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Quasi-isometry and deformations of Calabi–Yau manifolds

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In memory of Professor Andrey Todorov

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Abstract We prove several formulas related to Hodge theory and the Kodaira–Spencer–Kuranishi deformation theory of Kähler manifolds. As applications, we present a construction of globally convergent power series of integrable Beltrami differentials on Calabi–Yau manifolds and also a construction of global canonical family of holomorphic $(n, 0)$-forms on the deformation spaces of Calabi–Yau manifolds. Similar constructions are also applied to the deformation spaces of compact Kähler manifolds.

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1 Introduction

In this paper, we will present several results about Hodge theory and the deformation theory of Kodaira–Spencer–Kuranishi on compact Kähler manifolds. Our main observations include a simple L^2-quasi-isometry result for bundle valued differential forms, an explicit formula for the deformed $\bar{\partial}$-operator, and an iteration method to construct global Beltrami differentials on Calabi–Yau (CY) manifolds and holomorphic $(n, 0)$-forms on the deformation spaces of compact Kähler manifolds of dimension n. We will present an alternative simple method to solve the $\bar{\partial}$-equation, prove global convergence of the formal power series of the Beltrami differentials and the holomorphic $(n, 0)$-forms constructed from the Kodaira–Spencer–Kuranishi theory. These series previously were only proved to converge in an arbitrarily small neighborhood. We will discuss more applications to the Torelli problem and the extension of twisted pluricanonical sections in a sequel to this paper.

Let us first fix some notations to be used throughout this paper. All manifolds in this paper are assumed to be compact and Kähler, though some results still hold for complete Kähler manifolds; a Calabi–Yau, or CY manifold, is a compact projective manifold with trivial canonical line bundle. By Yau's solution to the Calabi conjecture, there is a CY metric on X such that the holomorphic $(n, 0)$-form Ω_0 on X is parallel with respect to the metric connection. For a complex manifold (X, ω) and a Hermitian holomorphic vector bundle (E, h) on X, we denote by $A^{p,q}(X)$ the space of smooth (p, q)-forms on X and by $A^{p,q}(E) = A^{p,q}(X, E)$ the space of smooth (p, q)-forms on X with values in E. Similarly, let $\mathbb{H}^{p,q}(X)$ be the space of the harmonic (p, q)-forms and let $\mathbb{H}^{p,q}(X, E)$ be the space of the harmonic (p, q)-forms with values in E. Let ∇ be the Chern connection on (E, h) with canonical decomposition $\nabla = \nabla' + \bar{\partial}$ where ∇' is the $(1, 0)$-part of the Chern connection ∇. Let $\mathbb{G}$ and $\mathbb{H}$ denote the Green operator and harmonic projection in the Hodge decomposition with respect to the operator $\bar{\partial}$, that is,

$$\mathbb{I} = \mathbb{H} + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\mathbb{G}.$$

A Beltrami differential is an element in $A^{0,1}(X, T_X^{1,0})$, where $T_X^{1,0}$ denotes the holomorphic tangent bundle of X. The L^2-norm $\|\cdot\| = \|\cdot\|_{L^2}^{\frac{1}{2}}$ is induced by the metrics ω and h. The $\mathscr{C}^k$-norm $\|\cdot\|_{\mathscr{C}^k}$ will be used on the Beltrami differentials.

Now we briefly describe the main results in this paper. The following quasi-isometry on compact Kähler manifolds is obtained in Sect. 2.

Theorem 1.1 (Quasi-isometry) *Let (E, h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X, ω).*

(1) *For any $g \in A^{n,\bullet}(X, E)$, we have the following estimate*

$$\|\overline{\partial}^* \mathbb{G} g\|^2 \leq \langle g, \mathbb{G} g \rangle.$$

(2) *If (E, h) is a strictly positive line bundle with Chern curvature Θ^E and $\omega = \sqrt{-1}\Theta^E$, for any $g \in A^{n-1,\bullet}(X, E)$ we obtain*

$$\|\overline{\partial}^* \mathbb{G} \nabla' g\| \leq \|g\|.$$

(3) *If E is the trivial line bundle, for any smooth $g \in A^{p,q}(X)$,*

$$\|\overline{\partial}^* \mathbb{G} \partial g\| \leq \|g\|.$$

In particular, if $\overline{\partial}\partial g = 0$ and g is ∂^*-exact, we obtain the isometry

$$\|\overline{\partial}^* \mathbb{G} \partial g\| = \|g\|.$$

Here the operator $\overline{\partial}^* \mathbb{G}$ can be viewed as the "inverse operator" of $\overline{\partial}$. More precisely, we can write down the explicit solutions of some $\overline{\partial}$-equations by using $\overline{\partial}^* \mathbb{G}$, which can also be considered as a bundle-valued version of the very useful $\partial\overline{\partial}$-lemma in complex geometry.

Proposition 1.2 *Let (E, h) be a Hermitian holomorphic vector bundle with semi-Nakano positive curvature tensor Θ^E over the compact Kähler manifold (X, ω). Then, for any $g \in A^{n-1,\bullet}(X, E)$ with $\overline{\partial}\nabla' g = 0$, the $\overline{\partial}$-equation $\overline{\partial} s = \nabla' g$ admits a solution*

$$s = \overline{\partial}^* \mathbb{G} \nabla' g,$$

such that

$$\|s\|^2 \leq \langle \nabla' g, \mathbb{G} \nabla' g \rangle.$$

Moreover, this solution is unique if we require $\mathbb{H}(s) = 0$ and $\overline{\partial}^* s = 0$.

Note that, in the proofs of Theorem 1.1 and Proposition 1.2, we only use basic Hodge theory, so they still hold on general Kähler manifolds as long as Hodge theory can be applied. On the other hand, in Proposition 1.2, the curvature Θ^E is only required to be semi-positive and it is significantly different from all variants of Hörmander's L^2-estimates. Moreover, Proposition 1.2 can also hold if h is a singular Hermitian metric, and the curvature Θ^E has certain weak positivity in the current sense.

In the following, we shall use i_ϕ and $\phi\lrcorner$ to denote the contraction operator with $\phi \in A^{0,1}(X, T_X^{1,0})$ alternatively if there is no confusion. For

$\phi \in A^{0,1}(X, T_X^{1,0})$, the Lie derivative can be lifted to act on bundle valued forms by

$$\mathcal{L}_\phi = -\nabla \circ i_\phi + i_\phi \circ \nabla.$$

There is also a canonical decomposition

$$\mathcal{L}_\phi = \mathcal{L}_\phi^{1,0} + \mathcal{L}_\phi^{0,1}$$

according to the types.

In Sect. 3, we prove some explicit formulas for the deformed differential operators on the deformation spaces of complex structures and one of our main results is

Theorem 1.3 *Let $\phi \in A^{0,1}(X, T_X^{1,0})$. Then on the space $A^{\bullet,\bullet}(X, E)$, we have*

$$e^{-i_\phi} \circ \nabla \circ e^{i_\phi} = \nabla - \mathcal{L}_\phi - i_{\frac{1}{2}[\phi,\phi]} = \nabla - \mathcal{L}_\phi^{1,0} + i_{\overline{\partial}\phi - \frac{1}{2}[\phi,\phi]}.$$

In particular, if $\sigma \in A^{n,\bullet}(X, E)$ and ϕ is integrable, i.e., $\overline{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0$, then

$$\left(e^{-i_\phi} \circ \nabla \circ e^{i_\phi}\right)(\sigma) = \overline{\partial}\sigma + \nabla'(\phi \lrcorner \sigma).$$

As applications of Theorems 1.1 and 1.3, we use ideas of recursive methods to construct Beltrami differentials in Kodaira–Spencer–Kuranishi deformation theory in Sect. 4. Similar methods are also presented in [1,2,4,8,10–13] and the references therein. At first, we present the following global convergence on the deformation space of CY manifolds:

Theorem 1.4 *Let X be a CY manifold and $\varphi_1 \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ with norm $\|\varphi_1\|_{\mathscr{C}^1} = \frac{1}{4C_1}$. Then for any nontrivial holomorphic $(n, 0)$ form Ω_0 on X, there exits a smooth globally convergent power series for $|t| < 1$,*

$$\Phi(t) = \varphi_1 t^1 + \varphi_2 t^2 + \cdots + \varphi_k t^k + \cdots \in A^{0,1}(X, T_X^{1,0}),$$

which satisfies:

(1) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$*;*
(2) $\overline{\partial}^* \varphi_k = 0$ *for each* $k \geq 1$*;*
(3) $\varphi_k \lrcorner \Omega_0$ *is* ∂*-exact for each* $k \geq 2$*;*
(4) $\|\Phi(t) \lrcorner \Omega_0\|_{L^2} < \infty$ *as long as* $|t| < 1$.

The key ingredient in Theorem 1.4 is that the convergent radius of the power series is at least 1, which was previously proved to be sufficiently small. We shall see that the L^2-estimate in Theorem 1.1 plays a key role in the proof of Theorem 1.4. The power series thus obtained is called an L^2-global canonical family of Beltrami differentials on the CY manifold X.

In Sect. 5, we obtain the following theorem to construct deformations of holomorphic $(n, 0)$-forms, which are globally convergent in the L^2-norm for CY manifolds.

Theorem 1.5 *Let Ω_0 be a nontrivial holomorphic $(n, 0)$-form on the CY manifold X and $X_t = (X_t, J_{\Phi(t)})$ be the deformation of the CY manifold X induced by $\Phi(t)$ as constructed in Theorem 1.4. Then for any $|t| < 1$,*

$$\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_0$$

defines an L^2-global canonical family of holomorphic $(n, 0)$-forms on X_t.

As a straightforward consequence of Theorem 1.5, we have the following global expansion of the canonical family of $(n, 0)$-forms on the deformation spaces of CY manifolds in cohomology classes. Similar ideas are also used in [4, Theorem 1.34]. This expansion also has interesting applications in studying the global Torelli problem.

Corollary 1.6 *With the same notations as in Theorem 1.5, there holds the following global expansion of $[\Omega_t^C]$ in cohomology classes for $|t| < 1$*

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega_0] t_i + O(|t|^2),$$

where $O(|t|^2) \in \bigoplus_{j=2}^{n} H^{n-j,j}(X)$ denotes the terms of orders at least 2 in t.

