# Duality and Yang–Mills fields on quaternionic Kähler manifolds

Krzysztof Galicki[a] and Yat Sun Poon
*Department of Mathematics, Rice University, P.O. Box 1892, Houston, Texas 77251*

(Received 22 August 1990; accepted for publication 5 December 1990)

The concept of a self-dual connection on a four-dimensional Riemannian manifold is generalized to the $4n$-dimensional case of any quaternionic Kähler manifold. The generalized self-dual connections are minima of a modified Yang–Mills functional. It is shown that our definitions give a correct framework for a mapping theory of quaternionic Kähler manifolds. The mapping theory is closely related to the construction of Yang–Mills fields on such manifolds. Some monopole-like equations are discussed.

## I. INTRODUCTION

A quaternionic Kähler manifold is a Riemannian manifold whose holonomy group can be reduced to a subgroup of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, $n>1$.[1,2] By definition, such manifold has dimension $4n$. As demonstrated by Salamon,[2,3] it can be also viewed as a higher-dimensional analogy of the anti-self-dual Einstein four-manifold. The bundle of two-forms on a quaternionic Kähler manifold $M$ has the following irreducible decomposition as representation of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$:

$$\Lambda^2T^*M=S^2\mathbb{H}\oplus S^2\mathbb{E}\oplus(S^2\mathbb{H}\oplus S^2\mathbb{E})^\perp, \tag{1.1}$$

where $\mathbb{H}$ and $\mathbb{E}$ are vector bundles associated to the standard representations of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$, respectively. This decomposition resembles the decomposition of $\Lambda^2T^*M$ into the direct sum of self-dual and anti-self-dual two-forms when $M$ is four dimensional. Just as in the four-dimensional case we are able to interpret the decomposition (1.1) in terms of the Hodge $*$-operator.

If the curvature of a connection $\nabla$ is in either the $S^2\mathbb{H}$ or the $S^2\mathbb{E}$ part of (1.1) then $\nabla$ is a minimum of the Yang–Mills functional and if the curvature is in the orthogonal complement of $S^2\mathbb{H}\oplus S^2\mathbb{E}$ then $\nabla$ is most likely a saddle point. We have found that the Yang–Mills functional can be modified so that whenever the curvature of $\nabla$ is in one and only one component of (1.1) the connection is its minimum.

We demonstrate that our definitions are compatible with the description of Yang–Mills fields on four-manifolds and that they give a correct framework for mapping theory of quaternionic Kähler manifolds. On the other hand, when the energy functional is interpreted as a classical Lagrangian, our quaternionic mapping theory yields many new examples of quantum field theories with SU(2) [or SO(3)] gauge symmetry and composite gauge fields: four-dimensional sigma models. We show that some fundamental properties of the well-known four-dimensional $\sigma$-models on the quaternionic projective spaces are shared by such models on arbitrary quaternionic Kähler manifolds. Finally, we demonstrate that our formalism provides a global picture for the generalized monopole equation of Pedersen and Poon.[4]

[a] Present address: Dept. of Math., The Univ. of New Mexico, Albuquerque, NM 87131.

## II. DUALITY

Let $M$ be a $4n$-dimensional Riemannian manifold whose holonomy group is contained in $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\subset\mathrm{SO}(4n)$. Then the cotangent bundle of $M$ can be identified with

$$T^*M=\mathbb{E}\otimes\mathbb{H},$$

where $\mathbb{E}$ and $\mathbb{H}$ are the standard representations of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$, respectively. Then $S^2\mathbb{H}$ is a real rank 3 subbundle of End $TM$. Locally, at each $x\in M$, $S^2\mathbb{H}$ has a basis $\{I,J,K\}$ satisfying

$$I^2=J^2=-\mathbb{1},\quad IJ=-JI=K. \tag{2.1}$$

The metric $g$ on $M$ is compatible with the bundle $S^2\mathbb{H}$ in the sense that for each $A\in S^2\mathbb{H}_x$, $g$ is Hermitian with respect to $A$, i.e., $g(AX,AY)=g(X,Y)$ for all $X,Y\in T_xM$. One can use the metric to define an isomorphism

$$\mathrm{End}\ TM\cong T^*M\otimes T^*M$$

under which $S^2\mathbb{H}$ is isometrically embedded in $\Lambda^2T^*M$. Explicitly, any element $A\in S^2\mathbb{H}_x$ is mapped into $\omega_A$ by

$$\omega_A(X,Y)=g(AX,Y),\quad X,Y\in T_xM.$$

Let $\{\omega_1,\omega_2,\omega_3\}$ be a local orthogonal frame of $S^2\mathbb{H}\subset\Lambda^2T^*M$. For convenience of further computations let us normalize $\{\omega_1,\omega_2,\omega_3\}$ to have length $2n$ and then define

$$\Omega=\omega_1\wedge\omega_1+\omega_2\wedge\omega_2+\omega_3\wedge\omega_3. \tag{2.2}$$

This $\Omega$ is a globally defined, nondegenerate four-form on $M$ and it is parallel. It is usually called the fundamental four-form or the quaternionic structure on $M$ as its parallelism determines reduction of the structure group on $M$. The condition $\nabla\Omega=0$ can be used to define quaternionic Kähler geometry in dimension bigger than 4. In dimension 4 we shall say that $M$ is quaternionic Kähler if it is self-dual and Einstein. The parallelism of $\Omega$ immediately implies that $d\Omega=0$. Recently, Swann[5] showed that the converse is also true provided dim $M\geqslant 12$.

