

SEMILINEAR EQUATIONS, THE γ_k FUNCTION, AND GENERALIZED GAUDUCHON METRICS

JIXIANG FU, ZHIZHANG WANG, AND DAMIN WU

ABSTRACT. In this paper, we generalize the Gauduchon metrics on a compact complex manifold and define the γ_k functions on the space of its hermitian metrics.

1. INTRODUCTION

Let X be a compact n -dimensional complex manifold. Let g be a hermitian metric on X and ω its hermitian form. It is well known that if $d\omega = 0$, then g or ω is called a Kähler metric and therefore X is called a Kähler manifold. When X is a non-Kähler manifold, one can consider the other conditions on ω such as

$$(1.1) \quad d\omega^k = 0, \quad 2 \leq k \leq n-1.$$

If $d(\omega^{n-1}) = 0$, then g or ω is called a balanced metric and so X is called a balanced manifold [19]. However, when $2 \leq k \leq n-2$, $d\omega^k = 0$ automatically yields $d\omega = 0$ [15]. Instead of (1.1), one can consider the k -Kähler condition [1]. A complex manifold is called k -Kähler if it admits a closed complex transverse (k, k) -form. By this definition, a complex manifold is 1-Kähler if and only if it is Kähler; it is $(n-1)$ -Kähler if and only if it is balanced.

One can also generalize the Kähler condition along other directions, for instance,

$$(1.2) \quad \partial\bar{\partial}\omega^k = 0, \quad 1 \leq k \leq n-1.$$

When $k = n-1$, the metric ω is called a *Gauduchon* metric. Gauduchon [11] proved an interesting result that, for any hermitian metric ω on a compact complex n -dimensional manifold X , there exists a unique (up to a constant) smooth function v such that

$$(1.3) \quad \partial\bar{\partial}(e^v \omega^{n-1}) = 0 \quad \text{on } X.$$

Thus, the Gauduchon metric always exists on a compact complex manifold. It is important in complex geometry since one can use such a metric to define the degree, and then make sense of the stability of holomorphic vector bundles over a non-Kähler complex manifold (see [18]).

When $k = n-2$, the metric ω satisfying (1.2) is called an *astheno-Kähler* metric. Jost and Yau [17] used this condition to study hermitian harmonic maps, and extended Siu's rigidity theorem to non-Kähler complex manifolds.

When $k = 1$, the metric ω in (1.2) is called a *pluriclosed* metric, which is also called strong KT (Kähler with torsion) metric (see [13, 7] and the references therein). Such a condition appeared in [6, 2] as a technical condition. Recently, Streets and

Tian [21] introduced a hermitian Ricci flow under which the pluriclosed metric is preserved.

It is important to find specific hermitian metrics on non-Kähler complex manifolds. J. Li, S.-T. Yau and Fu [8] have constructed balanced metrics on complex structures of manifolds $\#_{k \geq 2}(S^3 \times S^3)$ which are obtained from the conifold transition of Calabi-Yau threefolds. As a corollary, there exists no pluriclosed metric on such manifolds. We note here that the specific hermitian geometry of threefolds $\#_k(S^3 \times S^3)$ was first considered by Bozhkov [3, 4]. In this paper, we generalize (1.2) to weaker conditions:

$$(1.4) \quad \partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0, \quad 1 \leq k \leq n-1.$$

Definition 1. *Let ω be a hermitian metric on an n -dimensional complex manifold X , and k be an integer such that $1 \leq k \leq n-1$. We call ω the k -th Gauduchon metric if ω satisfies (1.4).*

Note that an $(n-1)$ -th Gauduchon metric is the classic Gauduchon metric. The natural question is whether there exists any k -th Gauduchon metric, $1 \leq k \leq n-2$, on a complex manifold. To answer this question, one way is to look for such a metric in the conformal class of a given hermitian metric ω on X :

$$(1.5) \quad \partial\bar{\partial}(e^v\omega^k) \wedge \omega^{n-k-1} = 0.$$

However, equation (1.5) in general needs not admit a solution (see below for reasons). In this paper, we solve the equation

$$(1.6) \quad \partial\bar{\partial}(e^v\omega^k) \wedge \omega^{n-k-1} = \gamma_k e^v \omega^n$$

for some constant γ_k satisfying the compatibility condition. The constant γ_k , if nonzero, can be viewed as an obstruction for the existence of a k -th Gauduchon metric in the conformal class of ω , for $1 \leq k < n-1$.

Equation (1.6) can be reformulated, in a slightly more general form, as follows: Let (X, ω) be an n -dimensional compact hermitian manifold, and B be a smooth real 1-form on X . For any smooth function f on X satisfying

$$(1.7) \quad \int_X f \omega^n = 0,$$

we consider the following semilinear equation

$$(1.8) \quad \Delta v + |\nabla v|^2 + \langle B, dv \rangle = f \quad \text{on } X.$$

Here Δ and ∇ are, respectively, the Laplacian and covariant differentiation associated with ω . Clearly, equation (1.8) needs not have a solution, due to the compatibility condition (1.7). For instance, let ω be balanced and $B = 0$, then in order that (1.8) has a solution the function f has to be zero. Nonetheless, we shall show that, there is a smooth function v so that equation (1.8) holds up to a unique constant c . More generally, we have the following result:

Theorem 2. *Let (X, ω) be a compact hermitian manifold, B be a smooth real 1-form on X , and $\psi \in C^\infty(\mathbb{R})$ satisfy*

$$(1.9) \quad \liminf_{t \rightarrow +\infty} \frac{\psi(t)}{t^\mu} \geq \nu > 0, \quad \text{where } \mu > 1/2 \text{ and } \nu \text{ are constants.}$$

Then, for each $f \in C^\infty(X)$ satisfying (1.7), there exists a unique constant c , and a smooth function v on X , unique up to a constant, such that

$$(1.10) \quad \Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle = f + c \quad \text{on } X.$$

Remark 3. The compatibility condition of (1.10) implies that

$$c = \frac{\int_X (\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle) \omega^n}{\int_X \omega^n},$$

which in general is nonzero.

Letting $\psi(t) = t$ on \mathbb{R} , we obtain an application of Theorem 2:

Corollary 4. Let (X, ω) be an n -dimensional compact hermitian manifold. For any integer $1 \leq k \leq n-1$, there exists a unique constant γ_k , and a function $v \in C^\infty(X)$ satisfying that

$$(1.11) \quad (\sqrt{-1}/2)\partial\bar{\partial}(e^v \omega^k) \wedge \omega^{n-k-1} = \gamma_k e^v \omega^n.$$

The solution v of (1.11) is unique up to a constant. In particular, when $k = n-1$ we have $\gamma_{n-1} = 0$. If ω is Kähler, then $\gamma_k = 0$ and v is a constant, for each $1 \leq k \leq n-1$.

Remark 5. When $k = n-1$, this corollary recovers the classical result of Gauduchon [11].

By Corollary 4, we can associate each hermitian metric ω a unique constant $\gamma_k(\omega)$. Clearly, $\gamma_k = \gamma_k(\omega)$ is invariant under biholomorphisms. Furthermore, we will prove that γ_k depends smoothly on the hermitian metric ω (see Proposition 9); and that $\gamma_k(\omega) = 0$ if and only if there exists a k -th Gauduchon metric in the conformal class of ω (Proposition 8).

We will prove in Proposition 11 that the sign of $\gamma_k(\omega)$, denoted by $(\text{sgn} \gamma_k)(\omega)$, is invariant in the conformal class of ω . We denote by $\Xi_k(X)$ the range of $\text{sgn} \gamma_k$. By definition $\Xi_k(X) \subset \{-1, 0, 1\}$ for each k , and by Corollary 4 we have $\Xi_{n-1}(X) = \{0\}$. A natural question is *whether* $\Xi_k(X) = \{-1, 0, 1\}$ for any $1 \leq k \leq n-2$ on any compact complex manifold X . Indeed, if $\Xi_k(X) \supset \{-1, 1\}$ then the answer is positive, by Proposition 9. Thus, there will be a k -th Gauduchon metric on X . We can also ask whether $\Xi_k(X)$ is invariant under the modification. These questions will be systematically studied later. As a first step, we obtain the following result.

