The Chern-Ricci flow and holomorphic bisectional curvature

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Abstract. In this note, we show that on Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$, the non-negativity of the holomorphic bisectional curvature is not preserved along the Chern-Ricci flow.

1. Introduction

The Chern-Ricci flow is an evolution equation for Hermitian metrics on complex manifolds, generalizing the Kähler-Ricci flow. Given an initial Hermitian metric $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}} dz^i \wedge d\bar{z}^j$, the Chern-Ricci flow is defined as

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0, \tag{1.1}$$

where $\mathrm{Ric}(\omega) := -\sqrt{-1}\partial\bar{\partial} \log \det g$ is the Chern-Ricci form of ω. In the case when ω_0 is Kähler, namely $d\omega_0 = 0$, (1.1) coincides with the Kähler-Ricci flow. The Chern-Ricci flow was first introduced by Gill [3] in the setting of manifolds with vanishing first Bott-Chern classes, and many fundamental properties are established by Tosatti and Weinkove [14] on more general manifolds. A variety of further results on Chern-Ricci flow are studied in [14, 15, 16, 4, 5, 17, 18] and some of them are analogues to classical results for the Kähler-Ricci flow (e.g.[7, 2, 9, 10, 13, 11, 12]).

It is proved by Mok [8] (see [1] for Kähler threefolds and also [6]) that the non-negativity of the holomorphic bisectional curvature is preserved along the Kähler-Ricci flow. However, we show that on Hermitian manifolds, the non-negativity of the holomorphic bisectional curvature is not necessarily preserved under the Chern-Ricci flow.

Theorem 1.1. *Let $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be a diagonal Hopf manifold. Fix $T_0 \geq 0$ and let*

$$\omega_0 = \frac{1}{|z|^4} \sum \left((1 + T_0)\delta_{ij}|z|^2 - T_0 \bar{z}^i z^j \right) \sqrt{-1} dz^i \wedge d\bar{z}^j.$$

Then the Chern-Ricci flow (1.1) has maximal existence time $T_{\max} = \frac{T_0+1}{n}$.

(1) *When $t \in \left[0, \dfrac{T_0}{n}\right]$, $\omega(t)$ has non-negative holomorphic bisectional curvature;*

(2) *However, when $t \in \left(\dfrac{2T_0+1}{2n}, \dfrac{T_0+1}{n}\right)$, the holomorphic bisectional curvature of $\omega(t)$ is no longer non-negative.*

Remark 1.2. It is worth to point out that the same proof as in the Kähler case (following Mok) fails for the Chern-Ricci flow since the evolution of the Riemann curvature tensor under the Chern-Ricci flow involves also some terms with the torsion (and its covariant derivatives), which are not there in the Kähler-Ricci flow, where the evolution of the curvature involves only the curvature tensor itself.

Remark 1.3. It is also interesting to investigate sufficient conditions on Hermitian manifolds such that the non-negativity of the holomorphic bisectional curvature is preserved under the Chern-Ricci flow.

2. The proof of Theorem 1.1

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$ with $|\alpha_1| = \cdots = |\alpha_n| \neq 1$, let M be the Hopf manifold $M = (\mathbb{C}^n \setminus \{0\})/\sim$, where

$$(z^1, \ldots, z^n) \sim \left(\alpha_1 z^1, \ldots, \alpha_n z^n\right).$$

It is easy to see that M is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. Fix $T_0 > 0$ and consider the Hermitian metric

$$\omega_0 = \frac{1}{|z|^4}\left((1+T_0)\delta_{ij}|z|^2 - T_0 \overline{z}^i z^j\right)\sqrt{-1}dz^i \wedge d\overline{z}^j.$$

where $|z|^2 = \sum_{j=1}^n |z^j|^2$. It is proved in [14] that

$$\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) \tag{2.1}$$

gives an explicit solution of the Chern-Ricci flow on M with initial metric ω_0. Indeed, by elementary linear algebra, we see $\det(\omega_0) = (1+T_0)^{n-1}|z|^{-2n}$ and so

$$\mathrm{Ric}(\omega_0) = n\sqrt{-1}\partial\overline{\partial}\log|z|^2 = \frac{n}{|z|^2}\left(\delta_{ij} - \frac{\overline{z}^i z^j}{|z|^2}\right)\sqrt{-1}dz^i \wedge d\overline{z}^j \geq 0.$$

