Appl Math Optim 52:365–399 (2005)
DOI: 10.1007/s00245-005-0832-3

Applied Mathematics & Optimization

The Rate of Convergence of Finite-Difference Approximations for Bellman Equations with Lipschitz Coefficients*

Nicolai V. Krylov

127 Vincent Hall, University of Minnesota,
Minneapolis, MN 55455, USA
krylov@math.umn.edu

Abstract. We consider parabolic Bellman equations with Lipschitz coefficients. Error bounds of order $h^{1/2}$ for certain types of finite-difference schemes are obtained.

Key Words. Finite-difference approximations, Bellman equations, Fully nonlinear equations.

AMS Classification. 65M15, 35J60, 93E20.

1. Introduction

Bellman equations arise in many areas of mathematics, say in control theory, differential geometry, and mathematical finance, to name a few. These equations typically are fully nonlinear second-order degenerate elliptic or parabolic equations. In the particular case of complete degeneration they become Hamilton–Jacobi first-order equations.

Quite naturally, the problem of finding numerical methods of approximating solutions to Bellman equations arises. First methods dating back some 30 years ago were based on the fact that the solutions are the value functions in certain problems for controlled diffusion processes, that can be approximated by controlled Markov chains. An account of the results obtained in this direction can be found in [5], [12], and [17]. These results are based, in part, on the theory of weak convergence of probability measures and, in part, on a remarkable result of [4].

Another approach is based on the notion of viscosity solution, which allows one to avoid using probability theory. We refer to [2] and [3] and the references therein for discussion of what is achieved in this direction.

---

* This work was partially supported by NSF Grant DMS-0140405.

We deal with degenerate *second-order* equations. There is a very extensive literature treating Hamilton–Jacobi equations or quasilinear equations and establishing the rate of convergence of various numerical approximations. The reader can find how much was done for them in [1], [6], [7], [11], and [19]. In contrast, until quite recently there were no results about the rate of convergence of finite-difference approximations for degenerate second-order fully nonlinear Bellman equations. The first results appeared only in 1997 for elliptic Bellman equations with constant "coefficients" (see [14]) and they were later extended to variable coefficients and parabolic equations in [2], [3], [15], and [16]. Surprisingly, as far as we know until now these are *the only* published results on the rate of convergence of finite-difference approximations even if the Bellman equation becomes a *linear* second-order degenerate equation. One has to notice however that there is vast literature about other types of numerical approximations for linear degenerate equations such as Galerkin or finite-element approximations (see, for instance, [18]). It is also worth noting that under a variety of conditions the first *sharp* estimates for finite-difference approximations in linear one-dimensional degenerate case are proved in [20].

Our approach is based on two ideas from [14]–[16] that the original equation and its finite-difference approximation should play symmetric roles and that one can "shake the coefficients" of the equation in order to be able to mollify under the sign of a nonlinear operator. While shaking the coefficients of the approximate equation we encounter a major problem of estimating how much the solution of the shaken equation differs from the original one. Solving this problem amounts to estimating the Lipschitz constant of the approximate solution. We prove this estimate on the basis of Theorem 5.2 and consider this theorem as the most important technical result of the present paper. Theorem 5.2 is new even if the equation is linear although in that case one can give a much simpler proof (see [9]).

Our main result says that for parabolic equations in a special form with $C^{1/2,1}$ coefficients, the rate of convergence is not less than $\tau^{1/4} + h^{1/2}$, where τ and h are the time and space steps, respectively. Simple examples show that under our conditions the estimate is sharp even for the case of linear first-order equations. For the elliptic case the rate becomes $h^{1/2}$, which under comparable conditions is slightly better than $h^{1/5}$ from [3].

The main emphasis of this paper is on *constructing* finite-difference approximations as good as possible for a given Bellman equation. There is another part of the story when one is interested in how more or less arbitrary consistent finite-difference type approximations converge to the true solution. In this direction the known results are somewhat weaker. We only know that for $\tau = h^2$ there is an estimate of order $h^{1/21}$, which sometimes becomes $h^{1/3}$ (see [14] and [15]).

It is often hard to write Bellman equations in nice analytical way. Therefore, it is worth reiterating that one particular case of Bellman equations is given by *linear* (degenerate) parabolic equations in which case the theory can be largely simplified and yields better results under additional assumptions. The interested reader can find more information about this in [9].

Another particular degenerate Bellman equation arises as an obstacle problem in PDEs or as an optimal stopping problem in stochastic control:

$$\max(\Delta u - u, -u + g) = 0,$$

where g is a given function. Here we have two operators, one is nondegenerate elliptic,

$\Delta u - u$, and the second one is completely degenerate elliptic, $-u$. One usually rewrites the above equation in an equivalent form:

$$\Delta u - u \le 0, \qquad g \le u, \qquad \Delta u - u = 0 \quad \text{on } \{u > g\}.$$

To conclude the Introduction, we set up some notation: $\mathbb{R}^d$ is a d-dimensional Euclidean space; $x = (x^1, x^2, \dots, x^d)$ is a typical point in $\mathbb{R}^d$. As usual the summation convention over repeated indices is enforced. For any $l \in \mathbb{R}^d$ and any differentiable function u on $\mathbb{R}^d$, we denote

$$D_l u = u_{x^i} l^i, \qquad D_l^2 u = u_{x^i x^j} l^i l^j,$$

etc. The symbol $D_t^n u$ stands for the nth derivative in t of $u = u(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $D_x^n u$ for the collection of all nth-order derivatives of u in x. We also use the notation

$$|u|_{0,Q} = \sup_Q |u|.$$

Various constants are denoted by N, their value may change from one occurrence to another. We set

$$a_\pm = a^\pm = \tfrac{1}{2}(|a| \pm a).$$

2. Main Results

Let A be a separable metric space, constants

$$T \in (0, \infty), \qquad K \in [1, \infty), \qquad \lambda \in [0, \infty), \qquad \text{integers} \quad d, d_1 \ge 1.$$

Suppose that we are given $\ell_k \in \mathbb{R}^d$ and real-valued

$$\sigma_k^\alpha(t, x), \quad b_k^\alpha(t, x), \quad c^\alpha(t, x), \quad f^\alpha(t, x), \quad g(x)$$

defined for $k = \pm 1, \dots, \pm d_1$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, and $\alpha \in A$ such that

$$\ell_k = -\ell_{-k}, \qquad \sigma_k^\alpha = \sigma_{-k}^\alpha, \qquad b_k^\alpha \ge 0, \qquad c^\alpha \ge \lambda, \qquad |\ell_k| \le K.$$

Assumption 2.1. For $\psi = \sigma_k^\alpha, b_k^\alpha, c^\alpha - \lambda, f^\alpha, g$, $k = 1, \dots, d_1$, $\alpha \in A$, for each $t \in [0, T]$ and $x, y \in \mathbb{R}^d$ we have

$$|\psi(t, x)| \le K, \qquad |\psi(t, x) - \psi(t, y)| \le K|x - y|.$$

We also assume that $\sigma_k^\alpha(t, x)$, $b_k^\alpha(t, x)$, $c^\alpha(t, x)$, $f^\alpha(t, x)$ are Borel in t and continuous in α.

Introduce

$$F(p_k, q_k, r, t, x) = \sup_{\alpha \in A} [a_k^\alpha(t, x) p_k + b_k^\alpha(t, x) q_k - c^\alpha(t, x) r + f^\alpha(t, x)]$$

with the summation in k performed before the supremum is taken.

Under the above assumptions there is a probabilistic solution v of the Bellman equation

$$\frac{\partial}{\partial t} u(t, x) + F(D_{\ell_k}^2 u(t, x), D_{\ell_k} u(t, x), u(t, x), t, x) = 0 \tag{2.1}$$

in

$$H_T := [0, T) \times \mathbb{R}^d$$

with terminal condition

$$u(T, x) = g(x), \qquad x \in \mathbb{R}^d. \tag{2.2}$$

This solution is constructed by means of control theory. The reader unfamiliar with control theory may consider v as the unique bounded viscosity solution of the above problem (see, for instance, [12] and [17]).

For $h, \tau > 0$, $l \in \mathbb{R}^d$, $(t, x) \in [0, T) \times \mathbb{R}^d$ introduce

$$\begin{aligned}
\delta_{h,l} u(t, x) &= \frac{u(t, x + hl) - u(t, x)}{h}, \\
\Delta_{h,l} u(t, x) &= -\delta_{h,l}\delta_{h,-l} u(t, x) \\
&= \frac{\delta_{h,l} + \delta_{h,-l}}{h} u(t, x) \\
&= \frac{u(t, x + hl) - 2u(t, x) + u(t, x - hl)}{h^2}, \\
\delta_\tau u(t, x) &= \frac{u(t + \tau, x) - u(t, x)}{\tau}, \\
\delta_\tau^T u(t, x) &= \frac{u(t + \tau_T(t), x) - u(t, x)}{\tau}, \qquad \tau_T(t) = \tau \wedge (T - t).
\end{aligned}$$

Just in case, notice that in the denominator of $\delta_\tau^T u$ we write τ and not $\tau_T(t)$. This is important in the proof of Lemma 6.2. Also note that

$$t + \tau_T(t) = (t + \tau) \wedge T.$$

Set

$$\delta_{0,l} u = 0, \qquad a_k^\alpha = \tfrac{1}{2}(\sigma_k^\alpha)^2.$$

In H_T consider the following equation with respect to a function u given in $\bar{H}_T$:

$$\delta_\tau^T u(t, x) + F(\Delta_{h,\ell_k} u(t, x), \delta_{h,\ell_k} u(t, x), u(t, x), t, x) = 0 \tag{2.3}$$

with terminal condition (2.2).

Equation (2.3) is an implicit finite-difference approximation for the Bellman equation (2.1). Existence of a unique bounded solution of problem (2.3)–(2.2), which we denote by $v_{\tau,h}$, is a standard fact proved by successive approximations in Lemma 3.1 (see also the comments before that lemma).

To proceed further we need the following.

Assumption 2.2. For $\psi = \sigma_k^\alpha, b_k^\alpha, c^\alpha - \lambda, f^\alpha$, $k = 1, \ldots, d_1$, $\alpha \in A$, for each $x \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$ we have

$$|\psi(t, x) - \psi(s, x)| \le K|t - s|^{1/2}.$$

Here are our main results.

Theorem 2.3. *Let $\tau, h \le 1$. Then there exists a constant N_1 depending only on d, d_1, T, and K (but not h or τ) such that*

$$|v - v_{\tau,h}| \le N_1(\tau^{1/4} + h^{1/2}) \tag{2.4}$$

in H_T. In addition, there exists a constant N_2 depending only on d_1 and K, such that if $\lambda \ge N_2$, then N_1 is independent of T.

Introduce

$$L^\alpha u = a_k^\alpha D_{\ell_k}^2 u + b_k^\alpha D_{\ell_k} u - c^\alpha u,$$
$$L_h^\alpha u = a_k^\alpha \Delta_{h,\ell_k} u + b_k^\alpha \delta_{h,\ell_k} u - c^\alpha u.$$

The operators L_h^α approximate L^α in certain sense (see (2.8) below) and they are automatically monotone in the sense that if a function $u(x)$ attains a nonnegative maximum at a point x_0, then $L_h^\alpha u(t, x_0) \le 0$ for all α, h, and t.

Theorem 2.4. *Suppose that σ, b, c, f are independent of t and $\lambda \ge N_2$, where N_2 is taken from Theorem 2.3. Let $\tilde{v}(x)$ be probabilistic or the unique bounded viscosity solution of*

$$\sup_{\alpha \in A}[L^\alpha u + f^\alpha] = 0$$

in $\mathbb{R}^d$. Let $\tilde{v}_h$ be the unique bounded solution of

$$\sup_{\alpha \in A}[L_h^\alpha u + f^\alpha] = 0 \tag{2.5}$$

in $\mathbb{R}^d$. Then for $h \le 1$,

$$|\tilde{v} - \tilde{v}_h| \le N h^{1/2}$$

in $\mathbb{R}^d$, where N depends only on d, d_1, and K.

The following result about semidiscretization allows one to use approximations of the time derivative different from the one in (2.3), in particular, explicit schemes could be used.

Theorem 2.5. *There exists a unique bounded solution $v_h(t, x)$ of*

$$\frac{\partial}{\partial t} u(t, x) + F(\Delta_{h,\ell_k} u(t, x), \delta_{h,\ell_k} u(t, x), u(t, x), t, x) = 0 \tag{2.6}$$

in H_T with terminal condition (2.2). Furthermore, there exists a constant N_1 depending only on K, T, d, and d_1 such that for $h \le 1$,

$$|v - v_h| \le N_1 h^{1/2}$$

in H_T. Finally, there is a constant N_2 depending only on K and d_1 such that if $\lambda \geq N_2$, then N_1 is independent of T.

We prove the above results in Section 7, after proving some auxiliary statements in Sections 3 and 4. Then comes the main estimate of the Lipschitz constant in x in Section 5 and finally the Hölder $\frac{1}{2}$ continuity in t in Section 6.

Remark 2.6. In a subsequent article we will show that Assumption 2.2 is not needed in Theorem 2.5. A few other possible extensions of the above results are discussed in Section 8.

For completeness we reproduce the following remark from [8], which in particular clarifies why we allow the set of directions $\{\ell_k\}$ to be different from the canonical Euclidean frame.

Remark 2.7. One may think that considering the operators L^α written in the form $a_k^\alpha D_{\ell_k}^2 + b_k^\alpha D_{\ell_k}$ is a severe restriction. However, it turns out that if we fix a finite subset $B \subset \mathbb{Z}^d$, such that $\operatorname{Span} B = \mathbb{R}^d$, and if an operator $Lu = a^{ij} u_{x^i x^j} + b^i u_{x^i}$ admits a finite-difference approximation

$$L_h u(x) = \sum_{y \in B} p_h(y) u(x + hy)$$

which is monotone, then automatically

$$L = \sum_{\substack{l \in B \\ l \neq 0}} a_l D_l^2 + \sum_{\substack{l \in B \\ l \neq 0}} b_l D_l$$

for some $a_l \geq 0$ and $b_l \in \mathbb{R}$.

