A CYCLIC GROUP ACTION ON FUKAYA CATEGORIES FROM MIRROR SYMMETRY

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ABSTRACT. Let (M,ω) be a compact symplectic manifold whose first Chern class $c_1(M)$ is divisible by a positive integer n. We construct a $\mathbb{Z}_{2n}$-action on its Fukaya category. In particular, it induces an action on its local mirror Landau-Ginzburg models.

1. INTRODUCTION

Let (M,ω) be a compact symplectic manifold . One studies a symplectic invariant $Fuk(M,\omega)$ which is an A_∞ category called the *Fukaya category* of (M,ω).

Let ζ be a complex number. Define a *ζ-twisted A_∞ functor* or simply *twisted A_∞ functor* to be an A_∞ functor of the form

$$\Phi : Fuk(M,\omega) \to Fuk(M,\omega)_{(\zeta)}$$

where $Fuk(M,\omega)_{(\zeta)}$ is the A_∞ category whose objects and morphism spaces are the same as those of $Fuk(M,\omega)$, and whose A_∞ product $(m_{(\zeta)})_k$ is defined by

$$(m_{(\zeta)})_k = \zeta^{k-2} m_k,\ k \geqslant 0$$

where m_k is the A_∞ product of $Fuk(M,\omega)$. Clearly, a twisted A_∞ functor can also be regarded as an A_∞ functor $Fuk(M,\omega)_{(\zeta^i)} \to Fuk(M,\omega)_{(\zeta^{i+1})}$ for any $i \in \mathbb{Z}$.

Our main result asserts the existence of a twisted cyclic group action on $Fuk(M,\omega)$.

Theorem 1.1. *Suppose the first Chern class $c_1(M) \in H^2(M;\mathbb{Z})$ of (M,ω) is divisible by a positive integer n. Put $\zeta = e^{\frac{2\pi i}{2n}}$. There exists a ζ-twisted A_∞ functor Φ on $Fuk(M,\omega)$ whose $(2n)$-th power is A_∞ homotopic to the identity functor $id_{Fuk(M,\omega)}$.*

The version of the Fukaya category we use is due to Akaho and Joyce [2] who constructed an A_∞ algebra over $\mathbb{Q}$ associated to an immersed Lagrangian submanifold L which could have clean self-intersection. We modify their construction to define an A_∞ category, and including relative spin structures σ and $\mathbb{C}^\times$-local systems $\mathcal{E}$. See Section 2 for details. Theorem 1.1 is expected to hold for other versions by similar arguments.

Following [15], we define $\mathcal{M}_{weak}(L)$ to be the space of all *weak bounding cochains* on a Lagrangian submanifold L of (M,ω) modulo the gauge equivalence. We call it a *local mirror*. By a formal argument, Theorem 1.1 implies the following

Corollary 1.2. *Φ induces a morphism $\tau_L : \mathcal{M}_{weak}(L) \to \mathcal{M}_{weak}(L)$ such that $\tau_L^n = id$ and*

$$m_0 \circ \tau_L = \zeta^2 m_0. \tag{1.1}$$

Now let X be a Fano manifold of index n[1]. Mirror symmetry [e.g. 21, 27] predicts that there exists a *mirror* of X, called the *Landau-Ginzburg model*, which is a pair $(\check{X}, W)$ consisting of a variety $\check{X}$ and a regular function W defined on $\check{X}$ such that the complex and symplectic geometry of X and $(\check{X}, W)$ are dual to each other. Corollary 1.2 is closely related to the following folklore

Conjecture 1.3. *There exists a $\mathbb{Z}_n$-action on $\check{X}$ with respect to which W is equivariant, i.e. we have*

$$W(\tau \cdot x) = e^{\frac{2\pi i}{n}} W(x) \text{ for any } x \in \check{X}$$

where τ is a generator of the action.

When X is toric Fano, it is well known [6, 11, 17] that its mirror LG model is given by

$$(\check{X}, W) = (\mathcal{M}_{weak}(L), m_0)$$

for a Lagrangian torus fiber L. In this case, Corollary 1.2 implies Conjecture 1.3.

In general, $(\check{X}, W)$ may be constructed using more than one L in which case one has to compute the *wall-crossing formulae* serving as the transition functions for the gluing of the local mirrors $\mathcal{M}_{weak}(L)$. See [1, 3, 5, 18, 22], and also [7, 8, 9, 10] where a gluing technique is developed. In this case, if one can show that the morphisms τ_L commute with the transition functions derived from the wall-crossing formulae, then they combine to give a morphism $\tau : \check{X} \to \check{X}$, verifying Conjecture 1.3. Examples for which this commutativity holds include $X = \mathbb{P}^2$, the complex projective plane [3] and $X = Gr(2, 2n)$, the complex Grassmannian of 2-planes in $\mathbb{C}^{2n}$ [20]. Hence, our result provides supporting evidence for this conjecture.

Remark 1.4. Conjecture 1.3 has also been mentioned by Kuznetsov and Smirnov [23, 24] who considered the *residual categories* associated to Lefschetz decompositions of the derived categories of coherent sheaves on Fano manifolds.

Let us discuss how Theorem 1.1 is proved.

On the object level, Φ sends an object $\mathbb{L} = (L, \sigma, \mathcal{E})$ to $\Phi(\mathbb{L}) = (L, \sigma, \mathcal{E} \otimes \mathcal{E}_L)$ where $\mathcal{E}_L$ is a $\mathbb{C}^\times$-local system on L which is defined as follows. Denote by $\mathcal{L}_M$ the Lagrangian Grassmannian bundle of (M, ω) parametrizing at every point $x \in M$ all Lagrangian subspaces of (T_xM, ω_x). By a lemma in [26], the condition $c_1(M) \equiv 0 \ (\mathrm{mod} n)$ implies that $\mathcal{L}_M$ admits a fiberwise $\mathbb{Z}_{2n}$-cover $\mathcal{L}'_M \to \mathcal{L}_M$. Let θ_L be a section of $\mathcal{L}_M|_L$ defined by $\theta_L(x) = T_xL$ for any $x \in L$. Then $\mathcal{E}_L$ is defined to be the inverse image of θ_L with respect to the fiberwise covering map $\mathcal{L}'_M|_L \to \mathcal{L}_M|_L$. It is a principal $\mathbb{Z}_{2n}$-bundle but we regard it as a $\mathbb{C}^\times$-local system via the inclusion $\mathbb{Z}_{2n} \hookrightarrow \mathbb{C}^\times : 1 \ (\mathrm{mod} 2n) \mapsto \zeta$.

On the morphism level, let $\mathbb{L}_i = (L_i, \sigma_i, \mathcal{E}_i),\ i = 0, 1$ be two objects. Let us assume for simplicity that L_0 and L_1 intersect transversely (instead of cleanly). The morphism space $\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1)$ of $\mathcal{A} := Fuk(M, \omega)$ is defined by

$$\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1) := \bigoplus_{x \in L_0 \cap L_1} Hom((\mathcal{E}_0)_x, (\mathcal{E}_1)_x).$$

[1]The *index* of a Fano manifold X is the greatest integer dividing its first Chern class $c_1(X)$.

Then we have

$$\begin{aligned}\mathcal{A}(\Phi(\mathbb{L}_0),\Phi(\mathbb{L}_1)) &= \bigoplus_{x\in L_0\cap L_1} Hom((\mathcal{E}_0\otimes\mathcal{E}_{L_0})_x,(\mathcal{E}_1\otimes\mathcal{E}_{L_1})_x)\\ &= \bigoplus_{x\in L_0\cap L_1} Hom((\mathcal{E}_0)_x,(\mathcal{E}_1)_x)\otimes Hom((\mathcal{E}_{L_0})_x,(\mathcal{E}_{L_1})_x).\end{aligned}$$

Thus, to describe $\Phi_1 : \mathcal{A}(\mathbb{L}_0,\mathbb{L}_1) \to \mathcal{A}(\Phi(\mathbb{L}_0),\Phi(\mathbb{L}_1))$, it suffices to specify, for each $x \in L_0\cap L_1$, an element of $Hom((\mathcal{E}_{L_0})_x,(\mathcal{E}_{L_1})_x)$ which we take to be the lift of the "canonical short path" [4], from $\theta_{L_0}(x)$ to $\theta_{L_1}(x)$ in $(\mathcal{L}_M)_x$, with respect to $\mathbb{Z}_{2n}$-covering map $(\mathcal{L}'_M)_x \to (\mathcal{L}_M)_x$. In the case of clean intersection, we use a family version of the canonical short path defined in Appendix B, and details are given in Section 3.2.

Define the higher maps $\Phi_{k>1}$ to be zero. Then it is clear that the $(2n)$-th power of Φ is equal to $\mathrm{id}_{\mathcal{A}}$. Our theorem follows if Φ satisfies the twisted version of A_∞ equations:

$$m_k\circ(\Phi_1^{\otimes k}) = \zeta^{2-k}\Phi_1\circ m_k,\ k\geqslant 0. \tag{1.2}$$

The proof of (1.2) is based on a geometric argument which we now illustrate by verifying the case $k=0$. For simplicity, we assume

(1) L is embedded; and
(2) m_0 counts Maslov index 2 *rigid* holomorphic disks only.

Since $\mathbb{L}$ and $\Phi(\mathbb{L})$ have the same underlying Lagrangian submanifold and relative spin structure, the moduli spaces involved are identical. The only difference is the weight associated to each disk being counted. By definition, the weight for $m_0(\mathbb{L})$ (resp. $m_0(\Phi(\mathbb{L}))$) is the holonomy of $\mathcal{E}$ (resp. $\mathcal{E}\otimes\mathcal{E}_L$) along the boundary of the disk.

Thus, to prove (1.2) for $k=0$, it suffices to show that for a disk $u:(D,\partial D)\to(M,L)$ representing the relative homotopy class β, the holonomy $hol_{\mathcal{E}_L}(\partial u)$ is equal to $\zeta^{\mu(\beta)}$ where $\mu(\beta)$ is the Maslov index of β. To see this, notice that the domain D of u is contractible, and hence the bundle $u^*\mathcal{L}'_M$ has a fiberwise $\mathbb{Z}$-cover $\mathcal{L}''\to u^*\mathcal{L}'_M$. Thus, over D we have three bundles

$$\mathcal{L}''\xrightarrow{\mathbb{Z}} u^*\mathcal{L}'_M\xrightarrow{\mathbb{Z}_{2n}} u^*\mathcal{L}_M.$$

Consider the lifts of $u^*\theta_L$ in $u^*\mathcal{L}'_M$ and in $\mathcal{L}''$ with respect to the fiberwise covering maps $u^*\mathcal{L}'_M\to u^*\mathcal{L}_M$ and $\mathcal{L}''\to u^*\mathcal{L}_M$ which are paths whose endpoints are related by some group elements $a\in\mathbb{Z}_{2n}$ and $b\in\mathbb{Z}$ respectively. It is clear that $\zeta^a=\zeta^b$. By definition, $hol_{\mathcal{E}_L}(\partial u)=\zeta^a$ and $\mu(\beta)=b$, and hence the result follows.

It should be emphasized that the construction of m_k in [2] does not rely on rigid but *abstract* counts of holomorphic disks. Moreover, these counts only give the *geometric product* $m_{k,geom}$ which is not an A_∞ product. One needs additional sophisticated algebraic arguments to turn $m_{k,geom}$ into the desired product m_k. Thus, in order to prove Theorem 1.1 rigorously, one has to examine each of these arguments to show that Φ constructed above can also be turned into an honest A_∞ functor. See Section 3.3 for the complete proof.

Remark 1.5. The idea of twisting objects by $\mathbb{C}^\times$-local systems has been used by Fukaya [12] in a different context. See also [16] and [25] for other applications of these local systems. An important difference is that their local systems are defined on the whole M whereas ours cannot be extended to an ambient one unless L is *monotone*.

Remark 1.6. In Appendix C, we will show that the complex conjugation of $\mathbb{C}$ gives rise to a *conjugate automorphism* R of $Fuk(M,\omega)$ which satisfies

$$R \circ \Phi \circ R \circ \Phi = \mathrm{id}_{Fuk(M,\omega)}, \tag{1.3}$$

where Φ is the A_∞ functor in Theorem 1.1. Thus, we have

Theorem 1.7. *There is an action on $Fuk(M,\omega)$ by the dihedral group D_{2n}.*

ACKNOWLEDGEMENTS

We thank Yong-Geun Oh for drawing our attention to [12, 16, 25]. C. H. Chow also thanks Kaoru Ono for useful comments and for teaching him a lot from the monumental book [15], as well as Cheuk Yu Mak and Weiwei Wu for helpful discussions.

This research was supported by grants of the Hong Kong Research Grants Council (Project No. CUHK14301117 & CUHK14303518) and a direct grant from the Chinese University of Hong Kong (Project No. 4053337).

2. FUKAYA CATEGORY OF IMMERSED LAGRANGIAN SUBMANIFOLDS

We need to define the Fukaya category before we can talk about any (twisted) A_∞ functors defined on it. The version we will take is the one given by Akaho and Joyce [2] with some modifications which are:

(1) An A_∞ category is constructed, instead of an A_∞ algebra;
(2) $\mathbb{C}^\times$-local systems on the Lagrangian submanifolds are introduced.

This section contains a sketch of how they are done. This is straightforward and involves no new ideas. See also [13] where a similar A_∞ category is constructed, using de Rham models.

The main result is

Theorem 2.1. *Let (M,ω) be a compact symplectic manifold. Let $\mathcal{S}$ be a finite collection of pairwise cleanly intersecting compact orientable immersed Lagrangian submanifolds of (M,ω) with clean self-intersection. There is an A_∞ category, denoted by $Fuk(M,\omega)$, whose objects are triples $\mathbb{L} = (L,\sigma,\mathcal{E})$ where $L \in \mathcal{S}$, σ is a relative spin structure on L and $\mathcal{E}$ is an isomorphism class of $\mathbb{C}^\times$-local systems on L. It is well defined up to a unique A_∞ homotopy class of A_∞ quasi-isomorphisms.*

Readers may now skip to Section 3 for the proof of Theorem 1.1, and return to this section for the definition of some notations.

Remark 2.2. When constructing the A_∞ algebra in [2], the notions of $A_{N,K}$ algebras, morphisms and homotopies were introduced. What we need is their categorical analogue, namely $A_{N,K}$ categories, functors and homotopies. A brief review of them is given in Appendix A.

