UNIFORMITY IN MORDELL–LANG FOR CURVES

VESSELIN DIMITROV, ZIYANG GAO AND PHILIPP HABEGGER

ABSTRACT. Consider a smooth, geometrically irreducible, projective curve of genus $g \geq 2$ defined over a number field of degree $d \geq 1$. It has at most finitely many rational points by the Mordell Conjecture, a theorem of Faltings. We show that the number of rational points is bounded only in terms of g, d, and the Mordell–Weil rank of the curve's Jacobian, thereby answering in the affirmative a question of Mazur. In addition we obtain uniform bounds, in g and d, for the number of geometric torsion points of the Jacobian which lie in the image of an Abel–Jacobi map. Both estimates generalize our previous work for 1-parameter families. Our proof uses Vojta's approach to the Mordell Conjecture, and the key new ingredient is the generalization of a height inequality due to the second- and third-named authors.

CONTENTS

1. INTRODUCTION

Let F be a field. By a smooth curve defined over F we mean a geometrically irreducible, smooth, projective curve defined over F. Let C be a smooth curve of genus at least 2 defined over a number field F. As was conjectured by Mordell and proved by Faltings [Fal83], $C(F)$, the set of F-rational points of C, is finite.

We let $\mathrm{Jac}(C)$ denote the Jacobian of C. Recall that $\mathrm{Jac}(C)(F)$ is a finitely generated abelian group by the Mordell–Weil Theorem.

The aim of this paper is to bound $\#C(F)$ from above. Here is our first result.

Theorem 1.1. *Let $g \geq 2$ and $d \geq 1$ be integers. Then there exists a constant $c = c(g,d) \geq 1$ with the following property. If C is a smooth curve of genus g defined over a*

2000 *Mathematics Subject Classification.* 11G30, 11G50, 14G05, 14G25.

number field F with $[F:\mathbb{Q}] \leq d$, then

$$\#C(F) \leq c^{1+\rho} \tag{1.1}$$

where ρ is the rank of $\mathrm{Jac}(C)(F)$.

This theorem gives an affirmative answer to a question posed by Mazur [Maz00, Page 223]. See also [Maz86, top of page 234] for an earlier question. Before this, Lang formulated a related conjecture [Lan78, page 140] on the number of integral points of elliptic curves.

The method of our theorem builds up on the work of many others. At the core we follow Vojta's proof [Voj91] of the Mordell Conjecture. Vojta's proof was later simplified by Bombieri [Bom90] and further developed by Faltings [Fal91]. Silverman [Sil93] proved a bound of the quality (1.1) if C ranges over twists of a given smooth curve. The bound by de Diego [dD97] is of the form $c(g)7^{\rho}$, where $c(g) > 0$ depends only on g; the value 7 had already arisen in Bombieri's work. But she only counts points whose height is large in terms of a height of C. Recently, and over the rationals, Alpoge [Alp18] improved 7 to 1.872 and, for g large enough, even to 1.311.

On combining the Vojta and Mumford Inequalities one gets an upper bound for the number of *large points* in $C(F)$; these are points whose height is sufficiently large relative to a suitable height of C. A lower bound for the Néron–Tate height, such as proved by David–Philippon [DP02], can be used to count the number of remaining points which we sometimes call *small points*. Indeed, Rémond [Rém00a] made the Vojta and Mumford Inequalities explicit and obtained explicit upper bounds for the number of rational points on curves embedded in abelian varieties. The resulting cardinality bounds depend on a suitable notion of height of C, an artifact of the lower bounds for the Néron–Tate height. Later, David–Philippon [DP07] proved stronger height lower bounds in a power of an elliptic curve. They then obtained uniform estimates of the quality (1.1) for a curve in a product of elliptic curves, thus providing evidence that Mazur's Question had a positive answer, see also David–Nakamaye–Philippon's work [DNP07].

We give an overview of the general method in more detail in §1.1 below.

The main innovation of this paper is to prove a lower bound for the Néron–Tate height that is sufficiently strong to eliminate the dependency on the height of C. This leads to the uniform estimate as in Theorem 1.1. In earlier work [DGH19] we applied the precursor [GH19] of the height lower bound presented here to recover a variant of Theorem 1.1 in a one-parameter family of smooth curves.

We now explain some further results that follow from the approach described above. For an integer $g \geq 1$, let $\mathbb{A}_{g,1}$ denote the coarse moduli space of principally polarized abelian varieties of dimension g. This is an irreducible quasi-projective variety which we can take as defined over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Suppose we are presented with an immersion $\iota\colon \mathbb{A}_{g,1} \to \mathbb{P}^m_{\overline{\mathbb{Q}}}$ into projective space. Let $h\colon \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the absolute logarithmic Weil height, or just Weil height, *cf.* [BG06, §1.5.1]. If C is a smooth curve of genus $g \geq 2$ defined over $\overline{\mathbb{Q}}$ and if $P_0 \in C(\overline{\mathbb{Q}})$, then we can consider $C - P_0$ as a curve in $\mathrm{Jac}(C)$ via the Abel–Jacobi map. We use $[\mathrm{Jac}(C)]$ to denote the point in $\mathbb{A}_{g,1}(\overline{\mathbb{Q}})$ parametrizing $\mathrm{Jac}(C)$.

Theorem 1.1 follows from the following theorem, which is more in the spirit of [Maz86, top of page 234].

Theorem 1.2. *Let $g \geq 2$ and let ι be as above. Then there exist two constants $c_1 = c_1(g,\iota) \geq 0$ and $c_2 = c_2(g,\iota) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over $\overline{\mathbb{Q}}$, let $P_0 \in C(\overline{\mathbb{Q}})$, and let Γ be a subgroup of* $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$ *of finite rank $\rho \geq 0$. If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c_2^{1+\rho}.$$

The following corollary follows from Theorem 1.2 when Γ is the group $\mathrm{Jac}(C)_{\mathrm{tors}}$ of points of finite order in $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$; it has rank 0.

Corollary 1.3. *Let $g \geq 2$ and let ι be as above. Then there exist two constants $c_1 = c_1(g,\iota) \geq 0$ and $c_2 = c_2(g,\iota) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over $\overline{\mathbb{Q}}$ and let $P_0 \in C(\overline{\mathbb{Q}})$. If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C)_{\mathrm{tors}} \leq c_2.$$

As in Theorem 1.1 we can drop the condition on the height of the Jacobian by working over a number field of bounded degree.

Theorem 1.4. *Let $g \geq 2$ and $d \geq 1$ be integers. Then there exists a constant $c = c(g,d) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over a number field $F \subseteq \overline{\mathbb{Q}}$ with $[F:\mathbb{Q}] \leq d$ and let $P_0 \in C(\overline{\mathbb{Q}})$, then*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C_{\overline{\mathbb{Q}}})_{\mathrm{tors}} \leq c.$$

Let us recall some previous results towards Mazur's Question for rational points, *i.e.*, towards Theorem 1.1. Based on the method of Vojta, Alpoge [Alp18] proved that the average number of rational points on a curve of genus 2 with a marked Weierstrass point is bounded. Let C be a smooth curve of genus $g \geq 2$ defined over a number field $F \subseteq \overline{\mathbb{Q}}$. The Chabauty–Coleman approach [Cha41, Col85] yields estimates under an additional hypothesis on the rank of Mordell–Weil group. For example, if $\mathrm{Jac}(C)(F)$ has rank at most $g-3$, Stoll [Sto19] showed that $\#C(F)$ is bounded solely in terms of $[F:\mathbb{Q}]$ and g if C is hyperelliptic; Katz–Rabinoff–Zureick-Brown [KRZB16] later, under the same rank hypothesis, removed the hyperelliptic hypothesis. Checcoli, Veneziano, and Viada [CVV17] obtain an effective height bound under a restriction on the Mordell–Weil rank.

As for algebraic torsion points, *i.e.*, in the direction of Theorem 1.4, DeMarco–Krieger–Ye [DKY20] proved a bound on the cardinality of torsion points on any genus 2 curve that admits a degree-two map to an elliptic curve when the Abel–Jacobi map is based at a Weierstrass point. Moreover, their bound is independent of $[F:\mathbb{Q}]$.

1.1. **Néron–Tate distance of algebraic points on curves.** Let C be a smooth curve defined over $\overline{\mathbb{Q}}$ of genus $g \geq 2$, let $P_0 \in C(\overline{\mathbb{Q}})$, and let Γ be a subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$ of finite rank r. For simplicity we identify C with its image under the Abel–Jacobi embedding $C \to \mathrm{Jac}(C)$ via P_0.

Our proof of Theorem 1.2 follows the spirit of the method of Vojta, later generalized by Faltings. Let $\hat{h}\colon \mathrm{Jac}(C)(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the Néron–Tate height attached to a symmetric and ample line bundle. We divide $C(\overline{\mathbb{Q}}) \cap \Gamma$ into two parts:

- Small points $\left\{P \in C(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}(P) \leq B(C)\right\}$;

- Large points $\left\{P \in C(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}(P) > B(C)\right\}$

where $B(C)$ is allowed to depend on a suitable height of C. Roughly speaking, we can take $B(C) = c_0 \max\{1, h(\iota([\mathrm{Jac}(C)]))\}$ for some $c_0 = c_0(g,\iota) > 0$. The constant c_0 is chosen in a way that accommodates both the *Mumford inequality* and the *Vojta inequality*. Combining these two inequalities yields an upper bound on the number of large points by $c_1(g)^{1+r}$, see for example Vojta's [Voj91, Theorem 6.1] in the important case where Γ is the group of points of $\mathrm{Jac}(C)$ rational over a number field or more generally in the work of David–Philippon [DP07] and Rémond [Rém00a].

Thus in order to prove Theorem 1.2, it suffices to bound the number of small points.

In this paper we achieve such a bound by studying the Néron–Tate distance of points in $C(\overline{\mathbb{Q}})$.

Roughly speaking, we find positive constants c_1, c_2, c_3, and c_4 that depend on g and ι, but not on C, such that if $h(\iota([\mathrm{Jac}(C)])) \geq c_1$ then for all $P \in C(\overline{\mathbb{Q}})$ we have the following alternative.

- Either P lies in a subset of $C(\overline{\mathbb{Q}})$ of cardinality at most c_2,
- or $\left\{Q \in C(\overline{\mathbb{Q}}) : \hat{h}(Q-P) \leq h(\iota([\mathrm{Jac}(C)]))/c_3\right\} < c_4$.

This dichotomy is stated in Proposition 7.1. In this paper, we make the statement precise by referring to the universal family of genus g smooth curves with suitable level structure, and the Néron–Tate height on each Jacobian attached to the tautological line bundle. The setting up is done in §6.

This proposition can be seen as a relative version of the Bogomolov conjecture for abelian varieties with large height. It has the following upshot: If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then the small points in $C(\overline{\mathbb{Q}}) \cap \Gamma$ lie in a set of uniformly bounded cardinality, or are contained in $(1 + c_0c_3)^r$ balls in the Néron–Tate metric, with each ball containing at most c_4 points. This will yield the desired bound in Theorem 1.2, as executed in §8.

1.2. **Height inequality and Non-degeneracy.** We follow the framework presented in our previous work [DGH19]. In *loc.cit.* we proved the result for 1-parameter families, as an application of the second- and third-named authors' height inequality [GH19, Theorem 1.4]. Passing to general cases requires generalizing this height inequality to higher dimensional bases. The generalization has two parts: generalizing the inequality itself under the non-degeneracy condition and generalizing the criterion of non-degenerate subvarieties. We execute the first part in the current paper while the second part was done by the second-named author in [Gao18]. Let us explain the setting up.

Let k be an algebraically closed subfield of $\mathbb{C}$. Let S be a regular, irreducible, quasi-projective variety defined over k that is Zariski open in a regular, irreducible projective variety $\overline{S} \subseteq \mathbb{P}^m_k$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. We suppose that we are presented with a closed immersion $\mathcal{A} \to \mathbb{P}^n_k \times S$ over S. On the generic fiber of π we assume that this immersion comes from a basis of the global sections of the l-th power of a symmetric and ample line bundle with $l \geq 4$. If $k = \overline{\mathbb{Q}}$ and as described in §3.1 we obtain two height functions, the restriction of the Weil height $h\colon \overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$ and the Néron–Tate height $\hat{h}_{\mathcal{A}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

For the purpose of our main applications, including Theorems 1.1 and 1.2, it suffices to work under the following hypothesis.

`(Hyp)`: $\mathcal{A} \to S$ carries a principal polarization and has level-ℓ-structure for some $\ell \geq 3$.
So in the main body of the paper, we will focus on the case `(Hyp)`. The general case where `(Hyp)` is not assumed will be handled in Appendix B.

The non-degenerate subvarieties of $\mathcal{A}$ are defined using the *Betti map* which we briefly describe here; the precise definition will be given by Proposition B.1 and in Proposition 2.1 under `(Hyp)`.

For any $s \in S(\mathbb{C})$, there exists an open neighborhood $\Delta \subseteq S^{\mathrm{an}}$ of s which we may assume is simply-connected. Then one can define a basis $\omega_1(s), \ldots, \omega_{2g}(s)$ of the period lattice of each fiber $s \in \Delta$ as holomorphic functions of s. Now each fiber $\mathcal{A}_s = \pi^{-1}(s)$ can be identified with the complex torus $\mathbb{C}^g/(\mathbb{Z}\omega_1(s) \oplus \cdots \oplus \mathbb{Z}\omega_{2g}(s))$, and each point $x \in \mathcal{A}_s(\mathbb{C})$ can be expressed as the class of $\sum_{i=1}^{2g} b_i(x)\omega_i(s)$ for real numbers $b_1(x), \ldots, b_{2g}(x)$. Then $b_\Delta(x)$ is defined to be the class of the $2g$-tuple $(b_1(x), \ldots, b_{2g}(x)) \in \mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$. We obtain with a real-analytic map

$$b_\Delta \colon \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g},$$

which is fiberwise a group isomorphism and where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$.

Definition 1.5. *An irreducible subvariety X of $\mathcal{A}$ is said to be non-degenerate if there exists an open non-empty subset Δ of S^{an}, with the Betti map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, such that*

$$\max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X$$

where $\mathrm{d}b_\Delta$ denotes the differential and $X^{\mathrm{sm,an}}$ is the analytification of the regular locus of X.

In Proposition 2.2(iii) we give another characterization of non-degenerate subvarieties. We can now formulate the height inequality.

Theorem 1.6. *Suppose that $\mathcal{A}$ and S are as above with $k = \overline{\mathbb{Q}}$; in particular, $\mathcal{A}$ satisfies* `(Hyp)`. *Let X be an irreducible subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$ that dominates S. Suppose X is non-degenerate, as defined in Definition 1.5. Then there exist constants $c_1 > 0$ and $c_2 \geq 0$ and a Zariski open dense subset U of X with*

$$\hat{h}_{\mathcal{A}}(P) \geq c_1 h(\pi(P)) - c_2 \quad \textit{for all} \quad P \in U(\overline{\mathbb{Q}}).$$

Note that [GH19, Theorem 1.4] is, up to some minor reduction, precisely Theorem 1.6 for $\dim S = 1$ together with the criterion for X to be non-degenerate when $\dim S = 1$. In general, the degeneracy behavior of X is fully studied in [Gao18]. See [Gao18, Theorem 1.1] for the criterion. However in practice, we sometimes still want to understand the height comparison on some degenerate X. One way to achieve this is by applying [Gao18, Theorem 1.3], which asserts the following statement: If X satisfies some reasonable properties, then we can apply Theorem 1.6 after doing some simple operations with X.

For the purpose of proving Proposition 7.1 and furthermore Theorem 1.2, we work in the following situation.

Let $\ell \geq 3$ be an integer and let $\mathbb{A}_{g,\ell}$ denote the moduli space of principally polarized g-dimensional abelian varieties with symplectic level-ℓ-structure. It is a classical fact that $\mathbb{A}_{g,\ell}$ is represented by an irreducible, regular, quasi-projective variety defined over a number field, see [MFK94, Theorem 7.9 and below] or [OS80, Theorem 1.9], so it is a fine moduli space. To simplify notation we take $\mathbb{A}_{g,\ell}$ as defined over $\overline{\mathbb{Q}}$ and when ℓ is fixed we often abbreviate $\mathbb{A}_{g,\ell}$ by $\mathbb{A}_g$. We will use the analytic description of $\mathbb{A}_{g,\ell}$ arising from a suitable quotient of Siegel's upper half space. Let $\mathbb{M}_g$ be the fine moduli space of smooth curves of genus g whose Jacobian is equipped with level-ℓ-structure; see [DM69, (5.14)] or [OS80, Theorem 1.8] for the existence.

Furthermore, let $\mathfrak{C}_g \to \mathbb{M}_g$ be the universal curve and $\mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety. Taking the Jacobian of a smooth curve leads to the Torelli morphism $\mathbb{M}_g \to \mathbb{A}_g$, which is finite-to-1 as we have level structure. Moreover, for $M \geq 2$ let $\mathscr{D}_M$ denote the M-th Faltings–Zhang morphism fiberwise defined by sending

$$(P_0, P_1, \ldots, P_M) \mapsto (P_1 - P_0, \ldots, P_M - P_0). \tag{1.2}$$

Roughly speaking, we will apply Theorem 1.6 to

$$X := \mathscr{D}_M(\underbrace{\mathfrak{C}_g \times_{\mathbb{M}_g} \ldots \times_{\mathbb{M}_g} \mathfrak{C}_g}_{(M+1)\text{-copies}}) \subseteq \underbrace{\mathfrak{A}_g \times_{\mathbb{A}_g} \ldots \times_{\mathbb{A}_g} \mathfrak{A}_g}_{M\text{-copies}}$$

for a suitable M. To verify non-degeneracy we will refer to the second-named author's work [Gao18, Theorem 1.2'] which applies if M is large in terms of g. So we can apply Theorem 1.6 to such X. This will eventually lead to Proposition 7.1.

The morphism and its variants are powerful tools in diophantine geometry, see [Fal91, Lemma 4.1]. It is closely connected to problems involving small Néron–Tate height, see [Zha98, Lemma 3.1] and [DP02, Proposition 4.1]. Stoll [Sto19] used a variant of (1.2) to show that a conjecture of Pink [Pin05b] on unlikely intersections implies Theorem 1.2 with the condition $h(\iota([\mathrm{Jac}(C)])) \geq c_1$ removed and with C allowed to be defined over $\mathbb{C}$.

1.3. **General Notation.** We collect here an overview of notation used throughout the text.

Let S be an irreducible, quasi-projective variety defined over an algebraically closed field k. Then S^{sm} denotes the regular locus of X. If $\pi\colon \mathcal{A} \to S$ is an abelian scheme then $[N]\colon \mathcal{A} \to \mathcal{A}$ is the multiplication-by-N morphism for all $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and if $s \in S(k)$, the fiber $\mathcal{A}_s = \pi^{-1}(s)$ is an abelian variety. If $k \subseteq \mathbb{C}$, then S^{an} denotes the analytification of S; it carries a natural topology that is Hausdorff.

We write $\mathbb{T}$ for the circle group $\{z \in \mathbb{C} : |z| = 1\}$.

Acknowledgements. The authors would like to thank Shou-wu Zhang for relevant discussions and Gabriel Dill for the argument involving torsion points to bound h_1 on page 28. We would also like to thank Lars Kühne and Ngaiming Mok for discussions on Hermitian Geometry; our paper is much influenced by Kühne's approach towards bounded height [Küh] and by Mok's approach to study the Mordell–Weil rank over function fields [Mok91]. We thank Gabriel Dill, Fabien Pazuki, and Joseph H. Silverman for corrections and comments on a draft of this paper. Vesselin Dimitrov would like to thank the NSF and the Giorgio and Elena Petronio Fellowship Fund II for financial support for this work. VD has received funding from the European Union's Seventh Framework Programme (FP7/2007–2013) / ERC grant agreement n° 617129. Both VD and ZG would like to thank the Institute for Advanced Study and the special year "Locally Symmetric Spaces: Analytical and Topological Aspects" for its hospitality during this work.

2. Betti map and Betti form

The goals of this section are to revisit the Betti map, the Betti form and make a link between them. In this paper we construct the Betti map using the universal family of principally polarized abelian varieties with level-ℓ-structure and bypass the ad-hoc construction found in [GH19].

In this section we will make the following assumptions. All varieties are defined over the field $\mathbb{C}$. Let S be an irreducible, regular, quasi-projective variety over $\mathbb{C}$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension g, that carries a principal polarization, and such that $\mathcal{A}$ is equipped with level-ℓ-structure, for some $\ell \geq 3$, *i.e.*, `(Hyp)` is satisfied.

Proposition 2.1. *Let $s_0 \in S(\mathbb{C})$. Then there exist a non-empty open neighborhood Δ of s_0 in S^{an}, and a map $b_\Delta\colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, called the Betti map, with the following properties.*

(i) *For each $s \in \Delta$ the restriction $b_\Delta|_{\mathcal{A}_s(\mathbb{C})}\colon \mathcal{A}_s(\mathbb{C}) \to \mathbb{T}^{2g}$ is a group isomorphism.*
(ii) *For each $\xi \in \mathbb{T}^{2g}$ the preimage $b_\Delta^{-1}(\xi)$ is a complex analytic subset of $\mathcal{A}_\Delta$.*
(iii) *The product $(b_\Delta, \pi)\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$ is a real analytic isomorphism.*

André, Corvaja, and Zannier [ACZ18] recently began the study of the maximal rank of the Betti map, especially the submersivity, using a slightly different definition. A full study of this maximal rank was realized in [Gao18]. Closely related to the Betti map is the Betti form, a semi-positive $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$, which was first introduced in Mok [Mok91].

