Compact Kähler manifolds homotopic to negatively curved Riemannian manifolds

Bing-Long Chen and Xiaokui Yang

Abstract. In this paper, we show that any compact Kähler manifold homotopic to a compact Riemannian manifold with negative sectional curvature admits a Kähler-Einstein metric of general type. Moreover, we prove that, on a compact symplectic manifold X homotopic to a compact Riemannian manifold with negative sectional curvature, for any almost complex structure J compatible with the symplectic form, there is no non-constant J-holomorphic entire curve $f : \mathbb{C} \to X$.

1. Introduction

In 1970s, S.-T. Yau proposed the following conjecture:

Conjecture. Let (X, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X > 1$. Suppose (X, ω) has negative Riemannian sectional curvature, then X is rigid, i.e. X has only one complex structure.

It is a fundamental problem on the rigidity of Kähler manifolds with negative curvature. Yau proved in [Yau77, Theorem 6] that when X is covered by a 2-ball, then any complex surface oriented homotopic to X must be biholomorphic to X. By using the terminology of "strongly negativity", Siu established in [Siu80, Theorem 2] that a compact Kähler manifold of the same homotopy type as a compact Kähler manifold (X, ω) with strongly negative curvature and $\dim_{\mathbb{C}} X > 1$ must be either biholomorphic or conjugate biholomorphic to X. Note that when $\dim_{\mathbb{C}} X = 2$, the above Yau's conjecture has been completely solved by Zheng [Zhe95]. It is well-known that the strongly negative curvature condition can imply the negativity of the Riemannian sectional curvature.

Based on Yau's conjecture, one can also ask a more general question: if a Kähler manifold–or complex manifold–X admits a Riemannian metric with negative sectional curvature, is there any restriction on the complex structure of X?

The first main result of our paper is one important step towards the question:

Theorem 1.1. *Let X be a compact manifold homotopic to a compact Riemannian manifold Y with negative sectional curvature. If X has a Kähler complex structure (J, ω), then (X, J) admits a Kähler-Einstein metric of general type. Moreover, each submanifold of (X, J) also admits a Kähler-Einstein metric of general type.*

Indeed, we prove in Theorem 3.1 a more general result. One of the main ingredients in the proofs is a notion called "Kähler hyperbolicity" introduced by Gromov in [Gro91], and our key observation is that every Kähler hyperbolic manifold has ample canonical bundle (see Theorem 2.11), which also answers a question asked by Gromov in [Gro91, p.267].

Recall that a compact complex manifold X is called Kobayashi (or Brody) hyperbolic if every holomorphic map $f : \mathbb{C} \to X$ is constant. Gromov pointed out in [Gro91] that every Kähler hyperbolic manifold is Kobayashi hyperbolic. On the other hand, one can also extend these terminologies to symplectic manifolds. Note that for a fixed symplectic form ω, there are many almost complex structures compatible with ω. Our second result is

Theorem 1.2. *Let (X, ω) be a compact symplectic manifold homotopic to a compact Riemannian manifold with negative sectional curvature. For any almost complex structure J on X compatible with ω, there exists no non-constant J-holomorphic map $f : \mathbb{C} \to X$.*

2. Background materials

2.1. **Positivity of line bundles.** Let (X, ω) be a smooth projective manifold of complex dimension n, $L \to X$ a holomorphic line bundle and $E \to X$ a holomorphic vector bundle. Let $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle of the projective bundle $\mathbb{P}(E^*)$ over X.

(1) L is said to be *ample* if L^k is very ample for some large k, i.e. the map $X \to \mathbb{P}(H^0(X, L^k)^*)$ defined by the global sections of L^k is a holomorphic embedding. L is called *semi-ample* if for some large positive integer k, L^k is generated by its global sections, i.e. the evaluation map $\iota : H^0(X, L^k) \to L^k$ is surjective. The vector bundle E is called *ample* if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is an ample line bundle.

(2) L is said to be *nef* (or *numerically effective*), if $L \cdot C \geq 0$ for any irreducible curve C in X.

(3) L is said to be *big*, if the Kodaira dimension $\kappa(L) = \dim X$ where

$$\kappa(L) := \limsup_{m \to +\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, L^m)}{\log m}.$$

Here we use the convention that the logarithm of zero is $-\infty$.

Definition 2.1. X is said to be *of general type* if the Kodaira dimension $\kappa(X) := \kappa(K_X)$ is equal to the complex dimension of X.

