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Geometric estimates for complex Monge–Ampère equations

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Abstract. We prove uniform gradient and diameter estimates for a family of geometric complex Monge–Ampère equations. Such estimates can be applied to study geometric regularity of singular solutions of complex Monge–Ampère equations. We also prove a uniform diameter estimate for collapsing families of twisted Kähler–Einstein metrics on Kähler manifolds of nonnegative Kodaira dimensions.

1. Introduction

Complex Monge–Ampère equations are a fundamental tool to study Kähler geometry and, in particular, canonical Kähler metrics of Einstein type on smooth and singular Kähler varieties. Yau's solution to the Calabi conjecture establishes the existence of Ricci flat Kähler metrics on Kähler manifolds of vanishing first Chern class by a priori estimates for complex Monge–Ampère equations [40].

Let (X, θ) be a Kähler manifold of complex dimension n equipped with a Kähler metric θ. We consider the following complex Monge–Ampère equation:

$$(\theta + i\partial\bar{\partial}\varphi)^n = e^{-f}\theta^n, \tag{1.1}$$

where $f \in C^\infty(X)$ satisfies the normalization condition

$$\int_X e^{-f}\theta^n = \int_X \theta^n = [\theta]^n.$$

In the deep work of Kolodziej [15], Yau's C^0-estimate for solutions of equation (1.1) is tremendously improved by applying the pluripotential theory and it has important applications for singular and degenerate geometric complex Monge–Ampère equations. More precisely, suppose the right-hand side of equation (1.1) satisfies the following L^p bound:

$$\int_X e^{-pf}\theta^n \leq K \quad \text{for some } p > 1;$$

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then there exists $C = C(X, \theta, p, K) > 0$ such that any solution φ of equation (1.1) satisfies the following L^∞-estimate:

$$\|\varphi - \sup_X \varphi\|_{L^\infty(X)} \le C.$$

In particular, equation (1.1) admits a unique continuous solution in $\mathrm{PSH}(X, \theta)$ as long as $e^{-f} \in L^p(X, \theta^n)$ without any additional regularity assumption for f. In [7, 16], it is shown that the bounded solution is also Hölder continuous and the Hölder exponent only depends only on n and p. However, in general the solution is not uniformly Lipschitz continuous (see e.g. [7]).

Complex Monge–Ampère equations are closely related to geometric equations of Einstein type, and in many geometric settings, one makes assumption on a uniform lower bound of the Ricci curvature. Therefore it is natural to consider the family of volume measures, whose curvature is uniformly bounded below. More precisely, we let $\Omega = e^{-f}\theta^n$ be a smooth volume form on X such that

$$\mathrm{Ric}(\Omega) = -i\partial\bar{\partial}\log\Omega \ge -A\theta \tag{1.2}$$

for some fixed constant $A \ge 0$. This is equivalent to saying,

$$i\partial\bar{\partial} f \ge -\mathrm{Ric}(\theta) - A\theta,$$

or

$$f \in \mathrm{PSH}(X, \mathrm{Ric}(\theta) + A\theta).$$

We will explain one of the motivations for condition (1.2) by some examples. Let $\{E_i\}_{i=1}^I$ and $\{F_j\}_{j=1}^J$ be two families of effective divisors of X. Let σ_{E_i} and σ_{F_j} be the defining sections for E_i and F_j, respectively, and h_{E_i} and h_{F_j} smooth hermitian metrics for the line bundles associated to E_i and F_j, respectively. In [40], Yau considers the following degenerate complex Monge–Ampère equations:

$$(\theta + i\partial\bar{\partial}\varphi)^n = \left(\frac{\sum_{i=1}^I |\sigma_{E_i}|_{h_{E_i}}^{2\beta_i}}{\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j}}\right)\theta^n, \tag{1.3}$$

where $\alpha_j, \beta_i > 0$, and various estimates are derived [40] assuming certain bounds on the degenerate right-hand side of equation (1.3).

If we consider the following case:

$$(\theta + i\partial\bar{\partial}\varphi)^n = \frac{\theta^n}{\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j}}, \tag{1.4}$$

the volume measure will blow up along common zeros of $\{F_j\}_{j=1}^J$. If the volume measure on the right-hand side of equation (1.4) is L^p-integrable for some $p > 1$, i.e.,

$$\Omega = \left(\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j}\right)^{-1}\theta^n$$

satisfies

$$\frac{\Omega}{\theta^n} = \left(\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j}\right)^{-1} \in L^p(X, \theta^n) \quad \text{for some } p > 1, \qquad \int_X \Omega = \int_X \theta^n,$$

then there exists a unique (up to a constant translation) continuous solution of (1.4). Furthermore, Ω can be approximated by smooth volume forms Ω_j (cf. [6]) satisfying

$$\mathrm{Ric}(\Omega_j) \geq -(A + A')\theta, \quad \left\| \frac{\Omega_j}{\theta^n} \right\|_{L^p(X,\theta^n)} \leq \left\| \frac{\Omega}{\theta^n} \right\|_{L^p(X,\theta^n)}, \quad \int_X \Omega_j = \int_X \theta^n$$

for some fixed $A' \geq 0$. Therefore condition (1.2) is a natural generalization of the above case. In the special case when $\{F_j\}_{j=1}^J$ is a union of smooth divisors with simple normal crossings and each $\alpha_j \in (0,1)$, the solution of equation (1.4) has conical singularities of cone angle of $2\pi(1-\alpha_j)$ along F_j, $j = 1, \ldots, J$.

We now state the first result of the paper.

Theorem 1.1. *Let (X, θ) be an Kähler manifold of complex dimension n equipped with a Kähler metric θ. We consider the following complex Monge–Ampère equation:*

$$(\theta + i\partial\bar{\partial}\varphi)^n = e^{\lambda\varphi}\Omega, \tag{1.5}$$

where $\lambda = 0$ or 1, and Ω is a smooth volume form satisfying $\int_X \Omega = \int_X \theta^n$. If

$$\int_X \left(\frac{\Omega}{\theta^n}\right)^p \theta^n \leq K, \quad \mathrm{Ric}(\Omega) = -i\partial\bar{\partial}\log\Omega \geq -A\theta, \tag{1.6}$$

for some $p > 1$, $K > 0$ and $A \geq 0$, then there exists a constant $C = C(X, \theta, p, K, A) > 0$ such that the solution φ of equation (1.5) and the Kähler metric g associated to the Kähler form $\omega = \theta + i\partial\bar{\partial}\varphi$ satisfy the following estimates:

(1) $\|\varphi - \sup_X \varphi\|_{L^\infty(X)} + \|\nabla_g \varphi\|_{L^\infty(X,g)} \leq C$,

(2) $\mathrm{Ric}(g) \geq -Cg$,

(3) $\mathrm{Diam}(X, g) \leq C$.

If we write $\Omega = e^{-f}\theta^n$, assumption (1.6) in Theorem 1.1 on Ω is equivalent to the following on f:

$$e^{-f} \in L^p(X,\theta), \quad \int_X e^{-f}\theta = [\theta]^n, \quad f \in \mathrm{PSH}(X, \mathrm{Ric}(\theta) + A\theta).$$

The function f is uniformly bounded above by the plurisubharmonicity and the Kähler metric g associated to $\omega = \theta + i\partial\bar{\partial}\varphi$ is bounded below by a fixed multiple of θ (see Lemma 2.2). However, one cannot expect that g is bounded from above since f is not uniformly bounded above as in the example of equation (1.4). Fortunately, we can bound the diameter of (X, g) uniformly by Theorem 1.1.

The gradient estimate in Theorem 1.1 is a generalization of the gradient estimate in [26]. The new insight in our approach is that one should estimate gradient and higher-order estimates of the potential functions with respect to the new metric instead of a fixed reference metric for geometric complex Monge–Ampère equations such as those studied in Theorem 1.1. We refer interested readers to [3, 22, 23] for gradient estimates for complex Monge–Ampère equations with respect to various background metrics.

Let $\mathcal{M}(X, \theta, p, K, A)$ be the space of all solutions of equation (1.5), where Ω satisfies assumption (1.6) in Theorem 1.1. We also identify $\mathcal{M}(X, \theta, p, K, A)$ with the space of

Kähler forms $\omega = \theta + i\partial\bar{\partial}\varphi$ for $\varphi \in \mathcal{M}(X, \theta, p, K, A)$. An immediate consequence of Theorem 1.1 is a uniform noncollapsing condition for $\mathcal{M}(X, \theta, p, K, A)$. More precisely, there exists a constant $C = C(X, \theta, p, K, A) > 0$ such that for all Kähler metric g associated to $\omega \in \mathcal{M}(X, \theta, p, K, A)$ and for any point $x \in X$, $0 < r < 1$,

$$C^{-1} r^{2n} \leq \mathrm{Vol}_g(B_g(x, r)) \leq C r^{2n}, \tag{1.7}$$

where $B_g(x, r)$ is the geodesic ball centered at x with radius r in (X, g).

Combining the lower bound of Ricci curvature and the noncollapsing condition (1.7), we can apply the theory of degeneration of Riemannian manifolds [5] so that any sequence of Kähler manifolds $(X, g_j) \in \mathcal{M}(X, \theta, p, K, A)$, after passing to a subsequence, converges to a compact metric space (X_∞, d_∞) with well-defined tangent cones of Hausdorff dimension $2n$ at each point in X_∞. In the case of equation (1.4), we believe the solution induces a unique Riemannian metric space homeomorphic to the original manifold X and all tangent cones are unique. If this is true, one might even be able to establish higher-order expansions for the solution. The ultimate goal of our approach is to construct canonical domains and equations on the blow-up of solutions for geometric degenerate complex Monge–Ampère equations, by degeneration of Riemannian manifolds.

We also remark that if we replace the lower bound for $\mathrm{Ric}(\Omega)$ by an upper bound

$$\mathrm{Ric}(\Omega) \leq A\theta$$

in assumption (1.6) of Theorem 1.1, we can still obtain a uniform diameter upper bound. This in fact easily follows from the argument for the second-order estimates of Yau [40] and Aubin [1].

We will also use similar techniques in the proof of Theorem 1.1 to obtain diameter estimates in more geometric settings. Before that, let us introduce a few necessary and well-known notions in complex geometry.

Definition 1.1. Let X be a Kähler manifold and $\alpha \in H^2(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{R})$. Then the class α is nef if $\alpha + \mathcal{A}$ is a Kähler class for any Kähler class $\mathcal{A}$.

Definition 1.2. Let X be a Kähler manifold of complex dimension n and let the class $\alpha \in H^2(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{R})$ be nef. The numerical dimension of the class α is given by

$$\nu(\alpha) = \max\{k = 0, 1, \dots, n : \alpha^k \neq 0 \text{ in } H^{2k}(X, \mathbb{R})\};$$

when $\nu(\alpha) = n$, the class α is said to be big.

The numerical dimension $\nu(\alpha)$ is always no greater than $\dim_{\mathbb{C}}(X)$.

When the canonical bundle K_X is nef, X is said to be a minimal model. The abundance conjecture in birational geometry predicts that the canonical line bundle is always semi-ample (i.e., a sufficiently large power of the canonical line bundle is globally generated) if it is nef.

Definition 1.3. Let ϑ be a smooth real-valued closed $(1, 1)$-form on a Kähler manifold X. The extremal function V associated to the form ϑ is defined by

$$V(z) = \sup\Big\{\phi(z) : \vartheta + i\partial\bar{\partial}\phi \geq 0,\ \sup_X \phi = 0\Big\}$$

for all $z \in X$.

Any $\psi \in \mathrm{PSH}(X, \vartheta)$ is said to have minimal singularities defined by Demailly (cf. [2]) if $\psi - V$ is bounded.

Let (X, θ) be a Kähler manifold of complex dimension n equipped with a Kähler metric θ. Suppose χ is a real-valued smooth closed $(1,1)$-form and its class $[\chi]$ is nef and of numerical dimension κ. We consider the following family of complex Monge–Ampère equations:

$$(\chi + t\theta + i\partial\bar{\partial}\varphi_t)^n = t^{n-\kappa} e^{\lambda\varphi_t + c_t}\Omega \quad \text{for } t \in (0,1], \tag{1.8}$$

where $\lambda = 0$, or 1, and c_t is a normalizing constant such that

$$\int_X t^{n-\kappa} e^{c_t}\Omega = \int_X (\chi + t\theta)^n. \tag{1.9}$$

Straightforward calculations show that c_t is uniformly bounded for $t \in (0,1]$. The following proposition generalizes the result in [4, 10, 15, 42] by studying a family of collapsing complex Monge–Ampère equations. It also generalizes the results in [8, 9, 17] for the case when the limiting reference form is semi-positive.

Proposition 1.1. *We consider equation* (1.8) *with the normalization condition* (1.9). *Suppose the volume measure* Ω *satisfies*

$$\int_X \left(\frac{\Omega}{\theta^n}\right)^p \theta^n \le K$$

for some $p > 1$ *and* $K > 0$. *Then there exists a unique* $\varphi_t \in \mathrm{PSH}(X, \chi + t\theta)$ *up to a constant translation solving equation* (1.8) *for all* $t \in (0,1]$. *Furthermore, there exists a constant* $C = C(X, \chi, \theta, p, K) > 0$ *such that for all* $t \in (0,1]$,

$$\left\| \left(\varphi_t - \sup_X \varphi_t\right) - V_t \right\|_{L^\infty(X)} \le C,$$

where V_t *is the extremal function associated to* $\chi + t\theta$ *as in Definition* 1.3.