Proposition 2.2. *There exists a closed $(1,1)$-form ω on $\mathcal{A}^{\mathrm{an}}$, called the Betti form, such that the following properties hold.*

(i) *The $(1,1)$-form ω is semi-positive, i.e., at each point the associated Hermitian form is positive semi-definite.*
(ii) *For all $N \in \mathbb{Z}$ we have $[N]^*\omega = N^2\omega$.*
(iii) *If X is an irreducible subvariety of $\mathcal{A}$ of dimension d and $\Delta \subseteq S^{\mathrm{an}}$ is open with $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$, then*
$$\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0 \quad \textit{if and only if} \quad \max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \operatorname{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2d.$$

We will prove both propositions during the course of this section using the universal abelian variety. A dynamical approach can be found in [CGHX18, §2].

2.1. **Betti map for the universal abelian variety.** We start to prove Proposition 2.1 for $S = \mathbb{A}_g$, the moduli space of principally polarized abelian variety of dimension g with level-ℓ-structure; it is a fine moduli space. Let $\pi^{\mathrm{univ}}\colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety.

The universal covering $\mathfrak{H}_g^+ \to \mathbb{A}_g^{\mathrm{an}}$, where $\mathfrak{H}_g^+$ is the Siegel upper half space, gives a polarized family of abelian varieties $\mathcal{A}_{\mathfrak{H}_g^+} \to \mathfrak{H}_g^+$ fitting into the diagram

$$\begin{tikzcd}
\mathcal{A}_{\mathfrak{H}_g^+} := \mathfrak{A}_g \times_{\mathbb{A}_g^{\mathrm{an}}} \mathfrak{H}_g^+ \arrow[r, "u_B"] \arrow[d] & \mathfrak{A}_g^{\mathrm{an}} \arrow[d, "\pi^{\mathrm{univ}}"] \\
\mathfrak{H}_g^+ \arrow[r] & \mathbb{A}_g^{\mathrm{an}}.
\end{tikzcd}$$

For the universal covering $u\colon \mathbb{C}^g\times\mathfrak{H}_g^+\to\mathcal{A}_{\mathfrak{H}_g^+}$ and for each $Z\in\mathfrak{H}_g^+$, the kernel of $u|_{\mathbb{C}^g\times\{Z\}}$ is $\mathbb{Z}^g+Z\mathbb{Z}^g$. Thus the map $\mathbb{C}^g\times\mathfrak{H}_g^+\to\mathbb{R}^g\times\mathbb{R}^g\times\mathfrak{H}_g^+\to\mathbb{R}^{2g}$, where the first map is the inverse of $(a,b,Z)\mapsto(a+Zb,Z)$ and the second map is the natural projection, descends to a *real* analytic map

$$b^{\mathrm{univ}}\colon \mathcal{A}_{\mathfrak{H}_g^+}\to\mathbb{T}^{2g}.$$

Now for each $s_0\in\mathbb{A}_g(\mathbb{C})$, there exists a contractible, relatively compact, open neighborhood Δ of s_0 in $\mathbb{A}_g^{\mathrm{an}}$ such that $\mathfrak{A}_{g,\Delta}:=(\pi^{\mathrm{univ}})^{-1}(\Delta)$ can be identified with $\mathcal{A}_{\mathfrak{H}_g^+,\Delta'}$ for some open subset Δ' of $\mathfrak{H}_g^+$. The composite $b_\Delta\colon\mathfrak{A}_{g,\Delta}\cong\mathcal{A}_{\mathfrak{H}_g^+,\Delta'}\to\mathbb{T}^{2g}$ is real analytic and satisfies the three properties listed in Proposition 2.1. Thus b_Δ is the desired Betti map in this case. Note that for a fixed (small enough) Δ, there are infinitely choices of Δ'; but for Δ small enough, if Δ_1' and Δ_2' are two such choices, then $\Delta_2'=\alpha\cdot\Delta_1'$ for some $\alpha\in\mathrm{Sp}_{2g}(\mathbb{Z})\subseteq\mathrm{SL}_{2g}(\mathbb{Z})$. Thus we have proved Proposition 2.1 for $\mathfrak{A}_g\to\mathbb{A}_g$.

2.2. **Betti form for the universal abelian variety.** Recall the universal covering $u\colon\mathbb{C}^g\times\mathfrak{H}_g^+\to\mathfrak{A}_g$. We will use (w,Z) to denote the coordinates on $\mathbb{C}^g\times\mathfrak{H}_g^+$.

Lemma 2.3. *Define*

$$\hat{\omega}^{\mathrm{univ}}:=\sqrt{-1}\partial\overline{\partial}\left(2(\mathrm{Im}w)^{\intercal}(\mathrm{Im}Z)^{-1}(\mathrm{Im}w)\right).$$

Then $\hat{\omega}^{\mathrm{univ}}$ is a closed semi-positive $(1,1)$-form on $\mathbb{C}^g\times\mathfrak{H}_g^+$ satisfying

$$\hat{\omega}^{\mathrm{univ}}=\sqrt{-1}\big(\mathrm{d}ZY^{-1}\mathrm{Im}(w)-\mathrm{d}w\big)^{\intercal}\wedge Y^{-1}\left(\mathrm{d}\overline{Z}Y^{-1}\mathrm{Im}(w)-\mathrm{d}\overline{w}\right)\tag{2.1}$$

with Y the imaginary part of an element of $\mathfrak{H}_g^+$. Moreover, if $N\in\mathbb{Z}$ and if we denote by $\widetilde{N}\colon\mathbb{C}^g\times\mathfrak{H}_g^+\to\mathbb{C}^g\times\mathfrak{H}_g^+$ the map $(w,Z)\mapsto(Nw,Z)$, then $\widetilde{N}^*\hat{\omega}^{\mathrm{univ}}=N^2\hat{\omega}^{\mathrm{univ}}$.

Proof. The $(1,1)$-form $\hat{\omega}^{\mathrm{univ}}$ is closed since $\mathrm{d}=\partial+\overline{\partial}$. We will prove the semi-positivity by direct computation.

Using

$$\begin{aligned}\overline{\partial}\mathrm{Im}w &= \frac{\sqrt{-1}}{2}\mathrm{d}\overline{w}, & \overline{\partial}Y^{-1} &= -\frac{\sqrt{-1}}{2}Y^{-1}\mathrm{d}\overline{Z}Y^{-1},\\ \partial\mathrm{Im}w &= -\frac{\sqrt{-1}}{2}\mathrm{d}w, & \partial Y^{-1} &= \frac{\sqrt{-1}}{2}Y^{-1}\mathrm{d}ZY^{-1}\end{aligned}$$

and the Leibniz rule (note that $Z=Z^{\intercal}$ and hence $\mathrm{d}Z=\mathrm{d}Z^{\intercal}$), we get

$$\begin{aligned}\hat{\omega}^{\mathrm{univ}} &= \sqrt{-1}\big((\mathrm{d}w)^{\intercal}Y^{-1}\wedge\mathrm{d}\overline{w}+(\mathrm{Im}w)^{\intercal}Y^{-1}\mathrm{d}Z\wedge Y^{-1}\mathrm{d}\overline{Z}Y^{-1}(\mathrm{Im}w)\\ &\qquad -(\mathrm{Im}w)^{\intercal}Y^{-1}\mathrm{d}ZY^{-1}\wedge\mathrm{d}\overline{w}-(\mathrm{d}w)^{\intercal}\wedge Y^{-1}\mathrm{d}\overline{Z}Y^{-1}(\mathrm{Im}w)\big).\end{aligned}$$

Rearranging yields the desired equality (2.1). The associated form is

$$H:\big((\xi_w,\xi_Z),(\eta_w,\eta_Z)\big)\mapsto\big(\xi_ZY^{-1}\mathrm{Im}(w)-\xi_w\big)^{\intercal}Y^{-1}\left(\overline{\eta_Z}Y^{-1}\mathrm{Im}(w)-\overline{\eta_w}\right),$$

for $\xi_w,\eta_w\in\mathbb{C}^g$ and $\xi_Z,\eta_Z\in\mathrm{Mat}_g(\mathbb{C})$ symmetric, is Hermitian and so $\hat{\omega}^{\mathrm{univ}}$ is real. Moreover,

$$H\big((\xi_w,\xi_Z),(\xi_w,\xi_Z)\big)=v^{\intercal}Y^{-1}\overline{v}\quad\text{with}\quad v=\xi_ZY^{-1}\mathrm{Im}(w)-\xi_w.$$

But Y is positive definite as a real symmetric matrix and thus positive definite as a Hermitian matrix. We see that H is positive semi-definite and this implies that $\hat{\omega}^{\mathrm{univ}}$ is positive semi-definite.

The "moreover" part of the lemma is clear. □

Next we want to show that $\hat{\omega}^{\mathrm{univ}}$ descends to a $(1,1)$-form on $\mathfrak{A}_g^{\mathrm{an}}$. To do this, we first show that $\hat{\omega}^{\mathrm{univ}}$ can be written in a simple form under an appropriate change of coordinates.

Define the complex space $\mathcal{X}_{2g,\mathrm{a}}$ of $\mathfrak{A}_g$ as follows:

- As a real algebraic space, $\mathcal{X}_{2g,\mathrm{a}} := \mathbb{R}^{2g} \times \mathfrak{H}_g^+$.
- The complex structure on $\mathcal{X}_{2g,\mathrm{a}}$ is given by

$$\mathbb{R}^{2g} \times \mathfrak{H}_g^+ = \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g^+ \cong \mathbb{C}^g \times \mathfrak{H}_g^+, \qquad (a,b,Z) \mapsto (a+Zb, Z). \tag{2.2}$$

Lemma 2.4. *Let $\hat{\omega}^{\mathrm{univ}}$ be as in Lemma 2.3. Then under the change of coordinates (2.2), we have $\hat{\omega}^{\mathrm{univ}} = 2(\mathrm{d}a)^\intercal \wedge \mathrm{d}b$.*

Proof. For the moment we write $Z = X + \sqrt{-1}Y$ with X and Y the real and imaginary part of $Z \in \mathfrak{H}_g^+$, respectively. Note that $w = a + Zb = (a + Xb) + \sqrt{-1}Yb$. Hence $Y^{-1}(\mathrm{Im} w) = b$ and $\mathrm{d}w = \mathrm{d}a + Z\mathrm{d}b + \mathrm{d}Zb$. Using this and noting that Z is symmetric, we have that (2.1) becomes

$$\begin{aligned}\hat{\omega}^{\mathrm{univ}} &= \sqrt{-1}\left(\sqrt{-1}(\mathrm{d}b)^\intercal Y + (\mathrm{d}b)^\intercal X + (\mathrm{d}a)^\intercal\right) \wedge Y^{-1}\left(\mathrm{d}a + X\mathrm{d}b - \sqrt{-1}Y\mathrm{d}b\right)\\ &= \sqrt{-1}\big(\sqrt{-1}(\mathrm{d}b)^\intercal \wedge \mathrm{d}a + (\mathrm{d}b)^\intercal \wedge Y\mathrm{d}b + (\mathrm{d}b)^\intercal X \wedge Y^{-1}\mathrm{d}a + (\mathrm{d}b)^\intercal X \wedge Y^{-1}X\mathrm{d}b +\\ &\qquad (\mathrm{d}a)^\intercal \wedge Y^{-1}\mathrm{d}a + (\mathrm{d}a)^\intercal \wedge Y^{-1}X\mathrm{d}b - \sqrt{-1}(\mathrm{d}a)^\intercal \wedge \mathrm{d}b\big).\end{aligned}$$

Many terms will vanish. Indeed, if M is a matrix, then $(\mathrm{d}b)^\intercal \wedge M\mathrm{d}a = -(\mathrm{d}a)^\intercal \wedge M^\intercal \mathrm{d}b$. As $(XY^{-1})^\intercal = Y^{-1}X$ and as $(\mathrm{d}b)^\intercal X \wedge Y^{-1}\mathrm{d}a = (\mathrm{d}b)^\intercal \wedge XY^{-1}\mathrm{d}a$ we find $(\mathrm{d}b)^\intercal X \wedge Y^{-1}\mathrm{d}a + (\mathrm{d}a)^\intercal \wedge Y^{-1}X\mathrm{d}b = 0$. Observe that Y is symmetric, and so $(\mathrm{d}b)^\intercal \wedge Y\mathrm{d}b = -(\mathrm{d}b)^\intercal \wedge Y\mathrm{d}b$ vanishes. Arguing along the same line and using that Y^{-1} and $XY^{-1}X$ are symmetric we find $(\mathrm{d}a)^\intercal \wedge Y^{-1}\mathrm{d}a = 0$ and $(\mathrm{d}b)^\intercal X \wedge Y^{-1}X = (\mathrm{d}b)^\intercal \wedge XY^{-1}X\mathrm{d}b = 0$. We are left with $\hat{\omega}^{\mathrm{univ}} = 2(\mathrm{d}a)^\intercal \wedge \mathrm{d}b$. □

Corollary 2.5. *Let $\hat{C}$ be an irreducible, 1-dimensional, complex analytic subset of an open subset of $\subseteq \mathcal{X}_{2g,\mathrm{a}} = \mathbb{R}^{2g} \times \mathfrak{H}_g^+$ and $\hat{C}^{\mathrm{sm}}$ its smooth locus. Then $\hat{\omega}^{\mathrm{univ}}$ restricted to $\hat{C}^{\mathrm{sm}}$ is trivial if and only if $\hat{C} \subseteq \{r\} \times \mathfrak{H}_g^+$ for some $r \in \mathbb{R}^{2g}$.*

Proof. First, assume that the coordinates (a,b) of $\mathbb{R}^{2g}$ are constant on $\hat{C}$. Then $\hat{\omega}^{\mathrm{univ}}$, which is simply $2(\mathrm{d}a)^\intercal \wedge \mathrm{d}b$ by Lemma 2.4, vanishes on $\hat{C}^{\mathrm{sm}}$.

Conversely, suppose that $\hat{\omega}^{\mathrm{univ}}$ vanishes identically on $\hat{C}^{\mathrm{sm}}$. This time we use (2.1) from Lemma 2.3. As Y^{-1} is positive definite we find $\mathrm{d}ZY^{-1}\mathrm{Im}(w) = \mathrm{d}w$ on $\hat{C}^{\mathrm{sm}}$. Using the change of coordinates $w = a + Zb$ we deduce $\mathrm{Im}(w) = Yb$ and $\mathrm{d}w = \mathrm{d}a + \mathrm{d}Zb + Z\mathrm{d}b$. So $\mathrm{d}Zb = \mathrm{d}ZY^{-1}w = \mathrm{d}a + \mathrm{d}Zb + Z\mathrm{d}b$ on $\hat{C}^{\mathrm{sm}}$. This equality simplifies to $\mathrm{d}a + Z\mathrm{d}b = 0$ on $\hat{C}^{\mathrm{sm}}$. As a and b are real valued and as $Z \in \mathfrak{H}_g^+$ we conclude $\mathrm{d}a = \mathrm{d}b = 0$ on $\hat{C}^{\mathrm{sm}}$. So a and b are constant on $\hat{C}$. □

Lemma 2.6. *Let $\hat{\omega}^{\mathrm{univ}}$ be as in Lemma 2.3. Then $\hat{\omega}^{\mathrm{univ}}$ descends to a semi-positive $(1,1)$-form ω^{univ} on $\mathfrak{A}_g$. Moreover, for $N \in \mathbb{Z}$ we have $[N]^*\omega^{\mathrm{univ}} = N^2\omega^{\mathrm{univ}}$.*

Proof. Let Sp_{2g} be the symplectic group defined over $\mathbb{Q}$, and let V_{2g} be the vector group over $\mathbb{Q}$ of dimension $2g$. Then the natural action of Sp_{2g} on V_{2g} defines a group $P_{2g,\mathrm{a}} := V_{2g} \rtimes \mathrm{Sp}_{2g}$.

We use the classical action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{H}_g^+$, it is transitive. The real coordinate on $\mathcal{X}_{2g,\mathrm{a}}$ on the left hand side of (2.2) has the following advantage. The group $P_{2g,\mathrm{a}}(\mathbb{R})$ acts transitively on $\mathcal{X}_{2g,\mathrm{a}}$ by the formula

$$(v,h)\cdot(v',Z) := (v+hv', hZ)$$

for $(v,h) \in P_{2g,\mathrm{a}}(\mathbb{R})$ and $(v',Z) \in \mathbb{R}^{2g} \times \mathfrak{H}_g^+ = \mathcal{X}_{2g,\mathrm{a}}$. The space $\mathfrak{A}_g^{\mathrm{an}}$ is then obtained as the quotient of $\mathcal{X}_{2g,\mathrm{a}}$ by a congruence subgroup of $P_{2g,\mathrm{a}}(\mathbb{Q})$. We refer to [Pin89, 10.5–10.9] or [Pin05a, Construction 2.9 and Example 2.12] for these facts.

It is clear that both $V_{2g}(\mathbb{R})$ and $\mathrm{Sp}_{2g}(\mathbb{R})$ preserve $2(\mathrm{d}a)^{\intercal} \wedge \mathrm{d}b$. Thus this 2-form is invariant under the action of $P_{2g,\mathrm{a}}(\mathbb{R})$ on $\mathcal{X}_{2g,\mathrm{a}}$.

So by Lemma 2.4, the previous two paragraphs imply that $\hat{\omega}^{\mathrm{univ}}$ descends to a $(1,1)$-form ω^{univ} on $\mathfrak{A}_g$. The semi-positivity of ω^{univ} follows from Lemma 2.3.

The property $[N]^*\omega^{\mathrm{univ}} = N^2\omega^{\mathrm{univ}}$ follows from the "moreover" part of Lemma 2.3 and the following commutative diagram

$$\begin{tikzcd} \mathbb{C}^g \times \mathfrak{H}_g^+ \arrow[r, "\widetilde{N}"] \arrow[d] & \mathbb{C}^g \times \mathfrak{H}_g^+ \arrow[d] \\ \mathfrak{A}_g^{\mathrm{an}} \arrow[r, "{[N]}"] & \mathfrak{A}_g^{\mathrm{an}}. \end{tikzcd}$$

□

This semi-positive $(1,1)$-form ω^{univ} will be the Betti form for $\mathfrak{A}_g \to \mathbb{A}_g$, as desired in Proposition 2.2. To show this, it suffices to establish property (iii) of Proposition 2.2. Hence it suffices to prove the following proposition.

Proposition 2.7. *Assume $\mathcal{A} \to S$ is $\mathfrak{A}_g \to \mathbb{A}_g$. Let X be an irreducible subvariety of $\mathfrak{A}_g$ of dimension d and let Δ be an open subset of S^{an} with $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$. Then*

$$\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0 \quad \textit{if and only if} \quad \max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2d. \tag{2.3}$$

Proof. We begin by reformulating Corollary 2.5. If C is an irreducible, 1-dimensional, complex analytic subset of an open subset of $\mathcal{A}_\Delta$, then

$$\omega^{\mathrm{univ}}|_{C^{\mathrm{sm}}} = 0 \quad \text{if and only if} \quad b_\Delta(C) \text{ is a point;} \tag{2.4}$$

indeed, this claim is local and it follows using the universal covering $u\colon \mathbb{C}^g \times \mathfrak{H}_g^+ \to \mathfrak{A}_g$.

We assume first that the right side of (2.3) is false, *i.e.*, the maximal rank is strictly less than $2d = 2\dim X$. So every $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ is a non-isolated point of $b_\Delta^{-1}(r) \cap X^{\mathrm{sm,an}}$ where $r = b_\Delta(x)$. Because $b_\Delta^{-1}(r)$ is a complex analytic subset of $\mathcal{A}_\Delta$ and $X^{\mathrm{sm,an}}$ is complex analytic in a neighborhood of x in $\mathcal{A}^{\mathrm{an}}$, there exists an irreducible complex analytic curve C in $b_\Delta^{-1}(r) \cap X^{\mathrm{sm,an}}$ passing through x. In particular, $b_\Delta(C)$ is a point and so $\omega^{\mathrm{univ}}|_{C^{\mathrm{sm}}} \equiv 0$ by (2.4).

The upshot of the previous paragraph is that the Hermitian form attached to the semi-positive $(1,1)$-form $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}$ vanishes along the tangent space of C^{sm}; it is degenerate. We can complete a tangent vector of C^{sm} to a basis of the tangent space of $X^{\mathrm{sm,an}}$. Considering holomorphic local coordinates we find $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}^{\wedge d} = 0$ at every point of

C^{sm}. By continuity it also vanishes at $x \in C$. Since $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ was arbitrary, we conclude $\omega^{\mathrm{univ}}|^{\wedge d}_{X^{\mathrm{sm,an}}} \equiv 0$.

For the converse we assume $\omega^{\mathrm{univ}}|^{\wedge d}_{X^{\mathrm{sm,an}}} \equiv 0$. So the Hermitian form attached to this semi-positive $(1,1)$-form is degenerate. Using holomorphic local coordinates any prescribed auxiliary point of $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ lies in an irreducible, 1-dimensional, complex analytic subset C of an open subset of $X^{\mathrm{sm,an}}$ with $\omega^{\mathrm{univ}}|_{C^{\mathrm{sm}}} \equiv 0$. So $b_\Delta(C)$ is a point by (2.4). As we can vary the auxiliary point we conclude that the rank on the right side of (2.3) is strictly less than $2d$. □

2.3. **General case.** We now prove Propositions 2.1 and 2.2 for $\pi\colon \mathcal{A} \to S$ as near the beginning of this section. In particular, we assume (`Hyp`). As $\mathbb{A}_g$ is a fine moduli space there exists a Cartesian diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\iota"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_g \arrow[d] \\ S \arrow[r, "\iota_S"] & \mathbb{A}_g. \end{tikzcd}$$

Now let $s_0 \in S(\mathbb{C})$. Applying Proposition 2.1 to the universal abelian variety $\mathfrak{A}_g \to \mathbb{A}_g$ and $\iota_S(s_0) \in \mathbb{A}_g(\mathbb{C})$, we obtain a non-empty open neighborhood Δ_0 of $\iota_S(s_0)$ in $\mathbb{A}_g^{\mathrm{an}}$ and a map

$$b_{\Delta_0}\colon \mathfrak{A}_g|_{\Delta_0} \to \mathbb{T}^{2g}$$

satisfying the properties listed in Proposition 2.1.