Theorem 6. For $n = 3$, we have $1 \in \Xi_1(X)$. Namely, for any 3-dimensional hermitian manifold X , there exists a hermitian metric ω such that $\gamma_1(\omega) > 0$. In particular, there is no 1-st Gauduchon metric in the conformal class of ω .

Then, we combine the above results to prove that, as an example, $\Xi_1 = \{-1, 0, 1\}$ on the three-dimensional complex manifolds constructed by Calabi [5]. As a consequence, there exists a 1-st Gauduchon metric on these manifolds. It is well-known that such manifolds are non-Kähler but admit balanced metrics. We do not know whether there exists any pluriclosed metric on them.

Another example we considered is $Y = S^5 \times S^1$, endowed with a complex structure so that the natural projection $\pi : S^5 \times S^1 \rightarrow \mathbb{P}^2$ is holomorphic. This would imply

that there is no balanced metrics on $S^5 \times S^1$. Moreover, we can prove that $S^5 \times S^1$ does not admit any pluriclosed metric. On the other hand, by considering a natural hermitian metric on $S^5 \times S^1$, we are able to show that $\Xi_1(S^5 \times S^1) = \{-1, 0, 1\}$. Thus, $S^5 \times S^1$ admits a 1-st Gauduchon metric.

We shall solve equation (1.10) by the continuity method. In Section 2, we set up the machinery and prove the openness. The closedness and *a priori* estimates are established in Section 3. In Section 4, we prove the uniqueness part in Theorem 2 and also prove Corollary 4. In Section 5, we discuss the relation between γ_k and the k -th Gauduchon metric. In section 6, we prove Theorem 6, and explicitly construct a metric with positive γ_1 on the complex 3-torus. As another example, we show that the natural balanced metric on the Iwasawa manifold has a positive γ_1 number. In section 7, we establish the existence of 1-st Gauduchon metric on Calabi's 3-dimensional non-Kähler manifold, by using Theorem 6 and proving that the balanced metric on the manifold has a negative γ_1 number. In the last section, we prove the existence of a 1-st Gauduchon metric on $S^5 \times S^1$. We also show the nonexistence of balanced metric and pluriclosed metric on $S^5 \times S^1$.

Acknowledgment. The authors would like to thank Professor S.-T. Yau for helpful discussion. Part of the work was done while the third named author was visiting Fudan University, he would like to thank their warm hospitality. Fu is supported in part by NSFC grants and LMNS.

2. NOTATION AND PRELIMINARIES

Throughout this note, we use the following convention: We write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Let $(g^{i\bar{j}})$ be the transposed inverse of the matrix $(g_{i\bar{j}})$. For any two real 1-forms A and B on X , locally given by

$$A = \sum_{i=1}^n (A_i dz_i + A_{\bar{i}} d\bar{z}_i) \quad \text{and} \quad B = \sum_{i=1}^n (B_i dz_i + B_{\bar{i}} d\bar{z}_i),$$

we denote

$$\langle A, B \rangle_\omega = \frac{1}{2} \sum_{i,j=1}^n g^{i\bar{j}} (A_i B_{\bar{j}} + A_{\bar{j}} B_i).$$

We may omit the subscript ω in $\langle \cdot, \cdot \rangle_\omega$ when it is understood from the context. In particular, we have

$$\langle dh, dh \rangle = \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial h}{\partial z_i} \frac{\partial h}{\partial \bar{z}_j} \equiv |\nabla h|^2, \quad \text{for all } h \in C^1(X).$$

The Laplacian Δ associated with ω is given by

$$\Delta h = \frac{n\omega^{n-1} \wedge (\sqrt{-1}/2)\partial\bar{\partial}h}{\omega^n} = \sum_{i,j=1}^n g^{i\bar{j}} h_{i\bar{j}}, \quad \text{for all } h \in C^2(X).$$

We use the continuity method to solve (1.10). Fix an integer $l \geq n + 4$ and a real number $0 < \alpha < 1$. We denote by $C^{l,\alpha}(X)$ the usual Hölder space on X . Let

$$S(u) = \Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle - \frac{\int_X (\Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle) \omega^n}{\int_X \omega^n},$$

for each $u \in C^{l,\alpha}(X)$. Consider the following family of equations,

$$(2.1) \quad S(v_t) = tf, \quad 0 \leq t \leq 1.$$

Let I be the subset of $[0, 1]$ consisting of t for which the equation (2.1) has a solution $v_t \in C^{l,\alpha}(X)$ satisfying

$$(2.2) \quad \int_X v_t \omega^n = 0.$$

Obviously, the set I is nonempty since $0 \in I$. The openness of I will follow from our previous results [9, Section 3]. Indeed, let

$$(2.3) \quad \mathcal{E}_\omega^{l,\alpha} = \left\{ h \in C^{l,\alpha}(X); \int_X h \omega^n = 0 \right\}.$$

Notice that $S : \mathcal{E}_\omega^{l+2,\alpha} \rightarrow \mathcal{E}_\omega^{l,\alpha}$. The linearization of S is

$$L_\omega(h) = \left. \frac{d}{dt} S(v + th) \right|_{t=0} = \Delta h + \langle \tilde{B}, dh \rangle - \frac{\int_X (\Delta h + \langle \tilde{B}, dh \rangle) \omega^n}{\int_X \omega^n},$$

where

$$\tilde{B} = B + 2\psi'(|\nabla v|^2) dv.$$

It follows from the proof of Lemma 13 in [9] that L_ω is a linear isomorphism from $\mathcal{E}_\omega^{l+2,\alpha}(X)$ to $\mathcal{E}_\omega^{l,\alpha}(X)$. Thus, by the implicit theorem we obtain the openness of I .

For the closedness of I we need the *a priori* estimate, which will be established in Section 3.

3. A PRIOR ESTIMATES

Let (X, ω) be an n -dimensional hermitian manifold, B a smooth 1-form on X , f a smooth function on X , c a constant, and $\psi \in C^\infty(\mathbb{R})$ satisfy (1.9). Consider the following semi-linear equation:

$$(3.1) \quad S(v) \equiv \Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle - c = f \quad \text{on } X,$$

where $v \in C^3(X)$ satisfies the normalization condition

$$(3.2) \quad \int_X v \omega^n = 0.$$

We shall first derive a uniform gradient estimate:

Lemma 7. *Let $v \in C^3(X)$ be a solution of (3.1). We have*

$$\sup_X |\nabla v| \leq C,$$

where $C > 0$ is a constant depending only on $B, f, \omega, \psi(0), \mu$ and ν .

Throughout this section, we always denote by $C > 0$ a generic constant depending only on $B, f, \omega, \psi(0), \mu$, and ν , unless otherwise indicated.

Proof. Since X is compact, we can assume that $|\nabla v|^2$ attains its maximum at some point $x_0 \in X$. Consider the following linear elliptic operator

$$L(h) = \Delta h + 2\psi'(|\nabla v|^2)\langle dh, dv \rangle_\omega = \Delta h + \psi'(|\nabla v|^2)g^{i\bar{j}}(h_i v_{\bar{j}} + h_{\bar{j}} v_i),$$

Here the summation convention is used, and we denote

$$h_i = \frac{\partial h}{\partial z^i}, \quad g_{,k}^{i\bar{j}} = \frac{\partial g^{i\bar{j}}}{\partial z^k}, \quad \dots$$

We compute that

$$\begin{aligned} L(|\nabla v|^2) &= \Delta(|\nabla v|^2) + \psi'g^{i\bar{j}}[(|\nabla v|^2)_i v_{\bar{j}} + v_i (|\nabla v|^2)_{\bar{j}}] \\ &= g^{i\bar{j}}g^{p\bar{q}}(v_{p\bar{i}}v_{\bar{q}\bar{j}} + v_{p\bar{j}}v_{i\bar{q}}) + g^{p\bar{q}}[(\Delta v)_p v_{\bar{q}} + v_p(\Delta v)_{\bar{q}}] + g^{i\bar{j}}g_{,i\bar{j}}^{p\bar{q}}v_p v_{\bar{q}} \\ &\quad + g^{i\bar{j}}g_{,i\bar{i}}^{p\bar{q}}(v_{p\bar{j}}v_{\bar{q}} + v_p v_{\bar{q}\bar{j}}) + g^{i\bar{j}}g_{,j\bar{j}}^{p\bar{q}}(v_{p\bar{i}}v_{\bar{q}} + v_p v_{i\bar{q}}) \\ &\quad - g^{p\bar{q}}(g_{,p\bar{p}}^{i\bar{j}}v_{i\bar{j}}v_{\bar{q}} + g_{,q\bar{q}}^{i\bar{j}}v_p v_{i\bar{j}}) + \psi'g^{i\bar{j}}[(|\nabla v|^2)_i v_{\bar{j}} + v_i (|\nabla v|^2)_{\bar{j}}]. \end{aligned}$$

