COMPACT EINSTEIN-WEYL MANIFOLDS WITH LARGE SYMMETRY GROUP

ANDERS BISBJERG MADSEN, HENRIK PEDERSEN, YAT SUN POON, AND ANDREW SWANN

ABSTRACT. A geometric classification of the compact four-dimensional Einstein-Weyl manifolds with at least four-dimensional symmetry group is given. Our results also sharpen previous results on four-dimensional Einstein metrics and correct Parker's topological classification of cohomogeneity-one four-manifolds.

1. INTRODUCTION

The Einstein-Weyl equations are a conformally invariant generalisation of the Einstein equations, introduced by Weyl [26]. They have been thoroughly studied in dimension three [4, 8, 10, 12, 23, 24, 25], where it is known that any solution on a compact manifold is either a compact quotient of hyperbolic three-space $\mathcal{H}^3$ or has a cohomogeneity-one action of the two-torus T^2. Furthermore, in any dimension, a compact Einstein-Weyl manifold which is not Einstein has a non-trivial symmetry [24]. To find new examples in higher dimensions, it is therefore natural to look for solutions with a high degree of symmetry.

In this paper we will give a full classification of the compact four-dimensional Einstein-Weyl structures for which the symmetry group is at least four-dimensional. Restricting to dimension four allows us to take advantage of various topological consequences of the Einstein-Weyl equations [22, 20, 7]. The assumption that the group of symmetries is at least four-dimensional implies that the solutions are either homogeneous or have cohomogeneity one. Our results also sharpen previous results [9, 2] on four-dimensional Einstein metrics (Theorem 3.1) and correct the topological classification [19] of cohomogeneity-one four-manifolds (Remark 6.4).

Let $(M,[g])$ be a conformal manifold. A torsion-free connection D preserving the conformal class $[g]$ is called a *Weyl connection*. Fixing a choice of Riemannian metric g in the conformal class, we obtain a one-form ω from the equation $Dg = \omega \otimes g$. Conversely, the one-form ω together with the Levi-Civita connection ∇ of g, determine D by

$$D = \nabla - \tfrac{1}{2}(\omega \veebar \operatorname{Id} - g \otimes \omega^\sharp),$$

where $\omega^\sharp$ is the vector field such that $\omega = g(\omega^\sharp, \cdot)$, and $(\omega \veebar \operatorname{Id})(X,Y) = \omega(X)Y + \omega(Y)X$. Under a conformal change $g \mapsto \exp(\lambda)g$, we have $\omega \mapsto \omega + d\lambda$ and so it makes sense to call D *closed* if $d\omega = 0$ and *exact* if ω is exact.

The Einstein-Weyl equations state

$$Sr^D = \Lambda g,$$

where Sr^D is the symmetric part of the Ricci curvature $r^D = \operatorname{Tr}(Z \mapsto R_{X,Z}Y)$ of D and $\Lambda\colon M \to \mathbb{R}$ is an arbitrary function. Suppose (g, D) satisfy the Einstein-Weyl equations. If D is exact, then g is conformal to an Einstein metric $\tilde{g}$ and D is the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$. If D is closed, then g is locally conformal to

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Einstein. A symmetry of $(M,[g],D)$ is a diffeomorphism preserving the conformal class $[g]$ and the connection D.

Main Theorem. *Let $(M,[g],D)$ be a four-dimensional Einstein-Weyl manifold whose symmetry group G is at least four-dimensional, then either*

(a) *D is exact, g is conformal to an Einstein metric $\tilde{g}$ of cohomogeneity at most one, D is the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ and $(M,\tilde{g})$ is given by Theorem 3.1, or*

(b) *D is closed, but not exact, and $(M,[g],D)$ is finitely covered by $S^1\times S^3$ with its standard Einstein-Weyl structure, or*

(c) *D is not closed, $(M,[g],D)$ is of cohomogeneity one and M is given in either Table 2 or 4 with $([g],D)$ described in §6 or §7, respectively.*

Let us comment on each part of this theorem. For part (a), Jensen [9] and Bérard Bergery [2] showed that the compact four-dimensional Einstein manifolds with symmetry group of dimension at least four, are finitely covered by either the flat metric on T^4, the symmetric metrics on S^4, $\mathbb{C}P^2$ or $S^2\times S^2$, or by the Page metric on $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. We determine which finite quotients occur.

In part (b), we referred to the standard Einstein-Weyl structure on $S^1\times S^3$. This is given as follows. Let g_{can} be the canonical metric on S^3 with sectional curvature one and let $S^1=\{\,\exp(i\theta):\theta\in[\,0,2\pi)\,\}$. Then for any constant s,

$$g=g_{\text{can}}+s^2d\theta^2,\quad \omega=2s\,d\theta$$

is Einstein-Weyl and is called the standard structure on $S^1\times S^3$. The constant s corresponds to reparameterisation of the circle S^1. Gauduchon [7] showed that any closed non-exact four-dimensional Einstein-Weyl structure is locally equivalent to this standard structure and says such manifolds are *of type* $S^1\times S^3$. He showed that these manifolds are finitely covered by a mapping torus of S^3. However, these mapping tori need not be finite quotients of $S^1\times S^3$. Thus the content of (b) is that the symmetry assumption restricts which manifolds of type $S^1\times S^3$ may arise.

In part (c), with one exception (on $\mathbb{C}P^2$), all the solutions come in one-dimensional families. Moreover, nearly all the solutions obtained are new: a few isolated cases were given in [22], but even for these metrics the information we obtain here is much more explicit. It is worth noting that the diffeomorphism types occurring in part (c) are those arising in the Einstein case (part (a)) except for the four-torus T^4. However, the list of equivariant diffeomorphism types is different (see §7, particularly Remark 7.3).

Having obtained some of these families of solutions, Einstein-Weyl structures were studied from the point of view of deformation theory [21]. We plan to study the limits of the one-dimensional families in future work. The results presented here are based in part on [15], where it was also shown that many of the new Einstein-Weyl structures obtained here have higher-dimensional generalisations. These will be presented elsewhere, together with various cohomogeneity-one structures on non-compact manifolds [13, 14].

The paper is organised as follows. We first show that if $(M,[g])$ is not a standard sphere, then the symmetry group acts as isometries with respect to a representative of $[g]$ known as the Gauduchon metric: when D is exact, this metric is the Einstein metric. We then identify precisely which manifolds occur in the Einstein case. In §4, we deal with homogeneous Einstein-Weyl manifolds, showing that they are either Einstein or finite quotients of $S^1\times S^3$. We then turn to cohomogeneity-one Einstein-Weyl structures. This reduces to three cases corresponding to the symmetry group being $SO(4)$, $S^1\times SO(3)$ or $U(2)$. The case of $SO(4)$ only gives $S^1\times S^3$-structures and Einstein metrics. The other two cases are covered in §§6 and 7 respectively. In each case, a certain amount of work can be done purely topologically. Thereafter, we explicitly solve the relevant system of ordinary differential equations over an open interval. The final step is to include the boundary conditions and either determine the solutions explicitly or at least determine the topology of the solution space.

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2. The Symmetry Group

We define the *symmetry group* of a Weyl manifold $(M^n, [g], D)$ to be the group of conformal transformations preserving the connection D. Note that for a conformal map $\phi\colon M \to M$ with $\phi^* g = \exp(f) g$, the pull-back connection is given by $D^\phi_X Y = \phi_*^{-1}(D_{\phi_* X}\phi_* Y)$ and satisfies $D^\phi g = (\phi^*\omega - df) \otimes g$. Hence D^ϕ is always a Weyl connection and ϕ lies in the symmetry group if and only if $\phi^*\omega = \omega + df$.

The following Lemma shows that one may equivalently define the symmetry group to consist of conformal transformations ϕ which are *projective*, that is ϕ preserves unparameterised geodesics and so $D^\phi - D = \alpha \veebar \mathrm{Id}$, for some one-form α.

Lemma 2.1. *Suppose D_1, D_2 are two Weyl connections on $(M, [g])$. Then $D_1 = D_2$ if and only if D_1 and D_2 are projectively equivalent.*

Proof. Assume D_1 and D_2 are projectively equivalent. Then there is a one-form α such that $D_1 - D_2 = \alpha \veebar \mathrm{Id}$. Now $D_i = \nabla - \frac{1}{2}(\omega_i \veebar \mathrm{Id} - g \otimes \omega_i^\sharp)$, where $D_i g = \omega_i \otimes g$, so $2\alpha \veebar \mathrm{Id} = -\omega \veebar \mathrm{Id} + g \otimes \omega^\sharp$, for $\omega = \omega_1 - \omega_2$. Evaluating ω on this gives $(2\alpha + \omega) \vee \omega = |\omega|^2 g$, which implies $\alpha = \omega = 0$. □

If M is compact, then the component of the identity of the group of conformal transformations preserves some metric in the conformal class provided M is not conformally equivalent to the Euclidean sphere S^n [11], cf. [17]. Thus, if M is not the Euclidean sphere, the symmetry group has dimension strictly smaller than $n(n+1)/2$.

Gauduchon [6] proved that a compact Weyl manifold admits a unique metric, up to homothety, such that $d^*\omega = 0$. We call this the *Gauduchon metric*. If M is in addition Einstein-Weyl, then for this metric $\omega^\sharp$ is a Killing vector [24] preserving ω. Thus, if M is not Einstein, the symmetry group is at least one-dimensional.

Lemma 2.2. *If M is a compact Weyl manifold which is not the Euclidean sphere, then the component of the identity G of the symmetry group of M preserves the Gauduchon metric.*

Proof. Let g be a metric in the conformal class preserved by G. To find the Gauduchon metric $\tilde g = \exp(f) g$ one solves the equation $\mathfrak{L}^* f = 0$, where $\mathfrak{L} = \Delta^g + (n-2)\omega^{\sharp_g}/2$ and the adjoint is taken with respect to the L^2-inner product defined by g [6, 24]. As G preserves g and ω, one sees that it preserves the kernel of $\mathfrak{L}^*$. Thus for any $a \in G$, $a^*\tilde g = \exp(f \circ a) g$ is also a Gauduchon metric, so $a^*\tilde g = \rho(a)\tilde g$, for some constant $\rho(a)$. This defines a homomorphism $\rho\colon G \to \mathbb{R}_{>0}$. But G is compact, so $\rho \equiv 1$. □

3. Einstein Four-Manifolds

Here we commence the proof of the Main Theorem. Let $(M, [g], D)$ be a compact Einstein-Weyl four-manifold whose symmetry group has dimension at least four. Let G denote the component of the identity of the symmetry group. If M is conformal to the Euclidean four-sphere, then Gauduchon [7, lemme 4] has shown that the Einstein-Weyl structure is that of the standard Einstein metric. Hence we may assume that M is not conformal to the Euclidean sphere. By Lemma 2.2, G preserves the Gauduchon metric g.