Proposition 1.1 can be applied to generalize Theorem 1.1, especially for minimal Kähler manifolds in a geometric setting.

Theorem 1.2. *Suppose* X *is a smooth minimal model equipped with a smooth Kähler form* θ. *For any* $t > 0$, *there exists a unique smooth twisted Kähler–Einstein metric* g_t *on* X *satisfying*

$$\mathrm{Ric}(g_t) = -g_t + t\theta. \tag{1.10}$$

There exists a constant $C = C(X, \theta) > 0$ *such that for all* $t \in (0,1]$,

$$\mathrm{Diam}(X, g_t) \le C.$$

Furthermore, for any $t_j \to 0$, *after passing to a subsequence, the twisted Kähler–Einstein manifolds* (X, g_{t_j}) *converge in Gromov–Hausdorff topology to a compact metric length space* $(\mathbf{Z}, d_{\mathbf{Z}})$. *The Kähler forms* ω_{t_j} *associated to* g_{t_j} *converge in distribution to a nonnegative closed current*

$$\widetilde{\omega} = \chi + i\partial\bar{\partial}\widetilde{\varphi}$$

for some $\widetilde{\varphi} \in \mathrm{PSH}(X, \chi)$ *of minimal singularities, where* $\chi \in [K_X]$ *is a fixed smooth closed* $(1,1)$-*form.*

Both Theorem 1.1 and Theorem 1.2 are generalization and improvement for the techniques developed in [26] for diameter and distance estimates. With the additional bounds on the volume measure, we transform Kolodziej's analytic L^∞-estimate to a geometric diameter estimate. The relation between analytic estimates of Kähler potentials and geometric estimates for distance functions was also studied in [20]. It is a natural question to ask how the metric space $(\mathbf{Z}, d_{\mathbf{Z}})$ is related to the current $\widetilde{\omega}$ on X. We conjecture $\widetilde{\omega}$ is smooth on an open dense set of X and its metric completion coincides with $(\mathbf{Z}, d_{\mathbf{Z}})$. However, at this moment, we do not even know the Hausdorff dimension or uniqueness of $(\mathbf{Z}, d_{\mathbf{Z}})$.

When X is a minimal model of general type, Theorem 1.2 is proved in [26, 27] and the result in [36] shows that the singular set is closed and of Hausdorff dimension no greater than $2n - 4$.

We can also replace the smooth Kähler form θ in Theorem 1.2 by Dirac measures along effective divisors. For example, if $\{E_j\}_{j=1}^J$ is a union of smooth divisors with normal crossings and

$$\sum_{j=1}^{J} a_j E_j$$

is an ample $\mathbb{Q}$-divisor with some $a_j \in (0, 1)$ for $j = 1, \dots, J$, then Theorem 1.2 also holds if we let $\theta = \sum_{j=1}^J a_j [E_j]$. In this case, the metric g_t is a conical Kähler–Einstein metric with cone angles of $2\pi(1 - a_j)$ along each complex hypersurface E_j.

A special case of the abundance conjecture is proved by Kawamata [14] for minimal models of general type. When X is a smooth minimal model of general type, it is recently proved by the third named author [27] that the limiting metric space $(\mathbf{Z}, d_{\mathbf{Z}})$ in Theorem 1.2 is unique and is homeomorphic to the algebraic canonical model X_{can} of X. This gives an analytic proof of Kawamata's result using complex Monge–Ampère equations, Riemannian geometry and geometric L^2-estimates. Theorem 1.2 also provides a Riemannian geometric model for the non-general type case. This analytic approach will shed light on the abundance conjecture if such a metric model is unique with reasonably good understanding of its tangle cones.

Theorem 1.2 can also be easily generalized to a Calabi–Yau manifold X equipped with a nef line bundle L over X of $\nu(L) = \kappa$.

Our final result assumes semi-ampleness for the canonical line bundle and aims to connect the algebraic canonical models to geometric canonical models. Let X be a Kähler manifold of complex dimension n. If the canonical bundle K_X is semi-ample, the pluricanonical system induces a holomorphic surjective map

$$\Phi : X \to X_{\mathrm{can}}$$

from X to its unique canonical model X_{can}. In particular, $\dim_{\mathbb{C}} X_{\mathrm{can}} = \nu(X)$. Let $\mathbf{S}$ be the set which consists of all singular fibers of Φ together with $\Phi^{-1}(S_{X_{\mathrm{can}}})$, where $S_{X_{\mathrm{can}}}$ is the singular set of X_{can}. The general fiber of Φ is a smooth Calabi–Yau manifold of complex dimension $n - \nu(X)$. It is proved in [26, 27] that there exists a unique twisted Kähler–Einstein current ω_{can} on X_{can} satisfying

$$\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}},$$

where $\Phi^*\omega_{\mathrm{can}} \in -c_1(X)$ and ω_{WP} is the Weil–Petersson metric for the variation of the Calabi–Yau fibers. In particular, ω_{can} has bounded local potentials and is smooth on $X_{\mathrm{can}} \setminus S_{\mathrm{can}}$. We let g_{can} be the smooth Kähler metric associated to ω_{can} on $X_{\mathrm{can}} \setminus S_{\mathrm{can}}$.

Theorem 1.3. *Suppose X is a projective manifold of complex dimension n equipped with a Kähler metric θ. If the canonical bundle K_X is semi-ample and $\nu(K_X) = \kappa \in \mathbb{N}$, then for the twisted Kähler–Einstein metrics g_t satisfying*

$$\mathrm{Ric}(g_t) = -g_t + t\theta, \quad t \in (0,1],$$

the following hold:

(1) *There exists $C > 0$ such that for all $t \in (0,1]$,*

$$\mathrm{Diam}(X, g_t) \leq C.$$

(2) *Let ω_t be the Kähler form associated to g_t. For any compact subset $K \subset\subset X \setminus \mathbf{S}$, we have*

$$\|g_t - \Phi^* g_{\mathrm{can}}\|_{C^0(K,\theta)} \to 0 \quad \text{as } t \to 0.$$

(3) *The rescaled metrics $t^{-1}\omega_t|_{X_y}$ converge uniformly to a Ricci-flat Kähler metric $\omega_{CY,y}$ on the fiber $X_y = \Phi^{-1}(y)$ for any $y \in X_{\mathrm{can}} \setminus \Phi(\mathbf{S})$, as $t \to 0$.*

(4) *For any sequence $t_j \to 0$, after passing to a subsequence, the manifolds (X, g_{t_j}) converge in Gromov–Hausdorff topology to a compact metric space $(\mathbf{Z}, d_{\mathbf{Z}})$. Furthermore, $X_{\mathrm{can}} \setminus S_{\mathrm{can}}$ is embedded as an open subset in the regular part $\mathcal{R}_{2\kappa}$ of $(\mathbf{Z}, d_{\mathbf{Z}})$ and the manifold $(X_{\mathrm{can}} \setminus S_{\mathrm{can}}, \omega_{\mathrm{can}})$ is locally isometric to its open image.*

In particular, if $\kappa = 1$, then $(\mathbf{Z}, d_{\mathbf{Z}})$ is homeomorphic to X_{can}, with the regular part being open and dense, and each tangent cone being a metric cone on $\mathbb{C}$ with cone angle less than or equal to 2π.

We remark that a special case of Theorem 1.3 is proved in [41] with a different approach for $\dim_{\mathbb{C}} X = 2$. In general, the collapsing theory in Riemannian geometry has not been fully developed except in lower dimensions. In the Kähler case, one hopes the rigidity properties can help us understand the collapsing behavior for Kähler metrics of Einstein type as well as long time solutions of the Kähler–Ricci flow on algebraic minimal models. Key analytic and geometric estimates in the proof of (2) in Theorem 1.3 are established in [29, 30] for the collapsing long time solutions of the Kähler–Ricci flow and its elliptic analogues. The proof for (3) and (4) is a technical modification of various local results of [11, 12, 37, 38], where collapsing behavior for families of Ricci-flat Calabi–Yau manifolds is comprehensively studied. Theorem 1.3 should also hold for Kähler manifolds with some additional arguments.

Finally, we will also apply our method to a continuity scheme proposed in [18] to study singularities arising from contraction of projective manifolds. This is an alternative approach for the analytic minimal model program developed in [29–31] to understand birational transformations via analytic and geometric methods [25, 28, 32–34]. Compared to the Kähler–Ricci flow, such a scheme has the advantage of prescribed Ricci lower bounds and so one can apply the Cheeger–Colding theory for degeneration of Riemannian manifolds, on the other hand, it loses the canonical soliton structure for the analytic transition of singularities corresponding to birational surgeries such as flips.

Let X be a projective manifold of complex dimension n. We choose an ample line bundle L on X and we can assume that $L - K_X$ is ample, otherwise we can replace L by a sufficiently large power of L. We choose $\theta \in [L - K_X]$ to be a smooth Kähler form and consider the following curvature equation:

$$\mathrm{Ric}(g_t) = -g_t + t\theta, \quad t \in [0,1]. \tag{1.11}$$

Let

$$t_{\min} = \inf\{t \in [0,1] : \text{equation (1.11) is solvable at } t\}.$$

It is straightforward to verify that $t_{\min} < 1$ by the usual continuity method (cf. [18]). The goal is to solve equation (1.11) for all $t \in (0,1]$, however, one might have to stop at $t = t_{\min}$ when K_X is not nef.

Theorem 1.4. *Let g_t the solution of equation* (1.11) *for $t \in (t_{\min}, 1]$. There exists a constant $C = C(X,\theta) > 0$ such that for any $t \in (t_{\min}, 1]$,*

$$\mathrm{Diam}(X, g_t) \leq C.$$

Theorem 1.2 is a special case of Theorem 1.4 when $t_{\min} = 0$ (cf. [26]). When $t_{\min} > 0$, Theorem 1.4 is also proved in [19] with the additional assumption that $t_{\min}L + (1 - t_{\min})K_X$ is semi-ample and big. The diameter estimate immediately allows one to identify the geometric limit as a compact metric length space when $t \to t_{\min}$. In particular, it is shown in [19] that the limiting metric space is homeomorphic to the projective variety from the contraction induced by the $\mathbb{Q}$-line bundle $t_{\min}L + (1 - t_{\min})K_X$ when it is big and semi-ample. One can also use Theorem 1.4 to obtain a weaker version of Kawamata's base point free theorem in the minimal model theory (cf. [13]). If $t_{\min}L + (1 - t_{\min})K_X$ is not big, our diameter estimate still holds and we conjecture the limiting collapsed metric space of (X, g_t) as $t \to t_{\min}$ is unique and is homeomorphic a lower dimensional projective variety from the contraction induced by $t_{\min}L + (1 - t_{\min})K_X$.

2. Proof of Theorem 1.1

Throughout this section, we let $\varphi \in \mathrm{PSH}(X,\theta)$ be the solution of equation (1.5) satisfying condition (1.6) in Theorem 1.1. We let $\omega = \chi + i\partial\bar\partial\varphi$ and let g be the Kähler metric associated to ω.

Lemma 2.1. *There exists a constant $C = C(X,\theta,p,K) > 0$ such that*

$$\|\varphi - \sup_X \varphi\|_{L^\infty(X)} \leq C.$$

Proof. The L^∞-estimate immediately follows from Kolodziej's theorem [15]. □

The following is a result similar to Schwarz lemma [39].

Lemma 2.2. *There exists a constant $C = C(X,\theta,p,K,A) > 0$ such that*

$$\omega \geq C\theta.$$

Proof. There exists a constant $C = C(X,\theta,A) > 0$ such that

$$\Delta_\omega \log \mathrm{tr}_\omega(\theta) \geq -C - C\,\mathrm{tr}_\omega(\theta),$$

where Δ_ω is the Laplace operator associated with ω. Then let

$$H = \log \mathrm{tr}_\omega(\theta) - B(\varphi - \sup_X \varphi)$$

for some $B > 2C$. Then

$$\Delta_\omega H \geq C\, \mathrm{tr}_\omega(\theta) - C.$$

It follows from maximum principle and the L^∞-estimate in Lemma 2.1 that

$$\sup_X \mathrm{tr}_\omega \theta \leq C. \qquad \square$$

Lemma 2.2 immediately gives the uniform Ricci lower bound.

Lemma 2.3. *There exists a constant* $C = (X, \theta, p, K, A) > 0$ *such that*

$$\mathrm{Ric}(g) \geq -Cg.$$

Proof. We calculate

$$\mathrm{Ric}(g) = -\lambda g + \mathrm{Ric}(\Omega) + \lambda\theta \geq -\lambda g - (A - \lambda)\theta \geq -Cg$$

for some fixed constant $C > 0$ by Lemma 2.2. $\square$

We will now prove the uniform diameter bound.

