Curvatures of moduli space of curves and applications

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Abstract. In this paper, we investigate the geometry of the moduli space of curves by using the curvature properties of direct image sheaves of vector bundles. We show that the moduli space $(\mathcal{M}_g, \omega_{\mathrm{WP}})$ of curves with genus $g > 1$ has dual-Nakano negative and semi-Nakano-negative curvature, and in particular, it has non-positive Riemannian curvature operator and also non-positive complex sectional curvature. We also prove that any submanifold in $\mathcal{M}_g$ which is totally geodesic in $\mathcal{A}_g$ with finite volume must be a ball quotient.

1. Introduction

In this paper, we study the curvature properties of the Weil-Petersson metric as well as its background Riemannian metric on the moduli space of curves.

On Riemannian manifolds, there are many curvature terminologies, e.g. curvature operator, sectional curvature, isotropic curvature and etc.. As it is well-known, the curvature relations are well understood on Riemannian manifolds (e.g. [3, p.100]). On the other hand, we also have some classical curvature concepts on Kähler manifolds, such as the holomorphic bisectional curvature, curvature in the sense of Siu and curvature in the sense of Nakano. At first, we obtain a list of curvature relations between a variety of curvature properties of the Kähler metric and its background Riemannian metric:

Theorem 1.1. *On a Kähler manifold (X, ω), the curvatures*

(1) *semi dual-Nakano-negative;*
(2) *non-positive Riemannian curvature operator;*
(3) *strongly non-positive in the sense of siu;*
(4) *non-positive complex sectional curvature;*
(5) *non-positive Riemannian sectional curvature;*
(6) *non-positive holomorphic bisectional curvature;*
(7) *non-positive isotropic curvature*

have the following relations

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (5) \Longrightarrow (6);$$
$$(1) \Longrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (7).$$

Let $f : \mathcal{T}_g \to \mathcal{M}_g$ be the universal curve with genus $g \geq 2$. Since it is a canonically polarized family, and

$$T^*\mathcal{M}_g \cong f_*(K^{\otimes 2}_{\mathcal{T}_g/\mathcal{M}_g}),$$

one can compute the curvature of the induced metric on $T^*\mathcal{M}_g$ by using the curvature formula of direct image sheaves (e.g. [2], [23] and [12]). This induced metric is actually conjugate dual to the Weil-Petersson metric ω_{WP} on $\mathcal{M}_g$. By adapting the methods in [11], we show that

$(\mathcal{M}_g, \omega_{\mathrm{WP}})$ has the similar curvature properties as the space form–the unit disk $(\mathbb{B}^{3g-3}, \omega_{\mathrm{B}})$ with the invariant Bergman metric ω_{B}, i.e. $(\mathcal{M}_g, \omega_{\mathrm{WP}})$ possesses the strongest curvature properties of complex manifolds.

Theorem 1.2. *The curvature of Weil-Petersson metric ω_{WP} on the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g \geq 2$ is dual-Nakano-negative and semi-Nakano-negative.*

As applications of Theorem 1.1 and Theorem 1.2, we obtain a variety of curvature properties of the moduli space of curves:

Theorem 1.3. *The moduli space $(\mathcal{M}_g, \omega_{\mathrm{WP}})$ has the following curvature properties:*

(1) *dual-Nakano-negative and semi Nakano-negative curvature;*
(2) *non-positive Riemannian curvature operator;*
(3) *non-positive complex sectional curvature;*
(4) *strongly-negative curvature in the sense of Siu;*
(5) *negative Riemannian sectional curvature;*
(6) *negative holomorphic bisectional curvature;*
(7) *non-positive isotropic curvature.*

Note that, part (2) is firstly obtained in [32] recently.

We describe another application of the curvature properties of $(\mathcal{M}_g, \omega_{\mathrm{WP}})$. Denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g and denote by

$$j : \mathcal{M}_g \to \mathcal{A}_g \tag{1.1}$$

the Torelli map associating to a curve its Jacobian with its natural principal polarization. We denote by Jac_g the image $j(\mathcal{M}_g)$ and let $\overline{Jac_g}$, the so-called Torelli locus, be the schematic closure of Jac_g in $\mathcal{A}_g$. Frans Oort asked in [21, Section 7] whether there exists any locally symmetric subvariety of $\mathcal{A}_g$ which is contained in $\overline{Jac_g}$ and intersects Jac_g, and he conjectured that nontrivial such subvarieties do not exist. This conjecture is extensively studied in the last decade by using algebraic geometry methods. In this paper, we use a differential geometric approach and obtain the following

Theorem 1.4. *Let $j : (\mathcal{M}_g, \omega_{\mathrm{WP}}) \to (\mathcal{A}_g, \omega_H)$ be the Torelli map where ω_H is the Hodge metric. Let V be a submanifold in $\mathcal{M}_g$ with $j(V)$ totally geodesic in $(\mathcal{A}_g, \omega_H)$. If $j(V)$ has finite volume, then V must be a ball quotient. In particular, any compact submanifold V in $\mathcal{M}_g$ with $j(V)$ totally geodesic in $(\mathcal{A}_g, \omega_H)$ must be a ball quotient.*

Remark 1.5. By using algebraic methods, Hain ([8, Theorem 1]) and de Jong-Zhang ([5, Theorem 1.1]) proved similar results under certain conditions. See also [17], [13] and [4]. For more progress on Oort's conjecture, we refer the reader to survey papers [21, 18].

As a special case, we show that there is no higher rank locally symmetric space in $\mathcal{M}_g$:

Corollary 1.6. *Let Ω be an irreducible bounded symmetric domain and $\Gamma \subset Aut(X)$ be a torsion-free cocompact lattice, $X := \Omega/\Gamma$. Let h be the canonical metric on X. If there exists a nonconstant holomorphic mapping $f : (X, h) \to (\mathcal{M}_g, \omega_{\mathrm{WP}})$, then Ω must be of rank 1, i.e. X must be a ball quotient.*

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2. Curvature relations on vector bundles

Let E be a holomorphic vector bundle over a Kähler manifold X and h a Hermitian metric on E. There exists a unique connection ∇ which is compatible with the metric h and the complex structure on E. It is called the Chern connection of (E,h). Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^\nabla \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has the form

$$R^\nabla = \frac{\sqrt{-1}}{2\pi} R^{\gamma}_{i\bar{j}\alpha} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma \tag{2.1}$$

where $R^{\gamma}_{i\bar{j}\alpha} = h^{\gamma\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \bar{z}^j}. \tag{2.2}$$

Here and henceforth we adopt the Einstein convention for summation.