If D is exact, then $\omega = 0$ for the Gauduchon metric and so g is Einstein with isometry group G. We then have (M, g) given by

Theorem 3.1. *Let (M, g) be a compact four-dimensional Einstein manifold whose isometry group has dimension at least four. Then either*

1. *M is homogeneous and so is either the flat torus T^4 or the standard, locally symmetric, metric on S^4, $\mathbb{R}P^4$, $\mathbb{C}P^2$, $S^2 \times S^2$, $S^2 \times \mathbb{R}P^2$, $\mathbb{R}P^2 \times \mathbb{R}P^2$ or $(S^2 \times S^2)/\{\pm(1,1)\}$, or*
2. *M is of cohomogeneity one and isometric to one of*
   (a) *the standard metric on $(S^2 \times_{(\zeta,-1)} S^2)$, where $\zeta = \mathrm{diag}(-1,1,1)$ is reflection in the equatorial plane,*
   (b) *the standard metric on $(S^2 \times_{(\delta,-1)} S^2)$ or $(\mathbb{R}P^2, \times_{(\delta,-1)} S^2)$, where $\delta = \mathrm{diag}(-1,-1,1)$ is a rotation through π,*
   (c) *$\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ with the Page metric, or its $\mathbb{Z}_2$-quotient $\mathbb{C}P^2 \# \mathbb{R}P^4$.*

This result is essentially due to Jensen [9] in the homogeneous case and Bérard Bergery [2] for metrics of cohomogeneity one. What is not discussed in these accounts is which of the finite quotients of the symmetric spaces occur. This we will now provide for the homogeneous case and the case of cohomogeneity one will follow from our later discussions. Note that cohomogeneity-one quotients of T^4 have three-dimensional symmetry group and so do not occur in the above theorem.

We need to determine the (non-trivial) finite groups Γ acting freely and isometrically on the symmetric spaces S^4, $\mathbb{C}P^2$ and $S^2 \times S^2$, such that the resulting quotients are homogeneous. Note that the order of Γ must divide the Euler characteristic of the symmetric space, because $\chi(M) = |\Gamma|\chi(M/\Gamma)$ for a free action. Thus for S^4, the only possibility is $\Gamma = \mathbb{Z}_2$ and $S^4/\Gamma = \mathbb{R}P^4$ (see [27]).

For $\mathbb{C}P^2$, Γ is $\mathbb{Z}_3 = \langle f \rangle$. If Ω is the Kähler form, then f maps a generator of $H^2(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}$ to another generator, so $f^*[\Omega] = [\Omega]$, since f has order 3. The Lefschetz number of f is now 3, so f must have a fixed point, contradicting the assumption that Γ acts freely.

Lemma 3.2. *The isometry group of $S^2 \times S^2$ is $(O(3) \times O(3)) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ swaps the two factors.*

Proof. Let Ω_1, Ω_2 be the pull-backs of the volume forms on the two S^2-factors. Then an isometry f maps $\{\Omega_1, \Omega_2\}$ to harmonic representatives for a pair of generators of $H^2(S^2 \times S^2, \mathbb{Z})$ and so acts as an element of $GL(2, \mathbb{Z})$. Also f preserves the inner products $g(\Omega_i, \Omega_j)$, so the action of f on (Ω_1, Ω_2) coïncides with the action of an element a of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ generated by $a_1(x, y) = (-x, y)$, $a_2(x, y) = (x, -y)$ and $s(x, y) = (y, x)$. Now replacing f by $f \circ a^{-1}$ we may assume f acts trivially on (Ω_1, Ω_2).

Let $i_z : S^2 \to S^2 \times S^2$ be the inclusion $i_z(x) = (x, z)$ and let p_1 be the projection to the first factor. Then $p_1 \circ f \circ i_z$ is an element of $PSL(2, \mathbb{C})$ for all z. But S^2 is simply-connected, so we can lift the map $z \mapsto p_1 \circ f \circ i_z$ to $SL(2, \mathbb{C}) \subset \mathbb{C}^4$ and hence conclude that $p_1 \circ f \circ i_z$ is independent of z. Repeating the argument with the other S^2-factor, shows that $f \in SO(3) \times SO(3)$. ☐

Lemma 3.3. *Let M be compact Riemannian manifold with isometry group G and suppose Γ is a discrete subgroup of G which acts freely on M. Then the dimension of the isometry group $\mathrm{Isom}(M/\Gamma)$ of M/Γ is the same as the dimension of the centraliser $C(\Gamma)$.*

Proof. Any Killing vector field on M/Γ lifts to a Killing vector field on M commuting with Γ, so $\dim \mathrm{Isom}(M/\Gamma) \leqslant \dim C(\Gamma)$. On the other hand, $C(\Gamma)$ is a subgroup of the normaliser $N_G(\Gamma)$ and elements of $N_G(\Gamma)/\Gamma$ act isometrically on M/Γ. □

Proposition 3.4. *If Γ is a non-trivial finite group acting freely and isometrically on $S^2 \times S^2$ and $(S^2 \times S^2)/\Gamma$ is homogeneous, then Γ is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\pm 1, \pm 1)\}$.*

Proof. Let f be a non-trivial element of Γ. First assume that f does not swap the S^2-factors, so $f(x,y) = (\alpha(x), \beta(y))$ for some $\alpha, \beta \in O(3)$. Now since α^2 and β^2 lie in $SO(3)$, $f^2 = (\alpha^2, \beta^2)$ has a fixed-point and so must be the identity, thus α and β are either $\pm\,\mathrm{Id}$ or conjugate to $\delta_\varepsilon = \mathrm{diag}(1, \varepsilon, -1)$, $\varepsilon = \pm 1$. Since f is fixed-point free, we have without loss of generality that $\beta = -\,\mathrm{Id}$. Now the centraliser $C(\delta_\varepsilon, -1)$ of $(\delta_\varepsilon, -1)$ in the isometry group of $S^2 \times S^2$ is $O(2) \times O(3)$ which acts with three-dimensional orbits on $S^2 \times S^2$. By the previous Lemma, the isometry group of the quotient has the same dimension as the centraliser $C(\Gamma)$. But $C(\Gamma) \leqslant C(f)$, so the quotient can only be homogeneous if $\alpha = \pm\,\mathrm{Id}$. Thus $f \in \{(\pm 1, \pm 1)\}$.

If f swaps the S^2-factors then $f(x,y) = (\alpha(y), \beta(x))$, for some $\alpha, \beta \in O(3)$. Now $f^2(x,y) = (\alpha\beta(x), \beta\alpha(y))$, which can not be the identity as $(x,y) \mapsto (\alpha(y), \alpha^{-1}(x))$ has $(\alpha(y), y)$ as a fixed-point. Now f^2 preserves the S^2-factors, so we may apply the above arguments to get $\alpha\beta = -\,\mathrm{Id}$. Thus $f(x,y) = (\alpha(y), -\alpha^{-1}(x))$. Now, if $((\gamma, \delta), \varepsilon)$ is an element of $C(f) = C((\alpha, -\alpha^{-1}), -1) \leqslant (O(3) \times O(3)) \rtimes \mathbb{Z}_2$, then $\delta = \varepsilon\alpha^{-1}\gamma\alpha^{\varepsilon}$, so the centraliser of f has dimension 3, which implies the quotient can not be homogeneous. □

This completes the proof of Theorem 3.1. □

4. Homogeneous Einstein-Weyl Four-Manifolds

The aim of this section is to prove:

Proposition 4.1. *A compact homogeneous Einstein-Weyl four-manifold is either finitely covered by $S^1 \times S^3$ or is a homogeneous Einstein manifold.*

Here the Einstein-Weyl structure on $S^1 \times S^3$ is given by the product metric, where the circle may have any prescribed length, and the pull-back of a one-form of appropriate constant length on the circle. These are special cases of what Gauduchon [7] calls manifolds of type $S^1 \times S^3$, see the introduction.

Proof. Assume D is not exact and let G be the symmetry group of $M = G/H$. Let $\mathfrak{m}$ be an Ad_H-invariant complement to $\mathfrak{h} \subset \mathfrak{g}$. Since G preserves the Weyl one-form ω, we have a further Ad_H-invariant splitting $\mathfrak{m} = \ker\omega \oplus (\ker\omega)^-$. As G acts effectively on M, we have that $\mathfrak{h}$ acts effectively on $\mathfrak{m}$, and so $\mathfrak{h} \leqslant \mathfrak{o}(3) \oplus \mathfrak{o}(1) \cong \mathfrak{su}(2)$. In particular, $\mathrm{rank}\,\mathfrak{h}$ is at most 1 and $\dim\mathfrak{g} = 4 + \dim\mathfrak{h} \leqslant 7$. Note that $\mathfrak{g}$ can not be Abelian, otherwise M is a torus and $\pi_1(M) = \mathbb{Z}^4$ forces the structure to be Einstein [22].

The classification of compact Lie groups now implies that are there are only four cases to consider: (A) $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$, (B) $\mathfrak{g} = 2\,\mathfrak{u}(1) \oplus \mathfrak{su}(2)$ with (a) $\mathfrak{h} = \mathfrak{u}(1) \leqslant \mathfrak{su}(2)$ or (b) $\mathfrak{h} = \mathfrak{u}(1)_\Delta \leqslant \mathfrak{u}(1) \oplus \mathfrak{su}(2)$, and (C) $\mathfrak{g} = \mathfrak{u}(1) \oplus 2\,\mathfrak{su}(2)$ with $\mathfrak{h} = \mathfrak{su}(2)_\Delta \leqslant 2\,\mathfrak{su}(2)$, where the subscript Δ indicates a subalgebra not contained in either factor.

In case (B)(a), M is finitely covered by $T^2 \times S^2$ which by the Einstein-Weyl inequality [20] can not admit an Einstein-Weyl structure. In the remaining three cases, M is finitely covered by $S^1 \times S^3$. The Einstein-Weyl inequality then implies that D is closed and Gauduchon's results [7] give that the structure on $S^1 \times S^3$ is standard. □

5. Cohomogeneity-one Einstein-Weyl Manifolds

Proceeding with the proof of the Main Theorem, the previous two sections imply that we may assume that M is neither Einstein nor homogeneous. Since the dimension of G is at least four, the principal orbits must be three-dimensional and thus G acts with cohomogeneity one on M. The only possibilities for G are now $SO(4)$, $S^1 \times SO(3)$ and $U(2)$ and the principal orbits are finitely covered by either $S^3 = SO(4)/SO(3)$, $S^1 \times S^2 = S^1 \times SO(3)/SO(2)$ or $S^3 = U(2)/U(1)$ [2, 1, 16]. The first of these cases is dealt with below after having established a result relevant to all three cases. The remaining two are the subject of the next two sections.

Suppose for the moment that M is not of type $S^1 \times S^3$ [7]. This implies that the conformal scalar curvature is strictly positive and that $\pi_1(M)$ is finite [22]. It now follows that the orbit space M/G is a closed interval $[0, \ell]$, since if it were a circle then the exact homotopy sequence together with connectedness of the principal orbits would give an infinite fundamental group.