There are many examples of compact complex manifolds of general type. For instance, manifolds with ample canonical bundles.

2.2. **Kähler hyperbolicity.** Let's recall some concepts introduced by Gromov in [Gro91]. Let (X, g) be a Riemannian manifold. A differential form α is called *d-bounded* if there exists a form β on X such that $\alpha = d\beta$ and

$$(2.1) \qquad \|\beta\|_{L^\infty(X,g)} = \sup_{x \in X} |\beta(x)|_g < \infty.$$

It is obvious that if X is compact, then every exact form is d-bounded. However, when X is not compact, there exist smooth differential forms which are exact but not d-bounded. For instance, on $\mathbb{R}^n$, $\alpha = dx^1 \wedge \cdots \wedge dx^n$ is exact, but it is not d-bounded.

Definition 2.2. Let (X, g) be a Riemannian manifold and $\pi : (\widetilde{X}, \widetilde{g}) \to (X, g)$ be the universal covering with $\widetilde{g} = \pi^* g$. A form α on X is called $\widetilde{d}$-*bounded* if $\pi^* \alpha$ is a d-bounded form on $(\widetilde{X}, \widetilde{g})$.

It is obvious that the $\widetilde{d}$-boundedness does not depend on the metric g when X is compact.

Lemma 2.3. *Let (X, g) be a compact Riemannian manifold. If α is $\widetilde{d}$-bounded on (X, g), then for any metric g_1 on X, α is also $\widetilde{d}$-bounded on (X, g_1).*

Proof. Since X is compact, any two smooth metrics on X are equivalent. $\qquad\square$

Note also that the $\widetilde{d}$-boundedness of a closed form α on a compact manifold X depends only on the cohomology class $[\alpha] \in H^*_{DR}(X, \mathbb{R})$.

Lemma 2.4. *Let (X, g) be a compact Riemannian manifold. Suppose α is $\widetilde{d}$-bounded, then $\alpha_1 = \alpha + d\gamma$ is also $\widetilde{d}$-bounded.*

Proof. Let $\pi : (\widetilde{X}, \widetilde{g}) \to (X, g)$ be the universal covering and β be the form on $\widetilde{X}$ such that $\pi^* \alpha = d\beta$ and $\|\beta\|_{L^\infty(\widetilde{X}, \widetilde{g})} < \infty$. Hence, we have

$$\pi^* \alpha_1 = d(\beta + \pi^* \gamma)$$

and

$$\begin{aligned}
\|\beta + \pi^* \gamma\|_{L^\infty(\widetilde{X}, \widetilde{g})} &\leq& \|\beta\|_{L^\infty(\widetilde{X}, \widetilde{g})} + \|\pi^* \gamma\|_{L^\infty(\widetilde{X}, \widetilde{g})} \\
&=& \|\beta\|_{L^\infty(\widetilde{X}, \widetilde{g})} + \|\gamma\|_{L^\infty(X, g)} < \infty.
\end{aligned}$$

$\qquad\square$

Definition 2.5. Let X be a Riemannian manifold. X has $\widetilde{d}$-*bounded i^{th} cohomology* if every class in $H^i_{\mathrm{DR}}(X, \mathbb{R})$ is $\widetilde{d}$-bounded.

Lemma 2.6. *Let $f : X \to Y$ be a smooth map between two compact Riemannian manifolds. Suppose α is $\widetilde{d}$-bounded on Y, then $f^*\alpha$ is $\widetilde{d}$-bounded on X.*

Proof. Let $\pi_X : \widetilde{X} \to X$ and $\pi_Y : \widetilde{Y} \to Y$ be the universal coverings of X and Y respectively. Since $\widetilde{X}$ is simply connected, there exists a lifting map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, such that the following diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{Y} \\
{\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

commutes. On the other hand, we know $\pi_Y^*\alpha = d\beta$ for some L^∞-bounded form β over $\widetilde{Y}$. Hence

$$(2.2) \qquad \pi_X^*(f^*\alpha) = \widetilde{f}^*(\pi_Y^*\alpha) = \widetilde{f}^*(d\beta) = d(\widetilde{f}^*\beta).$$

Since X and Y are compact, π_X and π_Y are local isometries,

$$(2.3) \qquad \|\widetilde{f}^*\beta\|_{L^\infty(\widetilde{X},\pi_X^* g_X)} \le C\|\beta\|_{L^\infty(\widetilde{Y},\pi_Y^* g_Y)} \cdot \|f\|^p_{C^1(g_X,g_Y)} < \infty$$

where p is the degree of β and C is a constant depending only on X and p.   □

In geometry, various notions of hyperbolicity have been introduced, and the typical examples are manifolds with negative curvature in suitable sense. The starting point for the present investigation is Gromov's notion of Kähler hyperbolicity [Gro91].