Pointwisely, $\Omega$ can be described as follows. At any point $x\in M$, $T_x^*M=\mathbb{E}_x\otimes\mathbb{H}_x$, where $\mathbb{E}_x$ is the $2n$-dimensional complex representation of $\mathrm{Sp}(n)$ and $\mathbb{H}_x$ is the two-dimensional complex representation of $\mathrm{Sp}(1)$. Let $\omega_E$ and $\omega_H$ be the symplectic forms on $\mathbb{E}_x$ and $\mathbb{H}_x$, respectively, and $j_E$ and $j$ the quaternionic structures. Then the metric $g$ on $T_x^*M$ can be expressed as

 0022-2488/91/051263-06$03.00 © 1991 American Institute of Physics 

Reuse of AIP Publishing content is subject to the terms: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 138.23.170.248 On: Fri, 22 Apr 2016 17:58:25

$$g = \omega_E \otimes \omega_H. \tag{2.3}$$

Let $\{e^j, j_E e^j\colon j = 1,\ldots,n\}$ be a symplectic basis on $\mathbb{E}_x$ and $\{h, jh\}$ a symplectic basis on $\mathbb{H}_x$. We define

$$\begin{aligned} \omega^j_0 &\doteq (1/\sqrt{2})(e^j \otimes h + j_E e^j \otimes jh),\\ \omega^j_1 &\doteq (i/\sqrt{2})(e^j \otimes h - j_E e^j \otimes jh),\\ \omega^j_2 &\doteq (1/\sqrt{2})(j_E e^j \otimes h - e^j \otimes jh),\\ \omega^j_3 &\doteq (i/\sqrt{2})(j_E e^j \otimes h + e^j \otimes jh). \end{aligned} \tag{2.4}$$

Now $\{\omega^j_0, \omega^j_1, \omega^j_2, \omega^j_3, j = 1,\ldots,n\}$ forms an orthonormal basis on $T^*_x M$. Let

$$\begin{aligned} \omega_1 &\doteq \sum_{j=1}^n (\omega^j_0 \wedge \omega^j_1 + \omega^j_2 \wedge \omega^j_3),\\ \omega_2 &\doteq \sum_{j=1}^n (\omega^j_0 \wedge \omega^j_2 - \omega^j_1 \wedge \omega^j_3),\\ \omega_3 &\doteq \sum_{j=1}^n (\omega^j_0 \wedge \omega^j_3 + \omega^j_1 \wedge \omega^j_2). \end{aligned} \tag{2.5}$$

Then $\{\omega_1, \omega_2, \omega_3\}$ forms an orthogonal basis on $S^2\mathbb{H}_x$. We shall choose $\Omega$ as in (2.2). The orthogonal basis for $S^2\mathbb{E}_x$ can be written as

$$\begin{aligned} \Sigma^{ij}_0 &\doteq (\omega^i_0 \wedge \omega^j_0 + \omega^i_1 \wedge \omega^j_1) + (\omega^i_2 \wedge \omega^j_2 + \omega^i_3 \wedge \omega^j_3),\\ &\qquad 1 \leqslant i < j \leqslant n,\\ \Sigma^{ij}_1 &\doteq (\omega^i_0 \vee \omega^j_1 - \omega^i_2 \wedge \omega^j_3) + (\omega^j_0 \wedge \omega^i_1 - \omega^j_2 \wedge \omega^i_3),\\ &\qquad 1 \leqslant i \leqslant j \leqslant n,\\ \Sigma^{ij}_2 &\doteq (\omega^i_0 \wedge \omega^j_2 + \omega^i_1 \wedge \omega^j_3) + (\omega^j_0 \wedge \omega^i_2 + \omega^j_1 \wedge \omega^i_3),\\ &\qquad 1 \leqslant i \leqslant j \leqslant n,\\ \Sigma^{ij}_3 &\doteq (\omega^i_0 \wedge \omega^j_3 - \omega^i_1 \wedge \omega^j_2) + (\omega^j_0 \wedge \omega^i_3 - \omega^j_1 \wedge \omega^i_2),\\ &\qquad 1 \leqslant i \leqslant j \leqslant n. \end{aligned} \tag{2.6}$$

Here, $\Sigma^{ij}_0$ give $n(n-1)/2$ basis elements and $\Sigma^{ij}_A$, $A = 1,2,3$, give $n(n+1)/2$ basis elements, respectively. One can easily check that

$$\operatorname{vol}(M) = [1/(2n+1)!]\,\Omega^n \tag{2.7}$$

$$\operatorname{vol}(M) = [1/12n(2n+1)]\,\Omega \wedge *\Omega, \tag{2.8}$$

where $\operatorname{vol}(M)$ is the volume form of $M$ and "$*$" is the Hodge $*$-operator. As a consequence we have

$$*\Omega = [6/(2n-1)!]\,\Omega^{n-1}. \tag{2.9}$$

Note that all these equations are valid even when $n$ is equal to 1.

*Definition 2.1:* A two-form $\omega$ on $M$ is $c$-self dual if

$$*\omega = c\,\omega \wedge \Omega^{n-1}. \tag{2.10}$$

When $n = 1$ then $c^2 = 1$, because $*^2 = 1$, and the above equation is reduced to the conformally invariant self-dual or anti-self-dual equations on a four-dimensional oriented Riemannian manifold. Notice that the above definition depends on the choice of both the fundamental four-form $\Omega$ and the constant $c$. In dimension higher than 4, as we shall now see, there are three different constants $c$ that give nontrivial solutions to (2.10). Similar equations were studied in Ref. 6.

**Theorem 2.2:** Let $\omega$ be a nonzero $c$-self-dual two-form. Then $c = c_i$, $i = 1,2,3$, where

$$c_1 = \frac{6n}{(2n+1)!}, \quad c_2 = \frac{-1}{(2n-1)!},$$
$$c_3 = \frac{3}{(2n-1)!}. \tag{2.11}$$

Moreover, when $c = c_1$ then $\omega \in S^2\mathbb{H}$, when $c = c_2$ then $\omega \in S^2\mathbb{E}$, and when $c = c_3$ then $\omega$ is in the orthogonal complement of $S^2\mathbb{H} \oplus S^2\mathbb{E}$ in $\Lambda^2 T^*M$.

