

## GEOMETRY OF HERMITIAN MANIFOLDS

KE-FENG LIU<sup>\*,†,‡</sup> and XIAO-KUI YANG<sup>\*,§</sup>

*\*Department of Mathematics, University of California  
Los Angeles, CA 90095-1555, USA*

*†Center of Mathematical Science  
Zhejiang University  
Hangzhou 310027, P. R. China*

*‡liu@math.ucla.edu*

*§xkyang@math.ucla.edu*

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On Hermitian manifolds, the second Ricci curvature tensors of various metric connections are closely related to the geometry of Hermitian manifolds. By refining the Bochner formulas for any Hermitian complex vector bundle (and Riemannian real vector bundle) with an arbitrary metric connection over a compact Hermitian manifold, we can derive various vanishing theorems for Hermitian manifolds and complex vector bundles by the second Ricci curvature tensors. We will also introduce a natural geometric flow on Hermitian manifolds by using the second Ricci curvature tensor.

*Keywords:* Chern connection; Levi-Civita connection; Bismut connection; second Ricci curvature; vanishing theorem; geometric flow.

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### 1. Introduction

It is well-known (see [5]) that on a compact Kähler manifold, if the Ricci curvature is positive, then the first Betti number is zero; if the Ricci curvature is negative, then there is no holomorphic vector field. The key ingredient for the proofs of such results is the Kähler symmetry. On the other hand, on a Hermitian manifold, we do not have such symmetry and there are several different Ricci curvatures. While on a Kähler manifold, all these Ricci curvatures coincide, since the Chern curvature on a Kähler manifold coincides with the curvature of the (complexified) Levi-Civita connection. We can see this more clearly on an abstract Hermitian holomorphic bundle  $(E, h)$ . The Chern connection  $\nabla^{CH}$  on  $E$  is the unique connection which is compatible with the holomorphic structure and the Hermitian metric  $h$  on  $E$ . Hence, the Chern curvature  $\Theta^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$ . There are two ways to take trace of  $\Theta^E$ . If we take trace of  $\Theta^E$  with respect to the Hermitian metric  $h$  on  $E$ , we get

a  $(1, 1)$ -form  $\text{Tr}_h \Theta^E \in \Gamma(M, \Lambda^{1,1} T^* M)$  on  $M$  which is called the first Ricci–Chern curvature of  $(E, h)$ . It is well-known that the first Ricci–Chern curvature represents the first Chern class of the bundle. On the other hand, if we take trace on the  $(1, 1)$ -part by using the metric of the manifold, we obtain an endomorphism of  $E$ ,  $\text{Tr}_\omega \Theta^E \in \Gamma(M, E^* \otimes E)$ . It is called the second Ricci–Chern curvature of  $(E, h)$ . The first and second Ricci–Chern curvatures have different geometric meanings, which were not clearly studied in some earlier literatures. We should point out that the nonexistence of holomorphic sections of a Hermitian holomorphic vector bundle  $E$  is characterized by the second Ricci–Chern curvature of  $E$ . Let  $E$  be the holomorphic tangent bundle  $T^{1,0} M$ . If  $M$  is Kähler, the first and second Ricci–Chern curvatures are the same by the Kähler symmetry. Unfortunately, on a Hermitian manifold, the Chern curvature is not symmetric, i.e. the first and second Ricci–Chern curvatures are different. Moreover, in general they cannot be compared. An interesting example is the Hopf manifold  $\mathbb{S}^{2n+1} \times \mathbb{S}^1$ . As is well-known the Hopf manifold is non-Kähler and has vanishing first Chern class. However, the canonical metric on it has strictly positive second Ricci–Chern curvature! Moreover, the first Ricci–Chern curvature is nonnegative and not identically zero, whereas it represents the zero first Chern class! For more details, see Proposition 6.1.

In this paper, we study the non-existence of holomorphic and harmonic sections of an abstract vector bundle over a compact Hermitian manifold. Let  $E$  be a holomorphic vector bundle over a compact Hermitian manifold  $(M, \omega)$ . Since the holomorphic section space  $H^0(M, E)$  is independent on the connections on  $E$ , we can choose any connection on  $E$  to detect  $H^0(M, E)$ . As mentioned above, the key part, is the second Ricci curvature of that given connection. For example, on the holomorphic tangent bundle  $T^{1,0} M$  of a Hermitian manifold  $M$ , there are three typical connections

- (1) the complexified Levi-Civita connection  $\nabla$  on  $T^{1,0} M$ ;
- (2) the Chern connection  $\nabla^{CH}$  on  $T^{1,0} M$ ;
- (3) the Bismut connection  $\nabla^B$  on  $T^{1,0} M$ .

It is well-known that if  $M$  is Kähler, all three connections are the same. However, in general, the relations among them are somewhat mysterious. In this paper, we derive certain relations about their curvatures on certain Hermitian manifolds.

Let  $E$  be a Hermitian *complex* (possibly *non-holomorphic*) vector bundle or a Riemannian *real* vector bundle over a compact Hermitian manifold  $(M, \omega)$ . Let  $\nabla^E$  be an arbitrary connection on  $E$  and  $\partial_E, \bar{\partial}_E$  the  $(1, 0), (0, 1)$  part of  $\nabla^E$ , respectively. The  $(1, 1)$ -curvature of  $\nabla^E$  is denoted by  $R^E \in \Gamma(M, \Lambda^{1,1} T^* M \otimes E^* \otimes E)$ . It can be viewed as a representation of the operator  $\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E$ . We can define harmonic section spaces associated to  $(E, \nabla^E)$  by

$$\mathcal{H}_{\partial_E}^{p,q}(M, E) = \{\varphi \in \Omega^{p,q}(M, E) \mid \bar{\partial}_E \varphi = \partial_E^* \varphi = 0\} \tag{1.1}$$

In general, on a complex vector bundle  $E$ , there is no terminology such as “holomorphic section of  $E$ ”. However, if the vector bundle  $E$  is holomorphic and  $\nabla^E$

is the Chern connection on  $E$ , i.e.  $\bar{\partial}_E = \bar{\partial}$ , then  $\mathcal{H}_{\bar{\partial}_E}^{p,q}(M, E)$  is isomorphic to the Dolbeault cohomology group  $H_{\bar{\partial}}^{p,q}(M, E)$  and  $H_{\bar{\partial}}^0(M, E)$  is the holomorphic section space  $H^0(M, E)$  of  $E$ .

If  $(E, h, \nabla^E)$  is a Hermitian complex vector bundle with a fixed connection  $\nabla^E$  over a compact Hermitian manifold  $(M, \omega)$ , we will call  $\text{Tr}_h R^E \in \Gamma(M, \Lambda^{1,1} T^*M)$  the first Hermitian–Ricci curvature of  $(E, h, \nabla^E)$  and  $\text{Tr}_\omega R^E \in \Gamma(M, E^* \otimes E)$  the second Hermitian–Ricci curvature. If  $\nabla^E$  is the Chern connection of a Hermitian holomorphic vector bundle  $(E, h)$ , they are called the first and second Ricci–Chern curvatures of  $(E, h)$  respectively.

**Theorem 1.1.** *Let  $E$  be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold  $(M, \omega)$  and  $\nabla^E$  be any metric connection on  $E$ .*

- (1) *If the second Hermitian–Ricci curvature  $\text{Tr}_\omega R^E$  is nonpositive everywhere, then every  $\bar{\partial}_E$ -closed section of  $E$  is parallel, i.e.  $\nabla^E s = 0$ ;*
- (2) *If the second Hermitian–Ricci curvature  $\text{Tr}_\omega R^E$  is nonpositive everywhere and negative at some point, then  $\mathcal{H}_{\bar{\partial}_E}^0(M, E) = 0$ ;*
- (3) *If the second Hermitian–Ricci curvature  $\text{Tr}_\omega R^E$  is  $p$ -nonpositive everywhere and  $p$ -negative at some point, then  $\mathcal{H}_{\bar{\partial}_E}^0(M, \Lambda^q E) = 0$  for any  $p \leq q \leq \text{rank}(E)$ .*

The proof of this theorem is based on generalized Bochner–Kodaira identities on vector bundles over Hermitian manifolds (Theorem 4.1). We prove that (Theorem 4.2) the torsion integral of the Hermitian manifold can be killed if the background Hermitian metric  $\omega$  on  $M$  is Gauduchon, i.e.  $\partial\bar{\partial}\omega^{n-1} = 0$ . On the other hand, in the conformal class of any Hermitian metric, the Gauduchon metric always exists (see [24]). So we can change the background metric in the conformal way. It is obvious that the positivity of the second Hermitian–Ricci curvature is preserved under conformal transformations. This method is very useful on Hermitian manifolds. Kobayashi–Wu (see [34]) and Gauduchon (see [22]) obtained similar result in the special case when  $\nabla^E$  is the Chern connection of the Hermitian holomorphic vector bundle  $E$ . Now we go back to the Hermitian manifold  $(M, \omega)$ .

**Corollary 1.1.** *Let  $(M, \omega)$  be a compact Hermitian manifold and  $\Theta$  is the Chern curvature of  $(T^{1,0}M, \omega)$ .*

- (1) *If the second Ricci–Chern curvature  $\text{Tr}_\omega \Theta$  is nonnegative everywhere and positive at some point, then  $H_{\bar{\partial}}^{p,0}(M) = 0$  for any  $1 \leq p \leq n$ . In particular, the arithmetic genus  $\chi(M, \mathcal{O}) = 1$ ;*
- (2) *If the second Ricci–Chern curvature  $\text{Tr}_\omega \Theta$  is nonpositive everywhere and negative at some point, then the holomorphic vector bundle  $\Lambda^p T^{1,0}M$  has no holomorphic vector field for any  $1 \leq p \leq n$ .*

As is well-known, if a Hermitian manifold has positive first Ricci–Chern curvature, it must be Kähler. However, we cannot draw the same conclusion if the

second Ricci–Chern curvature is positive, since the first and second Ricci–Chern curvatures of a Hermitian manifold cannot be compared. In fact, the first Ricci–Chern curvature is  $d$ -closed, but in general the second Ricci–Chern curvature is not  $d$ -closed and they are in the different  $(d, \bar{\partial}, \partial)$ -cohomology classes. For example, the Hopf manifold  $\mathbb{S}^{2n+1} \times \mathbb{S}^1$  with standard Hermitian metric has strictly positive second Ricci–Chern curvature and nonnegative first Ricci–Chern curvature, but it is non-Kähler. For more details, see Proposition 6.1.

Now we consider several special Hermitian manifolds. An interesting class of Hermitian manifolds is the balanced Hermitian manifolds, i.e. Hermitian manifolds with coclosed Kähler forms. It is well-known that every Kähler manifold is balanced. In some literatures, they are also called semi-Kähler manifolds. In complex dimensions 1 and 2, every balanced Hermitian manifold is Kähler. However, in higher dimensions, there exist non-Kähler manifolds which admit balanced Hermitian metrics. Such examples were constructed by Calabi (see [6]), see also [26, 39]. There are also some other important classes of non-Kähler balanced manifolds, such as: complex solvmanifolds, 1-dimensional families of Kähler manifolds (see [39]) and compact complex parallelizable manifolds (except complex torus) (see [48]). On the other hand, Alessandrini–Bassanelli (see [1]) proved that every Moishezon manifold is balanced and so balanced manifolds can be constructed from Kähler manifolds by modification. For more examples, see [2, 16–20, 39].

Every balanced metric  $\omega$  is Gauduchon. In fact,  $d^*\omega = 0$  is equivalent to  $d\omega^{n-1} = 0$  and so  $\partial\bar{\partial}\omega^{n-1} = 0$ . By [24], every Hermitian manifold has a Gauduchon metric. However, there are many manifolds which cannot support balanced metrics. For example, the Hopf surface  $\mathbb{S}^3 \times \mathbb{S}^1$  is non-Kähler, so it has no balanced metric. For more discussions, we refer the reader to [1, 2, 6, 39, 47] and references therein.

On a compact balanced Hermitian manifold  $M$ , we can also detect the holomorphic section spaces  $H^0(M, T^{1,0}M)$  and  $H_{\bar{\partial}}^{p,0}(M)$  by the Levi-Civita connection on  $(M, \omega_h)$ . Let  $\nabla$  be the complexified Levi-Civita connection on  $M$  and  $R$  the complexified Riemannian curvature. It is easy to see that  $R(X, Y, Z, W) = R(Z, W, X, Y)$  for any  $X, Y, Z, W \in \Gamma(M, T_{\mathbb{C}}M)$ . In the local holomorphic coordinates  $(z^1, \dots, z^n)$  of  $M$ , we set

$$R_{i\bar{j}k\bar{\ell}} = R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^{\bar{j}}}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^{\bar{\ell}}}\right), \quad R_{i\bar{j}} = h^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}} (= h^{k\bar{\ell}}R_{k\bar{\ell}i\bar{j}})$$

and call  $(R_{i\bar{j}})$  the Hermitian–Ricci curvature of  $(M, h)$ . Since  $\nabla$  is a connection on the complex vector bundle  $T_{\mathbb{C}}M$ , there is an induced connection on the Hermitian holomorphic vector bundle  $(T^{1,0}M, h)$  and we denote it still by  $\nabla$ . The curvature of  $(T^{1,0}M, h, \nabla)$  is denoted by  $\hat{R}$ . In general, the first and second Hermitian–Ricci curvatures of  $\hat{R}$  are different. Moreover,  $\hat{R}$  and  $R$  are different but they can be compared (see Proposition 2.1). This property can be viewed as a connection between Riemannian geometry and Hermitian geometry (or Symplectic geometry). For example, we can use it to study the non-existence of certain complex structures on complete Riemannian manifolds.

**Theorem 1.2.** *Let  $(M, \omega)$  be a compact balanced Hermitian manifold. Suppose the Hermitian–Ricci curvature  $(R_{i\bar{j}})$  of  $M$  is nonnegative everywhere.*

- (1) *If  $\varphi$  is a holomorphic  $p$ -form, then  $\Delta_{\partial\bar{\partial}}\varphi = 0$  and so  $\dim_{\mathbb{C}} H_{\partial\bar{\partial}}^{p,0}(M) \leq \dim_{\mathbb{C}} H_{\partial\bar{\partial}}^{0,p}(M)$  for any  $1 \leq p \leq n$ ;*
- (2) *If the Hermitian–Ricci curvature  $(R_{i\bar{j}})$  is positive at some point, then  $H_{\partial\bar{\partial}}^{p,0}(M) = 0$  for any  $1 \leq p \leq n$ . In particular, the arithmetic genus  $\chi(M, \mathcal{O}) = 1$ .*

Let  $\hat{R}_{i\bar{j}}^{(2)}$  be the components of the second Hermitian–Ricci curvature of  $\hat{R}$ . The dual version of Theorem 1.2 is

**Theorem 1.3.** *Let  $(M, \omega)$  be a compact balanced Hermitian manifold. If  $2\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}$  is nonpositive everywhere and negative at some point, there is no holomorphic vector field on  $M$ .*

**Remark 1.1.** It is easy to see that the Hermitian–Ricci curvature tensor  $(R_{i\bar{j}})$  and second Ricci–Chern curvature tensor  $\Theta^{(2)} := \text{Tr}_{\omega}\Theta$  cannot be compared. Therefore, Theorem 1.2 and Corollary 1.1 are independent of each other. For the same reason, Theorem 1.3 and Corollary 1.1 are independent. Balanced Hermitian manifolds with nonnegative Hermitian–Ricci curvatures are discussed in Proposition 3.2.