Definition 2.7. Let X be a compact complex manifold. X is called *Kähler hyperbolic* if it admits a Kähler metric ω such that ω is $\widetilde{d}$-bounded.

The typical examples of Kähler hyperbolic manifolds are locally Hermitian symmetric spaces of noncompact type.

As we mentioned before, a compact complex manifold is called Kobayashi hyperbolic if it contains no entire curves. A fundamental problem in complex geometry is Kobayashi's conjecture (e.g. Lang's survey paper [Lan86]):

Conjecture 2.8. Let X be a compact complex manifold. If X is Kobayashi hyperbolic, then the canonical bundle K_X is ample.

Along the same line, Gromov asked the following question in [Gro91, p.267].

Question 2.9. Let X be a compact Kähler hyperbolic manifold. Is the canonical bundle K_X ample? Is the cotangent bundle Ω_X^1 ample?

We first give a counter-example to the second part of Gromov's question.

Example 2.10. Let $X = C_1 \times C_2$ be the product of two smooth curves of genus at least 2. It is obvious that X is Kähler hyperbolic since both C_1 and C_2 are Kähler hyperbolic. The cotangent bundle is $\Omega_X^1 = \pi_1^*\Omega_{C_1}^1 \otimes \pi_2^*\Omega_{C_2}^1$, which is not ample. Indeed, its restriction to a curve $C_1 \times \{p\}$ has a trivial summand.

Next, we give an affirmative answer to the first part of Gromov's question based on an observation in algebraic geometry. To the readers' convenience, we include a straightforward proof here.

Theorem 2.11. *Let X be a compact Kähler hyperbolic manifold. Then the canonical bundle K_X is ample.*

Proof. If X is Kähler hyperbolic, then X contains no rational curves. Indeed, suppose $f : \mathbb{P}^1 \to X$ is a rational curve. We want to show f is a constant, i.e. $f^*\omega = 0$. Let $\pi_X : \widetilde{X} \to X$ be the universal covering. Then there is a lifting $\widetilde{f} : \mathbb{P}^1 \to \widetilde{X}$ such that $\pi_X \circ \widetilde{f} = f$. Since ω is $\widetilde{d}$-bounded, i.e. there exists a bounded 1-form β on $\widetilde{X}$ such that $\pi_X^*\omega = d\beta$,

$$(2.4) \qquad\qquad f^*\omega = \widetilde{f}^*(\pi_X^*\omega) = d(\widetilde{f}^*\beta).$$

It implies

$$\int_{\mathbb{P}^1} f^*\omega = \int_{\mathbb{P}^1} d(\widetilde{f}^*\beta) = 0,$$

and so $f^*\omega = 0$.

Gromov proved in [Gro91, Corollary 0.4C] that if X is Kähler hyperbolic, then K_X is a big line bundle, and so X is Moishezon. By Moishezon's theorem, the Kähler and Moishezon manifold X is projective. Since X contains no rational curves, Mori's cone theorem implies that K_X is nef. Since K_X is big and nef, by Kawamata-Reid-Shokurov base point free theorem, K_X is semi-ample. Then there exists m big enough such that $\varphi = |mK_X|$ is a morphism. Since K_X is big, there is a positive integer $\widetilde{m}$ such that

$$\widetilde{m}K_X = D + L$$

where D is an effective divisor and L is an ample line bundle. Suppose K_X is not ample, then there exists a curve C contracted by φ, i.e., $K_X \cdot C = 0$. Therefore,

$$D \cdot C = -L \cdot C < 0.$$

Let $\Delta = \varepsilon D$ for some small $\varepsilon > 0$, then (X, Δ) is a klt pair and $K_X + \Delta$ is not φ-nef. Then by the relative Cone theorem(e.g. [KM98, Theorem 3.25]) for log pairs, there exists a rational curve $\widetilde{C}$ contracted by the morphism φ. This is a contradiction since we have already proved that X contains no rational curves. Therefore, we conclude that K_X is ample. $\qquad\square$

3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1, which is based on the following result.

