tilde{\Delta}_j$, in the estimate of $\tilde{\Delta}_j I_5$ the condition that $s \geq 1$ can be removed. Moreover, in the estimates of I_3 , I_4 and I_5 , where Lemma 7.2 is applied, we may estimate $\|\tilde{\Delta}_j I_l\|_{\dot{H}^{m+s+t}}$, $l = 3, 4, 5$ instead of $\|\Delta_j I_l\|_{\dot{H}^m}$, $l = 3, 4, 5$ and still get the same bounds. Therefore, the condition $m \geq 0$ can also be removed too. Since these are the only places using these two conditions, we remark that to obtain (3.8) we only require $s < 2$, $t < 1$ and $m + t + s > 0$.

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