ON THE CONSTANT SCALAR CURVATURE KÄHLER METRICS (II) —EXISTENCE RESULTS

XIUXIONG CHEN, JINGRUI CHENG

Abstract. In this paper, we generalize our apriori estimates on cscK(constant scalar curvature Kähler) metric equation [15] to more general scalar curvature type equations (e.g., twisted cscK metric equation). As applications, under the assumption that the automorphism group is discrete, we prove the celebrated Donaldson's conjecture that the non-existence of cscK metric is equivalent to the existence of a destabilized geodesic ray where the K-energy is non-increasing. Moreover, we prove that the properness of K-energy in terms of L^1 geodesic distance d_1 in the space of Kähler potentials implies the existence of cscK metric. Finally, we prove that weak minimizers of the K-energy in $(\mathcal{E}^1, d_1)$ are smooth. The continuity path proposed in [14] is instrumental in this proof.

Contents

1. Introduction

This is the second of a series of papers discussing constant scalar curvature Kähler metrics. In this paper, for simplicity, we will only consider the case $Aut_0(M, J) = 0$. Here $Aut_0(M, J)$ denotes the identity component of the automorphism group and $Aut_0(M, J) = 0$ means the group is discrete. Under this assumption, we prove Donaldson's conjecture (mentioned above in the abstract) as well as the existence part of properness conjecture in this paper. Our main method is to adopt the continuity path introduced in [14] and we need to prove that the set of parameter $t \in [0, 1]$ the continuity path is both open (c.f. [14]) and closed under suitable geometric constraints. The apriori estimates obtained in [15] and their modifications (where the scalar curvature

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takes twisted form as in the twisted path introduced in [14]) are the crucial technical ingredients needed in this paper. In the sequel of this paper, we will prove a suitable generalization of both conjectures for general automorphism groups (i.e. no longer assume they are discrete).

We will begin with a brief review of history of this problem. In 1982 and 1985, E. Calabi published two seminal papers [8] [9] on extremal Kähler metrics where he proved some fundamental theorems on extremal Kähler metrics. His initial vision is that there should be a unique canonical metric in each Kähler class. Levine(c.f [47]) constructed examples that there there is no extremal metric on any Kähler class. More examples and obstructions are found over the last few decades and huge efforts are devoted to formulate the right conditions (in particular the algebraic conditions) under which we can "realize" Calabi's original dream in a suitable format. The well known Yau-Tian-Donaldson conjecture is one of the important formulations now which states that on projective manifolds, the cscK metrics exist in a polarized Kähler class if and only if this class is K-stable. It is widely expected among experts that the stability condition needs to be strengthened to a stronger notion such as uniform stability or stability through filtrations, in order to imply the existence of cscK metrics. We will have more in-depth discussions on this issue in the next paper in this series.

In a seminal paper [38], S. K. Donaldson proposed a beautiful program in Kähler geometry, aiming in particular to attack Calabi's renowned problem of existence of cscK metrics. In this celebrated program, Donaldson took the point of view that the space of Kähler metrics is formally a symmetric space of non-compact type and the scalar curvature function is the moment map from the space of almost complex structure compatible with a fixed symplectic form to the Lie algebra of certain infinite dimensional sympletic structure group which is exactly the space of all real valued smooth functions in the manifold. With this in mind, Calabi's problem of finding a cscK metric is reduced to finding a zero of this moment map in the infinite dimensional space setting. From this beautiful new point of view, S. K. Donaldson proposed a network of problems in Kähler geometry which have inspired many exciting developments over the last two decades, culminating in the recent resolution of Yau's stability conjecture on Kähler-Einstein metrics [18] [19] [20].

Let $\mathcal{H}$ denote the space of Kähler potentials in a given Kähler class $(M,[\omega])$. T. Mabuchi[52], S. Semmes [53] and S. K. Donaldson [38] set up an L^2 metric in the space of Kähler potentials:

$$\|\delta\varphi\|_\varphi^2 = \int_M (\delta\varphi)^2 \omega_\varphi^n, \qquad \forall\, \delta\varphi \in T_\varphi \mathcal{H}.$$

Donaldson [38] conjectured that $\mathcal{H}$ is a genuine metric space with the pathwise distance defined by this L^2 inner product. In [11], the first named author established the existence of $C^{1,1}$ geodesic segment between any two smooth Kähler potentials and proved this conjecture of S.K. Donaldson. He went on to prove (together with E. Calabi) that such a space is necessarily non-positively curved in the sense of Alexandrov[10]. More importantly, S. K. Donaldson proposed the following conjecture to attack the existence problem:

Conjecture 1.1. [38] *Assume $Aut_0(M,J)=0$. Then the following statements are equivalent:*

(1) There is no constant scalar curvature Kähler metric in $\mathcal{H}$;
(2) There is a potential $\varphi_0\in\mathcal{H}_0$ and there exists a geodesic ray $\rho(t)(t\in[0,\infty))$ in $\mathcal{H}_0$, initiating from φ_0 such that the K-energy is non-increasing;
(3) For any Kähler potential $\psi\in\mathcal{H}_0$, there exists a geodesic ray $\rho(t)(t\in[0,\infty))$ in $\mathcal{H}_0$, initiating from ψ such that the K-energy is non-increasing.

In the above, $\mathcal{H}_0=\mathcal{H}\cap\{\phi : I(\phi)=0\}$, where the functional I is defined by (2.7). The reason we need to use $\mathcal{H}_0$ is to preclude the trivial geodesic $\rho(t)=\varphi_0+ct$ where c is a constant.

In the original writing of S. K. Donaldson, he didn't specify the regularity of these geodesic rays in this conjecture. In this paper, we avoid this issue by working in the space $\mathcal{E}^1$ in which the potentials have only very weak regularity but the notion of geodesic still makes sense. Moreover, Theorem 4.7 of [6] shows the definition of K-energy can be extended to the space $\mathcal{E}^1$. The precise version of the result we prove is the following:

Theorem 1.1. *(Theorem 5.1) Assume $Aut_0(M,J)=0$. Then the following statements are equivalent:*

(1) There is no constant scalar curvature Kähler metric in $\mathcal{H}$;
(2) There is a potential $\varphi_0\in\mathcal{E}^1_0$ and there exists a locally finite energy geodesic ray $\rho(t)(t\in[0,\infty))$ in $\mathcal{E}^1_0$, initiating from φ_0 such that the K-energy is non increasing;
(3) For any Kähler potential $\psi\in\mathcal{E}^1_0$, there exists a locally finite energy geodesic ray $\rho(t)(t\in[0,\infty))$ in $\mathcal{E}^1_0$, initiating from ψ such that the K-energy is non increasing.

In the above, the space $\mathcal{E}^1$ is the abstract metric completion of the space $\mathcal{H}$ under the Finsler metric d_1 in $\mathcal{H}$ (see section 2 for more details) and the notion of finite energy geodesic segment was introduced in [4] (c.f. [31]). Also $\mathcal{E}^1_0=\mathcal{E}^1\cap\{\phi : I(\phi)=0\}$, where the functional I is defined as in (2.7). We learned about the idea of using locally finite energy geodesic ray from the recent beautiful work of Darvas-He [32] on Donaldson conjecture in Fano manifold where they use Ding functional instead of the K-energy functional. From our point of view, both the restriction to canonical Kähler class and the adoption of Ding functional are more of analytical nature.

Inspired by Donaldson's conjecture, the first named author introduced the following notion of geodesic stability [13].

Definition 1.1. *(c.f. Definition (3.10) in* [13]*) Let $\rho(t):[0,\infty)\to\mathcal{E}^1_0$ be a locally finite energy geodesic ray with unit speed such that $K(\rho(t))<\infty$ for $t\geq 0$. One can define an invariant $¥([\rho])$ as*

$$¥([\rho])=\lim_{k\to\infty} K(\rho(k+1))-K(\rho(k)).$$

One can check that this is well defined, due to the convexity of K-energy along geodesics (c.f. Theorem 2.5). Indeed, from the convexity of K-energy along locally finite energy geodesic ray, one actually has $K(\rho(k+1))-K(\rho(k))$ is increasing in k.

Definition 1.2. *(c.f. Definition (3.14) in* [13]*) Let $\varphi_0\in\mathcal{E}^1_0$ with $K(\varphi_0)<\infty$, $(M,[\omega])$ is called geodesic stable at φ_0(resp. geodesic-semistable) if for all locally finite energy* *geodesic ray initiating from φ_0, their ¥ invariant is always strictly positive(resp. non-negative). $(M,[\omega])$ is called geodesic stable(resp. geodesic semistable) if it is geodesic stable(resp. geodesic semistable) at any $\varphi \in \mathcal{E}_0^1$.*

Remark 1.3. *It is possible to define the ¥ invariant for a locally finite energy geodesic ray in $\mathcal{E}_0^p$ with $p > 1$. Note that a geodesic segment in $\mathcal{E}_0^p$ is automatically a geodesic segment in $\mathcal{E}_0^q$ for any $q \in [1,p]$. Following the preceding definition, one can also define geodesic stability in $\mathcal{E}_0^p(p > 1)$. Note that for a locally given finite energy geodesic ray in $\mathcal{E}_0^p(p > 1)$, the actual value of ¥ invariant in $\mathcal{E}_0^p$ might differ by a positive multiple from the ¥ invariant considered in $\mathcal{E}_0^1$. However, it will not affect the sign of the ¥ invariant for a particular locally finite energy geodesic ray. On the other hand, the collection of locally finite energy geodesic ray in $\mathcal{E}_0^p(p > 1)$ might be strictly contained in the collection of geodesic rays in $\mathcal{E}_0^1$. Therefore, the notion of geodesic stability in the $\mathcal{E}_0^1$ is strongest while the notion of geodesic stability in $\mathcal{E}_0^\infty$ is the weakest. Without going into technicality, we may define geodesic stability in $\mathcal{E}_0^\infty$ as the ¥ invariant being strictly positive for any locally finite energy geodesic ray which lies in $\bigcap_{p\geq 1} \mathcal{E}_0^p$.*

An intriguing question motivated from above remark is whether geodesic stability in $\mathcal{E}_0^\infty$ (in the sense defined in the above remark) implies geodesic stability in $\mathcal{E}_0^1$? The first named author believes the answer is affirmative. We will discuss this question and other stability notions in algebraic manifolds in greater detail in our next paper and refer interested readers to the following works and references therein: J. Ross [51], G. Székelyhidi [58], Berman-Boucksom-Jonsson [3], R. Dervan [37].

Using the notion of geodesic stability, we can re-formulate Theorem 1.1 as

Theorem 1.2. *Suppose $Aut_0(M,J) = 0$. Then $(M,[\omega])$ admits a cscK metric if and only if it is geodesic stable.*

Given the central importance of the notion of K-energy in Donaldson's beautiful program, the first named author proposed the following conjecture, shortly after [11]:

Conjecture 1.2. *Assume $Aut_0(M,J) = 0$. The existence of constant scalar curvature Kähler metric is equivalent to the properness of K-energy in terms of geodesic distance.*

Here "properness" means that the K-energy tends to $+\infty$ whenever the geodesic distance tends to infinity (c.f. Definition 3.1). The original conjecture naturally chose the distance introduced in [38] which we now call L^2 distance. After a series of fundamental work of T. Darvas on this subject (c.f [30] [31]), we now learn that the L^1 geodesic distance is a natural choice for the properness conjecture. Indeed, we prove

Theorem 1.3. *(Theorem 3.1 and 3.2) Assume $Aut_0(M,J) = 0$. The existence of constant scalar curvature Kähler metric is equivalent to the properness of K-energy in terms of the L^1 geodesic distance.*

Note that the direction that existence of cscK implies properness has been established by Berman-Darvas-Lu[5] recently. For the converse (namely the existence part), Darvas and Rubinstein have reduced this problem in [33] to a question of regularity of minimizers. In our paper, we will use continuity method to bypass this question and establish existence of cscK metrics.

For properness conjecture, we remark that there is a more well known formulation due to G. Tian where he conjectured that the existence of cscK metrics is equivalent to the propeness of K-energy in terms of Aubin functional J (c.f. Definition (2.7)). One may say that Tian's conjecture is more of analytical nature while Conjecture 1.2 above fits into Donaldson's geometry program in the space of Kähler potentials more naturally. According to T. Darvas (c.f. Theorem 5.5 of [30]), Aubin's J functional and the L^1 distance are equivalent. Therefore, these two properness conjectures are equivalent. Nonetheless, the formulation in conjecture 1.2 is essential to our proof.

Theorem 1.3 also holds for twisted cscK metric as well (c.f. Theorem 3.1 3.2), which is the solution to the equation

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi \chi - \underline{\chi}).$$

In the above, $0 < t \leq 1$, χ is a fixed Kähler form, and $\underline{R}$ is the average of scalar curvature, and $\underline{\chi} = \frac{\int_M n\chi \wedge \omega_0^{n-1}}{\int_M \omega_0^n}$. It is well-known that $\underline{R}$ and $\underline{\chi}$ depends only on the Kähler classes $[\omega_0]$ and $[\chi]$.