Proposition 5.1. *Let M be a compact manifold of cohomogeneity one with finite fundamental group. Let $\pi \colon M \to M/G = [0, \ell]$ be the projection. Then $\pi^{-1}(0, \ell)$ is a union of principal orbits G/H and there are two special orbits $\pi^{-1}(0) = G/K_1$ and $\pi^{-1}(\ell) = G/K_2$, where the subgroups K_i contain H and the quotients K_i/H are diffeomorphic to spheres.*

Suppose in addition that M is Einstein-Weyl. Let γ be a geodesic of the Gauduchon metric g orthogonal to one, and hence all, principal orbits G/H. Parameterise γ by arc length so that $\pi\gamma(t) = t$, for $t \in [0, \ell]$. Then the Gauduchon metric and one-form ω take the form $g = dt^2 + g_t$ and $\omega = \omega_t$, where (g_t, ω_t) are homogeneous Weyl structures on G/H.

Proof. The topological assertions may be found in [16]. The choice of γ implies that $g = dt^2 + g_t$ and $\omega = \alpha(t)dt + \omega_t$, for some function $\alpha \colon [0, \ell] \to \mathbb{R}$. For a fixed volume form vol on G/H, g_t has volume $vol_t = f(t)\, vol$. Note that (g_t, ω_t) is the Gauduchon gauge on $\pi^{-1}(t)$, because the conformal factor taking g_t to the Gauduchon metric is G-invariant and hence constant on G/H. We now have

$$0 = d^*\omega = - * d * (\alpha(t)dt) - d^{*\prime}\omega_t = -(\alpha f)'/f,$$

and thus it is sufficient to show $\alpha(0) = 0$. However, t is a radial coordinate on the disk bundle $(M \setminus (G/K_2)) \to G/K_1$, so α must vanish at 0 in order for ω to be smooth. □

Corollary 5.2. *Suppose M is a compact Einstein-Weyl four-manifold of cohomogeneity one under $G = SO(4)$. Then M is either Einstein or is a finite quotient of $S^1 \times S^3$.*

Proof. If M/G is an interval, then the fact that $S^3 = SO(4)/SO(3)$ is isotropy irreducible, implies that the G-invariant one-forms ω_t must be zero.

If M/G is a circle, then the topology of M is either $S^1 \times S^3$, $(S^1 \times S^3)/(-1, -1)$ or $S^1 \times \mathbb{R}P^3$, since the fibre has a transitive action of $SO(4)$ and so can only be S^3 or $\mathbb{R}P^3$. Again, the Einstein-Weyl structure must be standard by [20, 7]. □

Remark 5.3. In [22] it was shown that an Einstein-Weyl structure in the Gauduchon gauge (g, ω) is smooth as soon as g is C^2 and ω is C^1. This will be used repeatedly when finding boundary conditions later.

Notation 5.4. The topology of the manifolds M appearing in the Proposition is in general determined by the principal and special orbits together with a double coset of $(N_G(H) \cap N_G(K_1)) \backslash N_G(H) / (N_G(H) \cap N_G(K_2))$. However in all the cases we actually encounter $N_G(K_i)$ contains $N_G(H)$ and this double coset space is trivial. We will write $[\, G/K_1 \mid G/H \mid G/K_2 \,]$ for these manifolds M.

6. Symmetry Group $S^1 \times SO(3)$

Let M be a four-manifold of cohomogeneity one under an effective action of $G = S^1 \times SO(3)$ such that the orbit space is an interval $[0,\ell]$. The principal orbit G/H is then a finite quotient of $S^1 \times S^2$ and H is a one-dimensional subgroup of G. We will let $SO(2)$ denote the subgroup $\{1\} \times SO(2) \leqslant S^1 \times SO(3)$ consisting of the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)$ and let $\delta \in SO(3)$ be $\operatorname{diag}(-1,-1,1)$.

Lemma 6.1. (a) *The only proper Lie subgroup of $SO(3)$ strictly containing $SO(2)$ is $O(2) = S(O(1) \times O(2))$.*

(b) *The only one-dimensional Lie subgroups H of $S^1 \times SO(3)$ containing $SO(2)$ are $\mathbb{Z}_k \times SO(2)$, $\mathbb{Z}_k \times O(2)$ and $\mathbb{Z}_{2\ell} \ltimes SO(2)$, where $\mathbb{Z}_{2\ell}$ is generated by $(\exp(\pi i/\ell), \delta)$.*

Proof. Part (a) is well-known. For part (b), let $\rho_1, \rho_2 \colon S^1 \times SO(3) \to S^1, SO(3)$ be the projections. Then H is a subgroup of $\rho_1(H) \times \rho_2(H)$, part (a) implies $\rho_2(H)$ is either $SO(2)$ or $O(2)$ and the dimension restriction forces $\rho_1(H) = \mathbb{Z}_k$ for some k. Write $(\ker \rho_1) \cap H = \{1\} \times N \lhd \{1\} \times \rho_2(H)$. Then H is an extension of $\mathbb{Z}_k$ by N and N contains $SO(2)$. If N equals $\rho_2(H)$, then H is simply the product $\rho_1(H) \times \rho_2(H)$ giving the first two cases. Otherwise we have $\rho_2(H) = O(2)$, $N = SO(2)$ and that $\gamma = (\exp(2\pi i/k), \delta)$ is an element of H mapping to the generator of $\mathbb{Z}_k$. As $\gamma^k \in \ker \rho_1$, we necessarily have that k is even. □

Proposition 6.2. *If M^4 is of cohomogeneity one under $G = S^1 \times SO(3)$, then the principal orbit is either $S^1 \times S^2$, $S^1 \times \mathbb{R}P^2$ or $S^1 \times_\Delta S^2 = (S^1 \times S^2)/\{\pm(1,1)\}$ and the corresponding possible special orbits are given in Table 1.*

Proof. The principal orbits are G/H, where H is given by the previous Lemma. However, the factors $\mathbb{Z}_k$ and $\mathbb{Z}_\ell$ are central subgroups of both G and H and just shorten the S^1-factor. So by rescaling, we may assume H is either $SO(2)$, $O(2)$ or $\mathbb{Z}_2 \ltimes SO(2)$. The special orbits G/K are now determined by the condition that K/H be a sphere. Note that even though $\mathbb{R}P^1$ is just a circle, we use it to denote the quotient $S^1/\{\pm 1\}$, which has half the length of the S^1-factor in the principal orbit. □

H, G/H	K, G/K	Boundary conditions
$SO(2)$, $S^1 \times S^2$		
	$SO(3)$, S^1	$f > 0,\ f', h, h'' = 0,\ h' = 1$
	$S^1 \times SO(2)$, S^2	$h > 0,\ f, f'', h', \beta, \beta' = 0,\ f' = 1$
	$\mathbb{Z}_2 \times SO(2)$, $\mathbb{R}P^1 \times S^2$	$f, h > 0,\ f', h', \beta' = 0$
	$O(2)$, $S^1 \times \mathbb{R}P^2$	ditto
	$\mathbb{Z}_2 \ltimes SO(2)$, $S^1 \times_\Delta S^2$	ditto
$O(2)$, $S^1 \times \mathbb{R}P^2$ and		
$\mathbb{Z}_2 \ltimes SO(2)$, $S^1 \times_\Delta S^2$		
	$S^1 \times O(2)$, $\mathbb{R}P^2$	$h > 0,\ f, f'', h', \beta, \beta' = 0,\ f' = 1$
	$\mathbb{Z}_2 \times O(2)$, $\mathbb{R}P^1 \times \mathbb{R}P^2$	$f, h > 0,\ f', h', \beta' = 0$

Table 1. Principal orbits G/H, special orbits G/K and boundary conditions when $G = S^1 \times SO(3)$

Theorem 6.3. *Let M be a compact four-dimensional non-exact Einstein-Weyl manifold of cohomogeneity one under $G = S^1 \times SO(3)$. Then M/G is an interval and M is given in Table 2.*

G/H	G/K_1	G/K_2	M	Einstein-Weyl
$S^1\times S^2$				
	S^1	S^1	$S^1\times S^3$	type $S^1\times S^3$
	S^1	S^2	S^4	one-dimensional family
	S^1	$\mathbb{R}P^1\times S^2$	$S^1\times_{(-1,\zeta)} S^3$	type $S^1\times S^3$
	S^1	$S^1\times\mathbb{R}P^2$	$S^1\times\mathbb{R}P^3$	type $S^1\times S^3$
	S^1	$S^1\times_\Delta S^2$	$S^1\times_{(-1,-1)} S^3$	type $S^1\times S^3$
	S^2	S^2	$S^2\times S^2$	one-dimensional family
	S^2	$\mathbb{R}P^1\times S^2$	$\mathbb{R}P^2\times S^2$	one-dimensional family
	S^2	$S^1\times\mathbb{R}P^2$	$S^2\times_{(\zeta,-1)} S^2$	one-dimensional family
	S^2	$S^1\times_\Delta S^2$	$S^2\times_{(-1,-1)} S^2$	one-dimensional family
$S^1\times\mathbb{R}P^2$				
	$\mathbb{R}P^2$	$\mathbb{R}P^2$	$S^2\times\mathbb{R}P^2$	one-dimensional family
	$\mathbb{R}P^2$	$\mathbb{R}P^1\times\mathbb{R}P^2$	$\mathbb{R}P^2\times\mathbb{R}P^2$	one-dimensional family
$S^1\times_\Delta S^2$				
	$\mathbb{R}P^2$	$\mathbb{R}P^2$	$S^2\times_{(\delta,-1)} S^2$	one-dimensional family
	$\mathbb{R}P^2$	$\mathbb{R}P^1\times\mathbb{R}P^2$	$\mathbb{R}P^2\times_{(\delta,-1)} S^2$	one-dimensional family

Table 2. Topology of and Einstein-Weyl structures on four-manifolds M of cohomogeneity one under $G = S^1\times SO(3)$, with principal orbit G/H and special orbits G/K_1, G/K_2. Here $\zeta = \operatorname{diag}(-1,1,\dots,1)$ and $\delta = \operatorname{diag}(-1,-1,1)$.

The proof of Theorem 6.3 divides into two parts. First we shall show that M/G is not a circle and then determine via essentially topological arguments which four-manifolds M with cohomogeneity-one $S^1\times SO(3)$-actions and $M/G = [0,\ell]$ can not admit Einstein-Weyl structures or only admit structures of type $S^1\times S^3$. We will then construct all Einstein-Weyl solutions with the given symmetry on the remaining manifolds.

Topology. Suppose M/G is a circle. If G/H is $S^1\times S^2$ then M has the topology of $T^2\times S^2$ or a $\mathbb{Z}_2$-quotient. If G/H is $S^1\times\mathbb{R}P^2$, then the only topology for M is $T^2\times\mathbb{R}P^2$. In the case $G/H = S^1\times_\Delta S^2$, M has the topology of $(S^1\times_\Delta S^2)\times S^1$ or a $\mathbb{Z}_2$-quotient. In all these cases, M is finitely covered by $T^2\times S^2$, which does not satisfy the Einstein-Weyl inequality [20].