Lemma 2.4. *There exists a constant* $C = (X, \theta, p, K, A) > 0$ *such that*

$$\mathrm{Diam}(X, g) \leq C.$$

Proof. We first fix a sufficiently small $\epsilon = \epsilon(p) > 0$ so that $p - \epsilon > 1$. Without loss of generality we may assume $\mathrm{Diam}(X, g) = D$ for some $D \geq 100$. Let $\gamma : [0, D] \to X$ be a normal minimal geodesic with respect to the metric g and choose the points $\{x_i = \gamma(6i)\}_{i=0}^{[D/6]}$. The balls $\{B_g(x_i, 3)\}_{i=0}^{[D/6]}$ are disjoint, so

$$\sum_{i=0}^{[D/6]} \big(\mathrm{Vol}_{\theta^n}(B_g(x_i, 3)) + \mathrm{Vol}_\Omega(B_g(x_i, 3))\big) \leq \int_X \theta^n + \Omega = 2V,$$

hence there exists a geodesic ball $B_g(x_i, 3)$ such that

$$\mathrm{Vol}_{\theta^n}(B_g(x_i, 3)) + \mathrm{Vol}_\Omega(B_g(x_i, 3)) \leq 12VD^{-1}.$$

We fix such an x_i and construct a cut-off function $\eta(x) = \rho(r(x)) \geq 0$ with

$$r(x) = d_g(x, x_i)$$

such that

$$\eta = 1 \quad \text{on } B_g(x_i, 1), \qquad \eta = 0 \quad \text{outside } B_g(x_i, 2),$$

and

$$\rho \in [0, 1], \quad \rho^{-1}(\rho')^2 \leq 10, \quad |\rho''| \leq 10.$$

Define a piecewise linear continuous function $\tilde{F}:\mathbb{R}\to\mathbb{R}$ such that $F(t)=D^{\frac{\epsilon}{p(p-\epsilon)}}$ when $t\in[0,2]$, $\tilde{F}(t)\equiv a$ when $t\geq 3$ and $\tilde{F}(t)$ is linear when $t\in[2,3]$, where $a>0$ is a constant to be determined. Denote $F(x)=\tilde{F}(r(x))$; then $F\equiv a$ outside $B_g(x_i,3)$, $F=D^{\frac{\epsilon}{p(p-\epsilon)}}$ on $B_g(x_i,2)$. We choose the constant $a>0$ so that $\int_X F\Omega=[\theta]^n=V$. We observe that

$$V=a\,\mathrm{Vol}_\Omega(X\setminus B_g(x_i,3))+\int_{B_g(x_i,3)}F\Omega\geq V(1-12D^{-1})a\geq\frac{V}{2}a$$

so $0<a\leq 2$. Then it follows that

$$\int_X\left(\frac{F\Omega}{\theta^n}\right)^{p-\epsilon}\theta^n\leq\left(\int_X F^{\frac{p(p-\epsilon)}{\epsilon}}\theta^n\right)^{\frac{\epsilon}{p}}\left(\int_X\left(\frac{\Omega}{\theta^n}\right)^p\theta^n\right)^{\frac{p-\epsilon}{p}}\leq C$$

for some $C=C(X,\theta,p,K)>0$.

We now consider the equation

$$(\theta+i\partial\bar{\partial}\phi)^n=e^{\lambda\phi}F\Omega.$$

By similar arguments as before, $\|\phi-\sup_X\phi\|_{L^\infty}\leq C=C(X,\theta,p,K)$. Let $\hat{g}=\theta+i\partial\bar{\partial}\phi$. Then on $B_g(x_i,2)$,

$$\mathrm{Ric}(\hat{g})=-\lambda\hat{g}+\mathrm{Ric}(\Omega)+\lambda\theta,\quad \mathrm{Ric}(g)=-\lambda g+\mathrm{Ric}(\Omega)+\lambda\theta.$$

In particular,

$$\Delta_g\log\frac{\hat{\omega}^n}{\omega^n}=-\lambda n+\lambda\,\mathrm{tr}_g(\hat{g}),$$

where $\Delta_g=\Delta_\omega$. Let

$$H=\eta\Big(\log\frac{\hat{\omega}^n}{\omega^n}-\Big(\big(\varphi-\sup_X\varphi\big)-\big(\phi-\sup_X\phi\big)\Big)\Big).$$

On $B_g(x_i,2)$, we have

$$\Delta_g H=-(\lambda+1)n+(\lambda+1)\,\mathrm{tr}_g(\hat{g})\geq -2n+n\left(\frac{\hat{\omega}^n}{\omega^n}\right)^{\frac{1}{n}}.$$

In general, on the support of η, we have

$$\begin{aligned}\Delta_g H&\geq\eta\Big(-2n+n\left(\frac{\hat{\omega}^n}{\omega^n}\right)^{\frac{1}{n}}\Big)+2\eta^{-1}\,\mathrm{Re}(\nabla H\cdot\nabla\eta)-2\frac{H|\nabla\eta|^2}{\eta^2}+\eta^{-1}H\Delta_g\eta\\&\geq\eta^{-1}\Big(C\eta^2e^{\frac{H}{n\eta}}+2\,\mathrm{Re}(\nabla H\cdot\nabla\eta)-2\frac{H|\nabla\eta|^2}{\eta}+H\Delta_g\eta-2n\eta^2\Big).\end{aligned}$$

We may assume $\sup_X H>0$, otherwise we already have upper bound of H. The maximum of H must lie at $B_g(x_i,2)$ and at this point

$$\Delta_g H\leq 0,\quad |\nabla H|^2=0.$$

By Laplacian comparison we have

$$\Delta_g\eta=\rho'\Delta r+\rho''\geq -C,\quad\frac{|\nabla\eta|^2}{\eta}=\frac{(\rho')^2}{\rho}\leq C.$$

So at the maximum of H, it holds that

$$0 \geq C\eta^2 e^{\frac{H}{nn}} - CH - 2n \geq CH^2 - CH - 2n,$$

therefore we have $\sup_X H \leq C$. In particular, on the ball $B_g(x_i, 1)$ where $\eta \equiv 1$, it follows that $\frac{\hat{\omega}^n}{\omega^n} \leq C$. From the definition of $\hat{\omega}$ and ω,

$$C \geq \frac{\hat{\omega}^n}{\omega^n} = D^{\frac{\epsilon}{p(p-\epsilon)}} e^{\lambda(\phi-\varphi)}.$$

Combined with the L^∞-estimate of ϕ and φ, we conclude that

$$D \leq C = C(n, p, \theta, A, K). \qquad \square$$

Lemma 2.5. *There exists a constant* $C = (X, \theta, p, K, A) > 0$ *such that*

$$\sup_X |\nabla_g \varphi|_g \leq C.$$

Proof. Straightforward calculations show that

$$\begin{aligned} \Delta_g \varphi &= n - \operatorname{tr}_g(\theta), \\ \Delta_g |\nabla\varphi|_g^2 &= |\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2 + g^{i\bar{l}} g^{k\bar{j}} R_{i\bar{j}} \varphi_k \varphi_{\bar{l}} - 2\nabla\varphi \cdot \nabla \operatorname{tr}_g(\theta) \\ &\geq |\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2 - C|\nabla\varphi|^2 - 2\nabla\varphi \cdot \nabla \operatorname{tr}_g(\theta), \end{aligned}$$

and

$$\Delta_g \operatorname{tr}_g \theta = \operatorname{tr}_g \theta \cdot \Delta_g \log \operatorname{tr}_g \theta + \frac{|\nabla \operatorname{tr}_g \theta|^2}{\operatorname{tr}_g \theta} \geq -C + c_0 |\nabla \operatorname{tr}_g \theta|^2$$

for some uniform constant $c_0, C > 0$. We choose constants α and B satisfying

$$\alpha > 4c_0^{-1} > 4, \ B > \sup_X \varphi + 1$$

and define

$$H = \frac{|\nabla\varphi|^2}{B - \varphi} + \alpha \operatorname{tr}_g \theta.$$

Then we have

$$\begin{aligned} \Delta H \geq{}& \frac{|\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2}{B-\varphi} - C\frac{|\nabla\varphi|^2}{B-\varphi} - \frac{|\nabla\varphi|^2(\operatorname{tr}_g \theta - n)}{(B-\varphi)^2} \\ & - 2(1+\alpha)\frac{\langle \nabla\varphi, \nabla \operatorname{tr}_g \theta\rangle}{B-\varphi} - \alpha C + \alpha c_0 |\nabla \operatorname{tr}_g \theta|^2 + 2\left\langle \frac{\nabla\varphi}{B-\varphi}, \nabla H\right\rangle. \end{aligned} \tag{2.1}$$

We may assume at the maximum point $z_{\max}$ of H, $|\nabla\varphi| > \alpha$ and $H > 0$, otherwise we are done. At $z_{\max}$,

$$\nabla H = 0, \quad \Delta H \leq 0$$

and so at $z_{\max}$,

$$\nabla|\nabla\varphi| = \frac{1}{2}\left(-H\frac{\nabla\varphi}{|\nabla\varphi|} - \alpha(B-\varphi)\frac{\nabla \operatorname{tr}_g \theta}{|\nabla\varphi|} + \alpha\frac{\operatorname{tr}_g \theta \nabla\varphi}{|\nabla\varphi|}\right).$$

By Kato's inequality

$$|\nabla|\nabla\varphi||^2 \leq \frac{|\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2}{2},$$

it follows that

$$
\begin{aligned}
(2.2)\qquad \frac{|\nabla\nabla\varphi|^2+|\nabla\bar\nabla\varphi|^2}{B-\varphi} &\ge \frac{1}{2(B-\varphi)}\bigg(H^2+\alpha^2(B-\varphi)^2(\mathrm{tr}_g\,\theta)^2+\alpha^2\frac{|\nabla\,\mathrm{tr}_g\,\theta|^2}{|\nabla\varphi|^2}\\
&\qquad -2\alpha H(B-\varphi)\,\mathrm{tr}_g\,\theta-2\alpha H\frac{|\nabla\,\mathrm{tr}_g\,\theta|}{|\nabla\varphi|}\\
&\qquad -2\alpha^2(B-\varphi)\,\mathrm{tr}_g\,\theta\frac{|\nabla\,\mathrm{tr}_g\,\theta|}{|\nabla\varphi|}\bigg)\\
&\ge \frac{H^2}{4(B-\varphi)}-CH-\frac{|\nabla\,\mathrm{tr}_g\,\theta|^2}{B-\varphi}-C|\nabla\,\mathrm{tr}_g\,\theta|
\end{aligned}
$$

for some uniform constant $C>0$. After substituting inequality (2.2) to (2.1) and applying Cauchy–Schwarz inequality, we have at $z_{\max}$

$$
\begin{aligned}
0 &\ge \frac{H^2}{4(B-\varphi)}-CH-C-\frac{2|\nabla\,\mathrm{tr}_g\,\theta|^2}{B-\varphi}-C|\nabla\,\mathrm{tr}_g\,\theta|+4|\nabla\,\mathrm{tr}_g\,\theta|^2\\
&\ge \frac{H^2}{4(B-\varphi)}-CH-C
\end{aligned}
$$

for some uniform constant $C>0$. Therefore $\max_X H\le C$ for some $C=C(X,\theta,\Omega,A,p,K)$. The lemma then immediately follows from Lemma 2.1 and Lemma 2.2. □

3. Proof of Proposition 1.1

In this section, we will prove Proposition 1.1 by applying the techniques in [4, 8, 10, 15].

Let X be a Kähler manifold of dimension n. Suppose α is nef class on X of numerical dimension $\kappa\ge 0$. Let $\chi\in\alpha$ be a smooth closed $(1,1)$-form. We define the extremal function V_χ by

$$V_\chi=\sup\{\phi:\chi+i\partial\bar\partial\phi\ge 0,\ \phi\le 0\}.$$

Let θ be a fixed smooth Kähler metric on X. Then we define the perturbed extremal function V_t for $t\in(0,1]$ by

$$V_t=\sup\{\phi:\chi+t\theta+i\partial\bar\partial\phi\ge 0,\ \phi\le 0\}.$$

The above extremal functions were introduced in [4] when α is big.

We first rewrite equation (1.8) for $\lambda=0$ as follows:

$$(3.1)\qquad (\chi+t\theta+i\partial\bar\partial\varphi_t)^n=t^{n-\kappa}e^{-f+c_t}\theta^n,\quad \sup_X\varphi_t=0,\quad t\in(0,1],$$

by letting $\Omega=e^{-f}\theta^n$, where c_t is the normalizing constant satisfying

$$t^{n-\kappa}\int_X e^{-f+c_t}\theta^n=\int_X(\chi+t\theta)^n.$$

The function f satisfies the following uniform bound:

$$\int_X e^{-pf}\theta^n\le K$$

for some $p>1$ and $K>0$.

The following definition is an extension of the capacity introduced in [4, 8, 10, 15].

Definition 3.1. We define the capacity $\mathrm{Cap}_{\chi_t}(\mathcal{K})$ for a subset $\mathcal{K} \subset X$ by

$$\mathrm{Cap}_{\chi_t}(\mathcal{K}) = \sup\left\{\int_{\mathcal{K}} (\chi_t + i\partial\bar{\partial}u)^n : u \in \mathrm{PSH}(X, \chi_t),\ 0 \leq u - V_t \leq 1\right\},$$

where $\chi_t = \chi + t\theta$ is the reference metric in (3.1). We also define the extremal function $V_{t,\mathcal{K}}$ by

$$V_{t,\mathcal{K}} = \sup\{u \in \mathrm{PSH}(X, \chi_t) : u \leq 0 \text{ on } \mathcal{K}\}.$$

If $\mathcal{K}$ is open, then we have

(1) $V_{t,\mathcal{K}} \in \mathrm{PSH}(X, \chi_t) \cap L^\infty(X)$,

(2) $(\chi_t + i\partial\bar{\partial}V_{t,\mathcal{K}})^n = 0$ on $X \setminus \overline{\mathcal{K}}$.