Finally, we need to point out that on the deformation spaces of compact Kähler manifolds, if we assume the existence of a global family of Beltrami differentials $\Phi(t)$ as stated in Theorem 1.4, we can also construct L^2-global family of $(n, 0)$-forms on the deformation spaces of compact Kähler manifolds. For more details, see Theorem 5.5 and Corollary 5.6.

2 $\overline{\partial}$-Equations on non-negative vector bundles

In this section, we will prove a quasi-isometry result in L^2-norm with respect to the operator $\overline{\partial}^* \circ \mathbb{G}$ on a compact Kähler manifold. This gives a rather simple and explicit way to solve vector bundle valued $\overline{\partial}$-equations with L^2-estimates.

Let (E, h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X, ω) and $\nabla = \nabla' + \overline{\partial}$ be the Chern connection on it. With respect to metrics on E and X, we set

$$\overline{\Box} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial},$$
$$\Box' = \nabla'\nabla'^* + \nabla'^*\nabla'.$$

Accordingly, we associate the Green operators and harmonic projections $\mathbb{G}$, $\mathbb{H}$ and $\mathbb{G}'$, $\mathbb{H}'$ in Hodge decomposition to them, respectively. More precisely,

$$\mathbb{I} = \mathbb{H} + \overline{\Box} \circ \mathbb{G}, \quad \mathbb{I} = \mathbb{H}' + \Box' \circ \mathbb{G}'.$$

Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $\Theta^E \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has the form

$$\Theta^E = R_{i\bar{j}\alpha}^{\gamma} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma,$$

where $R_{i\bar{j}\alpha}^{\gamma} = h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}$ and

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}.$$

Here and henceforth we adopt the Einstein convention for summation.

Definition 2.1 A Hermitian vector bundle (E, h) is said to be *semi-Nakano-positive* (resp. *Nakano-positive*), if for any non-zero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} \geq 0, \quad (resp. > 0).$$

For a line bundle, it is strictly positive if and only if it is Nakano-positive.

Theorem 2.2 (Quasi-isometry) *Let (E, h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X, ω).*

(1) *For any $g \in A^{n,\bullet}(X, E)$, we have the following estimate*

$$\|\overline{\partial}^* \mathbb{G} g\|^2 \leq \langle g, \mathbb{G} g \rangle.$$

(2) *If (E, h) is a strictly positive line bundle and $\omega = \sqrt{-1}\Theta^E$, for any $g \in A^{n-1,\bullet}(X, E)$,*

$$\|\overline{\partial}^* \mathbb{G} \nabla' g\| \leq \|g\|.$$

(3) *If E is the trivial line bundle, for any smooth* $g \in A^{p,q}(X)$,

$$\|\bar{\partial}^*\mathbb{G}\partial g\|^2 = \|g\|^2 - \|\mathbb{H}(g)\|^2 - \left\langle \partial^* g, \mathbb{G}(\partial^* g)\right\rangle - \|\mathbb{G}(\bar{\partial}\partial g)\|^2 \le \|g\|^2.$$

In particular, if $\bar{\partial}\partial g = 0$ *and* g *is* ∂^**-exact, we obtain the isometry*

$$\|\bar{\partial}^*\mathbb{G}\partial g\| = \|g\|.$$

Proof (1). For $g \in A^{n,\bullet}(X, E)$,

$$\begin{aligned}\|\bar{\partial}^*\mathbb{G}g\|^2 &= \langle \overline{\partial\partial}^*\mathbb{G}g, \mathbb{G}g\rangle \\ &= \langle g, \mathbb{G}g\rangle - \langle \bar{\partial}^*\bar{\partial}\mathbb{G}g, \mathbb{G}g\rangle - \langle \mathbb{H}g, \mathbb{G}g\rangle \\ &= \langle g, \mathbb{G}\rangle - \langle \bar{\partial}\mathbb{G}g, \bar{\partial}\mathbb{G}g\rangle \\ &\le \langle g, \mathbb{G}g\rangle\end{aligned}$$

since the Green operator is self-adjoint and zero on the kernel of Laplacian by definition.

(2). If (E, h) is a strictly positive line bundle over X and $\omega = \sqrt{-1}\Theta^E$, for any $g \in A^{n-1,q}(X, E)$, by the well-known Bochner–Kodaira–Nakano identity $\overline{\square} = \square' + [\sqrt{-1}\Theta^E, \Lambda_\omega]$,

$$\overline{\square}(\nabla' g) = \square'(\nabla' g) + q(\nabla' g) = (\square' + q)(\nabla' g),$$

we obtain $\mathbb{H}(\nabla' g) = 0$ and thus $\overline{\square}\mathbb{G}(\nabla' g) = \nabla' g = \square'\mathbb{G}'(\nabla' g)$ since obviously $\mathbb{H}'(\nabla' g) = 0$ by Hodge decomposition. Moreover,

$$\begin{aligned}\langle \nabla' g, \mathbb{G}(\nabla' g)\rangle &= \langle \nabla' g, \overline{\square}^{-1}(\nabla' g)\rangle \\ &= \langle \nabla' g, (\square' + q)^{-1}(\nabla' g)\rangle \\ &\le \langle \nabla' g, \square'^{-1}(\nabla' g)\rangle \\ &= \langle \nabla' g, \mathbb{G}'(\nabla' g)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\bar{\partial}^*\mathbb{G}\nabla' g\|^2 &\le \langle \nabla' g, \mathbb{G}\nabla' g\rangle \\ &\le \langle \nabla' g, \mathbb{G}'\nabla' g\rangle \\ &= \langle g, \nabla'^*\nabla'\mathbb{G}' g\rangle \\ &= \langle g, g - \mathbb{H}'(g) - \nabla'\nabla'^*\mathbb{G}' g\rangle \\ &= \|g\|^2 - \|\mathbb{H}'(g)\|^2 - \langle \nabla'^* g, \mathbb{G}'\nabla'^* g\rangle \\ &\le \|g\|^2.\end{aligned}$$

(3). If E is the trivial line bundle, for any $g \in A^{p,q}(X)$, we have the following

$$\begin{aligned}
\|\overline{\partial}^*\mathbb{G}\partial g\|^2 &= \left\langle \overline{\partial}^*\mathbb{G}\partial g, \overline{\partial}^*\mathbb{G}\partial g \right\rangle = \left\langle \overline{\partial}\overline{\partial}^*\mathbb{G}\partial g, \mathbb{G}\partial g \right\rangle \\
&= \left\langle \overline{\Box}\mathbb{G}\partial g - \overline{\partial}^*\overline{\partial}\mathbb{G}\partial g, \mathbb{G}\partial g \right\rangle \\
&= \langle \partial g, \mathbb{G}\partial g\rangle - \left\langle \overline{\partial}^*\overline{\partial}\mathbb{G}\partial g, \mathbb{G}\partial g \right\rangle \\
&= \left\langle g, \partial^*\partial\mathbb{G}g \right\rangle - \left\langle \mathbb{G}\overline{\partial}\partial g, \mathbb{G}\overline{\partial}\partial g \right\rangle \\
&= \left\langle g, \Box'\mathbb{G}g - \partial\partial^*\mathbb{G}g \right\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \\
&= \left\langle g, g - \mathbb{H}(g) - \partial\partial^*\mathbb{G}g \right\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \\
&= \|g\|^2 - \|\mathbb{H}(g)\|^2 - \left\langle \partial^* g, \mathbb{G}(\partial^* g) \right\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \\
&\le \|g\|^2,
\end{aligned}$$

since the Green operator is nonnegative. In particular, if $\overline{\partial}\partial g = 0$ and g is ∂^*-exact, we have $\mathbb{H}(g) = 0$ and $\partial^* g = 0$. Hence, we obtain the isometry $\|\overline{\partial}^* G\partial g\| = \|g\|$. □

Proposition 2.3 ($\overline{\partial}$-Inverse formula) *Let (E, h) be a Hermitian holomorphic vector bundle with semi-Nakano positive curvature Θ^E over the compact Kähler manifold (X, ω). Then, for any $g \in A^{n-1,\bullet}(X, E)$,*

$$s = \overline{\partial}^*\mathbb{G}\nabla' g$$

is a solution to the equation $\overline{\partial}s = \nabla' g$ with $\overline{\partial}\nabla' g = 0$, such that

$$\|s\|^2 \le \langle \nabla' g, \mathbb{G}\nabla' g\rangle.$$

This solution is unique as long as it satisfies $\mathbb{H}(s) = 0$ and $\overline{\partial}^* s = 0$.

Proof By the well-known Bochner–Kodaira–Nakano identity $\overline{\Box} = \Box' + [\sqrt{-1}\Theta^E, \Lambda_\omega]$, one can see that for any $\phi \in A^{n,\bullet}(X, E)$,

$$\langle \sqrt{-1}[\Theta^E, \Lambda_\omega]\phi, \phi\rangle \ge 0$$

if E is semi-Nakano positive (e.g. [3]). It implies that, for any $\phi \in A^{n,\bullet}(X, E)$,

$$\langle \overline{\Box}\phi, \phi\rangle \ge \langle \Box'\phi, \phi\rangle.$$

Thus, on the space $A^{n,\bullet}(X, E)$,

$$\ker \overline{\Box} \subseteq \ker \Box' \text{ and } (\ker \Box')^\perp \subseteq (\ker \overline{\Box})^\perp. \tag{2.1}$$

By Hodge decomposition, we have

$$\bar{\partial} s = \bar{\partial}\bar{\partial}^* \mathbb{G}\nabla' g = \nabla' g - \mathbb{H}\nabla' g - \bar{\partial}^* \bar{\partial}\mathbb{G}\nabla' g = \nabla' g - \mathbb{H}\nabla' g = \nabla' g,$$

where the identity $\mathbb{H}\nabla' g = 0$ is used. Actually, we know $\nabla' g \perp \ker \square'$ and obviously $\nabla' g \perp \ker \bar{\square}$ by the first inclusion of (2.1).