Using equation (3.1) to the second term on the far right of above equalities and then using the Schwarz inequality, we find

$$L(|\nabla v|^2) \geq \frac{1}{2}g^{i\bar{j}}g^{p\bar{q}}(v_{p\bar{i}}v_{\bar{q}\bar{j}} + v_{p\bar{j}}v_{i\bar{q}}) - C|\nabla v|^2 - C.$$

To see more clearly, let us take a normal coordinate system around x_0 such that

$$g_{i\bar{j}}(x_0) = \delta_{ij}, \quad \text{for all } i, j = 1, \dots, n.$$

It follows that

$$\begin{aligned} L(|\nabla v|^2) &\geq \frac{1}{2} \sum_{i,p=1}^n |v_{p\bar{i}}|^2 - C|\nabla v|^2 - C \\ &\geq \frac{1}{2} \sum_{i=1}^n |v_{i\bar{i}}|^2 - C|\nabla v|^2 - C \\ &\geq \frac{1}{2n} |\Delta v|^2 - C|\nabla v|^2 - C \quad (\text{by Cauchy's inequality}) \\ &\geq \frac{1}{2n} |\psi(|\nabla v|^2) + \langle B, dv \rangle - f - c|^2 - C|\nabla v|^2 - C \quad (\text{by (3.1)}) \\ &\geq \frac{1}{4n} |\psi(|\nabla v|^2)|^2 - C|\nabla v|^2 - C(1 + |c|^2). \end{aligned}$$

We can assume, without loss of generality, that $|\nabla v|^2(x_0)$ is sufficiently large so that

$$\psi(|\nabla v|^2) \geq \frac{\nu}{2} |\nabla v|^2 \quad \text{at } x_0,$$

where $\mu > 1/2$ and $\nu > 0$ are constants, by (1.9). Now notice that

$$L(|\nabla v|^2) \leq 0 \quad \text{at } x_0,$$

because of

$$\Delta(|\nabla v|^2)(x_0) \leq 0, \quad \text{and} \quad \nabla(|\nabla v|^2)(x_0) = 0.$$

Hence, we obtain that

$$\sup_X |\nabla v|^2 = |\nabla v|^2(x_0) \leq C(1 + |c|^2).$$

It remains to bound the constant c in terms of f and $\psi(0)$: Apply the usual maximum principle to (3.1) to obtain that

$$(3.3) \quad \psi(0) - \sup_X f \leq c \leq -\inf_X f + \psi(0).$$

This finishes the proof. \square

Next, we establish the C^0 estimate: Noticing (3.2), there must exist some point $y_0 \in X$ such that $v(y_0) = 0$. Then, for any point $y \in X$, we take a geodesic curve γ connecting y_0 to y . We have by Lemma 7 that,

$$|v(y)| = |v(y) - v(y_0)| = \left| \int_0^1 \frac{d(v \circ \gamma)}{dt} dt \right| \leq \int_0^1 (|\nabla v| \circ \gamma) dt < C.$$

This settles the C^0 estimate of v .

We rewrite equation (3.1) as

$$\Delta v = -\psi(|\nabla v|^2) - \langle B, dv \rangle + f + c.$$

By $W^{2,p}$ theory of elliptic equations, we have for any $p > 1$,

$$\begin{aligned} \|v\|_{W^{2,p}} &\leq C(\|v\|_{L^p} + \|f + c - \psi(|\nabla v|^2) - \langle B, dv \rangle\|_{L^p}) \\ &\leq C_1, \end{aligned}$$

where in the last inequality we have used the C^0 and C^1 estimates of v , and (3.3). Here and below, we denote by C_1 a generic constant depending on B, f, ω, μ, ν , and also p , and $\max\{|\psi(t)|; 0 \leq t \leq \max |\nabla v|^2 \leq C\}$.

Fix a sufficiently large p such that $\alpha \equiv 2n/p < 1$. It follows from the Sobolev embedding theorem that

$$\|v\|_{C^{1,\alpha}} \leq C_1.$$

This allows us to apply Schauder's theory to obtain that

$$\|v\|_{C^{2,\alpha}} \leq C_1.$$

Thus, by the bootstrap argument, we have

$$(3.4) \quad \|v\|_{C^{k,\alpha}} \leq C_1, \quad \text{for any } k \geq 1.$$

This implies that the set I defined in Section 2 is closed. As a consequence, we have shown the existence part in Theorem 2.

4. UNIQUENESS AND COROLLARY

Let us prove the uniqueness in Theorem 2. Suppose that there exist c , v and \tilde{c} , \tilde{v} such that

$$\begin{aligned}\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle &= f + c, \\ \Delta \tilde{v} + \psi(|\nabla \tilde{v}|^2) + \langle B, d\tilde{v} \rangle &= f + \tilde{c}.\end{aligned}$$

Then,

$$(4.1) \quad c = \frac{\int_X (\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle) \omega^n}{\int_X \omega^n},$$

$$(4.2) \quad \tilde{c} = \frac{\int_X (\Delta \tilde{v} + \psi(|\nabla \tilde{v}|^2) + \langle B, d\tilde{v} \rangle) \omega^n}{\int_X \omega^n}.$$

Recall that we denote

$$S(u) = \Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle - \frac{\int_X (\Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle) \omega^n}{\int_X \omega^n},$$

for all $u \in C^2(X)$. It follows that

$$(4.3) \quad \begin{aligned}0 &= S(v) - S(\tilde{v}) = \int_0^1 \left[\frac{d}{dt} S(tv + (1-t)\tilde{v}) \right] dt \\ &= \Delta w + \langle \tilde{B}, dw \rangle - c_w.\end{aligned}$$

Here $w = v - \tilde{v}$,

$$\tilde{B} = B + 2 \int_0^1 \psi'(|t\nabla v + (1-t)\nabla \tilde{v}|^2) [tdv + (1-t)d\tilde{v}] dt,$$

and c_w is a constant given by

$$c_w = \frac{\int_X (\Delta w + \langle \tilde{B}, dw \rangle) \omega^n}{\int_X \omega^n}.$$

Applying the maximum principle to (4.3) yields

$$c_w = 0.$$

Then, by the strong maximum principle we conclude that w is equal to a constant. This shows that the solution of (1.10) is unique up to a constant. By (4.1) and (4.2) we have $c = \tilde{c}$. This completes the proof of Theorem 2.

Let us now prove Corollary 4. We define a smooth real 1-form on X

$$(4.4) \quad B_1 = \frac{\sqrt{-1}}{2} \frac{nk}{n-1} \frac{1}{n!} * (\partial(\omega^{n-1}) - \bar{\partial}(\omega^{n-1}))$$

and a smooth function

$$(4.5) \quad \varphi = \frac{n(\sqrt{-1}/2)\partial\bar{\partial}(\omega^k) \wedge \omega^{n-k-1}}{\omega^n}.$$

Then (1.11) is equivalent to

$$\Delta v + |\nabla v|^2 + \langle B_1, dv \rangle + \varphi = n\gamma_k.$$

Letting

$$\psi(t) = t \quad \text{and} \quad f = \frac{\int_X \varphi \omega^n}{\int_X \omega^n} - \varphi,$$

Corollary 4 then follows readily from Theorem 2.