Theorem 3.1. *Let X be a compact Kähler manifold and Y be a compact Riemannian manifold with $\widetilde{d}$-bounded $H_{\mathrm{DR}}^2(Y, \mathbb{R})$. Suppose there exist two smooth maps $f_1 : X \to Y$*

and $f_2 : Y \to X$ such that the image of the induced map

$$(f_2 \circ f_1)^* : H^2_{\mathrm{DR}}(X, \mathbb{R}) \to H^2_{\mathrm{DR}}(X, \mathbb{R})$$

contains at least one Kähler class. Then X admits a Kähler-Einstein metric of general type. Moreover, each submanifold of X is also a Kähler-Einstein manifold of general type.

Proof. Suppose $\widetilde{X}$ and $\widetilde{Y}$ are the universal coverings of X and Y respectively. Let $\widetilde{f}_1 : \widetilde{X} \to \widetilde{Y}$ and $\widetilde{f}_2 : \widetilde{Y} \to \widetilde{X}$ be the liftings of f_1 and f_2 respectively such that the following diagram

(3.1)
$$
\begin{array}{ccccc}
\widetilde{X} & \xrightarrow{\widetilde{f}_1} & \widetilde{Y} & \xrightarrow{\widetilde{f}_2} & \widetilde{X} \\
\pi_X \downarrow & & \downarrow \pi_Y & & \downarrow \pi_X \\
X & \xrightarrow{f_1} & Y & \xrightarrow{f_2} & X
\end{array}
$$

commutes. Let ω be a Kähler metric on X such that $[\omega]$ is contained in the image of

$$(f_2 \circ f_1)^* : H^2_{DR}(X, \mathbb{R}) \to H^2_{DR}(X, \mathbb{R}).$$

Then there exist a 1-form γ and a closed 2-form ω_1 on X such that

(3.2)
$$\omega = (f_2 \circ f_1)^* \omega_1 + d\gamma.$$

Since Y has $\widetilde{d}$-bounded $H^2_{\mathrm{DR}}(Y, \mathbb{R})$, for the 2-form ω_1 on X, there exists a 1-form β on $\widetilde{Y}$ such that

(3.3)
$$\pi_Y^* \circ f_2^* \omega_1 = d\beta$$

and β is d-bounded on $(\widetilde{Y}, \pi^* g_Y)$. It implies $\widetilde{f}_2^* \circ \pi_X^* \omega_1 = d\beta$ and

(3.4)
$$\widetilde{f}_1^* \circ \widetilde{f}_2^* \circ \pi_X^* \omega_1 = \widetilde{f}_1^* d\beta = d(\widetilde{f}_1^* \beta).$$

Moreover, by (3.2) and (3.4), we have

(3.5) $\pi_X^* \omega = d(\pi_X^* \gamma) + \pi_X^* \circ f_1^* \circ f_2^* \omega_1 = d(\pi_X^* \gamma) + \widetilde{f}_1^* \circ \widetilde{f}_2^* \circ \pi_X^* \omega_1 = d(\pi_X^* \gamma + \widetilde{f}_1^* \beta).$

By using a similar argument as in the proof of Lemma 2.6, we know $\widetilde{f}_1^* \beta$ is bounded on $\widetilde{X}$. Hence, $\pi_X^* \omega$ is d-bounded on $\widetilde{X}$. By definition 2.7, (X, ω) is Kähler hyperbolic. By Theorem 2.11, K_X is ample, i.e. $c_1(X) < 0$. Thanks to the Aubin-Yau theorem, there exists a smooth Kähler metric $\widetilde{\omega}$ on X such that $Ric(\widetilde{\omega}) = -\widetilde{\omega}$.

Suppose Z is a submanifold of X. Let ω_Z be the Kähler metric induced from (X, ω). By Lemma 2.6, (Z, ω_Z) is also Kähler hyperbolic. By Theorem 2.11 and the Aubin-Yau theorem again, Z is a Kähler-Einstein manifold of general type. $\square$

Before giving the proof of Theorem 1.1, we need the following result which refines a fact pointed out by Gromov[Gro91]:

Proposition 3.2. *Let (M, g) be a simply-connected n-dimensional complete Riemannian manifold with sectional curvature bounded from above by a negative constant, i.e.*

$$\sec \leq -K$$

for some $K > 0$. Then for any bounded and closed p-form ω on M, where $p > 1$, there exists a bounded $(p-1)$-form β on M such that

$$(3.6) \qquad \omega = d\beta \quad and \quad |\beta|_{L^\infty} \leq K^{-\frac{1}{2}} |\omega|_{L^\infty},$$

where the L^∞-norm is given by
$$(3.7)$$
$$|\beta|_{L^\infty} = \sup \left\{ |\beta(v_1, \cdots, v_{p-1})|(x) : x \in M, v_i \in T_x M, |v_i|_g = 1, i = 1, \cdots, p-1 \right\}.$$

(Note that the norms defined in (2.1) and (3.7) are equivalent.)