Now let $\Delta = \iota_S^{-1}(\Delta_0)$. Then Δ is an open neighborhood of s in S^{an}. Denote by $\mathcal{A}_\Delta = \pi^{-1}(\Delta)$ and define

$$b_\Delta = b_{\Delta_0} \circ \iota\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}.$$

Then b_Δ satisfies the properties listed in Proposition 2.1 for $\mathcal{A} \to S$. Hence b_Δ is our desired Betti map.

Next let us turn to the Betti form. Let ω^{univ} be the semi-positive $(1,1)$-form on $\mathfrak{A}_g$ as in Lemma 2.6. Define $\omega := \iota^*\omega^{\mathrm{univ}}$. We will show that ω satisfies the properties listed in Proposition 2.2.

The $(1,1)$-form ω is semi-positive as it is the pull-back of the semi-positive form ω^{univ}. Moreover, it satisfies $[N]^*\omega = N^2\omega$ since ω^{univ} has this property. Hence we have established properties (i) and (ii) of Proposition 2.2.

Let us verify (iii) of Proposition 2.2. Suppose X is an irreducible subvariety of $\mathcal{A}$ of dimension d. Let Δ be an open subset of S^{an} with $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$; we may shrink Δ subject to this condition. Let $Z = \overline{\iota(X)}$ and observe $\dim Z \leq d$.

Since $\omega = \iota^*\omega^{\mathrm{univ}}$, we have

$$\omega|^{\wedge d}_{X^{\mathrm{sm,an}}} \not\equiv 0 \quad \text{if and only if} \quad \omega^{\mathrm{univ}}|^{\wedge d}_{Z^{\mathrm{sm,an}}} \not\equiv 0. \tag{2.5}$$

Next by definition of b_Δ, we have the following property: For suitable non-empty open subsets Δ of S^{an} and Δ_0 of $\mathbb{A}_g^{\mathrm{an}}$ such that $\iota_S(\Delta) \subseteq \Delta_0$, we have

$$\max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x \leq \max_{x \in Z^{\mathrm{sm,an}} \cap \mathfrak{A}_{g,\Delta_0}} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta_0}|_{Z^{\mathrm{sm,an}}})_x \leq 2\dim Z \leq 2d. \tag{2.6}$$

Suppose first that $\omega|^{\wedge d}_{X^{\mathrm{sm,an}}} \not\equiv 0$, then (2.5) implies $\omega^{\mathrm{univ}}|^{\wedge d}_{Z^{\mathrm{sm,an}}} \not\equiv 0$ and in particular $d = \dim Z$. We can apply Proposition 2.2(ii) to Z and obtain $\max_{x \in Z^{\mathrm{sm,an}} \cap \mathfrak{A}_{g,\Delta_0}} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta_0}|_{Z^{\mathrm{sm,an}}})_x =$

$2d$. Now $\iota|_X\colon X\to Z$ is generically finite as $\dim X=\dim Z$, so the first inequality in (2.6) is an equality. We conclude

$$\max_{x\in X^{\mathrm{sm,an}}\cap\mathcal{A}_\Delta}\operatorname{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x=2d. \tag{2.7}$$

Conversely, assume (2.7) holds true. Then we have equalities throughout in (2.6). By Proposition 2.2(iii) applied to Z and by (2.5) we get $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d}\not\equiv 0$.

3. Setting-up and notations for the height inequality

In the next few sections we will prove Theorem 1.6. Let us first fix the setting.

All varieties are over an algebraically closed subfield k of $\mathbb{C}$. The ambient data is given as above Theorem 1.6. We repeat it here.

- Let S be a regular, irreducible, quasi-projective variety over k that is Zariski open in a regular, irreducible projective variety $\overline{S}\subseteq\mathbb{P}^m_k$.
- Let $\pi\colon\mathcal{A}\to S$ be an abelian scheme presented by a closed immersion $\mathcal{A}\to\mathbb{P}^n_k\times S$ over S.
- From the previous point, we get a closed immersion of the generic fiber A of $\mathcal{A}\to S$ into $\mathbb{P}^n_{k(S)}$. We assume that $A\to\mathbb{P}^n_{k(S)}$ arises from a basis of the global sections of the l-th power L of a symmetric ample line bundle with $l\geq 4$.
- Finally, we assume `(Hyp)` as on page 5.

If S is a regular, irreducible, quasi-projective variety of k, then $\overline{S}$ as in the first bullet point always exists due to Hironaka's Theorem. From the third bullet power, we see that the image of A is projectively normal in $\mathbb{P}^n_{k(S)}$, *cf.* [Mum70, Theorem 9]. By the fourth bullet point, Proposition 2.2 provides the Betti form ω on $\mathcal{A}^{\mathrm{an}}$.

For $s\in S(k)$ we write $\mathcal{A}_s$ for the abelian variety $\pi^{-1}(s)$.

Remark 3.1. *Let S be as in the first bullet point. Let $\pi\colon\mathcal{A}\to S$ be an abelian scheme. Suppose L_0 is a symmetric and ample line bundle on A, the generic fiber of π. An immersion of $\mathcal{A}$ as in the second bullet point can be obtained as follows. By [Ray70, Théorème XI 1.13] there exists an S-ample line bundle $\mathcal{L}$ on $\mathcal{A}$ whose restriction to the generic fiber of $\mathcal{A}\to S$ is isomorphic to $L_0^{\otimes l}$ for some integer $l\geq 4$. We may assume in addition that $\mathcal{L}$ satisfies $[-1]^*\mathcal{L}\cong\mathcal{L}$ and even that $\mathcal{L}$ becomes trivial when pulled back under the zero section $S\to\mathcal{A}$, see [Ray70, Remarque XI 1.3a]. After replacing $\mathcal{L}$ by a sufficiently high power, we may assume that $\mathcal{L}$ is very ample over S. We fix a basis of global sections of $L_0^{\otimes l}$ and, as $l\geq 4$, thereby realize the generic fiber of π as a projectively normal subvariety of $\mathbb{P}^n_{k(S)}$. Now we can take L in the third bullet point to be $L_0^{\otimes l}$, which is the restriction of $\mathcal{L}$ to A. A closed immersion $\mathcal{A}\to\mathbb{P}^n_k\times S$ as in the second bullet point arises as in [GW10, Summary 13.71] from $\mathcal{L}\otimes\pi^*\mathcal{M}$ for some very ample line bundle $\mathcal{M}$ on S. On restricting to a fiber of $\mathcal{A}\to S$ the induced closed immersion $\mathcal{A}_s\to\mathbb{P}^n_k$ comes from the restriction $\mathcal{L}|_{\mathcal{A}_s}$.*

Write $\overline{\mathcal{A}}$ for the Zariski closure of $\mathcal{A}$ in $\mathbb{P}^n_k\times\overline{S}$. Then $\overline{\mathcal{A}}$ is irreducible but not necessarily regular. On any product of r projective spaces and if $a_1,\ldots,a_r\in\mathbb{Z}$, we let $\mathcal{O}(a_1,\ldots,a_r)$ denote the tensor product over all $i\in\{1,\ldots,r\}$ of the pull-back under the i-th projection of $\mathcal{O}(a_i)$. We write $\mathcal{L}$ for the restriction of $\mathcal{O}(1,1)$ to $\overline{\mathcal{A}}$.

3.1. **Height functions on $\mathcal{A}$.** If $k = \overline{\mathbb{Q}}$ we have several height functions on $\mathcal{A}(\overline{\mathbb{Q}})$.

First for any $n \in \mathbb{N}$, we always consider the absolute logarithmic Weil height function $\mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \to \mathbb{R}$, or just Weil height, defined as in [BG06, §1.5.1].

Now say $P \in \mathcal{A}(\overline{\mathbb{Q}})$, we write $P = (P', \pi(P))$ with $P' \in \mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ and $\pi(P) \in \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$. By abuse of notation the sum of Weil heights

$$h(P) = h(P') + h(\pi(P)) \tag{3.1}$$

defines our first height $\mathcal{A}(\overline{\mathbb{Q}}) \to [0, \infty)$ which we call the *naive height* on $\mathcal{A}$. It depends on the fixed immersion of $\mathcal{A}$.

The line bundle $[-1]^*\mathcal{L}|_{\mathcal{A}} \otimes \mathcal{L}|_{\mathcal{A}}^{\otimes -1}$ of $\mathcal{A}$ restricted to the generic fiber A of $\mathcal{A} \to S$ equals $[-1]^*L \otimes L^{\otimes -1}$. By the third bullet point above this line bundle is trivial. So it equals $\pi^*\mathcal{K}$ for some line bundle $\mathcal{K}$ of S by [Gro67, Corollaire 21.4.13]. We conclude $[-1]^*\mathcal{L}|_{\mathcal{A}_s} \cong \mathcal{L}|_{\mathcal{A}_s}$ for all $s \in S(\overline{\mathbb{Q}})$. So the function (3.1) represents the height function, defined up-to $O(1)$, given by the Height Machine, *cf.* [BG06, Theorem 2.3.8], applied to $(\mathcal{A}_s, \mathcal{L}_s)$. As $\mathcal{L}_s$ is symmetric, the *fiberwise Néron–Tate* or *canonical height* $\hat{h}_{\mathcal{A}} \colon \mathcal{A}(\overline{\mathbb{Q}}) \to [0, \infty)$, defined by the convergent limit

$$\hat{h}_{\mathcal{A}}(P) = \lim_{N \to \infty} \frac{h([N](P))}{N^2}. \tag{3.2}$$

is a quadratic form on $\mathcal{A}_s(\overline{\mathbb{Q}})$. In the notation [BG06, Chapter 9] the height (3.2) is $\hat{h}_{\mathcal{A}_s, \mathcal{L}_s}$ where $s = \pi(P)$.

Remark 3.2. *We use here the notation of Remark 3.1. So that the immersion $\mathcal{A} \to \mathbb{P}^n_S$ arises via $L_0^{\otimes l}$. To normalize, we divide (3.2) by l and obtain the Néron–Tate height $\hat{h}_{\mathcal{A}, L_0} \colon \mathcal{A}(\overline{\mathbb{Q}}) \to [0, \infty)$.*

Let us verify that $\hat{h}_{\mathcal{A}, L_0}$ depends only on L_0. Suppose $\mathcal{L}'$ is another line bundle on $\mathcal{A}$ that restricts to $L_0^{\otimes l}$, then $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ is trivial on A. By [Gro67, Corollaire 21.4.13], this difference is the pull-back of some line bundle on S under $\mathcal{A} \to S$. So the restriction of $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ to $\mathcal{A}_s$ for each $s \in S(\mathbb{C})$ is trivial. Thus $\mathcal{L}|_{\mathcal{A}_s}$ and $\mathcal{L}'|_{\mathcal{A}_s}$ induce the same Néron–Tate height on $\mathcal{A}_s(\overline{\mathbb{Q}})$, see [BG06, §9.2].

3.2. **Integration against the Betti form.** Let $\mathcal{A}$ and S be as in the beginning of this section, so they are defined over an algebrically closed subfield k of $\mathbb{C}$. Recall that ω is the Betti form on $\mathcal{A}^{\mathrm{an}}$ as provided by Proposition 2.2. In particular, it is a semi-positive $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$ such that $[N]^*\omega = N^2\omega$ for all $N \in \mathbb{Z}$. In this section we discuss integrating against the Betti form.

Fix X to be an irreducible closed subvariety of $\mathcal{A}$ of dimension d, such that $\pi|_X \colon X \to S$ is dominant.

We are not allowed to integrate $\omega^{\wedge d}$ over $X^{\mathrm{sm,an}}$ as $\omega^{\wedge d}$ may not have compact support. So we modify ω in the following way.

Suppose we are provided with a base point $s_0 \in S^{\mathrm{an}}$. Let furthermore Δ be a relatively compact, contractible, open neighborhood of s_0 in S^{an}. Denote by $\mathcal{A}_\Delta$ the open subset $\pi^{-1}(\Delta)$ of $\mathcal{A}^{\mathrm{an}}$. Fix a smooth bump function $\vartheta \colon S^{\mathrm{an}} \to [0,1]$ with compact support $K \subseteq \Delta$ such that $\vartheta(s_0) = 1$. Finally, we define $\theta = \vartheta \circ \pi \colon \mathcal{A}^{\mathrm{an}} \to [0,1]$. Then $\theta\omega$ is a semi-positive smooth $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$; unlikely the Betti form, it may not be closed. By construction, the support of $\theta\omega$ lies in $\pi^{-1}(K)$ which is compact as π is proper and K is compact.

Remark 3.3. *Suppose X is non-degenerate, namely X satisfies one of the two equivalent conditions in property (iii) of Proposition 2.2. Then X^{an} contains a smooth point P_0 at which $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0$. Then we will take $s_0 = \pi(P_0)$.*

3.3. **The Graph Construction.** Let $N \in \mathbb{Z}$. The multiplication-by-N morphism $[N]\colon \mathcal{A} \to \mathcal{A}$ may not extend to a morphism $\overline{\mathcal{A}} \to \overline{\mathcal{A}}$. We overcome this by using the graph construction.

Recall that we have identified $\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq \mathbb{P}_k^n \times \overline{S} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$.

We write $\rho_1, \rho_2 \colon \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m \to \mathbb{P}_k^n \times \mathbb{P}_k^m$ for the two projections $\rho_1(P, Q, s) = (P, s)$ and $\rho_2(P, Q, s) = (Q, s)$.

Consider Γ_N the graph of $[N]$, determined by

$$\Gamma_N = \{(P, [N](P)) : P \in \mathcal{A}(k)\}.$$

We consider it as an irreducible closed subvariety of $\mathcal{A} \times_S \mathcal{A}$.

Let X be an irreducible closed subvariety of $\mathcal{A}$ of dimension d. The graph X_N of $[N]$ restricted to X is an irreducible closed subvariety of Γ_N determined by

$$\{(P, [N]P) : P \in X(k)\}.$$

Observe that $\rho_1|_{\Gamma_N} : \Gamma_N \to \mathcal{A}$ is an isomorphism; it maps $(P, [N](P))$ to P. So we can use $\rho_1|_{\Gamma_N}^{-1}$ to identify X with X_N.

Moreover, $\rho_2|_{\Gamma_N}$ maps $(P, [N](P))$ to $[N](P)$. Therefore

$$\rho_2|_{\Gamma_N} \circ \rho_1|_{\Gamma_N}^{-1} = [N]. \tag{3.3}$$

Let $\overline{X_N}$ be the Zariski closure of X_N in $\overline{\mathcal{A}} \times_{\overline{S}} \overline{\mathcal{A}} \subseteq \mathbb{P}_{\overline{S}}^n \times_{\overline{S}} \mathbb{P}_{\overline{S}}^n = \mathbb{P}_k^n \times \mathbb{P}_k^n \times \overline{S} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. Then $\overline{X_N}$ is an irreducible projective variety (which is not necessarily regular) with $\dim \overline{X_N} = \dim X_N = \dim X$.

In the next section, we will use the following line bundles on $\overline{X_N}$. Define

$$\mathcal{F} = \rho_2^* \mathcal{O}(1,1)|_{\overline{X_N}} = \mathcal{O}(0,1,1)|_{\overline{X_N}} \tag{3.4}$$

and

$$\mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}. \tag{3.5}$$

Let us close this subsection by relating the height functions defined by $\mathcal{F}$ and $\mathcal{M}$ with the ones in §3.1. Assume $k = \overline{\mathbb{Q}}$. Let $P \in X(\overline{\mathbb{Q}})$. Write $P = (P', \pi(P))$ with $P' \in \mathbb{P}_{\overline{\mathbb{Q}}}^n(\overline{\mathbb{Q}})$ and $\pi(P) \in \mathbb{P}_{\overline{\mathbb{Q}}}^m(\overline{\mathbb{Q}})$. We have $[N](P) = (P'_N, \pi(P))$ for some $P'_N \in \mathbb{P}_{\overline{\mathbb{Q}}}^n(\overline{\mathbb{Q}})$.

In view of the immersion $\overline{X_N} \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^m$, the point $(P, [N]P)$ in $\overline{X_N}(\overline{\mathbb{Q}})$ is $P_N = (P', P'_N, \pi(P)) \in (\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^m)(\overline{\mathbb{Q}})$. The function $P_N \mapsto h([N](P)) = h(P'_N) + h(\pi(P))$ defined in (3.1) represents the height attached by the Height Machine to $(\overline{X_N}, \mathcal{F})$ and $P_N \mapsto h(\pi(P))$ represents $(\overline{X_N}, \mathcal{M})$.

4. Intersection theory and height inequality on the total space

We keep the notation of §3. So we have a closed immersion $\mathcal{A} \to \mathbb{P}_k^n \times S$ over S satisfying the properties stated near the beginning of §3. Moreover, S is a Zariski open subset of an irreducible projective variety $\overline{S} \subseteq \mathbb{P}_k^m$. We assume in addition $k = \overline{\mathbb{Q}}$. Let X be a closed irreducible subvariety of $\mathcal{A}$ of dimension d defined over $\overline{\mathbb{Q}}$, such that $\pi|_X \colon X \to S$ is dominant. Let ω be the Betti form on $\mathcal{A}$ as defined in Proposition 2.2.

Proposition 4.1. *We keep the notation from above and suppose that X^{an} contains a smooth point at which $\omega|^{\wedge d}_{X^{\mathrm{sm,an}}} > 0$. Then there exists a constant $c_1 > 0$ satisfying the following property. Let $N \in \mathbb{N}$ be a power of 2, there exist a Zariski open dense subset U_N of X defined over $\overline{\mathbb{Q}}$ and a constant $c_2(N)$ such that*

$$h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N) \qquad \textit{for all } P \in U_N(\overline{\mathbb{Q}}).$$

4.1. **Bounding an Intersection Number from Below.** Let X be as in Proposition 4.1. For each $N \in \mathbb{N}$, let $\overline{X_N} \subseteq \mathbb{P}^n_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k$ be as in §3.3. In particular, $\dim \overline{X_N} = d$. Let $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$ be as in (3.4). The top self-intersection of $\mathcal{F}$ on $\overline{X_N}$ is bounded from below in the following proposition. To prove the next proposition we replace X by its base change to $k = \mathbb{C}$.

Proposition 4.2. *Suppose X^{an} contains a smooth point at which $\omega|^{\wedge d}_{X^{\mathrm{sm,an}}} > 0$. Then there exists a constant $\kappa > 0$, independent of N, such that $(\mathcal{F}^{\cdot d}) \geq \kappa N^{2d}$ for all $N \in \mathbb{N}$.*

Proof. We fix a point $P_0 \in X^{\mathrm{sm,an}}$ at which $\omega|^{\wedge d}_{X^{\mathrm{sm,an}}}$ is positive and let $s_0 = \pi(P_0), \Delta, \vartheta, \theta$, and K be as in §3.2, see Remark 3.3. We extend ϑ to a smooth function on $(\mathbb{P}^m_{\mathbb{C}})^{\mathrm{an}}$ by setting it 0 outside of the compact set $K \subseteq S^{\mathrm{an}}$. This extends $\theta = \vartheta \circ \pi$ to all of $(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})^{\mathrm{an}}$.

Let α be the pull-back of the Fubini–Study form under the analytification of the Segre morphism $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}} \to \mathbb{P}^{(n+1)(m+1)-1}_{\mathbb{C}}$. We replace α by its restriction to $\overline{\mathcal{A}}^{\mathrm{an}}$. Thus α represents the Chern class of $\mathcal{O}(1,1) \in \mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})$ restricted to $\overline{\mathcal{A}}$, using common notation.

Note that α is strictly positive on all of $\mathcal{A}^{\mathrm{an}}$. Since Δ is relatively compact we can find a constant $C > 0$ with

$$C\alpha|_{\mathcal{A}_\Delta} - \omega|_{\mathcal{A}_\Delta} \geq 0. \tag{4.1}$$

As the smooth and non-negative function $\theta = \vartheta \circ \pi$ on $\mathcal{A}^{\mathrm{an}}$ has support in $\pi^{-1}(K) \subseteq \pi^{-1}(\Delta) = \mathcal{A}_\Delta$ we have

$$C\theta\alpha - \theta\omega \geq 0.$$

We pull this $(1,1)$-form back under the holomorphic map $[N]\colon \mathcal{A}^{\mathrm{an}} \to \mathcal{A}^{\mathrm{an}}$ and get

$$C[N]^*(\theta\alpha) - N^2\theta\omega = C[N]^*(\theta\alpha) - [N]^*(\theta\omega) \geq 0 \tag{4.2}$$

where we used $[N]^*\omega = N^2\omega$ and $[N]^*\theta = \theta$; the former is a property of the Betti form, see Proposition 2.2(ii) and the latter holds as θ is the pull-back from the base of ϑ.

We define

$$\beta = C[N]^*(\theta\alpha) - N^2\theta\omega,$$

which is a $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$. It is semi-positive by (4.2). The support of θ is contained in $\pi^{-1}(K)$, which we have identified as compact above. So $C[N]^*(\theta\alpha)$ and $N^2\theta\omega$ have compact support on $\mathcal{A}^{\mathrm{an}}$.

We claim that $\int_{X^{\mathrm{sm,an}}} (C[N]^*(\theta\alpha))^{\wedge d} \geq \int_{X^{\mathrm{sm,an}}} (N^2\theta\omega)^{\wedge d}$.

First observe that both integrals are well-defined as both $[N]^*(\theta\alpha)$ and $N^2\theta\omega$ have compact support on $\mathcal{A}^{\mathrm{an}}$; this follows from work of Lelong [Lel57] which we use freely below. A textbook proof can be found in [Voi02, Theorem 11.21]. To prove the inequality let us write $\beta = \gamma - \delta$ with $\gamma = C[N]^*(\theta\alpha)$ and $\delta = N^2\theta\omega$. Then

(4.3)
$$\int_{X^{\mathrm{sm,an}}} \gamma^{\wedge d} - \int_{X^{\mathrm{sm,an}}} \delta^{\wedge d} = \int_{X^{\mathrm{sm,an}}} (\delta+\beta)^{\wedge d} - \int_{X^{\mathrm{sm,an}}} \delta^{\wedge d} = \sum_{i=0}^{d-1} \binom{d}{i} \int_{X^{\mathrm{sm,an}}} \delta^{\wedge i} \wedge \beta^{\wedge(d-i)}$$

as the exterior product is commutative on even degree forms. We know that $\beta \geq 0$ on $\mathcal{A}^{\mathrm{an}}$ and it is also crucial that $\delta \geq 0$ on $\mathcal{A}^{\mathrm{an}}$, the latter follows from $\omega \geq 0$, property (i) of Proposition 2.2, and from $\theta \geq 0$. Then $\delta^{\wedge i} \wedge \beta^{\wedge(d-i)}$ is semi-positive on $\mathcal{A}$; see [Dem12, Chapter III, Proposition 1.11][1]. Thus the right-hand side of (4.3) is non-negative and our claim is settled.