Now we recall an important notion introduced in [14]:

(1.1)

$$R([\omega_0], \chi) = \sup\{t_0 \in [0,1] : \text{ the above equation can be solved for any } 0 \leq t \leq t_0.\}$$

In the same paper, the first named author conjectured that this is an invariant of the Kähler class $[\chi]$. In this paper, as a consequence of Theorem 3.1 and 3.2, we will show that if χ_1 and χ_2 are two Kähler forms in the same class, then one has

$$R([\omega_0], \chi_1) = R([\omega_0], \chi_2),$$

so that the quantity $R([\omega_0], [\chi])$ is well-defined and gives rise to an invariant between two Kähler classes $[\omega_0]$, $[\chi]$. Moreover, when the K-energy is bounded from below, the twisted path (2.9) can be solved for any $t < 1$, as long as $t = 0$ can be solved. Thus in this case we have

Theorem 1.4. *Let χ be a Kähler form. If the K-energy is bounded from below on $(M, [\omega_0])$, then $R([\omega_0], [\chi]) = 1$ if and only if one can solve $tr_\varphi \chi = \underline{\chi}$.*

As noted in [14], it is interesting to understand geometrically for what Kähler classes this invariant is 1 but do not admit constant scalar curvature metrics. More broadly, it is interesting to estimate the upper and lower bound of this invariant. It is not hard to see the relation between the invariant introduced in [57] and the invariant introduced above when restricted to the canonical Kähler class in Fano manifold, where we take $[\chi]$ to be the first Chern class in (1.1) above. Hopefully, the method used there can be adapted to our setting to get estimate for this new invariant, in particular an upper bound.

T. Darvas and Y. Rubinstein conjectured in [33](Conjecture 2.9) that any minimizer of K-energy over the space $\mathcal{E}^1$ is actually a smooth Kähler potential. This is a bold and imaginative conjecture which might be viewed as a natural generalization of an earlier conjecture by the first named author that any $C^{1,1}$ minimizer of K-energy is smooth (c.f. [12], Conjecture 3). Under an additional assumption that there exists a smooth cscK metric in the same Kähler class, Darvas-Rubinstein conjecture is verified in [5]. In this paper, we establish this conjecture as an application of properness theorem. Note that Euler-Lagrange equation is not available apriori in our setting, so that the usual approach to the regularity problem in the calculus of variations does not immediately apply. Instead, we need to use the continuity path to overcome this difficulty.

Theorem 1.5. *(Theorem 4.1) Let $\varphi_* \in \mathcal{E}^1$ be such that $K(\varphi_*) = \inf_{\varphi\in\mathcal{E}^1} K(\varphi)$. Then φ_* is smooth and $\omega_{\varphi_*} := \omega_0 + \sqrt{-1}\partial\bar\partial\varphi_*$ is a cscK metric.*

We actually establish a more general result which allows us to consider more general twisted K-energy and we can show the weak minimizers of twisted K-energy are smooth as long as the twisting form is smooth, closed and nonnegative.

Remark 1.4. *W. He and Y. Zeng* [44] *proved Chen's conjecture on the regularity of $C^{1,1}$ minimizers of K-energy. Their original proof contains an unnecessary assumption that the $(1,1)$ current defined by the minimizer has a strictly positive lower bound which can be removed by adopting a weak Kähler-Ricci flow method initiated in Section 7 of Chen-Tian* [25]. *This will be discussed in an unpublished note* [16].

In view of Theorem 1.3, it is important to study, under what conditions, the K-energy functional is proper in a given Kähler class. In [12], the first named author proposed a decomposition formula for K-energy:

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_\varphi^n}{n!} + J_{-Ric}(\varphi). \tag{1.2}$$

where the functional J_{-Ric} is defined through its derivatives:

$$\frac{d\,J_{-Ric}}{d\,t} = \int_M \frac{\partial\varphi}{\partial t}(-Ric\wedge\frac{\omega_\varphi^{n-1}}{(n-1)!} + \underline{R}\frac{\omega_\varphi^n}{n!}). \tag{1.3}$$

One key observation in [12] (based on this decomposition formula) is that K-energy has a lower bound if the corresponding J_{-Ric} functional has a lower bound. Note that when the first Chern class is negative, one can choose a background metric such that $-Ric > 0$. Then, J_{-Ric} is convex along $C^{1,1}$ geodesics in $\mathcal{H}$ and is bounded from below if it has a critical point. In [54], Song-Weinkove further pointed out that, J_{-Ric} functional being bounded from below is sufficient to imply the properness of K-energy. The research in this direction has been very active and intense (c.f. Chen[12], Fang-Lai-Song-Weinkove [40], Song-Weikove [55], Li-Shi-Yao [48], R. Dervan [36], and references therein). Combining these results with Theorem 1.3, we have the following corollary.

Corollary 1.5. *There exists a cscK metric in $(M,[\omega])$ if any one of the following conditions holds:*

1. *There exists a constant $\epsilon \ge 0$ such that $\epsilon < \frac{n+1}{n}\alpha_M([\omega])$ and $\pi C_1(M) < \epsilon[\omega]$ such that*
$$\left(-n\frac{C_1(M)\cdot[\omega]^{n-1}}{[\omega]^n} + \epsilon\right)\cdot[\omega] + (n-1)C_1(M) > 0.$$
Here $\alpha_M(\omega)$ denotes the α-invariant of the Kähler class $(M,[\omega])$ (c.f. [59]*).*
2. *If*
$$\alpha_M([\omega]) > \frac{C_1(M)\cdot[\omega]^{n-1}}{[\omega]^n}\cdot\frac{n}{n+1}$$
and
$$C_1(M) \ge \frac{C_1(M)\cdot[\omega]^{n-1}}{[\omega]^n}\cdot\frac{n}{n+1}\cdot[\omega].$$

Here part (i) of Corollary 1.5 follows Theorem 1.3 and Li-Shi-Yao [48] (c.f. Fang-Lai-Song-Weinkove [40] Song-Weinkove [55]), part (ii) of Corollary 1.5 follows Theorem 1.3 and R. Dervan [36].

Following Donaldson's observation in [39], if a Kähler surface M admits no curve of negative self intersections and has $C_1(M) < 0$, then the condition

$$\frac{2[\omega]\cdot[-C_1(M)]}{[\omega]^2}\cdot[\omega] - [-C_1(M)] > 0$$

is satisfied automatically for any Kähler class $[\omega]$ (c.f. Song-Weinkove [54]). Consequently, on any Kähler surface M with $C_1(M) < 0$ with no curve of negative self-intersection, the K-energy is proper for any Kähler class (c.f. Song-Weinkove [55]). It follows that on these surfaces, every Kähler class admits a cscK metric. Albeit restrictive, this is indeed very close to the original vision of E. Calabi that every Kähler class should have one canonical representative. E. Calabi's vision has inspired generations of Kähler geometers to work on this exciting problem and without it, this very paper will never exist. To celebrate his vision, we propose to call such a manifold a *Calabi dream manifold.*

Definition 1.6. *A Kähler manifold is called* **Calabi dream manifold** *if every Kähler class on it admits an extremal Kähler metric.*

Clearly, all compact Riemann surfaces, complex projective spaces $\mathbb{CP}^n$ and all compact Calabi-Yau manifolds [64] are *Calabi dream manifolds.* Our discussion above asserts

Corollary 1.7. *Any Kähler surface with* $C_1 < 0$ *and no curve of negative self-intersection is a Calabi dream surface.*

It is fascinating to understand how large this family of Calabi dream surfaces is. We will delay more discussions on Calabi dream manifolds to the end of Section 2.

As a corollary of Theorem 5.3 of [15], we prove

Theorem 1.6. *The Calabi flow can be extended as long as the scalar curvature is uniformly bounded.*

Remark 1.8. *This is a surprising development. With completely different motivations in geometry, the first named author has a similar conjecture on Ricci flow which states that the only obstruction to the long time existence of Ricci flow is the* L^∞ *bound of scalar curvature. There has been significant progress in this problem, first by a series of works of B. Wang (c.f.* [61], [27]*) and more recently by the interesting and important work of Balmer-Zhang* [1] *and M. Simons* [50] *in dimension 4.*

Theorem 1.6 is a direct consequence of Theorem 5.3 of [15] and Chen-He short time existence theorem (c.f. Theorem 3.2 in [21]), where the authors proved the life span of the short time solution depends only on $C^{3,\alpha}$ norm of the initial Kähler potential and lower bound of the initial metric. By assumption, we know that $\partial_t\varphi$ remains uniformly bounded, hence φ is bounded on every finite time interval. On the other hand, since K-energy is decreasing along the flow, in particular K-energy is bounded from above along the flow. Due to (1.2) and that φ is bounded, we see that the entropy is bounded as well. Hence the flow remains in a precompact subset of $C^{3,\alpha}(M)$ on every finite time interval, hence can be extended.

In light of Theorem 1.7 and a compactness theorem of Chen-Darvas-He [17], a natural question is if one can extend the Calabi flow assuming only an upper bound on Ricci curvature. A more difficult question is whether one-sided bound of the scalar curvature is sufficient for the extension of Calabi flow. Ultimately, the remaining fundamental question is

Conjecture 1.3. *(Calabi, Chen) Initiating from any smooth Kähler potential, the Calabi flow always exists globally.*

Given the recent work by J. Street[56], Berman-Darvas-Lu[6], the weak Calabi flow always exists globally. Perhaps one can prove this conjecture via improving regularity of weak Calabi flow. On the other hand, one may hope to prove this conjecture on Kähler classes which already admit constant scalar curvature Kähler metrics and prove the flow will converges to such a metric as $t \to \infty$. An important and deep result in this direction is Li-Wang-Zheng's work [49].

Finally we explain the organization of the paper:

In section 2, we recall the necessary preliminaries needed for our proof, including the continuity path we will use to solve the cscK equation and the theory of geodesic metric spaces established by Darvas and others.

In section 3, we prove the equivalence between the existence of cscK metric and properness of K-energy, namely Theorem 1.3.

In section 4, we prove that a minimizer of K-energy over the space $\mathcal{E}^1$ is smooth. More general twisted K-energy is also considered and we show its minimizer is smooth as long as the twisting form is nonnegative, closed and smooth.

In section 5, we show that the existence of cscK metric is equivalent to geodesic stability, In particular, we verify the Donaldson's conjecture, Theorem 1.1.

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2. PRELIMINARIES

In this section, we will review some basic concepts in Kähler geometry as well as some fundamental results involving finite energy currents, which will be needed for our proof of Theorem 1.1 and 1.3. In particular, it includes the characterization of the space $(\mathcal{E}^1, d_1)$, a compactness result on bounded subsets of $\mathcal{E}^1$ with finite entropy. We also include results on the convexity of K-energy along $C^{1,1}$ geodesics as well as its extension to the space $\mathcal{E}^1$. For more detailed account on these topics, we refer to a recent survey paper by Demailly [35]. At the end of this section, we will discuss about Calabi dream manifolds.

2.1. *K***-energy and twisted** *K***-energy.** Let (M, ω_0) be a fixed Kähler class on M. Then we can define the space $\mathcal{H}$ of Kähler metrics cohomologous to ω_0 as:

$$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}. \tag{2.1}$$

We can introduce the K-energy in terms of its derivative:

$$\frac{dK}{dt}(\varphi) = -\int_M \frac{\partial \varphi}{\partial t}(R_\varphi - \underline{R})\frac{\omega_\varphi^n}{n!},\ \varphi \in \mathcal{H}. \tag{2.2}$$

Here R_φ is the scalar curvature of ω_φ, and

$$\underline{R} = \frac{[C_1(M)] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}} = \frac{\int_M R_\varphi \omega_\varphi^n}{\int_M \omega^n}.$$

Following [12], we can write down an explicit formula for $K(\varphi)$:

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_{\varphi^n}}{n!} + J_{-Ric}(\varphi), \tag{2.3}$$

where for a $(1,1)$ form χ, we define

$$\begin{aligned} J_\chi(\varphi) &= \int_0^1 \int_M \varphi\Big(\chi \wedge \frac{\omega_{\lambda\varphi}^{n-1}}{(n-1)!} - \underline{\chi}\frac{\omega_{\lambda\varphi}^n}{n!}\Big) d\lambda \\ &= \frac{1}{n!}\int_M \varphi \sum_{k=0}^{n-1} \chi \wedge \omega_0^k \wedge \omega_\varphi^{n-1-k} - \frac{1}{(n+1)!}\int_M \underline{\chi}\varphi \sum_{k=0}^{n} \omega_0^k \wedge \omega_\varphi^{n-k}. \end{aligned} \tag{2.4}$$

Here

$$\underline{\chi} = \frac{\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!}}{\int_M \frac{\omega_0^n}{n!}}.$$

Following formula (1.3), we have

$$\frac{dJ_\chi}{dt} = \int_M \partial_t\varphi(tr_\varphi\chi - \underline{\chi})\frac{\omega_\varphi^n}{n!}.$$

It is well-known that K-energy is convex along smooth geodesics in the space of Kähler potentials.