For each $1 \leq k \leq n-1$, the constant γ_k is given by

$$(4.6) \quad \gamma_k = \frac{\int_X e^{-v} (\sqrt{-1}/2) \partial \bar{\partial} (e^v \omega^k) \wedge \omega^{n-k-1}}{\int_X \omega^n}$$

$$(4.7) \quad = \frac{\int_X (\Delta v + |\nabla v|^2 + \langle B_1, dv \rangle + \varphi) \omega^n}{n \int_X \omega^n}.$$

On the other hand, directly integrating (1.11) over X yields that

$$(4.8) \quad \gamma_k = \frac{\int_X (\sqrt{-1}/2) \partial \bar{\partial} (e^v \omega^k) \wedge \omega^{n-k-1}}{\int_X e^v \omega^n}.$$

This together with (4.6) imposes some constraint on the constant γ_k . For instance, when $k = n-1$, by (4.8) we know that

$$\gamma_{n-1} = 0.$$

Thus, in this case Corollary 4 recovers the classic result of Gauduchon [11]. When ω is Kähler, by (4.8) again we have

$$\gamma_k = 0 \quad \text{for all } 1 \leq k \leq n-1.$$

Then, it follows from (4.7) that

$$\int_X |\nabla v|^2 \omega^n = 0.$$

This tells us that the solution v of (1.11) has to be a constant.

5. GENERALIZED GAUDUCHON METRICS AND γ_k

Let X be an n -dimensional complex manifold. We recall by Definition 1 that a hermitian metric ω on X is called k -th Gauduchon metric if

$$\partial \bar{\partial} (\omega^k) \wedge \omega^{n-k-1} = 0 \quad \text{on } X.$$

Then, the $(n-1)$ -th Gauduchon metric is the Gauduchon metric in the usual sense. By Corollary 4, each hermitian metric ω on X can be associated with a unique constant $\gamma_k(\omega)$, which is invariant under biholomorphisms. The induced function $\gamma_k = \gamma_k(\omega)$ can be used to characterize the k -th Gauduchon metric.

Proposition 8. *The hermitian manifold X admits a k -th Gauduchon metric if and only if there exists a hermitian metric ω on X such that*

$$(5.1) \quad \gamma_k(\omega) = 0.$$

Proof. If there is some hermitian metric ω satisfying (5.1), then Corollary 4 implies that the conformal metric $e^{v/k} \omega$ is a k -th Gauduchon metric on X . Conversely, if ω is a k -th Gauduchon metric, then the uniqueness of Corollary 4 implies that $\gamma_k(\omega) = 0$ and that v is a constant. \square

Let \mathfrak{M} be the set of all hermitian metrics on X . We shall prove that γ_k is a smooth function on \mathfrak{M} . Here \mathfrak{M} is viewed as an open subset in $C^{l+2,\alpha}(\Lambda_{\mathbb{R}}^{1,1}(X))$, for a nonnegative integer l and a real number $0 < \alpha < 1$. We denote by $C^{l,\alpha}(\Lambda_{\mathbb{R}}^{m,m}(X))$ the Hölder space of real (m,m) -forms on X , in which l and m are nonnegative integers, and $0 < \alpha < 1$ is a real number. In particular, $C^{l,\alpha}(\Lambda_{\mathbb{R}}^{0,0}(X)) = C^{l,\alpha}(X)$.

Proposition 9. *The function $\gamma_k = \gamma_k(\omega)$ is a smooth function on \mathfrak{M} , where \mathfrak{M} is viewed as an open subset in $C^{l+2,\alpha}(\Lambda_{\mathbb{R}}^{1,1}(X))$.*

Proof. It follows from Corollary 4 that, for each $\omega \in \mathfrak{M}$, there exists a unique constant γ_k and a function v such that

$$(5.2) \quad e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k) \wedge \omega^{n-k-1} - \gamma_k\omega^n = 0.$$

Then,

$$\gamma_k = \frac{\int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k) \wedge \omega^{n-k-1}}{\int_X \omega^n}$$

depends smoothly on v and ω . Thus, to show the result, it suffices to show that the solution v depends smoothly on ω . We shall use the implicit function theorem.

For each $\omega \in \mathfrak{M}$, the space $\mathcal{E}_{\omega}^{l,\alpha}$ is defined by (2.3). Fix $\omega_0 \in \mathfrak{M}$, for which we abbreviate $\mathcal{E}_0^{l,\alpha} = \mathcal{E}_{\omega_0}^{l,\alpha}$. We have two obvious linear isomorphisms from $\mathcal{E}_{\omega}^{l,\alpha}$ to $\mathcal{E}_0^{l,\alpha}$, given respectively by

$$(5.3) \quad h \mapsto h - \frac{\int_X h\omega_0^n}{\int_X \omega_0^n}, \quad \text{for all } h \in \mathcal{E}_{\omega}^{l,\alpha},$$

and

$$(5.4) \quad h \mapsto h \cdot \frac{\omega^n}{\omega_0^n} \quad \text{for all } h \in \mathcal{E}_{\omega}^{l,\alpha}.$$

Define a map $F : \mathfrak{M} \times \mathcal{E}_0^{l+2,\alpha} \rightarrow \mathcal{E}_0^{l,\alpha}$ by

$$F(\omega, v) = \frac{ne^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k) \wedge \omega^{n-k-1}}{\omega_0^n} - \frac{n \int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k) \wedge \omega^{n-k-1}}{\int_X \omega^n} \cdot \frac{\omega^n}{\omega_0^n}.$$

Obviously, F is a smooth map. Note that any $(\omega, v) \in \mathfrak{M} \times \mathcal{E}_0^{l+2,\alpha}$ satisfies (5.2) if and only if

$$F(\omega, v) = 0.$$

The Fréchet derivative of F with respect to the variable v is

$$D_v F(\omega, v)(h) = L_{\omega}(h) \frac{\omega^n}{\omega_0^n}.$$

Here

$$L_{\omega}(h) = \Delta h + \langle B_1 + 2dv, dh \rangle_{\omega} - \frac{\int_X (\Delta h + \langle B_1 + 2dv, dh \rangle_{\omega}) \omega^n}{\int_X \omega^n},$$

in which the Laplacian Δ is with respect to ω , and B_1 is the smooth real 1-form given by (4.4). By the proof of Lemma 13 in [9] and the isomorphism (5.3), the operator $L_\omega : \mathcal{E}_0^{l+2,\alpha} \rightarrow \mathcal{E}_\omega^{l,\alpha}$ is a linear isomorphism. This combining isomorphism (5.4) imply that $D_v F(\omega, v) : \mathcal{E}_0^{l+2,\alpha} \rightarrow \mathcal{E}_0^{l,\alpha}$ is a linear isomorphism. The result then follows by the Implicit Function Theorem. \square

A direct corollary of Proposition 9 is as below.

Corollary 10. *For $1 \leq k \leq n-2$, if there exists two hermitian metric ω_1, ω_2 on X such that*

$$\gamma_k(\omega_1) > 0 \quad \text{and} \quad \gamma_k(\omega_2) < 0,$$

then there exists a metric ω on X satisfying $\gamma_k(\omega) = 0$, i.e., ω is a k -th Gauduchon metric.

Proof. Let

$$\omega_t = t\omega_1 + (1-t)\omega_2, \quad \text{for all } 0 \leq t \leq 1.$$

Then ω_t is a hermitian metric for each t . The result follows immediately by applying the Mean Value Theorem to the function $\phi(t) = \gamma_k(\omega_t)$. \square

Proposition 11. *For any function $\rho \in C^2(M)$, we have*

$$(5.5) \quad e^{-\max_X \rho} \gamma_k(\omega) \leq \gamma_k(e^\rho \omega) \leq e^{-\min_X \rho} \gamma_k(\omega).$$

In particular, the sign of the function γ_k is a conformal invariant for hermitian metrics.