As we discuss in the above, on Hermitian manifolds, the second Hermitian–Ricci curvature tensors of various metric connections are closely related to the geometry of Hermitian manifolds. A natural idea is to define a flow by using second Hermitian–Ricci curvature tensors of various metric connections. For example,

$$\frac{\partial h}{\partial t} = -\Theta^{(2)} + \mu h, \quad \mu \in \mathbb{R} \tag{1.2}$$

on a general Hermitian manifold  $(M, h)$  by using the second Ricci–Chern curvature. This flow preserves the Kähler and the Hermitian structures and has short time solution on any compact Hermitian manifold. It is very similar to and closely related to the Hermitian Yang–Mills flow, the Kähler–Ricci flow and the harmonic map heat flow. It may be a bridge to connect them. In this paper, we only briefly discuss its basic properties. In a subsequent paper (see [35]), we will study its geometric and analytic property in detail.

## 2. Various Connections and Curvatures on Hermitian Manifolds

### 2.1. Complexified Riemannian curvature

Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ , the curvature  $R$  of  $(M, g, \nabla)$  is defined as

$$R(X, Y, Z, W) = g((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}Z), W). \tag{2.1}$$

On a Hermitian manifold  $(M, h)$ , let  $\nabla$  be the complexified Levi-Civita connection and  $g$  the background Riemannian metric. Two metrics  $g$  and  $h$  are related by

$$ds_h^2 = ds_g^2 - \sqrt{-1}\omega_h \tag{2.2}$$

where  $\omega_h$  is the fundamental  $(1, 1)$ -form (or Kähler form) associated to  $h$ . For any two holomorphic vector fields  $X, Y \in \Gamma(M, T^{1,0}M)$ ,

$$h(X, Y) = 2g(X, \bar{Y}). \tag{2.3}$$

*This formula will be used in several definitions.* In the local holomorphic coordinates  $\{z^1, \dots, z^n\}$  on  $M$ , the complexified Christoffel symbols are given by

$$\begin{aligned} \Gamma_{AB}^C &= \sum_E \frac{1}{2}g^{CE} \left( \frac{\partial g_{AE}}{\partial z^B} + \frac{\partial g_{BE}}{\partial z^A} - \frac{\partial g_{AB}}{\partial z^E} \right) \\ &= \sum_E \frac{1}{2}h^{CE} \left( \frac{\partial h_{AE}}{\partial z^B} + \frac{\partial h_{BE}}{\partial z^A} - \frac{\partial h_{AB}}{\partial z^E} \right), \end{aligned} \tag{2.4}$$

where  $A, B, C, E \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$  and  $z^A = z^i$  if  $A = i$ ,  $z^A = \bar{z}^i$  if  $A = \bar{i}$ . For example

$$\Gamma_{ij}^k = \frac{1}{2}h^{k\bar{\ell}} \left( \frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j} \right), \quad \Gamma_{ij}^{\bar{k}} = \frac{1}{2}h^{k\bar{\ell}} \left( \frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^{\bar{\ell}}} \right). \tag{2.5}$$

The complexified curvature components are

$$\begin{aligned} R_{ABCD} &:= 2\mathbf{g} \left( \left( \nabla_{\frac{\partial}{\partial z^A}} \nabla_{\frac{\partial}{\partial z^B}} - \nabla_{\frac{\partial}{\partial z^B}} \nabla_{\frac{\partial}{\partial z^A}} \right) \frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^D} \right) \\ &= \mathbf{h} \left( \left( \nabla_{\frac{\partial}{\partial z^A}} \nabla_{\frac{\partial}{\partial z^B}} - \nabla_{\frac{\partial}{\partial z^B}} \nabla_{\frac{\partial}{\partial z^A}} \right) \frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^D} \right). \end{aligned} \tag{2.6}$$

Hence

$$R_{ABC}^D = \sum_E R_{ABCE}h^{ED} = - \left( \frac{\partial \Gamma_{AC}^D}{\partial z^B} - \frac{\partial \Gamma_{BC}^D}{\partial z^A} + \Gamma_{AC}^F \Gamma_{FB}^D - \Gamma_{BC}^F \Gamma_{AF}^D \right). \tag{2.7}$$

By the Hermitian property, we have, for example

$$R_{ij\bar{k}}^l = - \left( \frac{\partial \Gamma_{ik}^l}{\partial \bar{z}^j} - \frac{\partial \Gamma_{jk}^l}{\partial z^i} + \Gamma_{ik}^s \Gamma_{js}^l - \Gamma_{jk}^s \Gamma_{is}^l - \Gamma_{jk}^{\bar{s}} \Gamma_{i\bar{s}}^l \right). \tag{2.8}$$

**Remark 2.1.** We have  $R_{ABCD} = R_{CDAB}$ . In particular,

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}}. \tag{2.9}$$

Unlike the Kähler case, we can define several Ricci curvatures.

**Definition 2.1.** (1) The *complexified Ricci curvature* on  $(M, h)$  is defined by

$$\mathbb{R}_{k\bar{\ell}} := h^{i\bar{j}}(R_{k\bar{j}i\bar{\ell}} + R_{ki\bar{j}\bar{\ell}}). \tag{2.10}$$

The complexified scalar curvature of  $h$  is defined as

$$s_h := h^{k\bar{\ell}} \mathbb{R}_{k\bar{\ell}}. \tag{2.11}$$

(2) The Hermitian–Ricci curvature is

$$R_{k\bar{\ell}} := h^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}. \tag{2.12}$$

The Hermitian-scalar curvature of  $h$  is given by

$$S := h^{k\bar{\ell}} R_{k\bar{\ell}}. \tag{2.13}$$

**Lemma 2.1.** *On a Hermitian manifold,*

$$\overline{R_{ABCD}} = R_{\overline{ABCD}}, \quad \overline{\mathbb{R}_{k\bar{\ell}}} = \mathbb{R}_{\ell\bar{k}}, \quad \overline{R_{k\bar{\ell}}} = R_{\ell\bar{k}} \tag{2.14}$$

and

$$\mathbb{R}_{k\bar{\ell}} = h^{i\bar{j}} (2R_{k\bar{j}i\bar{\ell}} - R_{k\bar{\ell}i\bar{j}}). \tag{2.15}$$

**Proof.** The Hermitian property of curvature tensors is obvious. By first Bianchi identity, we have

$$R_{ki\bar{j}\bar{\ell}} + R_{k\bar{j}\bar{\ell}i} + R_{k\bar{\ell}i\bar{j}} = 0.$$

That is  $R_{k\bar{i}j\bar{\ell}} = R_{k\bar{j}i\bar{\ell}} - R_{k\bar{\ell}i\bar{j}}$ . The curvature formula (2.10) turns out to be

$$\mathbb{R}_{k\bar{\ell}} = h^{i\bar{j}} (2R_{k\bar{j}i\bar{\ell}} - R_{k\bar{\ell}i\bar{j}}). \quad \square$$

**Definition 2.2.** The Ricci curvatures are called *positive* (respectively, *nonnegative*, *negative*, *nonpositive*) if the corresponding Hermitian matrices are positive (respectively, nonnegative, negative, nonpositive).

The following three formulas are used frequently in the sequel.

**Lemma 2.2.** *Assume  $h_{i\bar{j}} = \delta_{ij}$  at a fixed point  $p \in M$ , we have the following formula*

$$\begin{aligned} R_{i\bar{j}k\bar{\ell}} = & -\frac{1}{2} \left( \frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^\ell} \right) \\ & + \frac{1}{4} \left( \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^j} + \frac{\partial h_{i\bar{q}}}{\partial z^k} \frac{\partial h_{q\bar{j}}}{\partial \bar{z}^\ell} \right) + \frac{1}{4} \left( \frac{\partial h_{i\bar{q}}}{\partial z^k} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^j} + \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{q\bar{j}}}{\partial \bar{z}^\ell} \right) \\ & + \frac{1}{4} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \frac{\partial h_{k\bar{j}}}{\partial \bar{z}^q} + \frac{\partial h_{q\bar{j}}}{\partial z^k} \frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^q} \right) + \frac{1}{4} \left( \frac{\partial h_{i\bar{\ell}}}{\partial z^q} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} + \frac{\partial h_{k\bar{j}}}{\partial z^q} \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell} \right) \\ & - \frac{1}{4} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} + \frac{\partial h_{q\bar{j}}}{\partial z^k} \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell} \right) - \frac{1}{4} \left( \frac{\partial h_{i\bar{\ell}}}{\partial z^q} \frac{\partial h_{k\bar{j}}}{\partial \bar{z}^q} + \frac{\partial h_{k\bar{j}}}{\partial z^q} \frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^q} \right). \end{aligned} \tag{2.16}$$

By a linear transformation on the local holomorphic coordinates, one can get the following Lemma. For more details, we refer the reader to [44].

**Lemma 2.3.** *Let  $(M, h, \omega)$  be a Hermitian manifold. For any  $p \in M$ , there exist local holomorphic coordinates  $\{z^i\}$  centered at a point  $p$  such that*

$$h_{i\bar{j}}(p) = \delta_{ij} \quad \text{and} \quad \Gamma_{ij}^k(p) = 0 \tag{2.17}$$

By Lemma 2.3, we have a simplified version of curvatures:

**Lemma 2.4.** *Assume  $h_{i\bar{j}}(p) = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$  at a fixed point  $p \in M$ ,*

$$R_{i\bar{j}k\bar{\ell}} = -\frac{1}{2} \left( \frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^\ell} \right) - \sum_q \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} + \frac{\partial h_{q\bar{j}}}{\partial z^k} \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^\ell} \right). \tag{2.18}$$

For Hermitian-Ricci curvatures

$$\begin{aligned} R_{k\bar{\ell}} &= h^{i\bar{j}} R_{i\bar{j}k\bar{\ell}} \\ &= -\frac{1}{2} \sum_s \left( \frac{\partial^2 h_{s\bar{\ell}}}{\partial z^k \partial \bar{z}^s} + \frac{\partial^2 h_{k\bar{s}}}{\partial z^s \partial \bar{z}^\ell} \right) - \sum_{q,s} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^s} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{k\bar{q}}}{\partial z^s} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^s} \right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} h^{i\bar{j}} R_{k\bar{j}i\bar{\ell}} &= h^{i\bar{j}} R_{i\bar{\ell}k\bar{j}} \\ &= -\frac{1}{2} \sum_s \left( \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^s \partial \bar{z}^s} + \frac{\partial^2 h_{s\bar{s}}}{\partial z^k \partial \bar{z}^\ell} \right) - \sum_{q,s} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^k} \frac{\partial h_{s\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{q\bar{s}}}{\partial z^s} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^\ell} \right). \end{aligned} \tag{2.20}$$

For complexified Ricci curvature,

$$\begin{aligned} \mathbb{R}_{k\bar{\ell}} &= \frac{1}{2} \sum_s \left( \frac{\partial^2 h_{s\bar{\ell}}}{\partial z^k \partial \bar{z}^s} + \frac{\partial^2 h_{k\bar{s}}}{\partial z^s \partial \bar{z}^\ell} \right) - \sum_s \left( \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^s \partial \bar{z}^s} + \frac{\partial^2 h_{s\bar{s}}}{\partial z^k \partial \bar{z}^\ell} \right) \\ &\quad + \sum_{q,s} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^s} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{k\bar{q}}}{\partial z^s} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^s} \right) - 2 \sum_{q,s} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^k} \frac{\partial h_{s\bar{q}}}{\partial \bar{z}^s} + \frac{\partial h_{q\bar{s}}}{\partial z^s} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^\ell} \right). \end{aligned} \tag{2.21}$$

### 2.2. Curvature of complexified Levi-Civita connection on $T^{1,0}M$

Since  $T^{1,0}M$  is a subbundle of  $T_{\mathbb{C}}M$ , there is an induced connection  $\hat{\nabla}$  on  $T^{1,0}M$  given by

$$\hat{\nabla} = \pi \circ \nabla : T^{1,0}M \xrightarrow{\nabla} \Gamma(M, T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M) \xrightarrow{\pi} \Gamma(M, T_{\mathbb{C}}M \otimes T^{1,0}M). \tag{2.22}$$

The curvature  $\hat{R} \in \Gamma(M, \Lambda^2 T_{\mathbb{C}}M \otimes T^{*1,0}M \otimes T^{1,0}M)$  of  $\hat{\nabla}$  is given by

$$\hat{R}(X, Y)s = \hat{\nabla}_X \hat{\nabla}_Y s - \hat{\nabla}_Y \hat{\nabla}_X s - \hat{\nabla}_{[X, Y]} s \tag{2.23}$$



for any  $X, Y \in T_{\mathbb{C}}M$  and  $s \in T^{1,0}M$ . It has components

$$\hat{R}_{ABk}^l = \frac{\partial \Gamma_{Bk}^l}{\partial z^A} - \frac{\partial \Gamma_{Ak}^l}{\partial z^B} - \Gamma_{Ak}^s \Gamma_{Bs}^l + \Gamma_{Bk}^s \Gamma_{As}^l, \tag{2.24}$$

where

$$\hat{R} \left( \frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B} \right) \frac{\partial}{\partial z^k} = \sum_l \hat{R}_{ABk}^l \frac{\partial}{\partial z^l}. \tag{2.25}$$

For example,

$$\hat{R}_{i\bar{j}k}^l = - \left( \frac{\partial \Gamma_{ik}^l}{\partial \bar{z}^j} - \frac{\partial \Gamma_{\bar{j}k}^l}{\partial z^i} + \Gamma_{ik}^s \Gamma_{\bar{j}s}^l - \Gamma_{\bar{j}k}^s \Gamma_{si}^l \right). \tag{2.26}$$

With respect to the Hermitian metric  $h$  on  $T^{1,0}M$ , we can define

$$\hat{R}_{ABk\bar{l}} = \sum_{s=1}^n \hat{R}_{ABk}^s h_{s\bar{l}}. \tag{2.27}$$

**Definition 2.3.** The *first Hermitian–Ricci curvature* of the Hermitian vector bundle  $(T^{1,0}M, \hat{\nabla})$  is defined by

$$\hat{R}_{i\bar{j}}^{(1)} = h^{k\bar{l}} \hat{R}_{i\bar{j}k\bar{l}}. \tag{2.28}$$

The *second Hermitian–Ricci curvature* of it is

$$\hat{R}_{k\bar{l}}^{(2)} = h^{i\bar{j}} \hat{R}_{i\bar{j}k\bar{l}}. \tag{2.29}$$

The *scalar curvature* of  $\hat{\nabla}$  on  $T^{1,0}M$  is denoted by

$$S^{LC} = h^{i\bar{j}} h^{k\bar{l}} \hat{R}_{i\bar{j}k\bar{l}}. \tag{2.30}$$

By Lemma 2.3, we have the following formulas.