The uniqueness of this solution follows easily. In fact, if s_1 and s_2 are two solutions to $\bar{\partial} s = \nabla' g$ with $\mathbb{H}(s_1) = \mathbb{H}(s_2) = 0$ and $\bar{\partial}^* s_1 = \bar{\partial}^* s_2 = 0$, by setting $\eta = s_1 - s_2$, we see $\bar{\partial}\eta = 0$, $\mathbb{H}(\eta) = 0$ and $\bar{\partial}^* \eta = 0$. Therefore,

$$\eta = \mathbb{H}(\eta) + \bar{\square}\mathbb{G}(\eta) = \mathbb{H}(\eta) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^* \bar{\partial})\mathbb{G}(\eta) = 0.$$

□

3 Beltrami differentials and deformation theory

In this section we prove several new formulas to construct explicit deformed differential operators for bundle valued differential forms on the deformation spaces of Kähler manifolds. These formulas are applied to the deformation spaces of CY manifolds in later sections while more applications to the deformation theory of Kähler manifolds and holomorphic line bundles will be discussed in the sequel to this paper. Throughout this section, X is always assumed to be a complex manifold.

For $X_0 \in \Gamma(X, T_X^{1,0})$, the contraction operator is defined as

$$i_{X_0} : A^{p,q}(X) \to A^{p-1,q}(X)$$

by

$$(i_{X_0}\alpha)(X_1, \ldots, X_{p-1}, Y_1, \ldots, Y_q) = \alpha(X_0, X_1, \ldots, X_{p-1}, Y_1, \ldots Y_q)$$

for $\alpha \in A^{p,q}(X)$, $X_1, \ldots, X_{p-1} \in \Gamma(X, T_X^{1,0})$ and $Y_1, \ldots, Y_q \in \Gamma(X, T_X^{0,1})$. We will also use the notation '$\lrcorner$' to represent the contraction operator in the sequel, that is,

$$i_{X_0}(\alpha) = X_0 \lrcorner \alpha.$$

For $\phi \in A^{0,s}(X, T_X^{1,0})$, the contraction operator can be extended to

$$i_\phi : A^{p,q}(X) \to A^{p-1,q+s}(X). \tag{3.1}$$

For example, if $\phi = \eta \otimes Y$ with $\eta \in A^{0,q}(X)$ and $Y \in \Gamma(X, T_X^{1,0})$, then for any $\omega \in A^{p,q}(X)$,

$$(i_\phi)(\omega) = \eta \wedge (i_Y \omega).$$

The following result follows easily.

Lemma 3.1 *Let $\phi \in A^{0,q}(X, T_X^{1,0})$ and $\psi \in A^{0,s}(X, T_X^{1,0})$. Then*

$$i_\phi \circ i_\psi = (-1)^{(q+1)(s+1)} i_\psi \circ i_\phi.$$

For $Y \in \Gamma(X, T_X)$, the Lie derivative $\mathcal{L}_Y$ is defined as

$$\mathcal{L}_Y = d \circ i_Y + i_Y \circ d : A^s(X) \to A^s(X).$$

For any $\phi \in A^{0,q}(X, T_X^{1,0})$, we can define i_ϕ as (3.1) and thus extend $\mathcal{L}_\phi$ to be

$$\mathcal{L}_\phi = (-1)^q d \circ i_\phi + i_\phi \circ d.$$

According to the types, we can decompose

$$\mathcal{L}_\phi = \mathcal{L}_\phi^{1,0} + \mathcal{L}_\phi^{0,1},$$

where

$$\mathcal{L}_\phi^{1,0} = (-1)^q \partial \circ i_\phi + i_\phi \circ \partial$$

and

$$\mathcal{L}_\phi^{0,1} = (-1)^q \overline{\partial} \circ i_\phi + i_\phi \circ \overline{\partial}.$$

Let

$$\varphi^i = \frac{1}{p!} \sum \varphi^i_{\bar{j}_1,\ldots,\bar{j}_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p} \otimes \partial_i \quad \text{and}$$
$$\psi^i = \frac{1}{q!} \sum \psi^i_{\bar{k}_1,\ldots,\bar{k}_q} d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q} \otimes \partial_i.$$

Then, we write

$$[\varphi, \psi] = \sum_{i,j=1}^{n} (\varphi^i \wedge \partial_i \psi^j - (-1)^{pq} \psi^i \wedge \partial_i \varphi^j) \otimes \partial_j, \tag{3.2}$$

where

$$\partial_i \varphi^j = \frac{1}{p!} \sum \partial_i \varphi^j_{\bar{j}_1, \dots, \bar{j}_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p}$$

and similar for $\partial_i \psi^j$. In particular, if $\varphi, \psi \in A^{0,1}(X, T_X^{1,0})$,

$$[\varphi, \psi] = \sum_{i,j=1}^{n} (\varphi^i \wedge \partial_i \psi^j + \psi^i \wedge \partial_i \varphi^j) \otimes \partial_j.$$

Let (E, h) be a Hermitian holomorphic vector bundle over X and ∇ be the Chern connection on (E, h). Then the operators $i_\bullet$, $\mathcal{L}_\bullet$, $[\bullet, \bullet]$ can be extended to E-valued (p, q)-forms in the canonical way. For example, for any $\phi \in A^{0,k}(X, T_X^{1,0})$, on $A^{p,q}(X, E)$ we can define

$$\mathcal{L}_\phi = (-1)^k \nabla \circ i_\phi + i_\phi \circ \nabla.$$

Then we have the following general commutator formula.

Lemma 3.2 (cf. [6]) *For* $\varphi \in A^{0,k}(X, T_X^{1,0})$, $\varphi' \in A^{0,k'}(X, T_X^{1,0})$ *and* $\alpha \in A^{p,q}(X, E)$,

$$(-1)^{k'} \varphi \lrcorner \mathcal{L}_{\varphi'} \alpha + (-1)^{k'k+1} \mathcal{L}_{\varphi'} (\varphi \lrcorner \alpha) = [\varphi, \varphi'] \lrcorner \alpha,$$

or equivalently,

$$[\mathcal{L}_{\varphi'}, i_\varphi] = i_{[\varphi', \varphi]}.$$

In particular, if $\varphi, \varphi' \in A^{0,1}(X, T_X^{1,0})$, *then*

$$[\varphi, \varphi'] \lrcorner \alpha = -\nabla'(\varphi' \lrcorner (\varphi \lrcorner \alpha)) - \varphi \lrcorner (\varphi' \lrcorner \nabla' \alpha) + \varphi \lrcorner \nabla'(\varphi' \lrcorner \alpha) + \varphi' \lrcorner \nabla'(\varphi \lrcorner \alpha) \tag{3.3}$$

and

$$0 = -\bar{\partial}(\varphi' \lrcorner (\varphi \lrcorner \alpha)) - \varphi \lrcorner (\varphi' \lrcorner \bar{\partial} \alpha) + \varphi \lrcorner \bar{\partial}(\varphi' \lrcorner \alpha) + \varphi' \lrcorner \bar{\partial}(\varphi \lrcorner \alpha). \tag{3.4}$$

Proof Since the formulas are all local and $\mathbb{C}$-linear, without loss of generality, we can assume that

$$\varphi = \eta \otimes \chi, \quad \varphi' = \eta' \otimes \chi',$$

where $\eta \in A^{0,k}(X)$, $\eta' \in A^{0,k'}(X)$, $\chi, \chi' \in \Gamma(X, T_X^{1,0})$ and $d\eta = d\eta' = 0$. Since $d\eta = d\eta' = 0$, we have $\chi'(\eta) = \chi(\eta') = 0$. Hence, we obtain

$$[\varphi, \varphi'] = \eta \wedge \eta'[\chi, \chi'].$$

On the other hand, for any $\alpha \in A^{p,q}(X, E)$,

$$\begin{aligned}
\mathcal{L}_\varphi \alpha &= \eta \wedge (\chi \lrcorner \nabla \alpha) + (-1)^k \nabla(\eta \wedge (\chi \lrcorner \alpha)) \\
&= \eta \wedge (\chi \lrcorner \nabla \alpha) + (-1)^k (d\eta \wedge (\chi \lrcorner \alpha) + (-1)^k \eta \wedge \nabla(\chi \lrcorner \alpha)) \\
&= \eta \wedge (\chi \lrcorner \nabla \alpha + \nabla(\chi \lrcorner \alpha)) \\
&= \eta \wedge \mathcal{L}_\chi \alpha.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\varphi \lrcorner \mathcal{L}_{\varphi'} \alpha &= \eta \wedge \chi \lrcorner (\eta' \wedge \mathcal{L}_{\chi'} \alpha) \\
&= (-1)^{k'} \eta \wedge \eta' (\chi \lrcorner \mathcal{L}_{\chi'} \alpha) \\
&= (-1)^{k'} \eta \wedge \eta' \left(\mathcal{L}_{\chi'}(\chi \lrcorner \alpha) - [\chi', \chi] \lrcorner \alpha \right) \\
&= (-1)^{k'} \left(\eta \wedge \mathcal{L}_{\varphi'}(\chi \lrcorner \alpha) - \eta \wedge \eta' \wedge ([\chi', \chi] \lrcorner \alpha) \right) \\
&= (-1)^{k'} [\varphi, \varphi'] \lrcorner \alpha + (-1)^{k'(1+k)} \mathcal{L}_{\varphi'}(\eta \wedge (\chi \lrcorner \alpha)) \\
&= (-1)^{k'} [\varphi, \varphi'] \lrcorner \alpha + (-1)^{k'(1+k)} \mathcal{L}_{\varphi'}(\varphi \lrcorner \alpha),
\end{aligned}$$

where we apply the formula

$$[\chi', \chi] \lrcorner \alpha = \mathcal{L}_{\chi'}(\chi \lrcorner \alpha) - \chi \lrcorner \mathcal{L}_{\chi'} \alpha,$$

which is proven in [6], and

$$\mathcal{L}_{\varphi'}(\varphi \lrcorner \alpha) = (-1)^{k'k} \eta \wedge \mathcal{L}_{\varphi'}(\chi \lrcorner \alpha).$$

In fact,

$$\begin{aligned}
&\mathcal{L}_{\varphi'}(\varphi \lrcorner \alpha) \\
&= \mathcal{L}_{\varphi'}(\eta \wedge (\chi \lrcorner \alpha)) \\
&= \varphi' \lrcorner \nabla(\eta \wedge (\chi \lrcorner \alpha)) + (-1)^{k'} \nabla \circ \varphi \lrcorner (\eta \wedge (\chi \lrcorner \alpha)) \\
&= \varphi' \lrcorner (d\eta \wedge (\chi \lrcorner \alpha)) + (-1)^k \varphi' \lrcorner (\eta \wedge \nabla(\chi \lrcorner \alpha)) \\
&\quad + (-1)^{k' + k(k'-1)} \nabla(\eta \wedge (\varphi' \lrcorner (\chi \lrcorner \alpha))) \\
&= (-1)^{k + k(k'-1)} \eta \wedge (\varphi' \lrcorner (\nabla(\chi \lrcorner \alpha))) + (-1)^{k' + k(k'-1) + k} \eta \wedge \nabla(\varphi' \lrcorner (\chi \lrcorner \alpha)) \\
&(-1)^{k'k} \eta \wedge \mathcal{L}_{\varphi'}(\chi \lrcorner \alpha).
\end{aligned}$$

□

As an easy corollary, we have the following result which was known as Tian–Todorov lemma.