Let $\beta \geq 0$ be a smooth closed $(1,1)$ form, we define a "*twisted K-energy with respect to β*" by

$$K_\beta(\varphi) = K(\varphi) + J_\beta(\varphi). \tag{2.5}$$

The critical points of $K_\beta(\varphi)$ satisfy the following equations:

$$R_\varphi - \underline{R} = tr_\varphi\beta - \underline{\beta}, \quad \text{where } \underline{\beta} = \frac{\int_M \beta \wedge \frac{\omega_0^{n-1}}{(n-1)!}}{\int_M \frac{\omega_0^n}{n!}}. \tag{2.6}$$

For later use, we also define the functionals $I(\varphi), J(\varphi)$, given by

$$I(\varphi) = \frac{1}{(n+1)!}\int_M \varphi \sum_{k=0}^{n} \omega_0^k \wedge \omega_\varphi^{n-k}, \qquad J(\varphi) = \int_M \varphi(\omega_0^n - \omega_\varphi^n). \tag{2.7}$$

We also need to consider the more general twisted K-energy, which is defined to be

$$K_{\chi,t} = tK + (1-t)J_\chi. \tag{2.8}$$

Following [12], we can write down Euler-Lagrange equation for twisted K-energy:

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi\chi - \underline{\chi}), \; t \in [0,1]. \tag{2.9}$$

Following [14], for $t > 0$, we can rewrite this into two coupled equations:

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det g_{i\bar{j}}, \tag{2.10}$$

$$\Delta_\varphi F = -(\underline{R} - \frac{1-t}{t}\underline{\chi}) + tr_\varphi(Ric - \frac{1-t}{t}\chi). \tag{2.11}$$

In the following, we will assume $\chi > 0$, that is, χ is a Kähler form. The equation (2.9) with $t \in [0,1]$ is the continuity path proposed in [14] to solve the cscK equation. More generally, one can consider similar twisted paths in order to solve (2.6). Namely we consider

$$t(R_\varphi - \underline{R}) = t(tr_\varphi\beta - \underline{\beta}) + (1-t)(tr_\varphi\chi - \underline{\chi}). \tag{2.12}$$

The solution to (2.12) is a critical point of $tK_\beta + (1-t)J_\chi$. We will see later that it is actually a minimizer. For $t > 0$, this again can be equivalently put as

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det g_{i\bar{j}}, \tag{2.13}$$

$$\Delta_\varphi F = -(\underline{R} - \underline{\beta} - \frac{1-t}{t}\underline{\chi}) + tr_\varphi\big(Ric - \beta - \frac{1-t}{t}\chi\big). \tag{2.14}$$

An important question is whether the set of t for which (2.12) can be solved is open. The cited result is only for (2.9), but the same argument would work for (2.12).

Lemma 2.1. *([14], [63], [45]): Let $\beta \geq 0$ be nonegative closed smooth $(1,1)$ form and χ be a Kähler form. Suppose that for some $0 \leq t_0 < 1$, (2.12) has a solution $\varphi \in C^{4,\alpha}(M)$ with $t = t_0$, then for some $\delta > 0$, (2.12) has a solution in $C^{4,\alpha}$ for any $t \in (t_0-\delta, t_0+\delta)\bigcap[0,1)$.*

We observe that we can always make sure (2.9) or (2.12) can be solved for $t = 0$ by choosing $\chi = \omega_0$ or any Kähler form in $[\omega_0]$.

Remark 2.2. *Clearly if χ is smooth, it is easy to see by bootstrap that a $C^{4,\alpha}$ solution to (2.9) is actually smooth.*

Hence Lemma 2.1 shows the set of t for which (2.9) has a smooth solution is relatively open in $[0,1)$.

From the Theorem 5.3 of [15], we can conclude that

Proposition 2.3. *Let φ be a smooth solution to (2.9) or (2.12) with $t > \delta_0 > 0$, normalized so that $\sup_M \varphi = 0$. Then the higher derivatives of φ can be estimated in terms of an upper bound of entropy, defined as $\int_M \log(\frac{\omega_\varphi^n}{\omega_0^n})\omega_\varphi^n$, as well as δ_0.*

Proof. This follows directly from Theorem 5.3 of [15], by taking $f = \underline{R} - \underline{\beta} - \frac{1-t}{t}\underline{\chi}$, and $\eta = Ric(\omega_0) - \beta - \frac{1-t}{t}\chi$. Note that the assumption t being bounded below by δ_0 guarantees f and η is bounded. □

2.2. **The complete geodesic metric space $(\mathcal{E}^p, d_p)$.** In section 3.3 of [41] introduced the following space for any $p \geq 1$:

$$\mathcal{E}^p = \{\varphi \in PSH(M,\omega_0) : \int_M \omega_\varphi^n = \int_M \omega_0^n,\ \int_M |\varphi|^p\omega_\varphi^n < \infty\}. \tag{2.15}$$

In the above, $\varphi \in PSH(M,\omega_0)$ means that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ in the sense of currents. A fundamental conjecture of V. Guedj [42] stated that the completion of the space $\mathcal{H}$ of smooth potentials equipped with the L^2 metric is precisely the space $\mathcal{E}^2(M,\omega_0)$ of potentials of finite energy. This has been shown by Darvas [31], [30], in which he has shown similar characterization holds for general L^p metric. Note that the extension to the L^1 metric is essential and fundamental to our work.

Following Mabuchi, T. Darvas [31] introduced the notion of d_1 on $\mathcal{H}$.

$$||\xi||_\varphi = \int_M |\xi| \frac{\omega_\varphi^n}{n!}, \forall \ \xi \in T_\varphi \mathcal{H} = C^\infty(M). \tag{2.16}$$

Using this, we can define the path-length distance d_1 on the space $\mathcal{H}$, i.e. $d_1(u_0, u_1)$ equals the infimum of length of all smooth curves in $\mathcal{H}$, with $\alpha(0) = u_0$, $\alpha(1) = u_1$. Following Chen [11], T. Darvas proved ([31], Theorem 2) that $(\mathcal{H}, d_1)$ is a metric space.

We have the following characterization for $(\mathcal{E}^1, d_1)$:

Theorem 2.1. *([31], Theorem 5.5)Define*

$$I_1(u,v) = \int_M |u-v| \frac{\omega_u^n}{n!} + \int_M |u-v| \frac{\omega_v^n}{n!}, \ u, v \in \mathcal{H}.$$

Then there exists a constant $C > 0$ depending only on n, such that

$$\frac{1}{C} I_1(u,v) \leq d_1(u,v) \leq C I_1(u,v), \ \text{for any } u, v \in \mathcal{H}. \tag{2.17}$$

For later use, here we describe how to obtain "finite energy geodesics" from the $C^{1,1}$ geodesics between smooth potentials.

Theorem 2.2. *([31], Theorem 2) The metric completion of $(\mathcal{H}, d_1)$ equals $(\mathcal{E}^1, d_1)$ where*

$$d_1(u_0, u_1) =: \lim_{k \to \infty} d_1(u_0^k, u_1^k),$$

for any smooth decreasing sequence $\{u_i^k\}_{k \geq 1} \subset \mathcal{H}$ converging pointwise to $u_i \in \mathcal{E}^1$. Moreover, for each $t \in (0,1)$, define

$$u_t := \lim_{k \to \infty} u_t^k, \ t \in (0,1),$$

where u_t^k is the $C^{1,1}$ geodesic connecting u_0^k and u_1^k (c.f. [11]). We have $u_t \in \mathcal{E}^1$, the curve $[0,1] \ni t \mapsto u_t$ is independent of the choice of approximating sequences and is a d_1-geodesic in the sense that for some $c > 0$, $d_1(u_t, u_s) = c|t-s|$, for any s, $t \in [0,1]$.

The above limit is pointwise decreasing limit. Since the sequence $\{u_i^k\}_{k \geq 1}$ is decreasing sequence for $i = 0, 1$, we know $\{u_t^k\}_{k \geq 1}$ is also decreasing for $t \in (0,1)$, by comparison principle.

We say $u_t : [0,1] \ni t \to \mathcal{E}^1$ connecting u_0, u_1 is a finite energy geodesic if it is given by the procedure described in Theorem 2.2. The following result shows the limit of finite energy geodesics is again a finite energy geodesic.

Proposition 2.4. *([6] , Proposition 4.3) Suppose $[0,1] \ni t \to u_t^i \in \mathcal{E}^1$ is a sequence of finite energy geodesic segments such that $d_1(u_0^i, u_0)$, $d_1(u_1^i, u_1) \to 0$. Then $d_1(u_t^i, u_t) \to 0$, for any $t \in [0,1]$, where $[0,1] \ni t \mapsto u_t \in \mathcal{E}^1$ is the finite energy geodesic connecting u_0, u_1.*

Finally we record the following compactness result which will be useful later. This result was first established in [4]. The following version is taken from [6], which is the form most convenient to us.

Lemma 2.5. *([4], Theorem 2.17, [6], Corollary 4.8) Let $\{u_i\}_i \subset \mathcal{E}^1$ be a sequence for which the following condition holds:*

$$\sup_i d_1(0, u_i) < \infty, \ \sup_i K(u_i) < \infty.$$

Then $\{u_i\}_i$ contains a d_1-convergent subsequence.

2.3. **Convexity of K-energy.** In this subsection, we record some known results about the convexity of K-energy and J_χ functional along $C^{1,1}$ geodesics and also finite energy geodesics. In [12], the first named author proved the following result about the convexity of the functional J_χ.

Theorem 2.3. *([12], Proposition 2) Let $\chi \geq 0$ be a closed $(1,1)$ form. Let u_0, $u_1 \in \mathcal{H}$. Let $\{u_t\}_{t\in[0,1]}$ be the $C^{1,1}$ geodesic connecting u_0, u_1. Then $[0,1] \ni t \mapsto J_\chi(u_t)$ is convex.*

The convexity of K-energy along smooth geodesics was first observed by T. Mabuchi, c.f. [52]. However, such convexity over non-smooth geodesics is more challenging, and is conjectured by the first named author:

Conjecture 2.1. *(Chen) Let u_0, $u_1 \in \mathcal{H}$. Let $\{u_t\}_{t\in[0,1]}$ be the $C^{1,1}$ geodesic connecting u_0, u_1. Then $[0,1] \ni t \mapsto K(u_t)$ is convex.*

This conjecture was verified by the fundamental work of Berman and Berndtsson [2] (c.f. Chen-Li-Paun [23] also).

Theorem 2.4. *Conjecture 2.1 is true.*

It turns out that the K-energy and also the fuctional J_χ can be extended to the space $(\mathcal{E}^1, d_1)$ and is convex along finite energy geodesics. More precisely,

Theorem 2.5. *([6], Theorem 4.7) The K-energy defined in (2.3) can be extended to a functional $K : \mathcal{E}^1 \to \mathbb{R} \cup \{+\infty\}$. Besides, the extended functional $K|_{\mathcal{E}^1}$ is the greatest d_1-lower semi-continuous extension of $K|_{\mathcal{H}}$. Moreover, $K|_{\mathcal{E}^1}$ is convex along finite energy geodesics of $\mathcal{E}_1$.*

Theorem 2.6. *([6], Proposition 4.4 and 4.5) The functional J_χ as defined by (2.4) can be extended to be a d_1-continuous functional on $\mathcal{E}^1$. Besides, J_χ is convex along finite energy geodesics.*

2.4. **Calabi dream Manifolds.** Every example of a *Calabi dream surface* M that we discusse here is constructed from the "outside in". We begin with an ambient manifold that satisfies a weaker hypothesis making it easier to construct. Then we construct M as an appropriate complete intersections of ample hypersurfaces inside the ambient manifold and we encourage interested readers to Demailly-Peternell-Schneider[34] for further readings on this topic.

For a smooth, projective surface M, the "ample cone" equals the "big cone" if and only if the self-intersection of every irreducible curve is nonnegative. In analytic terms, the "ample cone" equals the "big cone" if and only if every holomorphic line bundle admitting a singular Hermitian metric of positive curvature current admits a regular Hermitian metric of positive curvature.

(1) For every smooth, projective variety P of dimension n at least 3 such that the ample cone equals the big cone, for every (n-2)-tuple of divisors $D_1, ..., D_{n-2}$. If the divisor classes of D_i are each globally generated, and if the D_i are "general"

in their linear equivalence classes, then the surface $M = D_1 \cap ... \cap D_{n-2}$ is smooth and connected by Bertini's theorems. If also every D_i is ample, if $K_P + (D_1 + ... + D_{n-2})$ is globally generated, and if the divisors D_i are "very general" in their linear equivalence classes, then the surface M has ample cone equal to the big cone, cf. the Noether-Lefschetz article of Ravindra and Srinivas. Finally, if also the divisor class $K_P + (D_1 + ... + D_{n-2})$ is ample, then K_M is ample. In that case, the smooth, projective surface M has $c_1(TM)$ negative, and the self-intersection of every irreducible curve is nonnegative, and thus are Calabi dream surfaes.

(2) If P and Q are projective manifolds whose ample cones equal the big cones, and if there is no nonconstant morphism from the (pointed) Albanese variety of P to the (pointed) Albanese variety of Q, then also the product $P \times Q$ is a projective manifold whose ample cone equals the big cone. In particular, if P and Q are compact Riemann surfaces of (respective) genera at least 2, and if there is no nonconstant morphism from the Jacobian of P to the Jacobian of Q, then the product $M = P \times Q$ is a Calabi dream manifold.

(3) There are many examples of smooth, projective varieties P as in item 1. When the closure of the ample cone equals the semiample cone and is finitely generated, then such a variety is precisely a "Mori dream space" that has only one Mori chamber, yet there are examples arising from Abelian varieties where the cone is not finitely generated. For instance, all projective varieties of Picard rank 1 trivially satisfy this property. The next simplest class consists of all varieties that are homogeneous under the action of a complex Lie group. This class includes all Abelian varieties. It also includes the "projective homogeneous varieties", e.g., projective spaces, quadratic hypersurfaces in projective space, Grassmannians, (classical) flag varieties,etc. This class is also stable for products and is Calabi dream manifolds.

(4) The next simplest class consists of every projective manifold P of "cohomogeneity one", i.e., those projective manifolds that admit a holomorphic action of a complex Lie group G whose orbit space is a holomorphic map from P to a compact Riemann surface. These are also Calabi dream surfaces.

