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Abstract

Let X be a smooth projective manifold with $\dim_{\mathbb{C}} X = n$. We show that if a line bundle L is $(n-1)$-ample, then it is $(n-1)$-positive. This is a partial converse to the Andreotti–Grauert theorem. As an application, we show that a projective manifold X is uniruled if and only if there exists a Hermitian metric ω on X such that its Ricci curvature $\mathrm{Ric}(\omega)$ has at least one positive eigenvalue everywhere.

1. Introduction

One of the most fundamental topics in algebraic geometry is the characterizing ampleness of line bundles by using numerical and cohomological vanishing theorems. The theorem of Cartan–Serre–Grothendieck is a milestone in this direction. The following statements are equivalent:

(i) L is an ample line bundle over a projective manifold X;

(ii) for every coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_0 = m_0(X, \mathcal{F}, L)$ such that

$$H^i(X, \mathcal{F} \otimes L^{\otimes m}) = 0$$

for all $i > 0$ and all $m \geqslant m_0$.

On complex projective manifolds, ampleness is also equivalent to the existence of a smooth metric with positive curvature, thanks to the celebrated Kodaira embedding theorem.

In [AG62], Andreotti and Grauert considered the case of line bundles with curvature of mixed signature. Now it is formulated into the following definition.

Definition 1.1. Let L be a holomorphic line bundle over a compact complex manifold X with $\dim_{\mathbb{C}} X = n$.

(i) L is called *q-positive*, if there exists a smooth Hermitian metric h on L such that the Chern curvature $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial} \log h$ has at least $(n-q)$ positive eigenvalues at every point on X.

(ii) L is called *q-ample*, if for any coherent sheaf $\mathcal{F}$ on X there exists a positive integer $m_0 = m_0(X, L, \mathcal{F}) > 0$ such that

$$H^i(X, \mathcal{F} \otimes L^m) = 0 \quad \text{for } i > q,\ m \geqslant m_0. \tag{1}$$

It is obvious that L is 0-positive if and only if L is positive, and L is 0-ample if and only if L is ample. Hence the 0-positivity and 0-ampleness are equivalent. In [AG62, Theorem 14], Andreotti and Grauert proved the following fundamental theorem (see also [DPS96, Proposition 2.1]).

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Theorem 1.2 (Andreotti–Grauert). *A q-positive line bundle is q-ample.*

Historically, the Andreotti–Grauert theorem is the first result on the relationship between (partially) positive line bundles and the cohomological vanishing theorems. In the pioneer work [DPS96], Demailly, Peternell and Schneider systematically investigated partial vanishing theorems and proposed the following problem on the converse to the Andreotti–Grauert theorem.

Problem 1.3. On a projective manifold, if a line bundle is q-ample, is it q-positive?

This is a long-standing open problem. The key difficulty arises from constructing a precise metric according to the formal partial vanishing theorem (1). Recently, there has been some progress on this problem, mainly contributed by Demailly, Totaro, Ottem, Küronya, Matsumura, Brown and so on (see [Dem11, Tot13, Mat13, Ott12, Bro12, Kür13, GK15] and also the references therein). Totaro proved in [Tot13] that the notion of q-ampleness is equivalent to others previously studied in [DPS96]. As a result, the q-ampleness of a line bundle depends only on its numerical class, and the cone of such bundles is open. In particular, Totaro established that the $(n-1)$-ample cone of an n-dimensional projective manifold X is equal to the negative of the complement of the pseudo-effective cone of X (see also Corollary 1.6). In dimension two, Demailly proved in [Dem11] an asymptotic version of this converse to the Andreotti–Grauert theorem using tools related to the holomorphic Morse inequality and asymptotic cohomology; subsequently, Matsumura obtained in [Mat13, Theorem 1.3] a positive answer to the question for projective surfaces. However, there exist higher-dimensional counter-examples to the converse Andreotti–Grauert problem in the range $(\dim X)/2 - 1 < q < \dim X - 2$, constructed by Ottem [Ott12, Theorem 10.3]. Our main result in this paper is a partial converse to the Andreotti–Grauert theorem on smooth projective manifolds.