Lemma 3.3 ([12,13]) *If* $\varphi, \psi \in A^{0,1}(X, T_X^{1,0})$ *and* $\Omega \in A^{n,0}(X)$*, then one has*

$$[\varphi, \psi]\lrcorner\Omega = -\partial(\psi\lrcorner(\varphi\lrcorner\Omega)) + \varphi\lrcorner\partial(\psi\lrcorner\Omega) + \psi\lrcorner\partial(\varphi\lrcorner\Omega).$$

In particular, if X *is a CY manifold and* Ω_0 *is a nontrivial holomorphic* $(n, 0)$ *form on* X*, then for any* $\varphi, \psi \in \mathbb{H}^{0,1}(X, T_X^{1,0})$,

$$[\varphi, \psi]\lrcorner\Omega_0 = -\partial(\psi\lrcorner(\varphi\lrcorner\Omega_0)).$$

Note that, here both $\varphi\lrcorner\Omega_0$ *and* $\psi\lrcorner\Omega_0$ *are harmonic.*

Let $\phi \in A^{0,1}(X, T_X^{1,0})$ and i_ϕ be the contraction operator. Define an operator

$$e^{i_\phi} = \sum_{k=0}^{\infty} \frac{1}{k!} i_\phi^k,$$

where $i_\phi^k = \underbrace{i_\phi \circ \cdots \circ i_\phi}_{k \text{ copies}}$. Since the dimension of X is finite, the summation in the above formulation is also finite.

The following theorem gives explicit formulas for the deformed differential operators on the deformation spaces of complex structures. It also explains why it is relatively easy to construct extension of sections of the bundle $K_X + E$ where K_X is the canonical bundle of X. We remark that this result is motivated by [2] where a special case was proved.

Theorem 3.4 *Let* $\phi \in A^{0,1}(X, T_X^{1,0})$*. Then on the space* $A^{\bullet,\bullet}(E)$*, we have*

$$e^{-i_\phi} \circ \nabla \circ e^{i_\phi} = \nabla - \mathcal{L}_\phi - i_{\frac{1}{2}[\phi,\phi]},$$

or equivalently

$$e^{-i_\phi} \circ \overline{\partial} \circ e^{i_\phi} = \overline{\partial} - \mathcal{L}_\phi^{0,1} \tag{3.5}$$

and

$$e^{-i_\phi} \circ \nabla' \circ e^{i_\phi} = \nabla' - \mathcal{L}_\phi^{1,0} - i_{\frac{1}{2}[\phi,\phi]}. \tag{3.6}$$

Moreover, if $\overline{\partial}\phi = \frac{1}{2}[\phi, \phi]$*, then*

$$\overline{\partial} - \mathcal{L}_\phi^{1,0} = e^{-i_\phi} \circ (\overline{\partial} - \mathcal{L}_\phi) \circ e^{i_\phi}. \tag{3.7}$$

Proof (3.5) follows from (3.3) and the formula

$$[\overline{\partial}, i_\phi^k] = k i_\phi^{k-1} \circ [\overline{\partial}, i_\phi],$$

which can be proved by induction by using (3.3). Similarly, (3.6) follows from (3.4) and

$$[\nabla', i_\phi^k] = k i_\phi^{k-1} \circ [\nabla', i_\phi] - \frac{k(k-1)}{2} i_\phi^{k-2} \circ i_{[\phi,\phi]}, \quad k \geq 2. \tag{3.8}$$

Now we prove (3.8) by induction. It is obvious that (3.8) is equivalent to the statement that, for any $k \geq 2$,

$$\begin{aligned} F_k &:= -k i_\phi^{k-1} \circ \nabla' \circ i_\phi + (k-1) i_\phi^k \circ \nabla' + \nabla' \circ i_\phi^k + \frac{k(k-1)}{2} i_\phi^{k-2} i_{[\phi,\phi]} \\ &= 0. \end{aligned} \tag{3.9}$$

If $k = 2$, it is (3.4). As for $k = 3$,

$$\begin{aligned} 0 &= i_{[\phi,\phi]} \circ i_\phi - i_\phi \circ i_{[\phi,\phi]} \\ &= 3 i_\phi \circ \nabla' \circ i_\phi^2 - \nabla' \circ i_\phi^3 - 3 i_\phi^2 \circ \nabla' \circ i_\phi + i_\phi^3 \circ \nabla' \\ &= 3 i_\phi^2 \circ \nabla' \circ i_\phi - 2 i_\phi^3 \circ \nabla' - \nabla' \circ i_\phi^3 - 3 i_\phi \circ i_{[\phi,\phi]} \\ &= -F_3, \end{aligned}$$

where Lemma 3.2 is applied.

Now we assume that (3.9) is right for all integers less than k where $k \geq 4$. That is,

$$F_2 = F_3 = \cdots = F_{k-1} = 0.$$

We will show $F_k = 0$. Now we set

$$\begin{aligned} G_k &= F_k - i_\phi \circ F_{k-1} \\ &= -i_\phi^{k-1} \circ \nabla' \circ i_\phi + i_\phi^k \circ \nabla' + \nabla' \circ i_\phi^k - i_\phi \circ \nabla' \circ i_\phi^{k-1} + (k-1) i_\phi^{k-2} i_{[\phi,\phi]}. \end{aligned}$$

So, by induction, we have

$$\begin{aligned} & G_k - i_\phi \circ G_{k-1} \\ &= \nabla' \circ i_\phi^k - 2 i_\phi \circ \nabla' \circ i_\phi^{k-1} + i_\phi^2 \circ \nabla' \circ i_\phi^{k-2} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ &= (\nabla' \circ i_\phi^2 + i_\phi^2 \circ \nabla' - 2 i_\phi \circ \nabla' \circ i_\phi) \circ i_\phi^{k-2} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ &= -i_{[\phi,\phi]} \circ i_\phi^{k-2} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ &= -i_\phi \circ i_{[\phi,\phi]} \circ i_\phi^{k-3} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ &= -i_\phi^2 \circ i_{[\phi,\phi]} \circ i_\phi^{k-4} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \end{aligned}$$

$$= -i_\phi^{k-3} \circ i_{[\phi,\phi]} \circ i_\phi + i_\phi^{k-3} \circ i_\phi \circ i_{[\phi,\phi]}$$
$$= -i_\phi^{k-3} \circ (i_{[\phi,\phi]} \circ i_\phi - i_\phi \circ i_{[\phi,\phi]})$$
$$= 0$$

since $i_{[\phi,\phi]} i_\phi - i_\phi i_{[\phi,\phi]} = 0$. (Alternatively, we can also approach this equality directly by induction on the term $G_k - i_\phi \circ G_{k-1}$, i.e., $0 = G_{k-1} - i_\phi \circ G_{k-2} = -i_{[\phi,\phi]} \circ i_\phi^{k-3} + i_\phi^{k-3} \circ i_{[\phi,\phi]}$.) The proof of (3.8) is finished. From (3.8), it follows that

$$[\nabla', e^{i_\phi}] = e^{i_\phi} \circ [\nabla', i_\phi] - e^{i_\phi} \circ \frac{1}{2} i_{[\phi,\phi]}$$

by comparing degrees. Then, we have

$$\begin{aligned} e^{-i_\phi} \circ \nabla' \circ e^{i_\phi} &= e^{-i_\phi} \circ [\nabla', e^{i_\phi}] + \nabla' \\ &= [\nabla', i_\phi] + \nabla' - i_{\frac{1}{2}[\phi,\phi]} \\ &= \nabla' - \mathcal{L}_\phi^{1,0} - i_{\frac{1}{2}[\phi,\phi]}. \end{aligned}$$

Now we finish the proof of (3.6) while the proof of (3.5) is similar.

Finally, when $\overline{\partial}\phi = \frac{1}{2}[\phi,\phi]$, we have $[2\overline{\partial} - \mathcal{L}_\phi, i_\phi] = 0$ and thus

$$[2\overline{\partial} - \mathcal{L}_\phi, e^{i_\phi}] = 0,$$

which implies that

$$e^{-i_\phi} \circ (\overline{\partial} - \mathcal{L}_\phi) \circ e^{i_\phi} = 2\overline{\partial} - \mathcal{L}_\phi - e^{-i_\phi} \circ \overline{\partial} \circ e^{i_\phi} = \overline{\partial} - \mathcal{L}_\phi^{1,0}.$$

□

Corollary 3.5 *If $\sigma \in A^{n,\bullet}(X, E)$, we have*

$$\begin{aligned} \left(e^{-i_\phi} \circ \nabla \circ e^{i_\phi}\right)(\sigma) &= \overline{\partial}\sigma - \mathcal{L}_\phi^{1,0}(\sigma) + i_{\overline{\partial}\phi - \frac{1}{2}[\phi,\phi]}(\sigma) \\ &= \overline{\partial}\sigma + \nabla'(\phi \lrcorner \sigma) + \left(\overline{\partial}\phi - \frac{1}{2}[\phi,\phi]\right) \lrcorner \sigma. \end{aligned}$$

In particular, if ϕ is integrable, i.e., $\overline{\partial}\phi - \frac{1}{2}[\phi,\phi] = 0$, then

$$\left(e^{-i_\phi} \circ \nabla \circ e^{i_\phi}\right)(\sigma) = \overline{\partial}\sigma + \nabla'(\phi \lrcorner \sigma).$$

The above formula gives an explicit recursive formula to construct deformed cohomology classes for deformation of Kähler manifolds. When E is a trivial bundle, the above formula was used in [5] to study the global Torelli theorem.