*Proof:* As the basis for $S^2\mathbb{H}$ is given in (2.5) and the basis for $S^2\mathbb{E}$ in (2.6) the proof is an easy exercise in linear algebra. Therefore, we only spell out the constraints on the coefficients of the two-form $\omega$. Using the orthonormal basis $\{\omega^j_0, \omega^j_1, \omega^j_2, \omega^j_3 : j = 1,\ldots,n\}$ any two-form $\omega$ can be written as

$$\omega = \sum_{i,j,\alpha,\beta} \omega_{\binom{i}{\alpha}\binom{j}{\beta}}\, \omega^i_\alpha \wedge \omega^j_\beta.$$

Then $*\omega = c_1 \omega \wedge \Omega^{n-1}$ if and only if

$$\begin{aligned} \omega_{\binom{i}{0}\binom{i}{1}} &= \omega_{\binom{i}{2}\binom{i}{3}} = \omega_{\binom{j}{0}\binom{j}{1}} = \omega_{\binom{j}{2}\binom{j}{3}},\\ \omega_{\binom{i}{0}\binom{i}{2}} &= -\omega_{\binom{i}{1}\binom{i}{3}} = \omega_{\binom{j}{0}\binom{j}{2}} = -\omega_{\binom{j}{1}\binom{j}{3}},\\ \omega_{\binom{i}{0}\binom{i}{3}} &= \omega_{\binom{i}{1}\binom{i}{2}} = \omega_{\binom{j}{0}\binom{j}{3}} = \omega_{\binom{j}{1}\binom{j}{2}}, \end{aligned} \tag{2.12}$$

for all $i,j$

$$\omega_{\binom{i}{0}\binom{j}{0}} = \omega_{\binom{i}{1}\binom{j}{1}} = \omega_{\binom{i}{2}\binom{j}{2}} = \omega_{\binom{i}{3}\binom{j}{3}} = 0 \quad \forall i,j, \tag{2.13}$$

and

$$\omega_{\binom{i}{\alpha}\binom{j}{\beta}} = 0 \quad \forall i \neq j \quad \forall \alpha,\beta. \tag{2.14}$$

Similarly, $*\omega = c_2 \omega \wedge \Omega^{n-1}$ if and only if

$$\begin{aligned} &\omega_{\binom{i}{0}\binom{j}{1}} = -\omega_{\binom{i}{2}\binom{j}{3}},\ \omega_{\binom{i}{0}\binom{j}{2}} = \omega_{\binom{i}{1}\binom{j}{3}},\\ &\omega_{\binom{i}{0}\binom{j}{3}} = -\omega_{\binom{i}{1}\binom{j}{2}} \ \forall i,j,\\ &\omega_{\binom{i}{\alpha}\binom{j}{\alpha}} = \omega_{\binom{i}{\beta}\binom{j}{\beta}}, \quad \forall i,j,\alpha,\beta,\\ &\omega_{\binom{i}{\alpha}\binom{j}{\beta}} = \omega_{\binom{j}{\alpha}\binom{i}{\beta}}, \quad \forall i,j,\alpha,\beta,\alpha \neq \beta. \end{aligned} \tag{2.15}$$

Finally, $*\omega = c_3 \omega \wedge \Omega^{n-1}$ if and only if

$$\begin{aligned} &\sum_{i=1}^n \omega_{\binom{i}{0}\binom{i}{1}} = \sum_{i=1}^n \omega_{\binom{i}{0}\binom{i}{2}} = \sum_{i=1}^n \omega_{\binom{i}{0}\binom{i}{3}} = 0,\\ &\sum_{\alpha=0}^3 \omega_{\binom{i}{\alpha}\binom{j}{\alpha}} = 0 \quad \forall i,j,\\ &\omega_{\binom{i}{0}\binom{j}{1}} + \omega_{\binom{j}{0}\binom{i}{1}} = \omega_{\binom{i}{2}\binom{j}{3}} + \omega_{\binom{j}{2}\binom{i}{3}} \quad \forall i,j,\\ &\omega_{\binom{i}{0}\binom{j}{2}} + \omega_{\binom{j}{0}\binom{i}{2}} = -\left(\omega_{\binom{i}{1}\binom{j}{3}} + \omega_{\binom{j}{1}\binom{i}{3}}\right) \quad \forall i,j,\\ &\omega_{\binom{i}{0}\binom{j}{3}} + \omega_{\binom{j}{0}\binom{i}{3}} = \omega_{\binom{i}{1}\binom{j}{2}} + \omega_{\binom{j}{1}\binom{i}{2}} \quad \forall i,j. \end{aligned} \tag{2.16}$$

■

*Definition 2.3:* Let $P$ be a principal bundle on $M$ with connection $\nabla$. This connection is $c$-self-dual if its curvature two-form is $c$-self-dual.

*Definition 2.4:* For any real constant $c$, a generalized "Yang–Mills" functional on the space of connections on $P$ is defined by



Reuse of AIP Publishing content is subject to the terms: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 138.23.170.248 On: Fri, 22 Apr 2016 17:58:25

$$\mathrm{YM}_c(\nabla) \doteq \frac{1}{2}\int_M [\|F\|^2 + c^2\|F\wedge\Omega^{n-1}\|^2]\,\mathrm{vol}(M), \tag{2.17}$$

where $F$ is the curvature of the connection.