The claim implies

$$C^d \int_{X^{\mathrm{sm,an}}} [N]^*(\theta\alpha)^{\wedge d} \geq \kappa' N^{2d} \quad \text{where} \quad \kappa' = \int_{X^{\mathrm{sm,an}}} (\theta\omega)^{\wedge d}. \tag{4.4}$$

Next we want to relate the integral on the left in (4.4) with an intersection number. First we recall that $[N]$ is given in terms of the graph construction, *cf.* (3.3). So we may rewrite

$$\int_{X^{\mathrm{sm,an}}} [N]^*(\theta\alpha)^{\wedge d} = \int_{X^{\mathrm{sm,an}}} (\rho_2|_{\Gamma_N} \circ \rho_1|_{\Gamma_N}^{-1})^*(\theta\alpha)^{\wedge d} = \int_{X^{\mathrm{sm,an}}} (\rho_1|_{\Gamma_N}^{-1})^* \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d}, \tag{4.5}$$

here Γ_N, ρ_1, and ρ_2 are as defined in §3.3.

Because $\rho_1|_{\Gamma_N^{\mathrm{an}}} \colon \Gamma_N^{\mathrm{an}} \to \mathcal{A}^{\mathrm{an}}$ is biholomorphic we can change coordinates and integrate over X_N, which is a complex analytic subset of the graph Γ_N, itself a complex manifold. More precisely, we have

$$\int_{X^{\mathrm{sm,an}}} (\rho_1|_{\Gamma_N}^{-1})^* \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d} = \int_{X_N^{\mathrm{sm,an}}} \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d}. \tag{4.6}$$

Recall that α is the restriction to $\overline{\mathcal{A}}^{\mathrm{an}}$ of a $(1,1)$-form on $(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})^{\mathrm{an}}$. Moreover, ρ_2 is also defined on all of $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$. So $\rho_2|_{\Gamma_N}^*(\theta\alpha)$ is the restriction to Γ_N of a $(1,1)$-form defined on $(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})^{\mathrm{an}}$. Observe that $X_N^{\mathrm{sm,an}} \subseteq \overline{X_N}^{\mathrm{an}}$ and the difference has dimension strictly less than $d = \dim X_N$. This justifies the first inequality in

$$\int_{X_N^{\mathrm{sm,an}}} \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d} = \int_{\overline{X_N}^{\mathrm{an}}} \rho_2^*(\theta\alpha)^{\wedge d} \tag{4.7}$$

where we take $\overline{X_N}$ as a complex analytic subset of the analytification of $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$ and $\rho_2^*(\theta\alpha)$ as a $(1,1)$-form on this ambient space. Now θ takes values in $[0,1]$ and so

$$\int_{\overline{X_N}^{\mathrm{an}}} \rho_2^*(\theta\alpha)^{\wedge d} \leq \int_{\overline{X_N}^{\mathrm{an}}} (\rho_2^*\alpha)^{\wedge d}. \tag{4.8}$$

The pull-back $\rho_2^*\alpha$ represents $\rho_2^*\mathcal{O}(1,1) \in \mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})$ in the Picard group and has compact support as $(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})^{\mathrm{an}}$ is compact. But integration coincides with

[1]As our convention is somewhat different from Demailly's, let us explain how to apply [Dem12, Chapter III, Proposition 1.11]. Our definition of semi-positive $(1,1)$-form coincides with that of *positive* $(1,1)$*-form* of [Dem12, Chapter III] by Corollary 1.7 of *loc.cit.*, and thus are precisely the strongly positive $(1,1)$-forms of [Dem12, Chapter III] by Corollary 1.9 of *loc.cit.* Therefore we can apply the cited proposition.

the intersection pairing in the compact case; see [Voi02, Theorem 11.21]. In particular, we have

$$\int_{\overline{X_N}^{\mathrm{an}}} (\rho_2^*\alpha)^{\wedge d} = (\rho_2^*\mathcal{O}(1,1)^{\cdot d}[\overline{X_N}]) \tag{4.9}$$

where the intersection takes place in $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$. We recall (3.4) and apply the projection formula to obtain

$$(\rho_2^*\mathcal{O}(1,1)^{\cdot d}[\overline{X_N}]) = (\mathcal{O}(0,1,1)^{\cdot d}[\overline{X_N}]) = (\mathcal{F}^{\cdot d}). \tag{4.10}$$

The (in)equalities (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) yield

$$\int_{X^{\mathrm{sm,an}}} [N]^*(\theta\alpha)^{\wedge d} \leq (\mathcal{F}^{\cdot d}).$$

We recall the lower bound (4.4) and the definition (3.4) to obtain $(\mathcal{F}^{\cdot d}) \geq (\kappa'/C^d)N^{2d}$ where C comes from (4.1) and κ' comes from (4.4).The proposition follows with $\kappa = \kappa'/C^d$. □

4.2. **Bounding an Intersection Number from Above.** We keep the notation from the last subsection with $k = \overline{\mathbb{Q}}$. So X is as above Proposition 4.1 with $\dim X = d$ For each $N \in \mathbb{N}$, let $\overline{X_N} \subseteq \mathbb{P}^n_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k$ be the graph construction as in §3.3. In particular, $\dim \overline{X_N} = d$. Here we need $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$ as defined in (3.4) and also $\mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}$ as defined in (3.5).

Proposition 4.3. *Assume $d \geq 1$. There exists a constant $c > 0$ depending on the data introduced above with the following property. Say $N \geq 1$ is a power of 2, then*

$$(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)}) \leq cN^{2(d-1)}.$$

Let us make some preliminary remarks before the proof. A similar upper bound for the intersection number was derived by the third-named author in [Hab09, Hab13] using Philippon's version [Phi86] of Bézout's Theorem for multiprojective space. The approach here is similar but does not refer to Philippon's result. Rather, we rely on the following well-known positivity property of the intersection theory of multiprojective space: any effective Weil divisor on a multiprojective space is nef. This approach was motivated by Kühne's [Küh] work on semiabelian varieties.

Proof of Proposition 4.3. Recall that $[2]\colon A \to A$ is the multiplication-by-2 morphism on A. For the symmetric and ample line bundle L on A, we have $[2]^*L \cong L^{\otimes 4}$. Recall that A is projectively normal in $\mathbb{P}^n_{k(S)}$. By a result of Serre, [Wal87, Corollaire 2, Appendix II], the morphism [2] is represented by homogeneous polynomials $f_0, \ldots, f_n$ in the $n+1$ projective coordinates of $\mathbb{P}^n$ of degree 4, with coefficients in $k(S)$ and with no common zeros in A.

Recall that the family $\mathcal{A}$ is embedded in $\mathbb{P}^n_k \times S \subseteq \mathbb{P}^n_k \times \mathbb{P}^m_k$. We can spread out the $f_0, \ldots, f_n$. More precisely, there exist a Zariski closed, proper subset $Z \subsetneq \mathcal{A}$ and polynomials $f_0, \ldots, f_n \in k[\mathbf{S}, \mathbf{X}]$ that are bihomogeneous of degree $(D', 4)$ in the $(m+1)$-tuple of projective coordinates $\mathbf{S}$ of $\mathbb{P}^m_k$ and the $(n+1)$-tuple of projective coordinates $\mathbf{X}$ of $\mathbb{P}^n_k$, with the following properties:

(i) the polynomials $f_0, \ldots, f_n$ have no common zeros on $(\mathcal{A} \setminus Z)(k)$, and
(ii) if $(s, P) \in (\mathcal{A} \setminus Z)(k)$, then $[2](s, P) = \big(s, [f_0(s,P) : \cdots : f_n(s,P)]\big)$.

Moreover, as $f_0, \ldots, f_n$ have no common zero on the generic fiber, we may assume that $\pi(Z)$ is Zariski closed and proper in S. So we may assume that $Z = \pi^{-1}(\pi(Z)) \subsetneq \mathcal{A}$ and in particular, $[2]$ maps $\mathcal{A} \setminus Z$ to itself.

The degree 4 in the bidegree $(D', 4)$ comes from $2^2 = 4$. The degree D' with respect to the base coordinates $\mathbf{S}$ is more mysterious. However, by successively iterating we will get it under control.

For each integer $l \geq 1$ we require polynomials $f_0^{(l)}, \ldots, f_n^{(l)}$ to describe multiplication-by-2^l, *cf.* [GH19, §9]. In order to obtain information on the degree with respect to $\mathbf{S}$ we construct them by iterating the $f_0^{(1)} = f_0, \ldots, f_n^{(1)} = f_n$. For all $i \in \{0, \ldots, n\}$ we set

$$f_i^{(l+1)}(\mathbf{S}, \mathbf{X}) = f_i\left(\mathbf{S}, \left(f_0^{(l)}(\mathbf{S}, \mathbf{X}), \ldots, f_n^{(l)}(\mathbf{S}, \mathbf{X})\right)\right)$$

for all i; it is bihomogeneous in $\mathbf{S}$ and $\mathbf{X}$. So for all $l \geq 2$

(i) the polynomials $f_0^{(l)}, \ldots, f_n^{(l)}$ have no common zeros on $(\mathcal{A} \setminus Z)(k)$, and
(ii) if $(s, P) \in (\mathcal{A} \setminus Z)(k)$, then $[2^l](s, P) = \left(s, [f_0^{(l)}(s, P) : \cdots : f_n^{(l)}(s, P)]\right)$.

If for all i the polynomials $f_i^{(l)}$ are bihomogeneous of degree (D_l', D_l), then all $f_i^{(l+1)}$ are bihomogeneous of degree $(D' + 4D_l', 4D_l)$. This is the case for $l = 1$ with $(D_1', D_1) = (D', 4)$, thus it holds for all $l \geq 1$ with recurrence relations

$$D_{l+1}' = D' + 4D_l' \quad \text{and} \quad D_{l+1} = 4D_l.$$

Therefore,

$$D_l' = \frac{4^l - 1}{3} D' \leq 4^l D' \quad \text{and} \quad D_l = 4^l \tag{4.11}$$

for all $l \geq 1$.

Up-to the constant linear factor D' the bidegrees both grow at the same rate in l.

We proceed as follows to cut out the graph $\overline{X_N}$ where $N = 2^l$. We start out with $\overline{X} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$. As X dominates S but Z does not, there is an i such that $f_i^{(l)}$ does not vanish identically on X, without loss of generality we assume $i = 0$.

Then as j varies over $\{1, \ldots, n\}$ we obtain n trihomogeneous polynomials

$$g_j := Y_j f_0^{(l)}(\mathbf{S}, \mathbf{X}) - Y_0 f_j^{(l)}(\mathbf{S}, \mathbf{X})$$

where $Y_0, \ldots, Y_n$ are the projective coordinates on the middle factor of $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. The tridegree of these polynomials is $(D_l', 1, D_l)$. Their zero locus on $\overline{X} \times \mathbb{P}_k^n$ has the graph $\overline{X_N}$ as an irreducible component; by permuting coordinates we consider $\overline{X} \times \mathbb{P}_k^n$ as a subvariety of $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. Observe that $\dim \overline{X_N} = \dim X = \dim(\overline{X} \times \mathbb{P}_k^n) - n$, so $\overline{X_N}$ is a proper component in the intersection of $\overline{X} \times \mathbb{P}_k^n$ with the zero locus of $g_1, \ldots, g_n$. However, there may be further irreducible components in this intersection, some could even have dimension greater than $\dim X$.

This issue is clarified by the positivity result [Ful98, Corollary 12.2.(a)] which is applicable to $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ as the tangent bundle of a product of projective spaces is generated by its global sections, *cf.* [Ful98, Examples 12.2.1.(a) and (c)]. It follows that the cycle class attached to the intersection of $\overline{X} \times \mathbb{P}_k^n$ with the zero locus of $g_1, \ldots, g_n$ is represented by a positive cycle on $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$, one of whose components is $\overline{X_N}$. As

$\mathcal{O}(0,0,1)$ and $\mathcal{O}(0,1,1)$ are numerically effective we conclude

(4.12)
$$(\mathcal{O}(0,0,1)\mathcal{O}(0,1,1)^{\cdot(d-1)}[\overline{X_N}]) \leq \left(\mathcal{O}(0,0,1)\mathcal{O}(0,1,1)^{\cdot(d-1)}\mathcal{O}(D'_l,1,D_l)^{\cdot n}[\overline{X}\times\mathbb{P}^n_k]\right).$$

The cycle $[\overline{X}\times\mathbb{P}^n_k]$ is linearly equivalent to $\sum_{i+p=n+m-d} a_{ip}H_1^{\cdot i}H_2^{\cdot p}$, with H_1 and H_2 hyperplane sections of the factors $\mathbb{P}^n_k\times\mathbb{P}^m_k\supseteq\overline{X}$, respectively, and with a_{ip} non-negative integers that depend only on $\overline{X}$. Thus the left-hand side of (4.12) is at most

$$\sum_{i+p=n+m-d} a_{ip}\left(\mathcal{O}(0,0,1)\mathcal{O}(0,1,1)^{\cdot(d-1)}\mathcal{O}(D'_l,1,D_l)^{\cdot n}\mathcal{O}(1,0,0)^{\cdot i}\mathcal{O}(0,0,1)^{\cdot p}\right).$$

We can expand the sum using linearly of intersection numbers to find that it equals

$$\sum_{\substack{i+p=n+m-d\\ j'+p'=d-1\\ i''+j''+p''=n}} a_{ip}\binom{d-1}{j',p'}\binom{n}{i'',j'',p''}D_l'^{i''}D_l^{p''}\left(\mathcal{O}(1,0,0)^{\cdot(i+i'')}\mathcal{O}(0,1,0)^{\cdot(j'+j'')}\mathcal{O}(0,0,1)^{\cdot(1+p+p'+p'')}\right)$$

Only terms with $i+i''\leq n$ and $j'+j''\leq n$ and $1+p+p'+p''\leq m$ contribute to the sum. On the other hand, any term in the sum satisfies $i+i''+j'+j''+1+p+p'+p''=2n+m$. So we can assume $i+i''=n$ and $j'+j''=n$ and $1+p+p'+p''=m$ in the sum which thus equals

$$\sum_{\substack{i+p=n+m-d, i+i''=n\\ j'+p'=d-1, j'+j''=n\\ i''+j''+p''=n, p+p'+p''=m-1}} a_{ip}\binom{d-1}{j',p'}\binom{n}{i'',j'',p''}D_l'^{i''}D_l^{p''}.$$

Note $i''+p''=n-j''=j'=d-1-p'\leq d-1$. We recall (4.11) and conclude that the left-hand side of (4.12) is at most

$$(4.13)\qquad (4^lD')^{d-1}\sum_{\substack{i+p=n+m-d\\ j'+p'=d-1\\ i''+j''+p''=n}} a_{ip}\binom{d-1}{j',p'}\binom{n}{i'',j'',p''}\leq(4^lD')^{d-1}2^{d-1}3^n\sum_{i+p=n+m-d}a_{ip}.$$

We recall $N=2^l$ and use the projection formula with the estimates above to find

$$(\mathcal{O}(0,0,1)|_{\overline{X_N}}\mathcal{O}(0,1,1)|_{\overline{X_N}}^{\cdot(d-1)})\leq cN^{2(d-1)}$$

where $c>0$ depends only on X. Recall our definition $\mathcal{F}=\mathcal{O}(0,1,1)|_{\overline{X_N}}$ and $\mathcal{M}=\mathcal{O}(0,0,1)|_{\overline{X_N}}$. So we get $(\mathcal{M}\cdot\mathcal{F}^{\cdot(d-1)})\leq cN^{2(d-1)}$, as desired. □

4.3. **Proof of Proposition 4.1.** Now let us prove Proposition 4.1 by comparing the intersection number inequalities in Propositions 4.2 and 4.3.

Let X be of dimension d as in Proposition 4.1. The case $d=0$ is trivial. So we assume $d\geq1$. Take $l\in\mathbb{N}$, and let $N=2^l$. Let $\overline{X_N}\subseteq\mathbb{P}^n_k\times\mathbb{P}^n_k\times\mathbb{P}^m_k$ be as in §3.3. In particular $\dim\overline{X_N}=d$.

Let $\kappa>0$ be as in Proposition 4.2. Then $(\mathcal{F}^{\cdot d})\geq\kappa N^{2d}$. Let $c>0$ be as in Proposition 4.3. Then $(\mathcal{M}\cdot\mathcal{F}^{\cdot(d-1)})\leq cN^{2(d-1)}$. We have indicated how to obtain κ and c at the end of the proof of each one of the corresponding propositions.

Take any rational number c_1 whose denominator divides $q \in \mathbb{N}$ and such that

$$0 < c_1 c d < \kappa. \tag{4.14}$$

Using the bounds above and linearity of intersection numbers we get

$$d(\mathcal{M}^{\otimes q c_1 N^2} \cdot (\mathcal{F}^{\otimes q})^{\cdot (d-1)}) = q^d c_1 d N^2 (\mathcal{M} \cdot \mathcal{F}^{\cdot (d-1)}) \leq q^d c_1 d N^2 c N^{2(d-1)} < q^d \kappa N^{2d} \leq ((\mathcal{F}^{\otimes q})^{\cdot d}).$$

Then $\mathcal{F}^{\otimes q} \otimes \mathcal{M}^{\otimes -q c_1 N^2}$ is a big line bundle on $\overline{X_N}$ by a theorem of Siu [Laz04, Theorem 2.2.15]. So after possibly increasing q the line bundle $\mathcal{F}^{\otimes q} \otimes \mathcal{M}^{\otimes -q c_1 N^2}$ admits a non-zero global section. Say $h_{\overline{X_N}, \mathcal{F}}$ and $h_{\overline{X_N}, \mathcal{M}}$ are representives of heights on $\overline{X_N}$ attached by the Height Machine to $\mathcal{F}$ and $\mathcal{M}$, respectively. After canceling q we conclude that $h_{\overline{X_N}, \mathcal{F}} - c_1 N^2 h_{\overline{X_N}, \mathcal{M}}$ is bounded from below on a Zariski open and dense subset of $\overline{X_N}$. The image of this subset under the projection ρ_1 contains a Zariski open and dense subset U_N of X. It follows from the end of §3.3 that there exists $c_2(N)$ such that

$$h([N](P)) \geq c_1 N^2 h(\pi(P)) - c_2(N) \quad \text{for all } P \in U_N(\overline{\mathbb{Q}}). \qquad \square$$

5. Proof of the height inequality Theorem 1.6

We keep the notation of §3. In particular, S is a regular, irreducible, quasi-projective variety over $\overline{\mathbb{Q}}$ and $\pi \colon \mathcal{A} \to S$ is an abelian scheme of relative dimension $g \geq 1$. Moreover, we have immersions as in §3 and we assume `(Hyp)` from page 5. We use the heights introduced in §3.1. Let X be an irreducible closed subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$. We assume that X dominates S and is non-degenerate as defined in Definition 1.5

The upshot of `(Hyp)` is that we obtain from Proposition 2.2 the Betti form ω on $\mathcal{A}^{\mathrm{an}}$. Moreover, part (i) and (iii) of Proposition 2.2 implies that, for $d = \dim X$,

$$\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0 \quad \text{at some smooth point of } X^{\mathrm{an}}. \tag{5.1}$$

Our assumption (5.1) allows us to apply Proposition 4.1 to X. There exists a constant $c_1 > 0$ as in (4.14) such that the following holds. Let $N \in \mathbb{N}$ be a power of 2, there exists a Zariski open dense subset $U_N \subseteq X$ and a constant $c_2(N) \geq 0$ such that

$$h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N) \tag{5.2}$$

for all $P \in U_N(\overline{\mathbb{Q}})$; we stress that U_N and $c_2 \geq 0$ may depend on N in addition to $X, \mathcal{A}$, and the various immersions such as $\mathcal{A} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$.

By the Theorem of Silverman-Tate, see [Sil83, Theorem A] and Theorem A.1, there exist a constant $c_0 \geq 0$ such that

$$|\hat{h}_{\mathcal{A}}(P) - h(P)| \leq c_0 \max\{1, h(\pi(P))\} \leq c_0\big(1 + h(\pi(P))\big) \tag{5.3}$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$.

Next we kill Zimmer constants as in Masser's [Zan12, Appendix C]. For any $P \in U_N(\overline{\mathbb{Q}})$, we have

$$\begin{aligned}
\hat{h}_{\mathcal{A}}([N](P)) &\geq h([N](P)) - c_0\big(1 + h(\pi([N](P)))\big) && \text{(by (5.3))}\\
&= h([N](P)) - c_0\big(1 + h(\pi(P))\big) && \text{(as } \pi([N](P)) = \pi(P))\\
&\geq c_1 N^2 h(\pi(P)) - c_2(N) - c_0\big(1 + h(\pi(P))\big) && \text{(by (5.2)).}
\end{aligned}$$

But $\hat{h}_{\mathcal{A}}([N]P) = N^2\hat{h}_{\mathcal{A}}(P)$. Dividing by N^2 and rearranging yields

$$\hat{h}_{\mathcal{A}}(P) \geq \left(c_1 - \frac{c_0}{N^2}\right) h(\pi(P)) - \frac{c_2(N) + c_0}{N^2}$$

for all $N \in \mathbb{N}$ that are powers of 2 and all $P \in U_N(\overline{\mathbb{Q}})$.