Now that M/G is an interval will identify those of type $S^1\times S^3$. For $[\,S^1 \mid S^1\times S^2 \mid S^1\,]$ and $[\,S^1 \mid S^1\times S^2 \mid S^1\times\mathbb{R}P^2]$ the S^1-factors split off to give $S^1\times[\,*\mid S^2\mid *\,] = S^1\times S^3$ and $S^1\times[\,*\mid S^2\mid\mathbb{R}P^2\,] = S^1\times\mathbb{R}P^3$, respectively. The manifold $[\,S^1\mid S^1\times S^2\mid\mathbb{R}P^1\times S^2\,]$ is the $\mathbb{Z}_2$-quotient of $S^1\times S^3 = [\,S^1\mid S^1\times S^2\mid S^1\,]$ by $(z,x,t)\mapsto(-z,x,\ell-t)\in S^1\times S^2\times[0,\ell]$. Note that this $\mathbb{Z}_2$-action preserves the standard Einstein-Weyl structure on $S^1\times S^3$. Similarly $[\,S^1\mid S^1\times S^2\mid S^1\times_\Delta S^2\,]$ is the quotient by $(z,x,t)\mapsto(-z,-x,\ell-t)$.

We now show that those combinations of orbit types not appearing in the Table do not admit Einstein-Weyl structures. This will mainly be based on the fact that if M is Einstein-Weyl then so is any finite unbranched cover $M'\to M$.

The first case is $[\,\mathbb{R}P^1\times S^2\mid S^1\times S^2\mid\mathbb{R}P^1\times S^2\,]$, which we may rewrite as

$$\begin{aligned}[\,\mathbb{R}P^1\times S^2\mid S^1\times S^2\mid\mathbb{R}P^1\times S^2\,] &= [\,\mathbb{R}P^1\mid S^1\mid\mathbb{R}P^1\,]\times S^2\\ &= \big([\,\mathbb{R}P^1\mid S^1\mid *\,]\#[\,*\mid S^1\mid\mathbb{R}P^1\,]\big)\times S^2\\ &= (\mathbb{R}P^2\#\mathbb{R}P^2)\times S^2 = K^2\times S^2,\end{aligned}$$

where K^2 is the Klein bottle. However $K^2 \times S^2$ is double-covered by $T^2 \times S^2$ which does not satisfy the Einstein-Weyl inequality [20].

The manifolds $[\,\mathbb{R}P^1 \times S^1 \mid S^1 \times S^2 \mid S^1 \times \mathbb{R}P^2\,]$ and $[\,\mathbb{R}P^1 \times S^1 \mid S^1 \times S^2 \mid S^1 \times_\Delta S^2\,]$ are $\mathbb{Z}_2$-quotients of $[\,\mathbb{R}P^1 \times S^2 \mid S^1 \times S^2 \mid \mathbb{R}P^1 \times S^2\,]$ by $(z, x, t) \mapsto (z, -x, \ell - t)$ and $(z, x, t) \mapsto (-z, -x, \ell - t)$, respectively, so are not Einstein-Weyl.

Similarly, we have $[\,S^1 \times \mathbb{R}P^2 \mid S^1 \times S^2 \mid S^1 \times \mathbb{R}P^2\,] = S^1 \times (\mathbb{R}P^3 \# \overline{\mathbb{R}P}^3)$ which has oriented cover $T^2 \times S^2$ and so is not Einstein-Weyl. The $\mathbb{Z}_2$-action $(z, x, t) \mapsto (-z, -x, \ell - t)$ on $[\,S^1 \times \mathbb{R}P^2 \mid S^1 \times S^2 \mid S^1 \times \mathbb{R}P^2\,]$ gives $[\,S^1 \times \mathbb{R}P^2 \mid S^1 \times S^2 \mid S^1 \times_\Delta S^2\,]$ and so this latter space can not be Einstein-Weyl.

The one remaining case with principal orbit $S^1 \times S^2$ is

$$M_\Delta := [\,S^1 \times_\Delta S^2 \mid S^1 \times S^2 \mid S^1 \times_\Delta S^2\,].$$

There is a free involution σ on M_Δ induced by the map $(1, -1)$ on $S^1 \times S^2$. From (6.1), below, we see that this involution preserves any $S^1 \times SO(3)$-invariant Einstein-Weyl structure on M_Δ. However, $M_\Delta/\sigma = K^2 \times \mathbb{R}P^2$ and so M_Δ is not Einstein-Weyl.

For each of the other two possible principal orbits there is only one case to consider. However, $[\,\mathbb{R}P^1 \times \mathbb{R}P^2 \mid S^1 \times \mathbb{R}P^2 \mid \mathbb{R}P^1 \times \mathbb{R}P^2\,]$ is $K^2 \times \mathbb{R}P^2$ and $[\,\mathbb{R}P^1 \times \mathbb{R}P^2 \mid S^1 \times_\Delta S^2 \mid \mathbb{R}P^1 \times \mathbb{R}P^2\,]$ is the quotient of $K^2 \times S^2 = [\,\mathbb{R}P^1 \times S^2 \mid S^1 \times S^2 \mid \mathbb{R}P^1 \times S^2\,]$ by $(z, x, t) \mapsto (-z, -x, t)$, so neither of these is Einstein-Weyl.

Remark 6.4. A smooth classification of compact four-manifolds of cohomogeneity one has been given in [19]. However, the group $\mathbb{Z}_2 \ltimes SO(2)$ is not given as a possibility for either H or K_i and hence the above eight spaces involving the orbit $S^1 \times_\Delta S^2$ are missing from that classification, as are the two distinct manifolds with principal orbits $S^1 \times_\Delta S^2$ and orbit space S^1.

Explicit Solutions. For each of the three choices of stabiliser H, the Einstein-Weyl structure lifts to an $S^1 \times SO(3)$-invariant structure on $S^1 \times S^2 \times (0, \ell)$. Since S^2 is isotropy irreducible, S^2 admits no non-zero invariant one-forms and any invariant metric is a constant multiple of the canonical metric g_{can} of sectional curvature one. Thus Proposition 5.1 implies that the Einstein-Weyl structure takes the form

$$g = dt^2 + f(t)^2 d\theta^2 + h(t)^2 g_{\mathrm{can}}, \qquad \omega = \beta(t)\, d\theta, \tag{6.1}$$

where θ is the arc-length parameter on a circle of length 2π and f, h and β are smooth functions on $[0, \ell]$ with $f, h > 0$ on $(0, \ell)$.

Using formulæ for warped-product metrics [3] (cf. [2, 18, 22]), the Einstein-Weyl equations become

$$-\frac{f''}{f} - 2\frac{h''}{h} = \Lambda, \tag{6.2}$$

$$-\frac{f''}{f} - 2\frac{f'h'}{fh} + \frac{1}{2}\frac{\beta^2}{f^2} = \Lambda, \tag{6.3}$$

$$-\frac{h''}{h} - \frac{h'^2}{h^2} - \frac{f'h'}{fh} + \frac{1}{h^2} = \Lambda, \tag{6.4}$$

$$\beta' - 2\beta\frac{f'}{f} = 0, \tag{6.5}$$

where Λ is some function. In addition, at 0 and ℓ the functions f, h and β satisfy certain boundary conditions depending on the type of the principal and special orbits. These conditions at 0 are given in Table 1 and those at ℓ are the same, except that the value 1 is replaced by -1.

We will first find the general non-exact $S^1 \times SO(3)$-invariant Einstein-Weyl solutions and then impose the boundary conditions.

Equation (6.5) implies that $\gamma := \beta/f^2$ is a constant and since our structure is in the Gauduchon gauge, non-exactness implies $\gamma \neq 0$. Eliminating Λ from the remaining equations gives

$$\frac{h''}{h} - \frac{f'h'}{fh} + \frac{1}{4}\gamma^2 f^2 = 0, \tag{6.6}$$

$$\frac{f''}{f} - \frac{{h'}^2}{h^2} + \frac{1}{h^2} - \frac{1}{4}\gamma^2 f^2 = 0. \tag{6.7}$$

Multiplying (6.6) by $2hh'/f^2$ shows that $\delta := \frac{1}{4}\gamma^2 h^2 + ({h'}^2/f^2)$ is a strictly positive constant, and so writing $r := \sqrt{\gamma^2/4\delta} > 0$, we have

$$f^2 = \delta^{-1}{h'}^2(1 - r^2h^2)^{-1} \tag{6.8}$$

with $h(t) \in [0, 1/r]$.

If we change the metric g by a homothety $g \mapsto \tilde{g} = \lambda^2 g$ and let $\tilde{t} = t/\lambda$, then $\tilde{g}$ is still in the Gauduchon gauge and has the form (6.1). However, the new constants are given by $\tilde{\gamma} = \gamma/\lambda^2$, $\tilde{\delta} = \delta/\lambda^2$ and $\tilde{r} = r/\lambda$. Thus by rescaling we may assume $r = 1$.

As $f > 0$ on $(0, \ell)$, we have that $h'(t_0) = 0$ at some $t_0 \in (0, \ell)$ if and only if $h(t_0) = 1$. However, (6.6) shows that h is not constant on any open subinterval of $(0, \ell)$, so Rolle's Theorem implies that h' has at most one zero on $(0, \ell)$.

Let $H(h) := {h'}^2$ and let $\cdot$ denote differentiation with respect to h. Then substituting (6.8) into (6.7) gives

$$\ddot{H} + \frac{3h}{1-h^2}\dot{H} + \frac{4h^4 + 4h^2 - 2}{h^2(1-h^2)^2}H = -\frac{2}{h^2}. \tag{6.9}$$

The homogeneous equation has a solution $H_0(h) = (1-h^2)^{3/2}/h$ and the general solution of (6.9) is $H(h) = v(h)H_0(h)$, where v satisfies

$$\dot{v} = (1 + qh^2)(1 - h^2)^{-3/2}, \tag{6.10}$$

for some constant q.

Choose φ so that $\sin\varphi = h$. Since $h > 0$ on $(0, \ell)$, we may demand that $\varphi \in [0, \pi]$ and solve (6.10) to get

$$v(\varphi) = \begin{cases} V(\varphi) + v(0), & \text{for } 0 \leqslant \varphi < \pi/2, \\ -V(\varphi) + v(\pi) - q\pi, & \text{for } \pi/2 < \varphi \leqslant \pi, \end{cases} \tag{6.11}$$

where $V(\varphi) := (q+1)\tan\varphi - q\varphi$. These expressions are not defined at $\pi/2$. However, $\varphi(t_0) = \pi/2$ corresponds to $h(t_0) = 1$ and so occurs for at most one $t_0 \in (0, \ell)$. From (6.6), we have $h''(t_0) < 0$, and together with $h' = \varphi'\cos\varphi$, this implies that φ' is strictly positive on $(0, \ell)$. Without loss of generality we may assume $\varphi' > 0$ on $(0, \ell)$, even when there is no such t_0.