$\mathrm{YM}_c(\nabla)$ has the following Euler–Lagrange equations

$$d*F + c^2(d*(F\wedge\Omega^{n-1}))\wedge\Omega^{n-1} = 0. \tag{2.18}$$

Notice that

$$\begin{aligned}0 &\leqslant \|*F - cF\wedge\Omega^{n-1}\|^2\\ &= \|*F\|^2 - 2\langle *F, cF\wedge\Omega^{n-1}\rangle + c^2\|F\wedge\Omega^{n-1}\|^2\\ &= \|F\|^2 - 2c(\mathrm{tr}\,F\wedge F)\wedge\Omega^{n-1} + c^2\|F\wedge\Omega^{n-1}\|^2\\ &= \|F\|^2 - 16c\pi^2 p_1(P)\wedge\Omega^{n-1} + c^2\|F\wedge\Omega^{n-1}\|^2\end{aligned}$$

or

$$c(8\pi^2)p_1(P)\wedge\Omega^{n-1} \leqslant \tfrac{1}{2}[\|F\|^2 + c^2\|F\wedge\Omega^{n-1}\|^2],$$

where $p_1(P)$ is the first Pontrjagin class of the bundle $P$ on $M$. Hence, after integrating over $M$, we get

$$8\pi^2 c\int_M p_1(P)\wedge\Omega^{n-1}\,\mathrm{vol}(M) \leqslant \mathrm{YM}_c(\nabla). \tag{2.19}$$

The equality holds if and only if

$$*F = cF\wedge\Omega^{n-1},$$

i.e., if $F$ is $c$-self-dual. In such case we shall call the connection $\nabla$ itself a $c$-self-dual-connection. As $p_1(P)$ is a topological invariant of the bundle $P$, we define

$$Q(P) \doteq 8\pi^2\int_M p_1(P)\wedge\Omega^{n-1}\,\mathrm{vol}(M) \tag{2.20}$$

and call it a topological charge of the bundle $P$. We have just demonstrated the following proposition.

*Proposition 2.5:* Any $c$-self-dual connection is minimum of the Yang–Mills energy functional $\mathrm{YM}_c(\nabla)$.

The following result is due to Ref. 7.

*Proposition 2.6:* Any $c$-self-dual connection is an extremum of the Yang–Mills energy functional $\mathrm{YM}(\nabla)$. Moreover, $c_1$- and $c_2$-self-dual connections are minimizing.

*Proof:* Suppose $\nabla$ is a $c$-self-dual connection. Then

$$d*F = cd*(F\wedge\Omega^{n-1}) = 0$$

as $dF = d\Omega = 0$. Hence, $d*F = 0$ or $\nabla$ is a Yang–Mills connection.

Let us write $F(\nabla)\in\Lambda^2T^*M$ as

$$F(\nabla) = F_1 + F_2 + F_3,$$

where $F_1\in S^2\mathbb{H}$, $F_2\in S^2\mathbb{E}$, and $F_3\in(S^2\mathbb{H}\oplus S^2\mathbb{E})^\perp$. Then

$$\mathrm{YM}(\nabla) = \tfrac{1}{2}\int_M (\|F_1\|^2 + \|F_2\|^2 + \|F_3\|^2)\,\mathrm{vol}(M)$$

because (1.1) is an orthogonal decomposition with respect to the usual norm $\|\cdot\|$ on $\Lambda^2T^*M$. Notice that the topological charge of $P$ can be written in terms of the components of $F(\nabla)$:

$$\begin{aligned}Q(P) &= \int_M \mathrm{tr}(F\wedge F)\wedge\Omega^{n-1}\,\mathrm{vol}(M)\\ &= \int_M\left(\frac{1}{c_1}\|F_1\|^2 + \frac{1}{c_2}\|F_2\|^2\right.\\ &\quad\left. + \frac{1}{c_3}\|F_3\|^2\right)\mathrm{vol}(M).\end{aligned}$$

Hence, we can write $\mathrm{YM}(\nabla)$ as

$$\begin{aligned}2\mathrm{YM}(\nabla) &= c_1Q(P) + \int_M\left(\left(1 - \frac{c_1}{c_2}\right)\|F_2\|^2\right.\\ &\quad\left. + \left(1 - \frac{c_1}{c_3}\right)\|F_3\|^2\right)\mathrm{vol}(M)\\ &= c_1Q(P) + \int_M\left(\left(1 + \frac{3}{2n+1}\right)\|F_2\|^2\right.\\ &\quad\left. + \left(1 - \frac{1}{2n+1}\right)\|F_3\|^2\right)\mathrm{vol}(M),\end{aligned} \tag{2.21}$$

$$\begin{aligned}2\mathrm{YM}(\nabla) &= c_2Q(P) + \int_M\left(\left(1 - \frac{c_2}{c_1}\right)\|F_1\|^2\right.\\ &\quad\left. + \left(1 - \frac{c_2}{c_3}\right)\|F_3\|^2\right)\mathrm{vol}(M)\\ &= c_2Q(P) + \int_M\left(\left(1 + \frac{2n+1}{3}\right)\|F_1\|^2\right.\\ &\quad\left. + \frac{4}{3}\|F_3\|^2\right)\mathrm{vol}(M),\end{aligned} \tag{2.22}$$

or

$$\begin{aligned}2\mathrm{YM}(\nabla) &= c_3Q(P) + \int_M\left(\left(1 - \frac{c_3}{c_1}\right)\|F_1\|^2\right.\\ &\quad\left. + \left(1 - \frac{c_3}{c_2}\right)\|F_2\|^2\right)\mathrm{vol}(M)\\ &= c_3Q(P) + \int_M((-2n)\|F_1\|^2\\ &\quad + 4\|F_2\|^2)\,\mathrm{vol}(M).\end{aligned} \tag{2.23}$$

It follows now from (2.21), (2.22), and Theorem 2.2 that $c_1$- and $c_2$-self-dual connections are minima of $\mathrm{YM}(\nabla)$. ■

We do not know of any examples of $c_3$-self-dual connections but (2.23) seems to indicate that, if they exist, they will be unstable.