Proof. Fix $x_0 \in M$, let $\exp_{x_0} : T_{x_0} M \to M$ be the exponential map, which is a diffeomorphism by Cartan-Hadamard theorem. Let $\varphi_t : M \to M$, $t \in [0, 1]$, be a family of maps defined by $\varphi_t(x) = \exp_{x_0}(t \cdot \exp_{x_0}^{-1}(x))$, $x \in M$. We denote the distance function from x_0 by ρ, then

$$X_t \mid_{\varphi_t(x)} = \left(\frac{d}{dt} \varphi_t \right) \mid_{\varphi_t(x)} = \rho(x) \nabla \rho \mid_{\varphi_t(x)} .$$

It is clear that $\varphi_1 = id$ and $\varphi_0 \equiv x_0$. Then

$$(3.8) \qquad \omega(x) = \int_0^1 \left(\frac{d}{dt} \varphi_t^* \omega \right)(x) dt = \int_0^1 \varphi_t^* (\mathcal{L}_{X_t} \omega)(x) dt = d \left(\int_0^1 \varphi_t^* (i_{X_t} \omega) dt \right)$$

where we have used Cartan's homotopy formula $\mathcal{L}_{X_t} = d \circ i_{X_t} + i_{X_t} \circ d$ for differential forms. If we set

$$(3.9) \qquad \beta = \int_0^1 \varphi_t^* (i_{X_t} \omega) dt,$$

then $\omega = d\beta$. We show β has bounded L^∞-norm. Fix $x \in M$, $v_1, v_2, \cdots v_{p-1} \in T_x M$, $|v_i| = 1$, $\langle v_i, \nabla \rho \rangle = 0$, we have

$$(3.10) \qquad \beta(v_1, \cdots, v_{p-1})(x) = \int_0^1 \omega \left(X_t, (d\varphi_t)(v_1), \cdots, (d\varphi_t)(v_{p-1}) \right) (\varphi_t(x)) dt.$$

By the standard comparison theorem (e.g. [CE75, Theorem 1.28]), we have

$$(3.11) \qquad |(d\varphi_t)(v)| \leq \frac{\sinh(t\sqrt{K}\rho(x))}{\sinh(\sqrt{K}\rho(x))}$$

for $v \in T_x M$, $|v| = 1$ and $\langle v, \nabla \rho \rangle = 0$. Hence,

$$(3.12) \qquad |\beta(v_1, \cdots, v_{p-1})(x)| \leq |\omega|_{L^\infty} \int_0^1 \rho(x) \left[\frac{\sinh(t\sqrt{K}\rho(x))}{\sinh(\sqrt{K}\rho(x))} \right]^{p-1} dt.$$

If $\rho(x) \geq K^{-\frac{1}{2}}$, since

$$(3.13) \qquad \int_0^1 \sinh^{p-1}(t\sqrt{K}\rho(x))dt \leq \frac{\cosh(\sqrt{K}\rho(x)) - 1}{\sqrt{K}\rho(x)}(\sinh(\sqrt{K}\rho(x)))^{p-2},$$

we have

$$(3.14) \qquad |\beta(v_1,\cdots,v_{p-1})(x)| \leq \frac{|\omega|_{L^\infty}}{\sqrt{K}} \cdot \frac{\cosh(\sqrt{K}\rho(x)) - 1}{\sinh(\sqrt{K}\rho(x))} \leq K^{-\frac{1}{2}}|\omega|_{L^\infty}.$$

If $\rho(x) \leq K^{-\frac{1}{2}}$, we have

$$(3.15) \qquad \int_0^1 \rho(x)\left[\frac{\sinh(t\sqrt{K}\rho(x))}{\sinh(\sqrt{K}\rho(x))}\right]^{p-1} dt \leq \rho(x) \leq K^{-\frac{1}{2}}.$$

Combining two cases, we get $|\beta(v_1,\cdots,v_{p-1})(x)| \leq K^{-\frac{1}{2}}|\omega|_\infty$.