To be more precise assume that for any smooth $u(x)$ we have

$$Lu(0) = \lim_{h \downarrow 0} L_h u(0) \tag{2.7}$$

and $L_h u(0) < 0$ if u has a strict maximum at 0. The latter condition just means that $0 \in B$, $p_h(0) < 0$, and $p_h(y) \geq 0$ for $y \in B \setminus \{0\}$.

That the term $b^i u_{x^i}$ admits the stated representation follows from the fact that $\operatorname{Span} B = \mathbb{R}^d$. Furthermore, if in place of B we take the union $B \cup (-B)$, then b_l can be chosen nonnegative. By the way, notice that we do not require any connections between a_k^α and b_k^α, so that one can always assume that there is an equal number of a_k^α and b_k^α setting part of them to be zero if necessary. Also it is worth noting that we do not need b to be proportional to any element of B, so that the grid need not be aligned with the drift.

Now, it only remains to deal with $a^{ij} u_{x^i x^j}$. The monotonicity of L_h and (2.7) applied to $-(\lambda, x)^2$ show that $a^{ij} \lambda^i \lambda^j \geq 0$ for any constant $\lambda \in \mathbb{R}^d$. Also obviously we may assume that the matrix (a^{ij}) is symmetric and $a^{ij} \not\equiv 0$ so that

$$\bar{a} := \operatorname{Trace} a > 0.$$

Let $B = \{0, l_1, \ldots, l_m\}$, where $|l_k| > 0$, and observe that for $v(x) = u(x) + u(-x)$ we have

$$L_h v(0) = \sum_{k=1}^{m} p_h(l_k)[u(hl_k) + u(-hl_k)] + 2p_h(0)u(0).$$

Furthermore,

$$0 = L1 = \lim_{h\downarrow 0} L_h 1 = \lim_{h\downarrow 0} \left[\sum_{k=1}^{m} p_h(l_k) + p_h(0) \right],$$

so that

$$L_h v(0) = \sum_{k=1}^{m} p_h(l_k)[u(hl_k) - 2u(0) + u(-hl_k)] + o(1)$$

as $h \downarrow 0$. It follows that

$$2a^{ij} u_{x^i x^j}(0) = \lim_{h\downarrow 0} \sum_{k=1}^{m} h^2 p_h(l_k) \Delta_{h,l_k} u(0).$$

In particular, for $u(x) = |x|^2$,

$$\frac{1}{\bar{p}_h} := \sum_{k=1}^{m} h^2 p_h(l_k)|l_k|^2 \to 2\bar{a} > 0.$$

Also, obviously, $h^2 p_h(l_k)\bar{p}_h$ are bounded in h and along a sequence $h_n \downarrow 0$ converge to some $\bar{a}_k \geq 0$. Then we see that

$$a^{ij} u_{x^i x^j}(0) = \bar{a} \sum_{k=1}^{m} \bar{a}_k D_{l_k}^2 u(0),$$

which proves our claim.

There is also a very substantial advantage of using this particular form of L^α because for any smooth function η by Taylor's formula we have

$$D_l^2 \eta(y) = \Delta_{h,l}^2 \eta(y) - \frac{1}{6h^2} \int_{-h}^{h} D_l^4 \eta(y + sl)(h - |s|)^3 \, ds$$

and the second term on the right has order h^2. By considering similarly first-order terms we see that for any four times continuously differentiable function η,

$$|L^\alpha \eta(x) - L_h^\alpha \eta(x)| \leq N^* \left(h^2 \sup_{B_K(x)} |D_x^4 \eta| + h \sup_{B_K(x)} |D_x^2 \eta| \right), \tag{2.8}$$

where $B_K(x)$ is the ball of radius K centered at x and N^* depends only on K and d_1.

3. Solvability and Comparison Principle for Finite-Difference Equations

Problem (2.3)–(2.2) is, actually, a collection of disjoint problems given on each mesh associated with points $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$:

$$\{((t_0 + j\tau) \wedge T, x_0 + h(i_1\ell_1 + \cdots + i_{d_1}\ell_{d_1})):\\ j = 0, 1, \ldots, i_k = 0, \pm 1, \ldots, k = 1, \ldots, d_1\}. \tag{3.1}$$

Indeed, problem (2.3)–(2.2) on such a mesh makes perfect sense even if u is defined only on it. In the future we will see that it is extremely convenient to consider this collection of problems simultaneously. However, while obtaining certain estimates it is more convenient to work in a more traditional setting with each particular mesh separately. In this way even the results look more general and the continuity hypothesis in t on the coefficients often becomes superfluous. It is also worth noting that we do not assume that $\{\ell_k\}$ generates $\mathbb{R}^d$ so that the meshes (3.1) may be meshes on hyperplanes.

For fixed $\tau, h > 0$ introduce

$$\bar{\mathcal{M}}_T = \{(t, x) \in [0, T] \times \mathbb{R}^d\colon\ t = (j\tau) \wedge T, x = h(i_1\ell_1 + \cdots + i_{d_1}\ell_{d_1}),\\ j = 0, 1, \ldots, i_k = 0, \pm 1, \ldots, k = 1, \ldots, d_1\}.$$

Of course, results obtained for equations on subsets of $\bar{\mathcal{M}}_T$ automatically translate into the corresponding results for all other meshes like (3.1).

Take a nonempty set

$$Q \subset \mathcal{M}_T := \bar{\mathcal{M}}_T \cap ([0, T) \times \mathbb{R}^d).$$

We start with a solvability result.

Lemma 3.1. *Let $g(t, x)$ be a bounded function on $\bar{\mathcal{M}}_T$. Then there is a unique bounded function u defined on $\bar{\mathcal{M}}_T$ such that* (2.3) *holds in Q and $u = g$ on $\bar{\mathcal{M}}_T \setminus Q$.*

Proof. Take a constant $\gamma \in (0, 1)$ and define a function $\xi(t) = \xi(t, x)$ on $\bar{\mathcal{M}}_T$ recursively by

$$\xi(T) = 1, \qquad \xi(t) = \gamma^{-1}\xi(t + \tau_T(t)) \qquad \text{for} \quad t < T. \tag{3.2}$$

Notice that for any function v,

$$\delta^T_\tau(\xi v) = \gamma\xi\delta^T_\tau v - \nu\xi v, \qquad \nu = \frac{1-\gamma}{\tau}. \tag{3.3}$$

Obviously the function u we are looking for is to satisfy

$$u = \xi v,$$

$$v(t, x) = \xi^{-1}(t)g(t, x)I_{\bar{\mathcal{M}}_T \setminus Q}(t, x) + I_Q(t, x)G[v](t, x), \tag{3.4}$$

where

$$G[v](t, x) := v(t, x) + \varepsilon\xi^{-1}(t)[\delta^T_\tau u(t, x)\\ + F(\Delta_{h,\ell_k}u(t, x), \delta_{h,\ell_k}u(t, x), u(t, x), t, x)]$$

and ε is any number. Observe that for $\varepsilon > 0$,

$$G[v](t,x) = \sup_{\alpha \in A}[p_\tau v(t+\tau_T(t),x) + p_k^\alpha(t,x)v(t,x+h\ell_k) \\ + \; p^\alpha(t,x)v(t,x) + \varepsilon\xi^{-1}(t+\tau_T(t))f^\alpha(t,x)], \tag{3.5}$$

where

$$p_\tau = \varepsilon\gamma\tau^{-1}, \qquad p_k^\alpha = 2\varepsilon h^{-2}a_k^\alpha + \varepsilon h^{-1}b_k^\alpha,$$

$$p^\alpha := 1 - p_\tau - \sum_k p_k^\alpha - \varepsilon\nu - \varepsilon c^\alpha.$$

We choose ε and γ so that

$$p_k^\alpha, p^\alpha \geq 0, \qquad 0 \leq \sum_k p_k^\alpha + p^\alpha + p_\tau = 1 - \varepsilon\nu - \varepsilon c^\alpha \leq \delta < 1,$$

where δ is a constant.

Then we use the fact that the difference of sups is less than the sup of differences and easily conclude that for any functions v and w we have

$$|G[v](t,x) - G[w](t,x)| \leq \delta \sup_{\bar{\mathcal{M}}_T}|v - w|,$$

so that the operator G is a contraction in the space of bounded functions on $\bar{\mathcal{M}}_T$. The application of Banach's fixed point theorem to (3.4) proves the lemma.

Remark 3.2. Sometimes dealing with functions on $\bar{\mathcal{M}}_T$ the fact that T may not be a point of type $\tau, 2\tau, \ldots$ is quite inconvenient just because then we should take care of two cases, $t < T$ and $t = T$, separately. In addition, on few occasions in the article we do not use any continuity hypotheses in t. Therefore, we may move the points $(j\tau) \wedge T$ along the time axis preserving their order in any way we like provided that we carry along with them the values of the coefficients and other functions involved. In connection with this we introduce T' as the least point in the progression $\tau, 2\tau, \ldots$, which is $\geq T$ and notice that (2.3) on Q is rewritten as the following equation on Q relative to a function $\tilde{u}$ given on $\bar{\mathcal{M}}_{T'}$:

$$\delta_\tau \tilde{u}(t,x) + \sup_{\alpha \in A}[L_h^\alpha(t,x)\tilde{u}(t,x) + f^\alpha(t,x)] = 0,$$

where $\tilde{u}(t,x) = u(t,x)$ on $\mathcal{M}_{T'}$ and $\tilde{u}(T',x) = u(T,x)$. Observe that

$$\delta_\tau \tilde{u}(t,x) = \delta_\tau^{T'}\tilde{u}(t,x) = \delta_\tau^T u(t,x)$$

on $\mathcal{M}_{T'}$. Also note that the condition $u = g$ on $\bar{\mathcal{M}}_T \setminus Q$ translates into $\tilde{u} = \tilde{g}$ on $\bar{\mathcal{M}}_{T'} \setminus Q$, where $\tilde{g}(t,x) = g(t,x)$ on $\mathcal{M}_{T'}$ and $\tilde{g}(T',x) = g(T,x)$.

The following is a comparison result.

Lemma 3.3. *Let u_1, u_2 be functions on $\bar{\mathcal{M}}_T$, let $f_1^\alpha(t,x)$, $f_2^\alpha(t,x)$ be functions on $A \times \mathcal{M}_T$ and let C be a constant. Assume that in Q,*

$$\sup_\alpha f_2^\alpha < \infty, \qquad f_1^\alpha \le f_2^\alpha,$$

$$\delta_\tau^T u_1 + \sup_{\alpha\in A}[L_h^\alpha u_1 + f_1^\alpha] + C \ge \delta_\tau^T u_2 + \sup_{\alpha\in A}[L_h^\alpha u_2 + f_2^\alpha]. \tag{3.6}$$

Finally, let $h \le 1$ and $u_1 \le u_2$ on $\bar{\mathcal{M}}_T \setminus Q$ and assume that $u_i e^{-\mu|x|}$ are bounded on $\bar{\mathcal{M}}_T$, where $\mu \ge 0$ is a constant. We assert that there exists a constant $\tau^* > 0$, depending only on K, d_1, and μ, such that if $\tau \in (0, \tau^*)$ then on $\bar{\mathcal{M}}_T$,

$$u_1 \le u_2 + T'C_+. \tag{3.7}$$

Furthermore, $\tau^*(K, d_1, \mu) \to \infty$ as $\mu \downarrow 0$ and if u_1, u_2 are bounded on $\bar{\mathcal{M}}_T$, so that if $\mu = 0$, then (3.7) *holds without any constraints on h and τ.*

Proof. Obviously, one can replace f_1^α with f_2^α preserving (3.6). Then, according to Remark 3.2, we can pass from T to T' and thereby we may assume that $T = T'$. We get from (3.6) that

$$\delta_\tau u + \sup_{\alpha\in A} L_h^\alpha u + C \ge 0$$

on Q, where $u = u_1 - u_2$. Further, without losing generality we assume that $C \ge 0$ and for

$$w := u - C(T-t)$$

find that

$$\delta_\tau w + L_h^\alpha w = \delta_\tau u + L_h^\alpha u + C + c^\alpha C(T-t) \ge \delta_\tau u + L_h^\alpha u + C,$$

$$\delta_\tau w + \sup_{\alpha\in A} L_h^\alpha w \ge 0, \qquad w + \varepsilon\delta_\tau w + \varepsilon \sup_{\alpha\in A} L_h^\alpha w \ge w \qquad \text{on } Q,$$

where $\varepsilon > 0$ is any number.