## III. QUATERNIONIC MAPS AND SIGMA MODELS

In this chapter we introduce a new concept of quaternionic maps. We shall do it in such a way that it generalizes the theory of holomorphic mappings between Kähler manifolds. On the other hand we shall see that it is also very natural in studying instantons on four-manifolds and four-dimensional $\sigma$-models with composite SU(2) [or SO(3)] gauge fields and Yang–Mills fields on quaternionic Kähler manifolds.

It is well-known that, if one defines a quaternionic Kähler submanifold to be a submanifold with a quaternionic structure given by restriction, then it is automatically a totally geodesic submanifold.[4] We shall therefore not insist that the whole quaternionic structure be preserved by such mappings. Instead we adopt a weaker definition.

*Definition 3.1:* Let $M$, $N$ be quaternionic Kähler manifolds. A map $f$ from $M$ to $N$ is called quaternionic if $f^*S^2\mathbb{H}_N \subset S^2\mathbb{H}_M$.

The following theorem is in an obvious analogy to the



Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 130.... On: Fri, 22 Apr 2016 17:58:25

well-known result stating that holomorphic maps between Kähler manifolds are energy minimizing.

**Theorem 3.2:** On the space of diffferentiable mappings between two compact oriented quaternionic Kähler manifolds, $M$ and $N$ define the following functional:

$$E(f) \doteq \frac{1}{2}\sum_{i=1}^{3}\int_M (\|f^*\omega_i\|^2 + c^2\|f^*\omega_i\wedge\Omega^{m-1}\|^2)\mathrm{vol}(M), \tag{3.1}$$

where $c = c_1 = 6m/(2m+1)!$, $4m = \dim M$, and

$$Q(f) \doteq \int_M f^*\Omega_N \wedge \Omega_M^{m-1}. \tag{3.2}$$

Then $cQ(f) \leqslant E(f)$ and the equality holds if and only if the map $f$ is quaternionic.

*Proof:* Let $\Omega_M$, $\Omega_N$ be the fundamental four-forms on $M$ and $N$, respectively. Once they are fixed $Q(f)$ is a homotopy invariant. As usual, we shall call it the degree or the topological charge of $f$.

Let $\{\omega_1,\omega_2,\omega_3\}$ be a local orthogonal frame on $S^2\mathbb{H}_N$ such that

$$\Omega_N = \omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3.$$

We have to show that $E(f)$ is well defined. If $\omega_i = \Sigma_j\phi_{ij}\mu_j$ is an SO(3) rotation of the frame field on $S^2\mathbb{H}_N$ then pointwisely

$$f^*\omega_i = \sum_{j=1}^{3}(\phi_{ij})f^*\mu_j.$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^{3}\|f^*\omega_i\|^2 &= \sum_{i=1}^{3} f^*\omega_i\wedge *f^*\omega_i\\ &= \sum_{i=1}^{3}\sum_{j,k}(\phi_{ij}f^*\mu_j)\wedge *(\phi_{ik}f^*\mu_k)\\ &= \sum_{i=1}^{3}\sum_{j,k}(\phi_{ij}\phi_{ik})(f^*\mu_j\wedge *f^*\mu_k)\\ &= \sum_{i=1}^{3}\sum_{j,k}(\phi_{ji}^{-1}\phi_{ik})(f^*\mu_j\wedge *f^*\mu_k)\\ &= \sum_{j,k}\delta_{jk}f^*\mu_j\wedge *f^*\mu_k = \sum_{j=1}^{3} f^*\mu_j\wedge *f^*\mu_j\\ &= \sum_{j=1}^{3}\|f^*\mu_j\|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{i=1}^{3}\|f^*\omega_i\wedge\Omega^{m-1}\|^2\\ &= \sum_{i=1}^{3}(f^*\omega_i\wedge\Omega^{m-1})\wedge *(f^*\omega_i\wedge\Omega^{m-1})\\ &= \sum_{i=1}^{3}\sum_{j,k}(\phi_{ij}\phi_{ik})(f^*\mu_j\wedge\Omega^{m-1})\wedge *(f^*\mu_k\wedge\Omega^{m-1})\\ &= \sum_{j=1}^{3}(f^*\mu_j\wedge\Omega^{m-1})\wedge *(f^*\mu_j\wedge\Omega^{m-1})\\ &= \sum_{j=1}^{3}\|f^*\mu_j\wedge\Omega^{m-1}\|^2.\end{aligned}$$

Hence, $E(f)$ is independent of the choice of any normalized frame on $S^2\mathbb{H}_N$ and therefore well defined. Now the inequality $cQ(f)\leqslant E(f)$ follows from

$$0 \leqslant \|*f^*\omega_i - cf^*\omega_i\wedge\Omega^{m-1}\|^2$$

which can be written as

$$c\langle *f^*\omega_i, f^*\omega_i\wedge\Omega^{m-1}\rangle \leqslant \tfrac{1}{2}(\|f^*\omega_i\|^2 + c^2\|f^*\omega_i\wedge\Omega^{m-1}\|^2). \tag{3.3}$$

Since

$$\langle *f^*\omega_i, f^*\omega_i\wedge\Omega^{m-1}\rangle = f^*\omega_i\wedge f^*\omega_i\wedge\Omega^{m-1}$$

and

$$f^*\Omega_N = \sum_{i=1}^{3} f^*\omega_i\wedge f^*\omega_i,$$

the inequality $cQ(f)\leqslant E(f)$ is simply obtained by summation of (3.3) over $i$ and integration over $M$.