Theorem 1.4. *Let L be a line bundle over a smooth projective manifold X with $\dim_{\mathbb{C}} X = n$. If L is $(n-1)$-ample, then it is $(n-1)$-positive.*

In particular, when $\dim_{\mathbb{C}} X = 2$, the converse Andreotti–Grauert problem 1.3 is true (see also [Mat13, Theorem 1.3]). Actually, Theorem 1.4 is a straightforward application of the following result on general *compact complex manifolds*.

Theorem 1.5. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Then the following statements are equivalent:*

(i) *L is $(n-1)$-positive;*

(ii) *the dual line bundle L^{-1} is not pseudo-effective.*

Note that, Theorem 1.5 is also valid if we replace the line bundle L by a Bott–Chern class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ (see Theorem 4.2). The key ingredients in the proof of Theorem 1.5 rely on several results in our previous paper [Yan17] on geometric characterizations of pseudo-effective line bundles (respectively Bott–Chern classes). As an application of Theorems 1.4, 1.5 and 1.2, one has the following.

Corollary 1.6. *On a projective manifold X of complex dimension n, the following are equivalent:*

(i) *L is $(n-1)$-ample;*

(ii) *L is $(n-1)$-positive;*
(iii) *L^{-1} is not pseudo-effective.*

Note that the equivalence of (1) and (3) is also obtained in [DPS96, Proposition 2.5], [Tot13, Theorem 9.1] and [Mat13, Lemma 2.4] by different methods in one direction.

According to Ottem's counter-examples in [Ott12, Theorem 10.3], for $(\dim X)/2 - 1 < q < \dim X - 2$, the q-ampleness cannot imply the q-positivity. On the contrary, by Theorem 1.4, we obtain the following.

PROPOSITION 1.7. *Let X be a smooth projective manifold with $\dim_{\mathbb{C}} X = n$. Suppose L is q-ample, then the restriction of L to every codimension-$(n-q-1)$ smooth submanifold Y is q-positive.*

In particular, we have the following.

COROLLARY 1.8. *Let X be a smooth projective manifold with $\dim_{\mathbb{C}} X = n$. If L is $(n-2)$-ample, then the restriction of L to every codimension-1 smooth submanifold is $(n-2)$-positive.*

On the other hand, by using the classical result of [BDPP13], one obtains the following.

COROLLARY 1.9. *On a projective manifold X of complex dimension n, the following are equivalent:*

(i) *X is uniruled;*
(ii) *K_X^{-1} is $(n-1)$-positive, i.e. there exists a Hermitian metric ω on X such that its Ricci curvature $\mathrm{Ric}(\omega)$ has at least one positive eigenvalue everywhere;*
(iii) *K_X is not pseudo-effective;*
(iv) *K_X^{-1} is $(n-1)$-ample.*

Remark 1.10. From Theorems 1.4, 1.5, Corollaries 1.6, 1.9, Theorems 4.1 and 4.2, one can derive and formulate various cone dualities on compact complex manifolds. The vector bundle analogous of the main results and applications are obtained in [Yan18].

2. Partially positive line bundles on compact complex manifolds

In this section, we deal with two different notions on q-positive line bundles over compact complex manifolds.

DEFINITION 2.1. Let X be a compact complex manifold and L be a holomorphic line bundle over X.

(i) L is called *q-positive*, if there exists a smooth Hermitian metric h on L and a smooth Hermitian metric ω on X such that the Chern curvature $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$ has at least $(n-q)$ positive eigenvalues at any point on X (with respect to ω).
(ii) L is called *uniformly q-positive*, if there exists a Hermitian metric h on L and a smooth Hermitian metric ω on X such that the summation of any distinct $(q+1)$ eigenvalues (counting multiplicity) of the Chern curvature $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$ is positive at any point of X (with respect to ω).

The following result is obtained by changing the metric on the complex manifold and the background idea dates back to [AV65, § 5].