Recall that c_0 and c_1 are independent of N. We fix $N \in \mathbb{N}$ to be the least power of 2 such that $N^2 \geq 2c_0/c_1$. As $h(\pi(P))$ is non-negative we get

$$\hat{h}_{\mathcal{A}}(P) \geq \frac{c_1}{2} h(\pi(P)) - \frac{c_2(N) + c_0}{N^2}$$

for all $P \in U_N(\overline{\mathbb{Q}})$. Since N is fixed now, the Zariski open dense subset U_N of X is also fixed. The proposition follows after adjusting c_1 and c_2. □

Remark 5.1. *In the proof of Theorem 1.6 we can keep track of the process to compute the constant $c_1 > 0$. Use the notation in §3. In particular ω is the Betti form on $\mathcal{A}$, we have an immersion $\mathcal{A} \subseteq \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$, α is a $(1,1)$-form on $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$ representing the Chern class of $\mathcal{O}(1,1)$, $\Delta \subseteq S^{\mathrm{an}}$ is open and relative compact, and $\theta \colon \mathcal{A}^{\mathrm{an}} \to [0,1]$ (which factors through S^{an}) is a smooth function with compact support contained in $\mathcal{A}_\Delta := \pi^{-1}(\Delta)$. The function θ should furthermore satisfy $\theta(P_0) = 1$ for some $P_0 \in X^{\mathrm{sm}}(\mathbb{C})$ such that $(\omega|_{X^{\mathrm{sm}}})^{\wedge d}$ is positive at P_0, where $d = \dim X$.*

Assume $d \geq 1$. The proof of Theorem 1.6 tells us that one half of any rational number satisfying the inequality (4.14) *can be taken as c_1. So the constant $c_1 > 0$ can be taken to be any rational number in $(0, \kappa/(2cd))$, such that:*

- *$\kappa = \kappa'/C^d$, where $\kappa' = \int_{X^{\mathrm{sm,an}}} (\theta\omega)^{\wedge d}$, as in (4.4), and C satisfies $C\alpha|_{\mathcal{A}_\Delta} - \omega|_{\mathcal{A}_\Delta} \geq 0$, as in (4.1),*
- *c is a constant depending on a certain degree of X and coming from (4.13).*

6. Preparation for counting points

6.1. **The universal family and non-degeneracy.** In this section, we fix the basic setting-up to prove Proposition 7.1, described as the alternative on page 4, and our main results.

Fix an integer $g \geq 2$. Recall from §1.2 that $\mathbb{M}_g$ denotes the fine moduli space of smooth curves of genus g, with level-ℓ-structure where $\ell \geq 3$ is fixed, *cf.* [ACG11, Chapter XVI, Theorem 2.11 (or above Proposition 2.8)], [DM69, (5.14)], or [OS80, Theorem 1.8]. Moreover, $\mathbb{M}_g$ is a regular, quasi-projective variety of dimension $3g-3$, we consider it over $\overline{\mathbb{Q}}$. It is irreducible by [DM69, Theorem 5.15]. There exists a universal curve $\mathfrak{C}_g$ over $\mathbb{M}_g$, *i.e.* it is smooth and proper over $\mathbb{M}_g$ and its fibers are smooth curves of genus g. Moreover, $\mathfrak{C}_g \to \mathbb{M}_g$ is projective, *cf.* [DM69, Corollary to Theorem 1.2] or [BLR90, Remark 2, §9.3].

Denote by $\mathrm{Jac}(\mathfrak{C}_g)$ the relative Jacobian of $\mathfrak{C}_g \to \mathbb{M}_g$. It is an abelian scheme coming with a natural principal polarization and equipped with level-ℓ-structure, see [MFK94, Proposition 6.9].

Recall from §1.2 that $\mathbb{A}_g$ denotes the fine moduli space of principally polarized abelian varieties of dimension g, with level-ℓ-structure. Moreover, $\mathbb{A}_g$ is regular and quasi-projective see [MFK94, Theorem 7.9 and below] or [OS80, Theorem 1.9]. We regard it as defined over $\overline{\mathbb{Q}}$; it is irreducible. Let $\pi \colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety;

it is an abelian scheme. Note that π is projective, we refer to Remark 3.1 for this and other details.

As $\mathbb{A}_g$ is a fine moduli space we have the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Jac}(\mathfrak{C}_g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{A}_g \arrow[d, "\pi"] \\ \mathbb{M}_g \arrow[r, "\tau"'] & \mathbb{A}_g \end{tikzcd}$$

the bottom arrow is the Torelli morphism. As we have level structure, the Torelli morphism need not be injective on $\mathbb{C}$-points, but it is finite-to-one on such points, *cf.* [OS80, Lemma 1.11]. Assume from now on that ℓ is even. There is a canonical symmetric line bundle $\mathcal{L}$ on $\mathfrak{A}_g$ that is ample relative to $\mathbb{A}_g$. For each $s \in \mathbb{A}_g(\overline{\mathbb{Q}})$, the line bundle $\mathcal{L}$ restricted to the abelian variety $\mathfrak{A}_{g,s} = \pi^{-1}(s)$ is ample. It is known that $[-1]^*\mathcal{L} = \mathcal{L}$; see [Pin89, Proposition 10.8 and 10.9].

We also fix an ample line bundle $\mathcal{M}$ on $\overline{\mathbb{A}_g}$, where $\overline{\mathbb{A}_g}$ is a, possibly non-regular, projective compactification of $\mathbb{A}_g$. The Height Machine provides an equivalence class of height functions of which we fix a representative $h_{\overline{\mathbb{A}_g},\mathcal{M}} \colon \overline{\mathbb{A}_g}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

For any morphism $S \to \mathbb{M}_g$ of schemes, define $\mathfrak{C}_S = \mathfrak{C}_g \times_{\mathbb{M}_g} S$.

Let $M \geq 1$ be an integer. We write $\mathfrak{A}_g^{[M]}$ for the M-fold fibered power $\mathfrak{A}_g \times_{\mathbb{A}_g} \cdots \times_{\mathbb{A}_g} \mathfrak{A}_g$ over $\mathbb{A}_g$. Then $\mathfrak{A}_g^{[M]} \to \mathbb{A}_g$ is an abelian scheme. Similarly, we define $\mathfrak{C}_g^{[M]}$ to be the M-fold fibered power $\mathfrak{C}_g \times_{\mathbb{M}_g} \cdots \times_{\mathbb{M}_g} \mathfrak{C}_g$. If $S \to \mathbb{M}_g$ is a morphism of schemes then we set $\mathfrak{C}_S^{[M]} = \mathfrak{C}_g \times_{\mathbb{M}_g} S$. If S is irreducible, then so is $\mathfrak{C}_S^{[M]}$ by induction on M and a topological argument using that $\mathfrak{C}_S \to S$ is smooth and hence open.

Next we fix a projective embedding of $\mathfrak{A}_g$. Recall that $\mathcal{L}$ is a symmetric line bundle on $\mathfrak{A}_g$ which is ample relative over $\mathbb{A}_g$. After replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes N}$, with $N \geq 4$ large enough, we can assume that $\mathcal{L}$ is symmetric and very ample relative over $\mathbb{A}_g$. Each fiber $\mathcal{L}_s$ is an ample line bundle on the abelian variety $\mathfrak{A}_{g,s}$, so $H^1(\mathfrak{A}_{g,s}, \mathcal{L}_s) = 0$ for all $s \colon \operatorname{Spec} k(s) \to \mathbb{A}_g$. Then $\pi_*\mathcal{L}$ is a locally free sheaf of finite type on $\mathbb{A}_g$ by [Mum74, Corollary II.5.2]. By the same reference $s^*\pi_*\mathcal{L}$ is naturally isomorphic to $H^0(\mathfrak{A}_{g,s}, \mathcal{L}_s)$ for all s. Consider the immersion $\mathfrak{A}_g \to \mathbb{P}(\pi_*\mathcal{L})$ over $\mathbb{A}_g$ induced by the natural homomorphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ which is surjective by [GW10, Proposition 13.56]. It is a closed immersion as π is proper. On each fiber we obtain a closed immersion $\mathfrak{A}_{g,s} \to \mathbb{P}(s^*\pi_*\mathcal{L}) \cong \mathbb{P}(H^0(\mathfrak{A}_{g,s}, \mathcal{L}_s))$; here the bundle $\mathcal{O}(1)$ on the projective space pulls back to $\mathcal{L}_s$. As $N \geq 4$ the abelian variety $\mathfrak{A}_{g,s}$ is projectively normal in $\mathbb{P}(H^0(\mathfrak{A}_{g,s}, \mathcal{L}_s))$.

In this way we obtain an immersion for each fiber of $\mathfrak{A}_g$ and a global immersion into some projective bundle. We would rather have a global immersion into a product of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ with a variety. There is a finite collection of Zariski open affine subsets $V_1, \ldots, V_t$ of $\mathbb{A}_g$ such that $\pi_*\mathcal{L}|_{V_i}$ is a free sheaf of modules for all $1 \leq i \leq t$. So $\mathbb{P}(\pi_*\mathcal{L}|_{V_i}) \cong \mathbb{P}^n_{\overline{\mathbb{Q}}} \times V_i$ by flat base change. Moreover, for each i we get, by restriction, a closed immersion $\mathfrak{A}_{V_i} = \pi^{-1}(V_i) \colon \mathfrak{A}_{V_i} \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times V_i$. Each fiber above $s \in V_i(\overline{\mathbb{Q}})$ is realized as a projectively normal subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$.

Recall that $\mathcal{L}$ is symmetric and very ample on each fiber of $\mathfrak{A}_g$. By Tate's Limit Argument we obtain the Néron–Tate height $\mathfrak{A}_{V_i}(\overline{\mathbb{Q}}) \to [0, \infty)$. These height functions match on the overlap and we obtain the fiberwise Néron–Tate height

$$\hat{h}\colon \mathfrak{A}_g(\overline{\mathbb{Q}}) \to [0,\infty). \tag{6.1}$$

It will be necessary to work with products. Let $i \in \{1,\dots,r\}$. By taking the product we obtain closed immersions $\mathfrak{A}_{V_i}^{[M]} = \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} V_i \to (\mathbb{P}^n_{\overline{\mathbb{Q}}})^M \times V_i$. The fiber of $\mathfrak{A}_{V_i}^{[M]} \to V_i$ above $s \in V_i(\overline{\mathbb{Q}})$ is the m-fold power of $\mathfrak{A}_{g,s}$. The associated fiberwise Néron–Tate height $\hat{h}\colon \mathfrak{A}_g^{[M]}(\overline{\mathbb{Q}}) \to [0,\infty)$ is the sum of the Néron–Tate heights, as in (6.1), of the m coordinates.

Finally, let us consider the morphism

$$\mathfrak{A}_g^{[M+1]} \to \mathfrak{A}_g^{[M]}$$

defined fiberwise over $\mathbb{A}_g$ by

$$(P_0, P_1, \dots, P_M) \mapsto (P_1 - P_0, \dots, P_M - P_0). \tag{6.2}$$

In fact, we will need it as a morphism on $\mathfrak{C}_g^{[M+1]}$. Note that there is no canonical embedding of $\mathfrak{C}_g$ in $\mathfrak{A}_g$ as we may have no section to construct the Abel–Jacobi map. However, we do have a morphism $\mathfrak{C}_g \to \mathrm{Pic}^1(\mathfrak{C}_g/\mathbb{M}_g)$ to the line bundles of degree 1; see the proof of [MFK94, Proposition 6.9]. The difference of two sections of $\mathrm{Pic}^1(\mathfrak{C}_g/\mathbb{M}_g)$ lies in $\mathrm{Pic}^0(\mathfrak{C}_g/\mathbb{M}_g) = \mathrm{Jac}(\mathfrak{C}_g)$. For each integer $M \geq 2$ the map (6.2) yields a well-defined morphism

$$\mathscr{D}_M\colon \mathfrak{C}_g^{[M+1]} \to \mathfrak{A}_g^{[M]} \tag{6.3}$$

called the *M-th Faltings–Zhang* morphism. By abuse of notation we also write $\mathscr{D}_M$ for the morphism $\mathfrak{C}_S^{[M+1]} \to \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S$ induced by the base change by $S \to \mathbb{M}_g \xrightarrow{\tau} \mathbb{A}_g$.

Let $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$. Then $\mathfrak{C}_s$ is the curve parametrized by s, and $\mathfrak{A}_{g,\tau(s)}$ is its Jacobian. To embed $\mathfrak{C}_s$ into $\mathfrak{A}_{g,\tau(s)}$ we must work with a base point $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$. Then $\mathfrak{C}_s - P = \mathscr{D}_1(\mathfrak{C}_s \times \{P\})$ is an irreducible curve inside $\mathfrak{A}_g$ lying above $\tau(s)$.

Say $i \in \{1,\dots,t\}$ such that $\tau(s) \in V_i(\overline{\mathbb{Q}})$. Then the construction above provides a closed immersion $\mathfrak{C}_s - P \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}}$.

Let $\deg X$ denote the degree of an irreducible closed subvariety X of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ and let $h(X)$ denote its height, *cf.* [BGS94].

Lemma 6.1. *There exists a constant c depending on choices made above such that if $i \in \{1,\dots,t\}$ and $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ with $\tau(s) \in V_i$, then there exists $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ such that*

$$\deg(\mathfrak{C}_s - P_s) \leq c \quad \text{and} \quad h(\mathfrak{C}_s - P_s) \leq c \max\{1, h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))\}.$$

Proof. We need a quasi-section of $\mathfrak{C}_g \to \mathbb{M}_g$ as provided by [Gro67, Corollaire 17.16.3(ii)]. So there is an affine scheme S and a morphism $S \to \mathfrak{C}_g$ that factors through a surjective, quasi-finite, étale morphism $S \to \mathbb{M}_g$. We consider the product $\mathfrak{C}_g \times_{\mathbb{M}_g} S \to \mathfrak{C}_g^{[2]}$ composed with the Faltings–Zhang morphism $\mathscr{D}_1$, this is a morphism $\mathfrak{C}_g \times_{\mathbb{M}_g} S \to \mathfrak{A}_g$. Its image is a constructible subset of $\mathfrak{A}_g$. By considering the Zariski closure of the image we find a finite set of irreducible closed subvarieties $\{X_i\}_i$ of $\mathfrak{A}_g$ with the following property.

Given a point $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$, there is an i such that the fiber of $\pi|_{X_i}\colon X_i \to \mathbb{A}_g$ above $\tau(s)$ is a finite union of irreducible curves, one being $\mathfrak{C}_s - P_s$ with $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

We have a closed immersion $\mathfrak{A}_g \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$. Moreover, we can assume $\mathbb{A}_g$ is locally closed in $\mathbb{P}^m_{\overline{\mathbb{Q}}}$ for some m. We identify with X its image in $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$, a locally Zariski closed set. Of course, everything is contain in $\mathbb{P}^{(n+1)(m+1)-1}_{\overline{\mathbb{Q}}}$ by Segre. Then $\mathfrak{C}_s - P_s \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}}$ arises as an irreducible component of the intersection of X with $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times \{\tau(s)\}$.

By Bézout's Theorem, $\deg(\mathfrak{C}_s - P_s)$ is bounded from above uniformly in s. This yields (i).

For part (ii) we remark that by the Height Machine we can bound the absolute logarithmic Weil height $h(\tau(s))$, where $\tau(s)$ is understood as an element of $\mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$, from above linearly in terms of $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))$. Part (ii) follows as (i) but with using the Arithmetic Bézout Theorem. $\square$

The following non-degeneracy theorem proved by the second-named author is crucial to prove our main result. It confirms that Theorem 1.6 can be applied to $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$, for $M \geq 3g-2$, so that we obtain the height inequality on a Zariski open dense subset.

Theorem 6.2 ([Gao18, Theorem 1.2']). *Let S be an irreducible variety with a (not necessarily dominant) finite morphism $S \to \mathbb{M}_g$. Assume $g \geq 2$ and $M \geq 3g-2$. Then $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$, which is an irreducible subvariety of the abelian subscheme $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S \to S$, is non-degenerate in the sense of Definition 1.5.*

The fibered product in the theorem involves $S \to \mathbb{M}_g \xrightarrow{\tau} \mathbb{A}_g$.

6.2. **Technical lemmas.** The following lemma will be useful in the proofs of the desired bounds. Let for the moment k be an algebraically closed field and integers $M \geq 1, n \geq 1$. If Z is a Zariski closed subset of $(\mathbb{P}^n_k)^M$ we let $\deg Z$ denote the sum of the degrees of all irreducible components of Z with respect to $\mathcal{O}(1,\ldots,1)$.

Lemma 6.3. *Let $C \subseteq \mathbb{P}^n_k$ be a irreducible curve defined over k and let $Z \subseteq (\mathbb{P}^n_k)^M$ be a Zariski closed subset of $(\mathbb{P}^n_k)^M$ such that $C^M = C \times \cdots \times C \not\subseteq Z$. Then there exists a number B, depending only on $M, \deg C$, and $\deg Z$, satisfying the following property. If $\Sigma \subseteq C(k)$ has cardinality $\geq B$, then $\Sigma^M = \Sigma \times \cdots \times \Sigma \not\subseteq Z(k)$.*

Proof. Let us prove this lemma by induction on M. The case $M = 1$ follows easily from Bézout's Theorem, [Ful98, Example 8.4.6].

Assume the lemma is proved for $1, \ldots, M-1$. Let $q\colon (\mathbb{P}^n_k)^M \to \mathbb{P}^n_k$ be the projection to the first factor.

The number of irreducible components of $Z \cap C^M$ and their degrees are bounded from above in terms of $M, \deg C$, and $\deg Z$ by Bézout's Theorem. Let Z' be the union of all irreducible components Y of $Z \cap C^M$ with $\dim q(Y) \geq 1$, let Z'' be the union of all other irreducible components.

Note that $q(Z') \subseteq C$. For all $P \in C(k)$ the fiber $q|_{Z'}^{-1}(P) = Z' \cap (\{P\} \times (\mathbb{P}^n_k)^{M-1})$ has dimension at most $\dim Z' - 1 \leq M-2$. So the projection of $q|_{Z'}^{-1}(P)$ to the final factors $(\mathbb{P}^n_k)^{M-1}$ does not contain C^{M-1}. By Bézout's Theorem the degree of this projection is bounded in terms of $M, \deg C$, and $\deg Z$. We apply the induction hypothesis to the projection of $q|_{Y}^{-1}(P)$ to $(\mathbb{P}^n_k)^{M-1}$ and obtain a number B', depending only on $M, \deg C$, and $\deg Z$ satisfying the following property. If $\Sigma \subseteq C(k)$ has cardinality $\geq B'$, then $\{P\} \times \Sigma^{M-1} \not\subseteq Z'(k)$ for all $P \in C(k)$.

Now $\dim q(Z'') = 0$, so $q(Z'')$ is a finite set of cardinality at most B'', the number of irreducible components of $Z \cap C^M$.

The lemma follows with $B = \max\{B', B''+1\}$. □

In the next lemma we use the Faltings–Zhang morphism in a single abelian variety.

Lemma 6.4. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$ and suppose C is a smooth curve of genus $g \geq 2$ contained in A. If Z is irreducible and Zariski closed in A^M, then*

$$\#\{P \in C(\overline{\mathbb{Q}}) : (C-P)^M \subseteq Z \subsetneq \mathscr{D}_M(C^{M+1})\} \leq 84(g-1).$$

Proof. For simplicity denote by $\Xi = \{P \in C(\overline{\mathbb{Q}}) : (C-P)^M \subseteq Z \subsetneq \mathscr{D}_M(C^{M+1})\}$.

Fix $P_0 \in \Xi$. It suffices to prove that there are only $84(g-1)$ possibilities for $P_1 - P_0$ when P_1 runs over Ξ.

Say $P_1 \in \Xi$. Since $\dim(C-P_i)^M = M$ and as $\mathscr{D}_M(C^{M+1})$ is irreducible of dimension at most $M+1$, we must have $(C-P_i)^M = Z$ for $i=0$ and $i=1$.

By projecting to A we find $C - P_1 = C - P_0$. In other words, $P_1 - P_0$ stabilizes C. By Hurwitz's Theorem [Hur92], a smooth curve of genus $g \geq 2$ has at most $84(g-1)$ automorphisms. Hence we are done. □

7. Néron–Tate distance between points on curves

The goal of this section is to prove Proposition 7.1, below. Namely we will show that $\overline{\mathbb{Q}}$-points on smooth curves are rather "sparse", in the sense that the Néron–Tate distance between two $\overline{\mathbb{Q}}$-points on a smooth curve C is in general large compared with the Weil height of C.

We use the notation from §6.1. Recall that we have fixed a projective compactification $\overline{\mathbb{A}_g}$ of $\mathbb{A}_g$ over $\overline{\mathbb{Q}}$, an ample line bundle $\mathcal{M}$, and a height function $h_{\overline{\mathbb{A}_g},\mathcal{M}} \colon \mathbb{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$ attached to this pair. If $s \in \mathbb{A}_g(\overline{\mathbb{Q}})$ then $\mathfrak{C}_s$ is a smooth curve of genus g defined over $\overline{\mathbb{Q}}$. Moreover, if $P, Q \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then $P - Q$ is a well-defined element of $\mathfrak{A}_g(\overline{\mathbb{Q}})$ and so is its Néron–Tate height $\hat{h}(P-Q)$, see (6.1).

Proposition 7.1. *Let S be an irreducible closed subvariety of $\mathbb{M}_g$. There exist positive constants c_1, c_2, c_3, c_4 depending on the choices made above and on S with the following property. Let $s \in S(\overline{\mathbb{Q}})$ with $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \geq c_1$. There exists a subset $\Xi_s \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\#\Xi_s \leq c_2$ such that any $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ satisfies the following alternative.*

(i) *Either $P \in \Xi_s$;*
(ii) *or $\#\{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}(Q-P) \leq h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))/c_3\} < c_4$.*

Proof. We prove Proposition 7.1 by induction on $\dim S$.

If $\dim S = 0$, then the proposition follows by enlarging c_1.