Lemma 3.1. *Let φ_t be the solution to* (3.1)*. Then there exist $\delta = \delta(X, \chi, \theta) > 0$ and $C = C(X, \chi, \theta, p, K) > 0$ such that for any open set $\mathcal{K} \subset X$ and $t \in (0, 1]$,*

$$\frac{1}{[\chi_t^n]} \int_{\mathcal{K}} (\chi_t + i\partial\bar{\partial}\varphi_t)^n \leq Ce^{-\delta\left(\frac{[\chi_t^n]}{\mathrm{Cap}_{\chi_t}(K)}\right)^{\frac{1}{n}}}.$$

Proof. Since $[\chi^m] = 0$ for $\kappa + 1 \leq m \leq n$, it follows that

$$\begin{aligned}[\chi_t^n] &= \int_X \chi_t^n = \int_X \sum_{k=0}^{n} \binom{n}{k} \chi^k \wedge t^{n-k}\theta^{n-k} \\ &= \int_X \sum_{k=0}^{\kappa} \binom{n}{k} \chi^k \wedge t^{n-k}\theta^{n-k} = O(t^{n-\kappa}).\end{aligned}$$

It follows that the normalizing constant c_t in (3.1) is uniform bounded. Let $M_{t,\mathcal{K}} = \sup_X V_{t,\mathcal{K}}$. Then we have

$$\begin{aligned}\frac{1}{[\chi_t^n]} \int_{\mathcal{K}} (\chi_t + i\partial\bar{\partial}\varphi_t)^n &= \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]} \int_{\mathcal{K}} e^{-f}\theta^n \\ &\leq \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]} \int_{\mathcal{K}} e^{-f} e^{-\frac{\delta V_{t,\mathcal{K}}}{q}} \theta^n \quad (\text{since } V_{t,\mathcal{K}} \leq 0 \text{ on } \mathcal{K}) \\ &\leq \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]} e^{-\frac{\delta M_{t,\mathcal{K}}}{q}} \int_X e^{-f} e^{-\frac{\delta(V_{t,\mathcal{K}} - M_{t,\mathcal{K}})}{q}} \theta^n \\ &\leq \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]} e^{-\frac{\delta M_{t,\mathcal{K}}}{q}} \left(\int_X e^{-pf}\theta^n\right)^{\frac{1}{p}} \left(\int_X e^{-\delta(V_{t,\mathcal{K}} - M_{t,\mathcal{K}})}\theta^n\right)^{\frac{1}{q}} \\ &\leq Ce^{-\frac{\delta M_{t,\mathcal{K}}}{q}},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Obviously, there exists $\gamma = \gamma(X, \chi, \theta) > 0$ such that for all $t \in (0, 1]$,

$$V_{t,\mathcal{K}} \in \mathrm{PSH}(X, \gamma\theta).$$

We apply the global Hörmander's estimate ([35]) so that there exists $\delta = \delta(X, \chi, \theta) > 0$ such that

$$\int_X e^{-\delta(V_{t,\mathcal{K}} - \sup_X V_{t,\mathcal{K}})}\theta^n \leq C_\delta.$$

To complete the proof, it suffices to show

$$M_{t,\mathcal{K}} + 1 \geq \left(\frac{[\chi_t^n]}{\mathrm{Cap}_{\chi_t}(\mathcal{K})} \right)^{\frac{1}{n}}. \tag{3.2}$$

First we observe that by definition

$$\sup_X \Big((V_{t,\mathcal{K}} - \sup_X V_{t,\mathcal{K}}) - V_t \Big) \leq 0,$$

since $V_{t,\mathcal{K}} - \sup_X V_{t,\mathcal{K}} \in \mathrm{PSH}(X, \chi_t)$ is nonpositive. On the other hand, $V_{t,\mathcal{K}} \geq V_t$. This immediately implies that

$$0 \leq V_{t,\mathcal{K}} - V_t \leq \sup_X V_{t,\mathcal{K}} = M_{t,\mathcal{K}}. \tag{3.3}$$

We break the rest of the proof into two cases.

The case when $M_{t,K} > 1$. We let

$$\psi_{t,\mathcal{K}} = M_{t,\mathcal{K}}^{-1}(V_{t,\mathcal{K}} - V_t) + V_t.$$

Then

$$V_t \leq \psi_{t,\mathcal{K}} \leq V_t + 1$$

and by (3.3),

$$\begin{aligned} \frac{1}{M_{t,\mathcal{K}}^n} &= \frac{1}{M_{t,\mathcal{K}}^n} \frac{\int_X (\chi_t + i\partial\bar{\partial} V_{t,\mathcal{K}})^n}{[\chi_t^n]} \\ &= \frac{1}{[\chi_t^n]} \int_{\overline{\mathcal{K}}} \big(M_{t,\mathcal{K}}^{-1}\chi_t + i\partial\bar{\partial}(M_{t,\mathcal{K}}^{-1} V_{t,\mathcal{K}})\big)^n \\ &\leq \frac{1}{[\chi_t^n]} \int_{\overline{\mathcal{K}}} \big(M_{t,\mathcal{K}}^{-1}\chi_t + i\partial\bar{\partial}(M_{t,\mathcal{K}}^{-1} V_{t,\mathcal{K}}) + (1 - M_{t,\mathcal{K}}^{-1})(\chi_t + i\partial\bar{\partial} V_t)\big)\big)^n \\ &= \frac{1}{[\chi_t^n]} \int_{\overline{\mathcal{K}}} (\chi_t + i\partial\bar{\partial}\psi_{t,\mathcal{K}})^n \\ &\leq \frac{\mathrm{Cap}_{\chi_t}(\mathcal{K})}{[\chi_t^n]}. \end{aligned} \tag{3.4}$$

The case when $M_{t,K} \leq 1$. By (3.3),

$$0 \leq V_{t,\mathcal{K}} - V_t \leq \sup_X V_{t,\mathcal{K}} = M_{t,\mathcal{K}} \leq 1.$$

Now

$$[\chi_t^n] = \int_{\overline{\mathcal{K}}} (\chi_t + i\partial\bar{\partial} V_{t,\mathcal{K}})^n \leq \mathrm{Cap}_{\chi_t}(\overline{\mathcal{K}}). \tag{3.5}$$

So in this case

$$\frac{[\chi_t^n]}{\mathrm{Cap}_{\chi_t}(\mathcal{K})} \leq 1.$$

Combining (3.4) and (3.5), (3.2) holds and we complete the proof of Lemma 3.1. □

The following is an immediate corollary of Lemma 3.1.

Corollary 3.1. *There exists $C = C(X, \chi, \theta, p, K) > 0$ such that for all $t \in (0, 1]$, we have*

$$\frac{1}{[\chi_t^n]} \int_{\mathcal{K}} (\chi_t + i\partial\bar{\partial}\varphi_t)^n \le C \left(\frac{\mathrm{Cap}_{\chi_t}(\mathcal{K})}{[\chi_t^n]} \right)^2.$$

Proof. This follows from Lemma 3.1 and the elementary inequality that

$$x^2 e^{-\delta x^{\frac{1}{n}}} \le C$$

for some uniform $C > 0$ and all $x \in (0, \infty)$. □

Lemma 3.2. *Let $u \in \mathrm{PSH}(X, \chi_t) \cap L^\infty(X)$. For any $s > 0$, $0 \le r \le 1$ and $t \in (0, 1]$, we have*

$$r^n \,\mathrm{Cap}_{\chi_t}(u - V_t < -s - r) \le \int_{\{u - V_t < -s\}} (\chi_t + i\partial\bar{\partial}u)^n. \tag{3.6}$$

Proof. For any $\phi \in \mathrm{PSH}(X, \chi_t)$ with $0 \le \phi - V_t \le 1$, we have

$$\begin{aligned}
r^n & \int_{\{u - V_t < -s - r\}} (\chi_t + i\partial\bar{\partial}\phi)^n \\
&= \int_{\{u - V_t < -s - r\}} (r\chi_t + i\partial\bar{\partial}(r\phi))^n \\
&\le \int_{\{u - V_t < -s - r\}} (\chi_t + i\partial\bar{\partial}(r\phi) + i\partial\bar{\partial}(1 - r)V_t)^n \\
&\le \int_{\{u - V_t < -s - r + r(\phi - V_t)\}} (\chi_t + i\partial\bar{\partial}(r\phi + (1 - r)V_t - s - r))^n \\
&\le \int_{\{u < r\phi + (1 - r)V_t - s - r\}} (\chi_t + i\partial\bar{\partial}u)^n \\
&\le \int_{\{u < V_t - s\}} (\chi_t + i\partial\bar{\partial}u)^n.
\end{aligned}$$

The third inequality follows from the comparison principle and the last inequality follows from the fact that

$$r\phi + (1 - r)V_t - s - r = r(\phi - V_t - 1) + V_t - s < V_t - s.$$

Taking supremum of all $\phi \in \mathrm{PSH}(X, \chi_t)$ with $0 \le \phi - V_t \le 1$, we get (3.6). □

Lemma 3.3. *Let φ_t be the solution to equation* (3.1). *Then there exists a constant $C = C(X, \chi, \theta, p, K) > 0$ such that for all $s > 1$,*

$$\frac{1}{[\chi_t^n]} \mathrm{Cap}_{\chi_t}(\{\varphi_t - V_t < -s\}) \le \frac{C}{(s - 1)^{\frac{1}{q}}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Applying Lemma 3.2 to $u = \varphi_t$ and $r = 1$, we have

$$\begin{aligned}
\frac{1}{[\chi_t^n]} \operatorname{Cap}_{\chi_t}(\{\varphi_t - V_t < -s\}) &\le \frac{1}{[\chi_t^n]} \int_{\{\varphi_t - V_t < -(s-1)\}} (\chi_t + i\partial\bar\partial\varphi_t)^n \\
&= \frac{1}{[\chi_t^n]} \int_{\{\varphi_t - V_t < -(s-1)\}} t^{n-\kappa} e^{-f+c_t} \theta^n \\
&\le \frac{C}{(s-1)^{\frac{1}{q}}} \int_{\{\varphi_t - V_t < -(s-1)\}} (-\varphi_t + V_t)^{\frac{1}{q}} e^{-f} \theta^n \\
&\le \frac{C}{(s-1)^{\frac{1}{q}}} \bigg(\int_{\{\varphi_t - V_t < -(s-1)\}} e^{-pf} \theta^n \bigg)^{\frac{1}{p}} \\
&\qquad \times \bigg(\int_{\{\varphi_t - V_t < -(s-1)\}} (-\varphi_t + V_t) \theta^n \bigg)^{\frac{1}{q}} \\
&\le \frac{C}{(s-1)^{\frac{1}{q}}} \bigg(\int_X (-\varphi_t) \theta^n \bigg)^{\frac{1}{q}},
\end{aligned}$$

where in the last inequality we use the assumption that $e^{-f} \in L^p(\theta^n)$, $V_t \le 0$ and $\varphi_t \le 0$. On the other hand, since $\varphi_t \in \mathrm{PSH}(X, \chi_t) \subset \mathrm{PSH}(X, C\theta)$ for some large $C > 0$ and $\sup_X \varphi_t = 0$, it follows from Green's formula that

$$\int_X (-\varphi_t) \theta^n \le C$$

for some uniform constant C. The lemma follows by combining the inequalities above. □

The following lemma is well known and its proof can be found, e.g., in [10, 15].

Lemma 3.4. *Let* $F : [0, \infty) \to [0, \infty)$ *be a non-increasing right-continuous function satisfying* $\lim_{s\to\infty} F(s) = 0$. *If there exist* $\alpha, A > 0$ *such that for all* $s > 0$ *and* $0 \le r \le 1$,

$$rF(s + r) \le A(F(s))^{1+\alpha},$$

then there exists $S = S(s_0, \alpha, A)$ *such that*

$$F(s) = 0$$

for all $s \ge S$, *where* s_0 *is the smallest* s *satisfying* $(F(s))^\alpha \le (2A)^{-1}$.

Proof of Proposition 1.1. Define for each fixed $t \in (0, 1]$,

$$F(s) = \bigg(\frac{\operatorname{Cap}_{\chi_t}(\{\varphi_t - V_t < -s\})}{[\chi_t]^n} \bigg)^{\frac{1}{n}}.$$

By Corollary 3.1 and Lemma 3.2 applied to the function φ_t, we have

$$rF(s + r) \le AF(s)^2 \quad \text{for all } r \in [0, 1],\ s > 0,$$

for some uniform constant $A > 0$ independent of $t \in (0, 1]$.