Proposition 2.2. *The following statements are equivalent:*

(i) *L is q-positive;*
(ii) *L is uniformly q-positive.*

Proof. (2) $\Longrightarrow$ (1). Let ω be a Hermitian metric on X and h be a smooth Hermitian metric on L such that $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$ is uniformly q-positive. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of $R^{(L,h)}$ with respect to ω over some coordinate chart. We have $\lambda_{n-q} > 0$. Otherwise, the summation of $q+1$ eigenvalues $\lambda_{n-q} + \lambda_{n-q+1} + \cdots + \lambda_n \leqslant 0$.

(1) $\Longrightarrow$ (2). We assume that there exists a smooth Hermitian metric h on L such that the curvature $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$ has at least $(n-q)$ positive eigenvalues at each point $p \in X$. Let ω_0 be a fixed Hermitian metric on X. For simplicity, we denote by R and Ω the local matrix representations of the matrices $R^{(L,h)}$ and ω_0 respectively, in some local holomorphic frames of X. Let

$$\lambda_1(z) \geqslant \cdots \geqslant \lambda_n(z)$$

be the eigenvalues of $R^{(L,h)}$ with respect to ω_0. It is obvious that $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the matrix $R\Omega^{-1}$. Note that $R\Omega^{-1}$ represents a tensor in $\Gamma(X, \mathrm{End}(T^{1,0}X))$, and so the eigenvalues λ_i are independent of the choice of coordinates. Since λ_{n-q} is a continuous function, we set

$$\lambda_0 = \frac{\log(n+1)}{\inf_X \lambda_{n-q}}. \tag{2}$$

λ_0 is a positive number since L is q-positive and X is compact. We define a new Hermitian metric ω over X with local matrix representation $\widetilde{\Omega}$ by the following formula

$$\widetilde{\Omega}^{-1} = \Omega^{-1} \cdot \left(\mathrm{Id} + \sum_{k=1}^{\infty} \frac{\lambda_0^k (R\Omega^{-1})^k}{(k+1)!}\right). \tag{3}$$

Note that the matrix $\widetilde{\Omega}^{-1}$ is positive definite. Indeed, the eigenvalues of the matrix $\mathrm{Id} + \sum_{k=1}^{\infty} (\lambda_0^k (R\Omega^{-1})^k/(k+1)!)$ are given by

$$1 + \sum_{k=1}^{\infty} \frac{\lambda_0^k \lambda_i^k}{(k+1)!} = \frac{e^{\lambda_0\lambda_i} - 1}{\lambda_0\lambda_i} > 0 \quad \text{if } \lambda_i \neq 0.$$

It is not hard to see that the Hermitian metric ω is globally well-defined on X. Let $\kappa_1 \geqslant \cdots \geqslant \kappa_n$ be the eigenvalues of $R^{(L,h)}$ with respect to the new metric ω, i.e. they are the eigenvalues of $R\widetilde{\Omega}^{-1}$. Note also that

$$R\widetilde{\Omega}^{-1} = \lambda_0^{-1}\left(\sum_{k=0}^{\infty} \frac{\lambda_0^k (R\Omega^{-1})^k}{k!} - \mathrm{Id}\right).$$

A straightforward computation shows

$$\kappa_{n-q} = \frac{e^{\lambda_0\lambda_{n-q}} - 1}{\lambda_0} \quad \text{and} \quad \kappa_n = \frac{e^{\lambda_0\lambda_n} - 1}{\lambda_0}.$$

For any summation of $(q+1)$ (distinct) eigenvalues of $R^{(L,h)}$ with respect to the new metric ω, we have the inequality

$$\begin{aligned}\sum_{\ell=1}^{q+1}\kappa_{i_\ell} &\geqslant \kappa_{n-q}+\cdots+\kappa_n\\ &\geqslant \kappa_{n-q}+q\kappa_n\\ &\geqslant \lambda_0^{-1}(e^{\lambda_0\lambda_{n-q}}+qe^{\lambda_0\lambda_n}-(q+1))\\ &> \lambda_0^{-1}(e^{\lambda_0\lambda_{n-q}}-(q+1))>0\end{aligned}$$

since $e^{\lambda_0\lambda_{n-q}}=e^{\log(n+1)(\lambda_{n-q}/\inf_X\lambda_{n-q})}\geqslant n+1$ by (2).