Proof. Let $\tilde{\omega} = e^\rho \omega$. Then, there exists a function \tilde{v} and a number $\tilde{\gamma}_k = \gamma_k(\tilde{\omega})$ satisfying

$$(\sqrt{-1}/2) \partial \bar{\partial} (e^{\tilde{v}} \tilde{\omega}^k) \wedge \tilde{\omega}^{n-k-1} = \tilde{\gamma}_k e^{\tilde{v}} \tilde{\omega}^n,$$

that is,

$$(5.6) \quad (\sqrt{-1}/2) \partial \bar{\partial} (e^{\tilde{v}+k\rho} \omega^k) \wedge \omega^{n-k-1} = \tilde{\gamma}_k e^{\tilde{v}+k\rho} e^\rho \omega^n.$$

We can rewrite (5.6) as

$$(5.7) \quad \Delta(\tilde{v} + k\rho) + |\nabla(\tilde{v} + k\rho)|^2 + \langle B_1, d(\tilde{v} + k\rho) \rangle + \varphi = ne^\rho \tilde{\gamma}_k,$$

where the operators Δ and ∇ are with respect to ω , and B_1 and φ are given by (4.4) and (4.5), respectively. Subtracting (5.7) by

$$\Delta v + |\nabla v|^2 + \langle B_1, dv \rangle + \varphi = n\gamma_k(\omega)$$

and then applying the maximum principle yields (5.5). \square

Proposition 12. *For a hermitian metric ω , the number $\gamma_k(\omega) > 0$ ($= 0$, or < 0) if and only if there exists a metric $\tilde{\omega}$ in the conformal class of ω such that*

$$(5.8) \quad (\sqrt{-1}/2) \partial \bar{\partial} \tilde{\omega}^k \wedge \tilde{\omega}^{n-k-1} > 0 \text{ (} = 0, \text{ or } < 0 \text{)} \quad \text{on } X.$$

Proof. Suppose that $\gamma_k(\omega) > 0$ ($= 0$, or < 0). Let $\tilde{\omega} = e^{v/k}\omega$, where v is the smooth function associated with ω so that (1.11) holds. Then,

$$(\sqrt{-1}/2)\partial\bar{\partial}\tilde{\omega}^k \wedge \tilde{\omega}^{n-k-1} = \gamma_k(\omega)\omega^n e^{(n-k)v} > 0 \quad (= 0, \text{ or } < 0).$$

Conversely, if there is a metric $\tilde{\omega}$ in the conformal class of ω such that (5.8) holds, then we claim that $\gamma_k(\tilde{\omega}) > 0$ ($= 0$, or < 0). Indeed, by Corollary 4 there exists a smooth function \tilde{v} such that

$$(\sqrt{-1}/2)\partial\bar{\partial}(e^{\tilde{v}}\tilde{\omega}^k) \wedge \tilde{\omega}^{n-k-1} = \gamma_k(\tilde{\omega})e^{\tilde{v}}\tilde{\omega}.$$

This is equivalent to the following equation

$$(5.9) \quad \Delta\tilde{v} + |\nabla\tilde{v}|^2 + \langle \tilde{B}_1, d\tilde{v} \rangle + \tilde{\varphi} = n\gamma_k(\tilde{\omega}),$$

where the operators Δ and ∇ are with respect to $\tilde{\omega}$, and \tilde{B}_1 and $\tilde{\varphi}$ are given by (4.4) and (4.5), respectively, with $\tilde{\omega}$ replacing ω . By (5.8) we have $\tilde{\varphi} > 0$ ($= 0$, or < 0). The claim then follows immediately by applying the maximum principle to (5.9). By Proposition 11, we finish the proof. \square

Moreover, for the case of $\gamma_k > 0$, we have the following criteria on the integration, which is often easier to verify.

Lemma 13. *Suppose that n , the complex dimension of X , is an odd number. Let $k = (n - 1)/2$. Then, there is some metric ω satisfying $\gamma_k(\omega) > 0$ if and only if there is some semi-metric $\hat{\omega}$ (i.e., semi-positive real $(1, 1)$ -form on X) satisfying*

$$\frac{\sqrt{-1}}{2} \int_X \partial\bar{\partial}\hat{\omega}^k \wedge \hat{\omega}^{n-k-1} > 0$$

Proof. By Proposition 12, the necessary part is obvious. For the sufficient part, let $\hat{\omega}$ be any hermitian metric. Let

$$\omega_t = \hat{\omega} + t\hat{\omega}$$

for $t \in (0, 1)$. Then we have

$$(5.10) \quad \begin{aligned} & \int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega_t^k) \wedge \omega_t^{n-k-1} \\ &= \frac{\sqrt{-1}}{2} \int_X (\partial\bar{\partial}\omega_t^k \wedge \omega_t^{n-k-1} + \partial v \wedge \bar{\partial}v \wedge \omega_t^{n-1}) \\ & \quad + \frac{\sqrt{-1}}{2} \int_X \left[\partial\bar{\partial}v \wedge \omega_t^{n-1} + \frac{k}{n-1} (\partial\omega_t^{n-1} \wedge \bar{\partial}v + \partial v \wedge \bar{\partial}\omega_t^{n-1}) \right] \\ &= \frac{\sqrt{-1}}{2} \int_X (\partial\bar{\partial}\omega_t^k \wedge \omega_t^{n-k-1} + \partial v \wedge \bar{\partial}v \wedge \omega_t^{n-1}) \\ & \quad + \frac{\sqrt{-1}}{2} \left(1 - \frac{2k}{n-1}\right) \int_X v \partial\bar{\partial}\omega_t^{n-1}. \end{aligned}$$

Since $k = (n - 1)/2$, the second integral on the right of (5.10) vanishes. It follows that

$$\begin{aligned} \int_X e^{-v(\sqrt{-1}/2)} \partial \bar{\partial} (e^v \omega_t^k) \wedge \omega_t^{n-k-1} &\geq \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega_t^k \wedge \omega_t^k \\ &= \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \hat{\omega}^k \wedge \hat{\omega}^k + t \frac{\sqrt{-1}}{2} \int_X (\partial \bar{\partial} \hat{\omega}^k \wedge \Psi_t + \partial \bar{\partial} \Psi_t \wedge \hat{\omega}^k) \\ &\quad + t^2 \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \Psi_t \wedge \Psi_t > 0, \quad \text{for sufficiently small } t, \end{aligned}$$

where $\Psi_t = \hat{\omega} \wedge (\hat{\omega}^{k-1} + \hat{\omega}^{k-2} \wedge \omega_t + \cdots + \hat{\omega} \wedge \omega_t^{k-2} + \omega_t^{k-1})$. This implies that $\gamma_k(\omega_t) > 0$ for the sufficiently small t . \square

A similar argument works for the (classic) Gauduchon metrics, for any dimension n , and for all $1 \leq k \leq n - 2$.

Lemma 14. *Let X be an n -dimensional hermitian manifold, k an integer such that $1 \leq k \leq n - 2$. Then, a hermitian metric ω on X satisfies $\gamma_k(\omega) > 0$ if the Gauduchon metric $\tilde{\omega}$ in the conformal class of ω satisfies*

$$(5.11) \quad \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \tilde{\omega}^k \wedge \tilde{\omega}^{n-k-1} > 0.$$

Proof. By Proposition 11, we can assume that $\omega = \tilde{\omega}$, without loss of generality. By (5.10) with ω replacing ω_t , and applying $\partial \bar{\partial} \omega^{n-1} = 0$ yields

$$\int_X e^{-v(\sqrt{-1}/2)} \partial \bar{\partial} (e^v \omega^k) \wedge \omega^{n-1-k} \geq \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega^k \wedge \omega^{n-1-k} > 0.$$

\square

Corollary 15. *Let (X, ω) be an n -dimensional balanced manifold. Then, for each $1 \leq k \leq n - 2$, we have $\gamma_k(\omega) > 0$ if*

$$\frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega^k \wedge \omega^{n-1-k} > 0.$$

6. CONSTRUCTIONS ON HERMITIAN THREE-MANIFOLDS

We shall apply previous results to construct a hermitian metric with $\gamma_1 > 0$ on a complex three dimensional manifold. Theorem 6 will follow from Proposition 12, together with the following theorem.