Here is an interesting question about Calabi dream manifolds: how "far" is the class of Calabi dream surfaces from the class of all smooth minimal surfaces of general type?

3. K-energy proper implies existence of cscK

Let the functional I be as given by (2.7), we define

$$\mathcal{H}_0 = \{\varphi \in \mathcal{H} : I(\varphi) = 0\}.$$

Following [60] [33], we introduce the following notion of properness:

Definition 3.1. *We say the K-energy is proper with respect to L^1 geodesic distance if for any sequence $\{\varphi_i\}_{i \geq 1} \subset \mathcal{H}_0$, $\lim_{i\to\infty} d_1(0, \varphi_i) = \infty$ implies $\lim_{i\to\infty} K(\varphi_i) = \infty$.*

The goal of this section is to prove the following existence result of cscK metrics.

Theorem 3.1. *Let $\beta \geq 0$ be a smooth closed $(1,1)$ form. Let K_β be defined as in (2.5). Suppose K_β is proper with respect to geodesic distance d_1, then there exists a twisted cscK metric with respect to β(i.e, solves (2.6)).*

For the converse direction, we have

Theorem 3.2. *(main theorem of* [5] *and Theorem 4.13 of* [6]*) Let β be as in the previous theorem. Suppose that either*

(1) $\beta > 0$;
or
(2) $\beta = 0$ and $Aut_0(M, J) = 0$.
Suppose there exists a twisted cscK metric with respect to β(i.e solves (2.6)), then the functional K_β is proper with respect to geodesic distance d_1.

In this theorem, the case $\beta = 0$ and $Aut_0(M, J) = 0$ is the main result of [5], and the case with $\beta > 0$ follows from the uniqueness of minimizers of twisted K-energy when the twisting form is Kähler (c.f. [6], Theorem 4.13). For completeness, we will reproduce the proof in this paper.

First we prove Theorem 3.1. For this we will use the continuous path (2.12) to solve (2.6). Put $\chi = \omega_0$ in (2.12), define

$$S = \{t_0 \in [0, 1] : (2.12) \text{ has a smooth solution for any } t \in [0, t_0].\}. \tag{3.1}$$

Remark 3.2. *One may also consider the set S', consisting of $t_0 \in [0, 1]$ for which (2.12) has a solution with $t = t_0$. In general, $t_0 \in S'$ does not imply $[0, t_0] \subset S'$. For instance, in* [24]*, it is shown that if a cscK metric exists (i.e, (2.12) can be solved at $t = 1$.), then we can solve this equation for all t sufficiently close to 1, for any $\beta > 0$. However, we can always find a $\chi > 0$ such that (2.12) has no solution with $t = 0$. .*

By Lemma 2.1, we know the set S is relatively open in $[0, 1]$. Also when $t = 0$, (2.12) has a trivial solution, namely $\varphi = 0$. In particular $S \neq \emptyset$. The only remaining issue for the continuity method is the closedness of S. Due to Proposition 2.3, we can conclude the following criterion for closedness:

Lemma 3.3. *Suppose $t_i \in S$, $t_i \nearrow t_* > 0$, and let φ_i be a solution to (2.12) with $t = t_i$. Denote $F_i = \log \frac{\omega_{\varphi_i}^n}{\omega_0^n}$. Suppose that $\sup_i \int_M e^{F_i} F_i dvol_g < \infty$, then $t_* \in S$.*

Proof. We just need to show (2.12), or equivalently the coupled equations (2.13), (2.14) has a smooth solution with $t = t_*$. The assumption implies that we can assume $t_i \geq \delta_0 > 0$ for some $\delta_0 > 0$. Moreover, we can normalize the solution φ_i to (2.12) so that $\sup_M \varphi_i = 0$ and the assumption implies that we have a uniform upper bound of entropy. Then Proposition 2.3 implies that we have a uniform bound for all higher derivative bounds of φ_i. Hence we may take a subsequence of φ_i which converges smoothly. Say $\varphi_i \to \varphi_*$. Then we know that φ_* solves (2.12) with $t = t_*$. □

To connect this criterion with properness, we need some estimates connecting the L^1 geodesic distance d_1 and the I , J_χ functional defined in (2.7), (2.4).

Lemma 3.4. *There exists a constant $C > 0$, depending only on n and the background metric ω_0, such that for any $\varphi \in \mathcal{H}_0$, we have*

$$|\sup_M \varphi| \leq C(d_1(0, \varphi) + 1), \ |J_\chi(\varphi)| \leq C \max_M |\chi|_{\omega_0} d_1(0, \varphi). \tag{3.2}$$

Proof. This is well known in the literature and we give a proof for completeness here. We now prove the first estimate. Let $G(x, y)$ be the Green's function defined by the metric ω_0, then we can write:

$$\varphi(x) = \frac{1}{vol(M,\omega_0)}\int_M \varphi(y)\frac{\omega_0^n}{n!}(y) + \frac{1}{vol(M,\omega_0)}\int_M G(x,y)\Delta_{\omega_0}\varphi(y)\frac{\omega_0^n}{n!}(y). \tag{3.3}$$

We know that $\sup_{M\times M} G(x,y) \le C_{15}$, hence

$$\begin{aligned}
&\int_M G(x,y)\Delta_{\omega_0}\varphi(y)\frac{\omega_0^n}{n!}(y) = \int_M (G(x,y)-C_{15})(\Delta_{\omega_0}\varphi(y)+n)\frac{\omega_0^n}{n!}\\
&-\int_M nG(x,y)\frac{\omega_0^n}{n!} + C_{15}n\int_M \frac{\omega_0^n}{n!} \le -n\inf_{x\in M}\int_M G(x,y)\frac{\omega_0^n}{n!}\\
&+C_{15}n\int_M \frac{\omega_0^n}{n!} := C_{16}vol(M,\omega_0).
\end{aligned} \tag{3.4}$$

Take sup in (3.3),

$$\sup_M \varphi \le \frac{1}{vol(M,\omega_0)}\int_M \varphi\frac{\omega_0^n}{n!} + C_{16} \le Cd_1(0,\varphi) + C_{16}. \tag{3.5}$$

On the other hand, since $I(\varphi) = 0$, it follows from (2.7) that $\sup_M \varphi \ge 0$, so the first estimate follows. For the second estimate, first we can calculate

$$\begin{aligned}
&\int_M \varphi\sum_{k=0}^{n-1}\chi\wedge\omega_0^k\wedge\omega_\varphi^{n-1-k} - n\int_M \varphi\chi\wedge\omega_0^{n-1}\\
&= \int_M \varphi\sum_{k=0}^{n-2}\chi\wedge\omega_0^k\wedge(\omega_\varphi^{n-1-k}-\omega_0^{n-1-k})\\
&= \int_M -\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{n-2}(n-1-l)\chi\wedge\omega_0^{n-2-l}\wedge\omega_\varphi^l
\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}
&\Big|\int_M \varphi\sum_{k=0}^{n-1}\chi\wedge\omega_0^k\wedge\omega_\varphi^{n-1-k} - \int_M n\varphi\chi\wedge\omega_0^{n-1}\Big|\\
&\le n\max_M|\chi|_{\omega_0}\int_M \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{n-1}\omega_0^{n-1-l}\wedge\omega_\varphi^l\\
&= n\max_M|\chi|_{\omega_0}\int_M \varphi(\omega_\varphi^n-\omega_0^n).
\end{aligned}$$

Using Theorem 2.1, we conclude

$$\Big|\int_M \varphi\sum_{k=0}^{n-1}\chi\wedge\omega_0^k\wedge\omega_\varphi^{n-1-k} - \int_M n\varphi\chi\wedge\omega_0^{n-1}\Big| \le C_n\max_M|\chi|_{\omega_0}d_1(0,\varphi).$$

Similar calculation shows

$$\Big|\int_M \underline{\chi}\varphi\sum_{k=0}^{n}\omega_0^k\wedge\omega_\varphi^{n-k} - (n+1)\int_M \underline{\chi}\varphi\omega_0^n\Big| \le C_n\max_M|\chi|_{\omega_0}d_1(0,\varphi).$$

On the other hand, the quantities $\int_M n\varphi\chi\wedge\omega_0^{n-1}$ and $\int_M \underline{\chi}\varphi\omega_0^n$ can be bounded in terms of $\max_M|\chi|_{\omega_0}d_1(0,\varphi)$, again due to Theorem 2.1. Now the claimed estimate follows from (2.4). □

From Theorem 2.2, any two elements in $\mathcal{E}^1$ can be connected by a "locally finite energy geodesic" segment. On the other hand, from Theorem 4.7 in [6], we know K_β is convex along locally finite energy geodesic segment. This implies $tK_\beta + (1-t)J_{\omega_0}$ is convex along locally finite energy geodesics. In view of this, we can observe:

Corollary 3.5. *Let φ be a smooth solution to (2.12) for some $t \in [0,1]$, then φ minimizes the functional $tK_\beta + (1-t)J_{\omega_0}$ over $\mathcal{E}^1$.*

Proof. Observe that it is sufficient to show that φ minimizes $tK_\beta + (1-t)J_{\omega_0}$ over $\mathcal{H}$, in view of the fact that an element in $\mathcal{E}^1$ can be approximated(under distance d_1) using smooth potentials with convergent entropy, as proved in Theorem 3.2, [6], while the J_χ functional is continuous under d_1, as shown by Proposition 4.1 and Proposition 4.4 in [6].

Next we can write $tK_\beta + (1-t)J_{\omega_0} = tK + J_{t\beta+(1-t)\omega_0}$. Take $\psi \in \mathcal{H}$. Let $\{u_s\}_{s\in[0,1]}$ be the $C^{1,1}$ geodesic connection φ and ψ, with $u_0 = \varphi$, $u_1 = \psi$. From Lemma 3.5 of [2] and the convexity of K-energy along $C^{1,1}$ geodesics, we conclude:

$$K(\psi) - K(\varphi) \geq \lim_{s\to 0^+} \frac{K(u_s) - K(u_0)}{s} \geq \int_M (\underline{R} - R_\varphi)\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}. \tag{3.7}$$

The first inequality used the convexity of K-energy along $C^{1,1}$ geodesics, proved by Berman-Berndtsson, [2], and the second inequality is Lemma 3.5 of [2].

On the other hand, let $\{\varphi_s\}_{s\in[0,1]}$ be any smooth curve in $\mathcal{H}$ with $\varphi_0 = \varphi$, $\varphi_1 = \psi$, and let $\chi \geq 0$, we know from the calculation in [12], Proposition 2 that

$$\begin{aligned} J_\chi(\psi) - J_\chi(\varphi) &= \int_M (tr_\varphi\chi - \underline{\chi})\frac{d\varphi_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} + \int_0^1 (1-s)\frac{d^2}{ds^2}J_\chi(\varphi_s)ds \\ &= \int_M (tr_\varphi\chi - \underline{\chi})\frac{d\varphi_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} + \int_0^1 (1-s)ds\int_M \left(\frac{\partial^2\varphi}{\partial s^2} - |\nabla_{\varphi_s}\frac{\partial\varphi_s}{\partial s}|^2_{\varphi_s}\right)tr_{\varphi_s}\chi\frac{\omega_{\varphi_s}^n}{n!} \\ &+ \int_0^1 (1-s)ds\int_M g_{\varphi_s}^{i\bar{j}}g_{\varphi_s}^{k\bar{l}}\chi_{i\bar{l}}\big(\frac{\partial\varphi}{\partial s}\big)_{,k}\big(\frac{\partial\varphi}{\partial s}\big)_{,\bar{j}}\frac{\omega_{\varphi_s}^n}{n!}. \end{aligned} \tag{3.8}$$

Now we choose $\varphi_s = u_s^\varepsilon$, namely the ε-geodesic(which is smooth by [11]), which means

$$\left(\frac{\partial^2\varphi_s}{\partial s^2} - |\nabla_{\varphi_s}\frac{\partial\varphi_s}{\partial s}|^2_{\varphi_s}\right)\det g_{\varphi_s} = \varepsilon\det g_0 \geq 0.$$

Hence we obtain from (3.8) that

$$J_\chi(\psi) - J_\chi(\varphi) \geq \int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s^\varepsilon}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}. \tag{3.9}$$

Also we know that $u_s^\varepsilon \to u_s$ weakly in $W^{2,p}$ for any $p < \infty$ as $\varepsilon \to 0$. This implies $\frac{du_s^\varepsilon}{ds}|_{s=0}$, as a function on M, is uniformly bounded with its first derivatives. Hence we may conclude $\frac{du_s^\varepsilon}{ds}|_{s=0} \to \frac{du_s}{ds}|_{s=0}$ uniformly. This convergence is sufficient to imply

$$\int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s^\varepsilon}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} \to \int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}, \text{ as } \varepsilon \to 0.$$

Therefore,

$$J_\chi(\psi) - J_\chi(\varphi) \geq \int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}. \tag{3.10}$$

Take $\chi = t\beta + (1-t)\omega_0$ in (3.10). Then multiply (3.7) by t, add to (3.10), we conclude

$$\begin{aligned}(3.11)\quad &\big(tK_\beta + (1-t)J_{\omega_0}\big)(\psi) - \big(tK_\beta + (1-t)J_{\omega_0}\big)(\varphi)\\ &\geq \int_M \Big(t(\underline{R} - R_\varphi) + (tr_\varphi\chi - \underline{\chi})\Big)\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} = 0.\end{aligned}$$

The last equality used that φ solves (2.13), (2.14). □

Using this fact, we can obtain the following improvement of Lemma 3.3, which asserts that having control over the geodesic distance d_1 along the path of continuity ensures we can pass to limit.