Next, looking at the proof of Lemma 3.1 we see that we can choose ε so that, for $\gamma = 1$ and any $\alpha \in A$ in (3.5) we have $p_k^\alpha \ge 0$, $p^\alpha \ge 0$. Then for any function $\psi \ge w$ we have

$$\psi + \varepsilon\delta_\tau\psi + \varepsilon \sup_{\alpha\in A} L_h^\alpha \psi \ge w \qquad \text{on } Q. \tag{3.8}$$

Take a rather small constant $\gamma > 0$ to be specified later and take the function $\xi(t)$ from (3.2). Also introduce

$$\eta(x) = \cosh(\mu|x|), \qquad \zeta = \xi\eta, \qquad N_0 = \sup_{\bar{\mathcal{M}}_T} \frac{w_+}{\zeta}.$$

Notice that by (2.8) and by straightforward computations

$$\begin{aligned}\sup_{\alpha\in A} L_h^\alpha \eta(x) &\le \sup_{\alpha\in A} L^\alpha \eta(x) + N_0(h^2 \sup_{B_K(x)} |D_x^4\eta| + h \sup_{B_K(x)} |D_x^2\eta|)\\ &\le \sup_{\alpha\in A} L^\alpha \eta(x) + N_1(h^2+h)\cosh(\mu|x| + \mu K)\\ &\le N_2 \cosh(\mu|x| + \mu K),\end{aligned}$$

where N_i depend only on K, μ, and d_1. It is seen as well that one can take N_2, so that $N_2(K, d_1, \mu) \to 0$ as $\mu \downarrow 0$ and $N_2(K, d_1, 0) = 0$ even if $h > 1$. Also note that (see (3.3))

$$\delta_\tau \xi(t) = \xi(t)\tau^{-1}(\gamma - 1).$$

Therefore,

$$\delta_\tau \zeta + \sup_{\alpha\in A} L_h^\alpha \zeta \le \zeta(\tau^{-1}(\gamma-1) + N_3) = \kappa\zeta,$$

where

$$N_3 = N_2 \sup_x \frac{\cosh(\mu|x| + \mu K)}{\cosh(\mu|x|)} < \infty,$$

$$\kappa = \kappa(\gamma) := \tau^{-1}(\gamma - 1) + N_3.$$

Now set $\tau^* = N_3^{-1}$ and assume that $\tau < \tau^*$. Upon noticing that $\kappa(0) < 0$ and $\kappa(1) \ge 0$ we see that we can take γ so that $\kappa < 0$ and $1 + \kappa\varepsilon > 0$.

After that for $\psi = N_0\zeta$, (3.8) implies that

$$N_0\zeta(1 + \kappa\varepsilon) = N_0\zeta + \kappa\varepsilon N_0\zeta \ge w$$

on Q. Since the right-hand side is nonpositive on $\bar{\mathcal{M}}_T \setminus Q$, the inequality holds on $\bar{\mathcal{M}}_T$ and by the definition of N_0 implies that $N_0(1 + \kappa\varepsilon) \ge N_0$. By recalling that $\kappa < 0$ we obtain $N_0 = 0$, $w \le 0$ and (3.7) follows.

To prove the second assertion of the lemma it suffices to add that if $\mu = 0$, then $N_3 = N_2 = 0$. The lemma is proved. □

Three completely standard applications of the comparison principle follow.

Corollary 3.4. *Let a constant $c_0 \geq 0$ be such that*

$$\tau^{-1}(e^{c_0\tau} - 1) \leq \lambda.$$

Then

$$|v_{\tau,h}(t,x)| \leq K\frac{1 - e^{-\lambda(T+\tau)}}{\lambda} + e^{-c_0(T-t)} \sup_x |g|$$

on $\bar{H}_T$ with natural interpretation of this estimate if $\lambda = 0$, that is,

$$|v_{\tau,h}| \leq K(T+\tau) + \sup_x |g|.$$

To prove the corollary we observe that it suffices to concentrate on $\bar{\mathcal{M}}_T$. Then we pass from T to T' thus reducing the general case to the one with $T = n\tau$, where n is an integer. Next, define

$$N_1 = \sup|g|, \qquad \xi(t) = K\lambda^{-1}(1 - e^{-\lambda(T-t)}) + e^{-c_0(T-t)}N_1$$

if $\lambda > 0$ with natural modification for $\lambda = 0$. We have $\xi \geq g = v_{\tau,h}$ on $\bar{\mathcal{M}}_T \setminus \mathcal{M}_T$ whereas on $\mathcal{M}_T$,

$$\delta_\tau \xi(t) - \lambda\xi(t) = -K\left[e^{\lambda t - \lambda T}\left(\frac{e^{\lambda\tau} - 1}{\tau\lambda} - 1\right) + 1\right] + N_1\tau^{-1}(e^{c_0\tau} - 1)e^{-c_0(T-t)} - \lambda N_1 e^{-c_0(T-t)} \leq -K,$$

so that

$$\delta_\tau \xi + \sup_{\alpha\in A}[L_h^\alpha \xi + f^\alpha] \leq 0.$$

By the lemma $v_{\tau,h} \leq \xi$ on $\bar{\mathcal{M}}_T$. Similarly one proves that $v_{\tau,h} \geq -\xi$.

Corollary 3.5. *Let u_1 and u_2 be bounded solutions of* (2.3) *in H_T with terminal condition $u_1(T,x) = g_1(x)$ and $u_2(T,x) = g_2(x)$, where g_1 and g_2 are given bounded functions. Then under the conditions of Corollary* 3.4 *we have*

$$u_1(t,x) \leq u_2(t,x) + e^{-c_0(T-t)} \sup(g_1 - g_2)_+ \tag{3.9}$$

in $\bar{H}_T$.

To prove this it suffices to replace u_2 in Lemma 3.3 with the right-hand side of (3.9).

Corollary 3.6. *Assume that there is a constant R such that $f^\alpha(t,x) = g(x) = 0$ if $|x| \geq R$. Then*

$$\lim_{|x|\to\infty} \sup_{[0,T]} |v_{\tau,h}(t,x)| = 0.$$

For the proof take a unit $l \in \mathbb{R}^d$ and for small $\gamma \in (0, 1)$ consider

$$\zeta = \xi\eta, \qquad \eta = e^{\gamma(x,l)},$$

where ξ is taken from the proof of the lemma. It is a matter of very simple computations that $L_h^\alpha \eta \le N\gamma\eta$, where N is independent of l, γ, α, and t, x. It follows that

$$\delta_\tau^T \zeta + \sup_{\alpha\in A} L_h^\alpha \zeta \le [\tau^{-1}(\gamma - 1) + N\gamma]\zeta \le 0$$

if γ is sufficiently small. If needed we reduce further the value of γ to have $\tau < \tau^*(K, d_1, \gamma)$. Then on

$$Q = \{(t, x) \in \mathcal{M}_T\colon\ (x, l) \le -R\},$$

where $f^\alpha = 0$, we have

$$\delta_\tau^T N\zeta + \sup_{\alpha\in A}[L_h^\alpha N\zeta + f^\alpha] \le 0$$

for any constant $N > 0$. On $\bar{\mathcal{M}}_T \setminus Q$ it holds that

$$\zeta(t, x) \ge e^{-\gamma R} \qquad \text{if} \quad t \in [0, T),$$

$$\zeta(t, x) \ge e^{-\gamma|x|} \qquad \text{if} \quad t = T,$$

which shows that $N\zeta \ge v_{\tau,h}$ on $\bar{\mathcal{M}}_T \setminus Q$ for sufficiently large N. By Lemma 3.3 we obtain $v_{\tau,h} \le N\zeta$ in $\bar{\mathcal{M}}_T$ and due to the arbitrariness of l we conclude

$$v_{\tau,h} \le N\xi(0)e^{-\gamma|x|}.$$

Similarly, one proves that

$$v_{\tau,h} \ge -N\xi(0)e^{-\gamma|x|}$$

and the result follows if we restrict ourselves to considering $v_{\tau,h}$ only on $\bar{\mathcal{M}}_T$. However, since every mesh (3.1) can be treated in the same way and our constants stay the same, we get the result as stated.

Corollary 3.7. *Let $h, \tau \le K$. Let $(t_0, x_0), (s_0, x_0) \in \bar{\mathcal{M}}_T$ be such that $s_0 - 1 \le t_0 \le s_0$ and $(s_0 - t_0)/\tau$ is an integer. Set*

$$\nu = \sup_{(s_0,x)\in\bar{\mathcal{M}}_T} \frac{|v_{\tau,h}(s_0, x) - v_{\tau,h}(s_0, x_0)|}{|x - x_0|}.$$

Then

$$|v_{\tau,h}(s_0, x_0) - v_{\tau,h}(t_0, x_0)| \le N(\nu + 1)|s_0 - t_0|^{1/2},$$

where N depends only on K and d_1.

To prove this we may assume that $s_0 > 0$. Also, shifting the origin of the time axis allows us to assume that $t_0 = 0$, so that $s_0 \le 1$. Then fix a constant $\gamma > 0$ and set

$$\xi(t) = e^{s_0 - t}, \quad t < s_0, \qquad \xi(t) = 1, \quad t \ge s_0,$$

$$\eta = |x - x_0|^2, \qquad \zeta = \xi\eta,$$

$$\psi = \gamma\nu[\zeta + \kappa(s_0 - t)] + K(s_0 - t) + \gamma^{-1}\nu + v_{\tau,h}(s_0, x_0),$$

where $\kappa > 0$ is a constant to be specified later. It is easy to check that $\delta_\tau^{s_0}\xi = -\theta\xi$ on $Q = \mathcal{M}_{s_0}$, where

$$\theta := \tau^{-1}(1 - e^{-\tau}) \ge K^{-1}(1 - e^{-K}).$$

Also in $\mathcal{M}_{s_0}$

$$\begin{aligned} L_h^\alpha \eta(t,x) &= 2a_k^\alpha(t,x)|\ell_k|^2 + b_k^\alpha(t,x)(\ell_k, 2(x - x_0) + h\ell_k) - c^\alpha(t,x)\eta(t,x) \\ &\le N_1(1 + |x - x_0|), \end{aligned}$$

$$\begin{aligned} \delta_\tau^{s_0}\zeta(t,x) + L_h^\alpha\zeta(t,x) &\le [N_1(1 + |x - x_0|) - \theta|x - x_0|^2]\xi(t) \\ &\le N_2(1 + |x - x_0|) - \theta|x - x_0|^2, \end{aligned}$$

where the constants N_i depend only on K and d_1. It follows that in $\mathcal{M}_{s_0}$,

$$\delta_\tau^{s_0}\psi + L_h^\alpha\psi + f^\alpha \le \gamma\nu[N_2(1 + |x - x_0|) - \theta|x - x_0|^2 - \kappa].$$

As is easy to see there is $\kappa > 0$ depending only on N_2 such that the right-hand side is negative for all x.

Furthermore,

$$\begin{aligned} \psi(s_0, x) &= \nu(\gamma|x - x_0|^2 + \gamma^{-1}) + v_{\tau,h}(s_0, x_0) \\ &\ge \nu|x - x_0| + v_{\tau,h}(s_0, x_0) \ge v_{\tau,h}(s_0, x). \end{aligned}$$

By Lemma 3.3 applied to $\mathcal{M}_{s_0}$ in place of $\mathcal{M}_T$ we conclude

$$v_{\tau,h}(t, x_0) \le \psi(t, x_0) = \gamma\nu\kappa(s_0 - t) + \gamma^{-1}\nu + K(s_0 - t) + v_{\tau,h}(s_0, x_0).$$

Minimizing with respect to $\gamma > 0$ yields

$$v_{\tau,h}(t, x_0) - v_{\tau,h}(s_0, x_0) \le 2\nu\kappa^{1/2}|s_0 - t|^{1/2} + Ks_0^{1/2}|s_0 - t|^{1/2}.$$

Thus we obtain a one-sided estimate of $v_{\tau,h}(t, x_0) - v_{\tau,h}(s_0, x_0)$. The estimate from the other side is obtained similarly by considering

$$-\gamma\nu[\zeta + \kappa(s_0 - t)] - K(s_0 - t) - \gamma^{-1}\nu + v_{\tau,h}(s_0, x_0)$$

in place of ψ.

One more simple consequence of Lemma 3.1 and Corollary 3.4 is the following stability result.

Lemma 3.8. *Let functions f_n^α and g_n, $n = 1, 2, \ldots$, satisfy the same conditions as f^α, g with the same constants and let $v_{\tau,h}^n$ be the unique solutions of problem* (2.3)–(2.2) *with f_n^α and g_n in place of f^α and g, respectively. Assume that on $\bar{H}_T$,*

$$\lim_{n\to\infty} \sup_{\alpha\in A}(|f^\alpha - f_n^\alpha| + |g - g_n|) = 0.$$

Then $v_{\tau,h}^n \to v_{\tau,h}$ on $\bar{H}_T$.

Proof. It suffices again to concentrate on $\bar{\mathcal{M}}_T$ and observe that any subsequence of uniformly bounded functions $v_{\tau,h}^n$ which converges at any point of $\bar{\mathcal{M}}_T$ will converge to a solution of the original problem (2.3)–(2.2), which is unique and equals $v_{\tau,h}$. Therefore, the whole sequence converges to $v_{\tau,h}$. The lemma is proved.