Definition 2.1. A Hermitian vector bundle (E,h) is said to be *Griffiths-positive*, if for any nonzero vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^i \bar{u}^j v^\alpha \bar{v}^\beta > 0. \tag{2.3}$$

(E,h) is said to be *Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0. \tag{2.4}$$

(E,h) is said to be *dual-Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\beta} \bar{u}^{j\alpha} > 0. \tag{2.5}$$

The notions of semi-positivity, negativity and semi-negativity can be defined similarly. We say E is Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, $\cdots$), if it admits a Nakano-positive(resp. Griffiths-positive, dual-Nakano-positive, $\cdots$) metric.

Remark 2.2. It is easy to see that (E,h) is dual-Nakano-positive if and only if (E^*,h^*) is Nakano-negative.

As models of complex manifolds, one has the following well-known curvature properties:

Lemma 2.3. *Let $n > 1$.*

1. $(T\mathbb{P}^n, \omega_{FS})$ *is dual-Nakano-positive and semi-Nakano-positive.*
2. *Let X be a hyperbolic space form with dimension n. If ω_B is the canonical metric on X, then (TX, ω_B) is dual-Nakano-negative and semi-Nakano-negative.*

We shall use the following curvature monotonicity formulas frequently, in particular the explicit curvature formulas(e.g. (2.8)). Hence we include a detailed proof.

Lemma 2.4. *Let (E,h) be a Hermitian holomorphic vector bundle over a complex manifold X, S be a holomorphic subbudle of E and Q the corresponding quotient bundle, $0 \to S \to E \to Q \to 0$.*

(1) *If E is (semi-)Nakano-negative, then S is also (semi-)Nakano negative.*
(2) *If E is (semi-)dual-Nakano-positive, then Q is also (semi-)dual-Nakano-positive.*

Proof. This lemma is well-known(e.g.[6]). It is obvious that (2) is dual to (1). Let r be the rank of E and s the rank of S. Without loss of generality, we can assume, at a fixed point $p \in X$, there exists a local holomorphic frame $\{e_1, \cdots, e_r\}$ of E centered at point p such that $\{e_1, \cdots, e_s\}$ is a local holomorphic frame of S. Moreover, we can assume that $h(e_\alpha, e_\beta)(p) = \delta_{\alpha\beta}$, for $1 \leq \alpha, \beta \leq r$. Hence, the curvature tensor of S at point p is

$$R^S_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + \sum_{\gamma=1}^{s} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{2.6}$$

where $1 \leq \alpha, \beta \leq s$. The curvature tensor of E at point p is

$$R^E_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + \sum_{\gamma=1}^{r} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{2.7}$$

where $1 \leq \alpha, \beta \leq r$. By formula (2.4), it is easy to see that

$$R^E|_S - R^S = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} \sum_{\alpha,\beta=1}^{s} \left(\sum_{\gamma=s+1}^{r} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \right) dz^i \wedge d\overline{z}^j \otimes e^\alpha \otimes e^\beta \tag{2.8}$$

is semi-Nakano-positive. Hence (1) follows. □

3. Curvatures of direct image sheaves and the moduli space $\mathcal{M}_g$

3.1. **Curvature of direct image sheaves.** Let $\mathcal{X}$ be a Kähler manifold with dimension $d+n$ and S a Kähler manifold with dimension d. Let $f : \mathcal{X} \to S$ be a proper Kähler fibration. Hence, for each $s \in S$,

$$X_s := f^{-1}(\{s\})$$

is a compact Kähler manifold with dimension n. Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be a Hermitian holomorphic vector bundle. Consider the space of holomorphic $\mathcal{E}$-valued $(n,0)$-forms on X_s,

$$E_s := H^0(X_s, \mathcal{E}_s \otimes K_{X_s}) \cong H^{n,0}(X_s, \mathcal{E}_s)$$

where $\mathcal{E}_s = \mathcal{E}|_{X_s}$. It is well-known that, if the vector bundle $\mathcal{E}$ is "positive" in certain sense, by Grauert locally free theorem, there is a natural holomorphic structure on

$$E = \bigcup_{s \in S} \{s\} \times E_s$$

such that the vector bundle E is isomorphic to the direct image sheaf $f_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. Using the canonical isomorphism $K_{\mathcal{X}/S}|_{X_s} \cong K_{X_s}$, a *local smooth* section u of E over S can be identified as a family of $\mathcal{E}$-valued holomorphic $(n,0)$ form on X_s. By this identification, there is a natural metric on E. For any local smooth section u of E, one can define a Hermitian metric on E by

$$h(u,u) = c_n \int_{X_s} \{u, u\} \tag{3.1}$$

where $c_n = (\sqrt{-1})^{n^2}$. Here, we only use the Hermitian metric of $\mathcal{E}_s$ on each fiber X_s and we do not specify background Kähler metrics on the fibers.