The following special case of Proposition 2.2 is of particular interest in complex geometry.

Corollary 2.3. *The following statements are equivalent:*

(i) *L is $(n-1)$-positive;*

(ii) *there exists a smooth Hermitian metric h on L and a Hermitian metric ω on X such that the function*

$$\operatorname{tr}_\omega(-\sqrt{-1}\partial\overline{\partial}\log h)>0. \tag{4}$$

Remark 2.4. The function $\operatorname{tr}_\omega(-\sqrt{-1}\partial\overline{\partial}\log h)$ is globally defined on X and it is independent of the choice of coordinates. It is also called the scalar curvature of the Chern curvature $R^{(L,h)}=-\sqrt{-1}\partial\overline{\partial}\log h$ with respect to the Hermitian metric ω.

3. The pseudo-effective line bundles on compact complex manifolds

A line bundle L on a compact complex manifold X is called *pseudo-effective* if there exists a (possibly) singular Hermitian metric h on L such that its Chern curvature $R^{(L,h)}=-\sqrt{-1}\partial\overline{\partial}\log h\geqslant 0$ in the sense of current. In order to describe pseudo-effective line bundles in a differential geometric setting, we introduce the Bott–Chern cohomology on X:

$$H^{p,q}_{\mathrm{BC}}(X):=\frac{\operatorname{Ker} d\cap\Omega^{p,q}(X)}{\operatorname{Im}\partial\overline{\partial}\cap\Omega^{p,q}(X)}.$$

Let $\operatorname{Pic}(X)$ be the set of holomorphic line bundles over X. As similar as the first Chern class map $c_1:\operatorname{Pic}(X)\to H^{1,1}_{\overline{\partial}}(X)$, there is a *first Bott–Chern class* map

$$c_1^{\mathrm{BC}}:\operatorname{Pic}(X)\to H^{1,1}_{\mathrm{BC}}(X). \tag{5}$$

Given any holomorphic line bundle $L\to X$ and any Hermitian metric h on L, we define $c_1^{\mathrm{BC}}(L)$ to be the class of its curvature form $R^{(L,h)}=-\sqrt{-1}\partial\overline{\partial}\log h$ in $H^{1,1}_{\mathrm{BC}}(X)$ (modulo a constant 2π). A Hermitian metric ω is called a *Gauduchon metric* if $\partial\overline{\partial}\omega^{n-1}=0$ where $\dim_{\mathbb{C}}X=n$. It is proved by Gauduchon [Gau77a] that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to constant scaling).

Proposition 3.1. *The following statements are equivalent:*

(i) *L is pseudo-effective;*

(ii) *for any Gauduchon metric* ω_{G} *on* X, *one has*

$$\int_X c_1^{\mathrm{BC}}(L) \wedge \omega_{\mathrm{G}}^{n-1} \geqslant 0. \tag{6}$$

Proof. The proof is essentially contained in [Yan17, Theorem 1.1] or [Yan17, Theorem 3.4] which relies on Lamari's positivity criterion [Lam99] and an observation of Michelsohn [Mic82]. For readers' convenience, we include a proof here.

(1) $\Longrightarrow$ (2). Suppose L is pseudo-effective, it is well-known that there exist a smooth Hermitian metric h on L and a real valued function $\varphi \in \mathscr{L}^1(X, \mathbb{R})$ such that

$$R^{(L,h)} + \sqrt{-1}\partial\overline{\partial}\varphi \geqslant 0$$

in the sense of current where $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$. Then for any smooth Gauduchon metric ω_{G}, we have

$$\begin{aligned}\int_X c_1^{\mathrm{BC}}(L) \wedge \omega_{\mathrm{G}}^{n-1} &= \int_X R^{(L,h)} \wedge \omega_{\mathrm{G}}^{n-1} \\ &= (R^{(L,h)} + \sqrt{-1}\partial\overline{\partial}\varphi, \omega_{\mathrm{G}}^{n-1}) \geqslant 0\end{aligned}$$

since $\partial\overline{\partial}\omega_G^{n-1} = 0$ and $R^{(L,h)} + \sqrt{-1}\partial\overline{\partial}\varphi \geqslant 0$ in the sense of current.