**Lemma 2.5.** On a Hermitian manifold  $(M, h)$ , on a point  $p$  with  $h_{i\bar{j}}(p) = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$ ,

$$\hat{R}_{i\bar{j}k\bar{l}} = -\frac{1}{2} \left( \frac{\partial^2 h_{i\bar{l}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^l} \right) - \sum_q \frac{\partial h_{q\bar{l}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j}. \tag{2.31}$$

For the first and second Hermitian–Ricci curvatures,

$$\hat{R}_{i\bar{j}}^{(1)} = -\frac{1}{2} \sum_k \left( \frac{\partial^2 h_{i\bar{k}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^k} \right) - \sum_{k,q} \frac{\partial h_{q\bar{k}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^j} \tag{2.32}$$

and

$$\hat{R}_{i\bar{j}}^{(2)} = -\frac{1}{2} \sum_k \left( \frac{\partial^2 h_{i\bar{k}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^k} \right) - \sum_{k,q} \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^k} \frac{\partial h_{q\bar{j}}}{\partial z^k}. \tag{2.33}$$

Moreover,

$$\hat{R}_{i\bar{j}}^{(1)} - \hat{R}_{i\bar{j}}^{(2)} = h_{m\bar{j}} h^{\ell\bar{k}} \Gamma_{k\bar{i}}^{\bar{q}} \Gamma_{\ell\bar{q}}^m - \Gamma_{k\bar{j}}^{\bar{q}} \Gamma_{i\bar{q}}^k = \sum_{k,q} \left( \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^k} \frac{\partial h_{q\bar{j}}}{\partial z^k} - \frac{\partial h_{i\bar{q}}}{\partial z^k} \frac{\partial h_{q\bar{j}}}{\partial \bar{z}^k} \right). \quad (2.34)$$

### 2.3. Curvature of Chern connection on $T^{1,0}M$

On the Hermitian holomorphic vector bundle  $(T^{1,0}M, h)$ , the Chern connection  $\nabla^{CH}$  is the unique connection which is compatible with the complex structure and the Hermitian metric. Its curvature components are

$$\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{p\bar{q}} \frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j} \frac{\partial h_{k\bar{q}}}{\partial z^i}. \quad (2.35)$$

It is well-known that the first Ricci-Chern curvature

$$\Theta^{(1)} := \frac{\sqrt{-1}}{2\pi} \Theta_{i\bar{j}}^{(1)} dz^i \wedge d\bar{z}^j \quad (2.36)$$

represents the first Chern class of  $M$  where

$$\Theta_{i\bar{j}}^{(1)} = h^{k\bar{\ell}} \Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log \det(h_{k\bar{\ell}})}{\partial z^i \partial \bar{z}^j}. \quad (2.37)$$

The second Ricci-Chern curvature components are

$$\Theta_{i\bar{j}}^{(2)} = h^{k\bar{\ell}} \Theta_{k\bar{\ell}i\bar{j}}. \quad (2.38)$$

The scalar curvature of the Chern connection is defined by

$$S^{CH} = h^{i\bar{j}} h^{k\bar{\ell}} \Theta_{i\bar{j}k\bar{\ell}}. \quad (2.39)$$

### 2.4. Curvature of Bismut connection on $T^{1,0}M$

In [4], Bismut defined a class of connections on Hermitian manifolds. In this subsection, we choose one of them (see [38, p. 21]). The Bismut connection  $\nabla^B$  on the holomorphic tangent bundle  $(T^{1,0}M, h)$  is characterized by

$$\nabla^B = \nabla + S^B, \quad (2.40)$$

where  $\nabla$  is the Levi-Civita connection and  $S^B$  is a 1-form with values in  $\text{End}(T^{1,0}M)$  defined by

$$\mathbf{h}(S^B(X)Y, Z) = 2\mathbf{g}(S^B(X)Y, \bar{Z}) = \sqrt{-1}(\partial - \bar{\partial})\omega_h(X, Y, \bar{Z}) \quad (2.41)$$

for any  $Y, Z \in T^{1,0}M$  and  $X \in T_{\mathbb{C}}M$ . Let  $\tilde{\Gamma}_{i\alpha}^{\beta}$  and  $\tilde{\Gamma}_{j\alpha}^{\beta}$  be the Christoffel symbols of the Bismut connection where  $i, j, \alpha, \beta \in \{1, \dots, n\}$ . We use different types of letters since the Bismut connection is not torsion free.

**Lemma 2.6.** *We have the following relations between  $\tilde{\Gamma}$  and  $\Gamma$ ,*

$$\tilde{\Gamma}_{i\alpha\bar{\beta}}^{\gamma} (:= h_{\beta\bar{\gamma}} \tilde{\Gamma}_{i\alpha}^{\bar{\gamma}}) = \Gamma_{i\alpha\bar{\beta}} + \Gamma_{\alpha\bar{\beta}i} = \frac{\partial h_{i\bar{\beta}}}{\partial z^{\alpha}}, \quad \tilde{\Gamma}_{j\alpha\bar{\beta}} = 2\Gamma_{j\alpha\bar{\beta}}. \quad (2.42)$$

**Proof.** Let  $X = \frac{\partial}{\partial z^i}, Y = \frac{\partial}{\partial z^j}, Z = \frac{\partial}{\partial z^k}$ . Since  $\omega_h = \frac{\sqrt{-1}}{2} h_{m\bar{n}} dz^m \wedge d\bar{z}^n$ , we obtain

$$\begin{aligned} \sqrt{-1}(\partial - \bar{\partial})\omega_h(X, Y, \bar{Z}) &= -\frac{1}{2} \frac{\partial h_{m\bar{n}}}{\partial z^p} dz^p dz^m d\bar{z}^n \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) \\ &= \frac{1}{2} \left( \frac{\partial h_{i\bar{k}}}{\partial z^j} - \frac{\partial h_{j\bar{k}}}{\partial z^i} \right) \\ &= \Gamma_{j\bar{k}}^{\bar{s}} h_{i\bar{s}} = \Gamma_{j\bar{k}i}. \end{aligned}$$

On the other hand

$$h \left( \nabla_{\frac{\partial}{\partial z^i}}^B \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) = \tilde{\Gamma}_{ij\bar{k}}. \tag{2.43}$$

By the definition (2.40) of Bismut connection, we get

$$\begin{aligned} \tilde{\Gamma}_{i\alpha\bar{\beta}} &= h \left( \nabla_{\frac{\partial}{\partial z^i}}^B \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta} \right) = h \left( \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta} \right) + h \left( S^B \left( \frac{\partial}{\partial z^i} \right) \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta} \right) \\ &= \Gamma_{i\alpha\bar{\beta}} + \Gamma_{\alpha\bar{\beta}i} = \frac{\partial h_{i\bar{\beta}}}{\partial z^\alpha}. \end{aligned}$$

The proof of the other one is similar. □

The Bismut curvature  $B \in \Gamma(M, \Lambda^{1,1} T^* M \otimes \text{End}(T^{1,0} M))$  is given by

$$B_{i\bar{j}\alpha}^\beta = -\frac{\partial \tilde{\Gamma}_{i\alpha}^\beta}{\partial \bar{z}^j} + \frac{\partial \tilde{\Gamma}_{j\alpha}^\beta}{\partial z^i} - \tilde{\Gamma}_{i\alpha}^\gamma \tilde{\Gamma}_{j\bar{\gamma}}^\beta + \tilde{\Gamma}_{j\alpha}^\gamma \tilde{\Gamma}_{i\bar{\gamma}}^\beta. \tag{2.44}$$

**Lemma 2.7.** Assume  $h_{i\bar{j}}(p) = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$  at a fixed point  $p \in M$ ,

$$\begin{aligned} B_{i\bar{j}\alpha\bar{\beta}} &= -\left( \frac{\partial^2 h_{i\bar{\beta}}}{\partial \bar{z}^j \partial z^\alpha} + \frac{\partial^2 h_{\alpha\bar{j}}}{\partial z^i \partial \bar{z}^\beta} - \frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} \right) \\ &\quad + \sum_{\gamma} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} - 4 \sum_{\gamma} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial \bar{z}^j} \frac{\partial h_{\gamma\bar{\beta}}}{\partial z^i}. \end{aligned} \tag{2.45}$$

**Proof.** It follows by (2.42) and (2.44). □

We can define the first Ricci–Bismut curvature  $B_{ij}^{(1)}$ , the second Ricci–Bismut curvature  $B_{ij}^{(2)}$  and scalar curvature  $S^{BM}$  similarly.

### 2.5. Relations among four curvatures on Hermitian manifolds

**Proposition 2.1.** On a Hermitian manifold  $(M, h)$ , we have

$$R_{ij\bar{k}\bar{l}} = \hat{R}_{ij\bar{k}\bar{l}}, \quad R_{i\bar{j}k\bar{l}} = \hat{R}_{i\bar{j}k\bar{l}} \tag{2.46}$$

and for any  $u, v \in \mathbb{C}^n$ ,

$$(R_{i\bar{j}k\bar{l}} - \hat{R}_{i\bar{j}k\bar{l}}) u^i \bar{u}^j v^k \bar{v}^l \leq 0. \tag{2.47}$$

In particular,  $R_{i\bar{j}} \leq \hat{R}_{i\bar{j}}^{(1)}$  and  $R_{i\bar{j}} \leq \hat{R}_{i\bar{j}}^{(2)}$  in the sense of Hermitian matrices.

**Proof.** By formulas (2.8) and (2.26), we can set

$$T_{i\bar{j}k\bar{\ell}} := R_{i\bar{j}k\bar{\ell}} - \hat{R}_{i\bar{j}k\bar{\ell}} = \Gamma_{\bar{j}k}^{\bar{s}} \Gamma_{i\bar{s}}^t h_{t\bar{\ell}}. \tag{2.48}$$

Without loss generality, we assume  $h_{i\bar{j}} = \delta_{ij}$  at a fixed point, then

$$T_{i\bar{j}k\bar{\ell}} = \sum_s \Gamma_{\bar{j}ks} \Gamma_{i\bar{s}\bar{\ell}} = - \sum_s \Gamma_{i\bar{s}\bar{\ell}} \overline{\Gamma_{\bar{j}sk}}, \tag{2.49}$$

where

$$\Gamma_{i\bar{s}\bar{\ell}} = \frac{1}{2} \left( \frac{\partial h_{i\bar{\ell}}}{\partial z^s} - \frac{\partial h_{i\bar{s}}}{\partial z^{\bar{\ell}}} \right) = -\Gamma_{i\bar{\ell}s} \tag{2.50}$$

and so

$$T_{i\bar{j}k\bar{\ell}} u^i \bar{u}^j v^k \bar{v}^{\bar{\ell}} = - \sum_s \left( \sum_{i,\ell} \Gamma_{i\bar{s}\bar{\ell}} u^i \bar{v}^{\bar{\ell}} \right) \overline{\left( \sum_{k,j} \Gamma_{\bar{j}sk} u^j \bar{v}^k \right)} \leq 0. \quad \square$$

**Remark 2.2.** (1) Because of the second order terms in  $R, \hat{R}, \Theta$  and  $B$ , we cannot compare  $R, \hat{R}$  with  $\Theta, B$ .

(2) Since the third order terms of  $\partial\Theta^{(2)}$  are not zero in general, it is possible that  $\Theta^{(1)}$  and  $\Theta^{(2)}$  are not in the same  $(d, \partial, \bar{\partial})$ -cohomology class. For the same reason  $B^{(1)}$  and  $B^{(2)}$  are not in the same  $(d, \partial, \bar{\partial})$ -cohomology class.

(3) If the manifold  $(M, h)$  is Kähler, all curvatures are the same.

### 3. Curvature Relations on Special Hermitian Manifolds

#### 3.1. Curvatures relations on balanced Hermitian manifolds

The following lemma is well-known (for example [22]), and we include a proof here in our setting.

**Lemma 3.1.** *Let  $(M, \omega)$  be a compact Hermitian manifold. The following conditions are equivalent:*

- (1)  $d^* \omega = 0$ ;
- (2)  $d\omega^{n-1} = 0$ ;
- (3) For any smooth function  $f \in C^\infty(M)$ ,

$$\frac{1}{2} \Delta_d f = \Delta_{\bar{\partial}} f = \Delta_{\partial} f = -h^{i\bar{j}} \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}. \tag{3.1}$$

- (4)  $\Gamma_{i\bar{\ell}}^\ell = 0$  for any  $1 \leq i \leq n$ .

**Proof.** On a compact Hermitian manifold,  $d^* \omega = - * d * \omega = -c_n * d\omega^{n-1}$  where  $c_n$  is a constant depending only on the complex dimension  $n$  of  $M$ . On the other hand, the Hodge  $*$  is an isomorphism, and so (1) and (2) are equivalent. If  $f$  is a

smooth function on  $M$ ,

$$\begin{cases} \Delta_{\bar{\partial}} f = -h^{i\bar{j}} \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} + 2h^{i\bar{j}} \Gamma_{i\bar{j}}^{\bar{\ell}} \frac{\partial f}{\partial \bar{z}^{\ell}}, \\ \Delta_{\partial} f = -h^{i\bar{j}} \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} + 2h^{i\bar{j}} \Gamma_{j\bar{i}}^k \frac{\partial f}{\partial z^k}. \end{cases} \tag{3.2}$$

On the other hand,

$$h^{i\bar{j}} \Gamma_{i\bar{j}}^{\bar{\ell}} = -\Gamma_{k\bar{j}}^{\bar{j}} h^{k\bar{\ell}} \quad \text{and} \quad h^{i\bar{j}} \Gamma_{j\bar{i}}^k = -\Gamma_{\bar{\ell}i}^i h^{k\bar{\ell}}. \tag{3.3}$$

Therefore (3) and (4) are equivalent. For the equivalence of (1) and (4), see Lemma A.6.  $\square$

**Definition 3.1.** A Hermitian manifold  $(M, \omega)$  is called *balanced* if it satisfies one of the conditions in Lemma 3.1.

On a balanced Hermitian manifold, there are more symmetries on the second derivatives of the metric.

**Lemma 3.2.** Let  $(M, h)$  be a balanced Hermitian manifold. On a point  $p$  with  $h_{i\bar{j}}(p) = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$ , we have

$$\sum_s \frac{\partial h_{s\bar{i}}}{\partial \bar{z}^s} = \sum_s \frac{\partial h_{s\bar{s}}}{\partial \bar{z}^i} = 0 \tag{3.4}$$

and

$$\sum_i \frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^i} = \sum_i \frac{\partial^2 h_{k\bar{i}}}{\partial z^i \partial \bar{z}^{\ell}} = \sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} - 2 \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}. \tag{3.5}$$

**Proof.** At a fixed point  $p$ , if  $h_{i\bar{j}} = 0$  and  $\Gamma_{ij}^k = 0$ , then

$$\frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k} = -\frac{\partial h_{i\bar{k}}}{\partial \bar{z}^j}. \tag{3.6}$$

The balanced condition  $\sum_s \Gamma_{is}^s = 0$  is reduced to

$$\sum_s \frac{\partial h_{s\bar{s}}}{\partial \bar{z}^i} = \sum_s \frac{\partial h_{s\bar{i}}}{\partial \bar{z}^s} = 0$$

by formula (3.6). By the balanced condition, we get

$$\begin{aligned} 0 &= \frac{\partial \Gamma_{\bar{\ell}i}^i}{\partial z^k} = \frac{\partial}{\partial z^k} \left( \frac{1}{2} h^{i\bar{q}} \left( \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^{\ell}} - \frac{\partial h_{i\bar{\ell}}}{\partial \bar{z}^q} \right) \right) \\ &= \frac{1}{2} \sum_i \left( \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} - \frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^i} \right) - \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}. \end{aligned}$$

Hence, we obtain formula (3.5).  $\square$

**Proposition 3.1.** *Let  $(M, h)$  be a balanced Hermitian manifold. At a point  $p$  with  $h_{i\bar{j}}(p) = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$ , we have following formulas about various Ricci curvatures:*

$$\Theta_{k\bar{\ell}}^{(1)} = \hat{R}_{k\bar{\ell}}^{(1)} = B_{k\bar{\ell}}^{(1)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} + \sum_{q,i} \frac{\partial h_{q\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}, \tag{3.7}$$

$$\Theta_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \sum_{i,q} \frac{\partial h_{q\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}, \tag{3.8}$$

$$\hat{R}_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} + \sum_{i,q} \left( 2 \frac{\partial h_{q\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{q}}}{\partial z^i} \right), \tag{3.9}$$

$$B_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} + \sum_{i,q} \left( 5 \frac{\partial h_{q\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - 4 \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{q}}}{\partial z^i} \right), \tag{3.10}$$

$$R_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} + \sum_{i,q} \left( \frac{\partial h_{q\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{q}}}{\partial z^i} \right), \tag{3.11}$$

$$\mathbb{R}_{k\bar{\ell}} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} - \sum_{i,q} \left( \frac{\partial h_{q\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{q}}}{\partial z^i} \right). \tag{3.12}$$

**Proof.** In (2.32), (2.33), (2.37), (2.38), (2.19), (2.21), we get expressions for all Ricci curvatures on Hermitian manifolds. By balanced relations (3.4) and (3.5), we get simplified versions of all Ricci curvatures. □

**Proposition 3.2.** (1) *A balanced Hermitian manifold with positive Hermitian-Ricci curvature  $(R_{i\bar{j}})$  is Kähler.*  
 (2) *Let  $(M, h)$  be a compact balanced Hermitian manifold. If the Hermitian-Ricci curvature  $(R_{i\bar{j}})$  is nonnegative everywhere and positive at some point, then  $M$  is Moishezon.*

**Proof.** (1) On a balanced Hermitian manifold, we have

$$(\Theta_{i\bar{j}}^{(1)}) = (\hat{R}_{i\bar{j}}^{(1)}) \geq (R_{i\bar{j}}) \tag{3.13}$$

by Propositions 2.1 and 3.1. If  $(R_{i\bar{j}})$  is Hermitian positive, then  $\Theta_{i\bar{j}}^{(1)}$  is Hermitian positive, and so

$$\Omega = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \det(h_{k\bar{\ell}}) \tag{3.14}$$

is a Kähler metric.