In particular, we consider a canonically polarized family $f : \mathcal{X} \to S$. We define the Hodge metric on $E = f_*(K_{\mathcal{X}/S}^{\otimes m})$ where m is an integer with $m \geq 2$. Let L_t be the restriction of the line bundle $\mathcal{L} := K_{\mathcal{X}/S}^{\otimes(m-1)}$ on the fiber X_t. Let $U \subset S$ be a small open neighborhood of $t = 0$. Let s be a local smooth section of $f_*(K_{\mathcal{X}/S}^{\otimes m})$, by the very definition of direct image sheaf $f_*(K_{\mathcal{X}/S}^{\otimes m})$, for any $t \in U$

$$s(t) \in H^0(X_t, K_{X_t}^{\otimes m}) \tag{3.2}$$

By this identification, for any local smooth sections s_α, s_β of $f_*(K_{\mathcal{X}/S}^{\otimes m})$, we define the *Hodge metric* on $f_*(K_{\mathcal{X}/S}^{\otimes m})$ by

$$h(s_\alpha, s_\beta) = c_n \int_{X_t} \langle s_\alpha, s_\beta \rangle dV_t \tag{3.3}$$

where $\langle \bullet, \bullet \rangle$ is the pointwise inner product on $\Gamma(X_t, K_{X_t}^{\otimes m})$. More precisely, if $s_\alpha = \varphi_\alpha \otimes e$ and $s_\beta = \varphi_\beta \otimes e$, where e is a local holomorphic basis of $K_{X_t}^{\otimes m}$ and $\varphi_\alpha, \varphi_\beta$ are local smooth functions on X_t, the metric is

$$h(s_\alpha, s_\beta) = c_n \int_{X_t} \langle s_\alpha, s_\beta \rangle dV_t = c_n \int_{X_t} \langle \varphi_\alpha, \varphi_\beta \rangle |e|^2 dV_t$$

where $|e|^2$ is the canonical metric on $K_{X_t}^{\otimes m}$. Since the family is canonically polarized, the metrics on $E = f_*(K_{\mathcal{X}/S}^{\otimes m})$ defined by (3.1) and (3.3) are the same.

Let $(t_1, \cdots, t_d)$ be the local holomorphic coordinates centered at a point $p \in S$ and

$$\nu : T_p S \to H^1(X, TX)$$

the Kodaira-Spencer map on the center fiber $X = \pi^{-1}(p)$ and $\theta_i \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ the harmonic representatives of the images $\nu(\frac{\partial}{\partial t^i})$ for $i = 1, \cdots, d$. Let $\{\sigma_\alpha\}$ be a basis of $\mathbb{H}^{n,0}(X, L)$ where $L = K_{\mathcal{X}/S}^{\otimes(m-1)}|_X$. The following theorem is well-known (e.g. [23, Theorem IV] with $p = n$; for similar formulations see also [9, 10, 11, 2, 29, 28, 12].)

Theorem 3.1. *If $f : \mathcal{X} \to S$ is effectively parameterized and $m \geq 2$, at point p, the curvature tensor of the Hodge metric h on $f_*(K_{\mathcal{X}/S}^{\otimes m})$ is*

$$\begin{aligned} R_{i\bar{j}\alpha\bar{\beta}} &= (m-1)\left((\Delta + m - 1)^{-1}\left(\theta_i \lrcorner \sigma_\alpha\right), \theta_j \lrcorner \sigma_\beta\right) \\ &+ (m-1)\left((\Delta + 1)^{-1}(\langle \theta_i, \theta_j \rangle) \cdot \sigma_\alpha, \sigma_\beta\right). \end{aligned} \tag{3.4}$$

Remark 3.2. Note that there are two Green's operators in formula (3.4) and they have different geometric meanings. More precisely, $(\Delta + m - 1)^{-1}$ acts on sections and $(\Delta + 1)^{-1}$ acts on functions. When $m = 2$, the curvature formula (3.4) is, in fact, different from Wolpert's curvature formula ([31]) since in Wolpert's formula, two Green's operators are the same and both of them act on functions. We will analyze them carefully in the next subsection.

3.2. **Wolpert's curvature formula.** In this subsection, we will derive Wolpert's curvature formula of the Weil-Petersson metric on the moduli space $\mathcal{M}_k$ of Riemann surfaces with genus $k \geq 2$ from curvature formula (3.4).

Let (X_0, ω_g) be a Riemann surface with the Poincaré metric ω_g. The Weil-Petersson metric on the moduli space $\mathcal{M}_k$ is defined as

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right)_{WP} = \int_{X_0} \theta_i \cdot \overline{\theta}_j dV \tag{3.5}$$

where $\theta_i, \theta_j \in \mathbb{H}^{0,1}(X_0, T^{1,0}X_0)$ are the images of $\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}$ under the Kodaira-Spencer map: $T_0\mathcal{M}_k \to \mathbb{H}^{0,1}(X_0, T^{1,0}X_0)$ respectively. It is known that $f_*(K^{\otimes 2}_{\mathcal{T}_k/\mathcal{M}_k})$ is isomorphic to the holomorphic cotangent bundle $T^{*1,0}\mathcal{M}_k$. Hence, there are two Hermitian metrics on this bundle, one is the Weil-Petersson metric and the other one is the Hodge metric defined in (3.3). In order to discuss the relations between these two metrics, we can consider the natural isomorphism

$$T : \Omega^{1,0}(X_0, K_{X_0}) \to \Omega^{0,1}(X_0, T^{1,0}X_0)$$

given by

$$T\left(\eta dz \otimes dz\right) = g^{-1}\overline{\eta} d\overline{z} \otimes \frac{\partial}{\partial z}. \tag{3.6}$$

Lemma 3.3. (1) *The operator T is well-defined;*
(2) *$\sigma \in \mathbb{H}^{1,0}(X_0, K_{X_0})$ if and only if $T(\sigma) \in \mathbb{H}^{0,1}(X_0, T^{1,0}X_0)$.*

Let $\sigma_\alpha \in \mathbb{H}^{1,0}(X_0, K_{X_0})$. To simplify notations, $T(\sigma_\alpha)$ is denoted by θ_α. The local inner product on the space $\Omega^{0,1}(X_0, T^{1,0}X_0)$ is denoted by $\langle \bullet, \bullet \rangle$ and sometimes it is also denoted by $\cdot$. That is, if $\eta, \mu \in \Omega^{1,0}(X_0, T^{1,0}X_0)$, $\langle \eta, \mu \rangle = \eta \cdot \overline{\mu}$. It is easy to see that

$$g \cdot \langle \theta, T(\sigma) \rangle d\overline{z} \otimes dz = \theta \lrcorner \sigma \tag{3.7}$$