For $t < \frac{T_0+1}{n}$, we have the Hermitian metrics

$$\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) = \frac{1}{|z|^2}\left((1+T_0-nt)\delta_{ij} - (T_0-nt)\frac{\overline{z}^i z^j}{|z|^2}\right)\sqrt{-1}dz^i \wedge d\overline{z}^j. \tag{2.2}$$

Hence

$$\det(\omega(t)) = \frac{(1+T_0-nt)^{n-1}}{|z|^{2n}},$$

from which it follows that $\mathrm{Ric}(\omega(t)) = \mathrm{Ric}(\omega_0) = n\sqrt{-1}\partial\overline{\partial}\log|z|^2$. It also implies that $\omega(t)$ solves the Chern-Ricci flow on the maximal existence interval $\left[0, \dfrac{T_0+1}{n}\right)$.

Next, we compute the curvature tensor of the involving metric (2.2). For simplicity, we define a rescaled metric $\omega_\lambda = \sqrt{-1}h_{i\overline{j}}dz^i \wedge d\overline{z}^j$ on M with

$$h_{i\overline{j}} = \frac{1}{|z|^4}\left(\delta_{ij}|z|^2 - \lambda\overline{z}^i z^j\right), \qquad \lambda < 1. \tag{2.3}$$

Note that when

$$\lambda = \frac{T_0 - nt}{1 + T_0 - nt},$$

we have

$$\omega_\lambda = \frac{\omega(t)}{1 + T_0 - nt}. \tag{2.4}$$

Lemma 2.1. *Let $R_{k\bar{j}i\bar{q}}$ be the curvature components of ω_λ, then*

$$\begin{aligned} R_{k\bar{j}i\bar{q}} &= \frac{\delta_{iq}(\delta_{jk}|z|^2 - \overline{z}^k z^j)}{|z|^6} + \frac{\lambda\left(\delta_{ij}|z|^2 - \overline{z}^i z^j\right)\left(\delta_{kq}|z|^2 - \overline{z}^k z^q\right)}{|z|^8} \\ &\quad + \frac{(\lambda^2 - 2\lambda)\overline{z}^i z^q(\delta_{kj}|z|^2 - \overline{z}^k z^j)}{|z|^8}. \end{aligned}$$

Proof. By using elementary linear algebra, one has $\det(h_{i\bar{j}}) = (1-\lambda)|z|^{-2n}$ and so

$$\mathrm{Ric}(\omega_\lambda) = n\sqrt{-1}\partial\overline{\partial}\log|z|^2 \geq 0. \tag{2.5}$$

On the other hand, one can verify that the matrix $(h_{i\bar{j}})$ has (transpose) inverse matrix

$$h^{i\bar{j}} = |z|^2\left(\delta_{ij} + \frac{\lambda z^i \overline{z}^j}{(1-\lambda)|z|^2}\right). \tag{2.6}$$

By straightforward computation,

$$\frac{\partial h_{i\bar{j}}}{\partial z^k} = -\frac{\delta_{ij}\overline{z}^k}{|z|^4} - \frac{\lambda\delta_{jk}\overline{z}^i}{|z|^4} + \frac{2\lambda\overline{z}^i\overline{z}^k z^j}{|z|^6} = \frac{2\lambda\overline{z}^i\overline{z}^k z^j}{|z|^6} - \frac{\lambda\delta_{jk}\overline{z}^i + \delta_{ij}\overline{z}^k}{|z|^4} \tag{2.7}$$