On the other hand, if v_i is parallel to $\nabla\rho$ for some i, then $(d\varphi_t)(v_i)$ is parallel to $\nabla\rho$. By the explicit formula (3.10), we see $\beta(v_1,\cdots,v_{p-1})(x) = 0$. Hence we obtain $|\beta|_{L^\infty} \leq K^{-\frac{1}{2}}|\omega|_{L^\infty}$. $\qquad\square$

The proof of Theorem 1.1. By Proposition 3.2, we see that a compact Riemannian manifold Y with negative Riemannian sectional curvature has $\widetilde{d}$-bounded q^{th} cohomology for all $q \geq 2$. If X is homotopic to Y, there exist two smooth maps $f_1 : X \to Y$ and $f_2 : Y \to X$ such that the induced map $(f_2 \circ f_1)^*$ is identity on $H^*_{DR}(X,\mathbb{R})$. Hence, we can apply Theorem 3.1 to complete the proof of Theorem 1.1. $\qquad\square$

As an application of Theorem 1.1, we give a slightly shorter proof on the following rigidity theorem which was proved by S.-T. Yau in [Yau77, Theorem 6].

Corollary 3.3 (Yau). *Let N be a compact complex surface covered by the unit ball in $\mathbb{C}^2$. Then any complex surface M that is oriented homotopic to N is biholomorphic to N.*

Proof. Since M is homotopic to the Kähler manifold N with even first Betti number b_1, M also has even first Betti number b_1 and so it is Kähler. Since N has a smooth metric with strictly negative Riemannian sectional curvature, by Theorem 1.1, M is a Kähler-Einstein manifold of general type. Hence, the classical Chern-number inequality (e.g. [Yau77, Theorem 4]) implies ,

$$3c_2(M) \geq c_1^2(M).$$

Since M is oriented homotopic to N, the signature of M equals that of N. One can see $c_1^2(M) = c_1^2(N)$ and $c_2(M) = c_2(N)$. Therefore, $3c_2(M) = c_1^2(M)$. By [Yau77, Theorem 4], M is covered by the unit ball in $\mathbb{C}^2$. By Mostow's rigidity theorem ([Mos73]), M is in fact biholomorphic to N. $\qquad\square$

4. Hyperbolicity on compact symplectic manifolds

We begin by recalling some basic definitions. A symplectic form ω on a manifold X *tames* an almost complex structure J if at each point of X, $\omega(Z, JZ) > 0$ for all nonzero vectors Z. We can define a Riemannian metric by

$$(4.1) \qquad g_\omega(Y, Z) = \frac{1}{2}(\omega(Y, JZ) + \omega(Z, JY)).$$

If, in addition,

$$\omega(JY, JZ) = \omega(Y, Z)$$

for all vectors Y, Z, then we say that ω is *compatible* with J. We establish a more general result than Theorem 1.2.

Theorem 4.1. *Let X be a compact symplectic manifold and Y be a compact Riemannian manifold with $\widetilde{d}$-bounded $H^2_{\mathrm{DR}}(Y, \mathbb{R})$. Suppose there exist two smooth maps $f_1 : X \to Y$ and $f_2 : Y \to X$ such that the image of the induced map*

$$(f_2 \circ f_1)^* : H^2_{\mathrm{DR}}(X, \mathbb{R}) \to H^2_{\mathrm{DR}}(X, \mathbb{R})$$

contains at least one symplectic class $[\omega]$. Then for any almost complex structure J on X compatible with the symplectic form ω, there exists no non-constant J-holomorphic map $f : \mathbb{C} \to X$.

Proof. Let ω be a symplectic form on X such that $[\omega]$ is contained in the image of

$$(f_2 \circ f_1)^* : H^2_{DR}(X, \mathbb{R}) \to H^2_{DR}(X, \mathbb{R}).$$

Then there exist a 1-form γ and a closed 2-form ω_1 on X such that

$$(4.2) \qquad \omega = (f_2 \circ f_1)^* \omega_1 + d\gamma.$$

Now we use the same commutative diagram as in (3.1). There exists a 1-form β on $\widetilde{Y}$ such that $\pi_Y^* \circ f_2^* \omega_1 = d\beta$ and β is d-bounded on $(\widetilde{Y}, \pi^* g_Y)$. Moreover, we have $\pi_X^* \omega = d(\pi_X^* \gamma + \widetilde{f}_1^* \beta)$. If we set $\eta = \pi_X^* \gamma + \widetilde{f}_1^* \beta$, then η is bounded on $\widetilde{X}$, i.e., ω is $\widetilde{d}$-bounded.