(2) $\Longrightarrow$ (1). We define several sets:

- $\mathscr{E}$ is the set of real $\partial\overline{\partial}$-closed $(n-1, n-1)$ forms on X;
- $\mathscr{V}$ is the set of real positive $\partial\overline{\partial}$-closed $(n-1, n-1)$ forms on X;
- $\mathscr{G} = \{\omega^{n-1} \mid \omega \text{ is a Gauduchonmetric}\}$.

In [Mic82, pp. 279–280], Michelsohn observed that $\mathscr{V} = \mathscr{G}$. Let $\mathscr{W}$ be the space of smooth Gauduchon metrics on X. We also define $\mathscr{F} : \mathscr{W} \to \mathbb{R}$ by

$$\mathscr{F}(\omega) = \int_X c_1^{\mathrm{BC}}(L) \wedge \omega^{n-1}.$$

Hence, by the assumption of (2), we have $\mathscr{F}(\omega) \geqslant 0$ for every $\omega \in \mathscr{W}$. Fix an arbitrary smooth Hermitian metric h on L. Since $\mathscr{V} = \mathscr{G}$, for any $\partial\overline{\partial}$-closed positive $(n-1, n-1)$ form $\psi \in \mathscr{V}$, there exists a smooth Gauduchon metric ω such that $\omega^{n-1} = \psi$. Hence

$$\int_X R^{(L,h)} \wedge \psi = \int_X R^{(L,h)} \wedge \omega^{n-1} = \int_X c_1^{\mathrm{BC}}(L) \wedge \omega^{n-1} = \mathscr{F}(\omega) \geqslant 0. \tag{7}$$

That means, as a functional on $\mathscr{V}$, $R^{(L,h)}$ is non-negative. Note that $\mathscr{V}$ is a hyperplane in $\mathscr{E}$. As proved in [Lam99, Lemma 3.3], by Hahn-Banach theorem, $c_1^{\mathrm{BC}}(L)$ is pseudo-effective, and there exists a locally integrable function $\varphi \in \mathscr{L}^1(X, \mathbb{R})$ such that

$$R^{(L,h)} + \sqrt{-1}\partial\overline{\partial}\varphi \geqslant 0$$

in the sense of current. That means, L is pseudo-effective.

Of course, Proposition 3.1 has the following variant.

Proposition 3.2. *The following statements are equivalent:*

(i) *the dual line bundle* L^{-1} *is not pseudo-effective;*

(ii) *there exists a Gauduchon metric* ω_{G} *such that*

$$\int_X c_1^{\mathrm{BC}}(L) \wedge \omega_{\mathrm{G}}^{n-1} > 0.$$

4. The proofs of Theorems 1.4 and 1.5

In this section, we prove Theorems 1.4, 1.5 and Proposition 1.7.

The proof of Theorem 1.5. (1) $\Longrightarrow$ (2). By Corollary 2.3, there exist a smooth Hermitian metric h on L and a Hermitian metric ω on X such that the function

$$\mathrm{tr}_\omega(-\sqrt{-1}\partial\overline{\partial}\log h) > 0. \tag{8}$$

Let $\omega_{\mathrm{G}} = e^f\omega$ be a Gauduchon metric in the conformal class of ω [Gau77b], then by (8), we obtain

$$\mathrm{tr}_{\omega_{\mathrm{G}}} R^{(L,h)} = e^{-f}\cdot \mathrm{tr}_\omega R^{(L,h)} > 0,$$

where $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$. In particular, we have

$$\int_X \mathrm{tr}_{\omega_G} R^{(L,h)}\cdot \omega_{\mathrm{G}}^n = n\int_X R^{(L,h)}\wedge\omega_G^{n-1} = n\int_X c_1^{\mathrm{BC}}(L)\wedge\omega_G^{n-1} > 0. \tag{9}$$

By Proposition 3.2, the dual line bundle L^{-1} is not pseudo-effective.