and so

$$\begin{aligned} \Gamma^p_{ki} &= h^{p\bar{j}}\frac{\partial h_{i\bar{j}}}{\partial z^k} = |z|^2\left(\delta_{pj} + \frac{\lambda z^p \overline{z}^j}{(1-\lambda)|z|^2}\right)\left(\frac{2\lambda\overline{z}^i\overline{z}^k z^j}{|z|^6} - \frac{\lambda\delta_{jk}\overline{z}^i + \delta_{ij}\overline{z}^k}{|z|^4}\right) \\ &= \frac{2\lambda\overline{z}^i\overline{z}^k z^p}{|z|^4} - \frac{\lambda\delta_{pk}\overline{z}^i + \delta_{ip}\overline{z}^k}{|z|^2} + \frac{2\lambda^2\overline{z}^i\overline{z}^k z^p}{(1-\lambda)|z|^4} - \frac{\lambda^2\overline{z}^i\overline{z}^k z^p + \lambda\overline{z}^i\overline{z}^k z^p}{(1-\lambda)|z|^4} \\ &= \frac{\lambda\overline{z}^i\overline{z}^k z^p}{|z|^4} - \frac{\lambda\delta_{pk}\overline{z}^i + \delta_{ip}\overline{z}^k}{|z|^2}. \end{aligned}$$

The Chern curvature tensor of ω_λ is

$$\begin{aligned} & R^p_{k\bar{j}i} = -\frac{\partial \Gamma^p_{ki}}{\partial \overline{z}^j} \\ = & -\frac{\lambda\delta_{ij}\overline{z}^k z^p + \lambda\delta_{kj}\overline{z}^i z^p}{|z|^4} + \frac{2\lambda\overline{z}^i\overline{z}^k z^p z^j}{|z|^6} + \frac{\lambda\delta_{pk}\delta_{ij} + \delta_{ip}\delta_{kj}}{|z|^2} - \frac{\lambda\delta_{pk}\overline{z}^i z^j + \delta_{ip}\overline{z}^k z^j}{|z|^4} \\ = & \frac{\lambda\delta_{pk}\delta_{ij} + \delta_{ip}\delta_{kj}}{|z|^2} + \frac{2\lambda\overline{z}^i\overline{z}^k z^p z^j}{|z|^6} - \frac{\lambda\left(\delta_{ij}\overline{z}^k z^p + \delta_{kj}\overline{z}^i z^p + \delta_{pk}\overline{z}^i z^j\right) + \delta_{ip}\overline{z}^k z^j}{|z|^4}. \end{aligned}$$