If $\dim S \geq 1$, we fix $m = 3g-2$. Let S^o denote the regular locus of S. By Theorem 6.2, we can apply Theorem 1.6 to the irreducible subvariety $\mathscr{D}_M(\mathfrak{C}_{S^o}^{[M+1]})$ of $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S^o$ using the morphism (6.3). Thus there exist constants $c > 0$ and c' as well as a Zariski open dense subset U of $X = \mathscr{D}_M(\mathfrak{C}_{S^o}^{[M+1]})$, satisfying the following property. For all $s \in S^o(\overline{\mathbb{Q}})$ and all $P, Q_1, \ldots, Q_M \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, we have

$$c h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \leq \hat{h}(Q_1-P)+\cdots+\hat{h}(Q_M-P)+c' \quad \text{if } (Q_1-P,\ldots,Q_M-P) \in U(\overline{\mathbb{Q}}). \tag{7.1}$$

Observe that $\pi(X) = S^o$, where $\pi\colon \mathfrak{A}_g^{[M]} \times_{\mathbb{M}_g} S \to S$ is the structure morphism. Therefore, $S\setminus\pi(U)$ is not Zariski dense in S. Let $S'_1,\ldots,S'_r$ be the irreducible components of the Zariski closure of $S\setminus\pi(U)$ in S. Then $\dim S'_j \le \dim S - 1$ for all j.

By induction hypothesis, this proposition holds for all S'_j. Thus it remains to prove the conclusion of this proposition for curves above

$$s \in S(\overline{\mathbb{Q}}) \setminus \bigcup_{j=1}^{r} S'_j(\overline{\mathbb{Q}}) \subseteq \pi(U(\overline{\mathbb{Q}})). \tag{7.2}$$

First we construct Ξ_s and then we will show that we are in one of the two alternatives.

It is convenient to fix a base point $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ and consider $\mathfrak{C}_s - P_s$ as a curve inside $(\mathfrak{A}_g)_s$ via the Abel–Jacobi map.

Let us set $W = X\setminus U$, it is a Zariski closed and proper subset of X. By (7.2) we find $W_s \subsetneq X_s = \mathscr{D}_M(\mathfrak{C}_s^{[M+1]})$.

Let Z be an irreducible component of W_s. Consider the set

$$\Xi_Z := \{P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : (\mathfrak{C}_s - P)^M \subseteq Z\}.$$

Apply Lemma 6.4 to $A = (\mathfrak{A}_g)_s, C = \mathfrak{C}_s - P_s \subseteq A$, and Z. As $Z \subsetneq \mathscr{D}_M(\mathfrak{C}_s^{[M+1]})$ we have $\#\Xi_Z \le 84(g-1)$.

Let $\Xi_s = \bigcup_Z \Xi_Z$ where Z runs over all irreducible components of W_s. The number of irreducible components of W_s is bounded from above by a constant that is independent of s; but it may depend on W. We take c_2 to be such a bound multiplied with $84(g-1)$. Thus $\#\Xi_s \le c_2$ if (7.2) and with c_2 independent of s.

Say $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ and $P \notin \Xi_s$. So we are not in case (i) of the proposition. Then $(\mathfrak{C}_s - P)^M \not\subseteq W_s$. We want to apply Lemma 6.3 to $\mathfrak{C}_s - P$ and W_s.

Recall that $V_1,\ldots,V_t$ are Zariski open affine subsets that cover $\mathbb{A}_g$. The image of $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ under the Torelli morphism is $\tau(s) \in \mathbb{A}_g(\overline{\mathbb{Q}})$. Say $\tau(s) \in V_i(\overline{\mathbb{Q}})$ for some i. Recall that the abelian scheme $\mathfrak{A}_g$ is embedded in $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times V_i$ over V_i, *cf.* §6.1. We may take $\mathfrak{C}_s - P$ as a smooth curve in $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. The degree of $\mathfrak{C}_s - P$ as a subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ is bounded from above independently of s by Lemma 6.1. Moreover, W_s is Zariski closed in $X_s \subseteq \mathfrak{A}^{[M]}$. Still holding s fixed we may take W_s as a Zariski closed subset of $(\mathbb{P}^n_{\overline{\mathbb{Q}}})^M$. Being the fiber above s of a subvariety of $(\mathbb{P}^n_{\overline{\mathbb{Q}}})^M \times \mathbb{A}_g$, we find that the degree of W_s is bounded from above independently of s. From Lemma 6.3 we thus obtain a number c_4, depending only on these bounds and with the following property. Any subset $\Sigma \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with cardinality $\ge c_4$ satisfies $(\Sigma - P)^M \not\subseteq W_s$. It is crucial that c_4 is independent of s.

We work with $\Sigma = \{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}(Q-P) \le h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))/c_3\}$ with $c_3 = 2M/c$. If $\#\Sigma < c_4$, then we are in alternative (ii) of the proposition.

If $\#\Sigma \ge c_4$ we will show $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) < c_1$ with $c_1 = 3c'/c$. Indeed, the discussion above implies that there exist $Q_1,\ldots,Q_M \in \Sigma$ such that $(Q_1 - P,\ldots,Q_M - P) \in U(\overline{\mathbb{Q}})$. Thus we can apply (7.1) and obtain

$$h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \le \frac{1}{c}\left(M\frac{c h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))}{2M} + c'\right) = \frac{1}{2}h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) + \frac{c'}{c}. \tag{7.3}$$

Then (7.3) implies $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \le 2c'/c$. □

8. Proof of Theorems 1.1, 1.2, and 1.4

The goal of this section is to prove the theorems and the corollary in the introduction. To this end let $g \geq 2$; we retain the notation of §6.1. In particular, $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ is the universal family of principally polarized abelian varieties of dimension g with level-ℓ-structure where $\ell \geq 3$.

Proposition 8.1. *The exist constants $c_1 \geq 0, c_2 \geq 1$ depending on the choices made above with the following property. Let $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ with $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \geq c_1$. Suppose Γ is a finite rank subgroup of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ with rank $\rho \geq 0$. If $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then*

$$\#(\mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c_2^{1+\rho}.$$

The proof combines Vojta's approach to the Mordell Conjecture with the results obtain in §7. We will use Rémond's quantitative version [Rém00a, Rém00b] of Vojta's method. A similar approach was used in the authors's earlier work [DGH19] which also contains a review of Vojta's method in §2. Let us recall the fundamental facts before proving Proposition 8.1.

Suppose we are given an abelian variety A of dimension g that is defined over $\overline{\mathbb{Q}}$ that is presented with a symmetric and very ample line bundle L. We assume also that we have a closed immersion of A into some projective space $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ determined by a basis of the global sections of L. We assume that A becomes a projectively normal subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. This is the case if L is an at least fourth power of a symmetric and ample line bundle.

Suppose C is an irreducible curve in A. Then let $\deg C$ denote the degree of C considered as subvariety of $A = \mathbb{P}^n_{\overline{\mathbb{Q}}}$, *i.e.*, $\deg C = (C.L)$. Moreover, let $h(C)$ denote the height of C.

On the ambient projective space we have the Weil height $h\colon \mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \to [0,\infty)$. Tate's Limit Argument, compare (3.2), applied to h yields the Néron–Tate height $\hat{h}_L\colon A(\overline{\mathbb{Q}}) \to [0,\infty)$. It vanishes precisely on the points of finite order. Moreover, it follows from Tate's construction that there exists a constant $c_{\mathrm{NT}} \geq 0$ such that

$$|\hat{h}_L(P) - h(P)| \leq c_{\mathrm{NT}}$$

for all $P \in A(\overline{\mathbb{Q}})$.

Finally, we need a measure for the heights of homogeneous polynomials that define the addition and substraction on A, as required in Rémond's [Rém00b]. Consider the $n+1$ global sections of $\mathcal{O}(1)$ corresponding to the projective coordinates of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. They restrict to global sections $\xi_0,\ldots,\xi_n$ of L on A. Let $f\colon A\times A \to A\times A$ denote the morphism induced by $(P,Q)\mapsto(P+Q,P-Q)$, and let $p_1,p_2\colon A\times A\to A$ be the first and section projection, respectively. For all $i,j \in \{0,\ldots,n\}$ there are $P_{ij} \in \overline{\mathbb{Q}}[\mathbf{X},\mathbf{X}']$ with

$$f^*(p_1^*\xi_i \otimes p_2^*\xi_j) = P_{ij}((p_1^*\xi_0,\ldots,p_1^*\xi_n),(p_2^*\xi_0,\ldots,p_2^*\xi_n)) \tag{8.1}$$

and where P_{ij} is bihomogeneous of bidegree $(2,2)$ in $\mathbf{X} = (X_0,\ldots,X_n)$ and $\mathbf{X}' = (X_0',\ldots,X_n')$; see the proof of [Rém00b, Proposition 5.2] for the existence of the P_{ij}. Here we require that $\xi_0,\ldots,\xi_n$ constitute a basis of $H^0(A,L)$. Let h_1 denote the Weil height of the point in projective space whose coordinates are all coefficients of all P_{ij}.

In [DGH19, §2] h_1 must also involve both addition and subtraction on A, and not just the addition.

The lemma below is [DGH19, Corollary 2.3] which is itself a standard application of Rémond's explicit formulation of the Vojta and Mumford inequalities. We thus obtain a bound that is exponential in the rank of the subgroup Γ for points of sufficiently large Néron–Tate height.

Lemma 8.2. *There exists a constant $c = c(n, \deg C) \geq 1$ depending only on n and $\deg C$ with the following property. Suppose Γ is a subgroup of $A(\overline{\mathbb{Q}})$ of finite rank $\rho \geq 0$. If C is not the translate of an algebraic subgroup of A, then*

$$\#\left\{P \in C(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}_L(P) > c\max\{1, h(C), c_{\mathrm{NT}}, h_1\}\right\} \leq c^\rho.$$

Proof of Proposition 8.1. As in §6.1 we have finitely many Zariski open affine subsets $V_1, \ldots, V_t$ that cover $\mathbb{A}_g$. Moreover, we have a closed immersion $\mathfrak{A}_{V_i} = \pi^{-1}(V_i) \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times V_i$. Let $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ with

$$h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \geq c_1, \tag{8.2}$$

where c_1 comes from Proposition 7.1.

Fix $i \in \{1, \ldots, t\}$ such that $\tau(s) \in S_i(\overline{\mathbb{Q}})$. We now bound two quantities attached to the abelian variety $A = \mathfrak{A}_{g,\tau(s)}$ taken with its closed immersion into $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. Observe that this closed immersion satisfies the condition imposed at the beginning of this section with $L = \mathcal{L}|_A$ where $\mathcal{L}$ was construction in §6.1. These quantities may depend on s. Below, $c > 0$ denotes a constant that depends on the fixed data such as g, m, and the ambient objects such as $\mathfrak{A}_g$ but not on s. We will increase c freely and without notice.

Bounding c_{NT}. For this we require the Silverman–Tate Theorem, Theorem A.1, applied to $\pi|_{V_i} \colon \mathfrak{A}_{V_i} \to V_i$. Recall that $h(\cdot)$ is the Weil height on $\mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ we may use it in lieu of $h_{\overline{\mathcal{A}},\overline{\mathcal{L}}}$ in Theorem A.1. For all $P \in A(\overline{\mathbb{Q}})$ we have $|h(P) - \hat{h}(P)| \leq c\max\{1, h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))\}$. So we may take

$$c_{\mathrm{NT}} = c\max\{1, h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))\}. \tag{8.3}$$

Bounding h_1. Recall that $f \colon A^2 \to A^2$ sends (P, Q) to $(P + Q, P - Q)$. We will construct P_{ij} as above. To this end we consider points $P = [\zeta_0 : \cdots : \zeta_n], Q = [\eta_0 : \cdots : \eta_n] \in A(\overline{\mathbb{Q}})$. Then $f(P, Q) = ([\nu_0^+ : \cdots : \nu_n^+], [\nu_0^- : \cdots : \nu_n^-])$. Recall that $A = \mathfrak{A}_{g,s}$ is presented as a projectively normal subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ by the construction in §6.1. By (8.1) there is for each $i, j \in \{0, \ldots, n\}$ a bihomogeneous polynomial P_{ij} of bidegree $(2, 2)$ that is independent of P and Q, with

$$\nu_i^+\nu_j^- = \lambda P_{ij}((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_n)) \tag{8.4}$$

for some non-zero $\lambda \in \overline{\mathbb{Q}}$ that may depend on (P, Q). We eliminate λ and consider

$$\nu_i^+\nu_j^- P_{i'j'}((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_n)) - \nu_{i'}^+\nu_{j'}^- P_{ij}((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_n)) = 0 \tag{8.5}$$

as a system of homogeneous linear equations parametrized by $(i, j), (i', j') \in \{0, \ldots, n\}^2$, the unknowns are the coefficients of the P_{ij}. As each P_{ij} is bihomogeneous of bidegree $(2, 2)$, the number of unknowns is $N = (n + 1)^2\binom{n+2}{2}^2$ which is independent of s.

Each pair of points $(P,Q) \in A(\overline{\mathbb{Q}})^2$ yields one system of linear equations. We know that there is a non-trivial solution $(P_{ij})_{ij}$ that solves for all (P,Q) simultaneously and such that some P_{ij} does not vanish identically on $A \times A$. Our goal is to find such a common solution of controlled height.

First, observe that a common solution for when (P,Q) runs over all torsion points of $A^2(\overline{\mathbb{Q}})$ is a common solution for all pairs (P,Q). Indeed, this follows as torsion points of $A^2(\overline{\mathbb{Q}})$ lie Zariski dense in A^2. Second, observe that the full system has finite rank $M < N$ so it suffices to consider only finite many torsion points (P,Q).

Our task is thus to find a common solution to (8.5) for all $(i,j),(i',j')$, where some P_{ij} does not vanish identically on $A \times A$, and where $[\zeta_0 : \cdots : \zeta_n]$,$[\eta_0 : \cdots : \eta_n]$, and $[\nu_0^{\pm} : \cdots : \nu_n^{\pm}]$ are certain torsion points on $A(\overline{\mathbb{Q}})$. We may assume that some ζ_i is 1 and similarly for η_i and $\nu_i^{\pm}$. So all coordinates are in $\overline{\mathbb{Q}}$. Moreover, the height of each torsion point is at most c_{NT}. The resulting system of linear equations is represented by an $M \times N$ matrix with algebraic coefficients. By elementary properties of the height, each coefficient in the system has affine Weil height $c \cdot c_{\mathrm{NT}}$. It is tempting, but unnecessary, to invoke Siegel's Lemma to find a non-trivial solution. As M, N are bounded in terms of n, Cramer's Rule establishes the existence of a basis of non-zero solution such that the Weil height of the coefficient vector is at most $c \cdot c_{\mathrm{NT}}$. Among this basis there is one solution where one P_{ij} does not vanish identically on $A \times A$. By (8.3) we find

$$h_1 \leq c \max\{1, h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))\}$$

for the projective height of the tuple $(P_{ij})_{ij}$. Thus (8.5) holds and so we get (8.4) on all of $A^2(\overline{\mathbb{Q}})$, at least with λ a rational function on $A \times A$ that is not identically zero. But λ cannot vanish anywhere on $A \times A$, as otherwise the left-hand side of (8.4) would vanish at some point of $A \times A$ for all (i,j). Hence λ is a non-zero constant. Replacing P_{ij} by λP_{ij} does not change the projective height; we get (8.1) with the desired bound for h_1.

Bounding height and degree of a curve. By Lemma 6.1 we have

$$\deg(\mathfrak{C}_s - P_s) \leq c \quad \text{and} \quad h(\mathfrak{C}_s - P_s) \leq c \max\{1, h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))\} \tag{8.6}$$

for some $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

We now follow the argumentation in [DGH19]. Let Γ be a subgroup of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ for finite rank ρ. We first prove the proposition in the case $P_0 = P_s$. We apply Lemma 8.2 to the curve $C = \mathfrak{C}_s - P_s \subseteq \mathfrak{A}_{g,\tau(s)} = A$ and use the bounds (8.3), (8.5), and (8.6). Note that C is a smooth curve of genus $g \geq 2$. So it cannot be the translate of an algebraic subgroup of A. It follows that the number of points $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $P - P_s \in \Gamma$ and $\hat{h}(P - P_s) > R^2$ where

$$R = (c \max\{1, h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))\})^{1/2} \tag{8.7}$$

is at most $c^{\rho} \leq c^{1+\rho}$.

The burden of this paper is to find a bound of the same quality for the number of distinct points $P_1, P_2, P_3, \ldots$ in $\mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\hat{h}(P_i - P_s) \leq R^2$. This is where Proposition 7.1 enters. Recall our assumption on s (8.2). As c_4 from Proposition 7.1 is independent of s and as $\#\Xi_s \leq c_4$ we may assume $P_i \notin \Xi_s$ for all i. So we may assume that each P_i is in the second alternative of Proposition 7.1.

As in [DGH19] we use the Euclidean norm $|\cdot|$ defined by $\hat{h}^{1/2}$ on the ρ-dimensional $\mathbb{R}$-vector space $\Gamma \otimes \mathbb{R}$. Let $r \in (0, R]$. By an elementary ball packing argument, any subset of $\Gamma \otimes \mathbb{R}$ contained in a closed ball of radius R is covered by at most $(1 + 2R/r)^\rho$ closed balls of radius r centered at elements of the given set; see [Rém00a, Lemme 6.1]. We apply this geometric argument to R as in (8.7) and to r, the positive square-root of $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s))/c_3 \geq c_1/c_3 > 0$. The contribution from s cancels out in the quotient R/r. So the number of balls in the covering is at most $c^{1+\rho}$.

By Proposition 7.1(ii) the number of the P_i's that map to a single closed ball of radius r is at most c_4. Thus after increasing c we find that $\#\{P_i \in \mathfrak{C}_s(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}(P_i - P_s) \leq R^2\} \leq c_4 c^{1+\rho}$, as desired. This completes the proof of the proposition in the case $P_0 = P_s$ for sufficiently large c_2.

The case of a general base point follows easily as our estimates depend only on the rank ρ of Γ. Indeed, let $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ be an arbitrary point and let Γ' be the subgroup of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ generated by Γ and $P_0 - P_s$. Its rank is at most $\rho + 1$.

Now if $Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0$ lies in Γ, then $Q + P_0 - P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}}) - P_s$ lies in Γ'. The number of such Q is at most $c_2^{2+\rho}$ by what we already proved. The proposition follows as $c_2^{2+\rho} \leq (c_2^2)^{1+\rho}$ and since we may replace c_2 by c_2^2. ☐

Proof of Theorem 1.1. It is possible to deduce Theorem 1.1 from Theorem 1.2, which we prove below. However in view of the importance of Theorem 1.1, we hereby give it a complete proof.

Our curve C corresponds to an F-rational point s_F of $\mathbb{M}_{g,1}$, the coarse moduli space of smooth genus g curves without level structure.

Fix $\ell \geq 3$. The fine moduli space $\mathbb{M}_g$ of smooth genus g curves with level-ℓ-structure is a finite cover of $\mathbb{M}_{g,1}$. After replacing F by a finite extension whose degree is bounded soley in terms of g we may assume that $\mathbb{M}_g$ is defined over F. Moreover, there exists a finite extension F'/F with $F' \subseteq \overline{\mathbb{Q}}$ and $s \in \mathbb{M}_g(F')$ that maps to s_F. Moreover, we may assume that $[F' : F] \leq B(g)$ is bounded only in terms of g. We may identify $C_{F'} = C \otimes_F F'$ with $\mathfrak{C}_s$, the fiber of $\mathfrak{C}_g \to \mathbb{M}_g$ above s.

Constructing the Jacobian commutes with finite field extension. We thus view $\Gamma = \mathrm{Jac}(C)(F)$ as a subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}}) = \mathrm{Jac}(C_{F'})(\overline{\mathbb{Q}})$.

To prove the theorem we may assume $C(F) \neq \emptyset$. So fix $P_0 \in C(F)$. We consider the Abel–Jacobi embedding $C - P_0 \subseteq \mathrm{Jac}(C)$ defined over F. Then $\#C(F) \leq \#(C_{F'}(\overline{\mathbb{Q}}) - P_0) \cap \Gamma = \#(\mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$. If $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \geq c_1$, the theorem follows from Proposition 8.1. Note that in this case, the constant c in (1.1) is independent of d.

So we may assume that the height of $\tau(s)$ is less than c_1. As $[F' : \mathbb{Q}] \leq [F' : F][F : \mathbb{Q}] \leq B(g)d$, Northcott's Theorem implies that $\tau(s)$ comes from a finite set in $\mathbb{A}_g(\overline{\mathbb{Q}})$ that depends only on g and d. The same holds for s since the Torelli morphism τ is finite-to-1 and thus has fibers of bounded cardinality. We may also assume that F' comes from a finite set. This means that the remaining C are twists in finitely many F'-isomorphism classes. But then it suffices to apply Rémond's estimate [DP02, page 643] to a single $C_{F'}$ and use $\#C(F) \leq \#(C_{F'}(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$ to conclude the theorem. Note that in this case, the constant c in (1.1) is polynomial in d. ☐

Proof of Theorem 1.2. Let C be a smooth curve of genus $g \geq 2$ defined over $\overline{\mathbb{Q}}$, and let Γ be a finite rank subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$. Let $P_0 \in C(\overline{\mathbb{Q}})$.

The curve C corresponds to a $\overline{\mathbb{Q}}$-point s_{c} of $\mathbb{M}_{g,1}$, the coarse moduli space of smooth genus g curves without level structure.

Fix $\ell \geq 3$. The fine moduli space $\mathbb{M}_g$ of smooth genus g curves with level-ℓ-structure is a finite covering of $\mathbb{M}_{g,1}$. So there exists an $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ that maps to s_{c}. Thus C is isomorphic, over $\overline{\mathbb{Q}}$, to the fiber $\mathfrak{C}_s$ of the universal family $\mathfrak{C}_g \to \mathbb{M}_g$. We thus view Γ as a finite rank subgroup of $\mathrm{Jac}(\mathfrak{C}_s)(\overline{\mathbb{Q}})$, and $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

Consider the Abel–Jacobi embedding $C - P_0 \subseteq \mathrm{Jac}(C)$. Then $\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma = \#(\mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$. Then the theorem follows from Proposition 8.1. □

Proof of Theorem 1.4. Let C be a smooth curve of genus $g \geq 2$ defined over a number field $k \subseteq \overline{\mathbb{Q}}$.