(2) If the Hermitian-Ricci curvature is nonnegative everywhere and positive at some point, so is  $(\Theta_{i\bar{j}}^{(1)})$ . The Hermitian line bundle  $L = \det(T^{1,0}M)$  satisfies

$$\int_M c_1(L)^n > 0. \tag{3.15}$$

By Siu-Demailly’s solution of Grauert-Riemenschneider conjecture (see [9, 41]),  $M$  is Moishezon. □

**3.2. Curvature relations on Hermitian manifolds with  $\Lambda(\partial\bar{\partial}\omega) = 0$**

Now we consider a compact Hermitian manifold  $(M, \omega)$  with  $\Lambda(\partial\bar{\partial}\omega) = 0$ . The condition  $\Lambda(\partial\bar{\partial}\omega) = 0$  is equivalent to

$$\sum_k \left( \frac{\partial h_{i\bar{j}}}{\partial z^k \partial \bar{z}^k} + \frac{\partial h_{k\bar{k}}}{\partial z^i \partial \bar{z}^j} \right) = \sum_k \left( \frac{\partial h_{i\bar{k}}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{k\bar{j}}}{\partial z^i \partial \bar{z}^k} \right) \tag{3.16}$$

for any  $i, j$ . Similar to Proposition 3.1, we can use (3.16) to simplify Ricci curvatures and get relations among them.

**Proposition 3.3.** *Let  $(M, h)$  be a compact Hermitian manifold with  $\Lambda(\partial\bar{\partial}\omega) = 0$ . At a point  $p$  with  $h_{i\bar{j}}(p) = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$ , the following identities about Ricci curvatures hold:*

$$\Theta_{k\bar{\ell}}^{(1)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} + \sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}, \tag{3.17}$$

$$\Theta_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}, \tag{3.18}$$

$$\hat{R}_{k\bar{\ell}}^{(1)} = -\frac{1}{2} \sum_i \left( \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} \right) - \sum_{i,q} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}, \tag{3.19}$$

$$\hat{R}_{k\bar{\ell}}^{(2)} = -\frac{1}{2} \sum_i \left( \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} \right) - \sum_{i,q} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i}, \tag{3.20}$$

$$B_{k\bar{\ell}}^{(1)} = -\sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \sum_{i,q} \left( \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - 4 \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \right), \tag{3.21}$$

$$B_{k\bar{\ell}}^{(2)} = -\sum_i \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} + \sum_{i,q} \left( \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} - 4 \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \right), \tag{3.22}$$

$$R_{k\bar{\ell}} = -\frac{1}{2} \sum_i \left( \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} \right) - \sum_{i,q} \left( \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} + \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \right), \tag{3.23}$$

$$\begin{aligned} \mathbb{R}_{k\bar{\ell}} = & -\frac{1}{2} \sum_i \left( \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + \frac{\partial^2 h_{i\bar{i}}}{\partial z^k \partial \bar{z}^{\ell}} \right) + \sum_{i,q} \left( \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} + \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^i} \frac{\partial h_{q\bar{\ell}}}{\partial z^i} \right) \\ & - 2 \sum_{q,i} \left( \frac{\partial h_{q\bar{\ell}}}{\partial z^k} \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^i} + \frac{\partial h_{q\bar{i}}}{\partial z^i} \frac{\partial h_{k\bar{q}}}{\partial \bar{z}^{\ell}} \right). \end{aligned} \tag{3.24}$$

**Proposition 3.4.** *If  $(M, \omega)$  is a compact Hermitian manifold with  $\Lambda(\partial\bar{\partial}\omega) = 0$ , then*

$$B^{(2)} \leq \Theta^{(1)} \quad \text{and} \quad B^{(1)} \leq \Theta^{(2)}. \tag{3.25}$$

in the sense of Hermitian matrices and identities hold if and only if  $(M, \omega)$  is Kähler. Moreover,

$$\Theta^{(2)} + B^{(2)} = \Theta^{(1)} + R^{(1)}. \tag{3.26}$$

Finally, we would like to discuss the relations of special metrics on Hermitian manifolds. By [1], every Moishezon manifold is balanced, i.e. there exists a smooth Hermitian metric  $\omega$  such that  $d^*\omega = 0$ . On the other hand, by Demailly–Paun [10] (see also [31]), on each Moishezon manifold, there exists a singular Hermitian metric  $\omega$  such that  $\partial\bar{\partial}\omega = 0$  in the sense of current. However, these two conditions cannot be satisfied simultaneously in the smooth sense on a Hermitian non-Kähler manifold.

**Proposition 3.5.** *Let  $(M, \omega)$  be a compact Hermitian manifold. If  $d^*\omega = 0$  and  $\Lambda(\partial\bar{\partial}\omega) = 0$ , then  $d\omega = 0$ , i.e.  $(M, \omega)$  is Kähler. In particular, if a compact Hermitian manifold admits a smooth metric  $\omega$  such that  $d^*\omega = 0$  and  $\partial\bar{\partial}\omega = 0$ , then it is Kähler.*

**Proof.** Let  $(M, \omega)$  be a balanced Hermitian manifold with  $\Lambda(\partial\bar{\partial}\omega) = 0$ . The condition  $\Lambda(\partial\bar{\partial}\omega) = 0$  is equivalent to

$$\sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} + \sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} = \sum_i \frac{\partial h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^i} + \sum_i \frac{\partial^2 h_{k\bar{i}}}{\partial z^i \partial \bar{z}^\ell}. \tag{3.27}$$

By formula (3.5), at a point  $p$  with  $h_{i\bar{j}} = \delta_{ij}$  and  $\Gamma_{ij}^k(p) = 0$ , we have

$$\begin{aligned} \sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} + \sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} &= \sum_i \frac{\partial h_{i\bar{\ell}}}{\partial z^k \partial \bar{z}^i} + \sum_i \frac{\partial^2 h_{k\bar{i}}}{\partial z^i \partial \bar{z}^\ell} \\ &= 2 \sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} - 4 \sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}. \end{aligned}$$

That is

$$\sum_i \frac{\partial h_{i\bar{i}}}{\partial z^k \partial \bar{z}^\ell} = \sum_i \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^i} + 4 \sum_{q,i} \frac{\partial h_{q\bar{\ell}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i}. \tag{3.28}$$

By taking trace of it, we obtain

$$4 \sum_{q,i,k} \frac{\partial h_{q\bar{k}}}{\partial \bar{z}^i} \frac{\partial h_{k\bar{q}}}{\partial z^i} = 0 \Leftrightarrow \frac{\partial h_{k\bar{q}}}{\partial z^i} = 0 \tag{3.29}$$

at point  $p$ . Since  $p$  is arbitrary, we have  $d\omega \equiv 0$ , therefore,  $(M, \omega)$  is Kähler.  $\square$

**Remark 3.1.** This result is known in [3] and also [14] in the conditions of  $d^*\omega = 0$  and  $\partial\bar{\partial}\omega = 0$ . By carefully computations, we find that their method works also for  $d^*\omega = 0$  and  $\Lambda(\partial\bar{\partial}\omega) = 0$ . Our method is quite different from theirs.



**4. Bochner Formulas on Hermitian Complex and Riemannian Real Vector Bundles Over Compact Hermitian Manifolds**

Let  $(M, h, \omega)$  be a compact Hermitian manifold. The complexified Levi-Civita connection  $\nabla$  on  $T_{\mathbb{C}}M$  induces a linear connection on  $\Omega^{p,q}(M)$ :

$$\nabla : \Omega^{p,q}(M) \rightarrow \Omega^1(M) \otimes (\Omega^{p,q}(M) \oplus \Omega^{p-1,q+1}(M) \oplus \Omega^{p+1,q-1}(M)). \tag{4.1}$$

We consider the following two canonical components of  $\nabla$ ,

$$\begin{cases} \nabla' : \Omega^{p,q}(M) \rightarrow \Omega^{1,0}(M) \otimes \Omega^{p,q}(M), \\ \nabla'' : \Omega^{p,q}(M) \rightarrow \Omega^{0,1}(M) \otimes \Omega^{p,q}(M). \end{cases} \tag{4.2}$$

Note that  $\nabla \neq \nabla' + \nabla''$  if  $(M, h, \omega)$  is not Kähler. The following calculation rule follows immediately

$$\nabla'(\varphi \wedge \psi) = (\nabla'\varphi) \wedge \psi + \varphi \wedge \nabla'\psi \tag{4.3}$$

for any  $\varphi, \psi \in \Omega^\bullet(M)$ .

**Lemma 4.1.** *On a Hermitian manifold  $(M, h)$ , we have*

$$\begin{cases} \partial h(\varphi, \psi) = h(\nabla'\varphi, \psi) + h(\varphi, \nabla''\psi) \\ \bar{\partial} h(\varphi, \psi) = h(\nabla''\varphi, \psi) + h(\varphi, \nabla'\psi) \end{cases} \Leftrightarrow \begin{cases} \frac{\partial}{\partial z^i} h(\varphi, \psi) = h(\nabla'_i \varphi, \psi) + h(\varphi, \nabla''_i \psi) \\ \frac{\partial}{\partial \bar{z}^j} h(\varphi, \psi) = h(\nabla''_j \varphi, \psi) + h(\varphi, \nabla'_j \psi) \end{cases}$$

for any  $\varphi, \psi \in \Omega^{p,q}(M)$ .

**Remark 4.1.** (1) Here we use the compact notations

$$\nabla'_i = \nabla'_{\frac{\partial}{\partial z^i}}, \quad \nabla''_{\bar{j}} = \nabla''_{\frac{\partial}{\partial \bar{z}^j}}.$$

Note that  $\nabla'_{\bar{j}} = \nabla''_i = 0$  and  $\nabla_i \neq \nabla'_i, \nabla_{\bar{j}} \neq \nabla''_{\bar{j}}$ .

(2) If we regard  $\Lambda^{p,q}T^*M$  as an abstract vector bundle  $E$ , the above lemma says that  $\nabla'$  and  $\nabla''$  are compatible with the Hermitian metric on  $E$ .

Now we go to an abstract setting. Let  $(E, h)$  be a Hermitian *complex* (possibly *non-holomorphic*) vector bundle or a *Riemannian* real vector bundle over a compact Hermitian manifold  $(M, \omega)$ . Let  $\nabla^E$  be an arbitrary *metric connection* on  $(E, h)$ , i.e.

$$dh(s, t) = h(\nabla^E s, t) + h(s, \nabla^E t) \tag{4.4}$$

for any  $s, t \in \Gamma(M, E)$ . There is a natural decomposition

$$\nabla^E = \nabla'^E + \nabla''^E, \tag{4.5}$$

where

$$\begin{cases} \nabla'^E : \Gamma(M, E) \rightarrow \Omega^{1,0}(M, E), \\ \nabla''^E : \Gamma(M, E) \rightarrow \Omega^{0,1}(M, E). \end{cases} \quad (4.6)$$

$\nabla'^E$  and  $\nabla''^E$  induce two differential operators. The first one is  $\partial_E : \Omega^{p,q}(M, E) \rightarrow \Omega^{p+1,q}(M, E)$  defined by

$$\partial_E(\varphi \otimes s) = (\partial\varphi) \otimes s + (-1)^{p+q}\varphi \wedge \nabla'^E s \quad (4.7)$$

for any  $\varphi \in \Omega^{p,q}(M)$  and  $s \in \Gamma(M, E)$ . The other one is  $\bar{\partial}_E : \Omega^{p,q}(M, E) \rightarrow \Omega^{p,q+1}(M, E)$  defined by

$$\bar{\partial}_E(\varphi \otimes s) = (\bar{\partial}\varphi) \otimes s + (-1)^{p+q}\varphi \wedge \nabla''^E s \quad (4.8)$$

for any  $\varphi \in \Omega^{p,q}(M)$  and  $s \in \Gamma(M, E)$ . The following formula is well-known

$$(\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E)(\varphi \otimes s) = \varphi \wedge (\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E)s \quad (4.9)$$

for any  $\varphi \in \Omega^{p,q}(M)$  and  $s \in \Gamma(M, E)$ . The operator  $\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E$  is represented by its  $(1, 1)$  curvature tensor  $R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E)$ . For any  $\varphi, \psi \in \Omega^{\bullet,\bullet}(M, E)$ , there is a *sesquilinear pairing*

$$\{\varphi, \psi\} = \varphi^\alpha \wedge \bar{\psi}^\beta \langle e_\alpha, e_\beta \rangle \quad (4.10)$$

if  $\varphi = \varphi^\alpha e_\alpha$  and  $\psi = \psi^\beta e_\beta$  in the local frame  $\{e_\alpha\}$  of  $E$ . By the metric compatible property of  $\nabla^E$ ,

$$\partial\{\varphi, \psi\} = \{\partial_E \varphi, \psi\} + (-1)^{p+q}\{\varphi, \bar{\partial}_E \psi\} \quad (4.11)$$

if  $\varphi \in \Omega^{p,q}(M, E)$ .

Let  $\omega$  be the Kähler form of the Hermitian metric  $h$ , i.e.

$$\omega = \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j. \quad (4.12)$$

On the Hermitian manifold  $(M, h, \omega)$ , the norm on  $\Omega^{p,q}(M)$  is defined by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \frac{\omega^n}{n!} = \frac{2^n}{(p+q)!} \int_M h(\varphi, \psi) \frac{\omega^n}{n!} = \int_M \varphi \wedge * \bar{\psi}. \quad (4.13)$$

The norm on  $\Omega^{p,q}(M, E)$  is defined by

$$(\varphi, \psi) = \int_M \{\varphi, *\psi\} = \int_M (\varphi^\alpha \wedge *\bar{\psi}^\beta) \langle e_\alpha, e_\beta \rangle \quad (4.14)$$

for  $\varphi, \psi \in \Omega^{p,q}(M, E)$ . The dual operators of  $\partial, \bar{\partial}, \partial_E$  and  $\bar{\partial}_E^*$  are denoted by  $\partial^*, \bar{\partial}^*, \partial_E^*$  and  $\bar{\partial}_E^*$ , respectively.

The following lemma was firstly shown by Demailly using Taylor expansion method (e.g. [8]). For the convenience of the reader, we will take another approach which seems to be useful in local computations.

**Lemma 4.2.** *Let  $(M, h, \omega)$  be a compact Hermitian manifold. If  $\tau$  is the operator of type  $(1, 0)$  defined by  $\tau = [\Lambda, 2\partial\omega]$  on  $\Omega^\bullet(M, E)$ ,*

$$\begin{cases} [\Lambda, \partial] = \sqrt{-1}(\bar{\partial}^* + \bar{\tau}^*), \\ [\Lambda, \bar{\partial}] = -\sqrt{-1}(\partial^* + \tau^*). \end{cases} \tag{4.15}$$

For the dual equation, it is

$$\begin{cases} [\bar{\partial}^*, L] = \sqrt{-1}(\partial + \tau), \\ [\partial^*, L] = -\sqrt{-1}(\bar{\partial} + \bar{\tau}), \end{cases} \tag{4.16}$$

where  $L$  is the operator  $L\varphi = 2\omega \wedge \varphi$  and  $\Lambda$  is the adjoint operator of  $L$ .

**Proof.** See Lemma A.5 of the appendix. □

In the rest of this section,  $E$  is assumed to be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold  $M$ .

