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GC^n continuity conditions for adjacent rational parametric surfaces

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Abstract

In this paper, the constraints on the homogeneous surface belonging to a certain rational surface are derived which are both necessary and sufficient to ensure that the rational surface is nth-order geometric continuous. This gives up the strong restriction that requires the homogeneous surface to be as smooth as the rational surface. Further the conditions for the rectangular rational Bézier patches are developed, and some simple and practical sufficient conditions are presented which might give a valid means for the construction of GC^n connecting surfaces.

Keywords: Rational surfaces; Bézier patches; Geometric continuity; Total differential vectors; Connecting functions; Parameter transformation

1. Introduction

In recent years geometric continuity between parametric surfaces has been receiving considerable attention in the field of computer aided geometric design (CAGD) (Barnhill, 1985). There was much research done on the geometric continuity condition and its applications. Most of the work concerns the sufficient conditions of tangent plane continuity or curvature continuity for Bézier surfaces based on certain simplifying assumptions (Boehm, 1988; Farin, 1982; Kahmann, 1983; Veron et al., 1976).

One effort to generalize this is the study of necessary and sufficient conditions for tangent plane continuity (Liu, 1986, Liu and Hoschek, 1989). Degen (1990)

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deduced explicit representations for GC^1 and GC^2 continuity between adjacent Bézier patches. However, for some special applications such as finite element analysis and ship hull design, higher orders of geometric continuity are needed. Therefore another generalization is the work on the geometric continuity of order n. Hahn (1989) described the characterization of geometric continuity in terms of diffeomorphism. Liang (1990) gave a theoretical foundation for the geometric continuity of arbitrary order.

In geometric modeling, rational surfaces, such as rational Bézier surfaces, are widely used. This is due in part to the fact that they possess many nice properties, one of which is the capability of describing exactly the conic surfaces which are commonly used in engineering. Nonetheless, very little previous work has been done on the rational surface of geometric continuity (Vinacua and Brunet, 1989; DeRose, 1990; Liu, 1990; Zheng et al., 1992). In general, the geometric continuity of rational surfaces is often ensured by requiring the associated homogeneous surfaces to possess the same continuity. However, it is only sufficient. DeRose and Liu, respectively, presented a system of necessary and sufficient conditions to ensure tangent plane continuity (DeRose, 1990; Liu, 1990). (Zheng et al., 1992) discussed curvature continuity between rational Bézier patches and its solutions. In this paper, we will derive the necessary and sufficient conditions of nth-order geometric continuity for the rational surface which are represented by its associated homogeneous surface and the so-called connecting functions. Then the conditions for rational Bézier patches and smooth connection are further analyzed. The main idea is using homogeneous coordinates for rational surfaces to make derivations simple, as for rational Bézier curves in (Degen, 1988; Hohmeyer and Barsky, 1989).

2. Preliminaries

This section introduces some terms used in the rest of the paper.

Let E^m be the Euclidean space of dimension m, and Δ be a region in E^2. A C^n surface of E^m is defined as an n-times continuously differentiable mapping $\boldsymbol{r}: \Delta \to E^m$, and is expressed by

$$\boldsymbol{r} = \boldsymbol{r}(u, v) \in C^n(\Delta). \tag{2.1}$$

In this paper, a surface $\boldsymbol{r}$ of E^3 is always assumed to be regular, i.e.

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \neq 0. \tag{2.2}$$

Assume that $\boldsymbol{r}(u, v)$ and $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ are two C^n surfaces of E^3, and curve CB is the intersection curve of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ which is also C^n. We call CB the common boundary curve of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$. CB responds to edge $E_1(s) = \{u(s), v(s)\}$ in domain Δ_1 of $\boldsymbol{r}$ and edge $E_2(s) = \{\bar{u}(s), \bar{v}(s)\}$ in domain Δ_2 of $\bar{\boldsymbol{r}}$ respectively.