Let J be an almost complex structure on X which is compatible with the symplectic form ω and $f : \mathbb{C} \to X$ be a J-holomorphic map. We want to show f is a constant, i.e. $f^* \omega = 0$. There is a lifting $\widetilde{f} : \mathbb{C} \to \widetilde{X}$ of f such that $\pi_X \circ \widetilde{f} = f$. Let g be the induced Riemannian metric on (X, ω) defined as in (4.1). The induced almost complex structure, symplectic form and metric on $\widetilde{X}$ are denoted by $\widetilde{J}$, $\widetilde{\omega}$ and $\widetilde{g}$ respectively. On the other hand, for any tangent vectors v, w on $\mathbb{C}$,

$$
\begin{aligned}
\widetilde{g}\left(\widetilde{f}_* v, \widetilde{f}_* w\right) &= \widetilde{\omega}\left(\widetilde{f}_* v, \widetilde{J}\widetilde{f}_*(w)\right) = \widetilde{\omega}\left(\widetilde{f}_* v, \widetilde{f}_*(J_0 w)\right) = (f^* \omega)(v, J_0 w) \\
&= \omega(f_* v, f_* J_0 w) = \omega(f_* v, J f_* w) = (f^* g)(v, w),
\end{aligned}
$$

where J_0 is the standard complex structure of $\mathbb{C}$. Hence, we have

$$(4.3) \qquad \widetilde{f}^* \widetilde{g} = f^* g.$$

Since $\pi_X^* \omega = d\eta$,

$$(4.4) \qquad\qquad f^*\omega = d(\widetilde{f}^*\eta)$$

where η is bounded on $(\widetilde{X}, \widetilde{g})$. On the other hand, since J is compatible with the symplectic form ω and $f : \mathbb{C} \to X$ is J-holomorphic, if f is not a constant, then

$$f^*\omega = \frac{1}{2}\sqrt{-1}\mu(z)dz \wedge d\overline{z}$$

is a Kähler form which may degenerate at countably many points on $\mathbb{C}$. Moreover, the potential $\widetilde{f}^*\eta$ in (4.4) is still (f^*g)-bounded on $\mathbb{C}$. Indeed, since η is bounded over $(\widetilde{X}, \widetilde{g})$, for any tangent vector v on $\mathbb{C}$, we have

$$\left|\left(\widetilde{f}^*\eta\right)(v)\right|^2 = \left|\eta\left(\widetilde{f}_*v\right)\right|^2 \leq C\left|\widetilde{f}_*v\right|_{\widetilde{g}}^2 = C|v|_{f^*g}^2,$$

where we use (4.3) in the last step.

For any bounded domain $\Omega \subset \mathbb{C}$, we use $A_\mu(\Omega)$ and $L_\mu(\partial\Omega)$ to denote the area of Ω and the length of $\partial\Omega$ with respect to the measure $\mu(z)|dz|$. Then

$$(4.5) \qquad\qquad A_\mu(\Omega) = \int_\Omega f^*\omega = \int_{\partial\Omega} \widetilde{f}^*\eta \leq CL_\mu(\partial\Omega)$$

since $\widetilde{f}^*\eta$ is (f^*g)-bounded. Denote $B_r = \{z \in \mathbb{C} : |z| < r\}$, $S_r = \{z \in \mathbb{C} : |z| = r\}$. For any $r > 0$, we have

$$\begin{aligned}
A_\mu(B_r) &= \iint_{B(r)} \mu(z)dxdy = \int_0^r \left(\int_{S_t} \mu\right) dt \\
&\geq \int_0^r \left(\int_{S_t} \sqrt{\mu}\right)^2 (2\pi t)^{-1}dt = \frac{1}{2\pi}\int_0^r L_\mu^2(S_t)\frac{1}{t}dt \\
&\geq \frac{1}{2\pi C^2}\int_0^r A_\mu^2(B_t)\frac{1}{t}dt.
\end{aligned}$$