Lemma 3.3 implies that $\lim_{s\to\infty} F(s) = 0$ and the s_0 in Lemma 3.4 can be taken as less than $(2AC)^q$, which is a uniform constant. It follows from Lemma 3.4 that $F(s) = 0$ for all $s > S$, where $S \leq 2 + s_0$. On the other hand, if $\mathrm{Cap}_{\chi_t}(\{\varphi_t - V_t < -s\}) = 0$, by Lemma 3.1 and equation (3.1), we have

$$\int_{\{\varphi_t - V_t < -s\}} e^{-f}\theta^n = 0,$$

hence the set $\{\varphi_t - V_t < -s\} = \emptyset$. Thus $\inf_X(\varphi_t - V_t) \geq -S$. Thus we finish the proof of Proposition 1.1. □

Therefore we have proved Proposition 1.1 when $\lambda = 0$. We finish this section by proving the case when $\lambda = 1$. To this end, we consider the following complex Monge–Ampère equations for $t \in (0, 1]$:

$$(\chi + t\theta + i\partial\bar{\partial}\varphi_t)^n = t^{n-\kappa} e^{\varphi_t - f + c_t}\theta^n,$$

where $f \in C^\infty(X)$ and c_t is the normalizing constant satisfying

$$t^{n-\kappa}\int_X e^{-f+c_t}\theta^n = \int_X (\chi + t\theta)^n.$$

Corollary 3.2. *If* $\|e^{-f}\|_{L^p(X,\theta^n)} \leq K$, *for* $p > 1$ *and* $K > 0$, *Then there exists a constant* $C = C(X, \chi, \theta, p, K) > 0$ *such that*

$$\|\varphi_t - V_t\|_{L^\infty} \leq C.$$

Proof. Since for each $t > 0$, it is proved in [2] that V_t is $C^{1,\alpha}(X, \theta)$, we can always find $W_t \in C^\infty(X)$ such that $\sup_X |V_t - W_t| \leq 1$. Furthermore, V_t is uniformly bounded above for all $t \in (0, 1]$. We let ψ_t be the solution of

$$(\chi_t + i\partial\bar{\partial}\psi_t)^n = t^{n-\kappa} e^{-f+c_t+W_t}\theta^n, \quad \sup_X \psi_t = 0.$$

and

$$u_t = \varphi_t - \psi_t.$$

Then

$$\frac{(\chi_t + i\partial\bar{\partial}\psi_t + i\partial\bar{\partial}u_t)^n}{(\chi_t + i\partial\bar{\partial}\psi_t)^n} = e^{u_t + \psi_t - W_t}.$$

Since $\sup_X |\psi_t - W_t| \leq \sup_X |\psi_t - V_t| + 1$, the maximum principle immediately implies that

$$\|u_t\|_{L^\infty(X)} \leq \|\psi_t - V_t\|_{L^\infty(X)} + 1$$

and so

$$\|\varphi_t - V_t\|_{L^\infty(X)} \leq 2\|\psi_t - V_t\|_{L^\infty(X)} + 1.$$ □

4. Proof of Theorem 1.2

Let X be a Kähler manifold; X is said to be a minimal model if the canonical bundle K_X is nef. The numerical dimension of K_X is given by

$$\nu(K_X) = \max\{m = 0, \ldots, n : [K_X]^m \neq 0 \text{ in } H^{m,m}(X, \mathbb{C})\}.$$

Let θ be a smooth Kähler form on a minimal model X of complex dimension n. Let $\kappa = \nu(X)$, the numerical dimension of K_X. Let Ω be a smooth volume form on X. We let χ be defined by

$$\chi = i\partial\bar{\partial}\log\Omega \in K_X.$$

We consider the following Monge–Ampère equation for $t \in (0,\infty)$,

$$(\chi + t\theta + i\partial\bar{\partial}\varphi_t)^n = t^{n-\kappa}e^{\varphi_t}\Omega. \tag{4.1}$$

Since K_X is nef, $[\chi + t\theta]$ is a Kähler class for any $t > 0$. By Aubin and Yau's theorem, there exists a unique smooth solution φ_t solving (4.1) for all $t > 0$. Let $\omega_t = \chi + t\theta + i\partial\bar{\partial}\varphi$. Then ω_t satisfies

$$\mathrm{Ric}(\omega_t) = -\omega_t + t\theta.$$

In particular, any Kähler metric satisfying the above twisted Kähler–Einstein equation must coincide with ω_t.

Lemma 4.1. *There exists a constant $C > 0$ such that for all $t \in (0,1]$,*

$$C^{-1}t^{n-\kappa} \le [\chi + t\theta]^n \le Ct^{n-\kappa}.$$

Proof. First we note that $[\chi]^\kappa \cdot [\theta]^{n-\kappa} > 0$ because $[\chi]^\kappa \neq 0$ and $[\chi]$ is nef. Then

$$[\chi + t\theta]^n = t^{n-\kappa}\binom{n}{\kappa}[\chi]^\kappa \cdot [\theta]^{n-\kappa} + t^{n-\kappa+1}\left(\sum_{j=\kappa+1}^{n}\binom{n}{j}t^{j-\kappa-1}[\chi]^j \cdot [\theta]^{n-j}\right). \qquad \square$$

Lemma 4.2. *Let $V_t = \sup\{u : u \in \mathrm{PSH}(X, \chi + t\theta),\ u \le 0\}$. Then there exists a constant $C > 0$ such that for all $t \in (0,1]$,*

$$\|\varphi_t - V_t\|_{L^\infty(X)} \le C.$$

Proof. The lemma immediately follows by applying Proposition 1.1 to (4.1). $\square$

We now prove the main result in this section.

Lemma 4.3. *There exists a constant $C > 0$ such that for all $t \in (0,1]$,*

$$\mathrm{Diam}(X, g_t) \le C.$$

Proof. In this proof we apply a similar argument to that used in the proof of Theorem 1.1. Suppose $\mathrm{Diam}(X, g_t) = D$ for some $D \ge 6$. Let $\gamma : [0, D] \to X$ be a smoothing minimizing geodesic with respect to the metric g_t and choose the points $\{x_i = \gamma(6i)\}_{i=0}^{[D/6]}$. It is clear that the balls $\{B_{g_t}(x_i, 3)\}$ are disjoint so

$$\sum_{i=0}^{[\frac{D}{6}]}\mathrm{Vol}_\Omega(B_{g_t}(x_i, 3)) \le \int_X \Omega = V,$$

where $\mathrm{Vol}_\Omega(B_{g_t}(x_i, 3)) = \int_{B_{g_t}}(x_i, 3)\Omega$. Hence there exists a geodesic ball $B_{g_t}(x_i, 3)$ such that

$$\mathrm{Vol}_\Omega(B_{g_t}(x_i, 3)) \le 6VD^{-1}.$$

We fix such an x_i and construct a cut-off function $\eta(x) = \rho(r(x)) \geq 0$ with $r(x) = d_{g_t}(x, x_i)$ such that

$$\eta = 1 \quad \text{on } B_{g_t}(x_i, 1), \qquad \eta = 0 \quad \text{outside } B_{g_t}(x_i, 2)$$

and

$$\rho \in [0, 1], \quad \rho^{-1}(\rho')^2 \leq C, \quad |\rho''| \leq C.$$

Define a function $F_t > 0$ on X such that

$$F_t = 1 \quad \text{outside } B_{g_t}(x_i, 3), \qquad F_t = D^{\frac{1}{2}} \quad \text{on } B_{g_t}(x_i, 2)$$

and

$$C^{-1} \leq \int_X F_t \Omega \leq C, \quad \int_X F_t^2 \Omega \leq C.$$

We now consider the equation

$$(\chi + t\theta + \psi_t)^n = t^{n-\kappa} e^{\psi_t} F_t \Omega \quad \text{for all } t \in (0, 1].$$

Applying Corollary 3.2, there exists a uniform constant $C > 0$ such that for all $t \in (0, 1]$,

$$\|\psi_t - V_t\|_{L^\infty(X)} \leq C,$$

and so by Lemma 4.2,

$$\|\varphi_t - \psi_t\|_{L^\infty(X)} \leq C. \tag{4.2}$$

Let $\hat{g}_t = \chi + t\theta_t + i\partial\bar{\partial}\psi_t$. Then on $B_{g_t}(x_i, 2)$,

$$\mathrm{Ric}(\hat{g}_t) = -\hat{g}_t + t\theta, \quad \mathrm{Ric}(g_t) = -g_t + t\theta,$$

and so

$$\Delta_{g_t} \log \frac{\hat{\omega}_t^n}{\omega_t^n} = -n + \mathrm{tr}_{g_t}(\hat{g}_t) \geq -n + n\left(\frac{\hat{\omega}_t^n}{\omega_t^n}\right)^{\frac{1}{n}}.$$

Let

$$H = \eta \log \frac{\hat{\omega}_t^n}{\omega_t^n}.$$

We may suppose $\sup_X H = H(z_{\max}) > 0$, otherwise we are done. The point $z_{\max}$ must lie in the support of η, and at $z_{\max}$ we have

$$\begin{aligned} 0 \geq \Delta_{g_t} H &\geq \frac{1}{\eta}\left(H\Delta_{g_t}\eta + 2\langle\nabla\eta, \nabla H\rangle - 2\frac{H}{\eta}|\nabla\eta|^2 - n\eta^2 + n\eta^2 e^{\frac{H}{n\eta}}\right) \\ &\geq \frac{1}{\eta}\left(\frac{1}{2n}H^2 - CH\right) \end{aligned}$$

for some uniform constant $C > 0$ for all $t \in (0, 1]$. The maximum principle implies that

$$\sup_X H \leq C(n);$$

in particular on $B_{g_t}(x_i, 1)$ where $\eta \equiv 1$, there exists $C > 0$ such that for all $t \in (0, 1]$,

$$\frac{\hat{\omega}_t^n}{\omega_t^n} = D^{\frac{1}{2}} e^{\psi_t - \varphi_t} \leq C.$$

By the uniform L^∞-estimate (4.2), there exists $C = C(n, \chi, \Omega, \theta)$ such that $D \leq C$. □

Now we can complete the proof of Theorem 1.2. Gromov's pre-compactness theorem and the diameter bound in Lemma 4.3 immediately imply that, after passing to a subsequence, (X, g_{t_j}) converges to a compact metric space. Since $\varphi_t - V_t$ is uniformly bounded and V_t is uniformly bounded below by V_0, φ_{t_j} always converges weakly to some $\varphi_\infty \in \mathrm{PSH}(X, \chi)$, after passing to a subsequence. In particular, there exists $C > 0$ such that

$$\|\varphi_\infty - V_0\|_{L^\infty(X)} \leq C,$$

where V_0 is the extremal function on X with respect to χ. □

5. Proof of Theorem 1.3

Our proof is based on the arguments of [29, 37, 38].

We fix some notations first. Recall X_{can} has dimension κ and χ is the restriction of the Fubini–Study metric on X_{can} from the embedding $X_{\mathrm{can}} \hookrightarrow \mathbb{CP}^{N_m}$, where

$$N_m + 1 = \dim H^0(X, mK_X).$$

Hence $\Phi^*\chi$ is a smooth nonnegative $(1,1)$-form on X, and in the following we identify χ with $\Phi^*\chi$ for simplicity. Let θ be a fixed Kähler metric on X.

Define a function $H \in C^\infty(X)$ as

$$\chi^\kappa \wedge \theta^{n-\kappa} = H\theta^n$$

which is the modulus squared of the Jacobian of the map $\Phi : (X, \theta) \to (X_{\mathrm{can}}, \chi)$ and vanishes on S, the indeterminacy set of Φ, hence $H^{-\gamma} \in L^1(X, \theta^n)$ for some small $\gamma > 0$. We fix a smooth nonnegative function σ on X_{can} as defined in [37], which satisfies

$$0 \leq \sigma \leq 1, \quad 0 \leq \sqrt{-1}\partial\sigma \wedge \bar\partial\sigma \leq C\chi, \quad -C\chi \leq i\partial\bar\partial\sigma \leq C\chi,$$

for some dimensional constant $C = C(\kappa) > 0$. From the construction, σ vanishes exactly on $S' = \Phi(S)$. There exist $\lambda > 0$, $C > 1$ such that for any $y \in X^\circ_{\mathrm{can}} = X_{\mathrm{can}} \setminus S'$ (see [37]),

$$\sigma(y)^\lambda \leq C \inf_{X_y} H, \quad \text{here } X_y = \Phi^{-1}(y).$$

The twisted Kähler–Einstein metric g_t in (1.10) satisfies the following complex Monge–Ampère equation (with $\theta = \theta$):

$$(\chi + t\theta + i\partial\bar\partial\varphi_t)^n = t^{n-\kappa} e^{\varphi_t}\Omega \quad \text{for all } t \in (0,1]. \tag{5.1}$$

In case K_X is semi-ample, $V_t = 0$ hence Corollary 3.2 implies (see also [8, 9, 15]):

Lemma 5.1. *There is a uniform constant $C > 0$ such that $\|\varphi_t\|_{L^\infty(X)} \leq C$.*

We have the following Schwarz lemma whose proof is similar to that of Lemma 2.2, so we omit it.