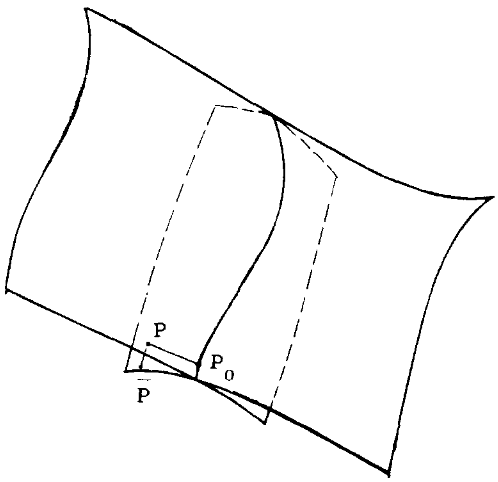

Fig. 1. Geometric interpretation of geometric continuity.

Definition 1. Let $\boldsymbol{r}(u, v)$ and $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ be two surfaces possessing a common boundary curve CB. If there exists a C^n-diffeomorphism φ:

$$\begin{cases} \bar{u} = \bar{u}(u, v) \\ \bar{v} = \bar{v}(u, v) \end{cases} \in C^n \tag{2.3}$$

such that

$$\left.\frac{\partial^s \boldsymbol{r}}{\partial u^i \partial v^{s-i}}\right|_{CB} = \left.\frac{\partial^s (\bar{\boldsymbol{r}} \circ \varphi)}{\partial u^i \partial v^{s-i}}\right|_{CB}, \quad i = 0, \ldots, s, \; s = 0, \ldots, n$$

then $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are said to meet along CB with geometric continuity of order n or GC^n. Meanwhile the diffeomorphism φ is called the parameter transformation of GC^n between $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$.

The definition has an intuitive geometric meaning (see Fig. 1). It expresses that if surfaces $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ meet along CB with GC^n, then there exists a transformation (2.3) so that in the neighbourhood of every point $\boldsymbol{P}_0$ along CB, any point $\boldsymbol{P}$ in surface $\boldsymbol{r}$ (or $\bar{\boldsymbol{r}}$) and its corresponding point $\bar{\boldsymbol{P}}$ in surface $\bar{\boldsymbol{r}}$ (or $\boldsymbol{r}$) under the transformation satisfy

$$\lim_{P \to P_0} \frac{\| \boldsymbol{P}\bar{\boldsymbol{P}} \|}{\| \boldsymbol{P}_0 \boldsymbol{P} \|^n} = 0,$$

where order n is the measure of contact between $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$.

By differential geometry, it is possible to show that this definition accords with tangent plane continuity for $n = 1$ and curvature continuity for $n = 2$.

Definition 2. The first-order total differential vector of surface $\boldsymbol{r}$ is defined by $\mathrm{d}\boldsymbol{r} = \boldsymbol{r}_u\,\mathrm{d}u + \boldsymbol{r}_v\,\mathrm{d}v$. The total differential vectors of higher order are defined recursively by $\mathrm{d}^0\boldsymbol{r} = \boldsymbol{r}$, and $\mathrm{d}^k\boldsymbol{r} = \mathrm{d}(\mathrm{d}^{k-1}\boldsymbol{r})$.

Lemma 1. *We have* $\mathrm{d}^0\boldsymbol{r} = \boldsymbol{r}$, *and*

$$\mathrm{d}^s\boldsymbol{r} = \sum_{k=1}^{s} \sum_{r_1+\cdots+r_k=s} \sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k} \cdot \mathrm{d}^{r_1}u \cdots \mathrm{d}^{r_h}u\ \mathrm{d}^{r_{h+1}}v \cdots \mathrm{d}^{r_k}v \cdot \frac{\partial^k \boldsymbol{r}}{\partial u^h \partial v^{k-h}},$$

$$s \geqslant 1, \tag{2.4}$$

where r_i are positive integers, and

$$A^{skh}_{r_1\ldots r_k} = \frac{s!}{h!(k-h)!r_1!\cdots r_k!}.$$

Proof. Formula (2.4) is obviously true for $s = 0$, 1. Assume it holds for s ($s \geqslant 1$), then