2.1. **Lagrangians with clean self-intersections.** Let (M,ω) be a compact symplectic manifold of dimension $2m$. Let (L,ι) be an immersed Lagrangian submanifold of (M,ω), i.e. a smooth manifold L together with an immersion $\iota : L \to M$ such that the image $d\iota(T_pL)$ of the differential $d\iota$ is a Lagrangian subspace of $(T_{\iota(p)}M, \omega_{\iota(p)})$ for every $p \in L$. We sometimes drop ι in our discussion. The following definition is taken from [13].

Definition 2.3. We say that (L,ι) has *clean self-intersection* if the fiber product

$$L\times_\iota L := \{(p,q)\in L\times L|\ \iota(p)=\iota(q)\}$$

is a smooth manifold such that for every point $(p,q)\in L\times_\iota L$,

$$T_{(p,q)}(L\times_\iota L) = T_pL\times_{d\iota} T_qL.$$

Let $\mathcal{S}$ be a finite collection of compact orientable immersed Lagrangian submanifolds of (M,ω) with the property that the disjoint union $\coprod_{L\in\mathcal{S}} L$ has clean self-intersection. In other words, each $L\in\mathcal{S}$ has clean self-intersection, and any two different $L_0,L_1\in\mathcal{S}$ intersect cleanly in the usual sense.

Next, we deal with the notion of *relative spin structure* which is used to orient the moduli spaces of holomorphic disks. Fix a triangulation of M and a triangulation of each $L\in\mathcal{S}$ such that $\bigcup_{L\in\mathcal{S}}\iota(L)$ is a sub-complex of M and $\iota: L\to M$ is a simplicial map for any $L\in\mathcal{S}$. Fix an oriented real vector bundle V on the 3-skeleton $M_{[3]}$ of M such that $\iota^*(w_2(V)) = w_2(TL)$ for any $L\in\mathcal{S}$. The following definition is due to [13].

Definition 2.4. A *V-relative spin structure* σ on $L\in\mathcal{S}$ consists of an orientation on L and a spin structure on $(TL\oplus\iota^*(V))|_{L_{[2]}}$ where $L_{[2]}$ is the 2-skeleton of L.

Notations 2.5.

(1) Denote by $Ob_{\mathcal{S}}$ the set of all triples $\mathbb{L}=(L,\sigma,\mathcal{E})$ where $L\in\mathcal{S}$, σ is a V-relative spin structure on L and $\mathcal{E}$ is an isomorphism class of $\mathbb{C}^\times$-local systems on L.
(2) For any $L_0,L_1\in\mathcal{S}$, put $C(L_0,L_1):=\pi_0(L_0\times_\iota L_1)$.
(3) For any $\mathbb{L}_i=(L_i,\sigma_i,\mathcal{E}_i)\in Ob_{\mathcal{S}},\ i=0,1$, put $\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1):=C(L_0,L_1)$ which we distinguish from $\mathcal{C}(\mathbb{L}_0',\mathbb{L}_1')$ even if the underlying Lagrangian of $\mathbb{L}_0'$ (resp. $\mathbb{L}_1'$) is equal to that of $\mathbb{L}_0$ (resp. $\mathbb{L}_1$).
(4) For $c\in C(L_0,L_1)$ (resp. $\gamma\in\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)$), define $L(c)$ (resp. $L(\gamma)$) to be the connected component of $L_0\times_\iota L_1$ represented by c (resp. γ).

For each $\gamma\in\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)$, there is a $\mathbb{Z}_2$-local system Θ_γ on $L(\gamma)$ which depends on the V-relative spin structures σ_0,σ_1 on L_0,L_1 respectively. See [13] for the construction of Θ_γ which is Θ_γ^- there. The use of Θ_γ is to describe the orientation bundles of the moduli spaces of holomorphic disks (Proposition 2.13).

Let $(\mathbb{L}_0,\mathbb{L}_1)\in Ob_{\mathcal{S}}^2$. Recall that for $i=0,1$, $\mathcal{E}_i$ is a $\mathbb{C}^\times$-local system on L_i as part of the data defining $\mathbb{L}_i$. By restriction, both $\mathcal{E}_0$ and $\mathcal{E}_1$ can be regarded as $\mathbb{C}^\times$-local systems on $L(\gamma)$, for each $\gamma\in\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)$. Define

$$\mathcal{E}_\gamma := \Theta_\gamma\otimes\mathcal{O}_{L(\gamma)}\otimes Hom(\mathcal{E}_0,\mathcal{E}_1)$$

where $\mathcal{O}_{L(\gamma)}$ is the orientation bundle of $L(\gamma)$. Notice that any $\mathbb{Z}_2$-local system can be regarded canonically as a $\mathbb{C}^\times$-local system, via the inclusion $\mathbb{Z}_2\simeq\{\pm1\}\hookrightarrow\mathbb{C}^\times$.

For any $L_0,L_1\in\mathcal{S}$, there is a diffeomorphism $\tau: L_0\times_\iota L_1\to L_1\times_\iota L_0$ defined by switching the coordinates. It induces maps $\tau: C(L_0,L_1)\to C(L_1,L_0)$ and $\tau: L(c)\to L(\tau(c))$ which we have also denoted by τ, by an abuse of notation. There are also analogous maps on $\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)$ and $L(\gamma)$ which we also denote by τ. It is clear that $\tau\circ\tau=\mathrm{id}$ in any sense.

Lemma 2.6. *[13, Lemma-Definition 3.10] For each $\gamma\in\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)$, we have*

$$\Theta_\gamma = \tau^*\Theta_{\tau(\gamma)}\otimes\tau^*\mathcal{O}_{L(\tau(\gamma))}.$$

2.2. **Singular homology with local coefficients.** Let X be a smooth manifold. Let $Sing^{sm}(X)$ be the set of all smooth singular simplices $f : \Delta^r \to X$ of arbitrary dimension r.

Let $\mathcal{X} \subseteq Sing^{sm}(X)$ be a subset with the property that all the faces of each $f \in \mathcal{X}$ lie in $\mathcal{X}$. Then $\mathcal{X}$ can be regarded as a Δ-complex whose geometric realization $|\mathcal{X}|$ is obtained by gluing the domain simplices of all $f \in \mathcal{X}$ along their common faces so that the simplices of $|\mathcal{X}|$ are in 1-1 correspondence with the elements of $\mathcal{X}$. This comes with a continuous map $f_{\mathcal{X}} : |\mathcal{X}| \to X$ whose restriction to a simplex is equal to the singular simplex f to which this simplex corresponds.

Recall the simplicial homology $H_\bullet(\mathcal{X}; \mathcal{E})$ of the Δ-complex $\mathcal{X}$ with coefficients in a local system $\mathcal{E}$ on X is defined to be the homology of the chain complex

$$\mathcal{C}_\bullet(\mathcal{X}; \mathcal{E}) := \left(\bigoplus_{f \in \mathcal{X}} \Gamma_{flat}(f^*(\mathcal{E})), \partial \right)$$

where $\Gamma_{flat}(f^*(\mathcal{E}))$ is the space of flat sections of $f^*(\mathcal{E})$ on the domain simplex of f and ∂ is the boundary operator, i.e. for any $f : \Delta^r \to X$ and $s \in \Gamma_{flat}(f^*(\mathcal{E}))$

$$\partial(s) := \sum_{i=0}^{r} (-1)^i s|_{\partial_i \Delta^r} \tag{2.1}$$

where $\partial_i \Delta^r$ is the i-th boundary face of Δ^r.

It is well known that $H_\bullet(\mathcal{X}; \mathcal{E}) \simeq H_\bullet(|\mathcal{X}|; f_{\mathcal{X}}^* \mathcal{E})$, the singular homology of $|\mathcal{X}|$ with coefficients in $f_{\mathcal{X}}^* \mathcal{E}$.

The following proposition is a slight generalization of [2, Proposition 2.13] which we need in order to extend the main results in *loc. cit.* which hold over $\mathbb{Q}$ to ones which hold over any $\mathbb{C}^\times$-local systems. [15] also contains a similar result where the outcome is a *countable infinite* set.

Proposition 2.7. *Let X be a compact smooth manifold. Let $\mathcal{X}$ be a finite subset of $Sing^{sm}(X)$ such that all the faces of each $f \in \mathcal{X}$ lie in $\mathcal{X}$. There exists a finite set $\mathcal{X}'$ with $\mathcal{X} \subseteq \mathcal{X}' \subseteq Sing^{sm}(X)$ such that all the faces of each $f \in \mathcal{X}'$ lie in $\mathcal{X}'$, and the map $f_{\mathcal{X}'} : |\mathcal{X}'| \to X$ is a homotopy equivalence. In particular, for any local system $\mathcal{E}$ on X, we have the isomorphism*

$$H_\bullet(\mathcal{X}'; \mathcal{E}) \simeq H_\bullet(X; \mathcal{E}).$$

Proof. The case when X is 0-dimensional is trivial. Assume from now on the dimension of X is positive. Triangulate X. Denote by $N^r(\mathcal{X})$ the r-th barycentric subdivision of $\mathcal{X}$. It is known that $N^2(\mathcal{X})$ is a simplicial complex. By the simplicial approximation theorem, for any sufficiently large r there is a simplical map $g : N^{r+2}(\mathcal{X}) \to X$ homotopic to $f_{\mathcal{X}}$.

Form the *simplicial mapping cylinder* $M(g)$ of g. (Recall that $M(g)$ is a simplicial complex whose geometric realization is homeomorphic to the usual mapping cylinder of g and which contains, as sub-complexes, $N^{r+3}(\mathcal{X})$ at one end and X at the other end. See [19].) Then the projection map $M(g) \to X$ is a homotopy equivalence.

Next by considering the iterated simplicial mapping cylinders of the identity on $\mathcal{X}$, we obtain a Δ-complex structure on $Y := |\mathcal{X}| \times [0, 1]$ which contains $\mathcal{X}$ at one end and $N^{r+3}(\mathcal{X})$ at the other end. Glue it to $M(g)$ along $N^{r+3}(\mathcal{X})$ and call the resulting Δ-complex $\mathcal{W}_{\mathcal{X}}$. By projecting the part belonging to Y down to X using g, we obtain a homotopy equivalence $G : \mathcal{W}_{\mathcal{X}} \to X$.

Now homotope G to a map F whose restriction to $\mathcal{X}$ is equal to $f_{\mathcal{X}}$, by the homotopy extension property of CW-pairs. Perturb F to get F' such that (I) the restriction of F' to each simplex is smooth and (II) different simplices correspond to different restrictions. This is possible since X has positive dimension. Then F' is also a homotopy equivalence since the target is a manifold. This map gives us a finite set $\mathcal{X}'$ of singular simplices of X. Condition (I) implies $\mathcal{X}' \subseteq Sing^{sm}(X)$. Condition (II) implies that the singular simplices in $\mathcal{X}'$ do not repeat so that we have $H_\bullet(\mathcal{X}';\mathcal{E}) \simeq H_\bullet(\mathcal{W}_\mathcal{X};\mathcal{E})$. □

Remark 2.8. In order to extend the main results in [2], we also need, as in [2], a generalization of Proposition 2.7 which allows X to have boundary and corners. In that case, the singular simplices involved are also required to satisfy some delicate transversality and combinatorial conditions near the boundary and corners.

2.3. **Moduli spaces of holomorphic disks.** Let $(M,\omega), \mathcal{S}, V$ be given as in Section 2.1.

Definition 2.9. Let $k \geqslant 0$ be an integer. A *Lagrangian label of length* $k+1$ is a pair $(\overrightarrow{\mathbb{L}}, \vec{\gamma})$ consisting of

- $\overrightarrow{\mathbb{L}} = (\mathbb{L}_0, \ldots, \mathbb{L}_k)$ with $\mathbb{L}_s = (L_s, \sigma_s, \mathcal{E}_s) \in Ob_\mathcal{S}$;
- $\vec{\gamma} = (\gamma_0, \ldots, \gamma_k)$ with $\gamma_s \in \mathcal{C}(\mathbb{L}_{s-1}, \mathbb{L}_s)$.

(Here the subscripts are considered modulo $k+1$.)

Let $(\overrightarrow{\mathbb{L}}, \vec{\gamma})$ be a Lagrangian label of length $k+1$. Let J be a compatible almost complex structure on (M,ω) and $\beta \in \pi_2(M, \cup_{s=0}^k \iota(L_s))$.

Definition 2.10.

(1) Define $\widetilde{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ to be the set of quintuples $(\Sigma, \vec{z}, u, \ell, \tilde{u})$ where
- Σ is a bordered Riemann surface of genus zero;
- $\vec{z} = (z_0, \ldots, z_k)$ are distinct non-singular marked points on $\partial\Sigma$;
- $u : \Sigma \to M$ is a J-holomorphic map such that $(\Sigma, \vec{z}, u)$ is stable in the usual sense;
- $\ell : S^1 \to \partial\Sigma$ is an orientation-preserving parametrization of $\partial\Sigma$ for which the preimages $\xi_s := \ell^{-1}(z_s) \in S^1,\ s = 0, \ldots, k$ are labelled in cyclic, counterclockwise order; and
- $\tilde{u} : S^1 - \{\xi_0, \ldots, \xi_k\} \to \coprod_{s=0}^k L_s$ is a continuous map such that $\iota \circ \tilde{u} = u \circ \ell$,

which satisfy
- $u_*([\Sigma]) = \beta$;
- for $s = 0, \ldots, k$, the image of $\tilde{u}|_{(\xi_s,\xi_{s+1})}$ lies in L_s where (ξ_s, ξ_{s+1}) denotes the interval in S^1 drawn from ξ_s to ξ_{s+1} in the counterclockwise direction; and
- $\tilde{u}(\xi_s) := \left(\lim_{\substack{\xi \to \xi_s \\ \xi \in (\xi_{s-1}, \xi_s)}} \tilde{u}(\xi), \lim_{\substack{\xi \to \xi_s \\ \xi \in (\xi_s, \xi_{s+1})}} \tilde{u}(\xi) \right)$ lies in $L(\gamma_s)$.