Apply Theorem 1.2 to $C_{\overline{\mathbb{Q}}}$, $P_0 \in C(\overline{\mathbb{Q}})$ and $\Gamma = \mathrm{Jac}(C_{\overline{\mathbb{Q}}})_{\mathrm{tors}}$, whose rank is 0. Then we obtain $c_1 \geq 0$ and $c_2 \geq 1$ such that

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C_{\overline{\mathbb{Q}}})_{\mathrm{tors}} \leq c_2$$

if $h(\iota([\mathrm{Jac}(C_{\overline{\mathbb{Q}}})])) \geq c_1$.

By the Northcott property and Torelli's Theorem, there are only finitely many $C_{\overline{\mathbb{Q}}}$'s defined over a number field k with $[k : \mathbb{Q}] \leq d$ such that $h(\iota([\mathrm{Jac}(C_{\overline{\mathbb{Q}}})])) < c_1$. By applying Raynaud's result on the Manin–Mumford Conjecture to each one of these finitely many curves separately, we obtain Theorem 1.4. □

Appendix A. The Silverman–Tate Theorem revisited

Our goal in this appendix is to present a treatment of the Silverman–Tate Theorem, [Sil83, Theorem A], using the language of Cartier divisors. Using Cartier divisors as opposed to Weil divisors allows us to relax the flatness hypotheses imposed on π in the notation of [Sil83, §3]. Apart from this minor tweak we closely follow the original argument presented by Silverman.

Suppose S is a regular, irreducible, quasi-projective variety over $\overline{\mathbb{Q}}$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme. We write η for the generic point of S and $\mathcal{A}_\eta$ for the generic fiber of π. Then $\mathcal{A}_\eta$ is an abelian variety defined over $\overline{\mathbb{Q}}(\eta)$.

Suppose we are presented with a closed immersion $\mathcal{A} \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$ over S and with a projective variety $\overline{S}$ containing S as a Zariski open and dense subset. We will assume that $\overline{S}$ is embedded into $\mathbb{P}^m_{\overline{\mathbb{Q}}}$.

We do not assume that $\overline{S}$ is regular. Let $\mathcal{M}$ be a fixed ample line bundle on $\overline{S}$.

We identify $\mathcal{A}$ with a subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$. Moreover, let $\overline{\mathcal{A}}$ denote the Zariski closure of $\mathcal{A}$ in $\mathbb{P}^n_{\overline{S}} = \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \overline{S}$.

We set $\overline{\mathcal{L}} = \mathcal{O}(1,1)|_{\overline{\mathcal{A}}}$ and $\mathcal{L} = \overline{\mathcal{L}}|_{\mathcal{A}}$. We will assume in addition that $[-1]^*\mathcal{L}_\eta \cong \mathcal{L}_\eta$ where $\mathcal{L}_\eta$ is the restriction of $\mathcal{L}$ to $\mathcal{A}_\eta$. This implies $[2]^*\mathcal{L}_\eta \cong \mathcal{L}_\eta^{\otimes 4}$.

Given these immersions, we have several height functions. For $(P,s) \in \overline{\mathcal{A}}(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \times \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ we define $h(P,s) = h(P) + h(s)$ using the Weil height. Moreover, for $s \in \overline{S}(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ we define $h_{\overline{S}}(s) = h(s)$. Finally, for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ we denote by

$$\hat{h}_{\mathcal{A}}(P) = \lim_{N \to \infty} \frac{h([N](P))}{N^2}$$

the Néron–Tate height with respect to $\mathcal{L}$; it is well-known that the limit converges, *cf.* arguments around (3.2).

We will prove the following variant of the Silverman–Tate Theorem.

Theorem A.1. *There exists a constant $c > 0$ such that for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ we have*

$$\left|\hat{h}_{\mathcal{A}}(P) - h(P)\right| \leq c \max\{1, h_{\overline{S}}(\pi(P))\}.$$

The constant c depends on $\mathcal{A}$ and on the various immersions but not on P. The proof is distributed over the next subsections.

A.1. **Extending multiplication-by-2.** We keep the notation from the previous subsection. We have constructed a (very naive) projective model $\overline{\mathcal{A}}$ of $\mathcal{A}$. Note that $\overline{\mathcal{A}}$ and $\overline{S}$ may fail to be regular. Moreover, the natural morphism $\overline{\mathcal{A}} \to \overline{S}$, which we also denote by π, may fail to be smooth or even flat.

Multiplication-by-2 is a morphism $[2]\colon \mathcal{A} \to \mathcal{A}$ that extends to a rational map $\overline{\mathcal{A}} \dashrightarrow \overline{\mathcal{A}}$. We consider the graph of $[2]$ on $\mathcal{A}$ as a subvariety of $\mathcal{A} \times_S \mathcal{A}$. Let $\overline{\mathcal{A}}'$ be the Zariski closure of this graph inside $\overline{\mathcal{A}} \times_{\overline{S}} \overline{\mathcal{A}}$. Write $\rho\colon \overline{\mathcal{A}}' \to \overline{\mathcal{A}}$ for the restriction of the projection onto the first factor and $[2]$ for the restriction onto the second factor. We may identify $\mathcal{A}$ with a Zariski open subset of $\overline{\mathcal{A}}'$. Under this identification, ρ restricts to the identity on $\mathcal{A}$ and $[2]$ restricts to multiplication-by-2 on $\mathcal{A}$.

The following diagram commutes

$$\begin{tikzcd}
\overline{\mathcal{A}} & \overline{\mathcal{A}}' \arrow[l, "\rho"'] \arrow[r, "{[2]}"] & \overline{\mathcal{A}} \\
\mathcal{A} \arrow[u, phantom, "\cup" sloped] \arrow[r, equal] & \mathcal{A} \arrow[u, phantom, "\cup" sloped] \arrow[r, "{[2]}"] & \mathcal{A} \arrow[u, phantom, "\cup" sloped]
\end{tikzcd}$$

where the first and third inclusions are equal and the middle one comes from the identification involved in the graph construction.

A.2. **Proof of the Silverman–Tate Theorem.** We keep the notation from the previous subsection.

Proposition A.2. *There exists a constant $c_1 > 0$ such that*

$$|h([2](P)) - 4h(P)| \leq c_1 \max\{1, h_{\overline{S}}(\pi(P))\} \tag{A.1}$$

holds for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$.

Proof. We define

$$\mathcal{F}' = [2]^*\overline{\mathcal{L}} \otimes \rho^*\overline{\mathcal{L}}^{\otimes(-4)} \in \operatorname{Pic}(\overline{\mathcal{A}}'). \tag{A.2}$$

Recall that we have identified $\mathcal{A}$ with a Zariski open subset of $\overline{\mathcal{A}}'$. The restriction of $[2]^*\overline{\mathcal{L}}$ to the generic fiber $\mathcal{A}_\eta \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}}'$ coincides with $[2]^*\mathcal{L}_\eta$ and the restriction of $\rho^*\overline{\mathcal{L}}$ to $\mathcal{A}_\eta$ is identified with $\mathcal{L}_\eta$. Using our assumption $[2]^*\mathcal{L}_\eta \cong \mathcal{L}_\eta^{\otimes 4}$ on the generic fiber $\mathcal{A}_\eta$ we see that $\mathcal{F}'$ is trivial on $\mathcal{A}_\eta$.

By [Gro67, Corollaire 21.4.13] applied to $\mathcal{A} \to S$ there exists a line bundle $\mathcal{M}$ on S such that $\pi|_{\mathcal{A}}^*\mathcal{M} \cong \mathcal{F}'|_{\mathcal{A}}$.

Let us first desingularize the compactified base $\overline{S}$ by applying Hironaka's Theorem. Thus there is a proper, birational morphism $b\colon \overline{S}' \to \overline{S}$ that is an isomorphism above

S such that $\overline{S}'$ is regular. We consider S as Zariski open in $\overline{S}'$. Note that b is even projective and $\overline{S}'$ is integral. So $\overline{S}'$ is an irreducible, regular, projective variety.

Now consider the base change $\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. This new scheme may fail to be irreducible or even reduced. However, recall that b is an isomorphism above the regular $S \subseteq \overline{S}$. So $(\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}')_S = \overline{\mathcal{A}}' \times_{\overline{S}} S$ is isomorphic to $\mathcal{A}$ and thus integral. We may consider $\mathcal{A}$ as an open subscheme of $\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. It must be contained in an irreducible component of $\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. We endow this irreducible component with the reduced induced structure and obtain an integral, closed subscheme $\overline{\overline{\mathcal{A}}} \subseteq \overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. We get a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\pi"'] & \subseteq & \overline{\overline{\mathcal{A}}} \arrow[r, "f"] \arrow[d, "\overline{\pi}"'] & \overline{\mathcal{A}}' \arrow[d, "\pi\circ\rho"'] \\
S & \subseteq & \overline{S}' \arrow[r] & \overline{S}
\end{tikzcd}$$

The horizontal morphisms compose to the identity on the domain.

We consider S as a Zariski open subset of $\overline{S}'$. As $\overline{S}'$ is regular, we can extend $\mathcal{M}$ to a line bundle on the regular $\overline{S}'$, *cf.* [GW10, Corollary 11.41]. The pull-back $f^*\mathcal{F}' \otimes \overline{\pi}^*\mathcal{M}^{\otimes(-1)}$ is trivial on $\mathcal{A} \subseteq \overline{\overline{\mathcal{A}}}$.

By Hironaka's Theorem there is a proper, birational morphism $\beta\colon \widetilde{\mathcal{A}} \to \overline{\overline{\mathcal{A}}}$ that is an isomorphism above $\mathcal{A}$ (which is regular) such that $\widetilde{\mathcal{A}}$ is regular. We may identify $\mathcal{A}$ with a Zariski open subset of $\widetilde{\mathcal{A}}$.

Now we pull everything back to the regular $\widetilde{\mathcal{A}}$. More precisely, we set $\mathcal{F} = \beta^* f^* \mathcal{F}'$. Then $\mathcal{F} \otimes \beta^*\overline{\pi}^*\mathcal{M}^{\otimes(-1)}$ is trivial when restricted to $\mathcal{A}$.

To a Cartier divisor D we attach its line bundle $\mathcal{O}(D)$. As $\widetilde{\mathcal{A}}$ is integral we may fix a Cartier divisor D on $\widetilde{\mathcal{A}}$ with $\mathcal{O}(D) \cong \mathcal{F} \otimes \beta^*\overline{\pi}^*\mathcal{M}^{\otimes(-1)}$. Let $\mathrm{cyc}(D)$ denote the Weil divisor of $\widetilde{\mathcal{A}}$ attached to D. The linear equivalence class of $\mathrm{cyc}(D)$ restricted to $\mathcal{A}$ is trivial. By [GW10, Proposition 11.40] $\mathrm{cyc}(D)$ is linearly equivalent to a Weil divisor $\sum_{i=1}^r n_i Z_i$ with $Z_i \subseteq \widetilde{\mathcal{A}} \setminus \mathcal{A}$ irreducible and of codimension 1 in $\widetilde{\mathcal{A}}$.

We let $\widetilde{\pi}$ denote the composition $\widetilde{\mathcal{A}} \to \overline{\overline{\mathcal{A}}} \to \overline{S}'$. Let us consider $\widetilde{\pi}(Z_i) = Y_i$. As $\widetilde{\pi}$ is proper, each Y_i is an irreducible closed subvariety of $\overline{S}'$. Moreover, $Y_i \subseteq \widetilde{\pi}(\widetilde{\mathcal{A}}\setminus\mathcal{A}) \subseteq \overline{S}'\setminus S$. So Y_i has dimension at most $\dim \overline{S}' - 1$. But Y_i could have codimension at least 2 and thus fail to be the support of a Weil divisor. On the regular $\overline{S}'$ a Cartier divisor is the same thing as a Weil divisor; see [GW10, Theorem 11.38(2)]. For each i we fix a Cartier divisor E_i of $\overline{S}'$ such that $\mathrm{cyc}(E_i)$ equals a prime Weil divisor supported on an irreducible subvariety containing Y_i. Since $\mathrm{cyc}(E_i)$ is effective, we find that E_i is effective, see [GW10, Theorem 11.38(1)] and its proof. An effective Cartier divisor and its image under the cycle map $\mathrm{cyc}(\cdot)$ have equal support. So the subscheme of $\overline{S}'$ attached to E_i contains Y_i.

The pull-back $\widetilde{\pi}^*E_i$ is well-defined as a Cartier divisor, we do not require that π is flat, *cf.* [GW10, Proposition 11.48(b)]. By [GW10, Corollary 11.49] the inverse image $\widetilde{\pi}^{-1}(E_i)$, taken as a subscheme of $\widetilde{\mathcal{A}}$ is the subscheme attached to $\widetilde{\pi}^*E_i$ and $\widetilde{\pi}^*E_i$ is effective.

Note that $\widetilde{\pi}^{-1}(E_i) \supseteq \widetilde{\pi}^{-1}(Y_i) \supseteq Z_i$. The support satisfies $\mathrm{Supp}(\widetilde{\pi}^* E_i) \supseteq Z_i$. Moreover, as $\widetilde{\pi}^* E_i$ is effective, $\mathrm{cyc}(\widetilde{\pi}^* E_i)$ is effective and $\mathrm{Supp}(\mathrm{cyc}(\widetilde{\pi}^* E_i)) = \mathrm{Supp}(\widetilde{\pi}^* E_i)$. Thus

$$\pm \sum_{i=1}^{r} n_i Z_i \leq \mathrm{cyc}\left(\widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i\right).$$

Recall that $\mathrm{cyc}(D) = \mathrm{cyc}(\mathrm{div}\phi) + \sum_{i=1}^{r} n_i Z_i$ for some rational function ϕ on $\widetilde{\mathcal{A}}$. Therefore,

$$0 \leq \mathrm{cyc}\left(\pm(D - \mathrm{div}\phi) + \widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i\right).$$

Since $\widetilde{\mathcal{A}}$ is regular and in particular normal, we find that

$$\pm (D - \mathrm{div}\phi) + \widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i \tag{A.3}$$

is an effective Cartier divisor for both signs; see [GW10, Theorem 11.38(1)] and its proof. Moreover, its support equals the support of

$$0 \leq \mathrm{cyc}\left(\pm(D - \mathrm{div}\phi) + \widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i\right) = \pm\mathrm{cyc}(D - \mathrm{div}\phi) + \sum_{i=1}^{r} |n_i| \mathrm{cyc}(\widetilde{\pi}^* E_i).$$

Thus the support of (A.3) lies in $\bigcup_{i=1}^{r} \mathrm{Supp}(\widetilde{\pi}^* E_i)$.

We apply $\mathcal{O}(\cdot)$ and pass again to line bundles. Let us denote $\mathcal{E} = \mathcal{O}(\sum_{i=1}^{r} |n_i| E_i)$, a line bundle on $\overline{S}'$. The line bundle attached to (A.3) is $(\mathcal{F} \otimes \beta^* \overline{\pi}^* \mathcal{M}^{\otimes(-1)})^{\otimes(\pm 1)} \otimes \widetilde{\pi}^* \mathcal{E}$. Since (A.3) is effective, both $(\mathcal{F} \otimes \beta^* \overline{\pi}^* \mathcal{M}^{\otimes(-1)})^{\otimes(\pm 1)} \otimes \widetilde{\pi}^* \mathcal{E}$ have a non-zero global section.

By the Height Machine this translates to

$$h_{\widetilde{\mathcal{A}}, (\mathcal{F} \otimes \beta^* \overline{\pi}^* \mathcal{M}^{\otimes(-1)})^{\otimes(\pm 1)} \otimes \widetilde{\pi}^* \mathcal{E}}(P) \geq O(1)$$

for all $\widetilde{P} \in \widetilde{\mathcal{A}}(\overline{\mathbb{Q}})$ with $\widetilde{\pi}(\widetilde{P}) \notin \bigcup_i \mathrm{Supp}(E_i)$. By functoriality properties of the Height Machine we obtain

$$|h_{\overline{\mathcal{A}}', \mathcal{F}'}(f(\beta(\widetilde{P})))| \leq h_{\overline{S}', \mathcal{E}}(\widetilde{\pi}(\widetilde{P})) + |h_{\overline{S}', \mathcal{M}}(\overline{\pi}(\beta(\widetilde{P})))| + O(1)$$

for the same $\widetilde{P}$. We recall (A.2) and again use the Height Machine to find

$$|h([2](P')) - 4h(\rho(P'))| \leq h_{\overline{S}', \mathcal{E}}(\widetilde{\pi}(\widetilde{P})) + |h_{\overline{S}', \mathcal{M}}(\widetilde{\pi}(\widetilde{P}))| + O(1)$$

where $P' = f(\beta(\widetilde{P}))$. Observe that all points of $\mathcal{A}(\overline{\mathbb{Q}})$ are in the image of $f \circ \beta$.

We recall that the desingularization morphism $\overline{S}' \to \overline{S}$ is an isomorphism above S and that we have identified $\mathcal{A}$ with a Zariski open subset of $\overline{\mathcal{A}}'$ and of $\overline{\mathcal{A}}$. Under these identifications and if P' corresponds to $P \in \mathcal{A}(\overline{\mathbb{Q}})$, then $[2](P')$ is the duplicate of P, $\rho(P') = P$, and $\widetilde{\pi}(\widetilde{P}) = \pi(\rho(P')) = \pi(P)$. We apply the Height Machine a final time and use that $h_{\overline{S}}$ arises from the Weil height restricted to $\overline{S}(\overline{\mathbb{Q}})$. We find

$$|h([2](P)) - 4h(P)| \leq c_1 \max\{1, h_{\overline{S}}(\pi(P))\}$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ with $\widetilde{\pi}(\widetilde{P}) \notin \bigcup_i \mathrm{Supp}(E_i)$, under the identifications above.

Let $P \in \mathcal{A}(\overline{\mathbb{Q}})$. As the Y_i lie in $\widetilde{\pi}(\widetilde{\mathcal{A}} \setminus \mathcal{A})$ we can choose all E_i above to avoid $\pi(P)$. After doing this finitely often (using noetherian induction) and replacing the E_i from before and adjusting c_1, we find

$$|h([2](P)) - 4h(P)| \le c_1 \max\{1, h_{\overline{S}}(\pi(P))\}$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ where $c_1 > 0$ is independent of P. □

Proof of Theorem A.1. Having (A.1) at our disposal the proof follows a well-known argument. Indeed, say $l \ge k \ge 0$ are integers. Then applying the triangle inequality to the appropriate telescoping sum yields

$$\begin{aligned}\left|\frac{h([2^l](P))}{4^l} - \frac{h([2^k](P))}{4^k}\right| &\le \sum_{m=k}^{l-1} \left|\frac{h([2^{m+1}](P))}{4^{m+1}} - \frac{h([2^m](P))}{4^m}\right| \\ &\le \sum_{m=k}^{l-1} 4^{-(m+1)} \left|h([2^{m+1}](P)) - 4h([2^m](P))\right|.\end{aligned}$$

We apply (A.1) to $[2^m](P)$ and find that the sum is bounded by $c_1 x \sum_{m=k}^{l-1} 4^{-(m+1)} \le c_1 x 4^{-k}$ where $x = \max\{1, h_{\overline{S}}(\pi(P))\}$. So $\left(h([2^l](P))/4^l\right)_{l \ge 1}$ is a Cauchy sequence with limit $\hat{h}_{\overline{\mathcal{A}}}(P)$. Taking $k = 0$ and $l \to \infty$ we obtain from the estimates above that $|\hat{h}_{\overline{\mathcal{A}}}(P) - h(P)| \le c_1 x$, as desired. □

Appendix B. Full version of Theorem 1.6

The goal of this section is to prove the full version of Theorem 1.6, *i.e.*, without assuming `(Hyp)`. Let S be a regular, irreducible, quasi-projective variety defined over $\overline{\mathbb{Q}}$ and let $\pi \colon \mathcal{A} \to S$ is an abelian scheme of relative dimension $g \ge 1$. By [GN06, §2.1], $\mathcal{A} \to S$ carries a polarization of type $D = \mathrm{diag}(d_1, \ldots, d_g)$ for some positive integers $d_1 | d_2 | \cdots | d_g$.

B.1. **Betti map.** In this subsection, we extend Proposition 2.1 to an arbitrary $\mathcal{A} \to S$.

Proposition B.1. *Let $s_0 \in S(\mathbb{C})$. Then there exist a non-empty open neighborhood Δ of s_0 in S^{an}, and a map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, called the Betti map, with the following properties.*

(i) *For each $s \in \Delta$ the restriction $b_\Delta|_{\mathcal{A}_s(\mathbb{C})} \colon \mathcal{A}_s(\mathbb{C}) \to \mathbb{T}^{2g}$ is a group isomorphism.*
(ii) *For each $\xi \in \mathbb{T}^{2g}$ the preimage $b_\Delta^{-1}(\xi)$ is a complex analytic subset of $\mathcal{A}_\Delta$.*
(iii) *The product $(b_\Delta, \pi) \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$ is a real analytic isomorphism.*

Proof. **Case: Moduli space with level structure.** Fix $\ell \ge 3$ with $(\ell, d_g) = 1$. We start by proving Proposition B.1 for $S = \mathbb{A}_{g,D,\ell}$, the moduli space of abelian varieties of dimension g polarized of type D with level-ℓ-structure. It is a fine moduli space; see [GN06, Theorem 2.3.1]. Let $\pi_D^{\mathrm{univ}} \colon \mathfrak{A}_{g,D,\ell} \to \mathbb{A}_{g,D,\ell}$ be the universal abelian variety.