4. Some Technical Tools

Set

$$T_{h,l}u(x) := u(x + hl).$$

Lemma 4.1. *For any functions $u(x)$, $v(x)$, $h > 0$, and $l \in \mathbb{R}^d$, we have*

$$T_{h,-l}T_{h,l}u = u, \tag{4.1}$$

$$T_{h,l}\delta_{h,-l} = \delta_{h,-l}T_{h,l} = -\delta_{h,l}, \qquad T_{h,-l}\delta_{h,l} = \delta_{h,l}T_{h,-l} = -\delta_{h,-l}, \tag{4.2}$$

$$\delta_{h,l}(uv) = (\delta_{h,l}u)v + (T_{h,l}u)\delta_{h,l}v = v\delta_{h,l}u + u\delta_{h,l}v + h(\delta_{h,l}u)\delta_{h,l}v, \tag{4.3}$$

$$\Delta_{h,l}(uv) = v\Delta_{h,l}u + u\Delta_{h,l}v + (\delta_{h,l}u)\delta_{h,l}v + (\delta_{h,-l}u)\delta_{h,-l}v. \tag{4.4}$$

In particular,

$$\Delta_{h,l}(u^2) = 2u\Delta_{h,l}u + (\delta_{h,l}u)^2 + (\delta_{h,-l}u)^2. \tag{4.5}$$

Proof. Equations (4.1)–(4.3) are almost trivial. They yield (4.5) because

$$\begin{aligned} -\Delta_{h,l}(u^2) &= \delta_{h,-l}[(\delta_{h,l}u)u] + \delta_{h,-l}[(T_{h,l}u)\delta_{h,l}u] \\ &= [-(\Delta_{h,l}u)u + (T_{h,-l}\delta_{h,l}u)\delta_{h,-l}u] \\ &\quad + [(\delta_{h,-l}T_{h,l}u)\delta_{h,l}u - (T_{h,-l}T_{h,l}u)\Delta_{h,l}u]. \end{aligned}$$

Equation (4.4) is obtained by polarizing (4.5), that is, by comparing the coefficient of λ in (4.5) applied to $u + \lambda v$ in place of u. The lemma is proved. □

Lemma 4.2. *Let u, v, w be functions on $\mathbb{R}^d$, l, $x_0 \in \mathbb{R}^d$, $h > 0$. Assume that $v(x_0) \le 0$ and $w(x_0) \le 0$. Then at x_0 it holds that*

$$-\delta_{h,l}v \le \delta_{h,l}(v_-), \qquad -\Delta_{h,l}v \le \Delta_{h,l}(v_-), \tag{4.6}$$

$$-\delta_{h,l}(u_-) \le [\delta_{h,l}((u + v)_-)]_- + [\delta_{h,l}(v_-)]_+, \tag{4.7}$$

$$(\Delta_{h,l}u)_- \leq [\delta_{h,-l}((\delta_{h,l}u+v)_-)]_- + [\delta_{h,l}((\delta_{h,-l}u+w)_-)]_- \\ + [\delta_{h,-l}(v_-)]_+ + [\delta_{h,l}(w_-)]_+, \tag{4.8}$$

$$|\Delta_{h,l}u| \leq |\delta_{h,-l}((\delta_{h,l}u)_-)| + |\delta_{h,l}((\delta_{h,-l}u)_-)|, \tag{4.9}$$

$$|\Delta_{h,l}u| \leq |\delta_{h,-l}((\delta_{h,l}u)_+)| + |\delta_{h,l}((\delta_{h,-l}u)_+)|. \tag{4.10}$$

Proof. We use the formulas $-\alpha \leq \alpha_-$ and $v(x_0) = -v_-(x_0)$ and get

$$-h\delta_{h,l}v(x_0) = v(x_0) - v(x_0+hl) \leq -v_-(x_0) + v_-(x_0+hl),$$

which is the first inequality in (4.6). The second one is obtained by summing up the first inequality corresponding to l and $-l$.

While proving (4.7) we may assume that $u(x_0) < 0$ since otherwise the left-hand side is negative. In that case by noting that by subadditivity, $(\alpha+\beta)_- \leq \alpha_- + \beta_-$, we have

$$-u_- \leq -(u+v)_- + v_-, \qquad -T_{h,l}u_- \leq -T_{h,l}(u+v)_- + T_{h,l}v_-$$

everywhere, whereas since $u(x_0) \leq 0$, $v(x_0) \leq 0$, we have at x_0,

$$u_- = (u+v)_- - v_-.$$

We conclude that at x_0,

$$-\delta_{h,l}u_- \leq -\delta_{h,l}(u+v)_- + \delta_{h,l}v_-$$

and (4.7) follows.

In the proof of (4.8) we may assume that $\Delta_{h,l}u(x_0) \leq 0$. Then owing to (4.2) at x_0,

$$\begin{aligned}(\Delta_{h,l}u)_- &= \delta_{h,-l}\delta_{h,l}u = \delta_{h,-l}((\delta_{h,l}u)_+) - \delta_{h,-l}((\delta_{h,l}u)_-) \\ &= T_{h,l}\delta_{h,-l}((-\delta_{h,-l}u)_+) - \delta_{h,-l}((\delta_{h,l}u)_-) \\ &= -\delta_{h,l}((\delta_{h,-l}u)_-) - \delta_{h,-l}((\delta_{h,l}u)_-).\end{aligned}$$

This and (4.7) imply (4.9).

If $\Delta_{h,l}u(x_0) \leq 0$, (4.9) follows from (4.8) with $v \equiv w \equiv 0$. Therefore, we may concentrate on the case that $\Delta_{h,l}u(x_0) \geq 0$. By applying (4.8) with $v \equiv w \equiv 0$ to $-u$ in place of u and using (4.2) we get at x_0 that

$$\begin{aligned}|\Delta_{h,l}u| &\leq |\delta_{h,-l}((-\delta_{h,l}u)_-)| + |\delta_{h,l}((-\delta_{h,-l}u)_-)| \\ &= |\delta_{h,-l}(T_{h,l}(\delta_{h,-l}u)_-)| + |\delta_{h,l}(T_{h,-l}(\delta_{h,l}u)_-)| \\ &= |T_{h,l}\delta_{h,-l}((\delta_{h,-l}u)_-)| + |T_{h,-l}\delta_{h,l}((\delta_{h,l}u)_-)| \\ &= |\delta_{h,l}((\delta_{h,-l}u)_-)| + |\delta_{h,-l}((\delta_{h,l}u)_-)|.\end{aligned}$$

This proves (4.9).

Equation (4.10) is obtained from (4.9) by substituting $-u$ in place of u. The lemma is proved. □

5. Main Estimates

We take τ, h, T, and $\mathcal{M}_T$ from Section 3, fix an $\varepsilon \in [0, Kh]$ and a unit vector $l \in \mathbb{R}^d$ and introduce

$$\bar{\mathcal{M}}_T(\varepsilon) := \{(t, x + i\varepsilon l)\colon (t, x) \in \bar{\mathcal{M}}_T, i = 0, \pm 1, \dots\}.$$

Let $Q \subset \bar{\mathcal{M}}_T(\varepsilon)$ be a nonempty *finite* set and let u be a function on $\bar{\mathcal{M}}_T(\varepsilon)$ satisfying (2.3) in $Q' := Q \cap ([0, T) \times \mathbb{R}^d)$.

Set

$$Q^o_\varepsilon = \{(t, x) \in Q'\colon (t + \tau_T(t), x), (t, x \pm h\ell_k), (t, x \pm \varepsilon l) \in Q, \\ \forall k = 1, \dots, d_1\},$$

$$\partial_\varepsilon Q = Q \setminus Q^o_\varepsilon.$$

Instead of Assumptions 2.1 and 2.2 in this section we use the following.

Assumption 5.1. For

$$\psi = b^\alpha_k, c^\alpha - \lambda, f^\alpha, \qquad k = \pm 1, \dots, \pm d_1, \quad \alpha \in A,$$

we have in Q^o_ε that

$$|\psi| \le K, \qquad |\delta_{h,\ell_k}\psi|, |\delta_{\varepsilon,\pm l}\psi| \le K, \qquad b^\alpha_k, c^\alpha - \lambda, \lambda \ge 0, \tag{5.1}$$

$$0 \le a^\alpha_k \le K, \qquad |\delta_{h,\ell_k} a^\alpha_k|, |\delta_{\varepsilon,\pm l} a^\alpha_k| \le K\sqrt{a^\alpha_k} + Kh. \tag{5.2}$$

Theorem 5.2. *There is a constant $N \in (0, \infty)$ depending only on K and d_1, such that if for a number $c_0 \ge 0$ it holds that*

$$\lambda + \frac{1 - e^{-c_0\tau}}{\tau} > N, \tag{5.3}$$

then for $\varepsilon \in (0, Kh]$ on Q,

$$|\delta_{\varepsilon,\pm l} u| \le N e^{c_0(T+\tau)} \left(1 + |u|_{0,Q} + \max_{\partial_\varepsilon Q}(\max_k |\delta_{h,\ell_k} u| + |\delta_{\varepsilon,l} u| + |\delta_{\varepsilon,-l} u|)\right). \tag{5.4}$$

Before proving the theorem we do some preparations. Denote

$$h_k = h, \qquad k = \pm 1, \dots, \pm d_1, \qquad h_{\pm(d_1+1)} = \varepsilon, \qquad \ell_{\pm(d_1+1)} = \pm l,$$

and let r be an index running through $\{\pm 1, \dots, \pm(d_1 + 1)\}$ and k through $\{\pm 1, \dots, \pm d_1\}$.

Take a constant $c_0 \geq 0$ and introduce T' as the least $n\tau$, $n = 1, 2, \ldots$, such that $n\tau \geq T$,

$$\xi(t) = e^{c_0 t}, \qquad t < T, \qquad \xi(T) = e^{c_0 T'},$$

$$v = \xi u, \qquad v_r = \delta_{h_r,\ell_r} v, \qquad v_r^- = (v_r)_-,$$

$$M_0 = \max_Q |v|, \qquad M_1 = \max_{Q,r} |v_r|.$$

Let (t_0, x_0) be a point in Q at which

$$V := \sum_r (v_r^-)^2$$

attains its maximum value in Q.

Observe that for each $(t, x) \in Q_\varepsilon^o$ and r we have

$$(t, x + h_r \ell_r) \in Q$$

and

$$\text{either} \quad v_r(t, x) \leq 0 \quad \text{or} \quad -v_r(t, x) = v_{-r}(t, x + h_r \ell_r) \leq 0.$$

In the first case

$$|v_r(t, x)| \leq V^{1/2}(t, x) \leq V^{1/2}(t_0, x_0),$$

whereas in the second case

$$|v_r(t, x)| \leq V^{1/2}(t, x + h_r \ell_r) \leq V^{1/2}(t_0, x_0).$$

It follows that

$$M_1 \leq \max_{\partial_\varepsilon Q, r} |v_r| + V^{1/2}(t_0, x_0), \tag{5.5}$$

$$|\delta_{\varepsilon,\pm l} u| \leq e^{c_0 T'} \max_{\partial_\varepsilon Q, r} |\delta_{h_r,\ell_r} u| + V^{1/2}(t_0, x_0) \tag{5.6}$$

on Q and we need only estimate $V^{1/2}(t_0, x_0)$.

Furthermore, obviously

$$V^{1/2}(t, x) \leq 2d_1 \max_r |v_r(t, x)| \leq 2d_1 e^{c_0(T+\tau)} \max_r |\delta_{h_r,\ell_r} u(t, x)|,$$

so that while estimating $V^{1/2}(t_0, x_0)$ we may assume that

$$(t_0, x_0) \in Q_\varepsilon^o. \tag{5.7}$$

Notice that there is a sequence $\alpha_n \in A$ such that

$$\begin{aligned}
&\delta_\tau^T u(t_0, x_0) + \lim_{n\to\infty} [a_k^{\alpha_n}(t_0, x_0) \Delta_{h,\ell_k} u(t_0, x_0) + b_k^{\alpha_n}(t_0, x_0) \delta_{h,\ell_k} u(t_0, x_0) \\
&\qquad - c^{\alpha_n}(t_0, x_0) u(t_0, x_0) + f^{\alpha_n}(t_0, x_0)] \\
&= \delta_\tau^T u(t_0, x_0) + F(\Delta_{h,\ell_k} u(t_0, x_0), \delta_{h,\ell_k} u(t_0, x_0), u(t_0, x_0), t_0, x_0) = 0.
\end{aligned}$$

Owing to Assumption 5.1 there is a subsequence $\{n'\} \subset \{1, 2, \dots\}$ and functions $\bar{a}_k(t,x), \bar{b}_k(t,x), \bar{c}(t,x), \bar{f}(t,x)$ such that they satisfy Assumption 5.1 changed in an obvious way and

$$\begin{aligned}&(\bar{a}_k(t,x), \bar{b}_k(t,x), \bar{c}(t,x), \bar{f}(t,x))\\ &\quad = \lim_{n'\to\infty}(a_k^{\alpha_{n'}}(t,x), b_k^{\alpha_{n'}}(t,x), c^{\alpha_{n'}}(t,x), f^{\alpha_{n'}}(t,x))\end{aligned}$$

on Q. Obviously, at (t_0, x_0) we have

$$\delta_\tau^T u + \bar{a}_k \Delta_{h_k,\ell_k} u + \bar{b}_k \delta_{h_k,\ell_k} u - \bar{c}u + \bar{f} = 0, \tag{5.8}$$

and for any $r\ (= \pm 1, \dots, \pm(d_1+1))$ owing to (5.7)

$$T_{h_r,\ell_r}[\delta_\tau^T u + \bar{a}_k \Delta_{h_k,\ell_k} u + \bar{b}_k \delta_{h_k,\ell_k} u - \bar{c}u + \bar{f}] \le 0, \tag{5.9}$$

where and below for simplicity of notation we drop (t_0, x_0) in the arguments of functions we are dealing with.