(2) Define $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ to be the quotient of $\widetilde{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ by isomorphisms: $(\Sigma, \vec{z}, u, \ell, \tilde{u})$ and $(\Sigma', \vec{z}', u', \ell', \tilde{u}')$ are *isomorphic* if there is a biholomorphism $\phi : \Sigma \to \Sigma'$ and an orientation-preserving homeomorphism $\psi : S^1 \to S^1$ such that
- $u' \circ \phi = u$;
- $\phi(z_s) = z'_s$ for $s = 0, \ldots, k$;
- $\phi \circ \ell = \ell' \circ \psi$; and
- $\tilde{u} = \tilde{u}' \circ \psi$ on $S^1 - \{\xi_0, \ldots, \xi_k\}$.

Elements of $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ are denoted by $[(\Sigma, \vec{z}, u, \ell, \tilde{u})]$.

Definition 2.11. For $s = 0, \dots, k$, the map

$$\begin{aligned} \mathrm{ev}_s : \overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J) &\to L(\gamma_s) \\ [(\Sigma, \vec{z}, u, \ell, \tilde{u})] &\mapsto \tilde{u}(\xi_s) \end{aligned}$$

is called the *s-th evaluation map.*

By [15], $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ is a compact Kuranishi space with tangent bundle and the evaluation maps $\mathrm{ev}_s,\ s = 0, \dots, k$ are strongly smooth and weakly submersive.

Remark 2.12. We require that the Kuranishi structure on $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ depends only on the underlying $(\overrightarrow{L}, \vec{c})$ of $(\overrightarrow{\mathbb{L}}, \vec{\gamma})$, i.e. independent of the $\mathbb{C}^\times$-local systems $\mathcal{E}_s$.

Now given $\vec{f} := (f_1, \dots, f_k)$ where for each $s = 1, \dots, k$, $f_s : \Delta^{r_s} \to L(\gamma_s)$ is a smooth singular simplex. The fiber product

$$\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}) := \overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J) \times_{\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_k} (f_1 \times \cdots \times f_k)$$

is also a compact Kuranishi space with tangent bundle and the evaluation map

$$\mathrm{ev}_0 : \overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}) \to L(\gamma_0)$$

is also strongly smooth and weakly submersive.

In [14], Fukaya and Ono introduce the notion of *perturbation data* for a pair $(\mathcal{M}, e)$ where $\mathcal{M}$ is a compact oriented Kuranishi space with tangent bundle and $e : \mathcal{M} \to K$ is a strongly smooth map from $\mathcal{M}$ to an orbifold K. This allows us to perturb $\mathcal{M}$, in an abstract way, to a nearby compact oriented smooth non-Hausdorff manifold $\mathcal{M}'$ on which e remains well-defined. A triangulation of $\mathcal{M}'$ gives the so-called *virtual chain* which is an element in $Sing^{sm}(K; \mathbb{Q})$.

We are going to apply this to the pair $\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \tau \circ \mathrm{ev}_0\right)$. However, it should be pointed out that in our situation, the moduli spaces are not necessarily oriented and $\mathbb{C}^\times$-local systems are present. As a result, the virtual chains should not be defined over $\mathbb{Q}$ but over certain local systems, and in order to make sense of it, it is necessary to have knowledge of the orientation bundle of $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f})$.

The following proposition describes the orientation bundle of $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ in terms of $\Theta_{\gamma_s},\ s = 0, \dots, k$ which are defined in the previous section.

Proposition 2.13. *[13, Proposition 3.29] There is an isomorphism of $\mathbb{Z}_2$-local systems*

$$\mathcal{O}_{\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)} \simeq \bigotimes_{s=0}^{k} ev_s^* \Theta_{\gamma_s}$$

which depends on the V-relatively spin structures as part of the data defining $\overrightarrow{\mathbb{L}}$.

Let $f_s : \Delta^{r_s} \to L(\gamma_s)$ be given as before. By the standard formula for the orientation bundles of fiber products, we have

$$\mathcal{O}_{\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f})} \simeq \mathcal{O}_{\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)} \otimes \bigotimes_{s=1}^{k} \left(\mathcal{O}_{L(\gamma_s)} \otimes \mathcal{O}_{\Delta^{r_s}} \right). \tag{2.2}$$

Notice that this isomorphism depends on a convention which we shall follow [2].

It follows from Proposition 2.13 that

$$\mathcal{O}_{\overline{\mathcal{M}}_{k+1}(\vec{\mathbb{L}},\vec{\gamma},\beta,J;\vec{f})} \simeq \left(\bigotimes_{s=0}^{k} \mathrm{ev}_s^* \Theta_{\gamma_s} \right) \otimes \bigotimes_{s=1}^{k} \left(\mathcal{O}_{L(\gamma_s)} \otimes \mathcal{O}_{\Delta^{r_s}} \right). \tag{2.3}$$

We would like to simplify the right-hand side of (2.3). However, this is impossible unless we fix additional data which we now describe.

First of all, it is easy to eliminate the term $\mathcal{O}_{\Delta^{r_s}}$ by specifying an orientation on the domain simplex Δ^{r_s} of f_s which we have already done when we define singular homologies. Second, to eliminate the terms $\mathrm{ev}_s^*\Theta_{\gamma_s}$ and $\mathcal{O}_{L(\gamma_s)}$, we need to specify, for each $s = 1, \dots, k$, a trivialization (as local systems) of $f_s^*(\Theta_{\gamma_s} \otimes \mathcal{O}_{L(\gamma_s)})$ over Δ^{r_s}. But since we would like to introduce $\mathbb{C}^\times$-local systems in the Floer cochain complexes, we trivialize $f_s^*\mathcal{E}_{\gamma_s}$ instead, where $\mathcal{E}_{\gamma_s} := \Theta_{\gamma_s} \otimes \mathcal{O}_{L(\gamma_s)} \otimes Hom(\mathcal{E}_{s-1}, \mathcal{E}_s)$ is defined in Section 2.1. This is equivalent to specifying a flat section $s_{f_s} \in \Gamma_{flat}(f_s^*(\mathcal{E}_{\gamma_s}))$ which we fix from now on.

Finally, we need a further simplification of the $\mathbb{C}^\times$-local systems we have introduced. This is done by considering parallel transports of these local systems along the boundary of the holomorphic disks: for $s = 0, \dots, k$ and $[(\Sigma, \vec{z}, u, \ell, \tilde{u})] \in \overline{\mathcal{M}}_{k+1}(\vec{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f})$, the parallel transport of $u^*\mathcal{E}_s$ along the segment $[\xi_s, \xi_{s+1}]$ gives rise to an isomorphism of $\mathbb{C}^\times$-local systems

$$\mathbb{1} \simeq Hom(\mathrm{ev}_s^*\mathcal{E}_s, \mathrm{ev}_{s+1}^*\mathcal{E}_s),$$

where $\mathbb{1}$ is the trivial local system.

Combining these isomorphisms for all s, we obtain an isomorphism

$$\mathbb{1} \simeq \bigotimes_{s=0}^{k} Hom(\mathrm{ev}_s^*\mathcal{E}_s, \mathrm{ev}_{s+1}^*\mathcal{E}_s) \tag{2.4}$$

or equivalently,

$$\bigotimes_{s=1}^{k} Hom(\mathrm{ev}_s^*\mathcal{E}_{s-1}, \mathrm{ev}_s^*\mathcal{E}_s) \simeq Hom(\mathrm{ev}_0^*\mathcal{E}_0, \mathrm{ev}_0^*\mathcal{E}_k). \tag{2.5}$$

Now we are ready for the simplification:

$$\begin{aligned}
\mathcal{O}_{\overline{\mathcal{M}}_{k+1}(\vec{\mathbb{L}},\vec{\gamma},\beta,J;\vec{f})} &\simeq \left(\bigotimes_{s=0}^{k} \mathrm{ev}_s^* \Theta_{\gamma_s} \right) \otimes \bigotimes_{s=1}^{k} \left(\mathcal{O}_{L(\gamma_s)} \otimes \mathcal{O}_{\Delta^{r_s}} \right) \\
&\simeq \mathrm{ev}_0^*\Theta_{\gamma_0} \otimes \bigotimes_{s=1}^{k} \left(\mathrm{ev}_s^*\Theta_{\gamma_s} \otimes \mathcal{O}_{L(\gamma_s)} \right) \\
&\simeq \mathrm{ev}_0^*\Theta_{\gamma_0} \otimes \bigotimes_{s=1}^{k} Hom(\mathrm{ev}_s^*\mathcal{E}_{s-1}, \mathrm{ev}_s^*\mathcal{E}_s) \\
&\simeq \mathrm{ev}_0^*\Theta_{\gamma_0} \otimes Hom(\mathrm{ev}_0^*\mathcal{E}_0, \mathrm{ev}_k^*\mathcal{E}_k) \\
&\simeq (\tau \circ \mathrm{ev}_0)^* \left(\Theta_{\tau(\gamma_0)} \otimes \mathcal{O}_{L(\tau(\gamma_0))} \otimes Hom(\mathcal{E}_0, \mathcal{E}_k) \right) \\
&\simeq (\tau \circ \mathrm{ev}_0)^* \mathcal{E}_{\tau(\gamma_0)}
\end{aligned}$$

The first isomorphism is (2.3). The second involves a rearrangement of terms and the standard orientation on Δ^{r_s}. The third is induced by the given flat sections s_{f_s}. The fourth follows from (2.5). The fifth is given by Lemma 2.6 and, finally, the sixth follows from the definition of $\mathcal{E}_{\tau(\gamma_0)}$.

Overall, we see that every singular simplex in the triangulation as part of the given perturbation data for $\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \tau \circ \mathrm{ev}_0\right)$ is given a flat section in $(\tau \circ \mathrm{ev}_0)^* \mathcal{E}_{\tau(\gamma_0)}$. Hence, these simplices define an element in $C_\bullet(L(\tau(\gamma_0)); \mathcal{E}_{\tau(\gamma_0)})$, denoted by $VC\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \vec{s}\right)$ where $\vec{s} = (s_{f_1}, \ldots, s_{f_k})$. (Notice that this element depends on the chosen perturbation data which we shall drop from the notation.)

To conclude this subsection, we remark that there is a generalization of the moduli spaces $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J)$ and $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f})$ to any smooth family $\mathcal{J} = \{J_\tau\}_{\tau \in \mathcal{T}}$ of compatible almost complex structures on (M, ω) parametrized by a compact oriented smooth manifold $\mathcal{T}$, possibly with boundary and corners. We denote these moduli spaces by $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, \mathcal{J})$ and $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, \mathcal{J}; \vec{f})$ respectively. Notice that the singular chain simplices f_s are now singular chain simplices in $L(\gamma_s) \times \mathcal{T}$. See Section 4.5 in [2] for more details. All the results we have covered, namely the orientation and virtual chains, have natural analogues for this family version.

2.4. A_∞ **structure.** Let $(M, \omega), \mathcal{S}, V$ be given as in Section 2.1. Fix a compatible almost complex structure J on (M, ω).

Definition 2.14. Let $\mathcal{G}$ be a submonoid of $\mathbb{R}_{\geqslant 0} \times \mathbb{Z}$ such that $\mathcal{G} \cap (\{0\} \times \mathbb{Z}) = \{(0,0)\}$ and $\mathcal{G} \cap ([0, C] \times \mathbb{Z})$ is finite for any $C \geqslant 0$. Define $||\cdot|| : \mathcal{G} \to \mathbb{Z}_{\geqslant 0}$ by $||(0,0)|| = 0$ and

$$||(\lambda, \mu)|| := \sup\left\{ m \,\middle|\, (\lambda, \mu) = \sum_{i=1}^{m} (\lambda_i, \mu_i),\ (\lambda_i, \mu_i) \in \mathcal{G} - \{(0,0)\} \right\} + \lfloor \lambda \rfloor,\ (\lambda, \mu) \neq 0.$$

By Gromov compactness, we can choose $\mathcal{G}$ in Definition 2.14 such that it contains all elements of the form $\left(\int_\beta \omega, \mu(\beta)\right)$ for any β with $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J) \neq \emptyset$ for some Lagrangian label $(\overrightarrow{\mathbb{L}}, \vec{\gamma})$. It is also possible to choose such a $\mathcal{G}$ if J is allowed to vary within a compact family. However, it is impossible if J is arbitrary. To see how this issue is addressed, see [2, Theorem 11.2].

Our goal is to construct the Fukaya category $Fuk(M, \omega)$ which is an A_∞ category. The construction consists of geometric inputs and algebraic inputs. Let us start with the geometric inputs. The following theorem plays the key role:

Theorem 2.15. *[2, Theorem 6.1] For any integer $N \geqslant 0$, and for $i = 0, \ldots, N$, there exist finite sets*

$$\mathcal{X}_{i,N} = \coprod_{(L_0, L_1) \in \mathcal{S}^2} \coprod_{c \in C(L_0, L_1)} \mathcal{X}_{i,N}(c)$$

such that

(1) *for any c, $\mathcal{X}_{0,N}(c) \subseteq \cdots \subseteq \mathcal{X}_{N,N}(c) \subseteq Sing^{sm}(L(c))$;*
(2) *for any c and i, all the faces of each $f \in \mathcal{X}_{i,N}(c)$ lie in $\mathcal{X}_{i,N}(c)$;*
(3) *for any c and i, the map $f_{\mathcal{X}_{i,N}(c)} : |\mathcal{X}_{i,N}(c)| \to L(c)$ is a homotopy equivalence (so $H_\bullet(\mathcal{X}_{i,N}(c); \mathcal{E}) \simeq H_\bullet(L(c); \mathcal{E})$ for any local system $\mathcal{E}$ on $L(c)$); and*
(4) *for any $k \geqslant 0$, $i_1, \ldots, i_k \geqslant 0$ and $(\lambda, \mu) \in \mathcal{G}$ such that*

$$i_1 + \cdots + i_k + ||(\lambda, \mu)|| + k - 1 \leqslant N,$$

and for any Lagrangian label $(\overrightarrow{\mathbb{L}},\vec{\gamma})$ *of length* $k+1$, $\vec{f}=(f_1,\dots,f_k)$ *with* $f_s\in\mathcal{X}_{i_s,N}(\gamma_s)$, $s=1,\dots,k$ *and* $\beta\in\pi_2(M,\cup_{s=0}^k\iota(L_s))$ *with* $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}},\vec{\gamma},\beta,J;\vec{f})\neq\emptyset$*, perturbation data for* $(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}},\vec{\gamma},\beta,J;\vec{f}),\tau\circ ev_0)$ *is chosen such that every singular simplex in the triangulation as part of this data lies in* $\mathcal{X}_{*,N}(\tau(\gamma_0))$ *where* $*=i_1+\cdots+i_k+||(\lambda,\mu)||+k-1$.
Moreover, the chosen perturbation data for $(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}},\vec{\gamma},\beta,J;\vec{f}),\tau\circ ev_0)$ *is compatible with the choice made for each of its boundary strata. (Some words are needed in order to make this sentence precise. See [2] for details.)*

The proof goes in exactly the same way as in *loc. cit.* except that we use Proposition 2.7 instead of Proposition 2.13 in *loc. cit.* so that the isomorphism in (3) above holds for any local system $\mathcal{E}$ (which holds over $\mathbb{Q}$ only in *loc. cit.*).