**Lemma 4.3.** *Let  $\nabla^E$  be a metric connection on  $E$  over a compact Hermitian manifold  $(M, \omega)$ . If  $\tau$  is the operator of type  $(1, 0)$  defined by  $\tau = [\Lambda, 2\partial\omega]$  on  $\Omega^\bullet(M, E)$ , then*

- (1)  $[\bar{\partial}_E^*, L] = \sqrt{-1}(\partial_E + \tau)$ ;
- (2)  $[\partial_E^*, L] = -\sqrt{-1}(\bar{\partial}_E + \bar{\tau})$ ;
- (3)  $[\Lambda, \partial_E] = \sqrt{-1}(\bar{\partial}_E^* + \bar{\tau}^*)$ ;
- (4)  $[\Lambda, \bar{\partial}_E] = -\sqrt{-1}(\partial_E^* + \tau^*)$ .

**Proof.** See Lemma A.8 of the appendix. □

**Theorem 4.1.** *Let  $\nabla^E$  be a metric connection  $E$  over a compact Hermitian manifold  $(M, \omega)$ .*

$$\Delta_{\bar{\partial}_E} = \Delta_{\partial_E} + \sqrt{-1} [\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E, \Lambda] + (\partial_E \tau^* + \tau^* \partial_E) - (\bar{\partial}_E \bar{\tau}^* + \bar{\tau}^* \bar{\partial}_E), \tag{4.17}$$

where

$$\begin{cases} \Delta_{\bar{\partial}_E} = \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E, \\ \Delta_{\partial_E} = \partial_E \partial_E^* + \partial_E^* \partial_E. \end{cases} \tag{4.18}$$

**Proof.** It follows from Lemma 4.3. □

We make a useful observation on the torsion  $\tau$ :

**Lemma 4.4.** *For any  $s \in \Gamma(M, E)$ , we have*

$$\tau(s) = -2\sqrt{-1}(\bar{\partial}^* \omega) \cdot s, \quad \bar{\tau}(s) = 2\sqrt{-1}(\partial^* \omega) \cdot s. \tag{4.19}$$

**Proof.** By definition

$$\begin{aligned} ([\Lambda, 2\partial\omega])s &= 2\Lambda((\partial\omega) \cdot s) \\ &= 2(\Lambda(\partial\omega)) \cdot s \\ &= -2\sqrt{-1}(\bar{\partial}^* \omega) \cdot s. \end{aligned}$$

Here we use the identity

$$\bar{\partial}^* \omega = \sqrt{-1}\Lambda(\partial\omega) \tag{4.20}$$

where the proof of it is contained in Lemma A.6 of the appendix. □

**Corollary 4.1.** *If  $(M, \omega)$  is a compact balanced Hermitian manifold, and  $\nabla^E$  a metric connection on  $E$  over  $M$ , then*

$$\|\bar{\partial}_E s\|^2 = \|\partial_E s\|^2 + (\sqrt{-1}[\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E, \Lambda]s, s) \tag{4.21}$$

for any  $s \in \Gamma(M, E)$ .

**Proof.** Since for any  $s \in \Gamma(M, E)$ ,  $\tau s = \bar{\tau} s = 0$  and  $\tau^* s = \bar{\tau}^* s = 0$  on a balanced Hermitian manifold, the result follows from formula (4.17). □

**Theorem 4.2.** *Let  $(M, \omega)$  be a compact Hermitian manifold with  $\partial\bar{\partial}\omega^{n-1} = 0$ . If  $\nabla^E$  is a metric connection on  $E$  over  $M$ , then*

$$0 = \|\bar{\partial}_E s\|^2 = \|\partial_E s\|^2 + (\sqrt{-1}[\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E, \Lambda]s, s) \tag{4.22}$$

for any  $s \in \Gamma(M, E)$  with  $\bar{\partial}_E s = 0$ .

**Proof.** By formula (4.17), we only have to prove that

$$((\partial_E \tau^* + \tau^* \partial_E)s - (\bar{\partial}_E \bar{\tau}^* + \bar{\tau}^* \bar{\partial}_E)s, s) = 0. \tag{4.23}$$

It is equivalent to  $(\partial_E s, \tau s) = 0$  since  $\tau^* s = \bar{\tau}^* s = \bar{\partial}_E s = 0$ . By formula (4.19) and Stokes' theorem,

$$\begin{aligned} (\tau^* \partial_E s, s) &= (\partial_E s, \tau s) = \int_M \{\partial_E s, *(\tau s)\} \\ &= 2\sqrt{-1} \int_M \{\partial_E s, *(\bar{\partial}^* \omega \cdot s)\} \\ &= 2\sqrt{-1} \int_M \{\partial_E s, (*\bar{\partial}^* \omega) \cdot s\} \end{aligned}$$

$$\begin{aligned} &= -2\sqrt{-1} \int_M \{s, \bar{\partial}_E((*\bar{\partial}^* \omega) \cdot s)\} \\ &= -2\sqrt{-1} \int_M \{s, (\bar{\partial} * \bar{\partial}^* \omega) \cdot s - (*\bar{\partial}^* \omega) \wedge \bar{\partial}_E s\}. \end{aligned}$$

It is easy to see that

$$\bar{\partial} * \bar{\partial}^* \omega = -\bar{\partial} * * \partial * \omega = c_n \bar{\partial} \partial \omega^{n-1} = 0 \tag{4.24}$$

since  $*\omega = c_n \omega^{n-1}$  where  $c_n$  is a constant depending only on the complex dimension of  $M$ . Hence

$$(\partial_E s, \tau s) = 2\sqrt{-1} \int_M \{s, (*\bar{\partial}^* \omega) \wedge \bar{\partial}_E s\} = 0 \tag{4.25}$$

since  $\bar{\partial}_E s = 0$ . □

**Remark 4.2.** By these formulas, we can obtain classical vanishing theorems on Kähler manifolds and rigidity of harmonic maps between compact Hermitian and compact Riemannian manifolds.

## 5. Vanishing Theorems on Hermitian Manifolds

### 5.1. Vanishing theorems on compact Hermitian manifolds

Let  $E$  be a Hermitian *complex* (possibly *non-holomorphic*) vector bundle or a Riemannian *real* vector bundle over a compact Hermitian manifold  $(M, \omega)$ . Let  $\nabla^E$  be an arbitrary connection on  $E$  and  $\partial_E, \bar{\partial}_E$  the  $(1, 0), (0, 1)$  part of  $\nabla^E$ , respectively. The  $(1, 1)$ -curvature of  $\nabla^E$  is denoted by  $R^E \in \Gamma(M, \Lambda^{1,1} T^* M \otimes E^* \otimes E)$ . It can be viewed as a representation of the operator  $\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E$ . We can define harmonic section spaces associated to  $(E, \nabla^E)$  by

$$\mathcal{H}_{\bar{\partial}_E}^{p,q}(M, E) = \{\varphi \in \Omega^{p,q}(M, E) \mid \bar{\partial}_E \varphi = \bar{\partial}_E^* \varphi = 0\}. \tag{5.1}$$

In general, on a complex vector bundle  $E$ , there is no terminology such as “holomorphic section of  $E$ ”. However, if the vector bundle  $E$  is holomorphic and  $\nabla^E$  is the Chern connection on  $E$ , i.e.  $\bar{\partial}_E = \bar{\partial}$ , then  $\mathcal{H}_{\bar{\partial}_E}^{p,q}(M, E)$  is isomorphic to the Dolbeault cohomology group  $H_{\bar{\partial}}^{p,q}(M, E)$  and  $H_{\bar{\partial}}^0(M, E)$  is the holomorphic section space  $H^0(M, E)$  of  $E$ .

**Definition 5.1.** Let  $A$  be an  $r \times r$  Hermitian matrix and  $\lambda_1 \leq \dots \leq \lambda_r$  be eigenvalues of  $A$ .  $A$  is said to be *p-nonnegative* (respectively, *positive, negative, nonpositive*) for  $1 \leq p \leq r$  if

$$\begin{aligned} \lambda_{i_1} + \dots + \lambda_{i_p} &\geq 0 \text{ (respectively, } > 0, < 0, \leq 0) \text{ for any} \\ &1 \leq i_1 < i_2 < \dots < i_p \leq n. \end{aligned} \tag{5.2}$$

**Theorem 5.1.** *Let  $E$  be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold  $(M, \omega)$  and  $\nabla^E$  be any metric connection on  $E$ .*

- (1) *If the second Hermitian–Ricci curvature  $\text{Tr}_\omega R^E$  is nonpositive everywhere, then every  $\bar{\partial}_E$ -closed section of  $E$  is parallel, i.e.  $\nabla^E s = 0$ ;*
- (2) *If the second Hermitian–Ricci curvature  $\text{Tr}_\omega R^E$  is nonpositive everywhere and negative at some point, then  $\mathcal{H}_{\bar{\partial}_E}^0(M, E) = 0$ ;*
- (3) *If the second Hermitian–Ricci curvature  $\text{Tr}_\omega R^E$  is  $p$ -nonpositive everywhere and  $p$ -negative at some point, then  $\mathcal{H}_{\bar{\partial}_E}^0(M, \Lambda^q E) = 0$  for any  $p \leq q \leq \text{rank}(E)$ .*

**Proof.** By [24], there exists a smooth function  $u : M \rightarrow \mathbb{R}$  such that  $\omega_G = e^u \omega$  is a Gauduchon metric, i.e.  $\partial \bar{\partial} \omega_G^{n-1} = 0$ . Now we replace the metric  $\omega$  on  $M$  by the Gauduchon metric  $\omega_G$ . By the relation  $\omega_G = e^u \omega$ , we get

$$\text{Tr}_{\omega_G} R^E = e^{-u} \text{Tr}_\omega R^E. \tag{5.3}$$

Therefore, the positivity conditions in the theorem are preserved. Let  $s \in \Gamma(M, E)$  with  $\bar{\partial}_E s = 0$ , by formula (4.22), we obtain

$$0 = \|\partial_E s\|^2 + (\sqrt{-1}[\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E, \Lambda_G]s, s) = \|\partial_E s\|^2 - (\text{Tr}_{\omega_G} R^E s, s), \tag{5.4}$$

where

$$R^E = \partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E = R_{ij\alpha}^\beta dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\beta. \tag{5.5}$$

Since the second Hermitian–Ricci curvature  $\text{Tr}_{\omega_G} R^E$  has components

$$R_{\alpha\bar{\beta}} = h_G^{i\bar{j}} R_{ij\alpha\bar{\beta}} \tag{5.6}$$

formula (5.4) can be written as

$$0 = \|\partial_E s\|^2 - \int_M R_{\alpha\bar{\beta}} s^\alpha \bar{s}^\beta. \tag{5.7}$$

Now (1) and (2) follow by identity (5.7) with the curvature conditions immediately. For (3), we set  $F = \Lambda^q E$  with  $p \leq q \leq r = \text{rank}(E)$ . Let  $\lambda_1 \leq \dots \leq \lambda_r$  be the eigenvalues of  $-\text{Tr}_{\omega_G} R^E$ , then we know

$$\lambda_1 + \dots + \lambda_p \geq 0 \tag{5.8}$$

and it is strictly positive at some point. If  $p \leq q \leq r$ , the smallest eigenvalue of  $-\text{Tr}_{\omega_G} R^E$  is  $\lambda_1 + \dots + \lambda_q \geq 0$  and it is strictly positive at some point. By (2), we know  $\mathcal{H}_{\bar{\partial}_E}^0(M, F) = 0$ . □

If  $\nabla^E$  is the Chern connection of the Hermitian holomorphic vector bundle  $E$ , we know

$$\mathcal{H}_{\bar{\partial}_E}^0(M, E) \cong H^0(M, E)$$

since  $\bar{\partial}_E = \nabla''^E = \bar{\partial}$  for the Chern connection.

**Corollary 5.1 (Kobayashi–Wu [34], Gauduchon [22]).** *Let  $\nabla^E$  be the Chern connection of a Hermitian holomorphic vector bundle  $E$  over a compact Hermitian manifold  $(M, h, \omega)$ .*

- (1) *If the second Ricci–Chern curvature  $\text{Tr}_\omega R^E$  is nonpositive everywhere, then every holomorphic section of  $E$  is parallel, i.e.  $\nabla^E s = 0$ ;*
- (2) *If the second Ricci–Chern curvature  $\text{Tr}_\omega R^E$  is nonpositive everywhere and negative at some point, then  $E$  has no holomorphic section, i.e.  $H^0(M, E) = 0$ ;*
- (3) *If the second Ricci–Chern curvature  $\text{Tr}_\omega R^E$  is  $p$ -nonpositive everywhere and  $p$ -negative at some point, then  $\Lambda^q E$  has no holomorphic section for any  $p \leq q \leq \text{rank}(E)$ .*

Now we can apply it to the tangent and cotangent bundles of compact Hermitian manifolds.

**Corollary 5.2.** *Let  $(M, \omega)$  be a compact Hermitian manifold and  $\Theta$  is the Chern curvature of the Chern connection  $\nabla^{CH}$  on the holomorphic tangent bundle  $T^{1,0}M$ .*

- (1) *If the second Ricci–Chern curvature  $\Theta^{(2)}$  is nonpositive everywhere and negative at some point, then  $M$  has no holomorphic vector field, i.e.  $H^0(M, T^{1,0}M) = 0$ ;*
- (2) *If the second Ricci–Chern curvature  $\Theta^{(2)}$  is nonnegative everywhere and positive at some point, then  $M$  has no holomorphic  $p$ -form for any  $1 \leq p \leq n$ , i.e.  $H_{\bar{\partial}}^{p,0}(M) = 0$ ; In particular, the arithmetic genus*

$$\chi(M, \mathcal{O}) = \sum (-1)^p h^{p,0}(M) = 1. \tag{5.9}$$

- (3) *If the second Ricci–Chern curvature  $\Theta^{(2)}$  is  $p$ -nonnegative everywhere and  $p$ -positive at some point, then  $M$  has no holomorphic  $q$ -form for any  $p \leq q \leq n$ , i.e.  $H_{\bar{\partial}}^{q,0}(M) = 0$ . In particular, if the scalar curvature  $S^{CH}$  is nonnegative everywhere and positive at some point, then  $H^0(M, mK_M) = 0$  for all  $m \geq 1$  where  $K_M$  is the canonical line bundle of  $M$ .*

**Proof.** Let  $E = T^{1,0}M$  and  $h$  be a Hermitian metric on  $E$  such that the second Ricci–Chern curvature  $\text{Tr}_{\omega_h} \Theta$  of  $(E, h)$  satisfies the assumption. It is obvious that all section spaces in consideration are independent of the choice of the metrics and connections.

The metric on the vector bundle  $E$  is fixed. Now we choose a Gauduchon metric  $\omega_G = e^u \omega_h$  on  $M$ . Then the second Ricci–Chern curvature  $\tilde{\Theta}^{(2)} = \text{Tr}_{\omega_G} \Theta = e^{-u} \text{Tr}_{\omega_h} \Theta$  shares the semi-definite property with  $\Theta^{(2)} = \text{Tr}_{\omega_h} \Theta$ . For the safety, we repeat the arguments in Theorem 5.1 briefly. If  $s$  is a holomorphic section of  $E$ , i.e.  $\bar{\partial}_E s = \bar{\partial} s = 0$ , by formula (4.22), we obtain

$$0 = \|\partial_E s\|^2 + (\sqrt{-1}[\partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E, \Lambda_G]s, s) = \|\partial_E s\|^2 - (\text{Tr}_{\omega_G} \Theta s, s). \tag{5.10}$$

If  $\text{Tr}_\omega \Theta$  is nonpositive everywhere, then  $\partial_E s = 0$  and so  $\nabla^E s = 0$ . If  $\text{Tr}_\omega \Theta$  is nonpositive everywhere and negative at some point, we get  $s = 0$ , therefore  $H^0(M, T^{1,0}M) = 0$ . The proofs of (2) and (3) are similar.  $\square$

**Remark 5.1.** It is well-known that the first Ricci–Chern curvature  $\Theta^{(1)}$  represents the first Chern class of  $M$ . But on a Hermitian manifold, it is possible that the second Ricci–Chern curvature  $\Theta^{(2)}$  is not in the same  $(d, \partial, \bar{\partial})$ -cohomology class as  $\Theta^{(1)}$ . For example,  $\mathbb{S}^3 \times \mathbb{S}^1$  with canonical metric has strictly positive second Ricci–Chern curvature but it is well-known that it has vanishing first Chern number  $c_1^2$ . For more details see Proposition 6.1. Therefore,  $\Theta^{(2)}$  in Corollary 5.2 cannot be replaced by  $\Theta^{(1)}$ . It seems to be an interesting question: if  $(M, \omega)$  is a compact Hermitian manifold and its first Ricci–Chern curvature is nonnegative everywhere and positive at some point, is the first Betti number of  $M$  zero? In particular, is it Kähler in dimension 2?