Lemma 3.6. *Suppose $t_i \in S$, $t_i \nearrow t_* > 0$, and let φ_i be the solution to (2.12) with $t = t_i$, normalized so that $I(\varphi_i) = 0$. Suppose $\sup_i d_1(0,\varphi_i) < \infty$, then $t_* \in S$.*

Proof. As before, we assume $t_i \geq \delta > 0$. First observe that $\sup_i (t_iK_\beta + (1-t_i)J_{\omega_0})(\varphi_i) < \infty$. Indeed, we know from Corollary 3.5 that φ_i are minimizers of $t_iK_\beta + (1-t_i)J_{\omega_0}$, hence

$$\begin{aligned}(3.12)\quad t_iK_\beta(\varphi_i) + (1-t_i)J_{\omega_0}(\varphi_i) &\leq K_{\chi,t_i}(0) = t_iK_\beta(0) + (1-t_i)J_{\omega_0}(0)\\ &\leq \max(K_\beta(0), J_{\omega_0}(0)).\end{aligned}$$

On the other hand, we know

$$(3.13)\quad t_iK_\beta(\varphi_i) + (1-t_i)J_{\omega_0}(\varphi_i) = t_i\int_M e^{F_i}F_i dvol_g + t_iJ_{-Ric+\beta}(\varphi_i) + (1-t_i)J_{\omega_0}(\varphi_i).$$

Since we assumed $\sup_i d_1(0,\varphi_i) < \infty$, Lemma 3.4 then implies that $\sup_i |J_{-Ric+\beta}(\varphi_i)| + |J_{\omega_0}(\varphi_i)| < \infty$. Consequently, $\sup_i \int_M e^{F_i}F_i dvol_g < \infty$ since $t_i \geq \delta > 0$. The result then follows from Lemma 3.3. □

Now we are ready to prove Theorem 3.1.

Proof. (of Theorem 3.1) Let S be defined as in (3.1), we just need to prove $S = [0,1]$. First we know from Lemma 2.1 that $t_* > 0$. We want to show that $t_* = 1$ and $1 \in S$. Indeed, if $t_* < 1$, then we can take a sequence $t_i \in S$, such that $t_i \nearrow t_*$. Let φ_i be the solution to (2.9) so that $I(\varphi_i) = 0$.

As observed in (3.12) above, $\sup_i \big(t_iK_\beta + (1-t_i)J_{\omega_0}\big)(\varphi_i) < \infty$. On the other hand, since $0 \in \mathcal{H}$ is a critical point of J_{ω_0}, we know from Corollary 3.5 that $J_{\omega_0}(\varphi_i) \geq J_{\omega_0}(0)$. Therefore we know $\sup_i K_\beta(\varphi_i) < \infty$. By properness, we can then conclude $\sup_i d_1(0,\varphi_i) < \infty$. From Lemma 3.6 we see $t_* \in S$. But then from Lemma 2.1 and Remark 2.2 we know $t_* + \delta' \in S$ for some $\delta' > 0$ small. This contradicts $t_* = \sup S$. Hence we must have $t_* = 1$. Repeat the argument in this paragraph, we can finally conclude $1 \in S$. □

For completeness, we also include here the proof of Theorem 3.2, following [5], [6].

Proof. (of Theorem 3.2) First we assume that $\beta = 0$ and $Aut_0(M,J) = 0$. Let $\varphi_0 \in \mathcal{H}_0$ be such that $\omega_{\varphi_0} := \omega_0 + \sqrt{-1}\partial\bar\partial\varphi_0$ is cscK. We will show that for some $\varepsilon > 0$, and for any $\psi \in \mathcal{H}_0$, $d_1(\varphi_0,\psi) \geq 1$, we have $K(\psi) \geq \varepsilon d_1(\psi,\varphi_0) + K(\varphi_0)$.

Indeed, if this were false, we will have a sequence of $\psi_i \in \mathcal{H}_0$, such that $d_1(\varphi_0,\psi_i) \geq 1$, but $\varepsilon_i := \frac{K(\psi_i) - K(\varphi_0)}{d_1(\psi_i,\varphi_0)} \to 0$. Let $c^i : t \in [0, d_1(\varphi_0,\psi_i)] \to \mathcal{E}^1$ be the unit speed $C^{1,1}$

geodesic segment connecting φ_0 and ψ_i [11]. Let $\phi_i = c^i(1)$, then $d_1(\phi_i, \varphi_0) = 1$. On the other hand, from the convexity of K-energy, we have

$$K(\phi_i) \leq \big(1 - \frac{1}{d_1(\psi_i, \varphi_0)}\big)K(\varphi_0) + \frac{1}{d_1(\psi_i, \varphi_0)}K(\psi_i) = K(\varphi_0) + \varepsilon_i. \tag{3.14}$$

By the compactness result Lemma 2.5, there exists a subsequence of $\{\phi_i\}_{i\geq 1} \subset \mathcal{E}^1$, denoted by ϕ_{i_j}, such that $\phi_{i_j} \xrightarrow{d_1} \phi_\infty$. Hence $d_1(\varphi_0, \phi_\infty) = 1$. From the lower semi-continuity of K-energy(Theorem 4.7 of [6]), we obtain:

$$K(\phi_\infty) \leq \lim_{j\to\infty} \inf K(\phi_{i_j}) \leq K(\varphi_0). \tag{3.15}$$

But since φ_0 is a minimizer of K-energy over $\mathcal{E}^1$, it follows that ϕ_∞ is also a minimizer. From Theorem 1.4 of [5], we know ϕ_∞ is also a smooth solution to cscK equation, and there exists $g \in Aut_0(M, J)$, such that $g^*\omega_{\phi_\infty} = \omega_{\varphi_0}$. But we assumed $Aut_0(M, J) = 0$, hence $\omega_{\phi_\infty} = \omega_{\varphi_0}$. Therefore $\phi_\infty - \varphi_0$ is constant. But from the normalization $I(\phi_\infty) = I(\varphi_0) = 0$, we know $\varphi_0 - \phi_\infty = 0$, this contradicts $d_1(\varphi_0, \phi_\infty) = 1$.

Next we assume $\beta > 0$. Let φ^β solves (2.12), normalized so that $I(\varphi^\beta) = 0$. We show that for some $\varepsilon > 0$, one has $K_\beta(\psi) \geq \varepsilon d_1(\varphi^\beta, \psi) + K_\beta(\varphi^\beta)$ for any $\psi \in \mathcal{H}_0$ with $d_1(\varphi^\beta, \psi) \geq 1$.

Indeed, if this were false, then there exists a sequence of $\psi_i \in \mathcal{H}_0$, such that $d_1(\varphi^\beta, \psi_i) \geq 1$, but $\varepsilon_i' := \frac{K_\beta(\psi_i) - K_\beta(\varphi^\beta)}{d_1(\psi_i, \varphi^\beta)} \to 0$. Note that K-energy is lower semi-continuous with respect to d_1 convergence and J_β is continuous([6], Proposition 4.4). Hence K_β is lower semicontinuous as well. So the same argument as last paragraph applies and we get a minimizer of K_β, denoted as $\psi_\infty \in \mathcal{H}_0$, such that $d_1(\psi_\infty, \varphi^\beta) = 1$. But by [6], Theorem 4.13, we know ψ_∞ and φ^β should differ by a constant. Because of the normalization $I(\psi_\infty) = I(\varphi^\beta) = 0$, we know that actually $\psi_\infty = \varphi^\beta$. This contradicts $d_1(\psi_\infty, \varphi^\beta) = 1$. □

As a corollary to this theorem, we show that the supremem of t for which (2.9) can be solved depends only on cohomology class of χ. More precisely,

Corollary 3.7. *Let χ_1, χ_2 be two Kähler forms in the same cohomology class. We define*

$$S_i = \{t_0 \in [0, 1] : \textit{(2.9) with } \chi = \chi_i \textit{ has a smooth solution for any } t \in [0, t_0].\}$$

Then $S_1 = S_2$. In particular, if we define $R([\omega_0], \chi_i) = \sup S_i$, then $R([\omega_0], \chi_1) = R([\omega_0], \chi_2)$.

Proof. First we know from [29], Proposition 21 and Proposition 22 that existence of smooth solutions to $tr_\varphi \chi_i = \underline{\chi}_i$, $i = 1, 2$ are equivalent. So we may assume both equations are solvable. Then it follows from Lemma 2.1 that $R([\omega_0], \chi_i) > 0$. In virtue of Theorem 3.1 and Theorem 3.2, we just need to show for any $0 < t_0 \leq 1$:

$$K_{\chi_1, t_0} \text{ is proper} \Leftrightarrow K_{\chi_2, t_0} \text{ is proper}. \tag{3.16}$$

Here K_{χ_i, t_0} is defined as in (2.8).

Indeed, suppose $t_0 \in S_1$ and $t_0 < 1$, then for any $0 < t \leq t_0$, (2.9) with $\chi = \chi_1$ has a solution. From Theorem 3.2 applied to $\beta = \frac{1-t}{t}\chi_1$, we know this implies $K_{\chi_1, t}$ is proper, for any $0 < t \leq t_0$. If (3.16) were true, then $K_{\chi_2, t}$ is proper for any $0 < t \leq t_0$. Use Theorem 3.1 again, we know (2.9) with $\chi = \chi_2$ is solvable for any $t \in [0, t_0]$. This means $t_0 \in S_2$.

If $t_0 \in S_1$ and $t_0 = 1$, then it means K-energy is bounded from below, hence $K_{\chi_2,t}$ will be proper for $0 \le t < 1$([29], Proposition 21). Then Theorem 3.1 implies (2.9) will be solvable for $\chi = \chi_2$ and any $0 \le t < 1$. While for $t = 1$, the solvability follows from the assumption that $t_0 = 1$, since the equation (2.9) for $t = 1$ does not involve χ_1 or χ_2. Therefore $1 \in S_2$.

Now we turn to the proof of (3.16), which is an elementary calculation (c.f. [57]). Since χ_1 and χ_2 are in the same Kähler class, we can write

$$\chi_1 - \chi_2 = \sqrt{-1}\partial\bar{\partial}\nu, \text{ for some smooth function } \nu.$$

From (2.4), we can compute for $\varphi \in \mathcal{H}_0$:

$$\begin{aligned} J_{\chi_1}(\varphi) - J_{\chi_2}(\varphi) &= \frac{1}{n!}\sum_{p=0}^{n-1}\int_M (-\varphi)\sqrt{-1}\partial\bar{\partial}\nu \wedge \omega_0^{n-p-1} \wedge \omega_\varphi^p \\ &= \frac{1}{n!}\sum_{p=0}^{n-1}\int_M -\nu\sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega_0^{n-p-1} \wedge \omega_\varphi^p \\ &= \frac{-1}{n!}\int_M \nu\omega_\varphi^n + \int_M \frac{1}{n!}\nu\omega_0^n. \end{aligned} \tag{3.17}$$

From this it is clear that

$$|J_{\chi_1}(\varphi) - J_{\chi_2}(\varphi)| \le c_n \sup_M |\nu|. \tag{3.18}$$

On the other hand,

$$|K_{\chi_1,t_0}(\varphi) - K_{\chi_2,t_0}(\varphi)| \le (1-t_0)|J_{\chi_1}(\varphi) - J_{\chi_2}(\varphi)| \le c_n \sup_M |\nu|. \tag{3.19}$$

From this (3.16) immediately follows. □

4. REGULARITY OF WEAK MINIMIZERS OF K-ENERGY

Our main goal in this section is to show the minimizers of K-energy over $\mathcal{E}^1$ are always smooth. The main ingredients are the continuity path as well as apriori estimates obtained in section 3. The strategy of the proof is somewhat different from the usual variational problem. Indeed, the usual strategy for variational problem will be first to take some smooth variation of the minimizer, and derive an Euler-Lagrange equation for the minimizer(in weak form). Then one works with the Euler-Lagrange equation to obtain regularity(or partial regularity).

However, the same strategy runs into difficulty here. Indeed, an Euler-Lagrange equation for minimizer is not apriori available, since an arbitrary smooth variation of φ_* does not necessarily preserve the condition that $\omega_\varphi \ge 0$.

To get around this difficulty, we will still use the continuity path and our argument is partly inspired from [5]. The difference here is that the properness theorem (Theorem 3.1) plays a central role. Here we sketch the argument. Take φ_j to be smooth approximations of φ_* (in the space $\mathcal{E}^1$), and we solve continuity path from φ_j. That K-energy is bounded from below ensures the continuity path is solvable for $t < 1$. We will show the existence of a minimizer ensures that for each fixed j, L^1 geodesic distance remains bounded as $t \to 1$. Hence we can take limit as $t \to 1$ and obtain a cscK potential u_j. Besides, such a sequence of u_j will also be uniformly bounded under L^1 geodesic distance, which follows from the uniform boundedness of φ_j under L^1 geodesic distance.

Our apriori estimates allow us to take smooth limit of u_j and conclude that $u_j \to \psi$ smoothly and ψ is a smooth cscK potential. The proof is then finished once we can show ψ and φ_* only differ by an additive constant.

First we show that the existence of minimizers implies existence of smooth cscK metric.