Denote $F(r) = \int_0^r A_\mu^2(B_t)\frac{1}{t}dt$, then

$$(4.6) \qquad\qquad t \cdot \frac{d}{dt}F(t) = A_\mu^2(B_t) \geq \frac{F(t)^2}{4\pi^2 C^4}$$

which implies

$$(4.7) \qquad\qquad \frac{d}{dt}\left(-\frac{1}{F(t)}\right) \geq \frac{1}{4\pi^2 C^4}\frac{1}{t}.$$

Integrating the above formula over interval $[a, b]$ with $b > a > 0$, we have

$$(4.8) \qquad\qquad \frac{1}{F(a)} \geq \frac{1}{F(b)} + \frac{1}{4\pi^2 C^4}\log\frac{b}{a}.$$

Let $b \to \infty$, we find $F(a) = 0$ for any $a > 0$. This is a contradiction. $\qquad\square$

The proof of Theorem 1.2. By Proposition 3.2, we see that a compact Riemannian manifold Y with strictly negative Riemannian sectional curvature has $\widetilde{d}$-bounded q^{th} cohomology for all $q \geq 2$. Hence, we can apply Theorem 4.1 to complete the proof of Theorem 1.2. □

As a special case of Theorem 4.1, one can see

Corollary 4.2. *Let* (X, ω) *be a compact Kähler manifold. If* X *is Kähler hyperbolic, then it is Kobayashi hyperbolic.*

On the other hand, the following result on Kobayashi hyperbolicity is fundamental (e.g. [Kob98, Theorem 3.6.21]):

Theorem 4.3. *Let* X *be a compact complex manifold. If the cotangent bundle* Ω^1_X *is ample, then* X *is Kobayashi hyperbolic.*

One may wonder whether a similar result holds for Kähler hyperbolicity. Unfortunately, we observe that

Corollary 4.4. *Let* X *be a complete intersection with ample cotangent bundle* Ω^1_X *and* $\dim X \geq 2$. *Then* X *is Kobayashi hyperbolic. However,*

(1) X *is not Kähler hyperbolic.*

(2) X *can not admit a Riemannain metric with non-positive sectional curvature.*

Proof. It is well-known that complete intersections are all simply connected (e.g. [Sha94, p.225-p.227]). □

For example, the intersection of two generic hypersurfaces in $\mathbb{P}^4$ whose degrees are greater than 35 has ample cotangent bundle ([Bro14, Corollary 4.13]). In particular, complete intersections are not strongly negative in the sense of Siu.

References

[Bro14]   Brotbek, D. Hyperbolicity related problems for complete intersection varieties. Compos. Math. **150** (2014), no. 3, 369–395.

[CE75]   Cheeger, J.; Ebin, D. *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[Dem10]   Demailly, J.-P. *Analytic Methods in Algebraic Geometry*, Higher Education Press, Surveys of Modern Mathematics, Vol. 1, 2010

[Gro91]   Gromov, M. Kähler hyperbolicity and L^2-Hodge theory, J. Differential Geom., **33** (1991), 263–292.

[Kob98]   Kobayashi, S. *Hyperbolic complex spaces*. Grundlehren der Mathematischen Wissenschaften, 318. Springer-Verlag, Berlin, 1998.

[KM98]   Kollár, J.; Mori, S. *Birational geometry of algebraic varieties*. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998.

[Lan86]   Lang, S. Hyperbolic and Diophantine analysis. Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 159–205.

[Laz04]   Lazarsfeld, R. *Positivity in algebraic geometry.* I. Springer-Verlag, Berlin, 2004.

[Mos73]   Mostow, G. D. Strong rigidity of locally symmetric spaces. Annals of Mathematics Studies, No. **78**., 1973.

[Sha94]   Shafarevich, R.I. *Basic algebraic geometry. 2. Schemes and complex manifolds.* Second edition. Springer-Verlag, Berlin, 1994.

[Siu80]   Siu, Y.-T. The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. Ann. of Math. (2) **112** (1980), no. 1, 73–111.

[Yau77]   Yau, S.-T. Calabi's conjecture and some new results in algebraic geometry. Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

[Yau78]   Yau, S.-T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Zhe95]   Zheng, F.-Y. First Pontrjagin form, rigidity and strong rigidity of nonpositively curved Kähler surface of general type. Math. Z. **220** (1995), no. 2, 159–169.

Address of Bing-Long Chen: Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China.

E-mail address: mcscbl@mail.sysu.edu.cn

Address of Xiaokui Yang: Morningside Center of Mathematics, Institute of Mathematics, Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.

E-mail address: xkyang@amss.ac.cn