Lemma 5.2. *There exists a constant $C > 0$ such that $\mathrm{tr}_{\omega_t}\chi \leq C$ for all $t \in (0,1]$.*

We denote $\theta_y = \theta|_{X_y}$ for $y \in X_{\mathrm{can}}^\circ$, the restriction of θ on the fiber X_y which is a smooth $(n-\kappa)$-dimensional Calabi–Yau submanifold of X. We will omit the subscript t in φ_t and simply write $\varphi = \varphi_t$, and define

$$\overline{\varphi}_y = \fint_{X_y} \varphi \theta_y^{n-\kappa}$$

to be the average of φ over the fiber X_y. Denote the reference metric $\hat{\omega}_t = \chi + t\theta$. We calculate

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi)|_{X_y} = (t\theta_y + i\partial\bar{\partial}(\varphi - \overline{\varphi}_y))|_{X_y} = \omega_t|_{X_y},$$

hence

$$(\theta_y + t^{-1} i\partial\bar{\partial}(\varphi - \overline{\varphi}_y)|_{X_y})^{n-\kappa} = t^{-n+\kappa}\omega_{t,y}^{n-\kappa}.$$

On the other hand,

$$\begin{aligned} t^{-n+\kappa}\frac{\omega_{t,y}^{n-\kappa}}{\theta_y^{n-\kappa}} &= t^{-n+\kappa}\frac{\omega_t^{n-\kappa} \wedge \chi^\kappa}{\theta^{n-\kappa} \wedge \chi^\kappa}\Big|_{X_y} \\ &\le C(\mathrm{tr}_{\omega_t}\chi)^\kappa \frac{\Omega}{\theta^{n-\kappa} \wedge \chi^\kappa}\Big|_{X_y} \\ &\le CH^{-1} \le C\sigma^{-\lambda}(y). \end{aligned}$$

Since the Sobolev constant of (X_y, θ_y) is uniformly bounded and the Poincaré constant of (X_y, θ_y) is bounded by $Ce^{B\sigma^{-\lambda}(y)}$ for some uniform constants $B, C > 0$ (see [37]), combined with the fact that

$$\fint_{X_y} (\varphi - \overline{\varphi}_y)\theta_y^{n-\kappa} = 0,$$

Moser iteration implies ([37, 40]):

Lemma 5.3. *There exist constants $B_1, C_1 > 0$ such that for any $y \in X_{\mathrm{can}}^\circ$,*

$$\sup_{X_y} t^{-1}|\varphi - \overline{\varphi}_y| \le C_1 e^{B_1\sigma^{-\lambda}(y)} \quad \textit{for all } t \in (0,1].$$

Proposition 5.1. *On any compact subset $K \Subset X \setminus S$, there exists $C = C(K) > 1$ such that for all $t \in (0,1]$,*

$$C^{-1}\hat{\omega}_t \le \omega_t \le C\hat{\omega}_t \quad \textit{on } K.$$

Given the C^0-estimate in Lemma 5.3, Proposition 5.1 can be proved by the C^2-estimate ([40]) for the Monge–Ampère equation together with a modification as in [29, 37, 38], so we omit the proof.

Let us recall the construction of the canonical metric ω_{can} on X_{can}° (see [29]). Define a function $F = \frac{\Phi_*\Omega}{\chi^\kappa}$ on X_{can}°, and F is in $L^{1+\varepsilon}$ for some small $\varepsilon > 0$ ([29]). The metric ω_{can} is obtained by solving the following complex Monge–Ampère equation on X_{can}:

$$(\chi + i\partial\bar{\partial}\varphi_\infty)^\kappa = \binom{n}{\kappa} F e^{\varphi_\infty}\chi^\kappa$$

for $\varphi_\infty \in \mathrm{PSH}(X_{\mathrm{can}}, \chi) \cap C^0(X_{\mathrm{can}}) \cap C^\infty(X_{\mathrm{can}}^\circ)$. Then $\omega_{\mathrm{can}} = \chi + i\partial\bar{\partial}\varphi_\infty$, and in the following we will write $\chi_\infty = \omega_{\mathrm{can}}$.

Any smooth fiber X_y with $y \in X_{\mathrm{can}}^\circ$ is a Calabi–Yau manifold hence there exists a unique Ricci flat metric $\omega_{\mathrm{SF},y} \in [\theta_y]$ such that $\omega_{\mathrm{SF},y} = \theta_y + i\partial\bar\partial\rho_y$ for some $\rho_y \in C^\infty(X_y)$ with normalization

$$\fint_{X_y} \rho_y \omega_{X,y}^{n-\kappa} = 0.$$

We write $\rho_{\mathrm{SF}}(x) = \rho_{\Phi(x)}$ if $\Phi(x) \in X_{\mathrm{can}}^\circ$. Then ρ_{SF} is a smooth function on $X \setminus S$ and may blow up near the singular set S. Denote $\omega_{\mathrm{SF}} = \theta + i\partial\bar\partial\rho_{\mathrm{SF}}$ which is smooth on $X \setminus S$, and by [29] we know that $\Omega/(\omega_{\mathrm{SF}}^{n-\kappa} \wedge \chi^\kappa)$ is constant on the smooth fibers X_y and is equal to $\Phi^* F$. For simplicity we will identify F with $\Phi^* F$. Our arguments below are motivated by [29, 38].

Denote $\mathcal{F} = e^{-e^{A\sigma - \lambda}}$ for suitably large constants $A, \lambda > 1$. From the proof of Proposition 5.1, we actually have that on $X \setminus S$ ([37]),

$$C^{-1}\mathcal{F}\hat\omega_t \le \omega_t \le C\mathcal{F}^{-1}\hat\omega_t \quad \text{for all } t \in (0, 1].$$

Next we are going to show $\varphi_t \to \varphi_\infty = \Phi^*\varphi_\infty$ as $t \to 0$. Proposition 5.2 below can proved by following similar argument as in [38], but we present a slightly different argument in establishing Claim 2 below.

Proposition 5.2. *There exists a positive function $h(t)$ with $h(t) \to 0$ as $t \to 0$ such that*

$$\sup_{X\setminus S} \mathcal{F}|\varphi_t - \varphi_\infty| \le h(t). \tag{5.2}$$

Proof. Let $D \subset X_{\mathrm{can}}$ be an ample divisor such that $X_{\mathrm{can}} \setminus X_{\mathrm{can}}^\circ \subset D$, where $D \in |\mu K_{X_{\mathrm{can}}}|$ for some $\mu \in \mathbb{N}$. Choose a continuous hermitian metric on $[D]$, $h_D = h_{\mathrm{FS}}^{\mu/m} e^{-\mu\varphi_\infty}$ and a smooth defining section s_D of $[D]$, where h_{FS} is the Fubini–Study metric induced from $\mathcal{O}_{\mathbb{CP}^{N_m}}(1)$. Clearly $i\partial\bar\partial \log h_D = \mu(\chi + i\partial\bar\partial\varphi_\infty) = \mu\chi_\infty$. For small $r > 0$, let

$$B_r(D) = \{x \in X_{\mathrm{can}} : d_\chi(x, D) \le r\}$$

be the tubular neighborhood of D under the metric d_χ, and denote $\mathcal{B}_r = \Phi^{-1}(B_r(D)) \subset X$.

Since both φ_t and φ_∞ are bounded in L^∞-norm, there exists r_ϵ with $\lim_{\epsilon\to 0} r_\epsilon = 0$ such that for all $t \in (0, 1]$,

$$\sup_{\mathcal{B}_{r_\epsilon}\setminus S} (\varphi_t - \varphi_\infty + \epsilon \log |s_D|_{h_D}^2) < -1,$$
$$\inf_{\mathcal{B}_{r_\epsilon}\setminus S} (\varphi_t - \varphi_\infty - \epsilon \log |s_D|_{h_D}^2) > 1.$$

Let η_ϵ be a smooth cut-off function on X_{can} such that $\eta_\epsilon = 1$ on $X_{\mathrm{can}} \setminus B_{r_\epsilon}(D)$ and $\eta_\epsilon = 0$ on $B_{r_\epsilon/2}(D)$. Write $\rho_\epsilon = (\Phi^*\eta_\epsilon)\rho_{\mathrm{SF}}$, and $\omega_{\mathrm{SF},\epsilon} = \omega_{\mathrm{SF}} + i\partial\bar\partial\rho_\epsilon$. Define the twisted differences of φ_t and φ_∞ by

$$\psi_\epsilon^\pm = \varphi_t - \varphi_\infty - t\rho_\epsilon \mp \epsilon \log |s_D|_{h_D}^2.$$

By similar arguments as in [29] we have:

Claim 1. *There exists an $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there exists a constant τ_ϵ such that for all $t \le \tau_\epsilon$, we have*

$$\sup_{X\setminus S} \psi_\epsilon^-(t, \cdot) \le 3\mu\epsilon, \quad \inf_{X\setminus S} \psi_\epsilon^+(t, \cdot) \ge -3\mu\epsilon.$$

Claim 2. *We have*

$$\int_X |\varphi_t - \varphi_\infty|\theta^n \to 0 \quad as\ t \to 0,$$

where φ_t is the Kähler potential of ω_t in (5.1).

Proof of Claim 2. For any $\eta > 0$, we may take $\mathcal{B}_{R_\eta} \subset X$ small enough so that

$$\int_{\mathcal{B}_{R_\eta}} \theta^n < \frac{\eta}{10}.$$

Take $\epsilon < \frac{\eta}{10\mu}$ small enough so that $r_\epsilon < R_\eta$. From Claim 1 when $t < \tau_\epsilon$,

$$\begin{aligned}\int_X |\varphi_t - \varphi_\infty|\theta^n &= \int_{\mathcal{B}_{R_\eta}} |\varphi_t - \varphi_\infty|\theta^n + \int_{X\setminus\mathcal{B}_{R_\eta}} |\varphi_t - \varphi_\infty|\theta^n \\ &\le C\eta + \int_{X\setminus\mathcal{B}_{R_\eta}} (t|\rho_{\mathrm{SF}}| + \epsilon|\log|s_D|^2_{h_D}|)\theta^n \\ &\le C\eta.\end{aligned}$$

□

Given Claim 2, Proposition 5.2 follows similarly as in [38], so we skip it. □

We will apply an argument in [38] with a slight modification to show the lemma below:

Lemma 5.4. *We have*

$$\lim_{t\to 0} \mathcal{F} t\dot{\varphi}_t = 0.$$

Proof. Denote $s = \log t$ for $t \in (0, 1]$. We have $t\dot{\varphi} = \frac{\partial\varphi}{\partial s}$. Taking derivatives on both sides of equation (5.1) and by maximum principle arguments, we then get (see also [38])

$$\frac{\partial^2\varphi}{\partial s^2} = t\dot{\varphi} + t^2\ddot{\varphi} \le C, \quad \text{here } \ddot{\varphi} = \frac{\partial^2\varphi}{\partial t^2}. \tag{5.3}$$

By the uniform convergence (5.2) of $\mathcal{F}\varphi(s) \to \mathcal{F}\varphi_\infty$ as $s \to -\infty$, for any $\epsilon > 0$, there is an S_ϵ such that for all $s_1, s_2 \le -S_\epsilon$, we have $\sup_X |\mathcal{F}\varphi(s_1) - \mathcal{F}\varphi(s_2)| \le \epsilon$. For any $s < -S_\epsilon - 1$ and $x \in X \setminus S$, by the mean value theorem

$$\mathcal{F}\partial_s\varphi(s_x, x) = \frac{1}{\sqrt{\epsilon}} \int_s^{s+\sqrt{\epsilon}} \partial_s(\mathcal{F}\varphi)\, ds \ge -\sqrt{\epsilon} \quad \text{for some } s_x \in [s, s + \sqrt{\epsilon}].$$

By the upper bound (5.3), it follows that $\mathcal{F}\partial_s\varphi(s, x) \ge -C\sqrt{\epsilon} - \sqrt{\epsilon}$. Similarly

$$\mathcal{F}\partial_s\varphi(\hat{s}_x, x) = \frac{1}{\sqrt{\epsilon}} \int_{s-\sqrt{\epsilon}}^{s} \partial_s(\mathcal{F}\varphi(\,\cdot\,, x))\, ds \le \sqrt{\epsilon} \quad \text{for some } \hat{s}_x \in [s - \sqrt{\epsilon}, s],$$

from (5.3) we get $\mathcal{F}\partial_s\varphi(s, x) \le C\sqrt{\epsilon} + \sqrt{\epsilon}$. Hence we show that for any $s \le -S_\epsilon - 1$ or $t = e^s \le e^{-S_\epsilon - 1}$, it holds that

$$\sup_{x\in X\setminus S} |\mathcal{F}\partial_s\varphi(s, x)| = \sup_{x\in X\setminus S} |\mathcal{F} t\partial_t\varphi(t, x)| \le C\sqrt{\epsilon},$$

so the lemma follows. □

Corollary 5.1. *There exists a positive decreasing function $h(t)$ with $h(t) \to 0$ as $t \to 0$ such that*

$$\sup_X \mathcal{F}(|\varphi_t - t\dot{\varphi}_t - \varphi_\infty| + t|\dot{\varphi}_t|) \le h(t).$$

From Corollary 5.1, by using a straightforward adaption of the arguments of [38], we have an improvement of the local C^2-estimate:

Lemma 5.5. *On any compact subset $K \subset\subset X \setminus S$, we have*

$$\limsup_{t\to 0}\Big(\sup_K(\operatorname{tr}_{\omega_t}\chi_\infty - \kappa)\Big) \le 0.$$

With the local C^2-estimate (see Proposition 5.1), we obtain the following standard local C^3-estimates ([21, 24, 40]):

Lemma 5.6. *For any compact $K \Subset X \setminus S$, there exists a constant $C = C(K) > 0$ such that*

$$\sup_K |\nabla_\theta \omega_t|^2 \le C t^{-1}.$$

We have built up all the necessary ingredients to prove Theorem 1.3, whose proof is almost identical to that of [38, Theorem 1.3]. For completeness, we sketch the proof below.