Lemma 4.1. *Suppose that for some $\varphi_* \in \mathcal{E}^1$, we have $K(\varphi_*) = \inf_{\varphi\in\mathcal{E}^1} K(\varphi)$, then there exists a smooth cscK in the class $[\omega_0]$.*

Proof. We consider the continuity path (2.9) with $\chi = \omega_0$. By assumption, K-energy over $\mathcal{E}^1$ is bounded from below. Therefore the twisted K-energy $K_{\omega_0,t}$, defined by (2.8) is proper for any $0 \le t < 1$. Hence we may invoke Theorem 3.1 with $\beta = \frac{1-t}{t}\omega_0$ to conclude that there exists a solution to (2.9) for any $0 < t < 1$. The only remaining issue is to see what happens in (2.9) as $t \to 1$.

Choose $t_i < 1$ and $t_i \to 1$, and let $\tilde{\varphi}_i$ be solutions to (2.9) with $t = t_i$, normalized up to an additive constant so that $I(\tilde{\varphi}_i) = 0$. Corollary 3.5 implies that $\tilde{\varphi}_i$ is the minimizer to K_{ω_0,t_i}. Therefore we have

$$t_iK(\varphi_*) + (1-t_i)J_{\omega_0}(\tilde{\varphi}_i) \le t_iK(\tilde{\varphi}_i) + (1-t_i)J_{\omega_0}(\tilde{\varphi}_i) \le t_iK(\varphi_*) + (1-t_i)J_{\omega_0}(\varphi_*). \tag{4.1}$$

Hence (4.1) implies that

$$J_{\omega_0}(\tilde{\varphi}_i) \le J_{\omega_0}(\varphi_*).$$

On the other hand, we know J_{ω_0} is proper, in the sense that $J_{\omega_0}(\varphi) \ge \delta d_1(0,\varphi) - C$, for $\varphi \in \mathcal{H}_0$ (c.f. [29], Proposition 22). This implies that

$$\sup_i d_1(0, \tilde{\varphi}_i) \le \frac{1}{\delta}\big(C + J_{\omega_0}(\varphi_*)\big) < \infty.$$

Now from Lemma 4.6 we conclude that (2.9) can be solved up to $t = 1$, and we obtain the existence of a cscK potential. ☐

The main result of [5] showed the following weak-strong uniqueness property: as long as a smooth cscK exists in the Kähler class $[\omega_0]$, then all the minimizers of K-energy over $\mathcal{E}^1$ are smooth cscK. Therefore, we can already conclude the following result:

Theorem 4.1. *Let $\varphi_* \in \mathcal{E}^1$ be such that $K(\varphi_*) = \inf_{\mathcal{E}^1} K(\varphi)$. Then φ_* is smooth, and ω_{φ_*} is a cscK metric.*

Next we will prove a more general version of Theorem 4.1. More precisely, we will prove:

Theorem 4.2. *Let $\chi \ge 0$ be a closed smooth $(1,1)$ form. Define $K_\chi(\varphi) = K(\varphi) + J_\chi(\varphi)$, where $J_\chi(\varphi)$ is defined by (2.4). Let $\varphi_* \in \mathcal{E}^1$ be such that $K_\chi(\varphi_*) = \inf_{\mathcal{E}^1} K_\chi(\varphi)$. Then φ_* is smooth and solves the equation $R_\varphi - \underline{R} = tr_\varphi\chi - \underline{\chi}$.*

Note that one can run the same argument as in Lemma 4.1 to show once there exists a minimizer to K_χ, then there exists a smooth solution to

$$R_\varphi - \underline{R} = tr_\varphi\chi - \underline{\chi}. \tag{4.2}$$

However, it is not clear to us whether the argument in [5] can be adapted to this case to show a weak-strong uniqueness result. Namely if there exists a smooth solution to $R_\varphi - \underline{R} = tr_\varphi\chi - \underline{\chi}$, can one conclude all minimizers of K_χ are smooth? Therefore, in the following, we will use a direct argument. This argument is motivated from [5], but now is more straightforward because of the use of properness theorem.

Let φ_* be a minimizer of K_χ. Then by [6], Lemma 1.3, we may take a sequence of $\varphi_j \in \mathcal{H}$, such that $d_1(\varphi_j, \varphi_*) \to 0$, and $K_\chi(\varphi_j) \to K_\chi(\varphi_*)$. Indeed, that lemma asserts the convergence of the entropy part, but the J_{-Ric} and J_χ are continuous under d_1 convergence, by [6], Proposition 4.4.

Since there exists a minimizer to K_χ, the functional K_χ is bounded from below. On the other hand, for each fixed j, by [29], Proposition 22, we know that $J_{\omega_{\varphi_j}}$ is proper. Therefore, for $0 \le t < 1$, the twisted K_χ-energy $K_{\chi,\omega_{\varphi_j},t} := tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ is proper. Hence we may invoke Theorem 3.1 to conclude there exists a smooth solution to the equation

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi \omega_{\varphi_j} - n) + t(tr_\varphi \chi - \underline{\chi}), \text{ for any } 0 \le t < 1. \tag{4.3}$$

Denote the solution to be φ_j^t, normalized up to an additive constant so that $\varphi_j^t \in \mathcal{H}_0$, namely $I(\varphi_j^t) = 0$.

Since $\chi \ge 0$ and closed, we know that J_χ is convex along $C^{1,1}$ geodesic(though not necessarily strictly convex). Hence the functional K_χ is convex along $C^{1,1}$ geodesic. This again implies the convexity of $tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ along $C^{1,1}$ geodesic. In particular, φ_j^t is a global minimizer of $tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ by Corollary 4.5.

Hence we know that

$$tK_\chi(\varphi_j^t)+(1-t)J_{\omega_{\varphi_j}}(\varphi_j) \le tK_\chi(\varphi_j^t)+(1-t)J_{\omega_{\varphi_j}}(\varphi_j^t) \le tK_\chi(\varphi_j)+(1-t)J_{\omega_{\varphi_j}}(\varphi_j). \tag{4.4}$$

The first inequality above uses that φ_j minimizes $J_{\omega_{\varphi_j}}$. Hence

$$\sup_{0<t<1,\, j} K_\chi(\varphi_j^t) \le \sup_j K_\chi(\varphi_j). \tag{4.5}$$

Next we will show that the family of solution φ_j^t are uniformly bounded in d_1. First we have

$$tK_\chi(\varphi_j^t)+(1-t)J_{\omega_{\varphi_j}}(\varphi_j^t) \le tK_\chi(\varphi_*)+(1-t)J_{\omega_{\varphi_j}}(\varphi_*) \le tK_\chi(\varphi_j^t)+(1-t)J_{\omega_{\varphi_j}}(\varphi_*). \tag{4.6}$$

The first inequality follows from that φ_j^t minimizes $tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ and the second inequality follows since φ_* minimizes K_χ. Therefore,

$$J_{\omega_{\varphi_j}}(\varphi_j) \le J_{\omega_{\varphi_j}}(\varphi_j^t) \le J_{\omega_{\varphi_j}}(\varphi_*). \tag{4.7}$$

The first inequality follows from that φ_j is a minimizer of $J_{\omega_{\varphi_j}}$. The second inequality follows from (4.6). As a first observation, we have

Lemma 4.2. *As $j \to \infty$,*

$$J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \to 0.$$

Proof. We can compute

$$
\begin{aligned}
J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) &= \int_0^1 \frac{d}{d\lambda}\big(J_{\omega_{\varphi_j}}(\lambda\varphi_* + (1-\lambda)\varphi_j)\big)d\lambda \\
&= \int_0^1 d\lambda \int_M (\varphi_* - \varphi_j)\frac{\omega^{n-1}_{\lambda\varphi_*+(1-\lambda)\varphi_j}\wedge\omega_{\varphi_j} - \omega^n_{\lambda\varphi_*+(1-\lambda)\varphi_j}}{(n-1)!} \\
&= \int_0^1 d\lambda \int_M \lambda(\varphi_* - \varphi_j)\wedge\sqrt{-1}\partial\bar\partial(\varphi_j - \varphi_*)\wedge\frac{\omega^{n-1}_{\lambda\varphi_*+(1-\lambda)\varphi_j}}{(n-1)!} \\
&= \int_0^1 d\lambda \int_M \lambda\sqrt{-1}\partial(\varphi_* - \varphi_j)\wedge\bar\partial(\varphi_* - \varphi_j)\wedge\frac{(\lambda\omega_{\varphi_*} + (1-\lambda)\omega_{\varphi_j})^{n-1}}{(n-1)!}.
\end{aligned} \tag{4.8}
$$

Define

$$
\begin{aligned}
I(\varphi_j,\varphi_*) &= \int_M \sqrt{-1}\partial(\varphi_j-\varphi_*)\wedge\bar\partial(\varphi_j-\varphi_*)\wedge\sum_{k=0}^{n-1}\omega^k_{\varphi_j}\wedge\omega^{n-1-k}_{\varphi_*} \\
&= \int_M(\varphi_j-\varphi_*)(\omega^n_{\varphi_*}-\omega^n_{\varphi_j}).
\end{aligned} \tag{4.9}
$$

Since we know $d_1(\varphi_j,\varphi_*) \geq \frac{1}{C}\int_M|\varphi_j-\varphi_*|(\omega^n_{\varphi_j}+\omega^n_{\varphi_*})$ for some dimensional constant C, by [31], Theorem 5.5, we have $I(\varphi_j,\varphi_*) \leq Cd_1(\varphi_j,\varphi_*) \to 0$. On the other hand, we have $J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \leq C'I(\varphi_j,\varphi_*)$ from (4.8) and (4.9). Hence $J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \leq C'Cd_1(\varphi_j,\varphi_*) \to 0$. □

Corollary 4.3. *Let $I(\varphi_j,\varphi_j^t)$ be defined similar to (4.9), then we have $\sup_{0<t<1} I(\varphi_j,\varphi_j^t) \to 0$ as $j\to\infty$.*

Proof. From previous lemma and (4.7), we know that as $j\to\infty$,

$$\sup_{0<t<1} J_{\omega_{\varphi_j}}(\varphi_j^t) - J_{\omega_{\varphi_j}}(\varphi_j) \leq J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \to 0.$$

On the other hand, we know from (4.8), (4.9) with φ_* replaced by φ_j^t, th following estimate holds:

$$\frac{1}{C_n}(J_{\omega_{\varphi_j}}(\varphi_j^t) - J_{\omega_{\varphi_j}}(\varphi_j)) \leq I(\varphi_j^t,\varphi_j) \leq C_n(J_{\omega_{\varphi_j}}(\varphi_j^t) - J_{\omega_{\varphi_j}}(\varphi_j)).$$

□

Next we would like to show the d_1 distance of φ_j^t remains uniformly bounded. For this we will need the following key lemma:

Lemma 4.4. *([4], Theorem 1.8 and Lemma 1.9) There exists a dimensional constant C_n, such that for any u, v, $w \in \mathcal{E}^1$, we have*

$$I(u,w) \leq C_n(I(u,v)+I(v,w)).$$

Besides, we have

$$\int_M \sqrt{-1}\partial(u-w)\wedge\bar\partial(u-w)\wedge\omega_v^{n-1} \leq C_n I(u,w)^{\frac{1}{2^{n-1}}}\big(I(u,v)^{1-\frac{1}{2^{n-1}}} + I(w,v)^{1-\frac{1}{2^{n-1}}}\big).$$

As an immediate consequence of this lemma and Corollary 4.3, we see that:

Corollary 4.5. $\sup_{0<t<1} I(\varphi_j^t,\varphi_*) \to 0$ *as* $j\to\infty$.

Proof. Indeed,

$$I(\varphi_j^t,\varphi_*)\le C_n(I(\varphi_j^t,\varphi_j)+I(\varphi_j,\varphi_*))\le C_n\big(I(\varphi_j^t,\varphi_j)+Cd_1(\varphi_j,\varphi_*)\big).$$

In the second inequality above, we again used Theorem 5.5 of [31]. □

Using Lemma 4.4, we can show the following:

Lemma 4.6. *There exists a constant C, depending only on $\sup_j d_1(0,\varphi_j)$, n, such that*

$$\sup_{j,0<t<1} d_1(0,\varphi_j^t)\le C.$$

Proof. Denote $d^c=\frac{\sqrt{-1}}{2}(\partial-\bar\partial)$, and let $\varepsilon>0$, we may calculate

$$\begin{aligned}
&J_{\omega_0}(\varphi_j^t)-J_{\omega_{\varphi_j}}(\varphi_j^t)\\
&=\int_0^1\frac{d}{d\lambda}\big(J_{\omega_0}(\lambda\varphi_j^t)-J_{\omega_{\varphi_j}}(\lambda\varphi_j^t)\big)d\lambda\\
&=\int_0^1\int_M\varphi_j^t\Big(\frac{\omega_0\wedge\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}-\frac{\omega_{\varphi_j}\wedge\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}\Big)d\lambda=\int_0^1\int_M d^c\varphi_j^t\wedge d\varphi_j\wedge\frac{\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}d\lambda\\
&\le\varepsilon\int_0^1\int_M d^c\varphi_j^t\wedge d\varphi_j^t\wedge\frac{\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}d\lambda+\frac{1}{\varepsilon}\int_0^1\int_M d^c\varphi_j\wedge d\varphi_j\wedge\frac{\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}d\lambda\\
&\le\varepsilon C_n\int_M d^c\varphi_j^t\wedge d\varphi_j^t\wedge\sum_{k=0}^{n-1}\omega_0^k\wedge\omega_{\varphi_j^t}^{n-1-k}+\frac{C_n}{\varepsilon}\int_M d^c\varphi_j\wedge d\varphi_j\wedge\frac{\omega_{\frac12\varphi_j^t}^{n-1}}{(n-1)!}\qquad(4.10)\\
&\le\varepsilon\tilde C_n d_1(0,\varphi_j^t)+\frac{\tilde C_n}{\varepsilon}I(\varphi_j,0)^{\frac{1}{2^{n-1}}}\Big(I(0,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}+I(\varphi_j,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}\Big)\\
&\le\varepsilon\tilde C_n d_1(0,\varphi_j^t)+\frac{\tilde C_n}{\varepsilon}I(0,\varphi_j)^{\frac{1}{2^{n-1}}}\Big(I(0,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}\\
&\qquad+D_nI(0,\varphi_j)^{1-\frac{1}{2^{n-1}}}+D_nI(0,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}\Big)\\
&\le\varepsilon\tilde C_n d_1(0,\varphi_j^t)+\varepsilon I(0,\frac12\varphi_j^t)+\varepsilon^{-2^n+1}\big(\tilde C_n(1+D_n)\big)^{2^{n-1}}I(0,\varphi_j).
\end{aligned}$$