Remark 2.16. Theorem 2.15 has a generalization to any smooth family $\mathcal{J}=\{J_\tau\}_{\tau\in\mathcal{T}}$ of compatible almost complex structures parametrized by a compact oriented smooth manifold $\mathcal{T}$, possibly with boundary and corners. Singular chains in $L(c)$ are replaced by singular chains in $L(c)\times\mathcal{T}$ and the moduli spaces are replaced by $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}},\vec{\gamma},\beta,\mathcal{J};\vec{f})$. A new feature of this generalization is the requirement of certain input which, however, can be taken to be the output of the theorem for the restriction of $\mathcal{J}$ to the codimension 1 strata of $\mathcal{T}$ which are compatible over the codimension 2 strata. See [2, Sections 8 and 10].

Let $N\geqslant 0$ and let $\mathcal{X}_{0,N}(c)\subseteq\cdots\subseteq\mathcal{X}_{N,N}(c)$ be the outcome of Theorem 2.15. For any $(\mathbb{L}_0,\mathbb{L}_1)\in Ob^2_{\mathcal{S}},\ 0\leqslant i\leqslant N$, define

$$\mathcal{A}^\bullet_{i,N}(\mathbb{L}_0,\mathbb{L}_1):=\bigoplus_{\gamma\in\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)}\mathcal{C}_{\dim L(\gamma)-\bullet}(\mathcal{X}_{i,N}(\gamma);\mathcal{E}_\gamma)=\bigoplus_{\gamma\in\mathcal{C}(\mathbb{L}_0,\mathbb{L}_1)}\bigoplus_{f\in\mathcal{X}_{i,N}(\gamma)}\Gamma_{flat}(f^*\mathcal{E}_\gamma).$$

Let $k\geqslant 0,\ i_1,\dots,i_k\geqslant 0$ and $(\lambda,\mu)\in\mathcal{G}$ such that

$$i_1+\cdots+i_k+||(\lambda,\mu)||+k-1\leqslant N.$$

Let $(\overrightarrow{\mathbb{L}},\vec{\gamma})$ be a Lagrangian label of length $k+1$. For any $\vec{f}=(f_1,\dots,f_s),\ \vec{s}=(s_{f_1},\dots,s_{f_k})$ where $f_s\in\mathcal{X}_{i_s,N}(\gamma_s),\ s_{f_s}\in\Gamma_{flat}(f_s^*\mathcal{E}_{\gamma_s}),\ s=1,\dots,k$. Define

$$m^{\lambda,\mu}_{k,geom}(s_{f_k},\dots,s_{f_1}):=\sum_{\substack{\beta\in\pi_2(M,\cup_{s=0}^k\iota(L_s))\\(\int_\beta\omega,\mu(\beta))=(\lambda,\mu)}}VC\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}},\vec{\gamma},\beta,J;\vec{f}),\vec{s}\right)\tag{2.6}$$

if $(k,\lambda,\mu)\neq(1,0,0)$ and $(-1)^m\partial(s_{f_1})$ if $(k,\lambda,\mu)=(1,0,0)$ where ∂ is the boundary operator (2.1). This extends to a graded $\mathbb{C}$-multilinear map

$$m^{\lambda,\mu}_{k,geom}:\mathcal{A}^\bullet_{i_k,N}(\mathbb{L}_{k-1},\mathbb{L}_k)\otimes\cdots\otimes\mathcal{A}^\bullet_{i_1,N}(\mathbb{L}_0,\mathbb{L}_1)\to\mathcal{A}^\bullet_{*,N}(\mathbb{L}_0,\mathbb{L}_k)\tag{2.7}$$

of degree $2-k-\mu$, where $*=i_1+\cdots+i_k+||(\lambda,\mu)||+k-1$.

Now replace N in the above discussion by $N(N+2)$. The multilinear maps $m^{\lambda,\mu}_{k,geom}$ allow us to define an $A_{N,0}$ category $(\mathcal{A}^J_N,m_N)$ by following the homological perturbation procedure in [2] which we now briefly describe.

Let $(\mathbb{L}_0,\mathbb{L}_1)\in Ob^2_{\mathcal{S}}$. Observe that the inclusion

$$(\mathcal{A}^\bullet_{N,N(N+2)}(\mathbb{L}_0,\mathbb{L}_1),m^{0,0}_{1,geom})\xrightarrow[\iota]{}(\mathcal{A}^\bullet_{N(N+2),N(N+2)}(\mathbb{L}_0,\mathbb{L}_1),m^{0,0}_{1,geom})$$

is a quasi-isomorphism. It follows that we can choose a pair (H,P) of $\mathbb{C}$-linear maps

$$(\mathcal{A}^{\bullet}_{N,N(N+2)}(\mathbb{L}_0,\mathbb{L}_1), m^{0,0}_{1,geom}) \underset{\iota}{\overset{P}{\rightleftarrows}} (\mathcal{A}^{\bullet}_{N(N+2),N(N+2)}(\mathbb{L}_0,\mathbb{L}_1), m^{0,0}_{1,geom}) \circlearrowleft H$$

of degree -1 and 0 respectively such that $\iota \circ P - \mathrm{id} = m^{0,0}_{1,geom} \circ H + H \circ m^{0,0}_{1,geom}$ and $P \circ \iota = \mathrm{id}$.

After fixing a choice of (H,P) for each pair $(\mathbb{L}_0,\mathbb{L}_1) \in Ob^2_{\mathcal{S}}$, we apply the "summing over planar trees" procedure. The outcome will be an $A_{N,0}$ category $(\mathcal{A}^J_N, m^J_N)$ whose set of objects is $Ob_{\mathcal{S}}$, whose morphism spaces are $\mathcal{A}^{\bullet}_{N,N(N+2)}(\mathbb{L}_0,\mathbb{L}_1)$, and whose $A_{N,0}$ structure $m^J_N = (m^{\lambda,\mu}_k)$ coincides with $(m^{\lambda,\mu}_{k,geom})$ over all tensor products

$$\mathcal{A}^{\bullet}_{i_k,N(N+2)}(\mathbb{L}_{k-1},\mathbb{L}_k) \otimes \cdots \otimes \mathcal{A}^{\bullet}_{i_1,N(N+2)}(\mathbb{L}_0,\mathbb{L}_1)$$

with $i_1 + \cdots + i_k + ||(\lambda,\mu)|| + k - 1 \leqslant N$. See [2] for more details.

The next step is to apply the generalization of Theorem 2.15 to three cases:

	$\mathcal{T}$	$\mathcal{J} = \{J_\tau\}_{\tau\in\mathcal{T}}$	Input of the theorem
1	$[0,1]$	$J_t \equiv J$	output of Theorem 2.15 (N replaced by $N(N+2)$) at $\{0\}$ and output of Theorem 2.15 (N replaced by $(N+1)(N+3)$) at $\{1\}$.
2	$[0,1]$	J_t = any smooth path connecting given J_0 and J_1	output of Theorem 2.15 (N replaced by $N(N+2)$, J by J_0) at $\{0\}$ and output of Theorem 2.15 (N replaced by $N(N+2)$, J by J_1) at $\{1\}$.
3	$[0,1]\times[0,1]$	$J_{s,t} = J_t$ where J_t is given in case 2	output of above cases (details omitted)

TABLE 1. Generalization of Theorem 2.15 applied to various cases.

What we obtain will be

(1) an $A_{N,0}$ quasi-isomorphism $F^J_{N,N+1} : (\mathcal{A}^J_N, m^J_N) \to (\mathcal{A}^J_{N+1}, m^J_{N+1})$;
(2) an $A_{N,0}$ quasi-isomorphism $F^{J_0\to J_1}_N : (\mathcal{A}^{J_0}_N, m^{J_0}_N) \to (\mathcal{A}^{J_1}_N, m^{J_1}_N)$;
(3) the commutativity (up to $A_{N,0}$ homotopy) of the diagram

$$\begin{array}{ccc} (\mathcal{A}^{J_0}_N, m^{J_0}_N) & \xrightarrow{F^{J_0}_{N,N+1}} & (\mathcal{A}^{J_0}_{N+1}, m^{J_0}_{N+1}) \\ {\scriptstyle F^{J_0\to J_1}_N}\downarrow & & \downarrow{\scriptstyle F^{J_0\to J_1}_{N+1}} \\ (\mathcal{A}^{J_1}_N, m^{J_1}_N) & \xrightarrow[F^{J_1}_{N,N+1}]{} & (\mathcal{A}^{J_1}_{N+1}, m^{J_1}_{N+1}) \end{array} . \quad (2.8)$$

Remark 2.17. The constriction of these functors is actually not completely geometric: Theorem A.11 has already been used to "invert" quasi-isomorphisms. See Lemma 3.8 in Section 3.3 for more details.

Now we come to the algebraic inputs which allow us to construct an A_∞ structure m^J on $\mathcal{A}^J_0$ using the $A_{N,0}$ structures m^J_N on $\mathcal{A}^J_N$ and the $A_{N,0}$ functors $F^J_{N,N+1}$. Using the $A_{N,0}$ functors $F^{J_0\to J_1}_N$ and the commutative diagram (2.8), we can show that the A_∞ categories thus constructed using J_0 and J_1 are A_∞ quasi-isomorphic.

By induction and applying Theorem A.12(1) at each inductive step, we obtain $A_{N,0}$ structures $m^J_{0,N}$ on $\mathcal{A}^J_0$ and $A_{N,0}$ quasi-isomorphisms $F^J_{0,N} : (\mathcal{A}^J_0, m^J_{0,N}) \to (\mathcal{A}^J_N, m^J_N)$ such that

(1) $m^J_{0,0} = m^J_0$;
(2) $m^J_{0,N+1}$ extends $m^J_{0,N}$;
(3) $F^J_{0,0}$ is the identity functor; and
(4) $F^J_{0,N+1}$ extends $F^J_{N,N+1} \circ F^J_{0,N}$.

By induction and applying Theorem A.12(2) at each inductive step, we obtain $A_{N,0}$ quasi-isomorphisms $F^{J_0\to J_1}_{0,N} : (\mathcal{A}^{J_0}_0, m^{J_0}_{0,N}) \to (\mathcal{A}^{J_1}_0, m^{J_1}_{0,N})$ such that

(1) $F^{J_0\to J_1}_{0,0} = F^{J_0\to J_1}_0$;
(2) $F^{J_0\to J_1}_{0,N+1}$ extends $F^{J_0\to J_1}_{0,N}$; and
(3) the diagram

$$\begin{array}{ccc} (\mathcal{A}^{J_0}_0, m^{J_0}_{0,N}) & \xrightarrow{F^{J_0}_{0,N}} & (\mathcal{A}^{J_0}_N, m^{J_0}_N) \\ {\scriptstyle F^{J_0\to J_1}_{0,N}}\downarrow & & \downarrow{\scriptstyle F^{J_0\to J_1}_N} \\ (\mathcal{A}^{J_1}_0, m^{J_1}_{0,N}) & \xrightarrow[F^{J_1}_{0,N}]{} & (\mathcal{A}^{J_1}_N, m^{J_1}_N) \end{array}$$

is commutative up to $A_{N,0}$ homotopy.

It follows that the sequence $\{m^J_{0,N}\}_{N\geqslant 0}$ induces an A_∞ structure m^J on $\mathcal{A}^J_0$ and that for any two compatible almost complex structures J_0, J_1, the sequence $\{F^{J_0\to J_1}_{0,N}\}_{N\geqslant 0}$ induces an A_∞ quasi-isomorphism $F^{J_0\to J_1} : (\mathcal{A}^{J_0}_0, m^{J_0}) \to (\mathcal{A}^{J_1}_0, m^{J_1})$. By similar arguments, it can also be shown that $F^{J_0\to J_1}$ is independent of any choices made throughout the construction up to A_∞ homotopy, and for any three compatible almost complex structures J_0, J_1, J_2, $F^{J_1\to J_2} \circ F^{J_0\to J_1}$ is A_∞ homotopic to $F^{J_0\to J_2}$. See [2] for more details.

Definition 2.18. Choose any compatible almost complex structure J. Define

$$Fuk(M,\omega) := (\mathcal{A}^J_0, m^J)$$

which, as we have just seen, is an A_∞ category well-defined up to a unique A_∞ homotopy class of A_∞ quasi-isomorphisms.

3. Main construction

In this section, we prove Theorem 1.1. Let $(M,\omega), \mathcal{S}, V$ be given as in Section 2.1. Assume $c_1(M)$ is divisible by a positive integer n.

Here is an outline of the construction of the twisted A_∞ functor Φ. In Section 3.1, we construct the map $\Phi_{ob} : Ob_{\mathcal{S}} \to Ob_{\mathcal{S}}$. In Section 3.2, we construct $\Phi^{0,0}_1$. Then we put $\Phi^{\lambda,\mu}_k = 0$ for $(k,\lambda,\mu) \neq (1,0,0)$. We show that this gives the desired twisted A_∞ functor.

To achieve this, recall that in the construction of $Fuk(M,\omega)$, we have made a number of choices including the compatible almost complex structure J, the outcome of Theorem 2.15, the pair (H,P) for the homological perturbation, the $A_{N,0}$ homotopy inverses which are used to construct $F^J_{N,N+1}$

(Remark 2.17) and the outcome of Theorem A.12(1). In Section 3.3, we show that for any set of these choices for the A_∞ structure on the *source* of Φ, there exists another set of choices for the A_∞ structure on the *target* of Φ such that Φ becomes a twisted A_∞ functor. Furthermore, the $(2n)$-th power

$$\Phi^{\circ 2n} := \Phi \circ \cdots \circ \Phi \quad (2n \text{ times})$$

of Φ is equal to the identity functor.

Remark 3.1. The sets of choices for different factors in the expression $\Phi^{\circ 2n} = \Phi \circ \cdots \circ \Phi$ should be compatible. For example, the one for the source of the second factor should equal the one for the target of the first factor. Moreover, the set of choices for the target of the last factor turns out to be equal to the one for the source of the first factor so that it makes sense to talk about the identity functor.