$$\begin{aligned}
\mathrm{d}^{s+1}\boldsymbol{r} &= \mathrm{d}(\mathrm{d}^s\boldsymbol{r}) \\
&= \sum_{k=1}^{s} \sum_{r_1+\cdots+r_k=s} \sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k} \cdot \mathrm{d}^{r_1}u \cdots \mathrm{d}^{r_k}v \\
&\quad \cdot \left(\frac{\partial^{k+1}\boldsymbol{r}}{\partial u^{h+1}\partial v^{k-h}}\mathrm{d}u + \frac{\partial^{k+1}\boldsymbol{r}}{\partial u^{h}\partial v^{k-h+1}}\mathrm{d}v \right) \\
&\quad + \sum_{k=1}^{s} \sum_{r_1+\cdots+r_k=s} \sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k} \cdot \frac{\partial^k \boldsymbol{r}}{\partial u^h \partial v^{k-h}} \cdot \sum_{i=1}^{k} (\mathrm{d}^{r_1}u \cdots \mathrm{d}^{r_i+1} \cdots \mathrm{d}^{r_k}v) \\
&= \sum_{k=2}^{s+1} \sum_{r_2+\cdots+r_k=s} \sum_{h=1}^{k} h \cdot A^{skh}_{1r_2\ldots r_k} \cdot \mathrm{d}u \cdot \mathrm{d}^{r_2}u \cdots \mathrm{d}^{r_k}v \cdot \frac{\partial^k \boldsymbol{r}}{\partial u^h \partial v^{k-h}} \\
&\quad + \sum_{k=2}^{s+1} \sum_{r_1+\cdots+r_{k-1}=s} \sum_{h=0}^{k-1} (k-h) \cdot A^{skh}_{r_1\ldots r_{k-1}1} \\
&\quad \cdot \mathrm{d}^{r_1}u \cdots \mathrm{d}^{r_{k-1}}v \cdot \mathrm{d}v \cdot \frac{\partial^k \boldsymbol{r}}{\partial u^h \partial v^{k-h}} \\
&\quad + \sum_{k=1}^{s} \sum_{i=1}^{k} \sum_{\substack{r_1+\ldots+r_k=s+1 \\ r_i>1}} \sum_{h=0}^{k} r_i A^{skh}_{r_1\ldots r_k} \cdot \mathrm{d}^{r_1}u \cdots \mathrm{d}^{r_k}v \cdot \frac{\partial^k \boldsymbol{r}}{\partial u^h \partial v^{k-h}} \\
&= \sum_{k=2}^{s+1} \sum_{i=1}^{k} \sum_{\substack{r_1+\cdots+r_k=s+1 \\ r_i=1}} \sum_{h=0}^{k} r_i A^{skh}_{r_1\ldots r_k} \cdot \mathrm{d}^{r_1}u \cdots \mathrm{d}^{r_k}v \cdot \frac{\partial^k \boldsymbol{r}}{\partial u^h \partial v^{k-h}}
\end{aligned}$$

$$+\sum_{k=1}^{s}\sum_{i=1}^{k}\sum_{\substack{r_1+\cdots+r_k=s+1\\ r_i>1}}\sum_{h=0}^{k} r_i A^{skh}_{r_1\ldots r_k}\cdot \mathrm{d}^{r_1}u\cdots \mathrm{d}^{r_k}v\cdot\frac{\partial^k \boldsymbol{r}}{\partial u^h\partial v^{k-h}}$$

$$=\sum_{k=1}^{s+1}\sum_{i=1}^{k}\sum_{r_1+\cdots+r_k=s+1}\sum_{h=0}^{k} r_i A^{skh}_{r_1\ldots r_k}\cdot \mathrm{d}^{r_1}u\cdots \mathrm{d}^{r_k}v\cdot\frac{\partial^k \boldsymbol{r}}{\partial u^h\partial v^{k-h}}$$

$$=(s+1)\cdot\sum_{k=1}^{s+1}\sum_{r_1+\cdots+r_k=s+1}\sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k}\cdot \mathrm{d}^{r_1}u\cdots \mathrm{d}^{r_k}v\cdot\frac{\partial^k \boldsymbol{r}}{\partial u^h\partial v^{k-h}}$$

$$=\sum_{k=1}^{s+1}\sum_{r_1+\cdots+r_k=s+1}\sum_{h=0}^{k} A^{(s+1)kh}_{r_1\ldots r_k}\cdot \mathrm{d}^{r_1}u\cdots \mathrm{d}^{r_k}v\cdot\frac{\partial^k \boldsymbol{r}}{\partial u^h\partial v^{k-h}}.$$

By induction, the formula (2.4) is obtained. □

By (2.4), the total differential of $\boldsymbol{r}$ to the sth-order is given by the partial derivatives of $\boldsymbol{r}$ and the total differentials of du, dv, all of which up to the sth-order.