Lemma 3.4. *The Hodge metric coincides with the Weil-Petersson metric on the moduli space $\mathcal{M}_k$. More precisely, $T : \left(\mathbb{H}^{1,0}(X_0, K_{X_0}), g_{\text{Hodge}}\right) \to \left(\mathbb{H}^{0,1}(X_0, T^{1,0}X_0), g_{\text{WP}}\right)$ is a conjugate-isometry.*

Proof. Let $\sigma_\alpha = f_\alpha dz \otimes e, \sigma_\beta = f_\beta dz \otimes e$, then

$$\langle \sigma_\alpha, \sigma_\beta \rangle_{\text{Hodge}} = \sqrt{-1} \int_{X_0} g^{-1} f_\alpha \overline{f}_\beta dz \wedge d\overline{z} = \int_{X_0} g^{-2} f_\alpha \overline{f}_\beta dV$$

Similarly,

$$\langle T(\sigma_\beta), T(\sigma_\alpha) \rangle_{\text{WP}} = \int_{X_0} g^{-2} f_\alpha \overline{f}_\beta dV = \langle \overline{T(\sigma_\alpha)}, \overline{T(\sigma_\beta)} \rangle_{\text{WP}}.$$

That is

$$\langle \sigma_\alpha, \sigma_\beta \rangle_{\text{Hodge}} = \langle \overline{T(\sigma_\alpha)}, \overline{T(\sigma_\beta)} \rangle_{\text{WP}}.$$

□

Let $\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be the Laplacian operator on the space $\Omega^{p,q}(X_0, L_0)$ and $\Delta_0 = \overline{\partial}^*\overline{\partial}$ the Laplacian operator on $C^\infty(X_0)$.

Lemma 3.5. *We have the following relation between two different Green's operators*

$$(\Delta+1)^{-1}(\theta_i \lrcorner \sigma_\alpha) = g(\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e \tag{3.8}$$

Proof. It is obvious that the right hand side of (3.8) is a well-defined tensor. Hence, without loss of generality, we can verify formula (3.8) in the normal coordinate of the Kähler-Einstein metric. Let $\{z\}$ be the normal coordinate centered at a fixed point p, i.e.,

$$g(p)=1, \frac{\partial g}{\partial z}(p) = \frac{\partial g}{\partial \overline{z}}(p) = 0$$

The Kähler-Einstein condition is equivalent to

$$\Delta_0 g = -1 \tag{3.9}$$

at the fixed point p. Let $s = f d\overline{z} \otimes e \in \Omega^{0,1}(X_0, K_{X_0})$, at p, we have

$$\begin{aligned}
\Delta s &= \overline{\partial}\overline{\partial}^* s = \overline{\partial}\left(\left(\overline{\partial}^*(f d\overline{z})\right) \otimes e + g^{-1} f \frac{\partial \log g}{\partial z} \otimes e\right) \\
&= (\Delta_0 f) d\overline{z} \otimes e + f d\overline{z} \otimes e \\
&= ((\Delta_0+1) f)\, d\overline{z} \otimes e
\end{aligned}$$

where we use the Kähler-Einstein condition (3.9). Hence, at point p,

$$\begin{aligned}
& \Delta\left(g(\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e\right) \\
= & \left((\Delta_0+1)\left(g(\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_\alpha)\right)\right) d\overline{z} \otimes e \\
= & g(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e + (\Delta_0 g)\left((\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e\right) \\
= & g(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e - \left((\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e\right) \\
= & \theta_i \lrcorner \sigma_\alpha - g\left((\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_\alpha) d\overline{z} \otimes e\right)
\end{aligned}$$

where we use (3.7) and $g(p)=1$ in the last step. That is, at the fixed point p, (3.8) holds. □

Now we obtain the well-known Wolpert formula:

Theorem 3.6 ([31])**.** *The curvature tensor of the Weil-Petersson metric on the* ***cotangent bundle*** *of the moduli space is:*

$$R_{i\overline{j}\alpha\overline{\beta}} = \int (\Delta_0+1)^{-1}\left(\theta_i \cdot \overline{\theta}_\alpha\right)\left(\overline{\theta}_j \cdot \theta_\beta\right) dV + \int (\Delta_0+1)^{-1}(\theta_i \cdot \overline{\theta}_j)\left(\overline{\theta}_\alpha \cdot \theta_\beta\right) dV \tag{3.10}$$

Proof. If we set $m=2$ in formula (3.4), (3.10) follows from formulas (3.4), (3.8) and (3.7). □

Theorem 3.7. *The Weil-Petersson metric is dual-Nakano-negative and semi-Nakano-negative.*

Proof. By duality (e.g. Remark 2.2), we only need to prove the curvature tensor (3.10) is Nakano-positive and semi-dual-Nakano-positive. At first, we prove the semi-dual-Nakano positive part. That is, for any nonzero matrix $u=(u^{i\alpha})$, it suffices to show

$$R_{i\overline{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} \geq 0 \tag{3.11}$$

Here we use similar ideas of [11, Section 4]. Let $G(z,w)$ be the kernel function of the integral operator $(\Delta_0+1)^{-1}$. It is well-known that G is strictly positive and in a neighborhood of the diagonal, $G(z,w)+\frac{1}{2\pi}\log|z-w|$ is continuous. So we obtain

$$\begin{aligned} R_{i\bar{j}\alpha\overline{\beta}}u^{i\beta}\overline{u}^{j\alpha} &= \int_{X_0}\int_{X_0} G(z,w)\theta_i(w)\overline{\theta}_\alpha(w)\overline{\theta}_j(z)\theta_\beta(z)u^{i\beta}\overline{u}^{j\alpha}dV_w dV_z \\ &+ \int_{X_0}\int_{X_0} G(z,w)\theta_i(w)\overline{\theta}_j(w)\overline{\theta}_\alpha(z)\theta_\beta(z)u^{i\beta}\overline{u}^{j\alpha}dV_w dV_z \end{aligned}$$