4 Global canonical family of Beltrami differentials

In this section, based on the techniques developed in Sects. 2 and 3, we shall construct the following globally convergent power series of Beltrami differentials in L^2-norm on CY manifolds. To avoid the bewildering notations, we just present the details on the one-parameter case and then give a sketch of the multi-parameter case.

The convergence of the power series in the following lemma is crucial in our proof of the global convergence and regularity results.

Lemma 4.1 *Let* $\{x_i\}_{i=1}^{+\infty}$ *be a series given by*

$$x_k := c\sum_{i=1}^{k-1} x_i \cdot x_{k-i}, \quad k \geq 2$$

inductively with real initial value x_1. *Then the power series* $S(\tau) = \sum_{i=1}^{\infty} x_i \tau^i$ *converges as long as* $|\tau| \leq \frac{1}{|4cx_1|}$.

Proof Setting $S := S(\tau) = \sum_{i=1}^{\infty} x_i \tau^i$, we have

$$cS^2 = c\left(\sum_{i=1}^{\infty} x_i \tau^i\right)\left(\sum_{j=1}^{\infty} x_j \tau^j\right) = \sum_{k=1}^{+\infty} x_k \tau^k - x_1\tau = S - x_1\tau. \tag{4.1}$$

It follows from (4.1) that

$$S = \frac{1 \pm \sqrt{1 - 4cx_1\tau}}{2c}.$$

Here we take $S(\tau) = \frac{1-\sqrt{1-4cx_1\tau}}{2c}$, since we have $S(0) = 0$ according to the assumption. Therefore, we have the following expansion for S

$$\begin{aligned} S &= \frac{1}{2c}\left(1 - \left(1 + \sum_{n\geq 1} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4cx_1\tau)^n\right)\right) \\ &= \sum_{n\geq 1} \frac{1}{2c}\left(\frac{\frac{1}{2}(1-\frac{1}{2})\cdots((n-1)-\frac{1}{2})}{n!}\right)(4cx_1)^n\tau^n, \end{aligned}$$

which implies that

$$x_n = \frac{\frac{1}{2}\left(1-\frac{1}{2}\right)\cdots\left((n-1)-\frac{1}{2}\right)}{2cn!}(4cx_1)^n, \quad \text{for } n \geq 2.$$

This is the explicit expression for each x_n. Now it is easy to check that the convergence radius of the power series $S = \sum_{i=1}^{\infty} x_i \tau^i$ is $(4|cx_1|)^{-1}$, and that this power series still converges when $\tau = \pm\frac{1}{4|cx_1|}$. □

Now we prove the global convergence of the Beltrami differential from the Kodaira–Spencer–Kuranishi theory. All sub-indices of the Beltrami differentials are at least 1.

The following result is contained in [12,13], we briefly recall here for the reader's convenience.

Lemma 4.2 *Assume that for* $\varphi_\nu \in A^{0,1}(X, T_X^{1,0})$, $\nu = 2, \ldots, K$,

$$\bar{\partial}\varphi_\nu = \frac{1}{2}\sum_{\alpha+\beta=\nu}[\varphi_\alpha, \varphi_\beta] \ \ \textit{and} \ \ \bar{\partial}\varphi_1 = 0. \tag{4.2}$$

Then one has

$$\bar{\partial}\left(\sum_{\nu+\gamma=K+1}[\varphi_\nu, \varphi_\gamma]\right) = 0.$$

Proof By the definition formula (3.2), one has

$$[\bar{\partial}\varphi, \varphi'] = -[\varphi', \bar{\partial}\varphi]. \tag{4.3}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\left(\sum_{\nu+\gamma=K+1}[\varphi_\nu, \varphi_\gamma]\right) &= \frac{1}{2}\sum_{\nu+\gamma=K+1}\left([\bar{\partial}\varphi_\nu, \varphi_\gamma] - [\varphi_\nu, \bar{\partial}\varphi_\gamma]\right)\\
&= \sum_{\nu+\gamma=K+1}[\bar{\partial}\varphi_\nu, \varphi_\gamma]\\
&= \frac{1}{2}\sum_{\nu+\gamma=K+1}\left[\sum_{\alpha+\beta=\nu}[\varphi_\alpha, \varphi_\beta], \varphi_\gamma\right]\\
&= \frac{1}{2}\sum_{\alpha+\beta+\gamma=K+1}\left[[\varphi_\alpha, \varphi_\beta], \varphi_\gamma\right],
\end{aligned}$$

where the second equality is implied by (4.3) and the third one follows from the assumption (4.2). When $\alpha = \beta = \gamma$, by Jacobi identity one has

$$3\left[\left[\varphi_\alpha, \varphi_\beta\right], \varphi_\gamma\right] = 0.$$

Otherwise, Jacobi identity implies that

$$\big[[\varphi_\alpha, \varphi_\beta], \varphi_\gamma\big] + \big[[\varphi_\beta, \varphi_\gamma], \varphi_\alpha\big] + \big[[\varphi_\gamma, \varphi_\alpha], \varphi_\beta\big] = 0.$$

□

We need some basic estimates. At first, let's recall the following estimate in [8, p.162], for any $\eta_1, \eta_2 \in A^{0,1}(X, T_X^{1,0})$,

$$\left\| \frac{1}{2}\overline{\partial}^* G[\eta_1, \eta_2] \right\|_{\mathscr{C}^1} \leq C_1 \|\eta_1\|_{\mathscr{C}^1} \cdot \|\eta_2\|_{\mathscr{C}^1}, \tag{4.4}$$

where C_1 is a constant independent of η_1, η_2. Next, for any $(n, 0)$-from s on X, we have

$$\|\eta_1 \lrcorner s\|_{L^2} \leq \|\eta_1\|_{\mathscr{C}^0} \cdot \|s\|_{L^2} \leq \|\eta_1\|_{\mathscr{C}^1} \cdot \|s\|_{L^2}.$$

This inequality follows by checking the local inner product by definition. Similarly,

$$\|\eta_1 \lrcorner \eta_2 \lrcorner s\|_{L^2} \leq C_2 \|\eta_1\|_{\mathscr{C}^1} \cdot \|\eta_2\|_{\mathscr{C}^1} \cdot \|s\|_{L^2}, \tag{4.5}$$

where C_2 is independent of η_1, η_2, s.

Theorem 4.3 *Let X be a CY manifold and $\varphi_1 \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ with norm $\|\varphi_1\|_{\mathscr{C}^1} = \frac{1}{4C_1}$. Then for any nontrivial holomorphic $(n, 0)$ form Ω_0 on X, there exits a smooth globally convergent power series for $|t| < 1$,*

$$\Phi(t) = \varphi_1 t^1 + \varphi_2 t^2 + \cdots + \varphi_k t^k + \cdots \in A^{0,1}(X, T_X^{1,0}),$$

which satisfies:

(a) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$*;*
(b) $\overline{\partial}^* \varphi_k = 0$ *for each* $k \geq 1$*;*
(c) $\varphi_k \lrcorner \Omega_0$ *is* ∂*-exact for each* $k \geq 2$*;*
(d) $\|\Phi(t) \lrcorner \Omega_0\|_{L^2} < \infty$ *as long as* $|t| < 1$.

Proof Let us first review the construction of the power series $\Phi(t)$ by induction from [12] and [13]. Suppose that we have constructed φ_k for $2 \leq k \leq j$ such that:

(a) $\overline{\partial}\varphi_k = \frac{1}{2}\sum_{i=1}^{k-1}[\varphi_{k-i}, \varphi_i]$;
(b) $\overline{\partial}^* \varphi_k = 0$;
(c) $\varphi_k \lrcorner \Omega_0$ is ∂-exact and thus $\partial(\varphi_k \lrcorner \Omega_0) = 0$.

Then we need to construct φ_{j+1} such that: a') $\bar\partial\varphi_{j+1} = \frac{1}{2}\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i]$; b') $\bar\partial^*\varphi_{j+1} = 0$; c') $\varphi_{j+1}\lrcorner\Omega_0$ is ∂-exact and thus $\partial(\varphi_{j+1}\lrcorner\Omega_0) = 0$. Actually, it follows from Lemma 3.3 and the assumption (c) that

$$\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i]\lrcorner\Omega_0 = -\partial\left(\sum_{i+k=j+1}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right).$$

Then, Lemma 4.2 and the assumption (a) imply

$$\bar\partial\partial\left(\sum_{i+k=j+1}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right) = \bar\partial\left(\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i]\right)\lrcorner\Omega_0 = 0. \tag{4.6}$$

So the formula (4.6) and Proposition 1.2 tell us that the equation

$$\bar\partial\Psi_{j+1} = -\partial\left(\sum_{i+k=j+1}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right)$$

has a solution $\Psi_{j+1} = -\bar\partial^*\mathbb{G}\partial\left(\sum_{i+k=j+1}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right)$. Hence, we define

$$\varphi_{j+1} = \frac{1}{2}\Psi_{j+1}\lrcorner\Omega_0^*,$$

where $\Omega_0^* := \frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}$ in local coordinates is the dual of Ω_0. It is easy to check that

$$\bar\partial^*(\Psi_{j+1}\lrcorner\Omega_0^*) = \bar\partial^*(\Psi_{j+1})\lrcorner\Omega_0^* + \Psi_{j+1}\lrcorner\bar\partial^*\Omega_0^* = 0,$$

since Ω_0 is parallel, and also $\bar\partial\varphi_{j+1} = \frac{1}{2}\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i]$. See [13, Lemma 1.2.2] for more details. Now we have completed the construction of $\varphi_{j+1} = \frac{1}{2}\Psi_{j+1}\lrcorner\Omega_0^*$, which is shown to satisfy Properties a'), b') and c'). To complete this induction, it suffices to work out the case $j = 2$. It is obvious that φ_2 can be constructed as

$$\varphi_2 = \frac{1}{2}\bar\partial^*\mathbb{G}\partial(\varphi_1\lrcorner\varphi_1\lrcorner\Omega_0)\lrcorner\Omega_0^*,$$

which satisfies (a), (b) and (c). Moreover, one has the following equality for each $k \geq 2$,

$$\varphi_k \lrcorner \Omega_0 = \frac{1}{2}\bar{\partial}^* \mathbb{G} \partial \sum_{i+j=k\geq 2} \varphi_i \lrcorner \varphi_j \lrcorner \Omega_0.$$