Finally, when $c = 6m/(2m+1)!$, the assertion that $cQ(f) = E(f)$ is equivalent to the requirement that

$$*f^*\omega = cf^*\omega\wedge\Omega_M^{m-1}$$

holds for all $\omega\in S^2\mathbb{H}_N$, or that $f^*\omega\in S^2\mathbb{H}_M$ by Theorem 2.2., i.e., $f$ is quaternionic. ■

*Example 3.3:* If $\dim M = 4$, $S^2\mathbb{H}_M \cong \Lambda_+^2$. As the Hodge $*$-operator is conformally invariant, any orientation preserving conformal automorphism is a quaternionic map in our sense.

In Ref. 8 Atiyah gave a geometric construction for all basic SU(2)-instantons, i.e., anti-self-dual Yang–Mills fields on the Euclidean four-sphere with topological charge $-1$, as follows: The Euclidean four-sphere is viewed as the quaternionic projective line $\mathbb{HP}^1$. The tautological bundle is the bundle $\mathbb{H}$ with charge $-1$. The natural connection $\nabla$ of $\mathbb{H}$ is anti-self-dual. Let $f$ be an orientation preserving conformal automorphism which is not an isometry. Then $f^*\nabla$, the pull-back connection of $f^*\mathbb{H}$, is a new anti-self-dual connection.

*Example 3.4:* The above example can be easily generalized as follows: The quaternionic projective space $\mathbb{HP}^n$ has a tautological bundle $\mathbb{H}$. By definition, any element of $GL^+(n+1,\mathbb{H})$ is an orientation preserving quaternionic linear map. In other words, if $f\in GL^+(n+1,\mathbb{H})$ is considered as an automorphism of $\mathbb{HP}^n$, then $f^*\mathbb{H}$ is isomorphic to $\mathbb{H}$. It follows that $f^*S^2\mathbb{H}\equiv S^2\mathbb{H}$ and hence $f$ is a quaternionic map. As the natural connection $\nabla$ on $\mathbb{H}$ is $c_1$-self-dual, so is $f^*\nabla$. Besides, as long as $f$ is not an isometry, $f^*\nabla$ is not gauge equivalent to $\nabla$. We do not know if these are all $c_1$-self-dual connections on $\mathbb{HP}^n$.

*Example 3.5:* Another well-known example of a mapping which in our language is quaternionic is a general SU(2)-instanton over four-sphere with the topological charge $k$.[8,9] The $S^2\mathbb{H}$ bundle on the quaternionic projective space $\mathbb{HP}^k$ has a canonical Sp(1)-connection and all instantons over $S^4$ are induced by an appropriate choice of $f: S^4\to\mathbb{HP}^k$. In fact $f$ can be described explicitly as follows: If $u\in\mathbb{HP}^k$ is a local (Fubini–Study) quaternionic coordinate on the quaternionic projective space and $x\in S^4$ is a local quaternionic coordinate on the four-sphere identified with the quaternionic projective line $\mathbb{HP}^1$ then



Reuse of AIP Publishing content is subject to the terms: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 138.23.170.248 On: Fri, 22 Apr 2016 17:58:25

$$\mathbf{u}(x) = [\boldsymbol{\lambda}\cdot(\mathbb{B} - x\mathbf{1})]^\dagger, \tag{3.4}$$

where $\boldsymbol{\lambda} = (\lambda_1,...,\lambda_k)$ is a quaternionic row vector, $\mathbf{u}$ is a quaternionic column vector, $\mathbb{B}$ is a symmetric quaternionic $k \times k$ matrix, $\dagger$ denotes quaternionic conjugation and transposition, and $(\boldsymbol{\lambda},\mathbb{B})$ are subject to the following two conditions:

$$\mathrm{Im}(\mathbb{B}^\dagger\mathbb{B} + \boldsymbol{\lambda}^\dagger\boldsymbol{\lambda}) = 0,$$

$$(\forall x \in \mathbf{HP}^1 (\mathbb{B} - x\mathbf{1})\xi = 0,\ \boldsymbol{\lambda}\cdot\xi = 0 \text{ where } \xi \in \mathbb{H}^k) \Rightarrow \xi = 0. \tag{3.5}$$

In the same way $k$-instantons over the complex projective plane can be generated by quaternionic maps from $\mathbf{CP}^2 \to \mathbf{HP}^{2k}$.[10,11]

The energy functional (3.1) may also be interpreted as an SO(3) locally gauge invariant Lagrangian of the interesting class of nonlinear field theories called $\sigma$-models. In particular, if $\dim M = 4$, one can think of $M$ as a physical, possibly curved, space-time and $f(x), x\in M$, becomes an $N$-valued classical field with the action functional given by $E(f)$. $E(f)$ is manifestly invariant with respect to the global coordinate transformation on $M$ (diffeomorphisms of $M$) as well as it is gauge invariant under the following gauge transformations

$$(f^*\omega_i)_x \to \sum_j \Phi_{ij}(x)(f^*\omega_j)_x, \tag{3.6}$$

where $\Phi_{ij}(x)$ is a local SO(3) transformation and $(f^*\omega_i)$ is the curvature two-form of a gauge field $A_j$ on $M$ defined as follows:

$$d(f^*\omega_i) = \sum_{j,k} \epsilon_{ijk} A_j \wedge f^*\omega_k. \tag{3.7}$$

The gauge potential one-form on $A_j$ transforms in the usual way

$$\delta(\epsilon_{ijk}A_k) = -d_A\Phi_{ij}(x). \tag{3.8}$$

$A_j(f)$ depends on the choice of $f(x)$, i.e., it is a composite gauge field. If $N = \mathbf{HP}^n$ and $\mathbf{u}\in\mathbf{HP}^n$ as before then

$$A(\mathbf{u}) = -\frac{1}{2}\frac{\mathbf{u}^\dagger\cdot d\mathbf{u} - d\mathbf{u}^\dagger\cdot\mathbf{u}}{1 + \mathbf{u}^\dagger\cdot\mathbf{u}} = iA_1 + jA_2 + kA_3.$$

This particular example was introduced and extensively studied by Gürsey and Tze.[12] Here we see that many interesting global and local properties of $\mathbf{HP}^n$-model are common for a large class of field theoretical models based on $E(f)$. All of them have duality equations built in and all possess global topological invariants.