If we set $H(w,z)=\theta_i(w)\theta_\beta(z)u^{i\beta}$,

$$\begin{aligned} R_{i\bar{j}\alpha\overline{\beta}}u^{i\beta}\overline{u}^{j\alpha} &= \int_{X_0}\int_{X_0} G(z,w)H(w,z)\overline{H(z,w)}dV_w dV_z \\ &\quad + \int_{X_0}\int_{X_0} G(z,w)H(w,z)\overline{H(w,z)}dV_w dV_z \end{aligned}$$

Since the Green's function is symmetric, i.e., $G(z,w)=G(w,z)$,

$$R_{i\bar{j}\alpha\overline{\beta}}u^{i\beta}\overline{u}^{j\alpha} = \frac{1}{2}\int_{X_0}\int_{X_0} G(z,w)\left(H(w,z)+H(z,w)\right)\overline{\left(H(w,z)+H(z,w)\right)}dV_w dV_z$$

which is non-negative. Hence we get (3.11).

For the Nakano-positivity, we can use the same method. It is easy to see that we can get strict Nakano-positivity since the Kodaira-Spencer map is injective. □

Remark 3.8. In virtue of Lemma 2.3, the moduli space $\mathcal{M}_k$ has the same curvature property as the unit disk with the invariant Bergman metric. It is optimal in the sense that the curvature can not be Nakano-negative at any point. In fact, it follows from the L^2-vanishing theorems on $\mathcal{M}_k$ (e.g. [20]).

4. Curvature properties of the moduli space of curves

In this section, we investigate the curvature properties of the Weil-Petersson metric as well as its background Riemannian metric on the moduli space of curves, based on very general curvature relations on Kähler manifolds.

4.1. **Curvatures on Riemannian manifold.** Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. The curvature tensor is defined as

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z \tag{4.1}$$

for any $X,Y,Z\in\Gamma(M,TM)$. In the local coordinates $\{x^i\}$ of M, we adopt the convention:

$$R(X,Y,Z,W)=g(R(X,Y)Z,W)=R_{ijk\ell}X^iY^jZ^kW^\ell. \tag{4.2}$$

The curvature operator is

$$\mathcal{R}:\Gamma(M,\Lambda^2TM)\to\Gamma(M,\Lambda^2TM)\quad\text{and}\quad g(\mathcal{R}(X\wedge Y),Z\wedge W)=R(X,Y,W,Z) \tag{4.3}$$

Note here, we change the orders of Z,W in the full curvature tensor. For Riemannian sectional curvature, we use

$$K(X,Y)=\frac{R(X,Y,Y,X)}{|X|_g^2|Y|_g^2-\langle X,Y\rangle_g^2} \tag{4.4}$$

for any linearly independent vectors X and Y.

Let (M, g) be a Riemannian manifold. $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ is the complexification of the real vector bundle TM. We can extend the metric g and ∇ to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way and still denote them by g and ∇ respectively.

Definition 4.1. Let (M, g) be a Riemannian manifold and R be the complexified Riemmanian curvature operator. We say (X, g) has non-positive (resp. non-negative) complex sectional curvature, if

$$R(Z, \overline{W}, W, \overline{Z}) \leq 0 \qquad (\text{resp. } \geq 0) \tag{4.5}$$

for any $Z, W \in T_{\mathbb{C}}M$.

Definition 4.2. A vector $v \in T_{\mathbb{C}}M$ is called isotropic if $g(v, v) = 0$. A subspace is called isotropic if every vector in it is isotropic. (M, g) is called to have non-positive (resp. non-negative) isotropic curvature if

$$g\left(\mathcal{R}(v \wedge w), v \wedge w\right) \leq 0 \ (\text{resp. } \geq 0), \tag{4.6}$$

for every pair of vectors $v, w \in T_{\mathbb{C}}M$ which span an isotropic 2-plane.

4.2. **Curvature relations on Kähler manifolds.** In [25], Siu introduced the following terminology:

Definition 4.3. Let (X, g) be a compact Kähler manifold. (X, g) has strongly negative curvature(resp. strongly positive) if

$$R_{i\bar{j}k\bar{\ell}}\left(A^i\overline{B}^j - C^i\overline{D}^j\right)\overline{\left(A^\ell\overline{B}^k - C^\ell\overline{D}^k\right)} \leq 0 \ \ (\text{resp. } \geq 0) \tag{4.7}$$

for any $A = A^i\frac{\partial}{\partial z^i}$, $B = B^j\frac{\partial}{\partial z^j}$, $C = C^i\frac{\partial}{\partial z^i}$, $D = D^j\frac{\partial}{\partial z^j}$ and the identity in the above inequality holds if and only if $A^i\overline{B}^j - C^i\overline{D}^j = 0$ for any i, j.

Theorem 4.4. *Let (X, g) be a Kähler manifold. Then g is a metric with strongly non-negative curvature (resp. strongly non-positive curvature) in the sense of Siu if and only if the complex sectional curvature is non-negative (resp. non-positive).*

Proof. Let $Z, W \in T_{\mathbb{C}}X$. In local holomorphic coordinates $\{z^i\}$ of X, one can write $Z = a^i\dfrac{\partial}{\partial z^i} + b^{\bar{i}}\dfrac{\partial}{\partial \overline{z}^i}$, $\quad W = c^j\dfrac{\partial}{\partial z^j} + d^{\bar{j}}\dfrac{\partial}{\partial \overline{z}^j}$. We can compute

$$\begin{aligned} &R(Z, \overline{W}, W, \overline{Z}) \\ &= R\left(a^i\frac{\partial}{\partial z^i} + b^{\bar{i}}\frac{\partial}{\partial \overline{z}^i}, \overline{c^j}\frac{\partial}{\partial \overline{z}^j} + \overline{d^{\bar{j}}}\frac{\partial}{\partial z^j}, c^k\frac{\partial}{\partial z^k} + d^{\bar{k}}\frac{\partial}{\partial \overline{z}^k}, \overline{a^\ell}\frac{\partial}{\partial \overline{z}^\ell} + \overline{b^{\bar{\ell}}}\frac{\partial}{\partial z^\ell}\right). \end{aligned} \tag{4.8}$$