Definition 3. Rational parametric surface $\boldsymbol{r}(u, v)$ is defined as

$$\boldsymbol{r}(u,v)=\boldsymbol{R}(u,v)/\omega(u,v),\quad (u,v)\in\Delta \tag{2.5}$$

where $\boldsymbol{R}(u, v)$ is a surface of E^3, $\omega(u, v)$ is a function $\Delta\to E^1$.

The rational surface $\boldsymbol{r}$ can be thought of as the composition of a surface $\boldsymbol{Q}(u, v)$ of E^4 with a projection function p, where

$$p:(\boldsymbol{R},\omega)\to\boldsymbol{R}/\omega, \tag{2.6}$$

$$\boldsymbol{Q}(u,v)=\rho(u,v)(\boldsymbol{R}(u,v),\omega(u,v)). \tag{2.7}$$

We refer to $\boldsymbol{Q}$ as the homogeneous surface associated with $\boldsymbol{r}$ and surface $\boldsymbol{r}$ as the projection of $\boldsymbol{Q}$. In general, we take $\rho\equiv 1$ and suppose that $\boldsymbol{Q}(u, v)$ is C^n.

Lemma 2. *If the rational surface $\boldsymbol{r}$ of E^3 is regular at point (u, v), then $\boldsymbol{Q}(u, v)$, $\boldsymbol{Q}_u(u, v)$ and $\boldsymbol{Q}_v(u, v)$ are linearly independent, i.e., there exists at least one i $(1\leqslant i\leqslant 3)$ such that $\langle \boldsymbol{Q}, \boldsymbol{Q}_u, \boldsymbol{Q}_v\rangle_i\neq 0$ at (u, v), where the notation $\langle\cdot,\cdot,\cdot\rangle_i$ means*

$$\langle \boldsymbol{R},\boldsymbol{S},\boldsymbol{T}\rangle_i=(-1)^{i+1}\det\begin{bmatrix}R_{i+1} & S_{i+1} & T_{i+1}\\ R_{i+2} & S_{i+2} & T_{i+2}\\ R_{i+3} & S_{i+3} & T_{i+3}\end{bmatrix},\quad i=1,2,3,4. \tag{2.8}$$

$\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$ are arbitrary vectors in E^4. R_i denotes the ith component of $\boldsymbol{R}$. If $i>4$, $R_i=R_{\mathrm{mod}(i-1,4)+1}$.

3. GC^n necessary and sufficient condition for rational surfaces

3.1. A general form of rational GC^n conditions

As pointed out in (DeRose, 1990), rational functions are much harder to differentiate than polynomials. Since the geometric continuity of higher order involved the higher-order partial derivatives, it is difficult to deal with the geometric continuity of arbitrary order for rational surfaces. To do that, we attempt to use the homogeneous surface instead of the rational surface itself as in (Degen, 1988; Hohmeyer and Barsky, 1989) with respect to rational Bézier curves. Our objective is to determine the exact conditions imposed on the associated homogeneous surfaces to ensure that two rational surfaces are GC^n continuous. This is given in the following theorem which is apparent from projective differential geometry (Bol, 1950).

Theorem 1. *Let $\boldsymbol{r}(u, v)$ and $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ be two regular rational surfaces which possess a common boundary curve CB. Then $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ meet with GC^n if and only if there exist a C^n-diffeomorphism $\varphi:(u, v) \to (\bar{u}, \bar{v})$ and a scalar function $e(u, v)$ such that*

$$\mathrm{d}^k(\bar{\boldsymbol{Q}} \circ \varphi)|_{CB} = \mathrm{d}^k(e(u, v)\boldsymbol{Q}(u, v))|_{CB}, \quad k = 0, \ldots, n \tag{3.1}$$

holds for any $\mathrm{d}^j u$, $\mathrm{d}^j v$, where $\boldsymbol{Q}$ and $\bar{\boldsymbol{Q}}$ are the homogeneous surfaces associated with $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ respectively.