Hence

$$
\begin{aligned}
& R_{k\bar{j}i\bar{q}} = h_{p\bar{q}} R^p_{k\bar{j}i} \\
= \; & \frac{\delta_{pq}|z|^2}{|z|^4}\left[\frac{\lambda\delta_{pk}\delta_{ij}+\delta_{ip}\delta_{kj}}{|z|^2}+\frac{2\lambda\bar{z}^i\bar{z}^k z^p z^j}{|z|^6}-\frac{\lambda\left(\delta_{ij}\bar{z}^k z^p+\delta_{kj}\bar{z}^i z^p+\delta_{pk}\bar{z}^i z^j\right)+\delta_{ip}\bar{z}^k z^j}{|z|^4}\right] \\
& -\frac{\lambda\bar{z}^p z^q}{|z|^4}\left[\frac{\lambda\delta_{pk}\delta_{ij}+\delta_{ip}\delta_{kj}}{|z|^2}+\frac{2\lambda\bar{z}^i\bar{z}^k z^p z^j}{|z|^6}-\frac{\lambda\left(\delta_{ij}\bar{z}^k z^p+\delta_{kj}\bar{z}^i z^p+\delta_{pk}\bar{z}^i z^j\right)+\delta_{ip}\bar{z}^k z^j}{|z|^4}\right] \\
= \; & \frac{\lambda\delta_{qk}\delta_{ij}+\delta_{iq}\delta_{jk}}{|z|^4}+\frac{2\lambda\bar{z}^i\bar{z}^k z^j z^q}{|z|^8}-\frac{\lambda\left(\delta_{ij}\bar{z}^k z^q+\delta_{kj}\bar{z}^i z^q+\delta_{kq}\bar{z}^i z^j\right)+\delta_{iq}\bar{z}^k z^j}{|z|^6} \\
& -\frac{\lambda^2\delta_{ij}\bar{z}^k z^q+\lambda\delta_{kj}\bar{z}^i z^q}{|z|^6}-\frac{2\lambda^2\bar{z}^i\bar{z}^k z^j z^q}{|z|^8} \\
& +\frac{\lambda^2\left(\delta_{ij}\bar{z}^k z^q|z|^2+\delta_{kj}\bar{z}^i z^q|z|^2+\bar{z}^i\bar{z}^k z^j z^q\right)+\lambda\bar{z}^i\bar{z}^k z^j z^q}{|z|^8} \\
= \; & \frac{\lambda\delta_{qk}\delta_{ij}+\delta_{iq}\delta_{jk}}{|z|^4}+\frac{(3\lambda-\lambda^2)\bar{z}^i\bar{z}^k z^j z^q}{|z|^8}-\frac{\lambda\delta_{qk}\bar{z}^i z^j}{|z|^6}-\frac{\lambda\delta_{ij}\bar{z}^k z^q}{|z|^6} \\
& +\frac{(\lambda^2-2\lambda)\delta_{kj}\bar{z}^i z^q}{|z|^6}+\frac{\delta_{iq}\bar{z}^k z^j}{|z|^6} \\
= \; & \frac{\delta_{iq}(\delta_{jk}|z|^2-\bar{z}^k z^j)}{|z|^6}+\frac{\lambda\delta_{ij}\left(\delta_{kq}|z|^2-\bar{z}^k z^q\right)}{|z|^6}+\frac{(\lambda^2-2\lambda)\bar{z}^i z^q(\delta_{kj}|z|^2-\bar{z}^k z^j)}{|z|^8} \\
& +\frac{\lambda\bar{z}^i z^j\left(\bar{z}^k z^q-\delta_{kq}|z|^2\right)}{|z|^8} \\
= \; & \frac{\delta_{iq}(\delta_{jk}|z|^2-\bar{z}^k z^j)}{|z|^6}+\frac{\lambda\left(\delta_{ij}|z|^2-\bar{z}^i z^j\right)\left(\delta_{kq}|z|^2-\bar{z}^k z^q\right)}{|z|^8}+\frac{(\lambda^2-2\lambda)\bar{z}^i z^q(\delta_{kj}|z|^2-\bar{z}^k z^j)}{|z|^8}.
\end{aligned}
$$

□

Lemma 2.2. *For any $\lambda\in[0,1)$, ω_λ has non-negative holomorphic bisectional curvature.*

Proof. For any $\xi=(\xi^1,\cdots,\xi^n)$ and $\eta=(\eta^1,\cdots,\eta^n)$, by Lemma 2.1 we have

$$
\begin{aligned}
R_{k\bar{j}i\bar{q}}\xi^k\bar{\xi}^j\eta^i\bar{\eta}^q \;=\; & \frac{|\eta|^2(|z|^2|\xi|^2-|\bar{z}\cdot\xi|^2)}{|z|^6}+\frac{\lambda\left|\left(\delta_{ij}|z|^2-\bar{z}^i z^j\right)\eta^i\bar{\xi}^j\right|^2}{|z|^8} \\
& +\frac{(\lambda^2-2\lambda)|\bar{z}\cdot\eta|^2(|z|^2|\xi|^2-|z\cdot\bar{\xi}|^2)}{|z|^8}.
\end{aligned}
$$

Since $|z|^2|\eta|^2\geq|\bar{z}\cdot\eta|^2$, we obtain

$$
R_{k\bar{j}i\bar{q}}\xi^k\bar{\xi}^j\eta^i\bar{\eta}^q\geq\frac{\lambda\left|\left(\delta_{ij}|z|^2-\bar{z}^i z^j\right)\eta^i\bar{\xi}^j\right|^2}{|z|^8}+\frac{(\lambda^2-2\lambda+1)|\bar{z}\cdot\eta|^2(|z|^2|\xi|^2-|z\cdot\bar{\xi}|^2)}{|z|^8}.
$$

The right hand side is non-negative when $\lambda\geq 0$. □

Corollary 2.3. *The initial metric ω_0 has non-negative holomorphic bisectional curvature.*