Lemma 5.3. *For all $k = \pm 1, \dots, \pm d_1$ at (t_0, x_0) we have*

$$v_r^- \Delta_{h_k,\ell_k} v_r \ge 0. \tag{5.10}$$

Furthermore, there is a constant $N \in (0, \infty)$ depending only on K and d_1, such that at (t_0, x_0),

$$\begin{aligned}&\hat{\lambda} V + \tfrac{1}{2} v_r^- \bar{a}_k \Delta_{h_k,\ell_k} v_r + \tfrac{1}{2} I + v_r^- (\delta_{h_r,\ell_r} \bar{a}_k) \Delta_{h_k,\ell_k} v + v_r^- h_r (\delta_{h_r,\ell_r} \bar{a}_k) \Delta_{h_k,\ell_k} v_r\\ &\qquad \le N(e^{c_0 T'} + M_0 + M_1) M_1,\end{aligned} \tag{5.11}$$

where

$$\hat{\lambda} = \lambda + \frac{1 - e^{-c_0 \tau}}{\tau}, \qquad I = \sum_r \bar{a}_k (\delta_{h_k,\ell_k} v_r^-)^2.$$

Proof. By Lemmas 4.1 and 4.2 (with v_r in place of v)

$$\begin{aligned}0 &\ge \Delta_{h_k,\ell_k} \sum_r (v_r^-)^2 = 2 v_r^- \Delta_{h_k,\ell_k} v_r^- + \sum_r [(\delta_{h_k,\ell_k} v_r^-)^2 + (\delta_{h_k,\ell_{-k}} v_r^-)^2]\\ &\ge -2 v_r^- \Delta_{h_k,\ell_k} v_r + \sum_r [(\delta_{h_k,\ell_k} v_r^-)^2 + (\delta_{h_k \ell_{-k}} v_r^-)^2].\end{aligned}$$

This obviously yields (5.10) and also that

$$I \le v_r^- \bar{a}_k \Delta_{h_k,\ell_k} v_r,$$

which in turn implies that to prove (5.11) it suffices to prove that

$$\begin{aligned}&\hat{\lambda} V + v_r^- \bar{a}_k \Delta_{h_k,\ell_k} v_r + v_r^- (\delta_{h_r,\ell_r} \bar{a}_k) \Delta_{h_k,\ell_k} v + v_r^- h_r (\delta_{h_r,\ell_r} \bar{a}_k) \Delta_{h_k,\ell_k} v_r\\ &\qquad \le N(e^{c_0 T'} + M_0 + M_1) M_1.\end{aligned} \tag{5.12}$$

By subtracting the inequalities (5.8) and (5.9) and using (4.3) we find

$$\delta^T_\tau(\xi^{-1}v_r) + \xi^{-1}\left[\bar a_k \Delta_{h_k,\ell_k} v_r + I_{1r} + I_{2r} + I_{3r} + I_{4r}\right] \le 0, \tag{5.13}$$

where (no summation in r)

$$\begin{aligned}
I_{1r} &= (\delta_{h_r,\ell_r}\bar a_k)\Delta_{h_k,\ell_k} v,\\
I_{2r} &= h_r(\delta_{h_r,\ell_r}\bar a_k)\Delta_{h_k,\ell_k} v_r,\\
I_{3r} &= (T_{h_r,\ell_r}\bar b_k)\delta_{h_k,\ell_k} v_r + (\delta_{h_r,\ell_r}\bar b_k)\delta_{h_k,\ell_k} v,\\
I_{4r} &= -(\delta_{h_r,\ell_r}\bar c)v - (T_{h_r,\ell_r}\bar c)v_r + \xi\delta_{h_r,\ell_r}\bar f.
\end{aligned}$$

We multiply (5.13) by ξv_r^- and sum up with respect to r.

Observe that in I_{4r},

$$\begin{aligned}
&\delta_{h_r,\ell_r}\bar f \ge -K, \qquad |\delta_{h_r,\ell_r}\bar c| \le K,\\
&-v_r^-(T_{h_r,\ell_r}\bar c)v_r = (T_{h_r,\ell_r}\bar c)[v_r^-]^2 \ge \lambda\sum_r [v_r^-]^2 = \lambda V,
\end{aligned}$$

since $\bar c \ge \lambda$. Therefore,

$$v_r^- I_{4r} \ge -KM_1(e^{c_0T'} + M_0) + \lambda V.$$

By using the fact that V attains its maximum in Q at $(t_0, x_0) \in Q^0_\varepsilon$ and using Lemma 4.2 (with v_r in place of v) we get

$$\begin{aligned}
0 \ge \delta_{h_k,\ell_k}\sum_r (v_r^-)^2 &= 2v_r^-\delta_{h_k,\ell_k}v_r^- + \sum_r h_k(\delta_{h_k,\ell_k}v_r^-)^2\\
&\ge 2v_r^-\delta_{h_k,\ell_k}v_r^- \ge -2v_r^-\delta_{h_k,\ell_k}v_r.
\end{aligned}$$

This result and the inequalities $b_k \ge 0$, $|\delta_{h_r,\ell_r}\bar b_k| \le K$ yield

$$-v_r^-(T_{h_r,\ell_r}\bar b_k)\delta_{h_k,\ell_k}v_r \le 0, \qquad v_r^- I_{3r} \ge -NM_1^2.$$

Similarly,

$$0 \le -\delta^T_\tau\sum_r (v_r^-)^2 \le 2v_r^-\delta^T_\tau v_r,$$

which implies that

$$\begin{aligned}
\xi v_r^-\delta^T_\tau(\xi^{-1}v_r) &= \xi v_r^-[\xi^{-1}(t_0 + \tau_T(t_0))\delta^T_\tau v_r + v_r\delta^T_\tau\xi^{-1}]\\
&= e^{-c_0\tau}v_r^-\delta^T_\tau v_r - V\xi\delta^T_\tau\xi^{-1} \ge -V\xi\delta^T_\tau\xi^{-1}\\
&= V\frac{1}{\tau}[1 - e^{-c_0\tau}].
\end{aligned}$$

By combining the above estimates we come to (5.12) and the lemma is proved. □

Proof of Theorem 5.2. By Lemma 5.3

$$\hat{\lambda} V \leq N(e^{c_0(T+\tau)} + M_0 + M_1)M_1 + J_1 + J_2, \tag{5.14}$$

where

$$J_1 := v_r^- |(\delta_{h_r,\ell_r}\bar{a}_k)\Delta_{h_k,\ell_k} v| - \tfrac{1}{4}\sum_r \bar{a}_k(\delta_{h_k,\ell_k} v_r^-)^2,$$

$$J_2 := J_3 - \tfrac{1}{2}\bar{a}_k v_r^- \Delta_{h_k,\ell_k} v_r - \tfrac{1}{4}\sum_r \bar{a}_k(\delta_{h_k,\ell_k} v_r^-)^2,$$

$$J_3 := h_r v_r^- |(\delta_{h_r,\ell_r}\bar{a}_k)\Delta_{h_k,\ell_k} v_r|.$$

First we estimate J_1. By Lemma 4.2

$$|\Delta_{h_k,\ell_k} v| \leq \sum_r |\delta_{h_k,\ell_k} v_r^-| + \sum_r |\delta_{h_k,\ell_{-k}} v_r^-|.$$

Also we recall (5.2) and use the inequality

$$h|\Delta_{h_k,\ell_k} v| \leq 2M_1.$$

Then we obtain

$$\begin{aligned} v_r^- |(\delta_{h_r,\ell_r}\bar{a}_k)\Delta_{h_k,\ell_k} v| &\leq NM_1|(\sqrt{\bar{a}_k} + h)\Delta_{h_k,\ell_k} v| \\ &\leq NM_1^2 + \tfrac{1}{4}\sum_r \bar{a}_k(\delta_{h_k,\ell_k} v_r^-)^2, \end{aligned}$$

$$J_1 \leq NM_1^2.$$

To estimate J_3 observe that

$$h_r \leq Kh, \qquad |a| = 2a_- + a, \qquad h^2|\Delta_{h_k,\ell_k} v_r| \leq 4M_1,$$

so that

$$\begin{aligned} J_3 &\leq N_1 v_r^- h\sqrt{\bar{a}_k}|\Delta_{h_k,\ell_k} v_r| + K^2 v_r^- h^2 \left|\sum_k \Delta_{h_k,\ell_k} v_r\right| \\ &\leq N_1 v_r^- h\sqrt{\bar{a}_k}|\Delta_{h_k,\ell_k} v_r| + N_2 M_1^2 \\ &= 2N_1 v_r^- h\sqrt{\bar{a}_k}(\Delta_{h_k,\ell_k} v_r)_- \\ &\quad + N_1 v_r^- h\sqrt{\bar{a}_k}\Delta_{h_k,\ell_k} v_r + N_2 M_1^2. \end{aligned}$$

Here the summation in r can be restricted to r such that

$$v_r^- \neq 0,$$

when by Lemma 4.2 it holds that

$$h(\Delta_{h_k,\ell_k} v_r)_- \le h|\Delta_{h_k,\ell_k}(v_r^-)| = |(\delta_{h_k,\ell_k} + \delta_{h_k\ell_{-k}})(v_r^-)| \\ \le |\delta_{h_k,\ell_k}(v_r^-)| + |\delta_{h_k\ell_{-k}}(v_r^-)|.$$

Therefore,

$$J_3 \le N_1 v_r^- h\sqrt{\bar{a}_k}\Delta_{h_k,\ell_k} v_r + N_2 M_1^2 + N_3 M_1 \left[\sum_r \bar{a}_k(\delta_{h_k,\ell_k} v_r^-)^2\right]^{1/2},$$

$$J_2 \le N M_1^2 - \tfrac{1}{2}(\bar{a}_k - 2N_1 h\sqrt{\bar{a}_k}) v_r^- \Delta_{h_k,\ell_k} v_r.$$

Finally, let

$$\mathcal{K} = \{k\colon\ \bar{a}_k - 2N_1 h\sqrt{\bar{a}_k} \ge 0\}.$$

Then, for $k \notin \mathcal{K}$ we have

$$\sqrt{\bar{a}_k} \le 2N_1 h, \qquad \bar{a}_k \le 4N_1^2 h^2, \qquad |\bar{a}_k - 2N_1 h\sqrt{\bar{a}_k}| \le Nh^2$$

and by using (5.10) and using again the fact that $h^2|\Delta_{h_k l}\varphi| \le 4\sup|\varphi|$ we conclude that

$$-\tfrac{1}{2}(\bar{a}_k - 2N_1 h\sqrt{\bar{a}_k}) v_r^- \Delta_{h_k,\ell_k} v_r \\ \le -\tfrac{1}{2}\sum_{k\in\mathcal{K}}(\bar{a}_k - 2N_1 h\sqrt{\bar{a}_k}) v_r^- \Delta_{h_k,\ell_k} v_r + N M_1^2 \le N M_1^2,$$

$$J_2 \le N M_1^2.$$

Returning to (5.14) we get

$$\hat{\lambda} V \le N(e^{c_0(T+\tau)} + M_0 + M_1) M_1,$$

which due to (5.5) leads to

$$\hat{\lambda} V \le N(e^{c_0(T+\tau)} + M_0 + \mu + V^{1/2})(\mu + V^{1/2}), \tag{5.15}$$

where

$$\mu := \sup_{\partial_\varepsilon Q, r} |v_r| \le e^{c_0(T+\tau)} \sup_{\partial_\varepsilon Q, r} |\delta_{h_r,\ell_r} u| =: e^{c_0(T+\tau)}\bar{\mu}.$$

Also introduce

$$\bar{M}_0 = |u|_{0,Q}, \qquad \bar{V} = e^{-2c_0(T+\tau)} V$$

and notice that

$$M_0 \le e^{c_0(T+\tau)}\bar{M}_0.$$

Then (5.15) yields

$$\begin{aligned}\hat{\lambda}\bar{V} &\le N(1+\bar{M}_0+\bar{\mu}+\bar{V}^{1/2})(\bar{\mu}+\bar{V}^{1/2})\\ &\le N^*(1+\bar{M}_0^2+\bar{\mu}^2+\bar{V}).\end{aligned}$$

If $\hat{\lambda} \ge N^*+1$, then we conclude that

$$\bar{V} \le N^*(1+\bar{M}_0^2+\bar{\mu}^2),$$

which along with (5.6) yields (5.4) and proves the theorem. □

The following theorem bears on estimates of how close two solutions of the Bellman finite-difference equations are if the coefficients are close. It is a generalization of Theorem 5.2.

In the rest of the section we take some objects $\hat{\sigma}_k^\alpha$, $\hat{b}_k^\alpha$, $\hat{c}^\alpha$, $\hat{\lambda}$, $\hat{f}^\alpha$ defined on $A \times [0,T]\times\mathbb{R}^d$ and having the same sense as in Section 2. We set

$$\hat{a}_k^\alpha = \tfrac{1}{2}|\hat{\sigma}_k^\alpha|^2.$$

Assumption 5.4. We are given a finite set $Q \subset \bar{\mathcal{M}}_T = \bar{\mathcal{M}}_T(0)$ and not only a_k^α, b_k^α, c^α, λ, f^α satisfy Assumption 5.1 with $\varepsilon = 0$ but $\hat{a}_k^\alpha$, $\hat{b}_k^\alpha$, $\hat{c}^\alpha$, $\hat{\lambda}$, $\hat{f}^\alpha$ satisfy Assumption 5.1 with $\varepsilon = 0$ as well. Moreover, $\lambda = \hat{\lambda}$.

Theorem 5.5. *Let u be a function on $\bar{\mathcal{M}}_T$ satisfying* (2.3) *in $Q \cap ([0,T)\times\mathbb{R}^d)$ and let $\hat{u}$ be a function on $\bar{\mathcal{M}}_T$ satisfying* (2.3) *in $Q\cap([0,T)\times\mathbb{R}^d)$ with $\hat{a}_k^\alpha$, $\hat{b}_k^\alpha$, $\hat{c}^\alpha$, $\hat{f}^\alpha$ in place of a_k^α, b_k^α, c^α, f^α, respectively. Assume that there is an $\varepsilon \in (0, Kh]$ such that for $k = \pm1,\dots,\pm d_1$ on Q_0^o we have*

$$|b_k^\alpha - \hat{b}_k^\alpha| + |c^\alpha - \hat{c}^\alpha| + |f^\alpha - \hat{f}^\alpha| \le K\varepsilon, \tag{5.16}$$

$$|a_k^\alpha - \hat{a}_k^\alpha| \le K\varepsilon\sqrt{a_k^\alpha \wedge \hat{a}_k^\alpha} + K\varepsilon h. \tag{5.17}$$

We assert that there is a constant $N \in (0,\infty)$ depending only on K and d_1, such that if for a number $c_0 \ge 0$, (5.3) *holds, then in Q,*

$$|u-\hat{u}| \le N\varepsilon e^{c_0(T+\tau)}\Big(1+|u|_{0,Q}+|\hat{u}|_{0,Q} + \sup_{\partial_0 Q}(\max_k|\delta_{h,\ell_k}u| + \max_k|\delta_{h,\ell_k}\hat{u}| + \varepsilon^{-1}|u-\hat{u}|)\Big). \tag{5.18}$$

Proof. We want to apply Theorem 5.2 to appropriate objects. Consider $\mathbb{R}^d$ as a subspace of

$$\mathbb{R}^{d+1} = \{x = (x', x^{d+1}) \colon x' \in \mathbb{R}^d, x^{d+1} \in \mathbb{R}\}.$$

Take an integer $m \geq 1/\varepsilon$ and introduce $l = (0, \dots, 0, 1) \in \mathbb{R}^{d+1}$. Then

$$\bar{\mathcal{M}}_T(\varepsilon) = \{(t, x', x^{d+1})\colon\ (t, x') \in \bar{\mathcal{M}}_T,\ x^{d+1} = 0, \pm\varepsilon, \pm 2\varepsilon, \dots\}.$$

For

$$\tilde{Q} := \{(t, x', x^{d+1})\colon\ (t, x') \in Q,\ x^{d+1} = 0, \pm\varepsilon, \dots, \pm m\varepsilon\},$$

we have

$$\tilde{Q}^o_\varepsilon = Q^o_0 \times \{0, \pm\varepsilon, \dots, \pm(m-1)\varepsilon\},$$

$$\partial_\varepsilon \tilde{Q} = (\partial_0 Q \times \{0, \pm\varepsilon, \dots, \pm m\varepsilon\}) \cup (Q^o_0 \times \{m\varepsilon, -m\varepsilon\}).$$

Next, define

$$\tilde{a}^\alpha_k(t, x', x^{d+1}) = \begin{cases} a^\alpha_k(t, x') & \text{if} \quad x^{d+1} > 0, \\ \hat{a}^\alpha_k(t, x') & \text{if} \quad x^{d+1} \leq 0, \end{cases}$$

and similarly introduce $\tilde{b}^\alpha_k$ and $\tilde{c}^\alpha$. Let

$$\tilde{f}^\alpha(t, x', x^{d+1}) = \begin{cases} f^\alpha(t, x')[1 - (x^{d+1} - \varepsilon)/(\varepsilon m)] & \text{if} \quad x^{d+1} > 0, \\ \hat{f}^\alpha(t, x')[1 + x^{d+1}/(\varepsilon m)] & \text{if} \quad x^{d+1} \leq 0, \end{cases}$$

and similarly define $\tilde{u}(t, x', x^{d+1})$.