Proof of Theorem 1.3. Fix a compact subset $K' \subset X^\circ_{\mathrm{can}}$ and let $K = \Phi^{-1}(K')$. By the Calabi C^3-estimate in Lemma 5.6, it follows that

$$\|t^{-1}\omega_t|_{X_y}\|_{C^1(X_y,\theta_y)} \le C, \quad t^{-1}\omega_t|_{X_y} \ge c\,\theta_y,$$

for all $y \in K'$ and $\theta_y = \theta|_{X_y}$.

Step 1. Define a function f on X_y by

$$\begin{aligned} f &= \frac{(t^{-1}\omega_t|_{X_y})^{n-\kappa}}{\omega_{\mathrm{SF},y}^{n-\kappa}} \\ &= \binom{n}{\kappa}\frac{(\omega_t|_{X_y})^{n-\kappa}\wedge\chi_\infty^\kappa}{\omega_t^n}e^{\varphi_t-\varphi_\infty} \\ &\le e^{h(t)}\Big(\frac{\operatorname{tr}_{\omega_t}\chi_\infty}{\kappa}\Big)^\kappa \le 1+\tilde{h}(t) \end{aligned}$$

for some $\tilde{h}(t) \to 0$ as $t \to 0$ (here $\tilde{h}(t)$ depends on K), where in the first inequality we use the Newton–Maclaurin inequality. The function f also satisfies that

$$\int_{X_y}(f-1)\omega_{\mathrm{SF},y}^{n-\kappa} = 0, \quad \lim_{t\to\infty}\int_{X_y}|f-1|\omega_{\mathrm{SF},y}^{n-\kappa} = 0. \tag{5.4}$$

The Calabi estimate implies that $\sup_{X_y}|\nabla f|_{\theta_y} \le C$ for all $y \in K'$, and (X_y, θ_y) have uniformly bounded diameter and volume for $y \in K'$. So it follows that f converges to 1 uniformly on K as $t \to 0$. That is,

$$\|(t^{-1}\omega_t|_{X_y})^{n-\kappa} - \omega_{\mathrm{SF},y}^{n-\kappa}\|_{C^0(X_y,\theta_y)} \to 0 \quad \text{as } t \to 0$$

uniformly on K'. Since $t^{-1}\omega_t|_{X_y}$ converges in $C^\alpha(X_y,\theta_y)$-topology to some limit metric $\omega_{\infty,y}$ which satisfies the Monge–Ampère equation (weakly) on X_y, $\omega_{\infty,y}^{n-\kappa} = \omega_{\mathrm{SF},y}^{n-\kappa}$, by the uniqueness of complex Monge–Ampère equations, it follows that $\omega_{\infty,y} = \omega_{\mathrm{SF},y}$ and $t^{-1}\omega_t|_{X_y}$ converge in C^α to $\omega_{\mathrm{SF},y}$, for any $y \in K'$. Next we show the convergence is uniform in K'.

Step 2. Define a new f on $X \setminus S$ which takes the form

$$f|_{X_y} = \frac{t^{-1}\omega_t|_{X_y} \wedge (\omega_{\mathrm{SF},y})^{n-\kappa-1}}{\omega_{\mathrm{SF},y}^{n-\kappa}} \geq \left(\frac{(t^{-1}\omega_t|_{X_y})^{n-\kappa}}{\omega_{\mathrm{SF},y}^{n-\kappa}}\right)^{\frac{1}{n-\kappa}},$$

and the right-hand side tends to 1 uniformly on K as $t \to \infty$. Then we have similar equations as in (5.4) for this new f. This implies

$$\left\| \frac{1}{n-\kappa} \operatorname{tr}_{\omega_{\mathrm{SF},y}} (t^{-1}\omega_t)|_{X_y} - 1 \right\|_{L^\infty(K)} \to 0 \quad \text{as } t \to 0.$$

So $t^{-1}\omega_t|_{X_y} \to \omega_{\mathrm{SF},y}$ uniformly for any $y \in K'$.

Step 3. Define

$$\tilde{\omega} = t\omega_{\mathrm{SF}} + \chi_\infty.$$

From a result of [38] (see [38, proof of Theorem 1.1]), we have $|\operatorname{tr}_{\omega_t}(\omega_{\mathrm{SF}} - \omega_{\mathrm{SF},y})| \leq Ct^{-\frac{1}{2}}$; then

$$\operatorname{tr}_{\omega_t} \tilde{\omega} \leq \operatorname{tr}_{\omega_t}(t\omega_{\mathrm{SF},y} + \chi_\infty) + C\sqrt{t} = n + \tilde{h}(t)$$

for some $\tilde{h}(t) \to 0$ when $t \to 0$. Moreover, it can be checked that

$$\lim_{t\to 0} \frac{\tilde{\omega}^n}{\omega_t^n} = 1 \quad \text{on } K.$$

Hence we see that $\omega_t \xrightarrow{C^0(K)} \chi_\infty$ as $t \to 0$.

We finish the proof of (1), (2) and (3) of Theorem 1.3.

Remark 5.1. From Steps 1, 2 and 3, we see that for any compact subset $K \subset X \setminus S$, there exists an $\varepsilon(t) = \varepsilon_K(t) \to 0$ as $t \to 0$ such that when t is small,

$$\Phi^*\chi_\infty - \varepsilon(t)\theta \leq \omega_t \leq \Phi^*\chi_\infty + \varepsilon(t)\theta \quad \text{on } K \tag{5.5}$$

and

$$\Phi^*\chi_\infty \leq (1+\varepsilon(t))\omega_t \quad \text{on } K. \tag{5.6}$$

From the uniform convergence of $t^{-1}\omega_t|_{X_y}$ to $\omega_{\mathrm{SF},y}$ for any $y \in \Phi(K)$, we see that there is a uniform constant $C_0 = C_0(K) > 0$ such that

$$\omega_t|_{X_y} \leq C_0 t\omega_{\mathrm{SF},y} \quad \text{for all } y \in \Phi(K).$$

Choose a sequence $t_k \to 0$. The metric spaces (X, ω_{t_k}) satisfy $\mathrm{Ric}(\omega_{t_k}) \geq -1$ and $\mathrm{diam}(X, \omega_{t_k}) \leq D$ for some constant $D < \infty$. By Gromov's pre-compactness theorem up to a subsequence we have

$$(X, \omega_{t_k}) \xrightarrow{d_{\mathrm{GH}}} (\mathbf{Z}, d_{\mathbf{Z}}),$$

for some compact metric length space $\mathbf{Z}$ with diameter bounded by D. The idea of the proof of (4) in Theorem 1.3 is motivated by [11], and we present below a slightly different argument from theirs.

Step 4. We will show:

Claim 3. *There exist an open subset* $\mathbf{Z}_0 \subset \mathbf{Z}$ *and a homeomorphism* $f : X^\circ_{\text{can}} \to \mathbf{Z}_0$ *which is a local isometry.*

Proof of Claim 3. By Lemma 5.2, the maps $\Phi = \Phi_k : (X, \omega_{t_k}) \to (X_{\text{can}}, \chi)$ are uniformly Lipschitz with respect to the given metrics, and the target space is compact, so up to a subsequence $\Phi_k \to \Phi_\infty : (\mathbf{Z}, d_{\mathbf{Z}}) \to (X_{\text{can}}, \chi)$ along the GH convergence $(X, \omega_{t_k}) \to (\mathbf{Z}, d_{\mathbf{Z}})$ which is also Lipschitz and the convergence is in the sense that for any $x_k \to (X, \omega_{t_k})$ which converges to $z \in \mathbf{Z}$, then $\Phi_\infty(z) = \lim_{k\to\infty} \Phi_k(x_k)$, and there is a constant $C > 0$ such that $d_\chi(\Phi_\infty(z_1), \Phi_\infty(z_2)) \le C d_{\mathbf{Z}}(z_1, z_2)$ for all $z_i \in \mathbf{Z}$.

We denote $\mathbf{Z}_0 = \Phi_\infty^{-1}(X^\circ_{\text{can}})$ which is an open subset of $\mathbf{Z}$ since Φ_∞ is continuous. We will show that $\Phi_\infty|_{\mathbf{Z}_0} : \mathbf{Z}_0 \to X^\circ_{\text{can}}$ is a bijection and a local isometry. Hence

$$f = (\Phi_\infty|_{\mathbf{Z}_0})^{-1} : X^\circ_{\text{can}} \to \mathbf{Z}_0$$

is the desired map.

$\Phi_\infty|_{\mathbf{Z}_0}$ *is injective.* Suppose $\Phi_\infty(z_1) = \Phi_\infty(z_2)$ for $z_1, z_2 \in \mathbf{Z}_0 = \Phi_\infty^{-1}(X^\circ_{\text{can}})$. Denote $y = \Phi_\infty(z_1) = \Phi_\infty(z_2) \in X^\circ_{\text{can}}$. Since $(X^\circ_{\text{can}}, \chi_\infty)$ is an (incomplete) smooth Riemannian manifold, there exists a small $r = r_y > 0$ such that $(B_{\chi_\infty}(y, 2r), \chi_\infty)$ is geodesic convex. Choose sequences $z_{1,k}$ and $z_{2,k} \in (X, \omega_{t_k})$ converging z_1 and z_2, respectively, along the GH convergence. By the definition of $\Phi_k = \Phi \to \Phi_\infty$ it follows that $d_\chi(\Phi(z_{1,k}), \Phi_\infty(z_1)) \to 0$ and $d_\chi(\Phi(z_{2,k}), \Phi_\infty(z_2)) \to 0$. Since d_χ and d_{χ_∞} are equivalent on $B_{\chi_\infty}(y, 2r)$, it follows that $d_{\chi_\infty}(\Phi(z_{1,k}), \Phi(z_{2,k})) \to 0$ and hence we can find minimal χ_∞-geodesics γ_k connecting $\Phi(z_{1,k})$ and $\Phi(z_{2,k})$ with $\gamma_k \subset B_{\chi_\infty}(y, r)$ and $L_{\chi_\infty}(\gamma_k) \to 0$. By the locally uniform convergence (5.5) on $\Phi^{-1}(\overline{B_{\chi_\infty}(y, 2r)})$ there exists a lift of γ_k, $\tilde{\gamma}_k$ in $\Phi^{-1}(\overline{B_{\chi_\infty}(y, 2r)})$ such that

$$L_{\omega_{t_k}}(\tilde{\gamma}_k) \le L_{\chi_\infty}(\gamma_k) + \epsilon(t_k) L_\omega(\tilde{\gamma}_k) \to 0 \quad \text{as } t_k \to 0.$$

Note that $\tilde{\gamma}_k$ connects $z_{1,k}$ and $z_{2,k}$ hence

$$d_{\omega_{t_k}}(z_{1,k}, z_{2,k}) \le L_{\omega_{t_k}}(\tilde{\gamma}_k) \to 0,$$

which implies by the convergence of $z_{i,k} \to z_i$ that $d_{\mathbf{Z}}(z_1, z_2) = 0$ and $z_1 = z_2$.