Proof. When $t=0$, or equivalently $\lambda = \frac{T_0}{1+T_0}$, we know $\omega_\lambda = \frac{\omega_0}{1+T_0}$. Since $\lambda = \frac{T_0}{1+T_0} \in [0,1)$, by Lemma 2.2, ω_0 has non-negative holomorphic bisectional curvature. $\square$

Lemma 2.4. *When $\lambda < -1$, the holomorphic sectional curvature of the metric ω_λ is no longer non-negative. In particular, the holomorphic bisectional curvature of the metric ω_λ is no longer non-negative.*

Proof. For any $\xi = (\xi^1, \cdots, \xi^n)$, we have

$$\begin{aligned} R_{k\bar{j}i\bar{q}}\xi^k\overline{\xi}^j\xi^i\overline{\xi}^q &= \frac{|\xi|^2(|z|^2|\xi|^2 - |\overline{z}\cdot\xi|^2)}{|z|^6} + \frac{\lambda(|z|^2|\xi|^2 - |\overline{z}\cdot\xi|^2)^2}{|z|^8} \\ &\quad + \frac{(\lambda^2 - 2\lambda)|\overline{z}\cdot\xi|^2(|z|^2|\xi|^2 - |\overline{z}\cdot\xi|^2)}{|z|^8} \\ &= \frac{(3\lambda-\lambda^2)|\overline{z}\cdot\xi|^4 + (\lambda+1)(|z|^2|\xi|^2)^2 + (\lambda^2-4\lambda-1)|\overline{z}\cdot\xi|^2|z|^2\cdot|\xi|^2}{|z|^8}. \end{aligned}$$

Let $a = |\overline{z}\cdot\xi|^2$ and $b = |z|^2|\xi|^2$, then

$$\begin{aligned} R_{k\bar{j}i\bar{q}}\xi^k\overline{\xi}^j\xi^i\overline{\xi}^q &= \frac{(3\lambda-\lambda^2)a^2 + (\lambda^2-4\lambda-1)ab + (\lambda+1)b^2}{|z|^8} \\ &= \frac{(b-a)a(\lambda-1)^2 + (b-a)^2(\lambda+1)}{|z|^8}. \end{aligned}$$

It is easy to see that, $b \geq a \geq 0$ and so for any $-1 \leq \lambda < 1$

$$R_{k\bar{j}i\bar{q}}\xi^k\overline{\xi}^j\xi^i\overline{\xi}^q \geq 0.$$

However, when $\lambda < -1$, $R_{k\bar{j}i\bar{q}}\xi^k\overline{\xi}^j\xi^i\overline{\xi}^q$ is no longer nonnegative. Indeed, for any given $z = (z^1, \cdots, z^n)$, we choose a nonzero vector $\xi = (\xi^1, \cdots, \xi^n)$ such that $\overline{z}\cdot\xi = 0$, i.e. $\sum \overline{z}^i \cdot \xi^i = 0$. In this case, we have $a = |\overline{z}\cdot\xi| = 0$, but $b = |z|^2|\xi|^2 > 0$. Moreover,

$$R_{k\bar{j}i\bar{q}}\xi^k\overline{\xi}^j\xi^i\overline{\xi}^q = \frac{b^2(\lambda+1)}{|z|^8} < 0$$

since $\lambda < -1$. $\square$

The proof of Theorem 1.1. By (2.4), we see when $\lambda = \frac{T_0 - nt}{1+T_0-nt}$, $\omega_\lambda = \frac{\omega(t)}{1+T_0-nt}$. Hence,

(1) by Lemma 2.2, when $\lambda \in [0,1)$ or equivalently, $0 \leq t \leq \frac{T_0}{n}$, $\omega(t)$ has non-negative holomorphic bisectional curvature;
(2) by Lemma 2.4, when $\lambda < -1$, or equivalently, $\frac{2T_0+1}{2n} < t < \frac{T_0+1}{n}$, the holomorphic bisectional curvature of $\omega(t)$ is no longer non-negative. $\square$

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