In particular, if we let $e(u, v) \equiv 1$, then condition (3.1) becomes $\mathrm{d}^k(\bar{\boldsymbol{Q}} \circ \varphi)|_{CB} = \mathrm{d}^k\boldsymbol{Q}|_{CB}$, i.e. $\bar{\boldsymbol{Q}}(\bar{u}, \bar{v})$ and $\boldsymbol{Q}(u, v)$ are GC^n continuous along CB. This shows that if the homogeneous surfaces are GC^n then the rational surfaces themselves will have the same continuity, but the converse is not true.

Theorem 1 just gives a descriptive condition. In the following we derive the explicit geometric continuity condition and show how to determine its solution.

Theorem 2. *Suppose that rational surfaces $\boldsymbol{r}(u, v)$ and $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ are regular. Then $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are GC^n along their common boundary curve CB: $\boldsymbol{r}(E_1(s)) = \bar{\boldsymbol{r}}(E_2(s))$ if and only if, for every set of $\bar{p}_i(s)$, $\bar{q}_i(s)$, $i = 1, \ldots, n$, satisfying*

$$\begin{vmatrix} \bar{p}_1 & \bar{q}_1 \\ \dfrac{\partial \bar{u}}{\partial s} & \dfrac{\partial \bar{v}}{\partial s} \end{vmatrix} \neq 0,$$

there exist scalar functions $c_0(s)$, $c_i(s)$, $p_i(s)$ and $q_i(s)$, $i = 1, \ldots, n$, satisfying

$$\begin{vmatrix} p_1 & q_1 \\ \dfrac{\partial u}{\partial s} & \dfrac{\partial v}{\partial s} \end{vmatrix} \neq 0,$$

such that

$$\bar{Q}(\bar{u},\bar{v})|_{CB}=c_0(s)Q(u,v)|_{CB},$$

$$\sum_{k=1}^{m}\sum_{r_1+\cdots+r_k=m}\sum_{h=0}^{k}A^{mkh}_{r_1\ldots r_k}\cdot\bar{p}_{r_1}\cdots\bar{p}_{r_h}\bar{q}_{r_{h+1}}\cdots\bar{q}_{r_k}\cdot\left.\frac{\partial^k\bar{Q}}{\partial\bar{u}^h\partial\bar{v}^{k-h}}\right|_{CB}$$

$$=c_mQ|_{CB}+\sum_{j=1}^{m}\binom{m}{j}c_{m-j}(s)\sum_{k=1}^{j}\sum_{r_1+\cdots+r_k=j}\sum_{h=0}^{k}A^{jkh}_{r_1\ldots r_k}$$

$$\cdot p_{r_1}\cdots p_{r_h}q_{r_{h+1}}\cdots q_{r_k}\cdot\left.\frac{\partial^kQ}{\partial u^h\partial v^{k-h}}\right|_{CB},\qquad m=1,\ldots,n. \tag{3.2}$$

Proof. With the given $\bar{p}_i(s)$, $\bar{q}_i(s)$, we construct a parameter transformation:

$$\psi_1:\begin{cases}\bar{u}=\bar{u}(s)+\sum_{i=1}^{n}\bar{p}_it^i/i!,\\ \bar{v}=\bar{v}(s)+\sum_{i=1}^{n}\bar{q}_it^i/i!.\end{cases} \tag{3.3}$$

The transformation ψ_1 is non-singular because of the condition

$$\begin{vmatrix}\bar{p}_1 & \bar{q}_1\\ \dfrac{\partial\bar{u}}{\partial s} & \dfrac{\partial\bar{v}}{\partial s}\end{vmatrix}\neq 0.$$

If $\bar{r}$ meets r with GC^n along CB, by Theorem 1, there exist a parameter transformation φ: $\bar{u}=\bar{u}(u,v)$, $\bar{v}=\bar{v}(u,v)$ and a function $e(u,v)$ such that $\mathrm{d}^m(\bar{Q}\circ\varphi)|_{CB}=\mathrm{d}^m(e\cdot Q)|_{CB}$. This gives us that

$$\bar{Q}(\bar{u},\bar{v})|_{CB}=c_0(s)Q(u,v)|_{CB},\qquad c_0(s)=e(u,v)|_{CB},$$

$$\mathrm{d}^m(\bar{Q}\circ\varphi)|_{CB}=\sum_{j=0}^{m}\binom{m}{j}\mathrm{d}^{m-j}e|_{CB}\,\mathrm{d}^jQ|_{CB},\quad m=1,\ldots,n.$$