Remark 3.2. We will see very shortly that Φ_{ob} is bijective and $\Phi_1^{0,0}$ is a chain isomorphism, and hence Φ can be made into a twisted A_∞ functor artificially by pushing forward the A_∞ structure on the source of Φ to an A_∞ structure on the target. The point is to show that the latter can be realized as the outcome of the construction in Section 2.4 by taking a suitable set of choices described above, and this is the purpose of Section 3.3.

3.1. Φ **on objects.** Fix a primitive $(2n)$-th root of unity ζ. Recall we have imposed the condition that $c_1(M)$ is divisible by n. By Lemma B.1, the Lagrangian Grassmannian bundle $\mathcal{L}_M := LG(TM, \omega)$ on M admits a fiberwise $\mathbb{Z}_{2n}$-cover $\mathcal{L}'_M \to \mathcal{L}_M$ with deck transformation group isomorphic to $\mathbb{Z}_{2n}$.

Let $L \in \mathcal{S}$. The tangent spaces of points of L define a section θ_L of $\mathcal{L}_M|_L$ by

$$\theta_L(x) := T_x L \in (\mathcal{L}_M)_x,\ x \in L. \tag{3.1}$$

The inverse image of the subspace $\theta_L(L) \subseteq \mathcal{L}_M|_L$ under the fiberwise covering map $\mathcal{L}'_M|_L \to \mathcal{L}_M|_L$ is then a $\mathbb{Z}_{2n}$-local system on L, which we denote by $\mathcal{E}_L$. We may regard $\mathcal{E}_L$ as a $\mathbb{C}^\times$-local system via the inclusion $\mathbb{Z}_{2n} \hookrightarrow \mathbb{C}^\times : 1 \ (\mathrm{mod} 2n) \mapsto \zeta$.

Definition 3.3. Define $\Phi_{ob} : Ob_{\mathcal{S}} \to Ob_{\mathcal{S}}$ by

$$\Phi_{ob}(\mathbb{L}) := (L, \sigma, \mathcal{E} \otimes \mathcal{E}_L)$$

for any $\mathbb{L} = (L, \sigma, \mathcal{E}) \in Ob_{\mathcal{S}}$.

This map will be used to define Φ_N and Φ in the next subsection.

3.2. Φ **on morphisms.** Let $N \geqslant 0$ be an integer. Let $(\mathbb{L}_0, \mathbb{L}_1) \in Ob^2_{\mathcal{S}}$. Recall in Section 2.4, we defined

$$\mathcal{A}^\bullet_{i,N}(\mathbb{L}_0, \mathbb{L}_1) := \bigoplus_{\gamma \in \mathcal{C}(\mathbb{L}_0, \mathbb{L}_1)} C_{\dim L(\gamma) - \bullet}(\mathcal{X}_{i,N}(\gamma); \mathcal{E}_\gamma)$$

where $\mathcal{X}_{i,N}(\gamma)$ is the outcome of Theorem 2.15 applied to a compatible almost complex structure J on (M, ω).

For any $i = 0, \ldots, N$, we want to define a $\mathbb{C}$-linear map

$$(\Phi_{i,N})_1^{0,0} : \mathcal{A}^\bullet_{i,N}(\mathbb{L}_0, \mathbb{L}_1) \to \mathcal{A}^\bullet_{i,N}(\Phi_{ob}(\mathbb{L}_0), \Phi_{ob}(\mathbb{L}_1))$$

for any $\mathbb{L}_i = (L_i, \sigma_i, \mathcal{E}_i) \in Ob_{\mathcal{S}},\ i = 0, 1$.

First observe that if γ' is the unique element of $\mathcal{C}(\Phi_{ob}(\mathbb{L}_0), \Phi_{ob}(\mathbb{L}_1))$ equal to γ as elements of $C(L_0, L_1)$ (see Notations 2.5(3)), then we have

$$\mathcal{E}_{\gamma'} \simeq \mathcal{E}_\gamma \otimes Hom(\mathcal{E}_{L_0}, \mathcal{E}_{L_1})|_{L(\gamma)}. \tag{3.2}$$

Thus, it suffices to specify, for each $\gamma \in \mathcal{C}(\mathbb{L}_0, \mathbb{L}_1)$, a flat section of $Hom(\mathcal{E}_{L_0}, \mathcal{E}_{L_1})|_{L(\gamma)}$, or equivalently an isomorphism of $\mathbb{Z}_{2n}$-local systems $\mathcal{E}_{L_0}|_{L(\gamma)} \to \mathcal{E}_{L_1}|_{L(\gamma)}$.

This is done as follows: consider the "canonical short path" θ_t^γ, defined up to homotopy, which is constructed in Appendix B. The lift of this path with respect to the fiberwise $\mathbb{Z}_{2n}$-covering map $\mathcal{L}'_M|_{L(\gamma)} \to \mathcal{L}_M|_{L(\gamma)}$ then gives the desired isomorphism $\mathcal{E}_{L_0}|_{L(\gamma)} \to \mathcal{E}_{L_1}|_{L(\gamma)}$ which we denote by ϕ_γ. By an abuse of notation, we also denote its equivalence form which is a section $\mathbb{1} \to Hom(\mathcal{E}_{L_0}, \mathcal{E}_{L_1})|_{L(\gamma)}$ by the same symbol.

Definition 3.4.
(1) Define $(\Phi_{i,N})_1^{0,0} : \mathcal{A}^\bullet_{i,N}(\mathbb{L}_0, \mathbb{L}_1) \to \mathcal{A}^\bullet_{i,N}(\Phi_{ob}(\mathbb{L}_0), (\Phi_{ob}(\mathbb{L}_1))$ by

$$(\Phi_{i,N})_1^{0,0}(s_f) := \zeta^{r-\dim L(\gamma)}(\mathrm{id} \otimes \phi_\gamma)(s_f) \tag{3.3}$$

for any $f : \Delta^r \to L(\gamma)$ which lies in $\mathcal{X}_{i,N}(\gamma)$ and $s_f \in \Gamma_{flat}(f^*\mathcal{E}_\gamma)$. Here, $\mathrm{id} \otimes \phi_\gamma$ is the morphism of $\mathbb{C}^\times$-local systems

$$\mathcal{E}_\gamma \to \mathcal{E}_{\gamma'} \simeq \mathcal{E}_\gamma \otimes Hom(\mathcal{E}_{L_0}, \mathcal{E}_{L_1})|_{L(\gamma)}$$

induced by $\phi_\gamma : \mathbb{1} \to Hom(\mathcal{E}_{L_0}, \mathcal{E}_{L_1})|_{L(\gamma)}$ which we have just constructed. Since ϕ_γ is an isomorphism, $\mathrm{id} \otimes \phi_\gamma$ is also an isomorphism.
(2) Define $(\Phi_N)_1^{0,0} := (\Phi_{N,N(N+2)})_1^{0,0}$ and $(\Phi_N)_k^{\lambda,\mu} = 0$ for $(k, \lambda, \mu) \neq (1, 0, 0)$.
(3) Define $\Phi := \Phi_0$.

Then Φ_N and Φ act on the objects and the morphism spaces of $\mathcal{A}^J_N$ and $Fuk(M, \omega)$ respectively. We will show in the next subsection that they are twisted $A_{N,0}$ functor and twisted A_∞ functor respectively.

3.3. Φ **is an** A_∞ **functor of order** $2n$**.** Fix a set of choices described at the beginning of this section. Since the compatible almost complex structure J will be fixed throughout the discussion, we will drop it from all the notations such as $\mathcal{A}^J$, m^J, etc.

Proposition 3.5. *Let $k \geqslant 0,\ i_1, \ldots, i_k \geqslant 0$ and $(\lambda, \mu) \in \mathcal{G}$ such that*

$$* := i_1 + \cdots + i_k + ||(\lambda, \mu)|| + k - 1 \leqslant N.$$

Let $(\overrightarrow{\mathbb{L}}, \vec{\gamma})$ be a Lagrangian label of length $k+1$. Recall the $\mathbb{C}$-multilinear map $m^{\lambda,\mu}_{k,geom}$ (2.7). *We have*

$$m^{\lambda,\mu}_{k,geom} \circ \left((\Phi_{i_k,N})_1^{0,0} \otimes \cdots \otimes (\Phi_{i_1,N})_1^{0,0}\right) = \zeta^{2-k}(\Phi_{*,N})_1^{0,0} \circ m^{\lambda,\mu}_{k,geom}. \tag{3.4}$$

Proof. The case $(k, \lambda, \mu) = (1, 0, 0)$ follows from the functoriality of singular chain complexes with respect to change of local coefficients. Notice that the factor $\zeta^{2-1} = \zeta^1$ in (3.4) corresponds to the ratio of the factors $\zeta^{r-\dim L(\gamma)}$ and $\zeta^{(r-1)-\dim L(\gamma)}$ both coming from (3.3).

Assume now $(k, \lambda, \mu) \neq (1, 0, 0)$. Recall from (2.6) that

$$m^{\lambda,\mu}_{k,geom}(s_{f_k}, \ldots, s_{f_1}) := \sum_{\substack{\beta \in \pi_2(M, \cup_{s=0}^k \iota(L_s)) \\ (\int_\beta \omega, \mu(\beta)) = (\lambda, \mu)}} VC\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \vec{s}\right)$$

for any $\vec{f} = (f_1, \ldots, f_s)$, $\vec{s} = (s_{f_1}, \ldots, s_{f_k})$ where $f_s \in \mathcal{X}_{i_s,N}(\gamma_s)$, $s_{f_s} \in \Gamma_{flat}(f_s^*\mathcal{E}_{\gamma_s})$, $s = 1, \ldots, k$.

Let $\vec{s}\,' := ((\mathrm{id} \otimes \phi_{\gamma_1})(s_{f_1}), \ldots, (\mathrm{id} \otimes \phi_{\gamma_k})(s_{f_k}))$. Then

$$\begin{aligned} &m^{\lambda,\mu}_{k,geom}\left((\Phi_{i_k,N})^{0,0}_1(s_{f_k}), \ldots, (\Phi_{i_1,N})^{0,0}_1(s_{f_1})\right) \\ &= \sum_{\substack{\beta \in \pi_2(M, \cup_{s=0}^k \iota(L_s)) \\ (\int_\beta \omega, \mu(\beta)) = (\lambda,\mu)}} \zeta^{\sum_{s=1}^k (r_s - \dim L(\gamma_s))} VC\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \vec{s}\,'\right). \end{aligned}$$

Hence it suffices to show that

$$\begin{aligned} &VC\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \vec{s}\,'\right) \\ &= \zeta^{m - \dim L(\gamma_0) + \mu(\beta)} (\mathrm{id} \otimes \phi_{\tau(\gamma_0)}) \left(VC\left(\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f}), \vec{s}\right)\right), \end{aligned}$$

by the dimension formula (B.2). (Recall $\dim_{\mathbb{R}} M = 2m$.)

This is an immediate consequence of the following lemma. ◻

Lemma 3.6. *Let* $[(\Sigma, \vec{z}, u, \ell, \tilde{u})] \in \overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f})$. *For* $s = 1, \ldots, k+1$, *let* $\psi_{\xi_{s-1} \to \xi_s}$ *be the parallel transport map of* $\mathcal{E}_{L_{s-1}}$ *along the path* $\tilde{u}|_{[\xi_{s-1}, \xi_s]}$. *(Notice that the limits of* $\tilde{u}(\xi)$ *when* ξ *approaches the endpoints of* $[\xi_{s-1}, \xi_s]$ *exist, see Definition 2.10(1).) Then*

$$\psi_{\xi_k \to \xi_0} \circ \phi_{\gamma_k}(\tilde{u}(\xi_k)) \circ \cdots \circ \phi_{\gamma_1}(\tilde{u}(\xi_1)) \circ \psi_{\xi_0 \to \xi_1} = \zeta^{m - dimL(\gamma_0) + \mu(\beta)} \phi_{\tau(\gamma_0)}(\tau(\tilde{u}(\xi_0))) \tag{3.5}$$

Proof. By the formula $\phi_{\gamma_0} \circ \phi_{\tau(\gamma_0)} = \zeta^{-m + \dim L(\gamma_0)} \mathrm{id}_{\mathcal{E}_{L_0}}$, we see that (3.5) is equivalent to

$$\phi_{\gamma_0}(\tilde{u}(\xi_0)) \circ \psi_{\xi_k \to \xi_0} \circ \cdots \circ \phi_{\gamma_1}(\tilde{u}(\xi_1)) \circ \psi_{\xi_0 \to \xi_1} = \zeta^{\mu(\beta)} \mathrm{id}_{\mathcal{E}_{L_0}}. \tag{3.6}$$

Recall we have put $\mathcal{L}_M = LG(TM, \omega)$ and $\mathcal{L}'_M \to \mathcal{L}_M$ is a fiberwise $\mathbb{Z}_{2n}$-cover. Since $u^*\mathcal{L}'_M$ is a trivial bundle on Σ, there exists a fiberwise $\mathbb{Z}$-cover $\mathcal{L}'' \to u^*\mathcal{L}'_M$.

Consider the loop η in $u^*\mathcal{L}_M$ which is the concatenation of the following paths (in the given order):

$$\theta_{L_0}(\tilde{u}|_{[\xi_0,\xi_1]}), \theta_t^{\gamma_1}(\tilde{u}(\xi_1)), \ldots, \theta_{L_k}(\tilde{u}|_{[\xi_k,\xi_0]}), \theta_t^{\gamma_0}(\tilde{u}(\xi_0))$$

where θ_L and θ_t^γ are given in (3.1) and (B.1) respectively. Then the lifts of η in $u^*\mathcal{L}'_M$ and in $\mathcal{L}''$ are paths whose end points are related by some group elements $a \in \mathbb{Z}_{2n}$ and $b \in \mathbb{Z}$ respectively. It is easy to see that $\zeta^a = \zeta^b$. By definition, $\zeta^a \mathrm{id}_{\mathcal{E}_{L_0}}$ is equal to the left-hand side of (3.6), and by Definition B.3, $b = \mu(\beta)$. This completes the proof. ◻

According to Section 2.4, the $A_{N,0}$ structure m_N on $\mathcal{A}_N$ is defined by choosing, for each pair $(\mathbb{L}_0, \mathbb{L}_1) \in Ob^2_{\mathcal{S}}$, a homotopy operator H and a projection P

$$(\mathcal{A}^\bullet_{N,N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_{1,geom}) \underset{\iota}{\overset{P}{\rightleftarrows}} (\mathcal{A}^\bullet_{N(N+2),N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_{1,geom}) \circlearrowleft H$$

such that $\iota \circ P - \mathrm{id} = m^{0,0}_{1,geom} \circ H + H \circ m^{0,0}_{1,geom}$ and $P \circ \iota = \mathrm{id}$, and then applying the homological perturbation. (Recall $\mathcal{A}_N(\mathbb{L}_0, \mathbb{L}_1) := \mathcal{A}^\bullet_{N,N(N+2)}(\mathbb{L}_0, \mathbb{L}_1)$.)