In the first line above, we used that $J_{\omega_0}(0)=J_{\omega_{\varphi_j}}(0)=0$, which follows from (2.4). We used the second inequality of Lemma 4.4 in the passage from the 5th line to 6th line, and the first inequality in the passage from 6th line to 7th line. In the passage from 7th line to the last line, we used Young's inequality. Next observe that

$$\begin{aligned}
I(0,\frac12\varphi_j^t)&=\int_M\sqrt{-1}\partial\big(\frac12\varphi_j^t\big)\wedge\bar\partial\big(\frac12\varphi_j^t\big)\wedge\sum_{k=0}^{n-1}\omega_{\frac12\varphi_j^t}^k\wedge\omega_0^{n-1-k}\\
&=\int_M\sqrt{-1}\partial\big(\frac12\varphi_j^t\big)\wedge\bar\partial\big(\frac12\varphi_j^t\big)\wedge\sum_{k=0}^{n-1}\frac{1}{2^k}(\omega_0+\omega_{\varphi_j^t})^k\wedge\omega_0^{n-1-k}\qquad(4.11)\\
&\le C_n\int_M\sqrt{-1}\partial\varphi_j^t\wedge\bar\partial\varphi_j^t\wedge\sum_{k=0}^{n-1}\omega_0^k\wedge\omega_{\varphi_j^t}^{n-1-k}=C_n\int_M\varphi_j^t(\omega_0^n-\omega_{\varphi_j^t}^n)\\
&\le\tilde C_n d_1(0,\varphi_j^t).
\end{aligned}$$

Hence we obtain

$$J_{\omega_0}(\varphi_j^t) \le J_{\omega_{\varphi_j}}(\varphi_j^t) + \varepsilon \tilde{C}_n d_1(0,\varphi_j^t) + 2\varepsilon^{-2^n+1}\big(\tilde{C}_n(1+D_n)\big)^{2^{n-1}} I(0,\varphi_j). \tag{4.12}$$

On the other hand, since we know J_{ω_0} is proper in the following sense:

$$J_{\omega_0}(\varphi) \ge \delta d_1(0,\varphi) - C, \qquad \varphi \in \mathcal{H}_0.$$

Choose ε small enough so that

$$2\varepsilon \tilde{C}_n \le \frac{\delta}{2}.$$

Hence we obtain from (4.12) that

$$d_1(0,\varphi_j^t) \le \frac{2}{\delta}\big(J_{\omega_{\varphi_j}}(\varphi_j^t) + \varepsilon^{-2^n+1}\big(\tilde{C}_n(1+D_n)\big)^{2^{n-1}} I(0,\varphi_j) + C\big). \tag{4.13}$$

Since we know that $I(0,\varphi_j) \le C d_1(0,\varphi_j)$, and $d_1(0,\varphi_j)$ is uniformly bounded, it only remains to find an upper bound for $J_{\omega_{\varphi_j}}(\varphi_j^t)$. In order to bound $J_{\omega_{\varphi_j}}(\varphi_j^t)$ from above, we just need to find an upper bound for $J_{\omega_{\varphi_j}}(\varphi_*)$ thanks to (4.7). For this we can write:

$$\begin{aligned} J_{\omega_{\varphi_j}}(\varphi_*) &= \int_0^1 d\lambda \int_M \varphi_* \Big(\frac{\omega_{\lambda\varphi_*}^{n-1}\wedge \omega_{\varphi_j}}{(n-1)!} - \frac{\omega_{\lambda\varphi_*}^{n}}{(n-1)!}\Big) \\ &\le \int_0^1 d\lambda \int_M \varphi_* \sqrt{-1}\partial\bar{\partial}(\varphi_j - \lambda\varphi_*) \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} \\ &= \int_0^1 d\lambda \int_M \lambda d^c\varphi_* \wedge d\varphi_* \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} - \int_0^1 d\lambda \int_M d^c\varphi_* \wedge d\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!}. \end{aligned} \tag{4.14}$$

In the above, $d^c = \frac{\sqrt{-1}}{2}(\partial - \bar{\partial})$, hence $d^c d = \sqrt{-1}\partial\bar{\partial}$. For the first term above, it can be bounded in the following way:

$$\int_0^1 d\lambda \int_M \lambda d^c\varphi_* \wedge d\varphi_* \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} \le \int_M d^c\varphi_* \wedge d\varphi_* \wedge \sum_{k=0}^{n-1} \omega_0^k \wedge \omega_{\varphi_*}^{n-1-k} \le C d_1(0,\varphi_*). \tag{4.15}$$

For the second term on the right hand side of (4.14),

$$\begin{aligned} -\int_0^1 d\lambda \int_M d^c\varphi_* \wedge d\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} &\le \frac{1}{2}\int_0^1 d\lambda \int_M d^c\varphi_* \wedge d\varphi_* \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} \\ &+ \frac{1}{2}\int_0^1 d\lambda \int_M d^c\varphi_j \wedge d\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!}. \end{aligned} \tag{4.16}$$

The first term above can be estimated in the same way as in (4.15). For the second term above, we have

$$\begin{aligned}
&\int_0^1 d\lambda \int_M \sqrt{-1}\partial\varphi_j \wedge \bar\partial\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!}\\
&\le C_n \int_M \sqrt{-1}\partial\varphi_j \wedge \bar\partial\varphi_j \wedge \frac{\omega_{\frac{1}{2}\varphi_*}^{n-1}}{(n-1)!}\\
&\le C_n I(0,\varphi_j)^{\frac{1}{2^{n-1}}}\Big(I(0,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}} + I(\varphi_j,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}}\Big)\\
&\le C_n I(0,\varphi_j)^{\frac{1}{2^{n-1}}}\Big(I(0,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}} + D_n I(0,\varphi_j)^{1-\frac{1}{2^{n-1}}}\\
&\qquad + D_n I(0,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}}\Big).
\end{aligned} \tag{4.17}$$

By [31], Theorem 5.5, $I(0,\varphi_j)$ is controlled by $d_1(0,\varphi_j)$ and the calculation in (4.11) shows that that $I(0,\frac{1}{2}\varphi_*)$ can be controlled in terms of $d_1(0,\varphi_*)$ respectively. □

Next we are ready to pass to limit. From $\sup_{0<t<1} d_1(0,\varphi_j^t) < \infty$, we may conclude that $\sup_{j,\,0<t<1} |J_{-Ric}(\varphi_j^t)| < \infty$ and $\sup_{j,0<t<1} |J_\chi(\varphi_j^t)| < \infty$ by Lemma 4.4. By (4.5) and our definition of K_χ, we know that $\sup_{j,t} \int_M \log\big(\frac{\omega_{\varphi_j^t}^n}{\omega_0^n}\big)\omega_{\varphi_j^t}^n < \infty$. Hence we may use Lemma 3.3 (the same argument works for K_χ) to conclude that up to a subsequence of t, $\varphi_j^t \to u_j$ as $t \to 1$ and u_j solves (4.2) for each j with $I(u_j) = 0$. This convergence is smooth convergence due to our previous estimates. Again due to to the last lemma, we have $\sup_j d_1(0,u_j) \le \sup_{j,t} d_1(0,\varphi_j^t) \le C$ for some fixed constant C depending only on n and $\sup_j d_1(0,\varphi_j)$. Hence we may again assume that up to a subsequence of j, $u_j \to \psi$ smoothly as $j \to \infty$ and ψ is a smooth solution to (4.3). To finish the proof that φ_* is smooth, we just need the following lemma:

Lemma 4.7. *φ_* and ψ differ by an additive constant.*

Proof. By taking limit as $t \to 1$, we can conclude from Corollary 4.5 that $I(u_j,\varphi_*) \to 0$ as $j \to \infty$. On the other hand, since $u_j \to \psi$ smoothly, we have $I(u_j,\psi) \to 0$ as $j \to \infty$. Hence

$$I(\varphi_*,\psi) \le C_n(I(u_j,\varphi_*) + I(u_j,\psi)) \to 0, \text{ as } j \to \infty.$$

That is, $I(\varphi_*,\psi) = 0$. On the other hand, from Lemma 4.8, we know $\varphi_* \in H^1(M)$ and

$$I(\varphi_*,\psi) \ge \int_M |\nabla_\psi(\varphi_* - \psi)|_\psi^2 \omega_\psi^n.$$

Therefore ψ and φ_* differ only up to a constant. □

In the above lemma, we used the following fact.

Lemma 4.8. *Let $\varphi \in \mathcal{E}^1$, then $\varphi \in H^1(M,\omega_0^n)$. Moreover, for any $\psi \in \mathcal{H}$, we have*

$$I(\varphi,\psi) \ge \int_M |\nabla_\psi(\varphi - \psi)|_\psi^2 \omega_\psi^n. \tag{4.18}$$

In the above, $|\nabla_\psi(\varphi-\psi)|_\psi^2 = g_\psi^{i\bar j}(\varphi-\psi)_i(\varphi-\psi)_{\bar j}$.

Proof. First we assume that both φ, $\psi \in \mathcal{H}$. Then we know that

$$\begin{aligned} I(\varphi,\psi) &= \int_M (\varphi-\psi)(\omega_\psi^n - \omega_\varphi^n) \\ &= \int_M d^c(\varphi-\psi)\wedge d(\varphi-\psi)\wedge \sum_{k=0}^{n-1}\omega_\varphi^k\wedge\omega_\psi^{n-1-k} \\ &\geq \int_M d^c(\varphi-\psi)\wedge d(\varphi-\psi)\wedge\omega_\psi^{n-1} = \int_M |\nabla_\psi(\varphi-\psi)|_\psi^2\omega_\psi^n. \end{aligned}$$

So (4.18) holds as long as $\varphi \in \mathcal{H}$. If $\varphi \in \mathcal{E}^1$, then we can find a sequence $\phi_j \in \mathcal{H}$, such that ϕ_j decreases pointwisely to φ. Such approximation is possible due to the main result of [7]. Also due to Lemma 4.3 of [31], we know that $d_1(\phi_j,\varphi)\to 0$. This implies that $I(\phi_j,\psi)\to I(\phi,\psi)$.

Since (4.18) holds with φ replaced by φ_j, we see that

$$\int_M |\nabla_\psi(\phi_j-\psi)|_\psi^2\omega_\psi^n \leq I(\phi_j,\psi)\to I(\varphi,\psi). \tag{4.19}$$

From $\sup_j d_1(0,\phi_j)<\infty$, we know that $\sup_j\int_M|\phi_j|dvol_g<\infty$. Now (4.19) shows ϕ_j is uniformly bounded in $H^1(M,\omega_\psi^n)$. Hence we can find a subsequence of ϕ_j which converges weakly in $H^1(M,\omega_\psi^n)$, strongly in $L^2(M,\omega_\psi^n)$. Clearly this limit must be φ. This shows $\varphi\in H^1(M,\omega_\psi^n)$, hence also in $H^1(M,\omega_0^n)$. Also we can conclude from (4.19) that

$$\int_M |\nabla_\psi(\varphi-\psi)|_\psi^2\omega_\psi^n \leq \liminf_{j\to\infty}\int_M |\nabla_\psi(\phi_j-\psi)|^2\omega_\psi^n \leq \liminf_j I(\phi_j,\psi) = I(\varphi,\psi).$$

□

5. Existence of cscK and geodesic stability

In this section, we prove Theorem 1.1. Similar to the definition of $\mathcal{H}_0$, we define

$$\mathcal{E}_0^1 = \mathcal{E}^1\cap\{u : I(u)=0\}.$$

Here $I(u)$ for $u\in\mathcal{E}^1$ is understood as the continuous extension of the functional I from $\mathcal{H}$ to $\mathcal{E}^1$. This is possible because of Proposition 4.1 in [6]. Also we notice that for any u_0, $u_1\in\mathcal{E}_0^1$, the finite energy geodesic segment (defined by Theorem 2.2) $[0,1]\ni t\to\mathcal{E}^1$ will actually lie in $\mathcal{E}_0^1$. This follows from the fact that the I functional is affine on $C^{1,1}$ geodesics and I can be continuously extended to the space $\mathcal{E}^1$. As before, $\beta\geq 0$ is a smooth closed $(1,1)$ form. We will first prove the following result in this section, which covers Theorem 1.1.

Theorem 5.1. *Suppose that either*

(1) $\beta>0$ *everywhere;*
or
(2) $\beta=0$ *everywhere and* $Aut_0(M,J)=0$.