It has sixteen terms, but it is well-known that on a Kähler manifold $R_{ijk\ell} = 0$, $R_{\bar{i}jk\ell} = R_{i\bar{j}k\ell} = R_{ij\bar{k}\ell} = R_{ijk\bar{\ell}} = 0$, and their conjugates are also zero, i.e. $R_{\overline{ijk\ell}} = 0$, $R_{i\overline{jk\ell}} = R_{\bar{i}j\overline{k\ell}} = R_{\overline{ij}k\bar{\ell}} = R_{\overline{ijk}\ell} = 0$. Since g is Kähler, by Bianchi identity, we see $R_{ij\overline{k\ell}} = -R_{j\bar{k}i\bar{\ell}} - R_{\bar{k}ij\bar{\ell}} =$

$-R_{j\bar{k}i\bar{\ell}}+R_{i\bar{k}j\bar{\ell}}=0$. Similarly, we have $R_{\bar{i}\bar{j}k\bar{\ell}}=0$. Hence (4.8) contains four nonzero terms, i.e.,

$$\begin{aligned}
R(Z,\overline{W},W,\overline{Z}) &= R_{i\bar{j}k\bar{\ell}}\cdot a^i\cdot\overline{c^j}\cdot c^k\cdot\overline{a^\ell}+R_{i\bar{j}\bar{k}\ell}\cdot a^i\cdot\overline{c^j}\cdot d^{\bar{k}}\cdot\overline{b^{\bar{\ell}}}\\
&\quad +R_{\bar{i}\bar{j}\bar{k}\ell}\cdot b^{\bar{i}}\cdot\overline{d^{\bar{j}}}\cdot d^{\bar{k}}\cdot\overline{b^{\bar{\ell}}}+R_{\bar{i}jk\bar{\ell}}\cdot b^{\bar{i}}\cdot\overline{d^{\bar{j}}}\cdot c^k\cdot\overline{a^\ell}\\
&= R_{i\bar{j}k\bar{\ell}}\cdot a^i\cdot\overline{c^j}\cdot c^k\cdot\overline{a^\ell}-R_{i\bar{j}\ell\bar{k}}\cdot a^i\cdot\overline{c^j}\cdot d^{\bar{k}}\cdot\overline{b^{\bar{\ell}}}\\
&\quad +R_{j\bar{i}\ell\bar{k}}\cdot b^{\bar{i}}\cdot\overline{d^{\bar{j}}}\cdot d^{\bar{k}}\cdot\overline{b^{\bar{\ell}}}-R_{j\bar{i}k\bar{\ell}}\cdot b^{\bar{i}}\cdot\overline{d^{\bar{j}}}\cdot c^k\cdot\overline{a^\ell}\\
&= R_{i\bar{j}k\bar{\ell}}\cdot a^i\cdot\overline{c^j}\cdot c^k\cdot\overline{a^\ell}-R_{i\bar{j}k\bar{\ell}}\cdot a^i\cdot\overline{c^j}\cdot d^{\bar{\ell}}\cdot\overline{b^{\bar{k}}}\\
&\quad +R_{i\bar{j}k\bar{\ell}}\cdot b^{\bar{j}}\cdot\overline{d^{\bar{i}}}\cdot d^{\bar{\ell}}\cdot\overline{b^{\bar{k}}}-R_{i\bar{j}k\bar{\ell}}\cdot b^{\bar{j}}\cdot\overline{d^{\bar{i}}}\cdot c^k\cdot\overline{a^\ell}\\
&= R_{i\bar{j}k\bar{\ell}}\Big(a^i\cdot\overline{c^j}\cdot c^k\cdot\overline{a^\ell}-a^i\cdot\overline{c^j}\cdot d^{\bar{\ell}}\cdot\overline{b^{\bar{k}}}\\
&\qquad\qquad +b^{\bar{j}}\cdot\overline{d^{\bar{i}}}\cdot d^{\bar{\ell}}\cdot\overline{b^{\bar{k}}}-b^{\bar{j}}\cdot\overline{d^{\bar{i}}}\cdot c^k\cdot\overline{a^\ell}\Big)\\
&= R_{i\bar{j}k\bar{\ell}}\left(a^i\cdot\overline{c^j}-b^{\bar{j}}\cdot\overline{d^{\bar{i}}}\right)\left(c^k\cdot\overline{a^\ell}-d^{\bar{\ell}}\cdot\overline{b^{\bar{k}}}\right)\\
&= R_{i\bar{j}k\bar{\ell}}\left(a^i\cdot\overline{c^j}-b^{\bar{j}}\cdot\overline{d^{\bar{i}}}\right)\overline{\left(a^\ell\cdot\overline{c^k}-b^{\bar{k}}\cdot\overline{d^{\bar{\ell}}}\right)}.
\end{aligned}$$

Let $A^{i\bar{j}}=a^i\cdot\overline{c^j}-b^{\bar{j}}\cdot\overline{d^{\bar{i}}}$. We obtain

$$R(Z,\overline{W},W,\overline{Z})=R_{i\bar{j}k\bar{\ell}}A^{i\bar{j}}\cdot\overline{A^{\ell\bar{k}}}=R_{i\bar{j}k\bar{\ell}}A^{i\bar{\ell}}\cdot\overline{A^{j\bar{k}}}. \tag{4.9}$$

Therefore, the curvature is strongly non-negative in the sense of Siu if and only if the complex sectional curvature is non-negative. The proof for the equivalent on non-positivity is similar. □

Theorem 4.5. *If (X,g) is a Kähler manifold with semi dual-Nakano-positive curvature (resp. semi dual-Nakano-negative curvature), then its background Riemmanian curvature operator is non-negative (resp. non-positive).*