Lemma 3.7. *There exists a choice (H', P') which defines an $A_{N,0}$ structure m'_N on $\mathcal{A}_N$ such that*

$$\Phi_N : (\mathcal{A}_N, m_N) \to (\mathcal{A}_N, (m'_N)_{(\zeta)})$$

is an $A_{N,0}$ functor, where $(m'_N)_{(\zeta)}$ is defined in Definition A.8.

Proof. We have the following diagram

$$\begin{array}{ccc}
(\mathcal{A}^{\bullet}_{N,N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_{1,geom}) & \underset{\iota}{\overset{P}{\rightleftarrows}} & (\mathcal{A}^{\bullet}_{N(N+2),N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_{1,geom}) \circlearrowleft H \\
\Big\downarrow {\scriptstyle (\Phi_{N,N(N+2)})^{0,0}_1} & & \Big\downarrow {\scriptstyle (\Phi_{N(N+2),N(N+2)})^{0,0}_1} \\
(\mathcal{A}^{\bullet}_{N,N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), \zeta^{-1} m^{0,0}_{1,geom}) & \xrightarrow[\iota]{} & (\mathcal{A}^{\bullet}_{N(N+2),N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), \zeta^{-1} m^{0,0}_{1,geom})
\end{array} \tag{3.7}$$

such that $(\Phi_{N(N+2),N(N+2)})^{0,0}_1 \circ \iota = \iota \circ (\Phi_{N,N(N+2)})^{0,0}_1$.

Observe that $(\Phi_{N,N(N+2)})^{0,0}_1$ and $(\Phi_{N(N+2),N(N+2)})^{0,0}_1$ are chain isomorphisms. It follows that there is a unique choice (H', P') such that the diagram

$$\begin{array}{ccc}
(\mathcal{A}^{\bullet}_{N,N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_{1,geom}) & \underset{\iota}{\overset{P}{\rightleftarrows}} & (\mathcal{A}^{\bullet}_{N(N+2),N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_{1,geom}) \circlearrowleft H \\
\Big\downarrow {\scriptstyle (\Phi_{N,N(N+2)})^{0,0}_1} & & \Big\downarrow {\scriptstyle (\Phi_{N(N+2),N(N+2)})^{0,0}_1} \\
(\mathcal{A}^{\bullet}_{N,N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), \zeta^{-1} m^{0,0}_{1,geom}) & \underset{\iota}{\overset{P'}{\rightleftarrows}} & (\mathcal{A}^{\bullet}_{N(N+2),N(N+2)}(\mathbb{L}_0, \mathbb{L}_1), \zeta^{-1} m^{0,0}_{1,geom}) \circlearrowleft H'
\end{array} \tag{3.8}$$

is commutative. This, together with Proposition 3.5, shows that Φ_N is indeed an $A_{N,0}$ functor provided the $A_{N,0}$ structures on the source and the target are defined using choices (H, P) and (H', P') respectively. $\square$

Next, recall the $A_{N,0}$ functor $F_{N,N+1} : (\mathcal{A}_N, m_N) \to (\mathcal{A}_{N+1}, m_{N+1})$ in Section 2.4.

Lemma 3.8. *There exists a choice $F'_{N,N+1} : (\mathcal{A}_N, m'_N) \to (\mathcal{A}_{N+1}, m'_{N+1})$ such that the diagram*

$$\begin{array}{ccc}
(\mathcal{A}_N, m_N) & \xrightarrow{F_{N,N+1}} & (\mathcal{A}_{N+1}, m_{N+1}) \\
{\scriptstyle \Phi_N} \Big\downarrow & & \Big\downarrow {\scriptstyle \Phi_{N+1}} \\
\big(\mathcal{A}_N, (m'_N)_{(\zeta)}\big) & \xrightarrow[(F'_{N,N+1})_{(\zeta)}]{} & \big(\mathcal{A}_{N+1}, (m'_{N+1})_{(\zeta)}\big)
\end{array} \tag{3.9}$$

is commutative.

Proof. Before we prove this, let us say a few more words on how $F^J_{N,N+1}$ is constructed. According to [2], it is done by introducing a third $A_{N,0}$ category $\left(\mathcal{A}^{(0,1)}_{N,N+1}, m^{(0,1)}_{N,N+1}\right)$ which is constructed using the outcome of Theorem 2.15 applied to the first case of Table 1 in Section 2.4 and a pair (H, P) compatible with the ones chosen for the $A_{N,0}$ structure m_N on $\mathcal{A}_N$ and for the $A_{N+1,0}$ structure m_{N+1} on $\mathcal{A}_{N+1}$.

There are $A_{N,0}$ quasi-isomorphisms $F^{(0,1)\to i}_{N,N+1} : \mathcal{A}^{(0,1)}_{N,N+1} \to \mathcal{A}_{N+i},\ i = 0, 1$, and $F_{N,N+1}$ is defined to be $F^{(0,1)\to 1}_{N,N+1} \circ \widetilde{F}^{(0,1)\to 0}_{N,N+1}$, where $\widetilde{F}^{(0,1)\to 0}_{N,N+1}$ is an $A_{N,0}$ homotopy inverse of $F^{(0,1)\to 0}_{N,N+1}$ which exists by Theorem A.11.

Construct an $A_{N,0}$ functor $\Phi^{(0,1)}_{N,N+1} : \left(\mathcal{A}^{(0,1)}_{N,N+1}, m^{(0,1)}_{N,N+1}\right) \to \left(\mathcal{A}^{(0,1)}_{N,N+1}, (m'^{(0,1)}_{N,N+1})_{(\zeta)}\right)$ in the same way as how Φ_N is constructed. Since $\left(\Phi^{(0,1)}_{N,N+1}\right)^{0,0}_1$ is also a chain isomorphism, there are unique choices $F'^{(0,1)\to 0}_{N,N+1}$ and $F'^{(0,1)\to 1}_{N,N+1}$ such that the following diagram is commutative:

$$\begin{array}{ccccc} (\mathcal{A}_N, m_N) & \xleftarrow{F^{(0,1)\to 0}_{N,N+1}} & \left(\mathcal{A}^{(0,1)}_{N,N+1}, m^{(0,1)}_N\right) & \xrightarrow{F^{(0,1)\to 1}_{N,N+1}} & (\mathcal{A}_{N+1}, m_{N+1}) \\ \Big\downarrow{\scriptstyle \Phi_N} & & \Big\downarrow{\scriptstyle \Phi^{(0,1)}_{N,N+1}} & & \Big\downarrow{\scriptstyle \Phi_{N+1}} \\ \left(\mathcal{A}_N, (m'_N)_{(\zeta)}\right) & \xleftarrow[\left(F'^{(0,1)\to 0}_{N,N+1}\right)_{(\zeta)}]{} & \left(\mathcal{A}^{(0,1)}_{N,N+1}, \left(m'^{(0,1)}_{N,N+1}\right)_{(\zeta)}\right) & \xrightarrow[\left(F'^{(0,1)\to 1}_{N,N+1}\right)_{(\zeta)}]{} & \left(\mathcal{A}_{N+1}, (m'_{N+1})_{(\zeta)}\right) \end{array} \tag{3.10}$$

Moreover, the $A_{N,0}$ homotopy inverse $\widetilde{F}'^{(0,1)\to 0}_{N,N+1}$ of $F'^{(0,1)\to 0}_{N,N+1}$ can be chosen such that $\Phi^{(0,1)}_{N,N+1} \circ \widetilde{F}^{(0,1)\to 0}_{N,N+1} = \left(\widetilde{F}'^{(0,1)\to 0}_{N,N+1}\right)_{(\zeta)} \circ \Phi_N$. Define $F'_{N,N+1} := F'^{(0,1)\to 1}_{N,N+1} \circ \widetilde{F}'^{(0,1)\to 0}_{N,N+1}$. Then the commutativity of (3.9) follows. □

Now recall the $A_{N,0}$ structure $m_{0,N}$ on $\mathcal{A}_0$ and the $A_{N,0}$ quasi-isomorphism $F_{0,N} : (\mathcal{A}_0, m_{0,N}) \to (\mathcal{A}_N, m_N)$ in Section 2.4. Argued in a similar way and by induction, there exist choices $m'_{0,N}$ and $F'_{0,N}$ such that $\Phi : (\mathcal{A}_0, m_{0,N}) \to (\mathcal{A}_0, (m'_{0,N})_{(\zeta)})$ is an $A_{N,0}$ functor and the diagram

$$\begin{array}{ccc} (\mathcal{A}_0, m_{0,N}) & \xrightarrow{F_{0,N}} & (\mathcal{A}_N, m_N) \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi_N} \\ (\mathcal{A}_0, (m'_{0,N})_{(\zeta)}) & \xrightarrow[(F'_{0,N})_{(\zeta)}]{} & (\mathcal{A}_N, (m'_N)_{(\zeta)}) \end{array} \tag{3.11}$$

is commutative. It follows that the sequence $\{m'_{0,N}\}_{N\geqslant 0}$ induces an A_∞ structure m' on $\mathcal{A}_0$ for which $\Phi : (\mathcal{A}_0, m) \to (\mathcal{A}_0, (m')_{(\zeta)})$ is an A_∞ functor.

Finally, observe that the $\mathbb{Z}_{2n}$-local system $\mathcal{E}_L$ defined in Section 3.1 satisfies $\mathcal{E}_L^{\otimes 2n} = \mathbb{1}$ (in fact $\mathcal{E}_L^{\otimes n} = \mathbb{1}$ as L is oriented), and that the isomorphism $\phi_\gamma : \mathcal{E}_{L_0}|_{L(\gamma)} \to \mathcal{E}_{L_1}|_{L(\gamma)}$ defined in Section 3.2 satisfies

$$\phi_\gamma^{\otimes 2n} = \mathrm{id} \in Hom(\mathcal{E}_{L_0}^{\otimes 2n}|_{L(\gamma)}, \mathcal{E}_{L_1}^{\otimes 2n}|_{L(\gamma)}) \simeq Hom(\mathbb{1}, \mathbb{1}).$$

It follows that for any $N \geqslant 0$, $\Phi_N^{\circ 2n}$ is equal to the identity functor $\mathrm{id}_{\mathcal{A}_N}$.

By the commutativity of diagrams (3.9) and (3.11), we conclude that all the choices for the target of $\Phi^{\circ 2n}$ coincide with the ones for the source of $\Phi^{\circ 2n}$. In other words, the A_∞ structure on the source and the target coincide. This can also be seen by using the fact that $\Phi^{\circ 2n}$ is an A_∞ functor which acts on the objects and the morphism spaces in the same way as the identity functor $\mathrm{id}_{Fuk(M,\omega)}$. Hence $\Phi^{\circ 2n} = \mathrm{id}_{Fuk(M,\omega)}$. The proof of Theorem 1.1 is complete.

APPENDIX A. $A_{N,K}$ CATEGORIES, FUNCTORS AND HOMOTOPIES

Definition A.1. Let T and e be two formal variables. Define the universal Novikov ring over $\mathbb{C}$

$$\Lambda_0 := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \,\middle|\, a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geqslant 0}, \mu_i \in \mathbb{Z}, \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

We grade Λ_0 by declaring T and e to have degree 0 and 1 respectively.

Definition A.2. Define a total order $\prec$ on $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ by

$$(N, K) \prec (\overline{N}, \overline{K}) \iff (N + K < \overline{N} + \overline{K}) \text{ or } (N + K = \overline{N} + \overline{K} \text{ and } N < \overline{N}).$$

Definition A.3. Let $\mathcal{A}$ be given the following data:

- a set $Ob(\mathcal{A})$, called the *set of objects*; and
- an assignment of a $\mathbb{Z}$-graded $\mathbb{C}$-vector space $\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1)$, called the *morphism space*, to every pair $(\mathbb{L}_0, \mathbb{L}_1) \in Ob(\mathcal{A})^2$.

Let $\mathcal{A}'$ and $\mathcal{A}''$ be given the similar data. Let $F_{ob} : Ob(\mathcal{A}) \to Ob(\mathcal{A}')$ be a map. Let $\mathcal{G}$ be given as in Definition 2.14 and $(N, K) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$.

(1) Define the $A_{N,K}$ version of *Hochschild cochain complex*

$$CC^{\bullet}_{N,K}(\mathcal{A}, \mathcal{A}') = \bigoplus_{r\in\mathbb{Z}} CC^{r}_{N,K}(\mathcal{A}, \mathcal{A}')$$

where for each $r \in \mathbb{Z}$, $CC^{r}_{N,K}(\mathcal{A}, \mathcal{A}')$ is the Λ_0-module consisting of formal sums

$$F = \sum_{(\lambda,\mu)\in\mathcal{G}} F^{\lambda,\mu} T^{\lambda} e^{\mu}$$

with $F^{\lambda,\mu} = (F^{\lambda,\mu}_k)$ where

$$(F^{\lambda,\mu}_k) \in \prod_{\substack{k\geqslant 0 \\ (||(\lambda,\mu)||-1,k)\preceq(N,K) \\ \mathbb{L}_0,\ldots,\mathbb{L}_k\in Ob(\mathcal{A})}} Hom^{r-k-\mu}\left(\mathcal{A}(\mathbb{L}_0, \ldots, \mathbb{L}_k), \mathcal{A}(F_{ob}(\mathbb{L}_0), F_{ob}(\mathbb{L}_k))\right)$$

where $\mathcal{A}(\mathbb{L}_0, \ldots, \mathbb{L}_k) = \mathcal{A}(\mathbb{L}_{k-1}, \mathbb{L}_k) \otimes \cdots \otimes \mathcal{A}(\mathbb{L}_0, \mathbb{L}_1)$. (Notice that every element F is a finite sum.)