Next, we check that Theorem 5.2 is applicable to $\tilde{Q}$, $\tilde{u}$, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, and $\tilde{f}$. Obviously, $\tilde{u}$ in $\tilde{Q} \cap [0, T) \times \mathbb{R}^{d+1}$ satisfies (2.3) constructed on the basis of $\tilde{a}, \tilde{b}, \tilde{c}$, and $\tilde{f}$. In Assumption 5.1 inequalities (5.1) and (5.2) for δ_{h,ℓ_k} hold by assumption. To check them for $\delta_{\varepsilon,\pm l}$, observe that in $\tilde{Q}^o_\varepsilon$,

$$\delta_{\varepsilon,l}(\tilde{a}^\alpha_k, \tilde{b}^\alpha_k, \tilde{c}^\alpha, \tilde{f}^\alpha)(t, x) = \begin{cases} (0, 0, 0, -f^\alpha(t, x')/m) & \text{if} \quad x^{d+1} > 0, \\ (0, 0, 0, \hat{f}^\alpha(t, x')/m) & \text{if} \quad x^{d+1} < 0, \end{cases}$$

and

$$\delta_{\varepsilon,l}(\tilde{a}^\alpha_k, \tilde{b}^\alpha_k, \tilde{c}^\alpha, \tilde{f}^\alpha)(t, x', 0) = \varepsilon^{-1}(a^\alpha_k - \hat{a}^\alpha_k, b^\alpha_k - \hat{b}^\alpha_k, c^\alpha - \hat{c}^\alpha, f^\alpha - \hat{f}^\alpha)(t, x'),$$

where by virtue of (5.17)

$$\varepsilon^{-1}|a^\alpha_k(t, x') - \hat{a}^\alpha_k(t, x')| \leq K\sqrt{\tilde{a}^\alpha_k(t, x', 0)} + Kh.$$

Using the above formulas along with (5.16) and the inequality $\varepsilon m \geq 1$ we conclude that in our situation (5.1) and (5.2) hold for $\delta_{\varepsilon,l}$. The same is true for $\delta_{\varepsilon,-l} = -T_{\varepsilon,-l}\delta_{\varepsilon,l}$.

Now by Theorem 5.2 we obtain that for $(t, x') \in Q$,

$$\varepsilon^{-1}|u(t,x') - \hat{u}(t,x')| \tag{5.19}$$
$$= |\delta_{\varepsilon,l}\tilde{u}(t,x',0)|$$
$$\le Ne^{c_0(T+\tau)}\Bigg(1 + |u|_{0,Q} + |\hat{u}|_{0,Q}$$
$$+\max_{\partial_0 Q}\left(\max_k|\delta_{h,\ell_k}u|+\max_k|\delta_{h,\ell_k}\hat{u}|+\varepsilon^{-1}|u-\hat{u}|\right)+I_m\Bigg), \tag{5.20}$$

where

$$I_m := \max_{Q_0^o\times\{m\varepsilon,-m\varepsilon\}}\left(\max_k|\delta_{h,\ell_k}\tilde{u}| + |\delta_{\varepsilon,l}\tilde{u}| + |\delta_{\varepsilon,-l}\tilde{u}|\right).$$

Since on $Q_0^o \times \{r\varepsilon\colon\ r = m, m\pm 1\}$

$$|\tilde{u}(t,x)| \le N(1/m + |1 - |x^{d+1}|/(\varepsilon m)|) \le N/m,$$

where N is independent of m, we have $I_m \to 0$ as $m \to \infty$ and by letting $m \to \infty$ in (5.20), we arrive at (5.18). The theorem is proved. □

We also need a version of Theorem 5.5 in the case that $Q = \bar{\mathcal{M}}_T$. In the following theorem we abandon Assumption 5.4 and return to our basic assumptions. However, Assumption 2.2 about Hölder continuity is not used.

Theorem 5.6. *Let $\hat{\sigma}_k^\alpha, \hat{b}_k^\alpha, \hat{c}^\alpha, \hat{\lambda}, \hat{f}^\alpha$ satisfy the assumptions in Section* 2 *apart from Assumption* 2.2*. Suppose that $\hat{\lambda} = \lambda$. Let u be a function on $\bar{\mathcal{M}}_T$ satisfying* (2.3) *in $\mathcal{M}_T$ and let $\hat{u}$ be a function on $\bar{\mathcal{M}}_T$ satisfying* (2.3) *in $\mathcal{M}_T$ with $\hat{a}_k^\alpha, \hat{b}_k^\alpha, \hat{c}^\alpha, \hat{f}^\alpha$ in place of $a_k^\alpha, b_k^\alpha, c^\alpha, f^\alpha$, respectively. Assume that u and $\hat{u}$ are bounded on $\bar{\mathcal{M}}_T$ and*

$$|u(T,\cdot)|, |\hat{u}(T,\cdot)| \le K.$$

Introduce

$$\varepsilon = \sup_{\mathcal{M}_T, A, k}(|\sigma_k^\alpha - \hat{\sigma}_k^\alpha| + |b_k^\alpha - \hat{b}_k^\alpha| + |c^\alpha - \hat{c}^\alpha| + |f^\alpha - \hat{f}^\alpha|).$$

Then there is a constant N depending only on K and d_1 such that if for a number $c_0 \ge 0$, (5.3) *holds, then*

$$|u - \hat{u}| \le N\varepsilon e^{c_0(T+\tau)} I \tag{5.21}$$

on $\bar{\mathcal{M}}_T$, where

$$I = \sup_{(T,x)\in\bar{\mathcal{M}}_T}\left(1 + \left(\max_k|\delta_{h,\ell_k}u| + \max_k|\delta_{h,\ell_k}\hat{u}| + \varepsilon^{-1}|u-\hat{u}|\right)(T,x)\right).$$

Proof. First we show that we may assume that $\varepsilon \in (0, h]$. To this end for $\theta \in [0, 1]$ introduce u^θ as the unique bounded solution of

$$\delta_\tau^T u + \sup_{\alpha \in A} [a_k^{\theta\alpha} \Delta_{h,\ell_k} u + b_k^{\theta\alpha} \delta_{h,\ell_k} u + c^{\theta\alpha} u + f^{\theta\alpha}] = 0$$

in $\mathcal{M}_T$ and $u = (1-\theta)u + \theta\hat{u}$ in $\{(T, x) \in \bar{\mathcal{M}}_T\}$, where

$$[\sigma_k^{\theta\alpha}, b_k^{\theta\alpha}, c^{\theta\alpha}, f^{\theta\alpha}] = (1-\theta)[\sigma_k^\alpha, b_k^\alpha, c^\alpha, f^\alpha] + \theta[\hat{\sigma}_k^\alpha, \hat{b}_k^\alpha, \hat{c}^\alpha, \hat{f}^\alpha],$$

$$a_k^{\theta\alpha} = \tfrac{1}{2}|\sigma_k^{\theta\alpha}|^2.$$

Obviously, $u^0 = u$ and $u^1 = \hat{u}$. Also notice that for any $\theta_1, \theta_2 \in [0, 1]$,

$$|\sigma_k^{\theta_1\alpha} - \sigma_k^{\theta_2\alpha}| + |b_k^{\theta_1\alpha} - b_k^{\theta_2\alpha}| + |c^{\theta_1\alpha} - c^{\theta_2\alpha}| + |c^{\theta_1\alpha} - c^{\theta_2\alpha}| \le |\theta_1 - \theta_2|\varepsilon.$$

Therefore, if the present theorem holds true for $\varepsilon \in (0, h]$, then for any $\varepsilon > 0$ as long as $|\theta_1 - \theta_2|\varepsilon \le h$ we have

$$|u^{\theta_1} - u^{\theta_2}| \le N_1 |\theta_1 - \theta_2| \varepsilon e^{c_0 T} I(\theta_1, \theta_2),$$

where

$$I(\theta_1, \theta_2) = \sup_{(T,x)\in\bar{\mathcal{M}}_T} \bigg(1 + \Big(\max_k |\delta_{h,\ell_k} u^{\theta_1}| + \max_k |\delta_{h,\ell_k} u^{\theta_2}| + |\theta_1 - \theta_2|^{-1}\varepsilon^{-1}|u^{\theta_1} - u^{\theta_2}|\Big)(T, x)\bigg).$$

Obviously, $I(\theta_1, \theta_2) \le 4I$, so that

$$|u^{\theta_1} - u^{\theta_2}| \le 4N_1 |\theta_1 - \theta_2| \varepsilon e^{c_0 T} I.$$

By dividing the interval $(0, 1)$ into pieces of appropriate length and adding up these estimates we come to (5.21) with the constant N which is four times larger than the one which suits $\varepsilon \le h$.

Thus indeed the only important case is the one with $\varepsilon \in (0, h]$. In this case, actually, the theorem is a simple consequence of Theorem 5.5, Corollaries 3.4 and 3.6, and Lemma 3.8. Indeed, by Lemma 3.8 we can approximate both u and $\hat{u}$ with solutions such that f and $\hat{f}$ have compact support as well as the restriction of approximating functions to $\{t = T\}$. For approximating functions we get the result as in the proof of Theorem 5.5 by expanding finite sets Q and using that the contribution coming from the distant boundary becomes negligible due to Corollary 3.6. We also eliminate the terms $|u|_{0,Q}$ and $|\hat{u}|_{0,Q}$ on the basis of Corollary 3.4. However, to use Theorem 5.5 we also have to notice that due to the assumption that $\varepsilon \le h$ we have

$$|a_k^\alpha - \hat{a}_k^\alpha| \le (|\sigma_k^\alpha| \wedge |\hat{\sigma}_k^\alpha|)|\sigma_k^\alpha - \hat{\sigma}_k^\alpha| + |\sigma_k^\alpha - \hat{\sigma}_k^\alpha|^2 \le 2\varepsilon\sqrt{a_k^\alpha \wedge \hat{a}_k^\alpha} + \varepsilon^2$$

and $\varepsilon^2 \le \varepsilon h$. The theorem is proved. □

6. Hölder Continuity of v and $v_{\tau,h}$ in t

We use the method of "shaking" the coefficients introduced in [15] and [16]. Take a nonempty set

$$S \subset B_1 = \{x \in \mathbb{R}^d : |x| < 1\}$$

and for $\varepsilon \in \mathbb{R}^d$ introduce $v_{\tau,h}^{\varepsilon,S}$ as the unique solution of equation

$$\delta_\tau^T u + \sup_{(\alpha,y)\in A\times S} [L_h^\alpha(t, x+\varepsilon y)u(t,x) + f^\alpha(t, x+\varepsilon y)] = 0$$

in H_T with terminal condition

$$u(T,x) = \sup_{y\in S} g(x+\varepsilon y) \qquad \text{on } \mathbb{R}^d. \tag{6.1}$$

Also let $v^{\varepsilon,S}$ be a probabilistic solution of

$$\frac{\partial}{\partial t}u(t,x) + \sup_{(\alpha,y)\in A\times S} [L^\alpha(t, x+\varepsilon y)u(t,x) + f^\alpha(t, x+\varepsilon y)] = 0$$

in H_T with terminal condition (6.1). Observe that if S is a singleton $\{y\}$, then by uniqueness

$$v_{\tau,h}^{\varepsilon,S}(t,x) = v_{\tau,h}(t, x+\varepsilon y), \qquad v^{\varepsilon,S}(t,x) = v(t, x+\varepsilon y).$$

In the following lemma Assumption 2.2 is not needed.