We need to determine the conditions for $f = \sqrt{\delta^{-1}v(\varphi)|\cos\varphi|/\sin\varphi}$ to be C^2 at t_0. From (6.11), we have

$$\lim_{t\to t_0} v(h)(1-h^2)^{1/2} = \lim_{\varphi\to\pi/2} v(\varphi)|\cos\varphi| = q + 1.$$

Thus $f(t_0) > 0$ only if $q > -1$. Now write $v(\varphi) = \pm V(\varphi) + c_\pm$, where $\pm$ is the sign of $\cos\varphi$. We have $2ff' = \varphi' d(f^2)/d\varphi$, $\varphi'(t_0) > 0$ and

$$\frac{d(f^2)}{d\varphi} = \delta^{-1}(-q\cot\varphi + (q\varphi \mp c_\pm)\operatorname{cosec}^2\varphi),$$

so $c_+ = -c_- =: c$ and $v(0) = q\pi - v(\pi)$. It is straightforward to check that continuity of f'' at t_0 imposes no further conditions.

Using φ as a coordinate, we have

$$g = \frac{d\varphi^2}{W(\varphi)} + W(\varphi)\delta^{-1}d\theta^2 + \sin^2\varphi\, g_{\mathrm{can}}, \qquad \omega = \pm 2W(\varphi)\delta^{-1/2}d\theta,$$

where $W(\varphi) = (q + 1 + (c - q\varphi)\cot\varphi)$ and $q > -1$ if $\pi/2 \in (\varphi(0), \varphi(\ell))$. Thus the Einstein-Weyl structure depends on the five parameters δ, q, c, $\varphi(0)$, $\varphi(\ell)$. We also have

$$\begin{aligned} {h'}^2 &= W(\varphi)\cos^2\varphi, \qquad {\varphi'}^2 = W(\varphi) = \delta f^2, \\ f' &= \frac{\varphi'}{2f}\frac{d(f^2)}{d\varphi} = \frac{1}{2\delta^{1/2}}(-q\cot\varphi + (q\varphi - c)\operatorname{cosec}^2\varphi), \\ f'' &= \tfrac{1}{2}(2q + (q\varphi - c)\cot\varphi)\operatorname{cosec}^2\varphi\, f, \\ h'' &= -\sin\varphi\,{\varphi'}^2 + \cos\varphi\,\varphi'' = \delta^{1/2}(-\sin\varphi\,\delta^{1/2}f^2 + \cos\varphi\, f'), \\ \beta &= \pm 2\delta^{1/2}f^2, \qquad \beta' = \pm 4\delta^{1/2}ff'. \end{aligned}$$

Using these expressions one gets the following Lemma.

Lemma 6.5. *Let ε be 1 if $t = 0$ and -1 if $t = \ell$. Then the boundary conditions are equivalent to the following:*

(a) *special orbit S^1: $\sin\varphi = 0$ and $c = q\varphi$;*

(b) *special orbit S^2 or $\mathbb{R}\mathrm{P}^2$: $\sin\varphi > 0$, $q = \varepsilon 2\delta^{1/2}\cot\varphi - 1 - \cot^2\varphi$ and $c = (\varepsilon 2\delta^{1/2} - \cot\varphi)(\varphi\cot\varphi - 1) - \varphi$;*

(c) *special orbit $\mathbb{R}\mathrm{P}^1 \times S^2$, $S^1 \times \mathbb{R}\mathrm{P}^2$, $S^1 \times_\Delta S^1$ or $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^2$: $\varphi = \pi/2$, $c = q\pi/2$ and $q > -1$.*

From this one sees immediately that there are no solutions when both orbits are of type (c), in agreement with our topological results. Each of the other pairs of special orbits do give Einstein-Weyl structures. Write $A := \varphi(0)$, $B := \varphi(\ell)$, so $0 \leqslant A < B \leqslant \pi$ and observe that we may swap the special orbits via the transformation $t \mapsto \ell - t$, $\varphi \mapsto \pi - \varphi$.

For $[\, S^1 \mid S^1 \times S^2 \mid S^1\,]$, we have $A = 0$, $B = \pi$ and $c = 0 = q\pi$, so $q = 0$ and $W \equiv 1$. Thus $g = d\varphi^2 + \delta^{-1}d\theta^2 + \sin^2\varphi\, g_{\mathrm{can}}$ and $\omega = \pm 2\delta^{-1/2}d\theta$, which is the standard structure on $S^1 \times S^3$. Similarly, we get $W \equiv 1$ and structures of type $S^1 \times S^3$ if one special orbit is S^1 and the other is of type (c).

The only combination (a)–(b) possible is $S^4 = [\, S^1 \mid S^1 \times S^2 \mid S^2\,]$. Here we get $A = 0$ and $c = 0$ from the S^1-orbit, whereas $t = \ell$ gives $B < \pi$, $q = -2\delta^{1/2}\cot B - 1 - \cot^2 B$, $c = -(2\delta^{1/2} + \cot B)(B\cot B - 1) - B$. The condition $c = 0$ determines δ, and hence q and W, in terms of B and we get the following family of Einstein-Weyl structures on S^4 depending on the parameter $B \in (0, \pi)$:

$$\begin{aligned} g &= \frac{1 - B\cot B}{\varphi\cot\varphi - B\cot B}\, d\varphi^2 + 4\frac{(1 - B\cot B)(\varphi\cot\varphi - B\cot B)}{(B + B\cot^2 B - \cot B)^2}\, d\theta^2 \\ &\quad + \sin^2\varphi\, g_{\mathrm{can}}, \\ \omega &= \pm 4\frac{\varphi\cot\varphi - B\cot B}{B + B\cot^2 B - \cot B}\, d\theta, \end{aligned} \tag{6.12}$$

where the coordinate φ runs over $[0, B]$.

The remaining two combinations of special orbit types are (b)–(b) and (b)–(c). The simplest to compute is (b)–(c). Here $B = \pi/2$, $c = q\pi/2$ and $0 < A < \pi/2$ with $q = 2\delta^{1/2}\cot A - 1 - \cot^2 A$, $c = (2\delta^{1/2} - \cot A)(A\cot A - 1) - A$. These equations may be solved to get δ, q and c in terms of A. Letting $C = \pi/2 - A \in (0, \pi/2)$ and

$\psi = \pi/2 - \varphi \in [0, C]$, the Einstein-Weyl solutions become

$$\begin{aligned} g &= \frac{1 + C\tan C}{C\tan C - \psi\tan\psi}\,d\psi^2 + 4\frac{(1 + C\tan C)(C\tan C - \psi\tan\psi)}{(C + C\tan^2 C) + \tan C)^2}\,d\theta^2 \\ &\quad + \cos^2\psi\, g_{\mathrm{can}}, \\ \omega &= \pm 4\frac{C\tan C - \psi\tan\psi}{C + C\tan^2 C + \tan C}\,d\theta. \end{aligned} \tag{6.13}$$

For the combination (b)–(b), the boundary conditions give

$$c = (2\delta^{1/2} - \cot A)(A\cot A - 1) - A = -(2\delta^{1/2} + \cot B)(B\cot B - 1) - B, \tag{6.14}$$

$$q = 2\delta^{1/2}\cot A - 1 - \cot^2 A = -2\delta^{1/2}\cot B - 1 - \cot^2 B. \tag{6.15}$$

From the last equation (6.15), we get that either $2\delta^{1/2} = \cot A - \cot B$ or $\cot A + \cot B = 0$. The former is not possible, because putting this expression for $2\delta^{1/2}$ into (6.14) leads to $A - B = \tan(A - B)$, which has no solution with $0 < B - A < \pi$. Thus $\cot B = -\cot A$, which for $A, B \in (0, \pi)$ implies $B = \pi - A$. Setting $C = \pi/2 - A$ and $\psi = \pi/2 - \varphi$, we once again obtain the solutions (6.13) except that ψ now lies in the interval $[-C, C]$. Thus we see that the solutions on $S^2 \times S^2$ and $S^2 \times \mathbb{R}\mathrm{P}^2$ are simply the lifts of the solutions of type (b)–(c) to the double covers.

Note that the condition $q > -1$ is fulfilled by all the above solutions whenever φ runs over an interval containing $\pi/2$.

7. Symmetry Group $U(2)$

For the proof of the Main Theorem, the last remaining case is that of an effective action of $G = U(2)$ on M^4 such that $M/G = [0, \ell]$. The principal orbits are finite quotients of $S^3 = U(2)/\,U(1)$, where $U(1) = \{\,\mathrm{diag}(\exp(i\theta), 1)\,\}$. As the only one-dimensional subgroups of $U(2)$ containing $U(1)$ are $U(1) \times \mathbb{Z}_k = \{\,\mathrm{diag}(\exp(i\theta), \exp(2\pi i\ell/k)\,\}$, the principal orbits are thus the Lens spaces $L(k, 1)$. The possible special orbits are given in Table 3.

H, G/H	K, G/K	Boundary conditions
$U(1)$, S^3		
	$U(2)$, $*$	$f, f'', h, h'', \beta, \beta' = 0$, $f', h' = 1$
	$U(1) \times U(1)$, $\mathbb{C}\mathrm{P}^1$	$h > 0$, $f, f'', h', \beta, \beta' = 0$, $f' = 1$
	$U(1) \times \mathbb{Z}_2$, $\mathbb{R}\mathrm{P}^3$	$f, h > 0$, $f', h', \beta' = 0$
$U(1) \times \mathbb{Z}_k$, $L(k, 1)$, $k \geqslant 2$		
	$U(1) \times U(1)$, $\mathbb{C}\mathrm{P}^1$	$h > 0$, $f, f'', h', \beta, \beta' = 0$, $f' = 1$
	$U(1) \times \mathbb{Z}_{2k}$, $L(2k, 1)$	$f, h > 0$, $f', h', \beta' = 0$

Table 3. Principal orbits G/H, special orbits G/K and boundary conditions when $G = U(2)$.

Theorem 7.1. *Let M be a compact four-dimensional non-exact Einstein-Weyl manifold of cohomogeneity one under $G = U(2)$. Then either M/G is a circle and M is a finite quotient of $S^1 \times S^3$ or M/G is an interval and M is given in Table 4.*

As in the case of $S^1 \times SO(3)$-symmetry we first look at the problem topologically. However, in this case we can not deduce so much and will have to rely on the more involved analysis of the differential equations.