Composing the transformation $\tau=\varphi^{-1}\circ\psi_1:(s,t)\to(u,v)$, which is also non-singular, we have

$$\left.\frac{\partial^m(\bar{Q}\circ\psi_1)}{\partial t^m}\right|_{CB}=\sum_{j=0}^{m}\binom{m}{j}\left.\frac{\partial^{m-j}(e\circ\tau)}{\partial t^{m-j}}\right|_{CB}\left.\frac{\partial^j(Q\circ\tau)}{\partial t^j}\right|_{CB}. \tag{3.4}$$

Let $c_i(s)=\partial^i(e\circ\tau)/\partial t^i|_{CB}$. In terms of Lemma 1, we obtain

$$\left.\frac{\partial^m(\bar{Q}\circ\psi_1)}{\partial t^m}\right|_{CB}=\sum_{k=1}^{m}\sum_{r_1+\cdots+r_k=m}\sum_{h=0}^{k}A^{mkh}_{r_1\ldots r_k}\cdot\bar{p}_{r_1}\cdots\bar{p}_{r_h}\cdot\bar{q}_{r_{h+1}}\cdots\bar{q}_{r_k}$$

$$\cdot\left.\frac{\partial^k\bar{Q}}{\partial\bar{u}^h\partial\bar{v}^{k-h}}\right|_{CB},\qquad m=1,\ldots,n. \tag{3.5}$$

If $p_i = \partial^i u/\partial t^i|_{CB}$, $q_i = \partial^i v/\partial t^i|_{CB}$ are set, then

$$\begin{vmatrix} p_1 & q_1 \\ \dfrac{\partial u}{\partial s} & \dfrac{\partial v}{\partial s} \end{vmatrix} \neq 0.$$

Therefore (3.2) will be arrived at by substituting (3.5) into (3.4).

We now turn to prove the sufficiency. Define two non-singular transformations ψ_1 and ψ_2: ψ_1 is expressed by (3.3), and ψ_2 is expressed by

$$\psi_2: \begin{cases} u = u(s) + \sum_{i=1}^{n} p_i t^i/i!, \\ v = v(s) + \sum_{i=1}^{n} q_i t^i/i!. \end{cases} \tag{3.6}$$

For $c_i(s)$, it is easy to construct a function $e(u, v)$ such that

$$\left.\frac{\partial^i(e \circ \psi_2)}{\partial t^i}\right|_{CB} = c_i(s). \tag{3.7}$$

Thus from condition (3.2) and expression (3.5),

$$\begin{aligned} \left.\frac{\partial^m(\overline{Q} \circ \psi_1)}{\partial t^m}\right|_{CB} &= \sum_{j=0}^{m} \binom{m}{j} \left.\frac{\partial^{m-j}(e \circ \psi_2)}{\partial t^{m-j}}\right|_{CB} \left.\frac{\partial^j(Q \circ \psi_2)}{\partial t^j}\right|_{CB} \\ &= \left.\frac{\partial^m((eQ) \circ \psi_2)}{\partial t^m}\right|_{CB}. \end{aligned}$$

Taking the partial derivatives of both sides of the above expression with respect to s, yields

$$\left.\frac{\partial^{m+j}(\overline{Q} \circ \psi_1)}{\partial t^m \partial s^j}\right|_{CB} = \left.\frac{\partial^{m+j}((eQ) \circ \psi_2)}{\partial t^m \partial s^j}\right|_{CB}.$$

As m and j are arbitrary, we have

$$\mathrm{d}^m(\overline{Q} \circ \psi_1)|_{CB} = \mathrm{d}^m((e \cdot Q) \circ \psi_2)|_{CB}.$$

Therefore $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are GC^n continuous along CB, and $\psi_1 \circ \psi_2^{-1}$ is the parameter transformation of GC^n between $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$. □

From the above proof, we can also obtain the following result.