## IV. GENERALIZED BOGOMOLNY EQUATIONS

In this section we discuss some special solutions of the $c$-self-duality equations. If $M = \mathbb{R}^4 \ni (x_0,x_1,x_2,x_3)$ and $P$ is a principal bundle over $M$ then one can study $x_0$-invariant solutions to the usual self-dual equations. They are called time invariant instantons or monopoles. In our case, let $M = \mathbb{R}^{4n} \simeq \mathbb{R}^4 \otimes \mathbb{R}^n \ni \{x_\alpha^i\}_{\alpha=0,...,3}^{i=1,...,n}$, $P$ be a principal bundle over $M$, and let $\mathrm{YM}_c(\nabla)$ be our Yang–Mills functional. In an obvious analogy to the four-dimensional case we can study $x_0^i$ invariant $c$-self-dual connections on $M$ or "$c$-monopoles" on $\mathbb{R}^3 \otimes \mathbb{R}^n$. Let us start with the following observation.

*Proposition 4.1:* Let $M = \mathbb{R}^4 \otimes \mathbb{R}^n$ be the $4n$-dimensional Euclidean flat space with global linear coordinates $x_\alpha^i$, $\alpha = 0,1,2,3$; $i = 1,..., n$. For any $(x_1,...,x_n)$ in $\mathbb{R}^n$ we define

$$p\colon\ \mathbb{R}^4 \to \mathbb{R}^4 \otimes \mathbb{R}^n$$

by

$$(x_0,x_1,x_2,x_3) \to x_\alpha^i = x_\alpha x^i. \tag{4.1}$$

Suppose $P$ is a principal bundle over $M$ with connection $\nabla$ and curvature $F$. Then $p^*\nabla$ is an anti-self-dual connection on $p^*P$ if

$$*F = -[1/(2n-1)!]F\wedge\Omega^{n-1}, \tag{4.2}$$

i.e., $F$ is $c_2$-self-dual.

*Proof:* In the $x_\alpha^i$-coordinates $dx_\alpha^i$ is exactly the one-form $\omega_\alpha^i$ of (2.4). Now a two-form $F$ satisfies the equation

$$*F = -[1/(2n-1)!]F\wedge\Omega^{n-1}$$

if and only if

$$F = -\tfrac{1}{6}*(F\wedge *\Omega). \tag{4.3}$$

Using Theorem 2.2 we get the following equations

$$\begin{aligned}
&F_{\binom{i}{0}\binom{j}{1}} = -F_{\binom{i}{2}\binom{j}{3}},\\
&F_{\binom{i}{0}\binom{j}{2}} = F_{\binom{i}{1}\binom{j}{3}}, F_{\binom{i}{0}\binom{j}{3}} = -F_{\binom{i}{1}\binom{j}{2}}, \quad \forall i,j,\\
&F_{\binom{i}{\alpha}\binom{j}{\alpha}} = F_{\binom{i}{\beta}\binom{j}{\beta}}, \quad \forall i,j,\alpha,\beta,\\
&F_{\binom{i}{\alpha}\binom{j}{\beta}} = F_{\binom{j}{\alpha}\binom{i}{\beta}}, \quad \forall i,j,\alpha,\beta,\alpha\neq\beta.
\end{aligned} \tag{4.4}$$

Let us denote the components of $p^*F$ by $F_{\alpha\beta}$. As a consequence of the chain rule we get

$$F_{\alpha\beta} = \sum_{i,j} x^i x^j F_{\binom{i}{\alpha}\binom{j}{\beta}} \tag{4.5}$$

and therefore

$$\begin{aligned}
F_{01} &= \sum_{i,j} x^i x^j F_{\binom{i}{0}\binom{j}{1}} = -F_{23},\\
F_{02} &= \sum_{i,j} x^i x^j F_{\binom{i}{0}\binom{j}{2}} = F_{13},\\
F_{03} &= \sum_{i,j} x^i x^j F_{\binom{i}{0}\binom{j}{3}} = -F_{12}.
\end{aligned} \tag{4.6}$$

In other words, $p^*\nabla$ is an anti-self-dual connection. ■

Recently, Pedersen and Poon used twistorial approach to find a generalization of the Bogomolny equations.[5] They introduced Yang–Mills–Higgs equations $\mathbb{R}^3 \otimes \mathbb{R}^n$. If one considers monopoles on $\mathbb{R}^3$ as time invariant instantons on $\mathbb{R}^4$ the following simple geometric description of generalized monopoles comes with no surprise.