The universal covering $\mathfrak{H}_g^+ \to \mathbb{A}_{g,D,\ell}^{\mathrm{an}}$ [GN06, Proposition 1.3.2], where $\mathfrak{H}_g^+$ is the Siegel upper half space, gives a family of abelian varieties $\mathcal{A}_{\mathfrak{H}_g^+,D} \to \mathfrak{H}_g^+$ fitting into the diagram

$$\begin{tikzcd} \mathcal{A}_{\mathfrak{H}_g^+,D} := \mathfrak{A}_{g,D,\ell} \times_{\mathbb{A}_{g,D,\ell}} \mathfrak{H}_g^+ \arrow[r] \arrow[d] & \mathfrak{A}_{g,D,\ell}^{\mathrm{an}} \arrow[d, "\pi_D^{\mathrm{univ}}"] \\ \mathfrak{H}_g^+ \arrow[r] & \mathbb{A}_{g,D,\ell}^{\mathrm{an}}. \end{tikzcd}$$

The family $\mathcal{A}_{\mathfrak{H}_g^+,D} \to \mathfrak{H}_g^+$ is polarized of type D. For the universal covering $u\colon \mathbb{C}^g \times \mathfrak{H}_g^+ \to \mathcal{A}_{\mathfrak{H}_g^+,D}$ and for each $Z \in \mathfrak{H}_g^+$, the kernel of $u|_{\mathbb{C}^g \times \{Z\}}$ is $D\mathbb{Z}^g + Z\mathbb{Z}^g$. Thus the map $\mathbb{C}^g \times \mathfrak{H}_g^+ \to \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g^+ \to \mathbb{R}^{2g}$, where the first map is the inverse of $(a,b,Z) \mapsto (Da + Zb, Z)$ and the second map is the natural projection, descends to a real analytic map

$$b^{\mathrm{univ}}\colon \mathcal{A}_{\mathfrak{H}_g^+,D} \to \mathbb{T}^{2g}.$$

Now for each $s_0 \in \mathbb{A}_{g,D,\ell}(\mathbb{C})$, there exists a contractible, relatively compact, open neighborhood Δ of s_0 in $\mathbb{A}_{g,D,\ell}^{\mathrm{an}}$ such that $\mathfrak{A}_{g,D,\ell,\Delta} := (\pi_D^{\mathrm{univ}})^{-1}(\Delta)$ can be identified with $\mathcal{A}_{\mathfrak{H}_g^+,\Delta'}$ for some open subset Δ' of $\mathfrak{H}_g^+$. The composite $b_\Delta\colon \mathfrak{A}_{g,D,\ell,\Delta} \cong \mathcal{A}_{\mathfrak{H}_g^+,D,\Delta'} \to \mathbb{T}^{2g}$ clearly satisfies the three properties listed in Proposition B.1. Thus b_Δ is the desired Betti map in this case.

Case: With level structure. Assume that $\mathcal{A} \to S$ carries level-ℓ-structure for some $\ell \geq 3$ with $(\ell, d_g) = 1$. As $\mathbb{A}_{g,D,\ell}$ is a fine moduli space there exists a Cartesian diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\iota"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{A}_{g,D,\ell} \arrow[d] \\ S \arrow[r, "\iota_S"] & \mathbb{A}_{g,D,\ell}. \end{tikzcd}$$

Now let $s_0 \in S(\mathbb{C})$. Applying Proposition B.1 to the universal abelian variety $\mathfrak{A}_{g,D,\ell} \to \mathbb{A}_{g,D,\ell}$ and $\iota_S(s_0) \in \mathbb{A}_{g,D,\ell}(\mathbb{C})$, we obtain a non-empty open neighborhood Δ_0 of $\iota_S(s_0)$ in $\mathbb{A}_{g,D,\ell}^{\mathrm{an}}$ and a real analytic map

$$b_{\Delta_0}\colon \mathfrak{A}_{g,\Delta_0} \to \mathbb{T}^{2g}$$

satisfying the properties listed in Proposition B.1.

Now let $\Delta = \iota_S^{-1}(\Delta_0)$. Then Δ is an open neighborhood of s in S^{an}. Denote by $\mathcal{A}_\Delta = \pi^{-1}(\Delta)$ and define

$$b_\Delta = b_{\Delta_0} \circ \iota\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}.$$

Then b_Δ satisfies the properties listed in Proposition B.1 for $\mathcal{A} \to S$. Hence b_Δ is our desired Betti map.

Case: General case. Let $s_0 \in S(\mathbb{C})$.

There exists a finite and étale covering $\rho\colon S' \to S$ such that $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ has level-ℓ-structure for some $\ell \geq 3$ with $(\ell, d_g) = 1$. Note that $\mathcal{A}' \to S'$ is still polarized of type D.

Let $s_0' \in \rho^{-1}(s_0)$. Applying Proposition B.1 to $\mathcal{A}' \to S'$ and $s_0' \in S'(\mathbb{C})$, we obtain a non-empty open neighborhood Δ' of s_0' in $(S')^{\mathrm{an}}$ and a map $b_{\Delta'}\colon \mathcal{A}'_{\Delta'} \to \mathbb{T}^{2g}$ satisfying the properties listed in Proposition B.1.

Let $\Delta = \rho(\Delta')$. Up to shrinking Δ', we may assume that $\rho|_{\Delta'} \colon \Delta' \to \Delta$ is a homeomorphism and that Δ is an open neighborhood of s_0 in S^{an}. Thus $\mathcal{A}'_{\Delta'} \cong \mathcal{A}_\Delta$. Now define

$$b_\Delta \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$$

to be the composite of the inverse of $\mathcal{A}'_{\Delta'} \cong \mathcal{A}_\Delta$ and $b_{\Delta'}$. Then b_Δ is our desired Betti map. $\square$

Here is an easy property of the generic rank of the Betti map.

Lemma B.2. *Let $b_\Delta \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ be a Betti map as in Proposition B.1. Let X be an irreducible subvariety of $\mathcal{A}$ with $X^{\mathrm{an}} \cap \mathcal{A}_\Delta \neq \emptyset$. Let U be a Zariski open dense subset of X. Then*

$$\max_{x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = \max_{u \in U^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{U^{\mathrm{sm,an}}})_u. \tag{B.1}$$

Proof. The statement (B.1) is true on replacing "=" by "≥", as $X^{\mathrm{sm,an}} \supseteq U^{\mathrm{sm,an}}$.

For the converse inequality we set $\max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = r$ and pick $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ satisfying $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = r$. Then there exists an open neighborhood V of x in $X^{\mathrm{sm,an}}$ such that $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_u = r$ for all $u \in V$. But $U^{\mathrm{sm}}(\mathbb{C}) \cap V \neq \emptyset$ since $U^{\mathrm{sm}} \neq \emptyset$ is Zariski open in X and V is Zariski dense in X. Thus there exists a $u \in U^{\mathrm{sm}}(\mathbb{C}) \cap V$. Then we must have $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{U^{\mathrm{sm,an}}})_u = r$ and the lemma follows. $\square$

B.2. **Non-degenerate subvariety and Theorem 1.6.** We keep the notation as in the beginning of this appendix.

Definition B.3. *An irreducible subvariety X of $\mathcal{A}$ is said to be non-degenerate if there exists an open non-empty subset Δ of S^{an}, with the Betti map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$ as in Proposition B.1, such that*

$$\max_{x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X.$$

Now we are ready to state and prove the full version of Theorem 1.6. We give the statement in terms of the Height Machine, *cf.* [BG06, Chapter 2 and 9].

Let S be a regular, irreducible, quasi-projective variety over $\overline{\mathbb{Q}}$ and let $\pi \colon \mathcal{A} \to S$ be an abelian scheme. Let $\mathcal{L}$ be a relative ample line bundle on $\mathcal{A} \to S$, and let $\mathcal{M}$ be an ample line bundle on a compactification $\overline{S}$ of S. All data above are assumed to be defined over $\overline{\mathbb{Q}}$. Define $\hat{h}_{\mathcal{A},\mathcal{L}} \colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ to be $\hat{h}_{\mathcal{A},\mathcal{L}}(P) = \hat{h}_{\mathcal{A}_s,\mathcal{L}_s}(P)$ with $s = \pi(P)$, and $h_{S,\mathcal{M}} \colon S(\overline{\mathbb{Q}}) \to \mathbb{R}$ to a height function.

Theorem B.4. *Let X be an irreducible subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$. Suppose X is non-degenerate, as defined in Definition B.3. Then there exist constants $c > 0$ and $c' \geq 0$ and a Zariski open dense subset U of X with*

$$\hat{h}_{\mathcal{A},\mathcal{L}}(P) \geq c h_{S,\mathcal{M}}(\pi(P)) - c' \quad \text{for all} \quad P \in U(\overline{\mathbb{Q}}). \tag{B.2}$$

Proof. We will reduce the current theorem to Theorem 1.6, *i.e.*, where $\mathcal{A} \to S$ satisfies `(Hyp)` (carries a principal polarization and has level-ℓ-structure for some $\ell \geq 3$), and is equipped with a closed immersion $\mathcal{A} \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$ as in the beginning of §3.

First we claim: we may assume $\pi(X) = S$. Indeed for $S' = \pi(X)$ and $\mathcal{A}' = \mathcal{A} \times_S S' = \pi^{-1}(S')$, as X is non-degenerate, for a component Δ' of $\Delta \cap (S')^{\mathrm{an}}$ we have

$$\max_{x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}'_{\Delta'}} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta'}|_{X^{\mathrm{sm,an}}})_x = 2 \dim X.$$

Thus X is a non-degenerate subvariety of $\mathcal{A}'$. So it suffices to prove Theorem B.4 with $\mathcal{A} \to S$ replaced by $\mathcal{A}' \to S'$, $\mathcal{L}$ replaced by $\mathcal{L}|_{\mathcal{A}'}$ and $\mathcal{M}$ replaced by $\mathcal{M}|_{\overline{S'}}$, where $\overline{S'}$ is the Zariski closure of S' in $\overline{S}$.

Next we reduce to the case where $\pi\colon \mathcal{A} \to S$ satisfies:

`(Hyp')`: it has level-ℓ-structure for some $\ell \geq 3$, and for its generic fiber A there exists an isogenous $A_0 \to A$ defined over $\overline{\mathbb{Q}}(S)$ with A_0 principally polarized.

There exists a finite and étale covering $S' \to S$ such that $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ has level-ℓ-structure for some $\ell \geq 3$. By [Mum74, §23, Corollary 1], each abelian variety over an algebraic closed field is isogenous to a principally polarized one. Applying this to the generic fiber of $\mathcal{A}' \to S'$, we obtain a quasi-finite and étale covering $S'' \to S'$ with the following property: For the generic fiber A'' of $\mathcal{A}'' := \mathcal{A}' \times_{S'} S'' \to S''$, there exists an A''_0, isogenous to A'' over $\overline{\mathbb{Q}}(S'')$ with A''_0 principally polarized. Note that $\mathcal{A}'' \to S''$ also has level-ℓ-structure.

Denote by $\rho\colon S'' \to S$ the composite of $S' \to S$ and $S'' \to S'$ above. Then $\mathcal{A}'' = \mathcal{A}' \times_{S'} S'' = \mathcal{A} \times_S S''$. Denote by $\rho_{\mathcal{A}}\colon \mathcal{A}'' \to \mathcal{A}$ the natural projection.

Let X'' be an irreducible component of $\rho_{\mathcal{A}}^{-1}(X)$. Since X is non-degenerate, we have that X'' is non-degenerate by the construction of the Betti map; see the proof of Proposition B.1. By the Height Machine, it suffices to prove the height inequality (B.2) with $\mathcal{A} \to S$ replaced by $\mathcal{A}'' \to S''$, X replaced by X'', $\mathcal{L}$ replaced by $\rho_{\mathcal{A}}^*\mathcal{L}$, and $\mathcal{M}$ replaced by an ample line bundle $\mathcal{M}''$ on a compactification of S'' (up to modifying the constant c' in (B.2)). Thus we may assume `(Hyp')`.

Now by `(Hyp')`, there exists a Zariski open dense subset S_0 of S such that: There exists an abelian scheme $\mathcal{A}_0 \to S_0$, carrying a principal polarization, with an S_0-isogeny $\lambda\colon \mathcal{A}_0 \to \mathcal{A}_{S_0} := \pi^{-1}(S_0)$. Up to replacing S_0 by its smooth locus, we may furthermore assume that S_0 is regular. Moreover, $\mathcal{A}_0 \to S_0$ has level-ℓ-structure for some $\ell \geq 3$ because $\mathcal{A}_{S_0} \to S_0$ does.

By Lemma B.2, X is non-degenerate if and only if $X \cap \pi^{-1}(S_0)$ is non-degenerate. So in order to prove Theorem B.4 we may replace $\mathcal{A} \to S$ by $\mathcal{A}_{S_0} = \pi^{-1}(S_0) \to S_0$. Then for the S-isogeny $\lambda\colon \mathcal{A}_0 \to \mathcal{A}$, by a similar argument as above, it suffices to prove the height inequality (1.6) with $\mathcal{A} \to S$ replaced by $\mathcal{A}_0 \to S$, X replaced by an irreducible component of $\lambda^{-1}(X)$, and $\mathcal{L}$ replaced by $\lambda^*\mathcal{L}$. Thus we are reduced to the case where S is regular, and $\mathcal{A} \to S$ satisfies `(Hyp)`.

Hence it remains to prove that we can assume $\mathcal{A} \to S$ to be equipped with a closed immersion $\mathcal{A} \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$ as in the beginning of §3. Let η be the generic fiber of $\mathcal{A} \to S$. The desired immersion can be obtained by applying Remark 3.1 to $L_0 = \mathcal{L}_\eta$; we point out that the $\mathcal{L}$, $\mathcal{M}$ in Remark 3.1 are different from our $\mathcal{L}$, $\mathcal{M}$ here. By Remark 3.2, our $\hat{h}_{\mathcal{A},\mathcal{L}}$ is $(1/l)\hat{h}_{\mathcal{A}}$ for the integer l from Remark 3.1 and the height $\hat{h}_{\mathcal{A}}$ defined in (3.2). Thus up to replacing the constant c in (B.2) by c/l, we are reduced to Theorem 1.6. Hence we are done. $\square$

References

[ACG11] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of Algebraic Curves, II (with a contribution by J. Harris)*, volume 268 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2011.

[ACZ18] Yves André, Pietro Corvaja, and Umberto Zannier. The Betti map associated to a section of an abelian scheme (with an appendix by Z. Gao). *ArXiv: 1802.03204*, 2018.

[Alp18] L. Alpoge. The average number of rational points on genus two curves is bounded. *arXiv:1804.05859*, 2018.

[BG06] E. Bombieri and W. Gubler. *Heights in Diophantine Geometry*. Cambridge University Press, 2006.

[BGS94] J.-B. Bost, H. Gillet, and C. Soulé. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(2):903–1027, 1994.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990.

[Bom90] E. Bombieri. The Mordell conjecture revisited. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(4):615–640, 1990.

[CGHX18] S. Cantat, Z. Gao, P. Habegger, and J. Xie. The Geometric Bogomolov Conjecture. *arXiv:1809.00848*, 2018.

[Cha41] C. Chabauty. Sur les points rationnels des courbes algébriques de genre supérieur à l'unité. *C. R. Acad. Sci. Paris*, 212:882–885, 1941.

[Col85] R. .F. Coleman. Effective Chabauty. *Duke Math. J.*, 52(3):765–770, 1985.

[CVV17] S. Checcoli, F. Veneziano, and E. Viada. On the explicit torsion anomalous conjecture. *Trans. Amer. Math. Soc.*, 369(9):6465–6491, 2017.

[dD97] T. de Diego. Points rationnels sur les familles de courbes de genre au moins 2. *J. Number Theory*, 67(1):85–114, 1997.

[Dem12] J.P. Demailly. *Complex analytic and differential geometry*. 2012. Available at `https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

[DGH19] V. Dimitrov, Z. Gao, and P. Habegger. Uniform bound for the number of rational points on a pencil of curves. *Int. Math. Res. Not. IMRN*, (rnz248):https://doi.org/10.1093/imrn/rnz248, 2019.

[DKY20] L. DeMarco, H. Krieger, and H. Ye. Uniform Manin-Mumford for a family of genus 2 curves. *accepted to Ann. of Math.*, 2020.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[DNP07] S. David, M. Nakamaye, and P. Philippon. Bornes uniformes pour le nombre de points rationnels de certaines courbes. In *Diophantine geometry*, volume 4 of *CRM Series*, pages 143–164. Ed. Norm., Pisa, 2007.

[DP02] S. David and P. Philippon. Minorations des hauteurs normalisées des sous-variétés de variétés abeliennes. II. *Comment. Math. Helv.*, 77(4):639–700, 2002.

[DP07] S. David and P. Philippon. Minorations des hauteurs normalisées des sous-variétés des puissances des courbes elliptiques. *Int. Math. Res. Pap. IMRP*, (3):Art. ID rpm006, 113, 2007.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73:349–366, 1983.

[Fal91] G. Faltings. Diophantine approximation on abelian varieties. *Ann. of Math. (2)*, 133(3):549–576, 1991.

[Ful98] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gao18] Z. Gao. Generic rank of Betti map and unlikely intersections. *arXiv: 1810.12929*, 2018.

[GH19] Z. Gao and P. Habegger. Heights in Families of Abelian Varieties and the Geometric Bogomolov Conjecture. *Ann. of Math.*, 189(2):527–604, 2019.

[GN06] A. Genestier and B.C. Ngô. Lecture on Shimura varieties. *https://www.math.uchicago.edu/ ngo/Shimura.pdf*, 2006.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas I-IV. *Inst. Hautes Études Sci. Publ. Math.*, (20,24,28,32), 1964–1967.

[GW10] U. Görtz and T. Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Hab09] P. Habegger. On the bounded height conjecture. *Int. Math. Res. Not. IMRN*, (5):860–886, 2009.

[Hab13] P. Habegger. Special Points on Fibered Powers of Elliptic Surfaces. *J.Reine Angew. Math.*, 685:143–179, 2013.

[Hur92] A. Hurwitz. Über algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.*, 41(3):403–442, 1892.

[KRZB16] E. Katz, J. Rabinoff, and D. Zureick-Brown. Uniform bounds for the number of rational points on curves of small Mordell-Weil rank. *Duke Math. J.*, 165(16):3189–3240, 2016.

[Küh] L. Kühne. The Bounded Height Conjecture for Semiabelian Varieties. *To appear in Compos. Math, arXiv: 1703.03891.*

[Lan78] S. Lang. *Elliptic curves: Diophantine analysis*, volume 231 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1978.

[Laz04] R. Lazarsfeld. *Positivity in Algebraic Geometry I*. Springer, 2004.

[Lel57] P. Lelong. Intégration sur un ensemble analytique complexe. *Bull. Soc. Math. France*, 85:239–262, 1957.

[Maz86] B. Mazur. Arithmetic on curves. *Bulletin of the American Mathematical Society*, 14(2):207–259, 1986.

[Maz00] B. Mazur. Abelian varieties and the Mordell-Lang conjecture. In *Model theory, algebra, and geometry*, volume 39 of *Math. Sci. Res. Inst. Publ.*, pages 199–227. Cambridge Univ. Press, Cambridge, 2000.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Mok91] N. Mok. Aspects of Kähler geometry on arithmetic varieties. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 335–396. Amer. Math. Soc., Providence, RI, 1991.

[Mum70] D. Mumford. Varieties defined by quadratic equations. In *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, pages 29–100. Edizioni Cremonese, Rome, 1970.

[Mum74] D. Mumford. *Abelian Varieties, 2nd ed.* Oxford University Press, London, 1974.

[OS80] F. Oort and J. Steenbrink. The local Torelli problem for algebraic curves. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 157–204. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Phi86] P. Philippon. Lemmes de zéros dans les groupes algébriques commutatifs. *Bull. Soc. Math. France*, 114:355–383, 1986.

[Pin89] R. Pink. *Arithmetical compactification of mixed Shimura varieties*. PhD thesis, Bonner Mathematische Schriften, 1989.

[Pin05a] R. Pink. A combination of the conjectures of Mordell-Lang and André-Oort. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pages 251–282. Birkäuser, 2005.

[Pin05b] R. Pink. A Common Generalization of the Conjectures of André-Oort, Manin-Mumford, and Mordell-Lang. *Preprint*, page 13pp, 2005.

[Ray70] M. Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin-New York, 1970.

[Rém00a] G. Rémond. Décompte dans une conjecture de Lang. *Invent. Math.*, 142(3):513–545, 2000.

[Rém00b] G. Rémond. Inégalité de Vojta en dimension supérieure. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(1):101–151, 2000.

[Sil83] J.H. Silverman. Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.*, 342:197–211, 1983.

[Sil93] J. H. Silverman. A uniform bound for rational points on twists of a given curve. *J. London Math. Soc. (2)*, 47(3):385–394, 1993.

[Sto19] M. Stoll. Uniform bounds for the number of rational points on hyperelliptic curves of small mordell-weil rank. *J. Eur. Math. Soc. (JEMS)*, 21:923–956, 2019.

[Voi02] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, English edition, 2002.

[Voj91] P. Vojta. Siegel's theorem in the compact case. *Ann. of Math. (2)*, 133(3):509–548, 1991.

[Wal87] M. Waldschmidt. Nombres transcendants et groupes algébriques. *Astérisque*, (69-70):218, 1987. With appendices by Daniel Bertrand and Jean-Pierre Serre.

[Zan12] U. Zannier. *Some problems of unlikely intersections in arithmetic and geometry*, volume 181 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012. With appendixes by David Masser.

[Zha98] S. Zhang. Equidistribution of small points on abelian varieties. *Ann. of Math. (2)*, 147(1):159–165, 1998.

Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, Canada M5S 2E4

E-mail address: dimitrov@math.toronto.edu

CNRS, IMJ-PRG, 4 place Jussieu, 75005 Paris, France

E-mail address: ziyang.gao@imj-prg.fr

Department of Mathematics and Computer Science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland

E-mail address: philipp.habegger@unibas.ch