Then the following statements are equivalent:

(1) There exists no twisted cscK metric with respect to β *in* $\mathcal{H}_0$.
(2) There is an infinite geodesic ray ρ_t *with locally finite energy,* $t\in[0,\infty)$ *in* $\mathcal{E}_0^1$, *such that the functional* K_β *is non-increasing along the ray.*

(3) For any $\phi \in \mathcal{E}_0^1$ with $K(\phi) < \infty$, there is a locally finite energy geodesic ray starting at ϕ, such that the functional K_β is non-increasing along the ray.

In the case $\beta > 0$, then from (1) one can additionally conclude K_β is strictly decreasing in (2) and (3) above.

Definition 5.1. *Let $[0, \infty) \ni t \to u_t \in \mathcal{E}^1$ be a continuous curve. Then we say u_t is an infinite geodesic ray with locally finite energy, if the following hold:*

(1) $d_1(u_t, u_s) = c|t - s|$ for some constant $c > 0$ and any $s, t \in [0, \infty)$.
(2) For any $K > 0$, $[0, K] \ni t \to u_t$ is a finite energy geodesic segment in the sense defined by Theorem 2.2.

Remark 5.2. *Observe that the implication* (3) $\Rightarrow$ (2) *is trivial.* (2) $\Rightarrow$ (1) *follows from Theorem 3.2, which is already proved in* [5], [6]. *We will use our apriori estimates and the continuity path (2.9) to resolve the implication* (1) $\Rightarrow$ (3). *We are partly motivated from arguments in the proof of Theorem 6.5 of* [6].

Next we observe the following lemma:

Lemma 5.3. *Consider the continuity path (2.12). Suppose there is no twisted cscK metric with respect to β in Kähler class $[\omega_0]$. Denote $t_* = \sup S$, where the set S is defined in (3.1). Let $S \ni t_i \nearrow t_*$. Denote φ_i to be the solution to (2.9) with $t = t_i$, normalized so that $I(\varphi_i) = 0$. Then we have* $\sup_i d_1(0, \varphi_i) = \infty$.

Proof. Suppose otherwise, then $\sup_i d_1(0, \varphi_i) < \infty$. We can apply Lemma 3.6 to conclude $t_* \in S$. If $t_* < 1$, then we conclude from Lemma 2.1 that $t_* + \delta' \in S$ for some $\delta' > 0$ sufficiently small. This contradicts $t_* = \sup S$. If $t_* = 1$, then $1 \in S$. But this will contradict our assumption that there is no cscK metric in $[\omega_0]$. In either case, the contradiction shows one cannot have $\sup_i d_1(0, \varphi_i) < \infty$. □

With the help of above lemma, we are ready to prove (1) $\Rightarrow$ (3) in Theorem 5.1.

Proof. Consider the continuity path (2.12) as in Lemma 6.3, we know that $\sup_i d_1(0, \varphi_i) = \infty$. Hence we may take a subsequence φ_{i_j}, such that $d_1(0, \varphi_{i_j}) \nearrow \infty$. We will construct a geodesic ray as described in Theorem 5.1, point (2) out of this subsequence φ_{i_j}. For simplicity, we will still denote this subsequence by φ_i.

By Theorem 2.2, there exists a unit speed finite energy d_1-geodesic segment connectiong ϕ and φ_i, such that the functional I is affine on the segment. Indeed, one can check I is affine on $C^{1,1}$ geodesic and the extension to d_1-geodesic follows from continuity of the functional I(c.f [6], Proposition 4.1).

Denote this geodesic by $c^i : [0, d_1(\phi, \varphi_i)] \to \mathcal{E}^1$. Since $I(\phi) = I(\varphi_i) = 0$, we know $I = 0$ on c^i. In other words, $c^i : [0, d_1(\phi, \varphi_i)] \to \mathcal{E}_0^1$. As noted in (3.12), we have

$$\sup_i \big(t_i K_\beta + (1 - t_i) J_{\omega_0}\big)(\varphi_i) \leq \max(K_\beta(0), J_{\omega_0}(0)).$$

On the other hand, since the functional J_{ω_0} is convex along $C^{1,1}$ geodesic, and we know 0 is a critical point of J_{ω_0}, we see that

$$J_{\omega_0}(\varphi_i) \geq J_{\omega_0}(0). \tag{5.1}$$

Therefore

$$K_\beta(\varphi_i) \leq \frac{\max(K_\beta(0), J_{\omega_0}(0)) - (1 - t_i) J_{\omega_0}(0)}{t_i} \leq C. \tag{5.2}$$

Hence from the convexity of K_β-energy as remarked before, we obtain for any $l \in [0, d_1(\phi, \varphi_i)]$,

$$K_\beta(c^i(l)) \leq (1 - \frac{l}{d_1(\phi, \varphi_i)})K_\beta(\phi) + \frac{l}{d_1(\phi, \varphi_i)}K_\beta(\varphi_i) \leq \max(K_\beta(\phi), C). \tag{5.3}$$

Therefore, for each fixed l. if we consider the sequence $\{c^i(l)\}_{d_1(\phi,\varphi_i)\geq l} \subset \mathcal{E}^1$, it satisfies the assumption in Lemma 2.5. Indeed, $d_1(\phi, c^i(l)) = l$, $\forall i$, which implies $\sup_i |J_\beta(c^i(l))|$ uniformly bounded for fixed l(by Lemma 3.4). Therefore, we have K-energy is uniformly bounded and we may apply Lemma 2.5.

Hence we may take a subsequence $c^{i_j}(l)$, such that $c^{i_j}(l) \to c^\infty(l)$ for some element $c^\infty(l) \in \mathcal{E}^1$ as $j \to \infty$. Since the functional I is continuous under d_1 convergence, we obtain $c^\infty(l) \in \mathcal{E}^1_0$ as well. Clearly we may apply this argument to each $l \in \mathbb{Q}$, then by Cantor's diagnal sequence argument, we can take a subsequence of φ_i, denoted by φ_{i_j}, such that

$$c^{i_j}(l) \to c^\infty(l) \text{ in } d_1, \text{ as } j \to \infty, \text{ for any } l \in \mathbb{Q}. \tag{5.4}$$

Since c^{i_j} are unit speed geodesic segment, we see that for any r, $s \in \mathbb{Q}$, with $0 \leq r$, $s \leq d_1(\phi, \varphi_{i_j})$, we have $d_1(c^{i_j}(r), c^{i_j}(s)) = |r - s|$. Sending $j \to \infty$ gives

$$d_1(c^\infty(r), c^\infty(s)) = |r - s|, \text{ for any } 0 \leq r, s \in \mathbb{Q}. \tag{5.5}$$

We can then define $c^\infty(r)$ for all $r \in \mathbb{R}$ by requiring $c^\infty(r) = d_1 - \lim_{r_k\in\mathbb{Q}, r_k\to r} c^\infty(r_k)$. From property (5.5) it is easy to see this is well defined, i.e, the said limit exists and does not depend on our choice of sequence r_k. Hence $[0, \infty) \ni r \to c^\infty(r)$ is a unit speed geodesic ray in $\mathcal{E}^1_0$. Besides, if we apply Proposition 2.4 to $[0, r_k]$ for any $r_k > 0$, $r_k \in \mathbb{Q}$, we know $c^{i_j}(r) \to u_k(r)$ for any $r \in [0, r_k]$. Here $[0, r_k] \ni r \to u_k(r)$ is the finite energy geodesic segment connecting ϕ and $c^\infty(r_k)$. Hence we know $c^\infty(r) = u_k(r)$ for any $r \in [0, r_k] \cap \mathbb{Q}$, by (5.4). Therefore $c^\infty(r) = u_k(r)$ for any $r \in [0, r_k]$ by density. Therefore, we have shown $c^\infty|_{[0,d_1(\phi,c^\infty(r))]}$ is the finite energy geodesic segment connecting ϕ and $c^\infty(r)$ for $r \in \mathbb{Q}$. It is easy to extend this to all $r \in \mathbb{R}_+$ by rescaling in time and apply Proposition 2.4 again.

We can now invoke Theorem 4.7, Proposition 4.5 of [6] to conclude $r \longmapsto K(c^\infty(r))$, $r \longmapsto J_\beta(c^\infty(r))$ is convex. Hence $r \longmapsto K_\beta(c^\infty(r))$ is convex as well.

Now from the lower semi-continuity of K_β-energy under d_1-convergence, we obtain from (5.3) that

$$K_\beta(c^\infty(r)) \leq \liminf_{j\to\infty} K_\beta(c^{i_j}(r)) \leq \max(K_\beta(\phi), C), \text{ for all } r \in \mathbb{Q}. \tag{5.6}$$

Use the lower semi-continuity again, we deduce

$$K_\beta(c^\infty(r)) \leq \liminf_{k\to\infty} K_\beta(c^\infty(r_k)) \leq \max(K_\beta(\phi), C). \tag{5.7}$$

Therefore, $(0, \infty) \ni r \longmapsto K_\beta(c^\infty(r))$ is both convex and bounded, this forces K_β-energy must be decreasing along c^∞.

To see the "in addition" part, if K_β is not strictly decreasing, them from the convexity of $r \longmapsto K_\beta(c^\infty(r))$, we can conclude that for some $r_0 > 0$, $K_\beta(c^\infty(r))$ remains a constant for $r \geq r_0$. Since both K and J_β are convex, we know J_β remains linear for $r \geq r_0$. Now [6], Theorem 4.12 shows $c^\infty(r_1) = c^\infty(r_r) + \mathit{const}$ for any r_1, $r_2 \geq r_0$. Because of the normalization $I(c^\infty(r)) = 0$, we know $c^\infty(r_1) = c^\infty(r_2)$ for any r_1, $r_2 \geq r_0$. But this contradicts $d_1(c^\infty(r_1), c^\infty(r_2)) = |r_1 - r_2|$ for any r_1, $r_2 \geq 0$. □

Finally, the implication (2) $\Rightarrow$ (1) follows immediately from Theorem 3.2.

Proof. Suppose otherwise, namely there exists a twisted cscK metric with respect to β in $\mathcal{H}_0$, denoted by φ^β. Then we can conclude from Theorem 3.2 that the twisted K-energy K_β is proper. In particular, $K_\beta \to +\infty$ along any locally finite energy geodesic ray. This contradicts the assumption in (2). □

We can deduce the following immediate consequence of Theorem 5.1.

Corollary 5.4. *Let $0 < t_0 < 1$, and let χ be a Kähler form. Then the following statements are equivalent:*

1. *There is no twisted cscK metric with $t = t_0$ in $\mathcal{H}_0$(i.e solves (2.9) with $t = t_0$).*
2. *There is an infinite geodesic ray ρ_t of locally finite energy, $t \in [0,\infty)$ in $\mathcal{E}_0^1$, such that the twisted K-energy K_{χ,t_0}(defined by (2.8)) is strictly decreasing along the ray.*
3. *For any $\phi \in \mathcal{E}_0^1$ with $K(\phi) < \infty$, there is a locally finite energy geodesic ray starting at ϕ, such that the twisted K-energy K_{χ,t_0}(defined by (2.8)) is strictly decreasing along the ray.*

Also we can show Theorem 1.2 as a consequence.

Proof. (of Theorem 1.2) First we prove the necessary part. Assume $(M,[\omega_0])$ admits a cscK metric. Denote φ_0 be the corresponding cscK potential. Recall we have shown in the proof of Theorem 3.2(the direction existence implies properness) that for all $\psi \in \mathcal{E}_0^1$, with $d_1(\psi,\varphi_0) \geq 1$, one has $K(\psi) \geq \varepsilon d_1(\psi,\varphi_0) + K(\varphi_0)$. Let $\phi \in \mathcal{E}_0^1$ and $\rho : [0,\infty) \ni t \mapsto \mathcal{E}_0^1$ be a locally finite energy geodesic ray initiating from ϕ. We can assume $\rho(t)$ has unit speed. Then as long as $d_1(\rho(t),\varphi_0) \geq 1$, one has

$$\begin{aligned} \frac{K(\rho(t)) - K(\phi)}{t} &\geq \frac{\varepsilon d_1(\rho(t),\varphi_0) + K(\varphi_0) - K(\phi)}{t} \\ &\geq \frac{\varepsilon d_1(\rho(t),\phi) - \varepsilon d_1(\phi,\varphi_0) + K(\varphi_0) - K(\phi)}{t} \\ &= \varepsilon - \frac{\varepsilon d_1(\phi,\varphi_0) - K(\varphi_0) + K(\phi)}{t}. \end{aligned} \tag{5.8}$$

This implies

$$\liminf_{t\to\infty} \frac{K(\rho(t)) - K(\phi)}{t} \geq \varepsilon.$$

In particular this means $¥([\rho]) \geq \varepsilon$. Thus, $(M,[\omega_0])$ is geodesic stable.

Now we want to show the converse. We assume $(M,[\omega_0])$ is geodesic stable and we want to prove that there is a cscK metric in the Kähler class. Suppose otherwise, then according to Theorem 5.1 with $\beta = 0$, point (3), we know that there exists a locally finite energy geodesic ray $\rho : [0,\infty) \ni t \mapsto \mathcal{E}_0^1$, initiating from $\phi \in \mathcal{E}_0^1$ with $K(\phi) < \infty$, such that the K-energy is non-increasing. It is clear that for this geodesic ray, one has $¥([\rho]) \leq 0$. This contradicts the assumption of geodesic stability at φ. This finishes the proof. □

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Xiuxiong Chen
University of Science and Technology of China and Stony Brook University

Jingrui Cheng
University of Wisconsin at Madison.