As special cases of our results, the following results for Kähler manifolds are well-known, and we list them here for the convenience of the reader. Let  $(M, h, \omega)$  be a compact Kähler manifold.

- (1) If the Ricci curvature is nonnegative everywhere, then any holomorphic  $(p, 0)$  form is parallel;
- (2) If the Ricci curvature is nonnegative everywhere and positive at some point, then  $h^{p,0} = 0$  for  $p = 1, \dots, n$ . In particular, the arithmetic genus  $\chi(M, \mathcal{O}) = 1$  and  $b_1(M) = 0$ ;
- (3) If the scalar curvature is nonnegative everywhere and positive at some point, then  $h^{n,0} = 0$ ;
- (A) If the Ricci curvature is nonpositive everywhere, then any holomorphic vector field is parallel;
- (B) If the Ricci curvature is nonpositive everywhere and negative at some point, there is no holomorphic vector field.

### 5.2. Vanishing theorems on special Hermitian manifolds

Let  $(M, h, \omega)$  be a compact Hermitian manifold and  $\nabla$  be the Levi-Civita connection.

**Lemma 5.1.** *Let  $(M, \omega)$  be a compact balanced Hermitian manifold. For any  $(p, 0)$ -form  $\varphi$  on  $M$ ,*

- (1) *If  $\varphi$  is holomorphic, then  $\partial^* \varphi = 0$ ;*
- (2) *If  $\nabla' \varphi = 0$ , then  $\partial \varphi = 0$ .*

**Proof.** For simplicity, we assume  $p = 1$ . For the general case, the proof is the same. By Lemma A.3, we know, for any  $(1, 0)$ -form  $\varphi = \varphi_i dz^i$ ,

$$\partial^* \varphi = -h^{i\bar{j}} \frac{\partial \varphi_i}{\partial \bar{z}^j}, \tag{5.11}$$



where we use the balanced condition  $h^{i\bar{j}}\Gamma_{i\bar{j}}^s = 0$ . If  $\varphi$  is holomorphic, then  $\frac{\partial\varphi_i}{\partial\bar{z}^j} = 0$ , hence  $\partial^*\varphi = 0$ . On the other hand,

$$\nabla'\varphi = \left( \frac{\partial\varphi_i}{\partial z^j} - \Gamma_{j\bar{i}}^m\varphi_m \right) dz^j \otimes dz^i. \tag{5.12}$$

If  $\nabla'\varphi = 0$ , we obtain

$$\partial\varphi = \frac{\partial\varphi_i}{\partial z^j} dz^j \wedge dz^i = \Gamma_{j\bar{i}}^m\varphi_m dz^j \wedge dz^i = 0. \tag{5.13}$$

□

**Theorem 5.2.** *Let  $(M, \omega)$  be a compact balanced Hermitian manifold with Levi-Civita connection  $\nabla$ .*

- (1) *If the Hermitian–Ricci curvature  $(R_{i\bar{j}})$  is  $p$ -nonnegative everywhere, then any holomorphic  $(q, 0)$ -form ( $p \leq q \leq n$ ) is  $\partial$ -harmonic; in particular,  $\dim_{\mathbb{C}} H_{\bar{\partial}}^{q,0}(M) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,q}(M)$  for any  $p \leq q \leq n$ ;*
- (2) *If the Hermitian–Ricci curvature  $(R_{i\bar{j}})$  is  $p$ -nonnegative everywhere and  $p$ -positive at some point,  $H_{\bar{\partial}}^{q,0}(M) = 0$  for any  $p \leq q \leq n$ ;*

*In particular,*

- (3) *if the Hermitian–Ricci curvature  $(R_{i\bar{j}})$  is nonnegative everywhere and positive at some point, then  $H_{\bar{\partial}}^{p,0}(M) = 0$ , for  $p = 1, \dots, n$  and so the arithmetic genus  $\chi(M, \mathcal{O}) = 1$  and  $b_1(M) \leq h^{0,1}(M)$ ;*
- (4) *if the Hermitian–scalar curvature  $S$  is nonnegative everywhere and positive at some point, then*

$$H^0(M, mK_M) = 0 \quad \text{for any } m \geq 1,$$

where  $K_M = \det T^{*1,0}M$ .

**Proof.** At first, we assume  $p = 1$  for (1) and (2). Now we consider  $E = T^{*1,0}M$  with the induced metric connection  $\nabla^E = \hat{\nabla}$  for  $h$  (see (2.22)). By formula (4.1), we have

$$\|\bar{\partial}_E s\|^2 = \|\partial_E s\|^2 + \sqrt{-1}([R^E, \Lambda]s, s), \tag{5.14}$$

where  $R^E$  is the  $(1, 1)$ -part curvature of  $E$  with respect to the connection  $\nabla^E$ . More precisely,

$$R^E = \partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E = -\hat{R}_{i\bar{j}k}^\ell dz^i \wedge d\bar{z}^j \otimes \frac{\partial}{\partial z^\ell} \otimes dz^k \tag{5.15}$$

since  $E$  is the dual vector bundle of  $T^{1,0}M$  and the  $(1, 1)$ -part of the curvature of  $T^{1,0}M$  is

$$\hat{R}_{i\bar{j}k}^\ell dz^i \wedge d\bar{z}^j \otimes dz^k \otimes \frac{\partial}{\partial z^\ell}. \tag{5.16}$$

If  $s = f_i dz^i$  is a holomorphic 1-form, i.e.

$$\bar{\partial}s = \frac{\partial f_i}{\partial \bar{z}^j} d\bar{z}^j \wedge dz^i = 0 \tag{5.17}$$

then

$$\bar{\partial}_E s = \left( \frac{\partial f_i}{\partial \bar{z}^j} - f_k \Gamma_{ji}^k \right) d\bar{z}^j \otimes dz^i = -f_k \Gamma_{ji}^k d\bar{z}^j \otimes dz^i. \tag{5.18}$$

Without loss of generality, we assume  $h_{i\bar{j}} = \delta_{ij}$  at a given point. By Proposition 2.1, the quantity

$$|\bar{\partial}_E s|^2 = \sum_{i,j,t,n} f_i \bar{f}_n \Gamma_{ji\bar{t}} \bar{\Gamma}_{j\bar{t}i\bar{n}} = \sum_{i,n} (\hat{R}_{n\bar{i}}^{(2)} - R_{n\bar{i}}) f_i \bar{f}_n. \tag{5.19}$$

On the other hand

$$\sqrt{-1} \langle [R^E, \Lambda]s, s \rangle = \sum_{i,n} \hat{R}_{n\bar{i}}^{(2)} f_i \bar{f}_n. \tag{5.20}$$

That is

$$|\bar{\partial}_E s|^2 - \sqrt{-1} \langle [R^E, \Lambda]s, s \rangle = - \sum_{i,n} R_{n\bar{i}} f_i \bar{f}_n \leq 0 \tag{5.21}$$

if the Hermitian–Ricci curvature  $(R_{n\bar{i}})$  of  $(M, h, \omega)$  is nonnegative everywhere. Then we get

$$0 \leq \|\partial_E s\|^2 = \|\bar{\partial}_E s\|^2 - \sqrt{-1} \langle [R^E, \Lambda]s, s \rangle \leq 0. \tag{5.22}$$

That is  $\partial_E s = 0$ . Since

$$\partial_E s = \nabla'^E s = \hat{\nabla}' s = \nabla' s = \left( \frac{\partial f_i}{\partial z^j} - f_\ell \Gamma_{ij}^\ell \right) dz^j \otimes dz^i,$$

we obtain  $\nabla' s = 0$ . By Lemma 5.1, we know  $\Delta_\partial s = 0$ . In summary, we get

$$H_{\bar{\partial}}^{1,0}(M) \subset H_{\partial}^{1,0}(M) \cong H_{\bar{\partial}}^{0,1}(M). \tag{5.23}$$

If the Hermitian–Ricci curvature  $(R_{n\bar{i}})$  is nonnegative everywhere and positive at some point, then  $f_i = 0$  for each  $i$ , that is  $s = 0$ . Now we obtain  $H_{\bar{\partial}}^{1,0}(M) = 0$ . The general cases follow by the same arguments as Theorem 5.1 and Corollary 5.2. In part (3),  $b_1(M) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(M)$  follows from the Frölicher relation  $b_1(M) \leq h^{1,0}(M) + h^{0,1}(M)$ .  $\square$

The dual of Theorem 5.2 is the following.

**Theorem 5.3.** *Let  $(M, h, \omega)$  be a compact balanced Hermitian manifold.*

- (1) *If  $2\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}$  is nonpositive everywhere, then any holomorphic vector field is  $\nabla'$ -closed;*
- (2) *If  $2\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}$  is nonpositive everywhere and negative at some point, there is no holomorphic vector field.*

**Proof.** Let  $E = T^{1,0}M$  and  $\hat{\nabla}$  the induced connection on it. If  $s = f^i \frac{\partial}{\partial z^i}$  is a holomorphic section, then

$$\bar{\partial}_E s = f^i \Gamma_{j\bar{i}}^\ell d\bar{z}^j \otimes \frac{\partial}{\partial z^\ell} \in \Gamma(M, \Lambda^{0,1} T^* M \otimes E). \tag{5.24}$$

Without loss generality, we assume  $h_{i\bar{j}} = \delta_{ij}$  at a given point. By Proposition 2.1,

$$\begin{aligned} |\bar{\partial}_E s|^2 - \sqrt{-1} \langle [\hat{R}^{1,1}, \Lambda] s, s \rangle &= (\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}) f^i \bar{f}^j + \hat{R}_{i\bar{j}}^{(2)} f^i \bar{f}^j \\ &= (2\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}) f^i \bar{f}^j. \end{aligned}$$

By formula (4.17),

$$0 \leq \|\partial_E s\|^2 = \|\bar{\partial}_E s\|^2 - \sqrt{-1} \langle [\hat{R}^{1,1}, \Lambda] s, s \rangle. \tag{5.25}$$

So if  $2\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}$  is nonpositive everywhere,  $\partial_E s = \nabla' s = 0$ . If  $2\hat{R}_{i\bar{j}}^{(2)} - R_{i\bar{j}}$  is nonpositive everywhere and negative at some point, there is no holomorphic vector field.  $\square$

**Remark 5.2.** (1) It is obvious that the second Ricci–Chern curvature  $\Theta_{k\bar{l}}^{(2)}$  and Hermitian–Ricci curvature  $R_{k\bar{l}}$  cannot be compared. Therefore, Corollary 5.2 and Theorem 5.2 are independent of each other. For the same reason, Corollary 5.2 and Theorem 5.3 are independent.

(2) For a special case in Theorem 5.2, if the Hermitian–Ricci curvature  $R_{k\bar{l}}$  is nonnegative everywhere and positive at some point, by Proposition 3.2, the manifold  $(M, \omega)$  is Moishezon. It is well-known that every 2-dimensional Moishezon/balanced manifold is Kähler, but there are many Moishezon non-Kähler manifolds in higher dimension (see [39]).

The following result was firstly obtained in [29].

**Corollary 5.3.** *Let  $(M, \omega)$  be a compact Hermitian manifold with  $\Lambda(\partial\bar{\partial}\omega) = 0$ . Let  $\nabla^B$  be the Bismut connection on  $T^{1,0}M$ .*

- (1) *If the first Ricci–Bismut curvature  $B^{(1)}$  is nonnegative everywhere, then every holomorphic  $(p, 0)$ -form is parallel with respect to the Chern connection  $\nabla^{CH}$ ;*
- (2) *If the first Ricci–Bismut curvature  $B^{(1)}$  is nonnegative everywhere and positive at some point, then  $M$  has no holomorphic  $(p, 0)$ -form for any  $1 \leq p \leq n$ , i.e.  $H_{\bar{\partial}}^{p,0}(M) = 0$ ; in particular, the arithmetic genus  $\chi(M, \mathcal{O}) = 1$ ;*
- (3) *If the first Ricci–Bismut curvature  $B^{(1)}$  is  $p$ -nonnegative everywhere and  $p$ -positive at some point then  $M$  has no holomorphic  $(q, 0)$ -form for any  $p \leq q \leq n$ , i.e.  $H_{\bar{\partial}}^{q,0}(M) = 0$ . In particular, if the scalar curvature  $S^{BM}$  of the Bismut connection is nonnegative everywhere and positive at some point, then  $H^0(M, mK_M) = 0$  for any  $m \geq 1$ .*

**Proof.** By Proposition 3.4, if  $\Lambda(\partial\bar{\partial}\omega) = 0$ , then

$$B^{(1)} \leq \Theta^{(2)}. \tag{5.26}$$

Now we can apply Corollary 5.2 to get (1)–(3). □

**Remark 5.3.** For more vanishing theorems on special Hermitian manifolds, one can consult [3, 19, 20, 29] and references therein.

### 6. Examples of Non-Kähler Manifolds with Nonnegative Curvatures

Let  $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$  be the standard  $n$ -dimensional ( $n \geq 2$ ) Hopf manifold. It is diffeomorphic to  $\mathbb{C}^n - \{0\}/G$  where  $G$  is cyclic group generated by the transformation  $z \rightarrow \frac{1}{2}z$ . It has an induced complex structure of  $\mathbb{C}^n - \{0\}$ . For more details about such manifolds, we refer the reader to [33]. On  $M$ , there is a natural metric

$$h = \sum_{i=1}^n \frac{4}{|z|^2} dz^i \otimes d\bar{z}^i. \tag{6.1}$$

The following identities follow immediately

$$\frac{\partial h_{k\bar{\ell}}}{\partial z^i} = -\frac{4\delta_{k\ell}\bar{z}^i}{|z|^4}, \quad \frac{\partial h_{k\bar{\ell}}}{\partial \bar{z}^j} = -\frac{4\delta_{k\ell}z^j}{|z|^4} \tag{6.2}$$

and

$$\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} = -4\delta_{k\ell} \frac{\delta_{i\bar{j}}|z|^2 - 2\bar{z}^i z^j}{|z|^6}. \tag{6.3}$$

**Example 6.1 (Curvatures of Chern connection).** Straightforward computations show that, the Chen curvature components are

$$\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{p\bar{q}} \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j} = \frac{4\delta_{kl}(\delta_{ij}|z|^2 - z^j \bar{z}^i)}{|z|^6}, \tag{6.4}$$

and the first and second Ricci–Chern curvatures are

$$\Theta_{k\bar{\ell}}^{(1)} = \frac{n(\delta_{k\ell}|z|^2 - z^{\ell}\bar{z}^k)}{|z|^4}, \quad \Theta_{k\bar{\ell}}^{(2)} = \frac{(n-1)\delta_{k\ell}}{|z|^2}. \tag{6.5}$$

It is easy to see that the eigenvalues of  $\Theta^{(1)}$  are

$$\lambda_1 = 0, \quad \lambda_2 = \dots = \lambda_n = \frac{n}{|z|^2}. \tag{6.6}$$

Hence,  $\Theta^{(1)}$  is nonnegative and 2-positive everywhere.