*Proposition 4.2:* Let $x_\mu^i$, $\mu = 0,1,2,3$; $i = 1,...,n$ be a global linear coordinate on $\mathbb{R}^4 \otimes \mathbb{R}^n$ and let

$$p\colon\ \mathbb{R}^4 \otimes \mathbb{R}^n \to \mathbb{R}^3 \otimes \mathbb{R}^n$$

be a projection

$$(x_0^i,x_1^i,x_2^i,x_3^i) \to (x_1^i,x_2^i,x_3^i).$$

If $(\nabla, \Phi^i)$ is a generalized monopole then

$$\nabla' \doteq p^*\nabla + \sum_i \Phi^i\, dx_0^i \tag{4.7}$$



Reuse of AIP Publishing content is subject to the terms: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 138.23.170.248 On: Fri, 22 Apr 2016 17:58:25

is a $c_2$-self-dual connection. Conversely, any $c_2$-self-dual connection that is $x_0^i$-invariant determines a solution of the generalized monopole equation.

*Proof:* The curvature $F'$ of the connection $\nabla'$ is given by

$$F' = p^*F + \sum_j (\nabla\Phi^j) \wedge dx_0^j + \tfrac{1}{2} \sum_{i<j} [\Phi^i,\Phi^j] dx_0^i \wedge dx_0^j, \tag{4.8}$$

where $F$ is the curvature two-form of $\nabla$. Now, using Eqs. (4.4), we get

$$\begin{aligned} &\nabla_{\binom{j}{1}}\Phi^i = F_{\binom{i}{2}\binom{j}{3}}, \nabla_{\binom{j}{2}}\Phi^i = -F_{\binom{i}{1}\binom{j}{3}}, \nabla_{\binom{j}{3}}\Phi^i = F_{\binom{i}{1}\binom{j}{2}}, \quad \forall i,j \\ &F_{\binom{i}{\alpha}\binom{j}{\alpha}} = \tfrac{1}{2}[\Phi^i,\Phi^j], \quad \forall i,j;\ \alpha = 1,2,3, \\ &\nabla_{\binom{i}{\alpha}}\Phi^j = \nabla_{\binom{j}{\alpha}}\Phi^i, \quad \forall i,j;\ \alpha = 1,2,3, \end{aligned} \tag{4.9}$$

which can be written as

$$\begin{aligned} &F_{\binom{i}{\alpha}\binom{j}{\beta}} = \sum_\gamma \epsilon_{\alpha\beta\gamma} \nabla_{\binom{i}{\gamma}}\Phi^j + \tfrac{1}{2}\delta_{\alpha\beta}[\Phi^i,\Phi^j], \ \forall i,j;\forall\alpha,\beta = 1,2,3 \\ &\nabla_{\binom{i}{\alpha}}\Phi^j = \nabla_{\binom{j}{\alpha}}\Phi^i, \quad \forall i,j;\ \alpha = 1,2,3. \end{aligned} \tag{4.10}$$

The converse is obvious. ■

We can also obtain "monopole" analogs of $c$-self duality equations in the $c_1$ and $c_3$ cases. The first one is not interesting, however, because it yields $n$ decoupled self-dual Bogomolny equations. In the second case we can explicitly write down the set of equations

$$\begin{aligned} &F_{\binom{i}{\alpha}\binom{j}{\beta}} + F_{\binom{j}{\alpha}\binom{i}{\beta}} = \sum_\gamma \epsilon_{\alpha\beta\gamma}(\nabla_{\binom{i}{\gamma}}\Phi^j + \nabla_{\binom{j}{\gamma}}\Phi^i), \quad \forall i,j;\forall\alpha,\beta, \\ &\sum_{i=1}^n \nabla_{\binom{i}{\alpha}}\Phi^i = 0, \forall\alpha, \quad [\Phi^i,\Phi^j] = -\sum_{\alpha=1}^3 F_{\binom{i}{\alpha}\binom{j}{\alpha}}, \quad \forall i,j. \end{aligned} \tag{4.11}$$

For $n = 1$ these are just the usual Bogomolny equations with the reversed orientation. We do not know any nontrivial solutions of (4.11) for $n > 1$ at the moment. Finally, let us remark that we could introduce additional invariance and reduce the $c$-self-duality equation to $2n$ dimensions, assuming that the $c$-self-dual equations of $\mathbb{R}^4 \otimes \mathbb{R}^n$ be both $x_0^i$ and $x_1^i$ invariant. Then we obtain an analog of the well-known vortex equation of the two-dimensional Yang–Mills–Higgs theory. Again the $c_2$ case is the most natural generalization and we shall address this problem in a future work.


## ACKNOWLEDGMENT

The authors were partially supported by the NSF Grant DMS-8906806.



[1]S. Ishihara, J. Diff. Geom. **9**, 483 (1974).
[2]S. Salamon, Invent. Math. **67**, 143 (1982).
[3]S. Salamon, Ann. Scient. Éc. Norm. Sup. **19**, 31 (1986).
[4]H. Pedersen and Y. S. Poon, Commun. Math. Phys. **117**, 569 (1989).
[5]A. Swann, "On the Fundamental 4-Form of a Quaternion–Hermitean Manifold," Oxford preprint, 1989.
[6]E. Corrigan and P. Goddard, Commun. Math. Phys. **100**, 1 (1985).
[7]M. M. Capria and M. S. Salamon, Nonlinearity **1**, 517 (1988).
[8]M. F. Atiyah, *Geometry of Yang–Mills Fields* (Lezioni Fermiane, Pisa 1979).
[9]M. F. Atiyah, V. G. Drin'feld, N. J. Hitchin, and Yu. I. Manin, Phys. Lett. A**65**, 185 (1978).
[10]N. Buchdal, J. Diff. Geom. **24**, 19 (1986).
[11]S. K. Donaldson, "Vector bundles on flag manifolds and the Ward correspondence," in *Geometry Today*, Progress in Math., No. 60 (Birkhauser, Boston, 1985).
[12]F. Güsey and C. H. Tze, Ann. Phys. **128**, 29 (1980).





Reuse of AIP Publishing content is subject to the terms: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 138.23.170.248 On: Fri, 22 Apr 2016 17:58:25