(2) Let $F_i \in CC^{r_i}_{N,K}(\mathcal{A}, \mathcal{A}')$, $i = 1, 2$. Define the $A_{N,K}$ version of *Gerstenhaber product*

$$F_1 \odot F_2 \in CC^{r_1+r_2-1}_{N,K}(\mathcal{A}, \mathcal{A}')$$

by

$$\begin{aligned}(F_1 \odot F_2)^{\lambda,\mu}_k(a_k, \ldots, a_1) \\ := \sum_{\substack{(\lambda_i,\mu_i)\in\mathcal{G},\ i=1,2 \\ (\lambda,\mu)=(\lambda_1,\mu_1)+(\lambda_2,\mu_2) \\ k-1\geqslant j\geqslant 0,\ k_2\geqslant 0 \\ k+1=k_1+k_2}} & (-1)^{j+\sum_{i=1}^{j}|a_{k-i+1}|}(F_1)^{\lambda_1,\mu_1}_{k_1}\Big(a_k, \ldots, a_{k-j+1}, (F_2)^{\lambda_2,\mu_2}_{k_2}(a_{k-j}, \\ & \qquad \ldots, a_{k-j-k_2+1}), a_{k-j-k_2}, \ldots, a_1\Big)\end{aligned}$$

for any $(\lambda, \mu) \in \mathcal{G},\ k \geqslant 0$ with $(||(\lambda, \mu)|| - 1, k) \preceq (N, K),\ \mathbb{L}_0, \ldots, \mathbb{L}_k \in Ob(\mathcal{A})$ and $a_i \in \mathcal{A}^{|a_i|}(\mathbb{L}_{i-1}, \mathbb{L}_i),\ i = 1, \ldots, k$.

(3) Let $F'_{ob}: Ob(\mathcal{A}') \to Ob(\mathcal{A}'')$ be a map which gives rise to $CC^{\bullet}_{N,K}(\mathcal{A}', \mathcal{A}'')$. Let $F_i \in CC^{r_i}_{N,K}(\mathcal{A}, \mathcal{A}'),\ i = 1, \ldots, \ell$ and $F' \in CC^{r'}_{N,K}(\mathcal{A}', \mathcal{A}'')$.
Define $(F'|F_1, \ldots, F_\ell) \in CC^{r'-\ell+\sum_{r=1}^{\ell} r_i}_{N,K}(\mathcal{A}, \mathcal{A}'')$ by

$$\begin{aligned} &(F'|F_1, \ldots, F_\ell)^{\lambda,\mu}_k(a_k, \ldots, a_1) \\ := &\sum_{\substack{(\lambda_i,\mu_i)\in\mathcal{G},\ i=0,\ldots,\ell \\ (\lambda,\mu)=\sum_{i=0}^{\ell}(\lambda_i,\mu_i) \\ 0\leqslant k_\ell\leqslant\cdots\leqslant k_1=k}} (F')^{(\lambda_0,\mu_0)}_{\ell}\Big((F_1)^{(\lambda_1,\mu_1)}_{k_1-k_2}(a_{k_1}, \ldots, a_{k_2+1}), \ldots, (F_\ell)^{(\lambda_\ell,\mu_\ell)}_{k_\ell}(a_{k_\ell}, \ldots, a_1)\Big) \end{aligned}$$

for any $(\lambda, \mu) \in \mathcal{G},\ k \geqslant 0$ with $(||(\lambda, \mu)|| - 1, k) \preceq (N, K),\ \mathbb{L}_0, \ldots, \mathbb{L}_k \in Ob(\mathcal{A})$ and $a_i \in \mathcal{A}^{\bullet}(\mathbb{L}_{i-1}, \mathbb{L}_i),\ i = 1, \ldots, k$.

Definition A.4. A *non-unital curved $\mathcal{G}$-gapped filtered $A_{N,K}$ category* $(\mathcal{A}, m)$ consists of $\mathcal{A}$ given as in Definition A.3 and $m \in CC^2(\mathcal{A}, \mathcal{A})$ (where $F_{ob} = \mathrm{id}$) such that $m^{0,0}_0 = 0$ and

$$m \odot m = 0.$$

For $(N, K) \preceq (\overline{N}, \overline{K})$, we have the projection $CC^{\bullet}_{\overline{N},\overline{K}}(\mathcal{A}, \mathcal{A}) \to CC^{\bullet}_{N,K}(\mathcal{A}, \mathcal{A})$.

Definition A.5. A *non-unital curved $\mathcal{G}$-gapped filtered A_∞ category* $(\mathcal{A}, m)$ consists of $\mathcal{A}$ given as in Definition A.3 and $m = (m_N)_{N\geqslant 0}$ such that for each $N \geqslant 0$, $(\mathcal{A}, m_N)$ is a non-unital curved $\mathcal{G}$-gapped filtered $A_{N,0}$ category and m_{N+1} projects to m_N.

For simplicity, we shall call $(\mathcal{A}, m)$ in Definition A.4 (resp. Definition A.5) an *$A_{N,K}$ category* (resp. *A_∞ category*).

Let (A, m) be an $A_{N,K}$ category. Using $m^{0,0}_0 = 0$ and $(m \odot m)^{0,0}_1 = 0$, we have $m^{0,0}_1 \circ m^{0,0}_1 = 0$. In other words, for any $\mathbb{L}_0, \mathbb{L}_1 \in Ob(\mathcal{A})$, $(\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_1)$ is a cochain complex.

Definition A.6. Let $(\mathcal{A}, m), (\mathcal{A}', m')$ be $A_{N,K}$ categories.

(1) A *$\mathcal{G}$-gapped filtered $A_{N,K}$ functor* $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ consists of a map $F_{ob} : Ob(\mathcal{A}) \to Ob(\mathcal{A}')$ and $F \in CC^1(\mathcal{A}, \mathcal{A}')$ such that $F^{0,0}_0 = 0$ and

$$F \odot m = \sum_{\ell=0}^{\infty}(m'|\underbrace{F, \ldots, F}_{\ell \text{ times}}). \tag{A.1}$$

(2) The *identity functor* $\mathrm{id}_{\mathcal{A}} : (\mathcal{A}, m) \to (\mathcal{A}, m)$ is defined by putting $(\mathrm{id}_{\mathcal{A}})_{ob} = \mathrm{id},\ (\mathrm{id}_{\mathcal{A}})^{0,0}_1 = \mathrm{id}$ and $(\mathrm{id}_{\mathcal{A}})^{\lambda,\mu}_k = 0$ for $(k, \lambda, \mu) \neq (1, 0, 0)$.
(3) A $\mathcal{G}$-gapped filtered $A_{N,K}$ functor $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ is *strict* if $F^{\lambda,\mu}_k = 0$ for all $k \neq 1,\ (\lambda, \mu) \in \mathcal{G}$.

For simplicity, we shall call $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ in Definition A.6(1) an *$A_{N,K}$ functor*. We define an *A_∞ functor* of A_∞ categories in a similar fashion as in Definition A.5.

Let $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ be an $A_{N,K}$ functor of $A_{N,K}$ categories. Using $F^{0,0}_0 = 0$ and equation (A.1) for $(k, \lambda, \mu) = (1, 0, 0)$, we have $F^{0,0}_1 \circ m^{0,0}_1 = (m')^{0,0}_1 \circ F^{0,0}_1$. In other words, for any $\mathbb{L}_0, \mathbb{L}_1 \in Ob(\mathcal{A})$,

$$F^{0,0}_1 : (\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1), m^{0,0}_1) \to (\mathcal{A}'(F_{ob}(\mathbb{L}_0), F_{ob}(\mathbb{L}_1)), (m')^{0,0}_1) \tag{A.2}$$

is a chain map.

Definition A.7. Let $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ be an $A_{N,K}$ functor of $A_{N,K}$ categories. We call F an $A_{N,K}$ *quasi-isomorphism* if for any $\mathbb{L}_0, \mathbb{L}_1 \in Ob(\mathcal{A})$, the chain map (A.2) is a quasi-isomorphism.

Definition A.8. Let $\zeta \in \mathbb{C}$ be a complex number.

(1) Let $(\mathcal{A}, m)$ be an $A_{N,K}$ category (resp. A_∞ category). Write $m = (m_k^{\lambda,\mu})$. Define $m_{(\zeta)} := (\zeta^{k-2} m_k^{\lambda,\mu})$. Then $(\mathcal{A}, m_{(\zeta)})$ is also an $A_{N,K}$ category (resp. A_∞ category).
(2) Let $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ be an $A_{N,K}$ functor (resp. A_∞ functor). Write $F = (F_k^{\lambda,\mu})$. Define $F_{(\zeta)} := (\zeta^{k-1} F_k^{\lambda,\mu})$. Then $F_{(\zeta)} : (\mathcal{A}, m_{(\zeta)}) \to (\mathcal{A}', (m')_{(\zeta)})$ is also an $A_{N,K}$ functor (resp. A_∞ functor).

Definition A.9. Let $(\mathcal{A}, m), (\mathcal{A}', m'), (\mathcal{A}'', m'')$ be $A_{N,K}$ categories. Let $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ and $F' : (\mathcal{A}', m') \to (\mathcal{A}'', m'')$ be $A_{N,K}$ functors. Define the $A_{N,K}$ functor $F' \circ F : (\mathcal{A}, m) \to (\mathcal{A}'', m'')$, called the *composite*, by putting $(F' \circ F)_{ob} := F'_{ob} \circ F_{ob}$ and

$$F' \circ F := \sum_{\ell=0}^{\infty} (F' | \underbrace{F, \ldots, F}_{\ell \text{ times}}).$$

Definition A.10.

(1) Let $F_1, F_2 : (\mathcal{A}, m) \to (\mathcal{A}', m')$ be $A_{N,K}$ functors of $A_{N,K}$ categories such that $(F_1)_{ob} = (F_2)_{ob}$. An $A_{N,K}$ *homotopy from* F_1 *to* F_2 is $H \in CC^0(\mathcal{A}, \mathcal{A}')$ such that $H_0^{0,0} = 0$ and
$$F_1 - F_2 = H \odot m + \sum_{\ell_1, \ell_2 = 0}^{\infty} (m' | \underbrace{F_1 \ldots, F_1}_{\ell_1 \text{ times}}, H, \underbrace{F_2 \ldots, F_2}_{\ell_2 \text{ times}})$$
In this case, we say that F_1 is $A_{N,K}$ *homotopic to* F_2.
(2) We call an $A_{N,K}$ functor $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ a *homotopy equivalence* if it has an $A_{N,K}$ *homotopy inverse*, i.e. an $A_{N,K}$ functor $G : (\mathcal{A}', m') \to (\mathcal{A}, m)$ such that F_{ob} and G_{ob} are inverse to each other (in particular they are both bijective), $F \circ G$ is $A_{N,K}$ homotopic to $\mathrm{id}_{\mathcal{A}'}$, and $G \circ F$ is $A_{N,K}$ homotopic to $\mathrm{id}_{\mathcal{A}}$.
(3) We define an A_∞ *homotopy* in a similar fashion as in Definition A.5.

The following theorems are generalization of [2, Theorem 3.22(c) and Theorem 3.23] which come from [15]. See also Theorem 13.11 in [13].

Theorem A.11. *Let $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ be an $A_{N,K}$ functor of $A_{N,K}$ categories. Suppose F_{ob} is bijective. Then F is an $A_{N,K}$ quasi-isomorphism if and only if it is an $A_{N,K}$ homotopy equivalence.*

Theorem A.12. *Let $(N, K), (\overline{N}, \overline{K}) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ such that $(N, K) \preceq (\overline{N}, \overline{K})$.*

(1) Let $F : (\mathcal{A}, m) \to (\mathcal{A}', m')$ be an $A_{N,K}$ functor of $A_{N,K}$ categories which is an $A_{N,K}$ homotopy equivalence. Suppose m' is the restriction of an $A_{\overline{N},\overline{K}}$ structure $\overline{m}'$ on $\mathcal{A}'$. Then m is the restriction of an $A_{\overline{N},\overline{K}}$ structure $\overline{m}$ on $\mathcal{A}$ and F is the restriction of an $A_{\overline{N},\overline{K}}$ functor $\overline{F} : (\mathcal{A}, \overline{m}) \to (\mathcal{A}', \overline{m}')$ which is an $A_{\overline{N},\overline{K}}$ homotopy equivalence.
(2) Let $F : (\mathcal{A}, \overline{m}_{\mathcal{A}}) \to (\mathcal{A}', \overline{m}_{\mathcal{A}'})$ and $G : (\mathcal{B}, \overline{m}_{\mathcal{B}}) \to (\mathcal{B}', \overline{m}_{\mathcal{B}'})$ be $A_{\overline{N},\overline{K}}$ functors of $A_{\overline{N},\overline{K}}$ categories which are $A_{\overline{N},\overline{K}}$ homotopy equivalence. Let $\overline{\Phi}' : (\mathcal{A}', \overline{m}_{\mathcal{A}'}) \to (\mathcal{B}', \overline{m}_{\mathcal{B}'})$ be an $A_{\overline{N},\overline{K}}$ functor and $\Phi : (\mathcal{A}, \overline{m}_{\mathcal{A}}) \to (\mathcal{B}, \overline{m}_{\mathcal{B}})$ be an $A_{N,K}$ functor such that $\overline{\Phi}' \circ F$ is $A_{N,K}$ homotopic to $G \circ \Phi$. Then Φ is the restriction of an $A_{\overline{N},\overline{K}}$ functor $\overline{\Phi} : (\mathcal{A}, \overline{m}_{\mathcal{A}}) \to (\mathcal{B}, \overline{m}_{\mathcal{B}})$ such that $\overline{\Phi}' \circ F$ is $A_{\overline{N},\overline{K}}$ homotopic to $G \circ \overline{\Phi}$.

APPENDIX B. LAGRANGIAN GRASSMANNIAN AND MASLOV INDEX

Let (W_0, ω_0) be a symplectic vector space of real dimension $2m > 0$. Denote by $LG(W_0, \omega_0)$ the space of Lagrangian subspaces of (W_0, ω_0). It is well known that

$$\pi_1(LG(W_0, \omega_0)) \simeq \mathbb{Z}.$$

For each positive integer k, we call the covering space of $LG(W_0, \omega_0)$ with deck transformation group isomorphic to $\mathbb{Z}_k$ (resp. $\mathbb{Z}$) the *$\mathbb{Z}_k$-cover* (resp. *$\mathbb{Z}$-cover*) of $LG(W_0, \omega_0)$.