G/H	G/K_1	G/K_2	M	Einstein-Weyl
S^3				
	$*$	$*$	S^4	one-dimensional family
	$*$	$\mathbb{CP}^1$	$\mathbb{CP}^2$	one-dimensional family and an isolated solution
	$*$	$\mathbb{RP}^3$	$\mathbb{RP}^4$	one-dimensional family
	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$	one-dimensional family
	$\mathbb{CP}^1$	$\mathbb{RP}^3$	$\mathbb{CP}^2 \# \mathbb{RP}^4$	one-dimensional family
$L(2j,1)$, $j \geqslant 2$				
	$\mathbb{CP}^1$	$L(4j,1)$	$S^2 \times \mathbb{RP}^2$	one-dimensional family
	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$S^2 \times S^2$	one-dimensional family
$L(2j+1,1)$, $j \geqslant 2$				
	$\mathbb{CP}^1$	$L(4j+2,1)$	$\mathbb{CP}^2 \# \mathbb{RP}^4$	one-dimensional family
	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$	one-dimensional family

TABLE 4. Topology of and Einstein-Weyl structures on four-manifolds M of cohomogeneity one under $G = U(2)$, with principal orbit G/H and special orbits G/K_1, G/K_2. Note that the same diffeomorphism type can appear as several different G-manifolds, see Remark 7.3.

When M/G is a circle, then M is obtained from $G/H \times [0,\ell]$ by identifying the fibres over the endpoints via an isometry ϕ. Topologically the resulting manifolds correspond to the elements of $N_G(H)/H$. Taking an n-fold cover of the circle yields the manifold obtained using the identification ϕ^n. However, in our case this latter group is finite, so we can find n such that $\phi^n \in H$ and topologically M is finitely covered by $G/H \times S^1$. Now G/H is itself a finite quotient of S^3, so M is a finite quotient of $S^1 \times S^3$. As before, [20, 7] imply that the Einstein-Weyl structure is standard.

Now consider the case when M/G is an interval.

Notation 7.2. We will write $M(k)$ for $[\,\mathbb{CP}^1 \mid L(k,1) \mid \mathbb{CP}^1\,]$ and $M(k)/\mathbb{Z}_2$ for its $\mathbb{Z}_2$-quotient $[\,\mathbb{CP}^1 \mid L(k,1) \mid L(2k,1)\,]$.

Remark 7.3. The above notation was used in [22], $M(k)$ is the S^2-bundle over $\mathbb{CP}^1$ built from the circle bundle $P(k) = L(k,1) \to \mathbb{CP}^1$. The diffeomorphism type of $M(k)$ depends only on the parity of k: if k is even then $M(k)$ is diffeomorphic to $S^2 \times S^2$; for k odd, $M(k)$ is $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. This is proved by calculating the intersection form of $M(k)$ (see [5, p. 4]). However, the equivariant diffeomorphism types of $M(k)$ are distinct for all $k \geqslant 0$. This is reflected in the (non-)existence of Einstein-Weyl structures: $M(2)$ and $M(3)$ have no solutions, whilst for all other k, the family of $U(2)$-invariant Einstein-Weyl structures on $M(k)$ is one-dimensional. Thus $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ have countably infinitely many one-dimensional families of solutions.

From the topological point of view, two cases that can be excluded easily from the Einstein-Weyl classification are $\mathbb{RP}^4 \# \mathbb{RP}^4 = [\,\mathbb{RP}^3 \mid S^3 \mid \mathbb{RP}^3\,]$ and its $\mathbb{Z}_k$-quotients $[\,L(2k,1) \mid L(k,1) \mid L(2k,1)\,]$. We claim that $\mathbb{RP}^4 \# \mathbb{RP}^4$ and its $\mathbb{Z}_k$-quotients can not be Einstein-Weyl. The space $\mathbb{RP}^4 \# \mathbb{RP}^4$ has oriented double cover $S^1 \times S^3$, where $\mathbb{Z}_2$ acts by $(\exp(it), p) \mapsto (\exp(-it), -p)$. However, this action does not preserve dt and hence the Einstein-Weyl structures on $S^1 \times S^3$ are not $\mathbb{Z}_2$-invariant and do not descend to the quotient.

We now turn to the analysis of the differential equations. The manifolds $L(k,1)$ are circle bundles over S^2. Write g_{FS} for the Fubini-Study metric on $S^2 = \mathbb{C}\mathrm{P}^1$ and note that $4g_{\mathrm{FS}} = g_{\mathrm{can}} = r_{\mathrm{FS}}$. Let ω_{FS} be the Kähler form on $\mathbb{C}\mathrm{P}^1$ and let σ be the connection one-form on $\pi\colon L(k,1) \to \mathbb{C}\mathrm{P}^1$ such that $d\sigma = 2k\pi^*\omega_{\mathrm{FS}}$. Proposition 5.1 implies that the Einstein-Weyl structure takes the form

$$g = dt^2 + f(t)^2\sigma^2 + h(t)^2 g_{\mathrm{FS}}, \qquad \omega = \beta(t)\,\sigma, \tag{7.1}$$

where f, h and β are smooth functions on $[0,\ell]$ with $f, h > 0$ on $(0,\ell)$.

The Einstein-Weyl equations now take the form

$$-\frac{f''}{f} - 2\frac{h''}{h} = \Lambda, \tag{7.2}$$

$$-\frac{f''}{f} - 2\frac{f'h'}{fh} + 2k^2\frac{f^2}{h^4} + \frac{1}{2}\frac{\beta^2}{f^2} = \Lambda, \tag{7.3}$$

$$-\frac{h''}{h} - \frac{h'^2}{h^2} - \frac{f'h'}{fh} + \frac{4}{h^2} - 2k^2\frac{f^2}{h^4} = \Lambda, \tag{7.4}$$

$$\beta' - 2\beta\frac{f'}{f} = 0, \tag{7.5}$$

where Λ is some function. The boundary conditions at 0 are given in Table 3 and those at ℓ are the same, except that the value 1 is replaced by -1. Our choice of g_{FS} instead of g_{can} is partly motivated by the simple form of these boundary conditions.

We commence by finding the general non-exact Einstein-Weyl solutions, before imposing the boundary conditions. Formally this has many similarities to the previous case with $S^1 \times SO(3)$-symmetry, but various details differ.

Equation (7.5) implies that $\gamma := \beta/f^2$ is a constant and since our structure is in the Gauduchon gauge, non-exactness implies $\gamma \neq 0$. Eliminating Λ from the remaining equations gives

$$\frac{h''}{h} - \frac{f'h'}{fh} + k^2\frac{f^2}{h^4} + \frac{1}{4}\gamma^2 f^2 = 0, \tag{7.6}$$

$$\frac{f''}{f} - \frac{h'^2}{h^2} - 3k^2\frac{f^2}{h^4} + \frac{4}{h^2} - \frac{1}{4}\gamma^2 f^2 = 0. \tag{7.7}$$

Multiplying (7.6) by $2hh'/f^2$ shows that $\delta := \frac{1}{4}\gamma^2h^2 - (k^2/h^2) + (h'^2/f^2)$ is constant. Let $A = 2\delta/\gamma^2$ and $B = (2\sqrt{k^2\gamma^2 + \delta^2})/\gamma^2 > 0$, then we have

$$\tfrac{1}{4}\gamma^2 f^2(h^2 - A + B)(A + B - h^2) = h^2h'^2.$$

If we rescale the metric by a homothety $g \mapsto \lambda^2 g$, then $\gamma \mapsto \gamma/\lambda^2$, $\delta \mapsto \delta/\lambda^2$, $A \mapsto \lambda^2 A$ and $B \mapsto \lambda^2 B$. Thus we may rescale to have $B = 1$ and hence

$$f^2 = \frac{(1 - A^2)h^2h'^2}{k^2(1 - (A - h^2)^2)}. \tag{7.8}$$

Note that $A < 1$, so $h'(t_0) = 0$ for some $t_0 \in (0,\ell)$ if and only if $h(t_0)^2 = 1 + A$. Equation (7.6) and Rolle's Theorem imply that h' has at most one zero on $(0,\ell)$.

Putting $H(h) := h'^2$ and substituting (7.8) into (7.7) gives

$$\begin{aligned} &\ddot{H} + \frac{3(1 - A^2 + h^4)}{h(1 - (A - h^2)^2)}\dot{H} \\ &\quad + \frac{h^8 + 2Ah^6 + (7 - 9A^2)h^4 - 8A(1 - A^2)h^2 - 2(1 - A^2)^2}{h^2(1 - (A - h^2)^2)^2} H = -\frac{8}{h^2}. \end{aligned} \tag{7.9}$$

The homogeneous equation has a solution $H_0(h) = (1-(A-h^2)^2)^{3/2}/h^4$ and the general solution of (7.9) is $H(h) = v(h)H_0(h)$, where v satisfies

$$\dot{v} = 2h^3(2+qh^2)(1-(A-h^2)^2)^{-3/2}, \tag{7.10}$$

for some constant q.

Let $\varphi_0 = -\sin^{-1} A \in (-\pi/2, \pi/2)$ and write $h^2 = A+\sin\varphi$ with $\varphi \in [\varphi_0, \pi-\varphi_0]$. Then we have $1-(A-h^2)^2 = \cos^2\varphi$ and we may solve (7.10) to get

$$v(\varphi) = \epsilon V(\varphi) + c_\epsilon,$$

where ϵ is the sign of $\pi/2-\varphi$,

$$V(\varphi) = (2A+(1+A^2)q)\tan\varphi + 2(1+Aq)\sec\varphi - q\varphi,$$

$c_+ = v(0) - 2(1+Aq)$ and $c_- = v(\pi) - 2(1+Aq) - q\pi$. These expressions are not defined at $\pi/2$, but we may assume that $\varphi' > 0$ on $(0,\ell)$, as before.

If $h'(t_0) = 0$, we need to have that $f = \sqrt{k^{-2}(1-A^2)v(\varphi)|\cos\varphi|/(A+\sin\varphi)}$ is of class C^2 and strictly positive at $\varphi = \pi/2$. Now

$$\lim_{\varphi\to\pi/2} v(\varphi)|\cos(\varphi)| = (1+A)(2+(1+A)q),$$

so $f(t_0) > 0$ implies

$$q > -\frac{2}{1+A}. \tag{7.11}$$

Looking at the first derivative we have

$$\begin{aligned}\frac{d(f^2)}{d\varphi} &= \frac{1-A^2}{k^2(A+\sin\varphi)^2}\left[(2+q(A+\sin\varphi))(A+\sin\varphi)^2\sec\varphi - \epsilon(1+A\sin\varphi)v(\varphi)\right]\\ &\to k^{-2}(1-A)(q\tfrac{\pi}{2} - \epsilon c_\epsilon)\end{aligned}$$

as $\varphi \to \pi/2$. Since the limits from the left and right should agree, we have $c_+ = -c_- =: c$. Continuity of f'' at t_0 imposes no further restrictions.