Proof. Let $z^i=x^i+\sqrt{-1}y^i$ be the local holomorphic coordinates centered at a given point. Then from the relation $\dfrac{\partial}{\partial z^i}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial x^i}-\sqrt{-1}\dfrac{\partial}{\partial y^i}\right)$, $\dfrac{\partial}{\partial \bar{z}^i}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial x^i}+\sqrt{-1}\dfrac{\partial}{\partial y^i}\right)$ one obtains $\dfrac{\partial}{\partial x^i}=\dfrac{\partial}{\partial z^i}+\dfrac{\partial}{\partial \bar{z}^i}$, $\dfrac{\partial}{\partial y^i}=\sqrt{-1}\left(\dfrac{\partial}{\partial z^i}-\dfrac{\partial}{\partial \bar{z}^i}\right)$. On the background Riemannian manifold, any vector V in $\Lambda^2T_{\mathbb{R}}X$ can be written as

$$V=a^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}+b^{pq}\frac{\partial}{\partial x^p}\wedge\frac{\partial}{\partial y^q}+c^{mn}\frac{\partial}{\partial y^m}\wedge\frac{\partial}{\partial y^n}. \tag{4.10}$$

In the coordinates $\{z^i, \overline{z}^i\}$, we have

$$
\begin{aligned}
V &= a^{ij}\left(\frac{\partial}{\partial z^i}+\frac{\partial}{\partial \overline{z}^i}\right)\left(\frac{\partial}{\partial z^j}+\frac{\partial}{\partial \overline{z}^j}\right)+\sqrt{-1}b^{pq}\left(\frac{\partial}{\partial z^p}+\frac{\partial}{\partial \overline{z}^p}\right)\left(\frac{\partial}{\partial z^q}-\frac{\partial}{\partial \overline{z}^q}\right)\\
&\quad -c^{mn}\left(\frac{\partial}{\partial z^m}-\frac{\partial}{\partial \overline{z}^m}\right)\left(\frac{\partial}{\partial z^n}-\frac{\partial}{\partial \overline{z}^n}\right)\\
&= a^{ij}\left(\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j}+\frac{\partial}{\partial \overline{z}^i}\wedge\frac{\partial}{\partial z^j}+\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial \overline{z}^j}+\frac{\partial}{\partial \overline{z}^i}\wedge\frac{\partial}{\partial \overline{z}^j}\right)\\
&\quad +\sqrt{-1}b^{pq}\left(\frac{\partial}{\partial z^p}\wedge\frac{\partial}{\partial z^q}+\frac{\partial}{\partial \overline{z}^p}\wedge\frac{\partial}{\partial z^q}-\frac{\partial}{\partial z^p}\wedge\frac{\partial}{\partial \overline{z}^q}-\frac{\partial}{\partial \overline{z}^p}\wedge\frac{\partial}{\partial \overline{z}^q}\right)\\
&\quad -c^{mn}\left(\frac{\partial}{\partial z^m}\wedge\frac{\partial}{\partial z^m}-\frac{\partial}{\partial \overline{z}^m}\wedge\frac{\partial}{\partial z^n}-\frac{\partial}{\partial z^m}\wedge\frac{\partial}{\partial \overline{z}^n}+\frac{\partial}{\partial \overline{z}^m}\wedge\frac{\partial}{\partial \overline{z}^n}\right)\\
&= A^{ij}\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j}+B^{i\overline{j}}\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial \overline{z}^j}+C^{\overline{i}\overline{j}}\frac{\partial}{\partial \overline{z}^i}\wedge\frac{\partial}{\partial \overline{z}^j}
\end{aligned}
$$

where

$$A^{ij} := a^{ij}+\sqrt{-1}b^{ij}-c^{ij}, \qquad C^{\overline{i}\overline{j}} := a^{ij}-\sqrt{-1}b^{ij}-c^{ij}$$

and

$$B^{i\overline{j}} := a^{ij}-\sqrt{-1}b^{ij}+c^{ij}-a^{ji}-\sqrt{-1}b^{ji}-c^{ji}.$$

By the elementary facts that

$$R_{ij\ell k} = R\left(\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^k}\wedge\frac{\partial}{\partial z^\ell}\right)=0;$$

$$R_{ij\overline{\ell}\overline{k}} = R\left(\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j},\frac{\partial}{\partial \overline{z}^k}\wedge\frac{\partial}{\partial \overline{z}^\ell}\right)=0;$$

$$R_{i\overline{j}\ell k} = R\left(\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial \overline{z}^j},\frac{\partial}{\partial z^k}\wedge\frac{\partial}{\partial z^\ell}\right)=0;$$

and also their conjugates are all zero, we obtain

$$
\begin{aligned}
\mathcal{R}(V,V) &= R\left(B^{i\overline{j}}\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial \overline{z}^j}, B^{k\overline{\ell}}\frac{\partial}{\partial z^k}\wedge\frac{\partial}{\partial \overline{z}^\ell}\right)\\
&= R_{i\overline{j}\overline{\ell}k}B^{i\overline{j}}B^{k\overline{\ell}} = -R_{i\overline{j}k\overline{\ell}}B^{i\overline{j}}B^{k\overline{\ell}}.
\end{aligned}
$$

Let $E^{i\overline{j}} := a^{ij}+c^{ij}-a^{ji}-c^{ji}, F^{i\overline{j}} := -b^{ij}-b^{ji}$, then $B^{i\overline{j}} = E^{i\overline{j}}+\sqrt{-1}F^{i\overline{j}}$. Note that the matrix $(E^{i\overline{j}})$ is real and skew-symmetric; the matrix $(F^{i\overline{j}})$ is real and symmetric. Hence, by the curvature property

$$
\begin{aligned}
\mathcal{R}(V,V) &= -R_{i\overline{j}k\overline{\ell}}B^{i\overline{j}}B^{k\overline{\ell}}\\
&= -R_{i\overline{j}k\overline{\ell}}E^{i\overline{j}}E^{k\overline{\ell}}-\sqrt{-1}R_{i\overline{j}k\overline{\ell}}(E^{i\overline{j}}F^{k\overline{\ell}}+E^{k\overline{\ell}}F^{i\overline{j}})+R_{i\overline{j}k\overline{\ell}}F^{i\overline{j}}F^{k\overline{\ell}}.
\end{aligned}
$$