Now let (W, ω) be a symplectic vector bundle on a space X. Then we can form the fiber bundle $LG(W, \omega)$ whose fiber at a point $x \in X$ is equal to $LG(W_x, \omega_x)$.

Lemma B.1. *[26, Lemma 2.2] If the first Chern class $c_1(W) \in H^2(X; \mathbb{Z})$ of (W, ω) is divisible by a positive integer n, then $LG(W, \omega)$ admits a fiberwise $\mathbb{Z}_{2n}$-cover.*

Let $(M, \omega), \mathcal{S}$ be given as in Section 2.1. Put $\mathcal{L}_M := LG(TM, \omega)$. Let $L \in \mathcal{S}$. As in Section 3.1, denote by θ_L the section of $\mathcal{L}_M|_L$ parametrizing the tangent spaces of points of L:

$$\theta_L(x) := T_x L \in (\mathcal{L}_M)_x, \ x \in L.$$

Let $L_0, L_1 \in \mathcal{S}$ and $c \in C(L_0, L_1)$. We define the family version of the "canonical short path" [4] from θ_{L_0} to θ_{L_1} over $L(c)$ as follows. Consider the symplectic vector bundle $V_c := TL(c)^{\perp\omega}/TL(c)$ defined on $L(c)$ with the induced symplectic form $[\omega]$ and its associated Lagrangian Grassmannian bundle $\mathcal{L}_{L(c)} := LG(V_c, [\omega])$ which embeds into $\mathcal{L}_M|_{L(c)}$ through the quotient map $TL(c)^{\perp\omega} \twoheadrightarrow V_c$. Then the images of θ_{L_0} and θ_{L_1} lie in $\mathcal{L}_{L(c)}$. Choose a compatible almost complex structure J_c on $(V_c, [\omega])$ such that $J_c \cdot TL_0/TL(c) = TL_1/TL(c)$. Then the desired path θ_t^c is defined to be

$$\theta_t^c := e^{-\frac{\pi t}{2} J_c} \cdot TL_0/TL(c), \ t \in [0, 1]. \tag{B.1}$$

Notice that it is a path of sections of $\mathcal{L}_{L(c)}$, which are also sections of $\mathcal{L}_M|_{L(c)}$.

Remark B.2. One can show that θ_t^c is independent of the choice of J_c up to homotopy.

Now let $(\overrightarrow{\mathbb{L}}, \vec{\gamma})$ be a Lagrangian label (Definition 2.9). Consider the following set-up which is similar to the one in Definition 2.10(1): let D be the unit disk with $k+1$ marked points $\xi_0, \ldots, \xi_k$ on the boundary $\partial D \approx S^1$ arranged in the counterclockwise order. We also put $\xi_{k+1} = \xi_0$. Let $u : D \to M$ be a continuous map such that $u|_{\partial D - \{\xi_0, \ldots, \xi_k\}}$ has a continuous lift $\tilde{u}$ in $\coprod_{s=0}^k L_s$ with $\tilde{u}((\xi_s, \xi_{s+1})) \subseteq L_s$. Moreover, we assume that for each s, the limit

$$\tilde{u}(\xi_s) := \left(\lim_{\substack{\xi \to \xi_s \\ \xi \in (\xi_{s-1}, \xi_s)}} \tilde{u}(\xi), \lim_{\substack{\xi \to \xi_s \\ \xi \in (\xi_s, \xi_{s+1})}} \tilde{u}(\xi) \right)$$

exists and lies in $L(\gamma_s)$.

We now define the Maslov index of the homotopy class represented by u. Consider the bundle $\mathcal{L} := u^*\mathcal{L}_M$ on D. Since D is contractible, $\mathcal{L}$ admits a fiberwise $\mathbb{Z}$-cover $\mathcal{L}'$. Consider the loop η which is the concatenation of the following paths:

$$\theta_{L_0}(\tilde{u}|_{[\xi_0, \xi_1]}), \theta_t^{c_1}(\tilde{u}(\xi_1)), \ldots, \theta_{L_k}(\tilde{u}|_{[\xi_k, \xi_0]}), \theta_t^{c_0}(\tilde{u}(\xi_0)).$$

Then η has a lift in $\mathcal{L}'$ under the fiberwise $\mathbb{Z}$-covering map $\mathcal{L}' \to \mathcal{L}$ whose endpoints differ by a deck transformation group element which is an integer. One can show that this integer depends only on the homotopy class β represented by u.

Definition B.3. Define $\mu(\beta)$, the *Maslov index of* β, to be this integer.

Finally, let $\vec{f} = (f_1, \dots, f_k)$ where $f_s : \Delta^{r_s} \to L(\gamma_s)$ is a smooth singular simplex.

Lemma B.4. *(dimension formula) The virtual dimension of* $\overline{\mathcal{M}}_{k+1}(\overrightarrow{\mathbb{L}}, \vec{\gamma}, \beta, J; \vec{f})$ *is equal to*

$$m + \mu(\beta) - \sum_{s=1}^{k}(dim(L(\gamma_s) - r_s) + k - 2. \tag{B.2}$$

Appendix C. The dihedral group action

Let $(M,\omega), \mathcal{S}, V$ be given as in Section 2.1. We construct a *conjugate automorphism* R of $Fuk(M,\omega)$ which satisfies (1.3), proving Theorem 1.7. For any $\mathbb{C}^\times$-local system $\mathcal{E}$ on $L \in \mathcal{S}$, there is a unique $\mathbb{C}^\times$-local system $\overline{\mathcal{E}}$ whose holonomy is equal to the complex conjugate of the holonomy of $\mathcal{E}$. Given $\mathbb{L} = (L, \sigma, \mathcal{E}) \in Ob_{\mathcal{S}}$, we put

$$R_{ob}(\mathbb{L}) := (\mathbb{L}, \sigma, \overline{\mathcal{E}}).$$

In Section 2.4 the morphism space $\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1)$ associated to any two objects $\mathbb{L}_i = (L_i, \sigma_i, \mathcal{E}_i),\ i = 0, 1$ of $\mathcal{A} := Fuk(M,\omega)$ is defined by

$$\mathcal{A}(\mathbb{L}_0, \mathbb{L}_1) = \bigoplus_{\gamma \in \mathcal{C}(\mathbb{L}_0, \mathbb{L}_1)} \bigoplus_{f \in \mathcal{X}_{i,N}(\gamma)} \Gamma_{flat}(f^*\mathcal{E}_\gamma).$$

Roughly speaking, it is the direct sum, over a finite collection of smooth singular simplices in the connected components of $L_0 \times_\iota L_1$, of the spaces of flat sections of the pullbacks of certain $\mathbb{C}^\times$-local systems.

For each singular simplex $f \in \mathcal{X}_{i,N}(\gamma)$, there is a canonical complex conjugate isomorphism $\Gamma_{flat}(f^*\mathcal{E}_\gamma) \to \Gamma_{flat}(f^*\overline{\mathcal{E}_\gamma})$, and hence we obtain a complex conjugate isomorphism

$$R_1^{0,0} : \mathcal{A}(\mathbb{L}_0, \mathbb{L}_1) \to \mathcal{A}(R_{ob}(\mathbb{L}_0), R_{ob}(\mathbb{L}_1)).$$

Put also $R_k^{\lambda,\mu} = 0$ for $(k, \lambda, \mu) \neq (1, 0, 0)$. Then one checks easily that $R = (R_k^{\lambda,\mu})$ satisfies the A_∞ equation (A.1) so that it is almost an A_∞ functor except that $R_1^{0,0}$ is not complex linear. But obviously, it makes sense to talk about the composite of "A_∞ functors" of this kind and A_∞ functors in the original sense, following Definition A.9.

Proposition C.1. *Let* Φ *be constructed in Section 3. Then we have*

$$R \circ \Phi \circ R \circ \Phi = id_{\mathcal{A}}.$$

Proof. The proof is based on the following two observations.

1. For any $L \in \mathcal{S}$, there is a canonical isomorphism
$$\mathbb{1} \simeq \mathcal{E}_L \otimes \overline{\mathcal{E}_L}. \tag{C.1}$$
2. Recall the isomorphism $\phi_c : \mathcal{E}_{L_0}|_{L(c)} \to \mathcal{E}_{L_1}|_{L(c)}$ associated to any $L_0, L_1 \in \mathcal{S}$ and $c \in C(L_0, L_1)$ which is constructed in Section 3.2. The element
$$\phi_c \otimes \overline{\phi_c} \in Hom((\mathcal{E}_{L_0} \otimes \overline{\mathcal{E}_{L_0}})|_{L(c)}, (\mathcal{E}_{L_1} \otimes \overline{\mathcal{E}_{L_1}})|_{L(c)})$$
is equal to the identity $\in Hom(\mathbb{1}, \mathbb{1})$ via the identification (C.1).

To prove (1), notice that as $\mathbb{Z}_{2n}$-local systems, $\mathcal{E}_L$ and $\overline{\mathcal{E}_L}$ are equal. (The complex conjugation does not change the principal bundles but the representations.) To regard them as $\mathbb{C}^\times$-local systems, we use the inclusion $1 \pmod{2n} \mapsto \zeta$ for $\mathcal{E}_L$ and $1 \pmod{2n} \mapsto \zeta^{-1}$ for $\overline{\mathcal{E}_L}$. Hence the result follows.

To prove (2), it suffices to prove the following statement. Let G be a group. Let $\mathcal{P}_1, \mathcal{P}_2$ be principal G-bundles on a space X and $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ be an isomorphism. Let $\lambda : G \to \mathbb{C}^\times$ be a group homomorphism. Then the composite

$$X \times \mathbb{C} \xrightarrow{p} (\mathcal{P}_1 \times_X \mathcal{P}_1) \times_{\lambda\boxtimes\lambda^{-1}} \mathbb{C} \xrightarrow{q} (\mathcal{P}_2 \times_X \mathcal{P}_2) \times_{\lambda\boxtimes\lambda^{-1}} \mathbb{C} \xrightarrow{r^{-1}} X \times \mathbb{C}$$

is equal to the identity, where

$$\begin{aligned} p(x,v) &= [(z,z):v] \\ q([(z_1,z_2):v]) &= [(\phi(z_1),\phi(z_2)):v] \\ r(x,v) &= [(w,w):v] \end{aligned}$$

and z (resp. w) is any point on the fiber of $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) over $x \in X$. But this is obvious.

□

It follows that Theorem 1.7 is proved.

References

[1] Abouzaid, M., Auroux, D., and Katzarkov, L.: *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. Inst. Hautes Etudes Sci. 123 (2016), 199-282.

[2] Akaho, A., Joyce, D.: *Immersed Lagrangian Floer theory*, J. Differential Geom. 86 (2010), no. 3, 381-500.

[3] Auroux, D.: *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT 1 (2007), 51-91. MR 2386535.

[4] Auroux, D.: *A beginner's introduction to Fukaya categories*, Contact and symplectic topology, *Bolyai Soc. Math. Stud.*, vol. 26, pp.85-136. Janos Bolyai Math. Soc., Budapest (2014). DOI 10.1007/978-3-319-02036-5_ 3.

[5] Chan, K., Lau, S.-C., and Leung, N.-C.: *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Differential Geom. 90 (2012), no. 2, 177-250.

[6] Chan, K., Leung N.-C.: *Mirror symmetry for toric Fano manifolds via SYZ transformation*, Adv. Math. 223 (2010), no. 3, 797-839.

[7] Cho, C.-H., Hong, H., and Lau, S.-C.: *Gluing localized mirror functors*, arXiv:1810.02045.

[8] ________: *Noncommutative homological mirror functor*, arXiv:1512.07128.

[9] ________: *Localized mirror functor for Lagrangian immersions, and homological mirror symmetry for* $\mathbb{P}^1_{a,b,c}$, J. Differential Geom. 106 (2017), no.1, 45-126.

[10] ________: *Localized mirror functor constructed from a Lagrangian torus*, J. Geom. Phys. 136 (2019), 284-320.

[11] Cho, C.-H., Oh, Y.: *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*. Asian J. Math 10 (2006), no.4 , 773-814 (2006).

[12] Fukaya, K.: *Galois symmetry on Floer cohomology*. Turk. J. Math. 27 (2003), no.1, 11-32.

[13] Fukaya, K.: *Unobstructed immersed Lagrangian correspondence and filtered A infinity functor*, arXiv:1706.02131.

[14] Fukaya, K., Ono, K.: *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), 933-1048.

[15] Fukaya, K., Oh, Y., Ohta, H., Ono, K.: *Lagrangian intersection Floer theory - anomaly and obstruction. Part I and II*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.

[16] Fukaya, K., Oh, Y., Ohta, H., Ono, K.: *Anchored Lagrangian submanifolds and their Floer theory*, Mirror symmetry and tropical geometry, Contemp. Math., vol. 527, Amer. Math. Soc., Providence, RI, 2010, pp. 15-54.

[17] Fukaya, K., Oh, Y., Ohta, H., Ono, K.: *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. 151 (2010), no. 1, 23-174.

[18] Gross, M., Siebert, B.: *From real affine geometry to complex geometry*, Ann. of Math. (2) 174 (2011), no. 3, 1301-1428.

[19] Hatcher, A: *Algebraic topology*, Cambridge University Press, ISBN 978-0-521-79540-1 (2005).

[20] Hong, H., Kim, Y. and Lau, S.-C.: *Immersed two-spheres and SYZ with application to Grassmannians*, arXiv:1805.11738.

[21] Kontsevich, M.: *Homological algebra of mirror symmetry*, Proceddings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120-139.

[22] Kontsevich, M., Soibelman, Y.: *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 321-385.

[23] Kuznetsov, A., Smirnov, M.: *On residual categories for Grassmannians*, arXiv:1802.08097.

[24] Kuznetsov, A., Smirnov, M.: *Residual categories for (co)adjoint Grassmannians in classical types*, arXiv:2001.04148.

[25] Oh. Y.: *Mean curvature vector and symplectic topology of Lagrangian submanifolds in Einstein-Kahler manifolds*, Math. Z., 216 (1994), 471-482.

[26] Seidel, P.: *Graded Lagrangian submanifolds*, Bull. Soc. Math. France 128 (2000), 103-149.

[27] Strominger, A., Yau, S.-T., Zaslow, E.: *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243-259.

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