(2) $\Longrightarrow$ (1). If L^{-1} is not pseudo-effective, by Proposition 3.2, there exists a Gauduchon metric ω_{G} such that

$$\int_X c_1^{\mathrm{BC}}(L)\wedge\omega_{\mathrm{G}}^{n-1} > 0.$$

We shall use Gauduchon's conformal trick ([Gau77a, Gau84], see also [Yan16, Yan17]) to construct a smooth Hermitian metric h on L such that $\mathrm{tr}_{\omega_{\mathrm{G}}}(-\sqrt{-1}\partial\overline{\partial}\log h) > 0$. Hence, by Corollary 2.3, L is $(n-1)$-positive.

Fix a smooth Hermitian metric h_0 on L. Let

$$R_0 = -\sqrt{-1}\partial\overline{\partial}\log h_0$$

be the Chern curvature of (L, h_0). It is easy to see that

$$\int_X \mathrm{tr}_{\omega_G} R_0\cdot\omega_{\mathrm{G}}^n = n\int_X R_0\wedge\omega_G^{n-1} = n\int_X c_1^{\mathrm{BC}}(L)\wedge\omega_G^{n-1}. \tag{10}$$

Since ω_{G} is Gauduchon, i.e. $\partial\overline{\partial}\omega_{\mathrm{G}}^{n-1} = 0$ and the integration

$$\int_X\left(\mathrm{tr}_{\omega_G} R_0 - \frac{n\int_X c_1^{\mathrm{BC}}(L)\wedge\omega_G^{n-1}}{\int_X\omega_G^n}\right)\omega_G^n = 0,$$

the equation

$$\mathrm{tr}_{\omega_G}\sqrt{-1}\partial\overline{\partial}f = \mathrm{tr}_{\omega_G} R_0 - \frac{n\int_X c_1^{\mathrm{BC}}(L)\wedge\omega_G^{n-1}}{\int_X\omega_G^n} \tag{11}$$

has a solution $f\in C^\infty(X)$ which is well-known (e.g. [Gau77a] or [CTW16, Theorem 2.2]). Let $h = e^f\cdot h_0$ be a smooth Hermitian metric on L. The Hermitian line bundle (L,h) has Chern curvature

$$R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h = R_0 - \sqrt{-1}\partial\overline{\partial}f.$$

The scalar curvature of $R^{(L,h)}$ with respect to ω_{G} is

$$\mathrm{tr}_{\omega_{\mathrm{G}}} R^{(L,h)} = \mathrm{tr}_{\omega_G} R_0 - \mathrm{tr}_{\omega_G}\sqrt{-1}\partial\overline{\partial}f = \frac{n\int_X c_1^{\mathrm{BC}}(L)\wedge\omega_G^{n-1}}{\int_X\omega_G^n} > 0.$$

The proof of Theorem 1.5 is completed. □

By Corollary 2.3, Proposition 3.2 and Theorem 1.5, we obtain the following.

Theorem 4.1. *Let L be a line bundle over a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent:*

(i) *the dual line bundle L^{-1} is not pseudo-effective;*
(ii) *there exists a smooth Gauduchon metric ω_{G} on X such that*

$$\int_X c_1^{\mathrm{BC}}(L) \cdot \omega_{\mathrm{G}}^{n-1} > 0;$$

(iii) *there exist a smooth Hermitian metric h on L and a Hermitan metric ω on X such that the scalar curvature of the Chern curvature $R^{(L,h)} = -\sqrt{-1}\partial\bar{\partial}\log h$ with respect to ω is positive, i.e.*

$$s = \mathrm{tr}_\omega R^{(L,h)} > 0;$$

(iv) *L is $(n-1)$-positive.*

It is easy to see that Theorem 4.1 has the following version on Bott–Chern classes.