Theorem 16. *There always exists a hermitian metric ω on a complex three dimensional manifold X such that*

$$(\sqrt{-1}/2) \partial \bar{\partial} \omega \wedge \omega > 0.$$

Proof. By Lemma 13 and Proposition 12, it suffices to construct a semi-metric $\hat{\omega}$ such that

$$\frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \hat{\omega} \wedge \hat{\omega} > 0.$$

Fix a point $q \in X$ and a coordinate patch $U \ni q$. Let (z_1, z_2, z_3) be coordinates on U centered at q . Here $z_j = x_j + \sqrt{-1}y_j$ for $1 \leq j \leq 3$. We can assume $N = B \times B \times R \subset U$, where B is the unit ball in \mathbb{C} , and

$$R = \{z_3 \in \mathbb{C} \mid |x_3| \leq 1, |y_3| \leq 1\}.$$

Take a nonnegative cut-off function $\eta \in C_0^\infty(B)$ and two nonnegative functions $f, g \in C_0^\infty([-1, 1])$ to be determined later. On N , define

$$\phi = \eta(z_1)\eta(z_2)f(x_3)f(y_3), \quad \psi = \eta(z_1)\eta(z_2)g(x_3)g(y_3),$$

and then define

$$(6.1) \quad \dot{\omega} = \frac{\sqrt{-1}}{2} [\phi(z)dz_1 \wedge d\bar{z}_1 + \psi(z)dz_2 \wedge d\bar{z}_2].$$

Obviously, $\dot{\omega}$ is semi-positive and with compact support in N . So it can be viewed as a semi-metric on X . Clearly,

$$(6.2) \quad \frac{\sqrt{-1}}{2} \partial\bar{\partial}\dot{\omega} \wedge \dot{\omega} = \left(\phi \frac{\partial^2\psi}{\partial z_3 \partial \bar{z}_3} + \psi \frac{\partial^2\phi}{\partial z_3 \partial \bar{z}_3} \right) dV,$$

where

$$(6.3) \quad dV = \left(\frac{\sqrt{-1}}{2} \right)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3.$$

Since

$$\frac{\partial}{\partial z_3} = \frac{1}{2} \left(\frac{\partial}{\partial x_3} - \sqrt{-1} \frac{\partial}{\partial y_3} \right), \quad \frac{\partial}{\partial \bar{z}_3} = \frac{1}{2} \left(\frac{\partial}{\partial x_3} + \sqrt{-1} \frac{\partial}{\partial y_3} \right),$$

we have

$$\begin{aligned} & \phi \frac{\partial^2\psi}{\partial z_3 \partial \bar{z}_3} + \psi \frac{\partial^2\phi}{\partial z_3 \partial \bar{z}_3} \\ &= \frac{\phi}{4} \left(\frac{\partial^2\psi}{\partial x_3 \partial x_3} + \frac{\partial^2\psi}{\partial y_3 \partial y_3} \right) + \frac{\psi}{4} \left(\frac{\partial^2\phi}{\partial x_3 \partial x_3} + \frac{\partial^2\phi}{\partial y_3 \partial y_3} \right) \\ &= \frac{1}{4} \eta^2(z_1)\eta^2(z_2)f(y_3)g(y_3) [f(x_3)g''(x_3) + g(x_3)f''(x_3)] \\ & \quad + \frac{1}{4} \eta^2(z_1)\eta^2(z_2)f(x_3)g(x_3) [f(y_3)g''(y_3) + g(y_3)f''(y_3)]. \end{aligned}$$

We choose η so that

$$\int_B \eta^2(z) \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} = 1.$$

Then it follows that

$$\begin{aligned} \frac{\sqrt{-1}}{2} \int_X \partial\bar{\partial}\dot{\omega} \wedge \dot{\omega} &= \frac{1}{2} \int_{-1}^1 f(t)g(t)dt \int_{-1}^1 [f(t)g''(t) + f''(t)g(t)] dt \\ &= \int_{-1}^1 f(t)g(t)dt \int_{-1}^1 [-f'(t)g'(t)] dt. \end{aligned}$$

The result follows immediately from the proposition below. \square

Proposition 17. *There exist nonnegative functions $f, g \in C_0^\infty([-1, 1])$ such that*

$$-\int_{-1}^1 f'(t)g'(t)dt > 0.$$

Proof. For any two real numbers $a < b$, we denote

$$\chi_{a,b}(t) = \begin{cases} \exp\left(\frac{1}{t-b} - \frac{1}{t-a}\right), & \text{if } a < t < b, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have that $\chi_{a,b} \in C_0^\infty(\mathbb{R})$, that $\chi'_{a,b}(t) > 0$ for $a < t < (a+b)/2$, that $\chi'_{a,b}(t) < 0$ for $(a+b)/2 < t < b$, and that $\chi'_{a,b}(t) = 0$ when $t = (a+b)/2$. Letting

$$f(t) = \chi_{-1/3, 1/3}(t), \quad \text{and} \quad g(t) = \chi_{0, 2/3}(t)$$

yields that $-f'(t)g'(t) > 0$ for $0 < t < 1/3$ and otherwise $f'(t)g'(t) = 0$. This in particular implies the result. \square

Let us now consider some examples. We can directly construct a hermitian metric ω with $\gamma_1(\omega) > 0$ on T^3 , the 3-dimensional complex torus.

Proposition 18. *On the complex T^3 , there is a metric ω satisfying*

$$\sqrt{-1}/2\partial\bar{\partial}\omega \wedge \omega > 0.$$

Proof. Let (z_1, z_2, z_3) be the coordinates of T^3 induced from \mathbb{C}^3 . Let

$$\omega = \frac{\sqrt{-1}}{2} [\xi(x_3)dz_1 \wedge d\bar{z}_1 + \eta(x_3)dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3],$$

where ξ and η are two positive smooth functions on T^3 only depending on x_3 , which will be determined later. Then

$$(\sqrt{-1}/2)\partial\bar{\partial}\omega \wedge \omega = \left(\eta \frac{\partial^2 \xi}{\partial z_3 \partial \bar{z}_3} + \xi \frac{\partial^2 \eta}{\partial z_3 \partial \bar{z}_3} \right) dV > 0$$

if and only if

$$\eta \frac{\partial^2 \xi}{\partial z_3 \partial \bar{z}_3} + \xi \frac{\partial^2 \eta}{\partial z_3 \partial \bar{z}_3} = \frac{1}{4} \eta \frac{\partial^2 \xi}{\partial x_3^2} + \frac{1}{4} \xi \frac{\partial^2 \eta}{\partial x_3^2} > 0.$$

Here dV is defined by (6.3). So we need to look for two smooth, positive, 2π -periodic functions η and ξ such that

$$\frac{\eta''(t)}{\eta(t)} + \frac{\xi''(t)}{\xi(t)} > 0.$$

We define

$$(6.4) \quad \xi(t) = 1 + \kappa \sin t, \quad \text{for some } 0 < \kappa < 1.$$

We observe that

$$\int_0^{2\pi} \frac{\xi''}{\xi} dt = - \int_0^{2\pi} \frac{\kappa \sin t}{1 + \kappa \sin t} dt = -2\pi + \int_0^{2\pi} \frac{dt}{1 + \kappa \sin t}.$$

By Proposition 8 in [9], the value of above integral tends to $+\infty$ monotonically, as $\kappa \rightarrow 1^-$. Hence, for a constant $C > 0$, there is a unique real number κ , such that the function ξ given by (6.4) satisfies

$$\int_0^{2\pi} \frac{\xi''}{\xi} dt = \int_0^{2\pi} C dt.$$

It implies that equation

$$\zeta'' + \frac{\xi''}{\xi} = C$$

has a smooth 2π -periodic solution ζ on \mathbb{R} . Let $\eta = e^\zeta$. Thus,

$$\frac{\eta''(t)}{\eta(t)} + \frac{\xi''(t)}{\xi(t)} = (\zeta')^2 + \zeta'' + \frac{\xi''}{\xi} \geq C > 0.$$

□

As another example, we show that the natural balanced metric on the Iwasawa manifold has positive γ_1 . Recall (for example, [16, p. 444] and [20, p. 115]) that the Iwasawa manifold is defined to be the quotient space G/Γ , where

$$G = \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix}; z_1, z_2, z_3 \in \mathbb{C} \right\},$$

Γ is the discrete subgroup of G consisting of matrices where z_1, z_2, z_3 are Gaussian integers, i.e., $z_i \in \{a + b\sqrt{-1} \mid a, b \in \mathbb{Z}\}$ for $1 \leq i \leq 3$, and Γ acts on G by left multiplications. Clearly, the global holomorphic 1-forms

$$\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2$$

on G are invariant under the action of Γ , hence descend down to G/Γ . Observe that G/Γ does not admit any Kähler metric, because $d\varphi_3 = \varphi_2 \wedge \varphi_1 \neq 0$. Let

$$\omega = (\sqrt{-1}/2)(\varphi_1 \wedge \bar{\varphi}_1 + \varphi_2 \wedge \bar{\varphi}_2 + \varphi_3 \wedge \bar{\varphi}_3).$$

Then, ω is a balanced hermitian metric on G/Γ , for $d\omega^2 = 0$. Furthermore, we have

$$(\sqrt{-1}/2)\partial\bar{\partial}\omega \wedge \omega = (\sqrt{-1}/2)^3 \varphi_1 \wedge \bar{\varphi}_1 \wedge \varphi_2 \wedge \bar{\varphi}_2 \wedge \varphi_3 \wedge \bar{\varphi}_3 > 0$$

on G/Γ ; hence, by Proposition 12, we conclude that $\gamma_1(\omega) > 0$.