$\Phi_\infty|_{\mathbf{Z}_0}$ *is a local isometry.* Let $z \in \mathbf{Z}_0$ and $y = \Phi_\infty(z) \in X^\circ_{\text{can}}$. There is a small radius $r = r_y > 0$ such that $(B_{\chi_\infty}(y, 3r), \chi_\infty)$ is geodesic convex. Take $U = (\Phi_\infty|_{\mathbf{Z}_0})^{-1}(B_{\chi_\infty}(y, r))$ to be an open neighborhood of $z \in \mathbf{Z}$. We will show that $\Phi_\infty|_{\mathbf{Z}_0} : (U, d_{\mathbf{Z}}) \to (B_{\chi_\infty}(y, r), \chi_\infty)$ is an isometry. Fix any two points $z_1, z_2 \in U$ and $y_i = \Phi_\infty(z_i) \in B_{\chi_\infty}(y, r)$ for $i = 1, 2$. As before we choose $z_{i,k} \in (X, \omega_{t_k})$ such that $z_{i,k} \to z_i$ along the GH convergence for $i = 1, 2$. It follows then from $\Phi_k = \Phi \to \Phi_\infty$ that $d_{\chi_\infty}(\Phi(z_{i,k}), y_i) \to 0$, and when k is large, $\Phi(z_{i,k})$ lie in $B_{\chi_\infty}(y, 1.1r)$. Choose ω_{t_k}-minimal geodesics γ_k connecting $z_{1,k}$ and $z_{2,k}$ such that

$$d_{\omega_{t_k}}(z_{1,k}, z_{2,k}) = L_{\omega_{t_k}}(\gamma_k) \to d_{\mathbf{Z}}(z_1, z_2).$$

The curve $\bar{\gamma}_k = \Phi(\gamma_k)$ connects $\Phi(z_{1,k})$ with $\Phi(z_{2,k})$. If $\bar{\gamma}_k \subset B_{\chi_\infty}(y, 3r)$, from (5.6) it follows that

$$d_{\chi_\infty}(\Phi(z_{1,k}), \Phi(z_{2,k})) \le L_{\chi_\infty}(\bar{\gamma}_k) \le (1 + \epsilon(t_k)) L_{\omega_{t_k}}(\gamma_k) \to d_{\mathbf{Z}}(z_1, z_2).$$

In case $\bar{\gamma}_k \not\subset B_{\chi_\infty}(y, 3r)$, we have

$$d_{\chi_\infty}(\Phi(z_{1,k}), \Phi(z_{2,k})) \le 3.8r \le L_{\chi_\infty}(\bar{\gamma}_k \cap B_{\chi_\infty}(y, 3r)) \\ \le (1 + \epsilon(t_k)) L_{\omega_{t_k}}(\gamma_k) \to d_{\mathbf{Z}}(z_1, z_2).$$

Letting $k \to \infty$, we conclude that $d_{\chi_\infty}(y_1, y_2) \le d_{\mathbf{Z}}(z_1, z_2)$. To see the reverse inequality, we take χ_∞-minimal geodesics σ_k connecting $\Phi(z_{1,k})$ and $\Phi(z_{2,k})$. Clearly $\gamma_k \subset B_{\chi_\infty}(y, 3r)$. Take a lift of σ_k, $\tilde{\sigma}_k$ in $\Phi^{-1}(\overline{B_{\chi_\infty}(y, 3r)})$; it follows from (5.5) that

$$d_{\omega_{t_k}}(z_{1,k}, z_{2,k}) \le L_{\omega_{t_k}}(\tilde{\sigma}_k) \le L_{\chi_\infty}(\sigma_k) + \epsilon(t_k) L_{\omega}(\tilde{\sigma}_k) \to d_{\chi_\infty}(y_1, y_2).$$

Letting $k \to \infty$, we get

$$d_{\mathbf{Z}}(z_1, z_2) \le d_{\chi_\infty}(y_1, y_2).$$

Hence $d_{\mathbf{Z}}(z_1, z_2) = d_{\chi_\infty}(y_1, y_2)$ and $\Phi_\infty|_{\mathbf{Z}_0} : U \to B_{\chi_\infty}(y, r)$ is an isometry.

$\Phi_\infty|_{\mathbf{Z}_0}$ *is surjective.* This is almost obvious from the definition. Take any $y \in X^\circ_{\mathrm{can}}$ and any fixed point $x \in \Phi^{-1}(y) \subset (X, \omega_{t_k})$. Up to a subsequence,

$$x \xrightarrow{d_{\mathrm{GH}}} z \in (\mathbf{Z}, d_{\mathbf{Z}}).$$

It then follows from $\Phi_k \to \Phi_\infty$ that $d_\chi(y, \Phi_\infty(z)) = d_\chi(\Phi_k(x), \Phi_\infty(z)) \to 0$ as $k \to \infty$. Hence $\Phi_\infty(z) = y$ and $z \in \Phi_\infty^{-1}(X^\circ_{\mathrm{can}}) = \mathbf{Z}_0$. □

Step 5. In this step we will show $\mathbf{Z}_0 \subset \mathbf{Z}$ is dense. Fix a base point $\bar{x} \in \mathbf{Z}_0$, upon rescaling if necessary we may assume the metric ball $B_{\chi_\infty}(f^{-1}(\bar{x}), 2) \subset (X^\circ_{\mathrm{can}}, \chi_\infty)$ is geodesic convex. Choose a sequence of points $\bar{p}_k \in (X, \omega_{t_k})$ such that $\bar{p}_k \to \bar{x}$ along the GH convergence $(X, \omega_{t_k}) \to (\mathbf{Z}, d_{\mathbf{Z}})$. We define a function on $X \times [0, \infty)$ as the normalized volume ([5])

$$\underline{V}_k(x, r) = \frac{\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(x, r))}{\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(\bar{p}_k, 1))};$$

by standard volume comparison it is shown in [5] that $\underline{V}_k(\cdot, \cdot)$ is equi-continuous and uniformly bounded hence they converges (up to a subsequence) to a function

$$\underline{V}_\infty : \mathbf{Z} \times [0, \infty) \to [0, \infty)$$

in the sense that for any $x_k \to x$ along the GH convergence and $r \ge 0$,

$$\underline{V}_k(x_k, r) \to \underline{V}_\infty(x, r) \quad \text{as } k \to \infty.$$

And $\underline{V}_\infty$ satisfies similar estimates as in volume comparison, i.e., for $r_1 \le r_2$,

$$\frac{\underline{V}_\infty(x, r_1)}{\underline{V}_\infty(x, r_2)} \ge \mu(r_1, r_2) > 0,$$

where $\mu(\cdot, \cdot)$ is the quotient of volumes of balls in a space form. The function $\underline{V}_\infty$ induces a Radon ν on $(\mathbf{Z}, d_{\mathbf{Z}})$. More precisely, for any $K \subset \mathbf{Z}$, define

$$\hat{\nu}(K) = \lim_{\delta \to 0} \hat{\nu}_\delta(K) = \lim_{\delta \to 0} \inf \sum_i \underline{V}_\infty(x_i, r_i),$$

where the infimum is taken over all metric balls $B_{d_{\mathbf{Z}}}(x_i, r_i)$ with $r_i \le \delta$ whose union covers K.

Claim 4. *For any $x \in \mathbf{Z}_0$ and $r = r_x > 0$ such that $B_{\chi_\infty}(f^{-1}(x), 2r) \subset X^\circ_{\mathrm{can}}$ is geodesic convex, we have*

$$\underline{V}_\infty(x,r) = v_0 \int_{\Phi^{-1}(B_{\chi_\infty}(f^{-1}(x),r))} e^{-\varphi_\infty}\theta^n$$

for a fixed constant

$$v_0 = \left(\int_{\Phi^{-1}(B_{\chi_\infty}(f^{-1}(\bar{x}),1))} e^{\varphi_\infty}\theta^n\right)^{-1}.$$

Proof of Claim 4. The proof is parallel to that in [11], so we only provide a sketch. For the given $x \in \mathbf{Z}_0$, we choose a sequence of points $p_k \in (X, \omega_{t_k})$ such that $p_k \to x$. As in [11], due to (5.5) and that the metrics ω_{t_k} and θ are equivalent in $\Phi^{-1}(B_{\chi_\infty}(f^{-1}(x), 2r))$, it can be shown that

$$\Phi^{-1}(B_{\chi_\infty}(f^{-1}(x), r - \epsilon_k)) \subset B_{\omega_{t_k}}(p_k, r) \subset \Phi^{-1}(B_{\chi_\infty}(f^{-1}(x), r + \epsilon_k)) \tag{5.7}$$

when $k \gg 1$ and here $\epsilon_k \to 0$ as $k \to \infty$. It follows then that

$$\lim_{k\to\infty} \int_{B_{\omega_{t_k}}(p_k,r)} e^{\varphi_{t_k}}\theta^n = \int_{\Phi^{-1}(B_{\chi_\infty}(f^{-1}(x),r))} e^{\varphi_\infty}\theta^n.$$

From the equation $\omega_t^n = t^{n-\kappa}e^{\varphi_t}\theta^n$, we have

$$\underline{V}_k(p_k, r) = \frac{\int_{B_{\omega_{t_k}}(p_k,r)} t^{n-\kappa} e^{\varphi_{t_k}}\theta^n}{\int_{B_{\omega_{t_k}}(\bar{p}_k,1)} t_k^{n-\kappa} e^{\varphi_{t_k}}\theta^n} \to \frac{\int_{\Phi^{-1}(B_{\chi_\infty}(f^{-1}(x),r))} e^{\varphi_\infty}\theta^n}{\int_{\Phi^{-1}(B_{\chi_\infty}(f^{-1}(\bar{x}),1))} e^{\varphi_\infty}\theta^n},$$

where for the convergence of the denominators we use a similar relation as in (5.7) for $\bar{p}_k$, $\bar{x}$. From the definition that $\underline{V}_k(p_k, r) \to \underline{V}_\infty(x, r)$, we finish the proof of Claim 4. □

Since along the Gromov–Hausdorff convergence the diameters are uniformly bounded by $D < \infty$, we have

$$\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(p_k, D)) = \mathrm{Vol}(X, \omega_{t_k}^n).$$

So

$$\begin{aligned}\underline{V}_\infty(x, D) &= \lim_{k\to\infty} \frac{\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(p_k, D))}{\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(\bar{p}_k, 1))} \\ &= \lim_{k\to\infty} \frac{\int_X e^{\varphi_{t_k}}\theta^n}{\int_{B_{\omega_{t_k}}(\bar{p}_k,1)} e^{\varphi_{t_k}}\theta^n} \\ &= v_0 \int_X e^{\varphi_\infty}\theta^n.\end{aligned}$$

Therefore from $\mathbf{Z} = B_{d_{\mathbf{Z}}}(x, D)$, we have

$$\hat{\nu}(\mathbf{Z}) \le v_0 \int_X e^{\varphi_\infty}\theta^n.$$

Assume $\mathbf{Z}_0 \subset \mathbf{Z}$ were not dense; then there exists a metric ball $B_{d_{\mathbf{Z}}}(z, \rho) \subset \mathbf{Z} \setminus \mathbf{Z}_0$ such that, by volume comparison estimate for $\underline{V}_\infty$,

$$\hat{\nu}(B_{d_{\mathbf{Z}}}(z, \rho)) \ge \underline{V}_\infty(z, D)\mu(\rho, D) =: \eta_0 > 0.$$

Then for any compact subset $K \subset \mathbf{Z}_0$, $\hat{\nu}(K) \leq \hat{\nu}(\mathbf{Z}) - \eta_0$. On the other hand, for any open covering $B_{d_{\mathbf{Z}}}(x_i, r_i)$ of K with $B_{\chi_\infty}(f^{-1}(x_i), 2r_i)$ geodesic convex in $(X^\circ_{\mathrm{can}}, \chi_\infty)$ and $r_i < \delta$, we have

$$\sum_i \underline{V}_\infty(x_i, r_i) = \sum_i v_0 \int_{\Phi^{-1}(B_{\chi_\infty}(f^{-1}(x_i), r_i))} e^{\varphi_\infty}\theta^n \geq v_0 \int_{\Phi^{-1}(f^{-1}(K))} e^{\varphi_\infty}\theta^n.$$

Taking infimum over all such coverings and letting $\delta \to 0$, we get

$$\hat{\nu}(K) \geq v_0 \int_{\Phi^{-1}(f^{-1}(K))} e^{\varphi_\infty}\theta^n.$$

If we take K large enough so that $f^{-1}(K) \subset X^\circ_{\mathrm{can}}$ is large, we can achieve that

$$\hat{\nu}(K) \geq v_0 \int_{\Phi^{-1}(X^\circ_{\mathrm{can}})} e^{\varphi_\infty}\theta^n - \frac{\eta_0}{10} = v_0 \int_X e^{\varphi_\infty}\theta^n - \frac{\eta_0}{10} \geq \hat{\nu}(\mathbf{Z}) - \frac{\eta_0}{10}.$$

Hence we get a contradiction, and $\mathbf{Z}_0 \subset \mathbf{Z}$ is dense since $\hat{\nu}(\mathbf{Z} \setminus \mathbf{Z}_0) = 0$. □

6. Proof of Theorem 1.4

The proof of Theorem 1.4 is almost identical with that of Theorem 1.3. We give the sketch here. The solution g_t lies in the Kähler class $tL + (1-t)K_X$ for all $t \in (t_{\min}, 1]$. By definition and straightforward calculations from estimates of Yau [40] and Aubin [1], for any $t \in (t_{\min}, 1]$, the class $tL + (1-t)K_X$ is Kähler and so $t_{\min}L + (1-t_{\min})K_X$ is nef. We let Ω be a smooth volume form on X and let $\chi \in [t_{\min}L + (1-t_{\min})K_X]$ be a smooth closed $(1,1)$-form defined by

$$\chi = i\partial\bar{\partial}\log\Omega + \theta.$$

Then the twisted Kähler–Einstein equation (1.11) is equivalent to the following complex Monge–Ampère equation for $t \in (t_{\min}, 1]$:

$$(\chi + (t - t_{\min})\theta + i\partial\bar{\partial}\varphi_t)^n = (t - t_{\min})^{n-\kappa} e^{\varphi_t}\Omega, \tag{6.1}$$

where $\kappa = \nu(t_{\min}L + (1-t_{\min})K_X)$, which is the numerical dimension of the line bundle $t_{\min}L + (1-t_{\min})K_X$. By Proposition 1.1, there exists a constant $C = C(X, \chi, \theta) > 0$ such that for all $t \in (t_{\min}, 1]$,

$$\|\varphi_t - V_t\|_{L^\infty(X)} \leq C,$$

where V_t is the extremal function associated to $\chi + (t - t_{\min})\theta$. The rest of the proof for Theorem 1.4 is exactly the same as that of Theorem 1.2 and we leave it as an exercise for interested readers.

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