Next, let us prove the L^2-convergence and regularity of $\Phi(t)$. Without loss of generality we can assume $\|\Omega_0\|_{L^2} = 1$ and thus have for $|t| < 1$,

$$\begin{aligned}
\|\Phi(t)\lrcorner\Omega_0\|_{L^2} &= \left\|(\varphi_1\lrcorner\Omega_0)t + (\varphi_2\lrcorner\Omega_0)t^2 + \cdots + (\varphi_k\lrcorner\Omega_0)t^k + \cdots\right\|_{L^2} \\
&= \left\|(\varphi_1\lrcorner\Omega_0)t + \sum_{j=2}^{\infty} \frac{1}{2}\bar{\partial}^* \mathbb{G}\partial \left(\sum_{i+k=j} \varphi_i \lrcorner \varphi_k \lrcorner \Omega_0\right) t^j\right\|_{L^2} \\
\text{(Theorem 1.1)} &\leq \frac{1}{4C_1}|t| + \sum_{j=2}^{\infty} \frac{1}{2} \left(\sum_{i+k=j} \|\varphi_i \lrcorner \varphi_k \lrcorner \Omega_0\|_{L^2}\right) |t|^j \\
\text{(Using (4.5))} &\leq \frac{1}{4C_1}|t| + \sum_{j=2}^{\infty} \frac{C_2}{2} \sum_{i+k=j} \left(\|\varphi_i\|_{\mathscr{C}^1} \cdot \|\varphi_k\|_{\mathscr{C}^1} \cdot \|\Omega_0\|_{L^2}\right) |t|^j \\
&\leq \frac{1}{4C_1}|t| + \sum_{j=2}^{\infty} \frac{C_2}{2} \sum_{i+k=j} \left(\|\varphi_i\|_{\mathscr{C}^1} \cdot \|\varphi_k\|_{\mathscr{C}^1}\right) |t|^j.
\end{aligned}$$

Now we set a sequence $\{x_j\}$ as in Lemma 4.1:

$$x_1 = \frac{1}{4C_1}, \quad \text{and} \quad x_j := C_1 \sum_{i+k=j} x_i \cdot x_k, \quad \text{for} \quad j \geq 2.$$

Therefore by Lemma 4.1, $\sum_{j=1}^{\infty} x_j t^j$ has convergent radius

$$\frac{1}{4C_1|x_1|} = 1.$$

Next, we claim

$$\|\varphi_j\|_{\mathscr{C}^1} \leq x_j \quad \text{for} \quad j = 1, 2, \ldots. \tag{4.7}$$

By assuming (4.7), we have

$$\begin{aligned}
\|\Phi(t)\lrcorner\Omega_0\|_{L^2} &\leq \frac{1}{4C_1}|t| + \sum_{j=2}^{\infty} \frac{C_2}{2} \sum_{i+k=j} \left(\|\varphi_i\|_{\mathscr{C}^1} \ldots \|\varphi_k\|_{\mathscr{C}^1}\right) |t|^j \\
&\leq \frac{1}{4C_1}|t| + \sum_{j=2}^{\infty} \frac{C_2}{2} \sum_{i+k=j} (x_i \cdot x_k) |t|^j
\end{aligned}$$

$$\leq \frac{1}{4C_1}|t| + \frac{C_2}{2C_1}\sum_{j=2}^{\infty} x_j |t|^j$$
$$\leq \frac{1}{4C_1}|t| - \frac{C_2}{8C_1^2}|t| + \frac{C_2}{2C_1}\sum_{j=1}^{\infty} x_j |t|^j < \infty$$

for $|t| < 1$ by Lemma 4.1. In the following we shall prove (4.7) by induction. From the iteration relation,

$$\overline{\partial}\varphi_k = \frac{1}{2}\sum_{i=1}^{k-1}[\varphi_{k-i}, \varphi_i],$$

we see $\overline{\partial}\varphi_2 = \frac{1}{2}[\varphi_1, \varphi_1]$, or equivalently,

$$\varphi_2 = \frac{1}{2}\overline{\partial}^* G[\varphi_1, \varphi_1].$$

Hence, by (4.4), we get

$$\|\varphi_2\|_{\mathscr{C}^1} \leq C_1\|\varphi_1\|_{\mathscr{C}^1} \cdot \|\varphi_1\|_{\mathscr{C}^1} \leq C_1 x_1 \cdot x_1 = x_2$$

since $x_1 = \|\varphi_1\|_{\mathscr{C}^1}$. By induction, we assume

$$\|\varphi_j\|_{\mathscr{C}^1} \leq x_j \quad \text{for} \quad j = 1, \dots, k-1.$$

and we shall prove $\|\varphi_k\|_{\mathscr{C}^1} \leq x_k$. In fact, we have

$$\varphi_k = \frac{1}{2}\overline{\partial}^* G\left(\sum_{i=1}^{k-1}[\phi_{k-i}, \varphi_i]\right),$$

and so by (4.4) and induction conditions,

$$\|\varphi_k\|_{\mathscr{C}^1} \leq C_1\sum_{i=1}^{k-1}\|\varphi_{k-i}\|_{\mathscr{C}^1} \cdot \|\varphi_i\|_{\mathscr{C}^1}$$
$$\leq C_1\sum_{i=1}^{k-1} x_{k-i} \cdot x_i = x_k.$$

Hence, we complete the proof of (4.7).

For local regularity of $\Phi(t)$ (i.e., t sufficiently small) it follows from standard elliptic operator theory (e.g. [8]). But for global regularity($|t| < 1$), their proof does not work directly. Here we use a different approach to prove it. At first, we see that $\Phi(t) \lrcorner \Omega_0$ is ∂-closed in the distribution sense, i.e.

$$\partial(\Phi(t) \lrcorner \Omega_0) = 0, \quad \text{in the distribution sense} \tag{4.8}$$

by using the definition of Φ and the fact that $\varphi_k \lrcorner \Omega_0$ are all ∂-exact for $k \geq 2$, $\varphi_1 \lrcorner \Omega_0$ is harmonic. In fact, for any test form η on X,

$$(\Phi(t) \lrcorner \Omega_0, \partial^* \eta) = \lim_{k \infty} \left(\left(\sum_{i=1}^{k} \varphi_i t^i \right) \lrcorner \Omega_0, \partial^* \eta \right) = \lim_{k \infty} \left(\sum_{i=1}^{k} \partial(\varphi_i \lrcorner \Omega_0), \eta \right) = 0.$$

Since $e^{\Phi(t)} \lrcorner \Omega_0$ is a family of $(n, 0)$ forms on X_t, by Corollary 3.5 (for more complete argument, see Proposition 5.1), we obtain

$$\overline{\partial}_t \left(e^{\Phi(t)} \lrcorner \Omega_0 \right) = 0 \quad \text{in the distribution sense,}$$

where $\overline{\partial}_t$ is the $(0, 1)$-part of the differential operator d on X_t induced by the complex structure $J_{\Phi(t)}$. Therefore, by the hypoellipticity of $\overline{\partial}_t$ on $(n, 0)$ forms, we obtain $e^{\Phi(t)} \lrcorner \Omega_0$ is a holomorphic $(n, 0)$ form on X_t and so $e^{\Phi(t)} \lrcorner \Omega_0$ is smooth on X_t and so on X. Finally, by contracting Ω_0^* as above, we obtain that $e^{\Phi(t)}$ is smooth on X, and so is $\Phi(t)$. □

Now we state the following multi-parameter result, while we just sketch its proof since it is essentially the same as the one-parameter case.

Theorem 4.4 *Let X be a CY manifold and $\{\varphi_1, \ldots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ be a basis with norm $\|\varphi_i\|_{\mathscr{C}^1} = \frac{1}{8NC_1}$. Then for any nontrivial holomorphic $(n, 0)$ form Ω_0 on X, and $|t| < 1$, we can construct a smooth power series of Beltrami differentials on X as follows*

$$\Phi(t) = \sum_{|I| \geq 1} \varphi_I t^I = \sum_{\substack{\nu_1 + \cdots + \nu_N \geq 1, \\ \text{each } \nu_i \geq 0, i=1,2,\ldots}} \varphi_{\nu_1 \cdots \nu_N} t_1^{\nu_1} \cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0}),$$

where $\varphi_{0 \cdots \nu_i \cdots 0} = \varphi_i$. This power series has the following properties:

(a) $\overline{\partial} \Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$, *the integrability condition;*
(b) $\overline{\partial}^* \varphi_I = 0$ *for each multi-index I with $|I| \geq 1$;*
(c) $\varphi_I \lrcorner \Omega_0$ *is ∂-exact for each I with $|I| \geq 2$. And more importantly,*
(d) *global convergence: $\|\Phi(t) \lrcorner \Omega_0\| \leq \sum_I \|\varphi_I \lrcorner \Omega_0\| \cdot |t|^{|I|} < \infty$ as long as $|t| < 1$.*

Proof Let us construct the power series $\Phi(t)$ in multi-parameters by induction. Write

$$\mathcal{B}_{\substack{\geq \\ = \\ <} K} = \{\varphi_{\nu_1\cdots\nu_N} \in A^{0,1}(M, T_M^{1,0}) \mid \text{each integer } \nu_i \geq 0 \text{ and } \nu_1+\cdots+\nu_N \substack{\geq \\ = \\ <} K,\ K \geq 1\}.$$

It is easy to see that $\Phi(t)$ should satisfy:

(a) $\overline{\partial}\varphi_{\nu_1\cdots\nu_N} = \frac{1}{2}\sum_{\alpha_i+\beta_i=\nu_i}\left[\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}\right]$ for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2}$;
(b) $\overline{\partial}^*\varphi_{\nu_1\cdots\nu_N} = 0$ for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 1}$;
(c) $\varphi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0$ is ∂-exact and thus $\partial(\varphi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0) = 0$ for each $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2}$.