7. THE FIRST GAUDUCHON METRIC ON CALABI'S MANIFOLDS

In this section, we shall establish the existence of the 1-st Gauduchon metric on the non-Kähler manifold introduced by Calabi [5]. In view of Theorem 6 and Corollary 10, we need to find a hermitian metric with negative γ_1 value.

We first recall Calabi's construction of non-Kähler complex three dimensional manifolds. Let $\mathbb{O} \cong \mathbb{R}^8$ denotes the Cayley numbers. We fix a basis $\{I_1, \dots, I_7\}$ such that

- (1) $I_i \cdot I_j = \delta_{ij}$ with respect to the inner product.
(2) The table of the multiplication of the cross product $I_j \times I_k$ is the following

$$(7.1) \quad \begin{array}{c|ccccccc} \times & I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 \\ \hline I_1 & 0 & I_3 & -I_2 & I_5 & -I_4 & I_7 & -I_6 \\ I_2 & -I_3 & 0 & I_1 & I_6 & -I_7 & -I_4 & I_5 \\ I_3 & I_2 & -I_1 & 0 & -I_7 & -I_6 & I_5 & I_4 \\ I_4 & -I_5 & -I_6 & I_7 & 0 & I_1 & I_2 & -I_3 \\ I_5 & I_4 & I_7 & I_6 & -I_1 & 0 & -I_3 & -I_2 \\ I_6 & -I_7 & I_4 & -I_5 & -I_2 & I_3 & 0 & I_1 \\ I_7 & I_6 & -I_5 & -I_4 & I_3 & I_2 & -I_1 & 0 \end{array}$$

Via this basis, we have the isomorphism $\mathbb{R}^7 \cong \text{Im}(\mathbb{O})$.

Calabi considered a smooth oriented hypersurface $X^6 \hookrightarrow \mathbb{R}^7$. Fix a unit normal vector field N of X . There is a natural almost complex structure $J : TX \rightarrow TX$ induced by Cayley multiplication as follows. For any $x \in X$ and any $V \in T_x X$, define $J : T_x X \rightarrow T_x X$ as

$$J(V) = N \times V.$$

Calabi proved that J is integrable if and only if J anticommutes with the second fundamental form of X .

Calabi constructed compact complex manifolds as follows. Let Σ be a compact Riemann surface which admits 3 holomorphic differentials ϕ_1, ϕ_2, ϕ_3 with the following properties:

- (1) linear independent;
- (2) $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$;
- (3) $\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2 + \phi_3 \wedge \bar{\phi}_3 > 0$.

Lifting ϕ_1, ϕ_2, ϕ_3 to the universal covering $\tilde{\Sigma} \rightarrow \Sigma$ and setting

$$x^j(p) = \text{Re} \int_{p'}^p \phi_j, \quad j = 1, 2, 3$$

for a fixed point $p' \in \Sigma$, we obtain a conformal minimal immersion

$$\psi = (x^1, x^2, x^3) : \tilde{\Sigma} \rightarrow \mathbb{R}^3.$$

This mapping is regular, since the differentials ϕ_j satisfy (3); by the weierstrass representation, property (2) is equivalent to the statement that ψ is minimal; finally, because of property (1), it follows that $\tilde{\Sigma}$ is not mapped into a plane.

Calabi then considered the hypersurface of the type

$$(\psi, id) : \tilde{\Sigma} \times \mathbb{R}^4 \rightarrow \mathbb{R}^3 \times \mathbb{R}^4 = \text{Im}(\mathbb{O}),$$

where $\mathbb{R}^3 = \text{span}_{\mathbb{R}}\{I_1, I_2, I_3\}$ and $\mathbb{R}^4 = \text{span}_{\mathbb{R}}\{I_4, I_5, I_6, I_7\}$. Since $\psi : \tilde{\Sigma} \rightarrow \mathbb{R}^3$ is minimal, $\tilde{\Sigma} \times \mathbb{R}^4$ is the complex manifold. If $g : \tilde{\Sigma} \rightarrow \tilde{\Sigma}$ denotes a covering transformation, then $\psi(gp) = \psi(p) + t_g$ for some vector $t_g \in \mathbb{R}^3$. It follows that the complex structure on $\tilde{\Sigma} \times \mathbb{R}^4$ is invariant by the covering group of Σ and so descends to $\Sigma \times \mathbb{R}^4$. On the other hand, for \mathbb{R}^4 , we can further divide by a lattice Λ of translation of \mathbb{R}^4 , and thereby produce a compact complex manifold $X_\Lambda = \Sigma \times T^4$. We can view X_Λ as a family of complex tori, parameterized by the Riemann surface.

Calabi showed that such complex manifolds X_Λ are non-Kähler. However, there exists a balanced metric on these manifolds [14, 19]. Let us consider the *nature* metric.

Define a 2-form on X_Λ as

$$\omega_0(V, W) = N \cdot (V \times W)$$

for any $V, W \in T_x X_\Lambda$ at any $x \in X_\Lambda$. Then clearly we have

$$\omega_0(V, W) = -\omega_0(W, V);$$

and using the formula

$$N \cdot (V \times W) = (N \times V) \cdot W,$$

we also have

$$\omega_0(JV, JW) = \omega_0(V, W);$$

$$\omega_0(V, JV) = (N \times V) \cdot (N \times V) > 0, \quad \text{if } V \neq 0.$$

So ω_0 is the positive $(1, 1)$ -form on X_Λ and therefore defines a hermitian metric.

Next we check that ω_0 is a balanced metric. The unit normal vector field of X in \mathbb{R}^7 can be written as

$$(7.2) \quad N = \sum_{j=1}^3 a_j I_j, \quad \sum_{j=1}^3 a_j^2 = 1,$$

where a_j for $j = 1, 2, 3$ are functions on Σ . Let (x_4, x_5, x_6, x_7) be the coordinates of \mathbb{R}^4 . Then we can write the hermitian metric ω_0 as

$$\omega_0 = \omega_\Sigma + \varphi_0,$$

where ω_Σ is a Kähler metric on Σ and

$$\begin{aligned} \varphi_0 = & a_1 dx_4 \wedge dx_5 + a_2 dx_4 \wedge dx_6 - a_3 dx_4 \wedge dx_7 \\ & - a_3 dx_5 \wedge dx_6 - a_2 dx_5 \wedge dx_7 + a_1 dx_6 \wedge dx_7. \end{aligned}$$

By direct check, we have

$$\varphi_0^2 = 2dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7.$$

Therefore,

$$d(\omega_0^2) = d(2\omega_\Sigma \wedge \varphi_0 + \varphi_0^2) = 2d\omega_\Sigma \wedge \varphi_0 + 2\omega_\Sigma \wedge d\varphi_0 = 0,$$

since ω_Σ is a Kähler metric and all functions a_j are defined on Σ .