**Example 6.2 (Curvatures of Levi-Civita connection).** Similarly, we have

$$\Gamma_{ik}^\ell = -\frac{\delta_{i\ell}\bar{z}^k + \delta_{k\ell}\bar{z}^i}{2|z|^2}, \quad \Gamma_{\bar{j}k}^\ell = \frac{\delta_{jk}z^\ell - \delta_{k\ell}z^j}{2|z|^2} \tag{6.7}$$

and

$$\frac{\partial\Gamma_{ik}^\ell}{\partial\bar{z}^j} = -\frac{\delta_{k\ell}\delta_{ij} + \delta_{i\ell}\delta_{jk}}{2|z|^2} + \frac{\delta_{i\ell}z^j\bar{z}^k + \delta_{k\ell}z^j\bar{z}^i}{2|z|^4}, \tag{6.8}$$

$$\frac{\partial\Gamma_{\bar{j}k}^\ell}{\partial z^i} = \frac{\delta_{jk}\delta_{i\ell} - \delta_{k\ell}\delta_{ij}}{2|z|^2} - \frac{(\delta_{jk}z^\ell - \delta_{k\ell}z^j)\bar{z}^i}{2|z|^4}. \tag{6.9}$$

The complexified Riemannian curvature components are

$$\begin{aligned} R_{i\bar{j}k}^\ell &= -\left(\frac{\partial\Gamma_{ik}^\ell}{\partial\bar{z}^j} - \frac{\partial\Gamma_{\bar{j}k}^\ell}{\partial z^i} + \Gamma_{ik}^s\Gamma_{\bar{j}s}^\ell - \Gamma_{\bar{j}k}^s\Gamma_{is}^\ell - \Gamma_{\bar{j}k}^{\bar{s}}\Gamma_{i\bar{s}}^\ell\right) \\ &= \frac{\delta_{i\ell}\delta_{jk}}{2|z|^2} - \frac{\delta_{i\ell}z^j\bar{z}^k + \delta_{jk}z^\ell\bar{z}^i}{4|z|^4} \end{aligned} \tag{6.10}$$

and

$$R_{i\bar{j}k\bar{\ell}} = \frac{2\delta_{i\ell}\delta_{jk}}{|z|^4} - \frac{\delta_{i\ell}z^j\bar{z}^k + \delta_{jk}z^\ell\bar{z}^i}{|z|^6}, \quad R_{k\bar{\ell}} = \frac{\delta_{k\ell}|z|^2 - z^\ell\bar{z}^k}{2|z|^4}. \tag{6.11}$$

**Example 6.3 (Curvatures of Bismut connection).** By Definition (2.44) and Lemma 2.6, we obtain

$$B_{i\bar{j}k}^\ell = \frac{\delta_{jk}\delta_{i\ell} - \delta_{k\ell}\delta_{ij}}{|z|^2} + \frac{\delta_{ij}\bar{z}^k z^\ell + \delta_{k\ell}\bar{z}^i z^j - \delta_{i\ell}\bar{z}^k z^j - \delta_{jk}\bar{z}^i z^\ell}{|z|^4}. \tag{6.12}$$

Two Ricci curvatures are

$$B_{i\bar{j}}^{(1)} = B_{i\bar{j}}^{(2)} = \frac{(2-n)(\delta_{ij}|z|^2 - \bar{z}^i z^j)}{4|z|^2}. \tag{6.13}$$

On the other hand, by formula (6.3), it is easy to see  $\partial\bar{\partial}\omega = 0$  and  $B^{(1)} = 0$  for  $n = 2$ .

**Proposition 6.1.** *Let  $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$  be the standard  $n$ -dimensional ( $n \geq 2$ ) Hopf manifold with canonical metric  $h$ ,*

- (1)  $(M, h)$  has positive second Ricci-Chern curvature  $\Theta^{(2)}$ ;
- (2)  $(M, h)$  has vanishing first Chern class but has nonnegative first Ricci-Chern curvature  $\Theta^{(1)}$ . Moreover,

$$\int_M (\Theta^{(1)})^n = 0; \tag{6.14}$$

- (3)  $(M, h)$  is semi-positive in the sense of Griffiths, i.e.

$$\Theta_{i\bar{j}k\bar{\ell}} u^i \bar{u}^j v^k \bar{v}^\ell \geq 0 \tag{6.15}$$

for any  $u, v \in \mathbb{C}^n$ ;

- (4) The Hermitian–Ricci curvature  $(R_{k\bar{l}})$  is nonnegative and 2-positive everywhere;
- (5)  $(M, h)$  has nonpositive and 2-negative first Ricci–Bismut curvature. In particular,  $(\mathbb{S}^3 \times \mathbb{S}^1, \omega)$  satisfies  $\partial\bar{\partial}\omega = 0$  and has vanishing first Ricci–Bismut curvature  $B^{(1)}$ .

Although we know all Betti numbers of Hopf manifold  $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ ,  $h^{p,0}$  is not so obvious.

**Corollary 6.1.** *Let  $(M, h)$  be  $n$ -dimensional Hopf manifold with  $n \geq 2$ ,*

- (1)  $h^{p,0}(M) = 0$  for  $p \geq 1$  and  $\chi(M, \mathcal{O}) = 1$ .
- (2)  $\dim_{\mathbb{C}} H^0(M, mK) = 0$  for any  $m \geq 1$  where  $K = \det(T^{*1,0}M)$ .

**Remark 6.1.** By Leray–Borel spectral sequence, one can compute all Hodge numbers of all Hopf manifolds. For more details, one can see [28].

### 7. A Natural Geometric Flow on Hermitian Manifolds

As we discussed in the above sections, on Hermitian manifolds, the second Ricci curvature tensors of various metric connections are closely related to the geometry of Hermitian manifolds. A natural idea is to define a flow by using second Ricci curvature tensors of various metric connections. We describe it in the following.

Let  $(M, h)$  be a compact Hermitian manifold. Let  $\nabla$  be an arbitrary metric connection on the holomorphic tangent bundle  $(E, h) = (T^{1,0}M, h)$ .

$$\nabla : E \rightarrow \Omega^1(E). \tag{7.1}$$

It has two components  $\nabla'$  and  $\nabla''$ ,

$$\nabla = \nabla' + \nabla''. \tag{7.2}$$

$\nabla'$  and  $\nabla''$  induce two differential operators

$$\partial_E : \Omega^{p,q}(E) \rightarrow \Omega^{p+1,q}(E), \tag{7.3}$$

$$\bar{\partial}_E : \Omega^{p,q}(E) \rightarrow \Omega^{p,q+1}(E). \tag{7.4}$$

Let  $R^E$  be the  $(1, 1)$  curvature of the metric connection  $\nabla$ . More precisely  $R^E$  is a representation of  $\partial_E\bar{\partial}_E + \bar{\partial}_E\partial_E$ . It is easy to see that

$$R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes \text{End}(E)) \tag{7.5}$$

and locally, we can write it as

$$R^E = R_{jA}^B dz^i \wedge dz^j \otimes e^A \otimes e_B. \tag{7.6}$$

Here we set  $e_A = \frac{\partial}{\partial z^A}$ ,  $e^B = dz^B$  where  $A, B = 1, \dots, n$ , since the geometric meanings of  $j$  and  $A$  are different. It is well-known that a metric connection  $\nabla$  is determined by its Christoffel symbols

$$\nabla_{\frac{\partial}{\partial z^i}} e_A = \Gamma_{iA}^B e_B, \quad \nabla_{\frac{\partial}{\partial \bar{z}^j}} e_A = \Gamma_{jA}^B e_B. \tag{7.7}$$

In particular, we don't have notations such as  $\Gamma_{Ai}^B$ . It is obvious that

$$R_{ij}^A = -\frac{\partial \Gamma_{iA}^B}{\partial \bar{z}^j} + \frac{\partial \Gamma_{jA}^B}{\partial z^i} - \Gamma_{iA}^C \Gamma_{jC}^B + \Gamma_{jA}^C \Gamma_{iC}^B. \tag{7.8}$$

We set the second Hermitian–Ricci curvature tensor of  $(\nabla, h)$  as

$$R^{(2)} = h^{i\bar{j}} R_{i\bar{j}A\bar{B}} e^A \otimes \bar{e}^B \in \Gamma(M, E^* \otimes \bar{E}^*). \tag{7.9}$$

In general, we can study a new class of flows on Hermitian manifolds

$$\begin{cases} \frac{\partial h}{\partial t} = \mathcal{F}(h) + \mu h, \\ h(0) = h_0, \end{cases} \tag{7.10}$$

where  $\mathcal{F}$  can be a linear combination of the first and the second Hermitian–Ricci curvature tensors of different metric connections on  $(T^{1,0}M, h)$ . For examples,  $\mathcal{F}(h) = -\Theta^{(2)}$ , the second Ricci–Chern curvature tensor of the Chern connection, and  $\mathcal{F}(h) = -\hat{R}^{(2)}$ , the second Hermitian–Ricci curvature tensor of the complexified Levi-Civita connection, or the second Ricci curvature of any other Hermitian connection. Quite interesting is to take  $\mathcal{F}(h) = s\Theta^{(1)} + (1 - s)\Theta^{(2)}$  as the mixed Ricci–Chern curvature, or  $\mathcal{F}(h) = B^{(2)} - 2\hat{R}^{(2)}$  where  $B^{(2)}$  is the second Ricci curvature of the Bismut connection. More generally, we can set  $\mathcal{F}(h)$  to be certain suitable functions on the metric  $h$ .

The following result holds for quite general  $\mathcal{F}(h)$ , but here for simplicity we will only take  $\mathcal{F}(h) = -\Theta^{(2)}$  as an example.

$$\begin{cases} \frac{\partial h}{\partial t} = -\Theta^{(2)} + \mu h, \\ h(0) = h_0, \end{cases} \tag{7.11}$$

where  $\mu$  is a real parameter. By formula (2.38), the second Ricci–Chern curvature tensor has components

$$\Theta_{k\bar{\ell}}^{(2)} = h^{i\bar{j}} \Theta_{i\bar{j}k\bar{\ell}} = -h^{i\bar{j}} \frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{i\bar{j}} h^{p\bar{q}} \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j}. \tag{7.12}$$

**Theorem 7.1.** *Let  $(M, h_0)$  be a compact Hermitian manifold.*

- (1) *There exists small  $\varepsilon$  such that, the solution of flow (7.11) exists for  $|t| < \varepsilon$ , and it preserves the Hermitian structure;*
- (2) *The flow (7.11) preserves the Kähler structure, i.e. if the initial metric  $h_0$  is Kähler, then  $h(t)$  are also Kähler.*

**Proof.** (1) Let  $\Delta_c$  be the canonical Laplacian operator on the Hermitian manifold  $(M, h)$  defined by

$$\Delta_c = h^{p\bar{q}} \frac{\partial^2}{\partial z^p \partial \bar{z}^q}. \tag{7.13}$$

Therefore, the second Ricci–Chern curvature  $-\Theta_{i\bar{j}}^{(2)}$  has leading term  $\Delta_c h_{i\bar{j}}$  which is strictly elliptic. The local existence of the flow (7.11) follows by general theory of parabolic PDE, and the solution is a Hermitian metric on  $M$ .

(2) The coefficients of the tensor  $\partial\omega$  are given by

$$f_{i\bar{j}k} = \frac{\partial h_{i\bar{j}}}{\partial z^k} - \frac{\partial h_{k\bar{j}}}{\partial z^i}. \tag{7.14}$$

Under the flow (7.11), we have

$$\begin{cases} \frac{\partial f_{i\bar{j}k}}{\partial t} = \frac{\partial \Theta_{k\bar{j}}^{(2)}}{\partial z^i} - \frac{\partial \Theta_{i\bar{j}}^{(2)}}{\partial z^k} + \mu f_{i\bar{j}k}, \\ f_{i\bar{j}k}(0) = 0. \end{cases} \tag{7.15}$$

At first, we observe that  $f_{i\bar{j}k}(t) \equiv 0$  is a solution of (7.15). In fact, if  $f_{i\bar{j}k}(t) \equiv 0$ , then  $h_{i\bar{j}}(t)$  are Kähler metrics, and so

$$\Theta_{i\bar{j}}^{(2)} = \Theta_{i\bar{j}}^{(1)} = -\frac{\partial^2 \log \det(h_{m\bar{n}})}{\partial z^i \partial \bar{z}^j}.$$

Therefore,

$$\frac{\partial \Theta_{k\bar{j}}^{(2)}}{\partial z^i} - \frac{\partial \Theta_{i\bar{j}}^{(2)}}{\partial z^k} = -\frac{\partial^3 \log \det(h_{m\bar{n}})}{\partial z^i \partial z^k \partial \bar{z}^j} + \frac{\partial^3 \log \det(h_{m\bar{n}})}{\partial z^i \partial z^k \partial \bar{z}^j} = 0. \tag{7.16}$$

On the other hand,

$$\frac{\partial \Theta_{k\bar{j}}^{(2)}}{\partial z^i} - \frac{\partial \Theta_{i\bar{j}}^{(2)}}{\partial z^k} = \Delta_c(f_{i\bar{j}k}) + \text{lower order terms}. \tag{7.17}$$

Hence the solution of (7.15) is unique. □

**Remark 7.1.** Theorem 7.1 holds also for quite general  $\mathcal{F}(h)$  which we will study in detail in a subsequent paper [35].

The flow (7.11) has close connections to several important geometric flows:

- (1) It is very similar to the Hermitian Yang–Mills flow on holomorphic vector bundles. More precisely, if the flow (7.11) has long time solution and it converges to a Hermitian metric  $h_\infty$  such that

$$\Theta_{i\bar{j}}^{(2)} = \mu h_{i\bar{j}}. \tag{7.18}$$

The Hermitian metric  $h_\infty$  is Hermitian–Einstein. So, by [36], the holomorphic tangent bundle  $T^{1,0}M$  is stable. As shown in Example 6.1, the Hopf manifold  $\mathbb{S}^{2n+1} \times \mathbb{S}^1$  is stable for any  $n \geq 1$ . In fact, in the definition of  $\Theta_{i\bar{j}}^{(2)}$ , if we take trace by using the initial metric  $h_0$ , then we get the original Hermitian–Yang–Mills flow equation.

- (2) If the initial metric is Kähler, then this flow is reduced to the usual Kähler–Ricci flow (see [7]).



- (3) The flow (7.11) is similar to the harmonic map flow equation as shown in Theorem 7.1. It is strictly parabolic, and so the long time existence depends on certain curvature condition of the target manifold as discussed in the pioneering work of Eells–Sampson in [11]. The long time existence of this flow and other geometric properties of our new flow will be studied in our subsequent work.

Certain geometric flows and related results have been considered on Hermitian manifolds recently, we refer the reader to [25, 43–45].

### Appendix A. The Proof of the Refined Bochner Formulas

**Lemma A.1.** *On a compact Hermitian manifold  $(M, h, \omega)$ , we have*

$$[\Lambda, 2\partial\omega] = A + B + C, \tag{A.1}$$

where

$$\begin{cases} A = -h^{k\bar{\ell}} h_{i\bar{m}} \Gamma_{s\bar{\ell}}^{\bar{m}} dz^s \wedge dz^i I_k, \\ \bar{A}^* = -h^{s\bar{t}} \Gamma_{s\bar{k}}^{\bar{i}} dz^k \wedge dz^i I_{\bar{t}}, \end{cases} \tag{A.2}$$

$$\begin{cases} B = -2\Gamma_{i\bar{j}}^{\bar{\ell}} dz^i \wedge dz^j I_{\bar{\ell}}, \\ \bar{B}^* = 2h^{p\bar{q}} \Gamma_{\bar{\ell}j}^{\bar{s}} dz^{\bar{\ell}} I_p I_{\bar{s}}, \end{cases} \tag{A.3}$$

$$\begin{cases} C = \Lambda(2\partial\omega) = 2\Gamma_{j\bar{\ell}}^{\bar{\ell}} dz^j, \\ \bar{C}^* = 2h^{j\bar{i}} \Gamma_{j\bar{s}}^{\bar{\ell}} I_{\bar{\ell}} = -2h^{j\bar{i}} \Gamma_{j\bar{i}}^{\bar{\ell}} I_{\bar{\ell}}. \end{cases} \tag{A.4}$$

Moreover,

- (1)  $[\Lambda, A] = -\sqrt{-1}\bar{B}^*$ ;
- (2)  $[\Lambda, B] = -\sqrt{-1}(2\bar{A}^* + \bar{B}^* + \bar{C}^*)$ ;
- (3)  $[\Lambda, C] = -\sqrt{-1}\bar{C}^*$ .