Assuming that the above three assumptions hold for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2} \cap \mathcal{B}_{\leq K}$, then one can construct $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{K+1}$ such that it also satisfies these three assumptions. In fact, Lemma 3.3 and the assumption c) for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2} \cap \mathcal{B}_{\leq K}$ imply that

$$[\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}] \lrcorner \Omega_0 = -\partial(\varphi_{\alpha_1\cdots\alpha_N} \lrcorner \varphi_{\beta_1\cdots\beta_N} \lrcorner \Omega_0),$$

where $\sum_i \alpha_i + \sum_j \beta_j = K + 1$. Then, by multi-index Lemma 4.2 and the assumption (a) for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2} \cap \mathcal{B}_{\leq K}$, we have

$$\overline{\partial}\partial\left(\sum_{\alpha_i+\beta_i=\nu_i} \varphi_{\alpha_1\cdots\alpha_N} \lrcorner \varphi_{\beta_1\cdots\beta_N} \lrcorner \Omega_0\right) = \overline{\partial}\left(\sum_{\alpha_i+\beta_i=\nu_i} [\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}]\right) \lrcorner \Omega_0 = 0, \quad (4.9)$$

for any $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{K+1}$. Therefore, one can construct $\Psi_{\nu_1\cdots\nu_N}$ directly by $\overline{\partial}$-inverse formula 2.3 and (4.9) as

$$\Psi_{\nu_1\cdots\nu_N} = -\overline{\partial}^*\mathbb{G}\partial\left(\sum_{\alpha_i+\beta_i=\nu_i} \varphi_{\alpha_1\cdots\alpha_N} \lrcorner \varphi_{\beta_1\cdots\beta_N} \lrcorner \Omega_0\right).$$

Hence we define

$$\varphi_{\nu_1\cdots\nu_N} = \frac{1}{2}\Psi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0^* \in \mathcal{B}_{K+1},$$

where $\Omega_0^* := \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}$ is the dual of Ω_0. Then it is easy to check that

$$\overline{\partial}^*(\Psi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0^*) = \overline{\partial}^*(\Psi_{\nu_1\cdots\nu_N}) \lrcorner \Omega_0^* + \Psi_{\nu_1\cdots\nu_N} \lrcorner \overline{\partial}^*\Omega_0^* = 0$$

since Ω_0 is parallel, and also $\overline{\partial}\varphi_{\nu_1\cdots\nu_N} = \frac{1}{2}\sum_{\alpha_i+\beta_i=\nu_i}[\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}]$. To complete this induction, we construct $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_2$ as

$$\varphi_{\nu_1\cdots\nu_N} = \begin{cases} -\overline{\partial}^*\mathbb{G}\partial(\varphi_i\lrcorner\varphi_j\lrcorner\Omega_0)\lrcorner\Omega_0^*, & \text{if } \nu_i = \nu_j = 1, i \neq j, \\ -\frac{1}{2}\overline{\partial}^*\mathbb{G}\partial(\varphi_i\lrcorner\varphi_i\lrcorner\Omega_0)\lrcorner\Omega_0^*, & \text{if } \nu_i = 2, \text{ for some} i \in \{1, \ldots, N\}, \end{cases}$$

which obviously satisfies (a), (b) and (c).

Up to now we have completed the construction of the power series $\Phi(t)$ satisfying (a), (b) and (c) as in Theorem 4.3. By using similar arguments as in the proof of Theorem 4.3, we get the global convergence in L^2-norm and also the smoothness of $\Phi(t)$. □

5 Global canonical family of holomorphic $(n, 0)$-forms

Based on the construction of L^2-global canonical family $\Phi(t)$ of Beltrami differentials in Theorem 4.4, we can construct an L^2-global canonical family of holomorphic $(n, 0)$-forms on the deformation spaces of CY manifolds. By using a similar method, we can also construct L^2-global canonical family of holomorphic $(n, 0)$-forms on the deformation spaces of compact Kähler manifolds.

5.1 Global canonical family on Calabi–Yau manifolds

Let X be an n-dimensional compact Calabi–Yau manifold and $\{\varphi_1, \ldots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ a basis where $N = \dim \mathbb{H}^{0,1}(X, T_X^{1,0})$. As constructed in Theorem 4.4, there exists a smooth family of Beltrami differentials in the following form

$$\Phi(t) = \sum_{i=1}^{N}\varphi_i t_i + \sum_{|I|\geq 2}\varphi_I t^I = \sum_{\nu_1+\cdots+\nu_N\geq 1}\varphi_{\nu_1\cdots\nu_N}t_1^{\nu_1}\cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0})$$

for $t \in \mathbb{C}^N$ with $|t| < 1$. It is easy to check that the map

$$e^{\Phi(t)}\lrcorner : A^{n,0}(X) \to A^{n,0}(X_t)$$

is a well-defined linear isomorphism.

Proposition 5.1 *For any smooth $(n, 0)$-form $\Omega \in A^{n,0}(X)$, the section $e^{\Phi(t)}\lrcorner\Omega \in A^{n,0}(X_t)$ is holomorphic with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t if and only if*

$$\bar{\partial}\Omega + \partial(\Phi(t)\lrcorner\Omega) = 0.$$

Proof This is a direct consequence of Corollary 3.5. In fact,

$$\left(e^{-i_\Phi} \circ d \circ e^{i_\Phi}\right)(\Omega) = \bar{\partial}\Omega + \partial(\Phi\lrcorner\Omega),$$

if the vector bundle E is trivial and $\Phi(t)$ satisfies the integrability condition. The operator d, which is independent of the complex structures, can be decomposed as $d = \bar{\partial}_t + \partial_t$, where $\bar{\partial}_t$ and ∂_t denote the $(0,1)$-part and $(1,0)$-part of d, with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t. Note that $e^{\Phi(t)}\lrcorner\Omega \in A^{n,0}(X_t)$ and so

$$\partial_t(e^{i_\Phi}(\Omega)) = \partial_t(e^{\Phi(t)}\lrcorner\Omega) = 0.$$

Hence,

$$\left(e^{-i_\Phi} \circ \bar{\partial}_t \circ e^{i_\Phi}\right)(\Omega) = \bar{\partial}\Omega + \partial(\Phi\lrcorner\Omega),$$

which implies the assertion. (In case $\Phi(t)$ is just L^2-integrable, we also see from this formula that $\bar{\partial}_t(e^{\Phi(t)}\lrcorner\Omega) = 0$ in the distribution sense if $\bar{\partial}\Omega + \partial(\Phi\lrcorner\Omega) = 0$ in the distribution sense, and so by hypoellipticity of $\bar{\partial}_t$ on $(n,0)$-forms of X_t, we know $e^{\Phi(t)}\lrcorner\Omega$ is, in fact, a holomorphic $(n,0)$-form on X_t.) □

Theorem 5.2 *Let Ω_0 be a nontrivial holomorphic $(n,0)$-form on the CY manifold X and $X_t = (X_t, J_{\Phi(t)})$ be the deformation of the CY manifold X induced by the L^2-global canonical family $\Phi(t)$ of Beltrami differentials on X as constructed in Theorem 4.4. Then, for $|t| < 1$,*

$$\Omega_t^C := e^{\Phi(t)}\lrcorner\Omega_0$$

defines an L^2-global canonical family of holomorphic $(n,0)$-forms on X_t and depends on t holomorphically.

Proof Since Ω_0 is holomorphic, and $\Phi(t)$ is smooth, by (4.8), we obtain

$$\bar{\partial}\Omega_0 + \partial(\Phi(t)\lrcorner\Omega_0) = 0.$$

Hence, by Proposition 5.1 and Theorem 4.4, $\Omega_t^C = e^{\Phi(t)}\lrcorner\Omega_0$ defines an L^2-global canonical family of holomorphic $(n,0)$-forms on X_t for $|t| < 1$. The holomorphic dependence of $\Phi(t)$ on t implies that Ω_t^C depends on t holomorphically. □

Corollary 5.3 *Let $\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_0$ be the L^2-global canonical family of holomorphic $(n,0)$-forms as constructed in Theorem 5.2. Then for $|t| < 1$, there holds the following global expansion of $[\Omega_t^C]$ in cohomology classes,*

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega_0] t_i + O(|t|^2).$$

where $O(|t|^2)$ denotes the terms in $\bigoplus_{j=2}^{n} H^{n-j,j}(X)$ of orders at least 2 in t.

Proof From Theorem 5.2 and Hodge theory we can see that for $|t| < 1$,

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\mathbb{H}(\varphi_i \lrcorner \Omega_0)] t_i + \sum_{|I| \geq 2} [\mathbb{H}(\varphi_I \lrcorner \Omega_0)] t^I + \sum_{k \geq 2} \frac{1}{k!} \left[\mathbb{H} \left(\bigwedge^k \Phi(t) \lrcorner \Omega_0 \right) \right].$$

By Theorem 4.4, $\varphi_i \lrcorner \Omega_0$ is harmonic and that $\varphi_I \lrcorner \Omega_0$ is ∂-exact for each $|I| \geq 2$. Hence

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega_0] t_i + O(|t|^2)$$

where $O(|t|^2)$ denotes the term $\sum_{k \geq 2} \frac{1}{k!} \left[\mathbb{H} \left(\bigwedge^k \Phi(t) \lrcorner \Omega_0 \right) \right] \in \bigoplus_{j=2}^{n} H^{n-j,j}(X)$. □

5.2 Iteration procedure on deformation spaces of compact Kähler manifolds

In this subsection, we extend our constructions to the deformation spaces of compact Kähler manifolds. We shall use iteration procedure to construct holomorphic sections of the canonical line bundle K_{X_t} of the deformation X_t of a Kähler manifold X induced by the Beltrami differential $\Phi(t)$ satisfying the integrability condition. More precisely, our goal is to find a convergent power series for any holomorphic section $\Omega_0 \in H^0(X, K_X)$,

$$\Omega_t = \Omega_0 + \sum_{|I| \geq 1} t^I \Omega_I$$

such that $e^{\Phi(t)} \lrcorner \Omega_t \in H^0(X_t, K_{X_t})$ is holomorphic with respect to the induced complex structure $J_{\Phi(t)}$ by $\Phi(t)$.