Theorem 3. *Suppose that the regular rational surfaces* $\boldsymbol{r}(u, v)$ *and* $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ *share a common boundary curve* CB: $\boldsymbol{r}(E_1(s)) = \bar{\boldsymbol{r}}(E_2(s))$. *If equations* (3.2) *hold for one system of functions* $\bar{p}_i(s)$, $\bar{q}_i(s)$, $i = 1, \ldots, n$, *satisfying*

$$\begin{vmatrix} \bar{p}_1 & \bar{q}_1 \\ \dfrac{\partial u}{\partial s} & \dfrac{\partial v}{\partial s} \end{vmatrix} \neq 0,$$

then they hold for any other system of $\bar{p}_i(s)$, $\bar{q}_i(s)$, $i = 1, \ldots, n$, satisfying

$$\begin{vmatrix} \bar{p}_1 & \bar{q}_1 \\ \dfrac{\partial u}{\partial s} & \dfrac{\partial v}{\partial s} \end{vmatrix} \neq 0,$$

with new scalar functions $c_i(s)$, $p_i(s)$ and $q_i(s)$ obtained by the corresponding parameter transformation.

Generally, the scalar functions $c_i(s)$, $p_i(s)$ and $q_i(s)$ are called the connecting functions of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$. They are the solutions of differential equation (3.2). In order to determine them, we rewrite (3.2) as

$$\bar{\boldsymbol{Q}}\,|_{CB} = c_0(s)\boldsymbol{Q}\,|_{CB},$$
$$c_m\boldsymbol{Q}\,|_{CB} + c_0 p_m \boldsymbol{Q}_u\,|_{CB} + c_0 q_m \boldsymbol{Q}_v\,|_{CB} = F^m$$

where

$$F^m = \sum_{k=1}^{m} \sum_{r_1+\cdots+r_k=m} \sum_{h=0}^{k} A^{mkh}_{r_1\ldots r_k} \cdot \bar{p}_{r_1} \cdots \bar{p}_{r_h} \bar{q}_{r_{h+1}} \cdots \bar{q}_{r_k} \cdot \left.\frac{\partial^k \bar{Q}}{\partial \bar{u}^h \partial \bar{v}^{k-h}}\right|_{CB}$$
$$- c_0(s) \sum_{k=2}^{m} \sum_{r_1+\cdots+r_k=m} \sum_{h=0}^{k} A^{mkh}_{r_1\ldots r_k} \cdot p_{r_1} \cdots p_{r_h} q_{r_{h+1}} \cdots q_{r_k}$$
$$\cdot \left.\frac{\partial^k Q}{\partial u^h \partial v^{k-h}}\right|_{CB} - \sum_{j=1}^{m-1} \binom{m}{j} c_{m-j}(s) \sum_{k=1}^{j} \sum_{r_1+\cdots+r_k=j} \sum_{h=0}^{k} A^{jkh}_{r_1\ldots r_k}$$
$$\cdot p_{r_1} \cdots p_{r_h} q_{r_{h+1}} \cdots q_{r_k} \cdot \left.\frac{\partial^k Q}{\partial u^h \partial v^{k-h}}\right|_{CB},$$
$$m = 1, \ldots, n. \tag{3.8}$$

Then F^m are functions of s, $\bar{p}_i$, $\bar{q}_i$, p_j and q_j ($i = 1, \ldots, m$; $j = 1, \ldots, m-1$). According to Lemma 2, $\boldsymbol{Q}$, $\boldsymbol{Q}_u$, $\boldsymbol{Q}_v$ are independent. Thus we can verify whether equations (3.8) have solutions or not by recursively solving a set of linear equations. And further the solutions c_i, p_i, q_i can be found if they satisfy (3.8). So far the problem of checking the geometric continuity conditions can be simplified to solve a system of linear equations.

Example. Consider two surfaces $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ given by the equations:

$$\boldsymbol{r} = \frac{1}{1+v}\{u,\ v,\ 0\},$$

$$\bar{\boldsymbol{r}} = \frac{1}{1 + \bar{u} + \frac{1}{2}\bar{v} + (1+2\bar{u})(\bar{u} - \frac{1}{2}\bar{v})^3}$$
$$\times \Big\{\bar{u} - \tfrac{1}{2}\bar{v} + 1 + (\bar{u} - \tfrac{1}{2}\bar{v})^3,\ \bar{u} + \tfrac{1}{2}\bar{v} + (\bar{u} - \tfrac{1}{2}\bar{v})^3(\bar{u} + \tfrac{1}{2}\bar{v} - \sin(\bar{u} - \tfrac{1}{2}\bar{v})),$$
$$5(\bar{u} + \tfrac{1}{2}\bar{v})(\bar{u} - \tfrac{1}{2}\bar{v})^5\Big\}.$$