At last we prove that there exists a 1-Gauduchon metric on X_Λ . By direct computation, we have

$$\partial\bar{\partial}\omega_0 \wedge \omega_0 = \partial\bar{\partial}\varphi_0 \wedge \varphi_0 = 2 \sum_{j=1}^3 a_j \partial\bar{\partial}a_j \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7.$$

Condition (7.2) implies

$$\sum_{j=1}^3 a_j \partial\bar{\partial}a_j = - \sum_{j=1}^3 \partial a_j \wedge \bar{\partial}a_j,$$

Combining the above two equalities yields

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\omega_0 \wedge \omega_0 &= -2\sqrt{-1}\sum_{j=1}^3 \partial a_j \wedge \bar{\partial} a_j \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \\ &= -4\sum_{j=1}^3 |\partial a_j|^2 \omega_0^3,\end{aligned}$$

and therefore,

$$\sqrt{-1}\int_{X_\Lambda} \partial\bar{\partial}(e^v\omega_0) \wedge \omega_0 = \sqrt{-1}\int_{X_\Lambda} e^v\omega_0 \wedge \partial\bar{\partial}\omega_0 < 0.$$

Hence, we have $\gamma_1(\omega_0) < 0$, by Corollary 4; so $-1 \in \Xi_1(X_\Lambda)$.

Proposition 19. $\Xi_1(X_\Lambda) = \{-1, 0, 1\}$.

Proof. We have proven $-1 \in \Xi_1(X_\Lambda)$ and According to Theorem 6 we also have $1 \in \Xi_1(X_\Lambda)$. Then by Corollary 10, $0 \in \Xi_1(X_\Lambda)$. \square

Corollary 20. *There exists a 1-Gauduchon metric on X_Λ .*

8. THE FIRST GAUDUCHON METRIC ON $S^5 \times S^1$

Let $S^5 \rightarrow \mathbb{P}^2$ be the hopf fibration of the complex projective plane \mathbb{P}^2 . S^5 can be viewed as the circle bundle over \mathbb{P}^2 twisted by $\frac{\omega_{FS}}{2\pi} \in H^2(\mathbb{P}^2, \mathbb{Z})$. Here ω_{FS} is the Fubini-Study metric on \mathbb{P}^2 . We let $\pi : S^5 \times S^1 \rightarrow \mathbb{P}^2$ be the natural projection. Then using a canonical way (c.f. [10, 12]), we can define a complex structure on $S^5 \times S^1$ such that π is a holomorphic map. We can define a natural hermitian metric on $S^5 \times S^1$ as follows:

$$(8.1) \quad \omega_0 = \pi^*\omega_{FS} + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta},$$

where $\theta = \theta_1 + \sqrt{-1}\theta_2$ is a $(1,0)$ -form on $S^5 \times S^1$ such that $d\theta_1 = \pi^*\omega_{FS}$ and $d\theta_2 = 0$. So $\bar{\partial}\theta = \pi^*\omega_{FS}$ and $\partial\theta = 0$ which imply

$$(8.2) \quad \frac{\sqrt{-1}}{2}\partial\bar{\partial}\omega_0 = -\frac{1}{4}\pi^*\omega_{FS}^2.$$

Thus

$$(8.3) \quad \frac{\sqrt{-1}}{2}\partial\bar{\partial}\omega_0 \wedge \omega_0 = \left(\frac{\sqrt{-1}}{2}\right)^3 \pi^*\omega_{FS}^2 \wedge \theta \wedge \bar{\theta} = -\frac{\omega_0^3}{3!}$$

and therefore

$$\sqrt{-1}\int_{S^5 \times S^1} \partial\bar{\partial}(e^v\omega_0) \wedge \omega_0 = \sqrt{-1}\int_{S^5 \times S^1} e^v\omega_0 \wedge \partial\bar{\partial}\omega_0 < 0.$$

Hence, we have $\gamma_1(\omega_0) < 0$, by Corollary 4; so $-1 \in \Xi_1(S^5 \times S^1)$. Then by Corollary 10, $0 \in \Xi_1(S^5 \times S^1)$. That is we have

Proposition 21. *There exists a 1-Gauduchon metric on $S^5 \times S^1$.*

Using above natural metric ω_0 on $S^5 \times S^1$, we can also prove

Proposition 22. *There does not exist any pluri-closed metric on $S^5 \times S^1$.*

Proof. If there would exist a pluri-closed metric ω on $S^5 \times S^1$, then

$$(8.4) \quad 0 = \int_{S^5 \times S^1} \frac{\sqrt{-1}}{2} \partial \bar{\partial} \omega \wedge \omega_0 = -\frac{1}{4} \int_{S^5 \times S^1} \omega \wedge \pi^* \omega_{FS}^2 < 0$$

since $\omega \wedge \pi^* \omega_{FS}^2$ is the strictly positive definite $(3, 3)$ -form on $S^5 \times S^1$. That is a contradiction. \square

We also know that there does not exist any balanced metric on $S^5 \times S^1$. The proof is standard and is given here. There is an obstruction to the existence of a balanced metric on a compact complex manifold. Namely, on a compact complex manifold with a balanced metric no compact complex submanifold of codimension 1 can be homologous to 0 [19]. Now for $\pi : S^5 \times S^1 \rightarrow \mathbb{P}^2$, since π is a holomorphic, $\pi^{-1}(\mathbb{P}^2)$ is a complex hypersurface in $S^5 \times S^1$. Certainly $\pi^{-1}(\mathbb{P}^2)$ is homologous to zero in $S^5 \times S^1$ since $H^4(S^5 \times S^1, \mathbb{R}) = 0$. Therefor there exist no balanced metric on $S^5 \times S^1$.

REFERENCES

- [1] L. Alessandrini and M. Andreatta, *Closed transverse (p, p) -forms on compact complex manifolds*, Compositio Math. **61**(1987), 181–200.
- [2] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. **284**(1989), 681–699.
- [3] Y. Bozhkov, *The specific Hermitian geometry of certain three-folds*, Riv. Mat. Univ. Parma **4**(1995), 61–68.
- [4] Y. Bozhkov, *The geometry of certain three-folds*, Rend. Ist. Mat. Univ. Trieste **26**(1994), 79–93.
- [5] E. Calabi, *Construction and properties of some 6-dimensional almost complex manifolds*, Trans. Amer. Math. Soc. **87**(1958), 407–438.
- [6] J.-P. Demailly, *Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne*, 88–97, Lecture Notes in Math., **1198**, Springer, Berlin, 1986.
- [7] A. Fino and A. Tomassini, *A survey on strong KT structures*, Bull. Math. Soc. Sci. Math. Roumanie Tome **52**(2009), 99–116.
- [8] J.-X. Fu, J. Li and S.-T. Yau, *Balanced metrics on non-Kähler Calabi–Yau threefolds*, arXiv:0809.4748.
- [9] J.-X. Fu, Z. Wang and D. Wu, *Form-type Calabi–Yau equations*, Math. Res. Lett. **17**(2010), 887–903.
- [10] J.-X. Fu and S.-T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation*, J. Differential Geom. **78**(2008), 369–428.
- [11] P. Gauduchon, *Sur la 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267**(1984), 495–518.
- [12] P. Goldstein and S. Prokushkin, *Geometric model for complex non-Kähler manifolds with $SU(3)$ structures*, Commun. Math. Phys. **251**(2004), 65–78.
- [13] D. Grantcharov, G. Grantcharov and Y.-S. Poon, *Calabi-Yau connections with torsion on tori bundles*, J. Diff. Geom. **78**(2008), 13–32.
- [14] A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J. Math. **10**(1966), 353–366.
- [15] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Annali Mat. Pura Appl. **123**(1980), 35–58.
- [16] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.
- [17] J. Jost and S.-T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math. **170**(1993), 221–254.

- [18] J. Li and S.-T. Yau, *Hermitian–Yang–Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), 560–573, Adv. Ser. Math. Phys., **1**, World Sci. Publishing, Singapore, 1987.
- [19] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149**(1982), 261–295.
- [20] J. Morrow and K. Kodaira, *Complex Manifolds*, Holt, Rinehart and Winston, 1971.
- [21] J. Streets and G. Tian, *A parabolic flow of pluriclosed metrics*, arXiv:0903.4418.

INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA
E-mail address: majxfu@fudan.edu.cn

INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA
E-mail address: youxiang163wang@163.com

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 1179 UNIVERSITY DRIVE,
NEWARK, OH 43055, U.S.A.
E-mail address: dwu@math.ohio-state.edu