We now have

$$h'^2 = \frac{k^2\cos^2\varphi}{(A+\sin\varphi)(1-A^2)}f^2, \qquad \varphi' = \frac{2k}{(1-A^2)^{1/2}}f,$$

$$\begin{aligned}f' = \frac{k}{(1-A^2)^{1/2}}\Big[&-\frac{\cos\varphi}{A+\sin\varphi}f^2\\ &+\frac{1-A^2}{k^2(A+\sin\varphi)}\left\{A(2+Aq)\cos\varphi - c\sin\varphi + q\varphi\sin\varphi\right\}\Big],\end{aligned}$$

$$\begin{aligned}f'' = \frac{2k^2}{1-A^2}\Big[&\frac{\sin\varphi}{A+\sin\varphi}f^2 + \frac{2\cos^2\varphi}{(A+\sin\varphi)^2}f^2\\ &-\frac{2(1-A^2)\cos\varphi}{k^2(A+\sin\varphi)^2}\left\{A(2+Aq)\cos\varphi - c\sin\varphi + q\varphi\sin\varphi\right\}\\ &-\frac{1-A^2}{k^2(A+\sin\varphi)}\left\{(2A-(1-A^2)q)\sin\varphi + c\cos\varphi + q\varphi\cos\varphi\right\}\Big],\end{aligned}$$

$$2hh'' + 2h'^2 = \frac{2k}{(1-A^2)^{1/2}}f'\cos\varphi - \frac{4k^2}{1-A^2}f^2\sin\varphi,$$

$$\beta = \pm\frac{2k}{(1-A^2)^{1/2}}f^2, \qquad \beta' = \pm\frac{4k}{(1-A^2)^{1/2}}ff'.$$

The Einstein-Weyl structure is given by

$$g = \frac{d\varphi^2}{W} + \frac{1-A^2}{4k^2}W\sigma^2 + (A+\sin\varphi)g_{\mathrm{FS}}, \qquad \omega = \pm\frac{(1-A^2)^{1/2}}{2k}W\sigma, \tag{7.12}$$

where

$$W = \varphi'^2 = \frac{4}{A+\sin\varphi}\left((2A+(1+A^2)q)\sin\varphi + (c-q\varphi)\cos\varphi + 2(1+Aq)\right).$$

Regarding k as being given by the topology, we see that the Einstein-Weyl structure thus depends on the five constants A, q, c, $\varphi(0)$, $\varphi(\ell)$ with the constraint (7.11) which only applies if $\pi/2 \in (\varphi(0), \varphi(\ell))$. Using the above expressions, the boundary conditions are now given by

Lemma 7.4. *Let ε be 1 if $t=0$ and -1 if $t=\ell$. The boundary conditions at t are equivalent to the following:*

(a) *special orbit $*$ (implies $k=1$): $\sin\varphi = -A$ and $c = q(\varphi + \sin\varphi\cos\varphi) - 2\cos\varphi$;*

(b) *special orbit $\mathbb{C}\mathrm{P}^1$: if $\varphi = \pi/2$, then $q = -2/(1+A)$ and*

$$c = -\varepsilon\left(\frac{1+A}{1-A}\right)^{1/2} k - \frac{\pi}{1+A},$$

whereas for $\varphi \neq \pi/2$ we have

$$c = -(2A+(1+A^2)q)\tan\varphi - 2(1+Aq)\sec\varphi + q\varphi,$$

$$q = \left(\varepsilon\frac{k\cos\varphi}{(1-A^2)^{1/2}} - 2\right)\frac{1}{A+\sin\varphi};$$

(c) *special orbit $L(2k,1)$: $\varphi = \pi/2$, $c = q\pi/2$, $q > -2/(1+A)$.*

This immediately implies that there are no solutions when both special orbits are $L(2k,1)$. We now consider the other five cases in turn.

Case 1: $S^4 = [\,*\mid S^3 \mid *\,]$. The boundary conditions imply $\varphi(\ell) = \pi - \varphi(0)$ and, writing $\psi = \pi/2 - \varphi$ and $D = \psi(0)$,

$$q = -\frac{2\sin D}{D - \sin D\cos D}, \qquad c = -\frac{\pi\sin D}{D - \sin D\cos D}.$$

Hence, the Einstein-Weyl structure is given by (7.12) with

$$W = \frac{8(D + \sin D\cos D - (D\cos D + \sin D)\cos\psi - \psi\sin D\sin\psi)}{(\cos\psi - \cos D)(D - \sin D\cos D)}.$$

Case 2: $\mathbb{R}\mathbf{P}^4 = [\,*\mid S^3 \mid \mathbb{R}\mathbf{P}^3\,]$. Noting that in the S^4 solutions we had $c = q\pi/2$, we see that all the solutions on S^4 descend to $\mathbb{R}\mathrm{P}^4$, and we obtain the same expressions, except that φ runs over $[\varphi(0), \pi/2]$ instead of $[\varphi(0), \pi - \varphi(0)]$.

Case 3: $\mathbb{C}\mathbf{P}^2 = [\,*\mid S^3 \mid \mathbb{C}\mathbf{P}^1\,]$. Write $D = \varphi(0)$ and $E = \varphi(\ell)$. Note that $\sin E - \sin D$ is $h(\ell)$ and so is strictly positive. There are two cases to consider because of the different form of the boundary conditions.

Case 3a: $E = \pi/2$. From the boundary conditions we have

$$\begin{aligned} q &= -\frac{2}{1-\sin D} \\ c &= \left(\frac{1-\sin D}{1+\sin D}\right)^{1/2} - \frac{\pi}{1-\sin D} \\ &= (D + \sin D\cos D)q - 2\cos D. \end{aligned}$$

Substituting the first equation into the last, writing $\left((1-\sin D)/(1+\sin D)\right)^{1/2} = \cos D/(1+\sin D)$ and multiplying through by $\cos^2 D$, leads to

$$\frac{3+\sin D}{1+\sin D}\cos D + 2D - \pi = 0.$$

The left-hand-side of this equation $F(D)$ has derivative $F'(D) = \sin D\cos^2 D/(1+\sin D)^2$, which vanishes only at 0. Also $F(\pi/2) = 0$ and $F(D)$ tends to ∞ as D tends to $-\pi/2$. As $F(0)$ equals $3-\pi$ which is negative, we deduce that F has precisely one zero in $(-\pi/2, \pi/2)$ and hence there is one solution to the Einstein-Weyl equations with $E = \pi/2$.

Case 3b: $E \neq \pi/2$. The boundary conditions imply

$$\begin{aligned} q &= -\left(\frac{\cos E}{\cos D} + 2\right)\frac{1}{\sin E - \sin D}, \\ c &= (D + \sin D\cos D)q - 2\cos D \\ &= (2\sin D - (1+\sin^2 D)q)\tan E - 2(1 - q\sin D)\sec E + qE. \end{aligned}$$

Equating the last two expressions, substituting for q from the first and multiplying through by $\cos D\cos E(\sin E - \sin D)$ leads to

$$\bigl\{(E-D)(\cos E + 2\cos D) + 2(\sin D - \sin E) + \cos D\sin(D-E)\bigr\}\cos E = 0. \tag{7.13}$$

However, we have assumed $E \neq \pi/2$, so $\cos E \neq 0$ and the first factor in (7.13) must vanish.

Introduce new variables $\alpha = (E-D)/2$ and $\beta = (D+E)/2$. Equation (7.13) then implies

$$\tan\beta = \frac{(2+\cos^2\alpha)\sin\alpha - 3\alpha\cos\alpha}{\sin\alpha(\alpha - \sin\alpha\cos\alpha)}. \tag{7.14}$$

The variables D and E lie in the region specified by

1. $D \in (-\pi/2, \pi/2)$,
2. $\sin E > \sin D$ and
3. if $E > \pi/2$, then $\cos E/(1-\sin E) < -2\cos D/(1-\sin D)$.

The latter condition is equivalent to $E < \pi - E'$ where $E' \in (-\pi/2, \pi/2)$ satisfies $\cos E'/(1-\sin E') = 2\cos D/(1-\sin D)$ or equivalently

$$\sin E' = \frac{3+5\sin D}{5+3\sin D}.$$

Note that this is stronger than condition 2.

In terms of α and β these constraints become

1. $\alpha > 0$,
2. $\beta - \alpha > -\pi/2$ and
3. $\sin\alpha < 3\cos\beta$.

Note that one boundary of this region is $\{\,(\alpha,\beta) : \alpha = 0,\ \beta \in (-\pi/2, \pi/2)\,\}$ and that expanding (7.14) for small α shows that there are indeed solutions. A sketch of the region and the curve (7.14) is provided in Figure 1 and shows that there is a one-dimensional family of solutions. It is easily checked that the solution obtained in Case 3a does not have the property that the first factor of (7.13) vanishes and so does not lie in this family.

Case 4: $M(k)/\mathbb{Z}_2 = [\,\mathbb{C}\mathbf{P}^1 \mid L(k,1) \mid L(2k,1)\,]$. The boundary condition at ℓ implies $\varphi(\ell) = \pi/2$ and hence $\varphi(0) < \pi/2$. Writing $D = \pi/2 - \varphi(0)$, the boundary conditions become

$$\begin{gathered} (D\sin D + (1+A^2)\cos D + 2A)q = -2A\cos D - 2, \\ q = \left(\frac{k\sin D}{(1-A^2)^{1/2}} - 2\right)\frac{1}{A+\cos D}, \\ q > -2/(1+A). \end{gathered}$$

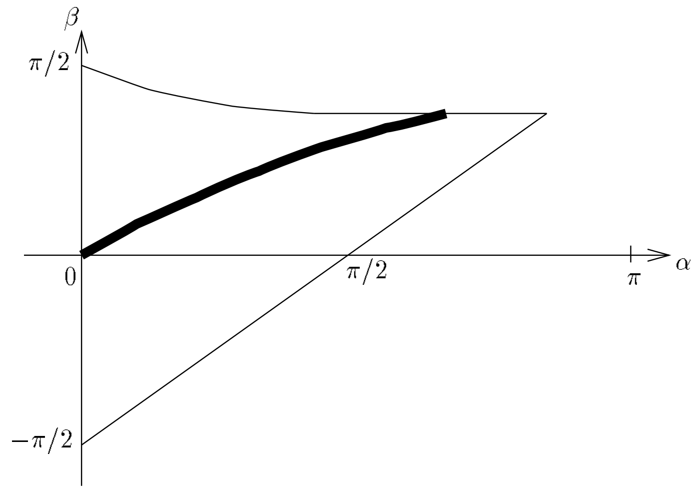

FIGURE 1. A one-dimensional family of Einstein-Weyl solutions on $\mathbb{C}P^2$. The curve (7.14) is sketched within the region specified by the constraints on α and β.