Lemma 6.1. *There is a constant N depending only on K and d_1 such that if for a number $c_0 \geq 0$, (5.3) holds, then for all $\varepsilon \in \mathbb{R}$,*

$$|v_{\tau,h}^{\varepsilon,S} - v_{\tau,h}| \leq N e^{c_0(T+\tau)}|\varepsilon| \qquad \text{on } \bar{H}_T, \tag{6.2}$$

$$|v^{\varepsilon,S} - v| \leq N e^{(N-\lambda)_+ T}|\varepsilon| \qquad \text{on } \bar{H}_T. \tag{6.3}$$

In particular (take $S = \{(y-x)/|y-x|\}$, $\varepsilon = |y-x|$),

$$|v_{\tau,h}(t,y) - v_{\tau,h}(t,x)| \leq N e^{c_0(T+\tau)}|y-x|, \qquad (t,y),(t,x) \in \bar{H}_T,$$

$$|v(t,y) - v(t,x)| \leq N e^{(N-\lambda)_+ T}|y-x|, \qquad (t,y),(t,x) \in \bar{H}_T. \tag{6.4}$$

Proof. While proving (6.2) we may concentrate on $\bar{\mathcal{M}}_T$. Then it suffices to use Theorem 5.6, where we take $A \times S$, $(\sigma, b, c, f)(t,x)$ and $(\sigma, b, c, f)(t, x+\varepsilon y)$ in place of A, (σ, b, c, f), and $(\hat\sigma, \hat b, \hat c, \hat f)$, respectively. We also use that the difference of sups is less than the sup of differences while estimating the boundary terms.

Estimate (6.4) is a particular case of Theorem 4.1.1 of [13] and (6.3) is, actually, a particular case of (6.4) since one can view ε as just another coordinate of the space variable. The lemma is proved. □

For $\Lambda \subset (-1,0)$ introduce $v^{\varepsilon,\Lambda,S}_{\tau,h}$ as the unique bounded solution of equation

$$\delta^T_\tau u(t,x) + \sup_{(\alpha,r,y)\in A\times\Lambda\times S} [L^\alpha_h(t+\varepsilon^2 r, x+\varepsilon y)u(t,x) + f^\alpha(t+\varepsilon^2 r, x+\varepsilon y)] = 0 \tag{6.5}$$

in H_T with terminal condition (2.2). Also let $v^{\varepsilon,\Lambda,S}$ be a probabilistic solution of

$$\frac{\partial}{\partial t}u(t,x) + \sup_{(\alpha,r,y)\in A\times\Lambda\times S} [L^\alpha(t+\varepsilon^2 r, x+\varepsilon y)u(t,x) + f^\alpha(t+\varepsilon^2 r, x+\varepsilon y)] = 0$$

in H_T with terminal condition (2.2).

Lemma 6.2. *There is a constant N depending only on K and d_1 such that if for a number $c_0 \geq 0$, (5.3) holds, then for all $\varepsilon \in \mathbb{R}$,*

$$|v^{\varepsilon,\Lambda,S}_{\tau,h} - v_{\tau,h}| \leq Ne^{c_0(T+\tau)}|\varepsilon| \qquad on \quad [0,T]\times\mathbb{R}^d, \tag{6.6}$$

If, additionally, $\tau, h \leq K$, then

$$|v^{\varepsilon,\Lambda,S}_{\tau,h}(t,x) - v^{\varepsilon,\Lambda,S}_{\tau,h}(t,y)| \leq Ne^{c_0(T+\tau)}|y-x|, \tag{6.7}$$

$$|v^{\varepsilon,\Lambda,S}_{\tau,h}(t,x) - v^{\varepsilon,\Lambda,S}_{\tau,h}(s,x)| \leq Ne^{c_0(T+\tau)}(|t-s|^{1/2}+\tau^{1/2}), \tag{6.8}$$

$$|v_{\tau,h}(t,x) - v_{\tau,h}(t,y)| \leq Ne^{c_0(T+\tau)}|y-x|, \tag{6.9}$$

$$|v_{\tau,h}(t,x) - v_{\tau,h}(s,x)| \leq Ne^{c_0(T+\tau)}(|t-s|^{1/2}+\tau^{1/2}) \tag{6.10}$$

for all $t,s \leq T$, $x,y \in \mathbb{R}^d$ with $|t-s| \leq 1$,

$$|v^{\varepsilon,\Lambda,S} - v| \leq Ne^{(N-\lambda)_+T}|\varepsilon| \qquad on \quad [0,T]\times\mathbb{R}^d, \tag{6.11}$$

$$|v^{\varepsilon,\Lambda,S}(t,x) - v^{\varepsilon,\Lambda,S}(s,y)| \leq Ne^{(N-\lambda)_+T}(|t-s|^{1/2}+|y-x|), \tag{6.12}$$

$$|v(t,x) - v(s,y)| \leq Ne^{(N-\lambda)_+T}(|t-s|^{1/2}+|y-x|) \tag{6.13}$$

for all $(t,x),(s,y) \in \bar{H}_T$ with $|t-s| \leq 1$.

Proof. Estimates (6.11) and (6.12) are proved in Corollary 3.2 of [15]. Estimate (6.13) is obtained from (6.12) by setting $\varepsilon = 0$.

The proof of (6.6) follows that of (6.2) and is left to the reader. Estimates (6.7) and (6.9) are taken from Lemma 6.1.

On the one hand, estimate (6.8) for $\varepsilon = 0$ implies (6.10) and, on the other hand, (6.5) is a particular case of (2.3), and therefore (6.8) is a particular case of (6.10). Hence to finish proving the lemma it only remains to prove (6.10), the left-hand side of which we denote by $I(t,s,x)$.

Observe that if $0 \le t \le s \le T$, $s - t \le 1$, and $s - t = n\tau + \gamma$, where $n = 0, 1, \ldots$, $\gamma \in [0, \tau)$, then by Lemma 6.1 and Corollary 3.7,

$$\begin{aligned} I(t, s, x) &\le |v_{\tau,h}(t, x) - v_{\tau,h}(t + n\tau, x)| + |v_{\tau,h}(t + n\tau, x) - v_{\tau,h}(s, x)| \\ &\le Ne^{c_0(T+\tau)}|t - s|^{1/2} + |v_{\tau,h}(t + n\tau, x) - v_{\tau,h}(s, x)|. \end{aligned}$$

Thus, it suffices to estimate $I(t, s, x)$ for $s = t + \gamma$ with $\gamma \in (0, \tau)$. By shifting the origin we reduce the problem to showing that for $\gamma \le T$,

$$I(0, \gamma, 0) \le Ne^{c_0(T+\tau)}\tau^{1/2}. \tag{6.14}$$

Introduce $S = \tau[T/\tau]$ and, first, additionally assume that $S \ge \tau$. In that case, set $u = v_{\tau,h}$, $\hat{u}(r, y) = v_{\tau,h}((r + \gamma) \wedge T, y)$, and

$$[\hat{\sigma}^\alpha_k, \hat{b}^\alpha_k, \hat{c}^\alpha, \hat{f}^\alpha](r, y) = [\sigma^\alpha_k, b^\alpha_k, c^\alpha, f^\alpha](r + \gamma, y).$$

Notice that for $(r, y) \in \mathcal{M}_S$ we have $r + \gamma < S \le T$,

$$\tau_S(r) = \tau, \qquad r + \tau_S(r) = r + \tau,$$

$$(r + \tau + \gamma) \wedge T = r + \gamma + \tau_T(r + \gamma),$$

$$\hat{u}(r + \tau_S(r), y) - \hat{u}(r, y) = v_{\tau,h}((r + \tau + \gamma) \wedge T, y) - v_{\tau,h}(r + \gamma, y).$$

It follows that relative to $\bar{\mathcal{M}}_S$ the function $\hat{u}$ in $\mathcal{M}_S$ satisfies (2.3) constructed from $\hat{\sigma}^\alpha_k, \hat{b}^\alpha_k, \hat{c}^\alpha, \hat{f}^\alpha$. By observing that the parameter ε in Theorem 5.6 is less than $NK\gamma^{1/2}$ owing to Assumption 2.2 and using again that $v_{\tau,h}$ is Lipschitz continuous in x we obtain from Theorem 5.6 that

$$\begin{aligned} I(0, \gamma, 0) &= |v_{\tau,h}(0, 0) - v_{\tau,h}(\gamma, 0)| = |u(0, 0) - \hat{u}(0, 0)| \\ &\le Ne^{c_0(T+\tau)}\gamma^{1/2} + \sup_{(S,y)\in\bar{\mathcal{M}}_S} |u(S, y) - \hat{u}(S, y)| \\ &= Ne^{c_0(T+\tau)}\gamma^{1/2} + \sup_y |v_{\tau,h}(S, y) - v_{\tau,h}((S + \gamma) \wedge T, y)|. \end{aligned}$$

Thus, after one more shift of the origin, bringing S to zero, we reduce the problem of estimating $I(0, \gamma, 0)$ to the situation when $T < \tau$, so that $t = 0$, $\tau_T(t) = T - t$, and $t + \tau_T(t) = T$ on

$$\mathcal{M}_T = \bar{\mathcal{M}}_T \cap \{t = 0\}.$$

If $\gamma < T$, then the function $\tilde{u}$, introduced on $\bar{\mathcal{M}}_T$ by

$$\tilde{u}(0, x) = v_{\tau,h}(\gamma, x), \qquad \tilde{u}(T, x) = g(x),$$

on $\mathcal{M}_T$ satisfies (2.3) corresponding to $\hat{\sigma}_k^\alpha, \hat{b}_k^\alpha, \hat{c}^\alpha, \hat{f}^\alpha$. By Theorem 5.6 we conclude that

$$I(0,\gamma,0) = |v_{\tau,h}(\gamma,0) - v_{\gamma,h}(0,0)| = |\tilde{u}(0,0) - u(0,0)| \le N\gamma^{1/2}.$$

Finally, if $T = \gamma < \tau$, then the function $\check{u}$ introduced on $\bar{\mathcal{M}}_\tau$ by

$$\check{u}(0,x) = v_{\tau,h}(0,x), \qquad \check{u}(\tau,x) = g(x)$$

on $\mathcal{M}_\tau$ satisfies (2.3). By Corollary 3.7

$$I(0,\gamma,0) = |v_{\tau,h}(\gamma,0) - v_{\gamma,h}(0,0)| = |\check{u}(\tau,0) - \check{u}(0,0)| \le N\tau^{1/2}.$$

Estimate (6.14) and the lemma are proved. □

7. Proof of Theorems 2.3–2.5

Proof of Theorem 2.3. We start by proving (2.4) with N which may depend on T. Observe that if

$$T \le 2\varepsilon^2, \qquad \varepsilon := (\tau + h^2)^{1/4},$$

then we have nothing to prove since then by (6.13) and (6.10),

$$\sup_{\bar{H}_T}|v_{\tau,h} - v| \le \sup_{\bar{H}_T}|v_{\tau,h} - g| + \sup_{\bar{H}_T}|v_{\tau,h} - g| \le N(T^{1/2} + \tau^{1/2})$$
$$\le N(\tau + h^2)^{1/4} \le N(\tau^{1/4} + h^{1/2}).$$

Therefore in the rest of the proof without loss of generality we assume that $T > 2\varepsilon^2$. By Corollary 3.4 we have $|v|$ and $|v_{\tau,h}|$ under control and therefore we may assume that τ is so small that there is a $c_0 = c_0(K, d_1)$ such that even with $\lambda = 0$ it satisfies condition (5.3) imposed in Lemma 6.2.

First we prove that

$$v \le v_{\tau,h} + N(\tau^{1/4} + h^{1/2}) \qquad \text{on } \bar{H}_T. \tag{7.1}$$

We take $\Lambda = (-1, 0)$ and $S = B_1$ and set

$$v_{\tau,h}^\varepsilon = v_{\tau,h}^{\varepsilon,\Lambda,S},$$

where the latter function is introduced before Lemma 6.2. Then for any $\alpha \in A$, $r \in (-1, 0)$, and $|y| < 1$,

$$\delta_\tau v_{\tau,h}^\varepsilon(t - \varepsilon^2 r, x - \varepsilon y) + L_h^\alpha(t,x) v_{\tau,h}^\varepsilon(t - \varepsilon^2 r, x - \varepsilon y) + f^\alpha(t,x) \le 0 \tag{7.2}$$

provided that

$$(t,x) \in \bar{H}_{T-2\varepsilon^2} \subset \bar{H}_{T-\tau-\varepsilon^2}.$$

Next take a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ with support in $(-1, 0) \times B_1$ and unit integral. For any function u for which it makes sense we set

$$u^{(\varepsilon)}(t, x) = \varepsilon^{-d-2} \int_{\mathbb{R}^{d+1}} u(s, y)\zeta((t-s)/\varepsilon^2, (x-y)/\varepsilon)\, ds\, dy.$$

By multiplying (7.2) by ζ and integrating we get that for any $\alpha \in A$ on $\bar{H}_{T-2\varepsilon^2}$ it holds that

$$\delta_\tau v_{\tau,h}^{\varepsilon(\varepsilon)} + L_h^\alpha v_{\tau,h}^{\varepsilon(\varepsilon)} + f^\alpha \le 0.$$

From here by Taylor's formula (see (2.8)) we infer

$$\frac{\partial}{\partial t} v_{\tau,h}^{\varepsilon(\varepsilon)} + L^\alpha v_{\tau,h}^{\varepsilon(\varepsilon)} + f^\alpha$$
$$\le N\left(\tau |D_t^2 v_{\tau,h}^{\varepsilon(\varepsilon)}|_{0,\bar{H}_{T-2\varepsilon^2}} + h^2 |D_x^4 v_{\tau,h}^{\varepsilon(\varepsilon)}|_{0,\bar{H}_{T-2\varepsilon^2}} + h|D_x^2 v_{\tau,h}^{\varepsilon(\varepsilon)}|_{0,\bar{H}_{T-2\varepsilon^2}}\right) =: I$$

in $\bar{H}_{T-2\varepsilon^2}$. It follows that

$$v_{\tau,h}^{\varepsilon(\varepsilon)} + (T - 2\varepsilon^2 - t)I \tag{7.3}$$

is a supersolution of (2.1) in $\bar{H}_{T-2\varepsilon^2}$ and either by Itô's formula or by properties of viscosity solutions we have in $\bar{H}_{T-2\varepsilon^2}$ that

$$v \le v_{\tau,h}^{\varepsilon(\varepsilon)} + (T - 2\varepsilon^2 - t)I + \sup_{\{T-2\varepsilon^2\}\times\mathbb{R}^d} |v - v_{\tau,h}^{\varepsilon(\varepsilon)}|. \tag{7.4}$$

Now use the fact that owing to (6.7), (6.8), inequality $\tau^{1/2} \le \varepsilon$, and well-known properties of convolutions we have in $\bar{H}_{T-2\varepsilon^2}$ that

$$|v_{\tau,h}^{\varepsilon(\varepsilon)} - v_{\tau,h}^{\varepsilon}| \le N\varepsilon$$

with N depending only on K, T, d, and d_1 and for any $n = 1, 2, \ldots,$

$$|D_t^n v_{\tau,h}^{\varepsilon(\varepsilon)}|_{0,\bar{H}_{T-2\varepsilon^2}} + |D_x^{2n} v_{\tau,h}^{\varepsilon(\varepsilon)}|_{0,\bar{H}_{T-2\varepsilon^2}} \le N/\varepsilon^{2n-1},$$

where N depends only on n, K, T, d, and d_1. Also, notice that

$$|v(T - 2\varepsilon^2, x) - v_{\tau,h}^{\varepsilon}(T - 2\varepsilon^2, x)|,$$

that appears from the last term in (7.4), is estimated through $N\varepsilon$ in the beginning of the proof. Then we conclude

$$v \le v_{\tau,h} + N[\varepsilon + (\tau + h^2)/\varepsilon^3 + h/\varepsilon]$$

in $\bar{H}_{T-2\varepsilon^2}$. Actually, the same estimate holds in $\bar{H}_T$ due to the argument in the beginning of the proof. Finally by observing that

$$\varepsilon + (\tau + h^2)/\varepsilon^3 + h/\varepsilon \le \varepsilon + (\tau + h^2)/\varepsilon^3 + (\tau + h^2)^{1/2}/\varepsilon$$

and recalling that $\varepsilon = (\tau + h^2)^{1/4}$ we come to (7.1).