Let X be an n-dimensional compact Kähler manifold and $\{\varphi_1, \ldots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ a basis with the norm $\|\varphi_i\| = C_N$, for each $i = 1, 2, \ldots$ where $N = \dim \mathbb{H}^{0,1}(X, T_X^{1,0})$. In general, on deformation spaces of compact Kähler manifolds, we can not construct Beltrami differentials $\Phi(t)$ as stated in Theorem 4.3 or Theorem 4.4, where we essentially use the non-where vanishing property of Ω_0 on Calabi–Yau manifolds. Hence, it is natural to make the following definition.

Definition 5.4 A power series of Beltrami differentials of the following form

$$\Phi(t) = \sum_{i=1}^{N} \varphi_i t_i + \sum_{|I|\geq 2} \varphi_I t^I = \sum_{\nu_1+\cdots+\nu_N\geq 1} \varphi_{\nu_1\cdots\nu_N} t_1^{\nu_1} \cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0})$$

with $\varphi_{0\cdots\nu_i\cdots 0} = \varphi_i$, is called an L^2-global canonical family of Beltrami differentials on the Kähler manifold X if it satisfies:

(1) the integrability condition: $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$;
(2) global convergence in the sense that

$$\|\Phi(t)\lrcorner\Omega_0\|_{L^2} \leq \sum_{|I|\geq 1} \|\varphi_I\|\|\Omega_0\| \cdot t^{|I|} < \infty$$

as long as $t \in \mathbb{C}^N$ with $|t| < R$, where the convergence radius R is a constant only depending on C_N and Ω_0 is a non-zero holomorphic $(n, 0)$-form.

As an analogue to Theorem 5.2 on deformation spaces of CY manifolds, we have the following result on deformation spaces of compact Kähler manifolds:

Theorem 5.5 *If there exists an L^2-global canonical family $\Phi(t)$ of Beltrami differentials on the Kähler manifold X with convergence radius R, and let $X_t = (X_t, J_{\Phi(t)})$ be the deformation of X induced by $\Phi(t)$, then for any holomorphic $(n, 0)$-form Ω, we can construct a smooth power series*

$$\Omega_t = \Omega_0 + \sum_{|I|\geq 1}^{\infty} \Omega_I t^I \in A^{n,0}(X) \tag{5.1}$$

such that $\Omega_0 = \Omega$ with the following properties: a) $\Omega_t^C := e^{\Phi(t)}\lrcorner\Omega_t \in H^0(X_t, K_{X_t})$ is holomorphic with respect to $J_{\Phi(t)}$; b) $\Omega_I \in A^{n,0}(X)$ is ∂-exact and also $\overline{\partial}^*$-exact for all $|I| \geq 1$.

Proof By the proof of Proposition 5.1, we see it also holds on compact Kähler manifold X. Hence by Proposition 5.1, we know that Ω_t must satisfy the equation

$$\overline{\partial}\Omega_t = -\partial(\Phi(t) \lrcorner \Omega_t). \tag{5.2}$$

By comparing the coefficients of $t_1^{\nu_1} \cdots t_N^{\nu_N}$ of both sides of (5.2), one knows that Eq. (5.2) is equivalent to

$$\begin{cases} \overline{\partial}\Omega_0 = 0, \\ \overline{\partial}\Omega_{\nu_1\cdots\nu_N} = -\partial\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0} \varphi_{\alpha_1\cdots\alpha_N} \lrcorner \Omega_{\beta_1\cdots\beta_N}\right), \end{cases} \tag{5.3}$$

where each $\nu_i \geq 0$ and $\Sigma\nu_i \geq 1$.

We first prove that the Eq. (5.3) has a ∂-exact solution by induction. Set

$$\eta_{\nu_1\cdots\nu_N} = -\partial\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0} \varphi_{\alpha_1\cdots\alpha_N} \lrcorner \Omega_{\beta_1\cdots\beta_N}\right),$$

which is clearly ∂-exact and thus $\mathbb{H}_{\overline{\partial}}(\eta) = 0$ by the Kähler identity $\Box_\partial = \Box_{\overline{\partial}}$. So by $\overline{\partial}$-inverse Lemma 2.3 it suffices to show that $\overline{\partial}\eta_{\nu_1\cdots\nu_N} = 0$.

For the initial case $\Sigma\nu_i = 1$, one has

$$\overline{\partial}\eta_{\nu_1\cdots\nu_N} = -\overline{\partial}\partial(\varphi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0) = \partial(\overline{\partial}\varphi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0 + \varphi_{\nu_1\cdots\nu_N} \lrcorner \overline{\partial}\Omega_0) = 0$$

since $\overline{\partial}\varphi_{\nu_1\cdots\nu_N} = 0$ and $\overline{\partial}\Omega_0 = 0$. Thus we have

$$\Omega_{\nu_1\cdots\nu_N} = \overline{\partial}^*\mathbb{G}\eta_{\nu_1\cdots\nu_N} = -\overline{\partial}^*\partial\mathbb{G}(\varphi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0) = \partial\overline{\partial}^*\mathbb{G}(\varphi_{\nu_1\cdots\nu_N} \lrcorner \Omega_0)$$

by $\overline{\partial}$-inverse Lemma 2.3 and Kähler identity.

Supposing that the $(n, 0)$-forms $\Omega_{\nu_1\cdots\nu_N}$ with $\Sigma\nu_i = K$ are constructed, we can also prove

$$\overline{\partial}\eta_{\nu_1\cdots\nu_N} = 0$$

for $\Sigma\nu_i = K + 1$ by induction and the commutator formula Lemma 3.3. This calculation is routine and left to the interested readers. Similar to the initial case, we can construct the $(n, 0)$-forms $\Omega_{\nu_1\cdots\nu_N}$ with $\Sigma\nu_i = K + 1$ as

$$\Omega_{\nu_1\cdots\nu_N} = -\overline{\partial}^*\partial\mathbb{G}\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0}\varphi_{\alpha_1\cdots\alpha_N}\lrcorner\Omega_{\beta_1\cdots\beta_N}\right)$$
$$= \partial\overline{\partial}^*\mathbb{G}\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0}\varphi_{\alpha_1\cdots\alpha_N}\lrcorner\Omega_{\beta_1\cdots\beta_N}\right).$$

Hence we have completed the construction of the power series Ω_t of $(n, 0)$-forms.

Finally, let us prove the global convergence of the formal power series. See the related parts in [7,9] By the global convergence of the canonical family of Beltrami differentials, we know that there exists a small constant $\xi > 0$ and a constant $R_1 \in (0, R]$ such that

$$\sum_{|I|=i}\|\varphi_I\|R_1^i \leq \xi$$

for all large $i > 0$. We may assume that this fact holds for all $i > 0$. Then we have the following estimate for each $i > 0$

$$\sum_{|I|=i}\|\Omega_I\| \leq \xi(\xi+1)^{i-1}R_1^{-i}, \tag{5.4}$$

which follows by induction and implies the convergence of power series (5.1) as long as $|t| < R_1$. We set $\|\Omega_0\| = 1$ for convenience. First for the initial case $i = 1$, one has

$$\sum_{|I|=1}\|\Omega_I\| \leq \|\Omega_0\|\sum_{|I|=1}\|\varphi_I\| \leq R_1^{-1}\xi,$$

where the quasi-isometry Theorem 1.1 is applied. Then, we assume that the estimate (5.4) is true for $l = 1, \ldots, i-1$ and try to prove the case $l = i$ as follows.

$$\sum_{|I|=i}\|\Omega_I\| \leq \sum_{\substack{|I|=i,|I_2|\geq 1,\\ I_1+I_2=I}}\|\Omega_{I_1}\|\cdot\|\varphi_{I_2}\|$$
$$\leq \xi R_1^{-1}\xi(\xi+1)^{i-2}R_1^{-(i-1)} + \cdots + \xi R_1^{-i}\xi + \xi R_1^{-i}$$
$$= (\xi R_1^{-i})\xi\frac{1-(\xi+1)^{i-1}}{1-(\xi+1)} + \xi R_1^{-i}$$
$$= \xi(\xi+1)^{i-1}R_1^{-i},$$

where the first inequality is also due to Theorem 1.1. Yet it is easy to check that the convergence domain for $|t|$ of $\sum_{i=1} \xi(\xi+1)^{i-1} R_1^{-i} |t|^i$ is obviously $[0, R_1)$.

The regularity of Ω_t follows by similar arguments as in the proof of Theorem 4.3. This completes the proof of Theorem 5.5. □

As similar as Corollary 5.3, we also obtain a global expansion of the canonical family of $(n, 0)$-forms on the deformation spaces of compact Kähler manifolds in cohomology classes.

Corollary 5.6 *Let $\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_t$ be the L^2-global canonical family of holomorphic $(n, 0)$-forms as constructed in Theorem 5.5. Then for $|t| < R$, there holds the following global expansion of the de Rham cohomology classes of it*

$$[\Omega_t^C] = [\Omega_0] + \sum_{|I|\geq 1} [\mathbb{H}(\varphi_I \lrcorner \Omega_0)] t^I + O(|t|^2),$$

where $O(|t|^2)$ denotes the terms in $\bigoplus_{j=2}^{n} H^{n-j,j}(X)$ of orders at least 2 in t.

Proof The proof is very similar to that of Corollary 5.3.

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\mathbb{H}(\varphi_i \lrcorner \Omega_0)] t_i + \sum_{|I|\geq 2} [\mathbb{H}(\varphi_I \lrcorner \Omega_0)] t^I + \sum_{k\geq 2} \frac{1}{k!} \left[\mathbb{H}\Big(\bigwedge^{k} \Phi(t) \lrcorner \Omega_0 \Big) \right]$$

The difference is that, $\varphi_i \lrcorner \Omega_0$ is not necessarily harmonic, and for $|I| \geq 2$ $\varphi_I \lrcorner \Omega_0$ is not ∂-exact in general. □

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