Theorem 4.2. *Let X be a compact complex manifold X with $\dim_{\mathbb{C}} X = n$ and $[\alpha] \in H^{1,1}_{\mathrm{BC}}(X)$. The following statements are equivalent:*

(i) *the class $-[\alpha]_{\mathrm{BC}}$ is not pseudo-effective;*
(ii) *there exists a smooth Gauduchon metric ω_{G} on X such that*

$$\int_X [\alpha]_{\mathrm{BC}} \cdot \omega_{\mathrm{G}}^{n-1} > 0;$$

(iii) *there exist a smooth $(1,1)$ form $\chi \in [\alpha]_{\mathrm{BC}}$ and a Hermitian metric ω on X such that*

$$\mathrm{tr}_\omega \chi > 0;$$

(iv) *$[\alpha]_{\mathrm{BC}}$ is $(n-1)$-positive.*

Now we are ready to prove our main theorem.

The proof of Theorem 1.4. The proof follows from Theorem 1.5 and a simple argument by the Serre duality.

Since L is $(n-1)$-ample, by Definition 1.1, for any ample line bundle A, there exists a positive number $m_0 = m_0(K_X \otimes A^{-1})$ such that when $m > m_0$, we have

$$H^n(X, K_X \otimes A^{-1} \otimes L^m) = 0$$

which is also equivalent to

$$H^0(X, A \otimes L^{-m}) = 0 \tag{12}$$

by the Serre duality.

We argue by contradiction, i.e. suppose L is $(n-1)$-ample, but L is not $(n-1)$-positive. In this case, by Theorem 1.5, we know the dual line bundle L^{-1} must be pseudo-effective. For the pseudo-effective line bundle L^{-1}, it is well-known that, there exists an ample line bundle A on X such that for every positive integer m

$$H^0(X, A \otimes L^{-m}) \neq 0. \tag{13}$$

This contradicts with (12).

For readers' convenience, we present a sketched analytical proof of (13) following [DPS96, Proposition 1.5]. Indeed, for a fixed very ample line bundle H, we choose an ample line bundle A such that $A \otimes K_X^{-1} \otimes H^{-n}$ is also ample. Hence, there exists a positive rational number ε_0, such that for all $m \geqslant 0$ and rational number $\varepsilon \in (0, \varepsilon_0)$,

$$c_1(L^{-m} \otimes A \otimes K_X^{-1} \otimes H^{-(n+\varepsilon)})$$

lies in the interior of the effective cone of X. It implies that $L^{-m} \otimes A \otimes K_X^{-1} \otimes H^{-(n+\varepsilon)}$ is linearly equivalent to an effective $\mathbb{Q}$-divisor D plus a numerically trivial line bundle T. Hence

$$\mathcal{O}_X(L^{-m} \otimes A) \simeq \mathcal{O}_X(K_X \otimes H^{(n+\varepsilon)} \otimes D \otimes T). \tag{14}$$

Now, fix a point $x_0 \notin D$. Let $\{s_j\} \subset H^0(X, H)$ be a basis such that all s_j vanish at point x_0. Fix a local holomorphic basis e_H of H and write $s_j = h_j e_H$. Then

$$h = \frac{1}{(\sum_j |h_j|^2)^n}$$

is a singular Hermitian metric on H^n with semi-positive curvature in the sense of current. Moreover, the weight function of h is not integrable at point x_0 and the Lelong number of the curvature current is $\geqslant n$. On the other hand, we can put a singular metric h_ε on $H^\varepsilon \otimes D \otimes T$ such that the curvature current of h_ε equals $\varepsilon\omega + [D]$ where ω is a Kähler form in $c_1(H)$ and $[D]$ is the current of integration over D. Since $x_0 \notin D$, the weight function of the singular metric hh_ε on $H^{(n+\varepsilon)} \otimes D \otimes T$ has isolated singularity at point x_0. Moreover,

$$-\sqrt{-1}\partial\overline{\partial}\log(hh_\varepsilon) \geqslant \varepsilon\omega$$

in the sense of current and the weight function of hh_ε has Lelong number $\geqslant n$ at point x_0. By Hörmander's L^2 existence theorem (e.g. [Dem92, Corollary 3.3]), we know

$$H^0(X, K_X \otimes H^{(n+\varepsilon)} \otimes D \otimes T) \neq 0.$$

By (14), we obtain (13). The proof of Theorem 1.4 is completed. □

The proof of Proposition 1.7. Let $f : Y \to X$ be the inclusion map. Using the projection formula and the Leray spectral sequence, one has

$$H^i(Y, \mathcal{F} \otimes (f^*L)^{\otimes m}) = H^i(X, f_*(\mathcal{F}) \otimes L^{\otimes m}).$$