It remains to prove that

$$v_{\tau,h} - v \le N(\tau^{1/4} + h^{1/2}). \tag{7.5}$$

Similarly to what was done with the discrete approximation above, on the basis of functions $v^{\varepsilon,\Lambda,S}$ in the proof of Theorem 2.1 of [16] an infinitely differentiable function u on $\bar{H}_T$ is constructed such that

$$\frac{\partial}{\partial t}u + \sup_{\alpha\in A}[L^\alpha u + f^\alpha] \le 0, \qquad |u - v| \le N\varepsilon \quad \text{on } \bar{H}_T,$$

with N depending only on K, T, d, and d_1 and for any $n = 1, 2, \dots,$

$$|D_t^n u|_{0,\bar{H}_T} + |D_x^{2n} u|_{0,\bar{H}_T} \le N/\varepsilon^{2n-1},$$

where N depends only on n, K, T, d, and d_1. As above, it follows by Taylor's formula that on $\bar{H}_{T-\tau}$ (where $\tau_T(t) = \tau$) we have

$$\delta_\tau^T u + \sup_{\alpha\in A}[L_h^\alpha u + f^\alpha] \le N(\tau + h^2)/\varepsilon^3 + Nh/\varepsilon.$$

Upon taking

$$u_1 = v_{\tau,h}, \qquad u_2 = u + \sup_{H_T\setminus H_{T-\tau}}(v_{\tau,h} - u)_+,$$
$$C = N(\tau + h^2)/\varepsilon^3 + Nh/\varepsilon$$

in Lemma 3.3, we obtain

$$v_{\tau,h} \le u + \sup_{H_T\setminus H_{T-\tau}}(v_{\tau,h} - u)_+ + N(\tau + h^2)/\varepsilon^3 + Nh/\varepsilon.$$

Here $u \le v + N\varepsilon$ and, owing to (6.13) and (6.10) and the above mentioned properties of u,

$$\sup_{H_T\setminus H_{T-\tau}}(v_{\tau,h} - u)_+ \le \sup_{H_T\setminus H_{T-\tau}}|v_{\tau,h} - g| + \sup_{H_T\setminus H_{T-\tau}}|g - v| + \sup_{H_T\setminus H_{T-\tau}}|v - u|$$
$$\le N(\tau^{1/2} + \varepsilon).$$

Thus,

$$v_{\tau,h} \le v + N[\varepsilon + \tau^{1/2} + (\tau + h^2)/\varepsilon^3 + h/\varepsilon]$$
$$\le N[\varepsilon + (\tau + h^2)/\varepsilon^3 + (\tau + h^2)^{1/2}/\varepsilon].$$

Recalling that $\varepsilon = (\tau + h^2)^{1/4}$ yields (7.5) and (2.4) with N perhaps depending on T.

However, if λ is large enough, $c_0 = 0$ satisfies condition (5.3) imposed in Lemma 6.2 and for any $\lambda > 0$, the functions v and $v_{\tau,h}$ are bounded by a constant depending only on K and λ owing to Corollary 3.4. In that case the estimates in Lemma 6.2 are also independent of T. Furthermore, we can replace $T - 2\varepsilon^2 - t$ in (7.3) with the constant N from (5.3). This allows us to check that in the above proof the constants are actually independent of T if $\lambda \geq N = N(K, d_1)$.

The theorem is proved. $\square$

Proof of Theorem 2.4. Take $g \equiv 0$ and denote the functions v and $v_{\tau,h}$ from Theorem 2.3 by v^T and $v^T_{\tau,h}$. Obviously, it suffices to prove that for all (t, x),

$$\tilde{v}(x) = \lim_{T\to\infty} v^T(t, x), \qquad \tilde{v}_h(x) = \lim_{T\to\infty} v^T_{\tau,h}(t, x), \tag{7.6}$$

whenever $\lambda > 0$ and τ is small enough.

The first relation in (7.6) is well known (see, for instance, [12] or [13]). To prove the second, it suffices to prove that for any sequence $T_n \to \infty$ such that $v^{T_n}_{\tau,h}(t, x)$ converges at all points of $\mathcal{M}_\infty$, the limit is independent of t and satisfies (2.5) on the grid

$$G = \{i_i h\ell_1 + \cdots + i_{d_1} h\ell_{d_1}\colon\ i_k = 0, \pm 1, \ldots, k = 1, \ldots, d_1\}.$$

Given the former, the latter is obvious. Also notice that the translation $t \to t + \tau$ brings any solution of (2.3) on $\mathcal{M}_\infty$ again to a solution. Therefore, it only remains to prove the uniqueness of the bounded solutions of (2.3) on $\mathcal{M}_\infty$.

Observe that if u_1 and u_2 are two solutions of (2.3) on $\mathcal{M}_\infty$, then they also solve (2.3) on $\mathcal{M}_T$ for any T with terminal condition u_1 and u_2, respectively. By the comparison result

$$|u_1 - u_2| \leq e^{-\lambda T/2} \sup|u_1 - u_2|$$

if τ is small enough. Sending $T \to \infty$ proves the uniqueness and the theorem. $\square$

Proof of Theorem 2.5. The unique solvability of (2.6)-(2.2) in the space of bounded functions is shown by rewriting the problem as

$$u(t, x) = g(x) + \int_t^T F(\Delta_{h,\ell_k} u(s, x), \delta_{h,\ell_k} u(s, x), u(s, x), s, x)\, ds \tag{7.7}$$

and using, say, the method of successive approximations.

Next, since $v_{\tau,h}$ are Hölder continuous in (t, x), for any sequence $\tau_n \downarrow 0$, one can find a subsequence $\tau_{n'} \downarrow 0$ such that $v_{\tau_{n'},h}(t, x)$ converge at each point of $\mathbb{R}^d$ uniformly in $t \in [0, T]$. Call u the limit of one of the subsequences and introduce

$$\kappa_{n'}(t) = i\tau_{n'} \qquad \text{for} \quad i\tau_{n'} \leq t < (i+1)\tau_{n'}, \quad i = 0, 1, \ldots.$$

Then for any smooth $\psi(t)$ vanishing at $t = T$ and $t = 0$,

$$\int_0^T [\psi F(\Delta_{h,\ell_k} v_{\tau_{n'},h}, \delta_{h,\ell_k} v_{\tau_{n'},h}, v_{\tau_{n'},h})](\kappa_{n'}(t), x)\, dt$$
$$= \int_0^T v_{\tau_{n'},h}(\kappa_{n'}(t), x) \frac{\psi(\kappa_{n'}(t), x) - \psi(\kappa_{n'}(t) - \tau_{n'}, x)}{\tau_{n'}}\, dt.$$

Since the integrands converge uniformly on $[0, T]$ to their natural limits, we conclude that u satisfies (2.6) in the weak sense. This is also a continuous function and $u(T, x) = g(x)$. It follows that u satisfies (7.7) and by uniqueness $u = v_h$. Now Theorem 2.5 follows directly from Theorem 2.3. □

8. Concluding Remarks

The methods of this article can also be applied to equations in cylinders like $Q = [0, T) \times D$, where D is a domain in $\mathbb{R}^d$ (see [10]). It is natural to consider (2.1) and (2.3) in Q with terminal condition $u(T, x) = g(x)$ in D and require v and $v_{\tau,h}$ to be zero in $[0, T] \times (\mathbb{R}^d \setminus D)$. If we also assume that $g = 0$ on ∂D, then to carry over our methods we only need to assume that there is a sufficiently smooth function ψ such that $\psi > 0$ in D, $\psi = 0$ on ∂D, $|\psi_x| \geq 1$ on ∂D, and $L^\alpha \psi < -1$ in Q. The reader who went through our proofs understands that the only use of ψ is in estimating the first-order finite-differences of $v_{\tau,h}$ near the lateral boundary of Q and the gradient of v on the lateral boundary of Q.

Elliptic problems and semidiscretization can also be considered in domains. Although these generalizations are almost straightforward, some additional work still needs to be done and to avoid overburdening the present article with technicalities we decided to put them in a subsequent article [10] along with a generalization of Theorem 2.5 to the case when Assumption 2.2 is dropped.

Finally, speaking about equations in domains it is worth noting that one can reduce a smooth nonzero lateral condition to zero by just subtracting the boundary function from the solution.

Acknowledgments

The author is sincerely grateful to Hongjie Dong who made valuable comments on the first version of the article and noticed a few errors. Also greatly appreciated are the comments by the referees of the paper, which certainly helped improve the presentation.

References

1. M. Bardi and I. Cappuzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations, Birkhäuser, Boston, MA, 1997.
2. G. Barles and E.R. Jakobsen, On the convergence rate of approximation schemes for Hamilton–Jacobi–Bellman equations, Math. Model. Numer. Anal., 36(1) (2002), 33–54.
3. G. Barles and E.R. Jakobsen, Error bounds for monotone approximation schemes for Hamilton–Jacobi–Bellman equations, SIAM J. Numer. Anal., to appear.
4. G. Barles and P.E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, Asymp. Anal., 4(3) (1991), 271–283.
5. F. Camilli and M. Falcone, An approximation scheme for optimal control of diffusion processes, Math. Model. Numer. Anal., 29 (1995), 97–122.
6. M.G. Crandall and P.-L. Lions, Convergent difference schemes for nonlinear parabolic equations and mean curvature motion, Numer. Math., 75 (1996), 17-41.

7. K. Deckelnick, Error analysis for a difference scheme approximating mean curvature flow, Interface Free Bound., 2 (2000), 117–142.
8. Hongjie Dong and N.V. Krylov, On the rate of convergence of finite-difference approximations for Bellman equations with constant coefficients, Algebra i Analiz, 17(2) (2005).
9. Hongjie Dong and N.V. Krylov, On the rate of convergence of finite-difference approximations for degenerate linear parabolic equations with C^1 and C^2 coefficients, submitted to Math. Comp.
10. Hongjie Dong and N.V. Krylov, On the rate of convergence of finite-difference approximations for parabolic Bellman equations with Lipschitz coefficients in cylindrical domains, in preparation.
11. M. Falcone and T. Giorgi, An approximation scheme for evolutive Hamilton–Jacobi equations, in Stochastic Analysis, Control, Optimization and Applications, pp. 289–303, Systems & Control: Foundations of Applications, Birkhäuser Boston, Boston, MA, 1999.
12. W. Fleming and M. Soner, Controlled Markov processes and viscosity solutions, Springer-Verlag, New York, 1993.
13. N.V. Krylov, Controlled Diffusion Processes, Nauka, Moscow, 1977; English transl., Springer-Verlag, New York, 1980.
14. N.V. Krylov, On the rate of convergence of finite-difference approximations for Bellman's equations, Algebra i Analiz, 9(3) (1997), 245–256. Also in St. Petersburg Math. J., 9(3) (1998), 639–650.
15. N.V. Krylov, Approximating value functions for controlled degenerate diffusion processes by using piece-wise constant policies, Electron. J. Probab., 4 (1999), paper no. 2, 1–19, http://www.math.washington.edu/ ejpecp/EjpVol4/paper2.abs.html
16. N.V. Krylov, On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients, Probab. Theory Related Fields, 117(1) (2000), 1–16.
17. H.J. Kushner and P.G. Dupuis, Numerical Methods for Stochastic Control Problems in Continuous Time, second edition, Springer-Verlag, Berlin, 2001.
18. S. Larsson and V. Thomée, Partial Differential Equations with Numerical Methods, Texts in Applied Mathematics, 45. Springer-Verlag, Berlin, 2003.
19. R.H. Nochetto and C. Verdi, Convergence past singularities for a fully discrete approximation of curvature driven interfaces, SIAM J. Numer. Anal., 34 (1997), 490–512.
20. J. Zhang, Rate of convergence of finite difference approximations for degenerate ODEs, submitted to Math. Comp.

Accepted 30 *April* 2005. *Online publication* 9 *September* 2005.