On the other hand $R_{i\overline{j}k\overline{\ell}}$ is skew-symmetric in the pairs (i,j) and (k,ℓ), we obtain

$$\sqrt{-1}R_{i\overline{j}k\overline{\ell}}(E^{i\overline{j}}F^{k\overline{\ell}}+E^{k\overline{\ell}}F^{i\overline{j}}) = R_{i\overline{j}k\overline{\ell}}F^{i\overline{j}}F^{k\overline{\ell}} = 0$$

since $(E^{i\bar{j}})$ is real and skew-symmetric and $(F^{i\bar{j}})$ is real and symmetric. Therefore,

$$\mathcal{R}(V,V) = -R_{i\bar{j}k\bar{\ell}}E^{i\bar{j}}E^{k\bar{\ell}} = -R_{i\bar{j}k\bar{\ell}}E^{i\bar{\ell}}E^{k\bar{j}} = R_{i\bar{j}k\bar{\ell}}E^{i\bar{\ell}}\overline{E^{j\bar{k}}}$$

where in the last step we use again the fact that $E^{k\bar{j}}$ is real and skew-symmetric, i.e. $E^{k\bar{\ell}} = -E^{j\bar{k}} = -\overline{E^{j\bar{k}}}$. Now, we see that if (X,g) is semi- dual-Nakano-positive (resp. semi dual-Nakano-negative), then the Riemannian curvature operator is non-negative (resp. non-positive). □

Remark 4.6. $(\mathbb{P}^2, \omega_{FS})$ is dual-Nakano-positive, but the Riemannian curvature operator of the background Riemannian metric is only non-negative. In fact, on any compact Kähler manifold, there does not exist a Riemannian metric with quasi-positive Riemannian curvature operator since it has nonzero second Betti number (e.g. [22, p.212]).

The proof of Theorem 1.1. (1) $\Longrightarrow$ (2) follows from Theorem 4.5, and (3) $\Longleftrightarrow$ (4) follows from Theorem 4.4. (2) $\Longrightarrow$ (4): let $Z, W \in T_{\mathbb{C}}M$. Let $Z \wedge \overline{W} = V + iU$, where V and U are real tensors. Then

$$\begin{aligned} R(Z,\overline{W},W,\overline{Z}) &= \mathcal{R}(Z\wedge\overline{W}, \overline{Z}\wedge W) \\ &= \mathcal{R}(V+iU, V-iU) \\ &= \mathcal{R}(V,V) + \mathcal{R}(U,U) \end{aligned}$$

where the last step follows since our curvature operator is extended to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way and $\mathcal{R}(U,V) = \mathcal{R}(V,U)$. The other relations follow from similar computations.

Remark 4.7. (1) Exactly the same relations hold for semi-positivity.

(2) There is another notion called "weakly $\frac{1}{4}$-pinched negative Riemannian sectional curvature". If (X,ω) is a compact Kähler manifold with weakly $\frac{1}{4}$-pinched negative Riemannian sectional curvature, then (X,ω) is semi dual-Nakano-negative. Indeed, Yau-Zheng proved in [34](see also [7]) that any compact Kähler manifold with weakly $\frac{1}{4}$-pinched negative Riemannian sectional curvature must be a ball quotient. However, Mostow-Siu surfaces ([19]) have dual-Nakano-negative curvature tensors, but they are not covered by a 2-ball.

The proof of Theorem 1.3. It follows from Theorem 1.1 and Theorem 1.2.

5. Totally geodesic submanifolds in Torelli locus.

In this section, we study the existence of certain locally symmetric submanifold in moduli space $\mathcal{M}_k$ of curves with genus $k \geq 2$ by using the curvature properties we obtained. As an application of Theorem 1.3 and Lemma 2.4, we derive

Corollary 5.1. *Let S be any submanifold of $(\mathcal{M}_k, \omega_{\mathrm{WP}})$ with the induced metric, then S has*

(1) *semi Nakano-negative curvature;*
(2) *strictly negative holomorphic bisectional curvature.*

We need the following rigidity result by W-K. To:

Theorem 5.2 ([30])**.** *Let (X, g) be a locally symmetric Hermitian manifold of finite volume uniformized by an irreducible bounded symmetric domain of rank ≥ 2. Suppose h is Hermitian metric on X such that (X, h) carries non-positive holomorphic bisectional curvature. Then $h = cg$ for some constant $c > 0$.*

The proof of Theorem 1.4. It is well-known that any totally geodesic submanifold of $\mathcal{A}_k$ is also locally symmetric. Suppose $j(V)$ has rank> 1. Then by Siu's computation in [24, Appendix, Theorem 4], the holomorphic bisectional curvatures of the canonical metrics on irreducible bounded symmetric domains of rank> 1 are non-positive but *not strictly negative*. Let h be the metric on $j(V)$ induced by the Hodge metric on $\mathcal{A}_k$. Hence, $(j(V), h)$ has non-positive holomorphic bisectional curvature by formula (2.8). Let g be the Hermitian metric on V induced by the Weil-Petersson metric on $\mathcal{M}_k$. By Corollary 5.1, (V, g) has strictly negative holomorphic bisectional curvature. Since the Torelli map j is holomorphic and injective, by Theorem 5.2, $h = cg$ for some positive constant c which is a contradiction. Hence, $j(V)$ is of rank 1, i.e. a ball quotient.

The proof of Corollary 1.6. Suppose Ω has rank > 1. Then by a result of Mok ([16, Theorem 4] or [14]), we see X must be totally geodesic in $(\mathcal{M}_k, \omega_{\mathrm{WP}})$. That is, the second fundamental form of the immersion must be zero. Since $(\mathcal{M}_k, \omega_{\mathrm{WP}})$ is dual-Nakano-negative, we see (X, h) is also dual-Nakano-negative, and in particular, (X, h) has strictly negative curvature in the sense of Siu which is a contradiction([24, Appendix, Theorem 4]). Hence Ω must be of rank 1, i.e. X must be a ball quotient.

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