Hence, if $L \to X$ is q-ample, $f^*L \to Y$ is also q-ample. On the other hand, since $\dim_{\mathbb{C}} Y = q + 1$ and by Theorem 1.4, the q-ample line bundle $L|_Y$ is q-positive. □

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References

AG62 A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

AV65 A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace–Beltrami equation on complex manifolds*, Publ. Math. Inst. Hautes Études Sci. **25** (1965), 81–130.

BDPP13 S. Boucksom, J.-P. Demailly, M. Paun and P. Peternell, *The pseudoeffective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), 201–248.

Bro12 M.-V. Brown, *Big q-ample line bundles*, Compos. Math. **148** (2012), 790–798.

CTW16 J.-C. Chu, V. Tosatti and B. Weinkove, *The Monge-Ampère equation for non-integrable almost complex structures*, J. Eur. Math. Soc. (JEMS), to appear. Preprint (2016), arXiv:1603.00706.

Dem92 J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, in *Complex algebraic varieties*, Lecture Notes in Mathematics, vol. 1507, eds K. Hulek, T. Peternell, M. Schneider and F. Schreyer (Springer, Berlin, 1992), 87–104.

Dem11 J.-P. Demailly, *A converse to the Andreotti–Grauert theorem*, Ann. Fac. Sci. Toulouse Math. (6) **20** (2011), 123–135; Fascicule Special.

DPS96 J.-P. Demailly, T. Peternell and M. Schneider, *Holomorphic line bundles with partially vanishing cohomology*, in *Proceedings of the Hirzebruch 65 conference on algebraic geometry*, Israel Mathematical Conference Proceedings, vol. 9 (Bar-Ilan University, Ramat Gan, 1996), 165–198.

Gau77a P. Gauduchon, *Le théorème de l'excentricitè nulle*, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), A387–A390.

Gau77b P. Gauduchon, *Fibrés hermitiens à endomorphisme de Ricci non-négatif*, Bull. Soc. Math. France **105** (1977), 113–140.

Gau84 P. Gauduchon, *La 1-forme de torsion d'une varietè ehermitienne compacte*, Math. Ann. **267** (1984), 495–518.

GK15 D. Greb and A. Küronya, *Partial positivity: geometry and cohomology of q-ample line bundles*, in *Recent advances in algebraic geometry*, London Mathematical Society of Lecture Note Series, vol. 417 (Cambridge University Press, Cambridge, 2015), 207–239.

Kür13 A. Küronya, *Positivity on subvarieties and vanishing of higher cohomology*, Ann. Inst. Fourier (Grenoble) **63** (2013), 1717–1737.

Lam99 A. Lamari, *Courants kähleriens et surfaces compactes*, Ann. Inst. Fourier (Grenoble) **49** (1999), 263–285.

Mat13 S. Matsumura, *Asymptotic cohomology vanishing and a converse to the Andreotti–Grauert theorem on surfaces*, Ann. Inst. Fourier (Grenoble) **63** (2013), 2199–2221.

Mic82 M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), 261–295.

Ott12 J.-C. Ottem, *Ample subvarieties and q-ample divisors*, Adv. Math. **229** (2012), 2868–2887.

Tot13 B. Totaro, *Line bundles with partially vanishing cohomology*, J. Eur. Math. Soc. (JEMS) **15** (2013), 731–754.

Yan16 X.-K. Yang, *Hermitian manifolds with semi-positive holomorphic sectional curvature*, Math. Res. Lett. **23** (2016), 939–952.

Yan17 X.-K. Yang, *Scalar curvature on compact complex manifolds.* Trans. Amer. Math. Soc., to appear. Preprint (2017), arXiv:1705.02672.

Yan18 X.-K. Yang, *RC-positivity, rational connectedness and Yau's conjecture*, Cam. J. Math. **6** (2018), 183–212.

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