Write $A = -\cos\chi$ with $\chi \in (D, \pi]$. Then eliminating q from the boundary conditions leads to

$$k(D\sin D + (1+\cos^2\chi)\cos D - 2\cos\chi) - 2\sin\chi(D - \cos\chi\sin D) = 0,$$
$$k\cot(D/2) > 2\cot(\chi/2).$$

Putting $\tau = \tan(\chi/2)$, we have

$$\begin{aligned} G(D,\tau) &:= k\big(\tau^4(D\sin D + 2\cos D + 2) + 2\tau^2 D\sin D \\ &\qquad + (D\sin D + 2\cos D - 2)\big) \\ &\qquad - 4\tau\big(D - \sin D + \tau^2(D+\sin D)\big) \\ &= 0, \\ \tau &> \frac{2}{k}\tan\frac{D}{2}. \end{aligned}$$

Lemma 7.5. *$G(D,\tau)$ is monotone increasing in τ in the region $\tau > 2\tan(D/2)/k$.*

In the following it will be convenient to write $\sigma = \tan(D/2)$. We will refer to the line $\tau = 2\sigma/k$ with $\sigma \in (0,\infty)$ as the *critical line*. The region $\tau > 2\sigma/k$ we will refer to as the region above the critical line.

Proof. Now $(\partial^4 G/\partial\tau^4)(D,\tau) = 24k(D\sin D + 2\cos D + 2)$ which is strictly positive.

On the critical line $\partial^3 G/\partial\tau^3$ is $24G_1(D)/(1+\sigma^2)$, where $G_1(D) := (3\sigma^2-1)D + 6\sigma$. However, $G_1'(D) = 3(1+\sigma^2)\sigma D + 6\sigma^2 + 2 > 0$ and $G_1(0) = 0$, so $G_1 > 0$ on the critical line. Hence $\partial^3 G/\partial\tau^3 > 0$ on and above the critical line.

Consider $\partial^2 G/\partial\tau^2$, which equals $8G_2(D)/(k(1+\sigma^2))$ on the critical line, where $G_2(D) := k^2\sigma D + 6\sigma[(\sigma^2-1)D + 2\sigma]$. The function G_2 satisfies $G_2'' > 0$, $G_2'(0) = 0$ and $G_2(0) = 0$, so we conclude that $G_2 > 0$ on the critical line. Thus $\partial^2 G/\partial\tau^2 > 0$ on and above the critical line.

Finally, on the critical line $\partial G/\partial\tau = 4G_3(D)/(k^2(1+\sigma^2))$, where $G_3(D) := (13\sigma^4 + 3k^2\sigma^2 - 3\sigma^2 - k^2)D + 26\sigma^3 + 2k^2\sigma$. This has $G_3' > 0$ and $G_3(0) = 0$, so we conclude that $\partial G/\partial\tau$ is strictly positive on and above the critical line, as required. □

The lemma implies that zeros of G, and hence Einstein-Weyl solutions, above the critical line are in one-to-one correspondence with points on the critical line where G is strictly negative.

On the critical line we have

$$G(D, 2\sigma/k) = \frac{2(k^2-4)\sigma}{k^3(1+\sigma^2)} G_4(D),$$

where $G_4(D) := (k^2 + 4\sigma^2)D - 2k^2\sigma$. We thus conclude that *for $k = 2$, there are no Einstein-Weyl solutions.*

The function G_4 satisfies $G_4'(D) = 4\sigma(1+\sigma^2)D + (4-k^2)\sigma^2$, $G_4'(0) = 0$, $G_4''(0) = 0$ and $2G_4'''(0) = 12 - k^2$. Thus for $k \geqslant 4$, G_4' is initially negative, whereas for $k \leqslant 3$ it is initially positive.

We claim G_4' has precisely one zero in $(0, \pi)$ if $k \geqslant 4$, and has no zero if $k \leqslant 3$. To prove this note that zeros of G_4' correspond to solutions of $k^2 - 4 = 4G_5(D)$, where $G_5(D) := (1 + \sigma^2)D/\sigma$. Now $G_5'(D) = (1 + \sigma^2)G_6(D)/(2\sigma^2)$, where $G_6(D) := 2\sigma + (\sigma^2 - 1)D$. But $G_6'(D) = 2\sigma^2 + (\sigma^2 + 1)\sigma D$ is strictly positive, and hence G_5 is monotone increasing. However, $G_5(D) \to 2$ as $D \to 0$ and $G_5(D) \to \infty$ as $D \to \pi$, so G_4' has the properties claimed.

Thus for $k = 3$, the function G_4, and hence G, is strictly positive and there are no Einstein-Weyl solutions. On the other hand for $k \geqslant 4$, G_4 is initially negative and its derivative has one zero, so G_4 is strictly negative on an open interval and we obtain a connected one-dimensional family of Einstein-Weyl solutions. Finally, for $k = 1$, G_4 remains strictly positive, but G contains the factor $k^2 - 4$ on the critical line, so one obtains an Einstein-Weyl solution for each $D \in (0, \pi)$.

Case 5: $M(k) = [\,\mathbb{CP}^1 \mid L(k,1) \mid \mathbb{CP}^1\,]$. We claim that the only cohomogeneity-one Einstein-Weyl structures on $M(k)$ are the pull-backs of those on $M(k)/\mathbb{Z}_2$. In particular, there is a one-dimensional family if $k \geqslant 4$ or $k = 1$ and no solutions for $k = 2$ or 3.

Without loss of generality we may assume $\varphi(0) < \pi/2$ and $\varphi(0) < \varphi(\ell)$. Let $D = \varphi(0)$, $E = \varphi(\ell)$, $\alpha := (\varphi(\ell) - \varphi(0))/2$ and $\beta := (\varphi(\ell) + \varphi(0))/2$. Write $A = -\sin\chi$ with $\chi \in (-\pi/2, \pi/2)$.

Lemma 7.6. *Either $E = \pi/2$ or $E = \pi - D$, and in the latter case the Einstein-Weyl structure is $\mathbb{Z}_2$-invariant.*

Proof. Assume that $E \neq \pi/2$. The boundary conditions imply

$$q = \left(\frac{k\cos D}{\cos\chi} - 2\right)\frac{1}{\sin D - \sin\chi} = \left(-\frac{k\cos E}{\cos\chi} - 2\right)\frac{1}{\sin E - \sin\chi}.$$

Multiplying through by $\cos\chi(\sin D - \sin\chi)(\sin E - \sin\chi)$ gives

$$k(\cos D\sin E + \cos E\sin D) = k(\cos D + \cos E)\sin\chi + 2(\sin E - \sin D)\sin\chi.$$

In terms of α and β this implies that either $\cos\beta = 0$ or

$$\sin\beta = \cos\alpha\sin\chi + \frac{2}{k}\sin\alpha\cos\chi. \tag{7.15}$$

If $\cos\beta = 0$, then $E + D = \pi$ and the boundary conditions reduce to the equations for the case $M(k)/\mathbb{Z}_2$.

If (7.15) holds, substitute this expression into the second boundary condition at ℓ to get

$$q = \left(-\frac{k\cos E}{\cos\chi} - 2\right)\frac{1}{\sin E - \sin\chi} = -\frac{k\cos\alpha}{\cos\chi\sin\alpha}.$$

Now using the boundary conditions on c, we have

$$0 = -(2A + (1 + A^2)q)(\tan D - \tan E) - 2(1 + Aq)(\sec D - \sec E) + q(D - E).$$

Multiplying through by $(\cos D \cos E)/2$ and writing in terms of α, β and χ gives

$$\begin{aligned} 0 = &(-2\sin\chi + (1+\sin^2\chi)q)\sin\alpha\cos\alpha + 2(1 - q\sin\chi)\sin\alpha\sin\beta \\ &- q(\cos^2\alpha - \sin^2\beta). \end{aligned}$$

Substituting (7.15) into this equation gives

$$0 = \frac{1}{k\tan\alpha}\left(4k\sin\chi\sin\alpha\cos\alpha + \cos\chi(4\sin^2\alpha - k^2\cos^2\alpha)\right)(\tan\alpha - \alpha).$$

Thus either $\alpha = \tan\alpha$ or

$$\tan\chi = \frac{k^2 - 4\tan^2\alpha}{4k\tan\alpha}. \tag{7.16}$$

Now D lies in $(-\pi/2, \pi/2)$ and E lies in $(D, 3\pi/2)$, so α is in $(0, \pi)$. Thus $\alpha = \tan\alpha$ has no relevant solutions.

If (7.16) holds, then

$$\sin\chi = \frac{k^2 - 4\tan^2\alpha}{k^2 + 4\tan^2\alpha} \qquad \text{and} \qquad \cos\chi = \frac{4k\tan\alpha}{k^2 + 4\tan^2\alpha}.$$

Substituting these expressions into (7.15) gives $\sin\beta = \cos\alpha$, so $\beta = \pi/2 \pm \alpha$ and either $D = \pi/2$ or $E = \pi/2$. But we have excluded the former and so are left with $E = \pi/2$. □

It now remains to show that there are no solutions when $E = \pi/2$. The boundary condition for q gives

$$q = -\frac{2}{1 - \sin\chi} = \left(\frac{k\cos D}{\cos\chi} - 2\right)\frac{1}{\sin D - \sin\chi},$$

which simplifies to

$$\frac{k\cos D}{1 - \sin D} = \frac{2\cos\chi}{1 - \sin\chi},$$

or equivalently

$$k\cot\left(\frac{1}{2}\left(\frac{\pi}{2} - D\right)\right) = 2\cot\left(\frac{1}{2}\left(\frac{\pi}{2} - \chi\right)\right). \tag{7.17}$$

The boundary condition for c gives

$$\begin{aligned} \frac{\pi}{2} - D &= \frac{(1-\sin\chi)^{3/2}}{(1+\sin\chi)^{1/2}}k + \frac{k}{2}\left(\frac{1-\sin\chi}{\cos\chi}\right)(1 + \sin\chi) \\ &= \frac{k}{2}\left(\frac{1-\sin\chi}{\cos\chi}\right)(3 - \sin\chi). \end{aligned}$$

Thus (7.17) implies

$$\cot\left(\frac{k}{4}\left(\frac{1-\sin\chi}{\cos\chi}\right)(3 - \sin\chi)\right) = \frac{2}{k}\cot\left(\frac{1}{2}\left(\frac{\pi}{2} - \chi\right)\right).$$

Substitute $x = \frac{k}{2}(1 - \sin\chi)/\cos\chi = \frac{k}{2}\tan((\pi/2 - \chi)/2)$ to get

$$\cot(xy) = 1/x, \tag{7.18}$$

where $y = (3 - \sin\chi)/2$ is strictly greater than 1. Note that x is bounded below by 0 and above by the requirement that $2xy = \pi/2 - D < \pi$. Thus, we need to show (7.18) has no solutions for $0 < x < \pi/(2y)$. However, cot is monotone decreasing, so $\cot(xy) < \cot(x)$. But $\cot(x) < 1/x$, since $\tan x > x$ on $(0, \pi/2)$. Thus there are no Einstein-Weyl solutions with $E = \pi/2$.

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(Madsen, Pedersen) Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark
E-mail address: andersbm@imada.ou.dk, henrik@imada.ou.dk

(Poon) Department of Mathematics, University of California, Riverside, CA 92521, USA
E-mail address: ypoon@math.ucr.edu

(Swann) School of Mathematical Sciences, University of Bath, Claverton Down, Bath, Avon BA2 7AY, England
E-mail address: A.F.Swann@maths.bath.ac.uk