**Proof.** All formulas follow by straightforward computations. □

**Definition A.1.** With respect to  $\nabla'$  and  $\nabla''$ , we define

$$\begin{cases} D' := dz^i \wedge \nabla'_i, \\ D'' := dz^j \wedge \nabla''_j. \end{cases} \tag{A.5}$$

The dual operators of  $\partial, \bar{\partial}, D', D''$  with respect to the norm in (4.13) are denoted by  $\partial^*, \bar{\partial}^*, \delta', \delta''$  and define

$$\begin{cases} \delta'_0 := -h^{i\bar{j}} I_i \nabla''_j, \\ \delta''_0 := -h^{j\bar{i}} I_{\bar{i}} \nabla'_j, \end{cases} \tag{A.6}$$

where  $I$  the contraction operator and  $I_i = I_{\frac{\partial}{\partial z^i}}$  and  $I_{\bar{i}} = I_{\frac{\partial}{\partial \bar{z}^i}}$ .

**Remark A.1.** It is obvious that these first order differential operators  $D', D'', \delta'_0$  and  $\delta''_0$  are well-defined and they do not depend on the choices of holomorphic frames. If  $(M, h)$  is Kähler,  $D' = \partial, D'' = \bar{\partial}, \delta'_0 = \delta' = \partial^*$  and  $\delta''_0 = \delta'' = \bar{\partial}^*$ .

**Lemma A.2.** *In the local holomorphic coordinates,*

$$\partial = D' - \frac{B}{2} \quad \text{and} \quad \bar{\partial} = D'' - \frac{\bar{B}}{2}. \tag{A.7}$$

**Proof.** We only have to check them on functions and 1-forms. □

**Lemma A.3.** *On a compact Hermitian manifold  $(M, h)$ , we have*

$$\begin{cases} \delta'' = \delta'_0 - \frac{\bar{C}^*}{2}, \\ \delta' = \delta'_0 - \frac{C^*}{2}. \end{cases} \tag{A.8}$$

For  $\partial$  and  $\bar{\partial}$ , we have

$$\begin{cases} \partial^* = \delta'_0 - \frac{B^* + C^*}{2}, \\ \bar{\partial}^* = \delta''_0 - \frac{\bar{B}^* + \bar{C}^*}{2}. \end{cases} \tag{A.9}$$

**Proof.** For any  $\varphi \in \Omega^{p,q-1}(M)$  and  $\psi \in \Omega^{p,q}(M)$ , by Stokes' theorem

$$\begin{aligned} 0 &= \int_M \bar{\partial}(\varphi \wedge * \bar{\psi}) \\ &= \int_M \frac{\partial}{\partial \bar{z}^j} (d\bar{z}^j \wedge \varphi \wedge * \bar{\psi}) \\ &= \int_M \frac{\partial}{\partial \bar{z}^j} \left( \langle d\bar{z}^j \wedge \varphi, \psi \rangle \frac{\omega^n}{n!} \right) \\ &= \int_M \frac{\partial}{\partial \bar{z}^j} \left( \langle \varphi, h^{j\bar{i}} I_{\bar{i}} \psi \rangle \frac{\omega^n}{n!} \right) \\ &= \int_M \left( \langle \nabla_j'' \varphi, h^{j\bar{i}} I_{\bar{i}} \psi \rangle + \langle \varphi, \nabla_j' h^{j\bar{i}} I_{\bar{i}} \psi \rangle + \langle \varphi, h^{j\bar{i}} I_{\bar{i}} \psi \rangle \frac{\partial \log \det(h_{m\bar{n}})}{\partial \bar{z}^j} \right) \frac{\omega^n}{n!} \\ &= \int_M \left( \langle d\bar{z}^j \wedge \nabla_j'' \varphi, \psi \rangle + \langle \varphi, h^{j\bar{i}} \nabla_j' I_{\bar{i}} \psi \rangle + \left\langle \varphi, \frac{\partial h^{j\bar{i}}}{\partial z^j} I_{\bar{i}} \psi \right\rangle \right) \frac{\omega^n}{n!} \\ &\quad + \int_M \langle \varphi, h^{j\bar{i}} I_{\bar{i}} \psi \rangle \frac{\partial \log \det(h_{m\bar{n}})}{\partial \bar{z}^j} \frac{\omega^n}{n!}. \end{aligned}$$

That is

$$\begin{aligned} (D'' \varphi, \psi) &= (d\bar{z}^j \wedge \nabla_j'' \varphi, \psi) \\ &= -(\varphi, h^{j\bar{i}} \nabla_j' I_{\bar{i}} \psi) - \left( \varphi, \left( \frac{\partial h^{j\bar{i}}}{\partial z^j} + h^{j\bar{i}} \frac{\partial \log \det(h_{m\bar{n}})}{\partial z^j} \right) I_{\bar{i}} \psi \right). \end{aligned} \tag{A.10}$$

Now we will compute the second and third terms on the right-hand side.

$$\frac{\partial h^{j\bar{i}}}{\partial z^j} + h^{j\bar{i}} \frac{\partial \log \det(h_{m\bar{n}})}{\partial z^j} = h^{j\bar{i}} h^{s\bar{t}} \left( \frac{\partial h_{s\bar{t}}}{\partial z^j} - \frac{\partial h_{j\bar{t}}}{\partial z^s} \right) = 2h^{j\bar{i}} \Gamma_{j\bar{t}}^{\bar{t}} = -2h^{j\bar{\ell}} \Gamma_{j\bar{\ell}}^{\bar{i}}. \tag{A.11}$$

On the other hand

$$\begin{aligned} -h^{j\bar{i}} \nabla'_j I_{\bar{i}} &= -h^{j\bar{i}} I_{\bar{i}} \nabla'_j - h^{j\bar{i}} I \left( \nabla'_j \frac{\partial}{\partial z^i} \right) \\ &= \delta''_0 - h^{j\bar{i}} \Gamma_{j\bar{i}}^{\bar{\ell}} I_{\bar{\ell}}. \end{aligned} \tag{A.12}$$

In summary, by formulas (A.10)–(A.12), the adjoint operator  $\delta''$  of  $D''$  is

$$\delta'' = (\delta''_0 - h^{j\bar{i}} \Gamma_{j\bar{i}}^{\bar{\ell}} I_{\bar{\ell}}) + 2h^{j\bar{i}} \Gamma_{j\bar{i}}^{\bar{\ell}} I_{\bar{\ell}} = \delta''_0 - \frac{\bar{C}^*}{2}.$$

Since  $\bar{\partial} = D'' - \frac{\bar{B}}{2}$ , we get

$$\bar{\partial}^* = \delta'' - \frac{\bar{B}^*}{2} = \delta''_0 - \frac{\bar{B}^* + \bar{C}^*}{2}. \quad \square$$

**Lemma A.4.** *On a compact Hermitian manifold  $(M, h)$ , we have*

$$\begin{cases} [\Lambda, D'] = \sqrt{-1} \left( \delta'' + \frac{\bar{C}^*}{2} \right) \\ [\Lambda, D''] = -\sqrt{-1} \left( \delta' + \frac{C^*}{2} \right) \end{cases} \quad \text{and} \quad \begin{cases} [\delta'', L] = \sqrt{-1} \left( D' + \frac{C}{2} \right) \\ [\delta', L] = -\sqrt{-1} \left( D'' + \frac{\bar{C}}{2} \right). \end{cases} \tag{A.13}$$

**Proof.** By definition

$$\begin{aligned} (\Lambda D')\varphi &= (\sqrt{-1} h^{i\bar{j}} I_i I_{\bar{j}})(dz^k \wedge \nabla'_k \varphi) \\ &= -\sqrt{-1} h^{i\bar{j}} I_i (dz^k \wedge I_{\bar{j}} \nabla'_k \varphi) \\ &= -\sqrt{-1} h^{i\bar{j}} I_{\bar{j}} \nabla'_i \varphi + \sqrt{-1} h^{i\bar{j}} dz^k I_i I_{\bar{j}} \nabla'_k \varphi \\ &= \sqrt{-1} \delta''_0 + dz^k \wedge \nabla'_k (\sqrt{-1} h^{i\bar{j}} I_i I_{\bar{j}} \varphi) \\ &= \sqrt{-1} \delta''_0 + D' \Lambda \varphi, \end{aligned}$$

where we use the metric compatible condition

$$\nabla' \omega = 0 \Rightarrow \nabla'_k (\Lambda \varphi) = \Lambda (\nabla'_k \varphi). \quad \square$$

**Lemma A.5.** *On a compact Hermitian manifold  $(M, h)$ , we have*

$$\begin{cases} [\Lambda, \partial] = \sqrt{-1} (\bar{\partial}^* + \bar{\tau}^*), \\ [\Lambda, \bar{\partial}] = -\sqrt{-1} (\partial^* + \tau^*). \end{cases} \tag{A.14}$$

For the dual case, it is

$$\begin{cases} [\bar{\partial}^*, L] = \sqrt{-1}(\partial + \tau), \\ [\partial^*, L] = -\sqrt{-1}(\bar{\partial} + \bar{\tau}). \end{cases} \tag{A.15}$$

**Proof.** By Lemmas A.1, A.2 and A.4,

$$\begin{aligned} [\Lambda, \partial] &= [\Lambda, D'] - \left[ \Lambda, \frac{B}{2} \right] \\ &= \sqrt{-1} \left( \delta_0'' + \frac{2\bar{A}^* + \bar{B}^* + \bar{C}^*}{2} \right) \\ &= \sqrt{-1} \left( \delta_0'' + \frac{\bar{C}^*}{2} + \frac{2\bar{A}^* + \bar{B}^* + \bar{C}^*}{2} \right) \\ &= \sqrt{-1}(\bar{\partial}^* + \bar{\tau}^*). \end{aligned}$$

The other relations follow by complex conjugate and adjoint operations. □

**Lemma A.6.** On a Hermitian manifold  $(M, h, \omega)$ ,

$$\bar{\partial}^* \omega = \sqrt{-1} \Lambda(\partial \omega) = \sqrt{-1} \Gamma_{\ell j}^{\bar{j}} dz^\ell. \tag{A.16}$$

**Proof.** We have

$$\frac{C}{2} = \Lambda(\partial \omega) = \Gamma_{\ell j}^{\bar{j}} dz^\ell.$$

On the other hand, by Lemma A.3 and  $\delta_0'' \omega = 0$

$$\begin{aligned} \bar{\partial}^* \omega &= \left( \delta_0'' - \frac{\bar{B}^* + \bar{C}^*}{2} \right) \omega = -\frac{\bar{B}^* \omega}{2} - \frac{\bar{C}^* \omega}{2} \\ &= (h_{\ell k} \bar{h}^{p \bar{j}} h^{i \bar{s}} \Gamma_{i j}^{\bar{k}} dz^\ell I_p I_{\bar{s}}) \left( \frac{\sqrt{-1}}{2} h_{m \bar{n}} dz^m \wedge d\bar{z}^n \right) - \frac{\bar{C}^* \omega}{2} \\ &= -\frac{\sqrt{-1}}{2} h_{\ell k} \bar{h}^{i \bar{j}} \Gamma_{i j}^{\bar{k}} dz^\ell - \frac{\bar{C}^* \omega}{2} \\ &= \frac{\sqrt{-1}}{2} \Gamma_{\ell j}^{\bar{j}} dz^\ell - \frac{\bar{C}^* \omega}{2} \\ &= \sqrt{-1} \Gamma_{\ell j}^{\bar{j}} dz^\ell \\ &= \sqrt{-1} \Lambda(\partial \omega). \end{aligned} \tag{A.17}$$

□

Now we assume  $E$  is a Hermitian complex vector bundle or a Riemannian vector bundle over a compact Hermitian manifold  $(M, h, \omega)$  and  $\nabla^E$  is a metric connection on  $E$ .

**Lemma A.7.** *We have the following formula:*

$$\bar{\partial}_E^*(\varphi \otimes s) = (\bar{\partial}^* \varphi) \otimes s - h^{i\bar{j}}(I_{\bar{j}}\varphi) \wedge \nabla_i^E s \tag{A.17}$$

for any  $\varphi \in \Omega^{p,q}(M)$  and  $s \in \Gamma(M, E)$ .

**Proof.** The proof of is the same as Lemma A.3. □

**Lemma A.8.** *If  $\tau$  is the operator of type  $(1,0)$  defined by  $\tau = [\Lambda, 2\partial\omega]$  on  $\Omega^\bullet(M, E)$ , then*

- (1)  $[\bar{\partial}_E^*, L] = \sqrt{-1}(\partial_E + \tau)$ ;
- (2)  $[\partial_E^*, L] = -\sqrt{-1}(\bar{\partial}_E + \bar{\tau})$ ;
- (3)  $[\Lambda, \partial_E] = \sqrt{-1}(\bar{\partial}_E^* + \bar{\tau}^*)$ ;
- (4)  $[\Lambda, \bar{\partial}_E] = -\sqrt{-1}(\partial_E^* + \tau^*)$ .

**Proof.** We only have to prove (3). For any  $\varphi \in \Omega^\bullet(M)$  and  $s \in \Gamma(M, E)$ ,

$$\begin{aligned} (\Lambda\partial_E)(\varphi \otimes s) &= \Lambda(\partial\varphi \otimes s + (-1)^{|\varphi|}\varphi \wedge \partial_E s) \\ &= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}\sqrt{-1}h^{k\bar{\ell}}I_k I_{\bar{\ell}}(\varphi \wedge \partial_E s) \\ &= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}\sqrt{-1}h^{k\bar{\ell}}I_k((I_{\bar{\ell}}\varphi) \wedge \partial_E s) \\ &= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}\sqrt{-1}h^{k\bar{\ell}}(I_k(I_{\bar{\ell}}\varphi)) \wedge \partial_E s \\ &\quad - \sqrt{-1}h^{k\bar{\ell}}I_{\bar{\ell}}(\varphi) \wedge I_k \partial_E s \\ &= (\Lambda\partial\varphi) \otimes s + (-1)^{|\varphi|}(\Lambda\varphi) \wedge \partial_E s - \sqrt{-1}h^{k\bar{\ell}}I_{\bar{\ell}}(\varphi) \wedge \nabla_k^E s. \end{aligned}$$

On the other hand

$$\begin{aligned} (\partial_E\Lambda)(\varphi \otimes s) &= \partial_E((\Lambda\varphi) \otimes s) \\ &= (\partial\Lambda\varphi) \otimes s + (-1)^{|\varphi|}(\Lambda\varphi) \wedge \partial_E s. \end{aligned}$$

Therefore

$$\begin{aligned} [\Lambda, \partial_E](\varphi \otimes s) &= ([\Lambda, \partial]\varphi) \otimes s - \sqrt{-1}h^{k\bar{\ell}}I_{\bar{\ell}}(\varphi) \wedge \nabla_k^E s \\ &= \sqrt{-1}((\bar{\partial}^* + \bar{\tau}^*)\varphi) \otimes s - \sqrt{-1}h^{k\bar{\ell}}I_{\bar{\ell}}(\varphi) \wedge \nabla_k^E s \\ &= \sqrt{-1}(\bar{\partial}_E^* + \bar{\tau}^*)(\varphi \otimes s) \end{aligned}$$

where the last step follows by formula (A.17). □

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