It can be verified that they are connected with GC^3 continuity along the common boundary CB: $E_1(s)=\{1, s\}$ and $E_2(s)=\{s/2, s\}$. If we choose $\bar{p}_1=0.5$, $\bar{q}_1=-1$ and $\bar{p}_i=\bar{q}_i=0$ for $i>1$, then the connecting functions are $c_0=1$, $c_3=6$, $p_1=1$, $c_i=p_j=q_k=0$ for $j>1$, $k>0$ and $i\neq 0, 3$. After the parameter transformations:

$$\psi_1: \begin{cases} \bar{u}=\frac{1}{2}(s+t) \\ \bar{v}=s-t \end{cases} \quad \text{and} \quad \psi_2: \begin{cases} u=t+1 \\ v=s \end{cases},$$

surfaces $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ become

$$\boldsymbol{r}\circ\psi_2=\frac{1}{1+s}\{t+1,\ s,\ 0\}$$

$$\bar{\boldsymbol{r}}\circ\psi_1=\frac{1}{1+s+(1+s+t)t^3}\{1+t+t^3,\ s+t^3(s-\sin t),\ 5st^5\}.$$

We have

$$\boldsymbol{r}\circ\psi_2\,|_{t=0}=\bar{\boldsymbol{r}}\circ\psi_1\,|_{t=0}=\frac{1}{1+s}\{1,\ s,\ 0\},$$

$$\left.\frac{\partial(\boldsymbol{r}\circ\psi_2)}{\partial t}\right|_{t=0}=\left.\frac{\partial(\bar{\boldsymbol{r}}\circ\psi_1)}{\partial t}\right|_{t=0}$$

$$=\left.\frac{\partial^k(\boldsymbol{r}\circ\psi_2)}{\partial t^k}\right|_{t=0}=\left.\frac{\partial^k(\bar{\boldsymbol{r}}\circ\psi_1)}{\partial t^k}\right|_{t=0}=\{0,\ 0,\ 0\},\quad k=2, 3.$$

Thus $\boldsymbol{r}\circ\psi_2$ and $\bar{\boldsymbol{r}}\circ\psi_1$ are C^3 continuous (see Figs. 2 and 3).

3.2. A simple form of GC^n conditions

Note that there are very few restrictions on the common boundary curve CB in the preceding sections. Here we consider a simple case in which the parametric representations of the common boundary in the domains of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are respectively $u=1$, $v=v$ and $\bar{u}=0$, $\bar{v}=v$. Meanwhile, we take $\bar{p}_1=1$, $\bar{p}_i=\bar{q}_j=0$ $(i=1,\dots,n;\ j=0,\dots,n)$. Clearly, the condition

$$\begin{vmatrix} \bar{p}_1 & \bar{q}_1 \\ \dfrac{\partial\bar{u}}{\partial s} & \dfrac{\partial\bar{v}}{\partial s} \end{vmatrix}\neq 0$$

is preserved. In this case, GC^n conditions of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ can be simplified.

Theorem 4. *Regular rational surfaces $\boldsymbol{r}(u, v)$ and $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ meet with GC^n along their common boundary curve CB: $\bar{\boldsymbol{r}}(0, \bar{v})=\boldsymbol{r}(1, v)$ $(\bar{v}=v)$ if and only if there exist functions $c_0(v)$, $c_i(v)$, $p_i(v)$ and $q_i(v)$ $(p_1\neq 0)$ such that*

$$\left.\frac{\partial^s\bar{Q}}{\partial\bar{u}^s}\right|_{CB}=c_sQ\,|_{CB}+\sum_{j=1}^{s}\binom{s}{j}c_{s-j}(v)\times\sum_{k=1}^{j}\sum_{r_1+\cdots+r_k=j}\sum_{h=0}^{k}A^{jkh}_{r_1\dots r_k}\cdot p_{r_1}\cdots p_{r_h}q_{r_{h+1}}\cdots q_{r_k}\cdot\left.\frac{\partial^kQ}{\partial u^h\partial v^{k-h}}\right|_{CB},\quad s=0,\dots,n. \tag{3.9}$$

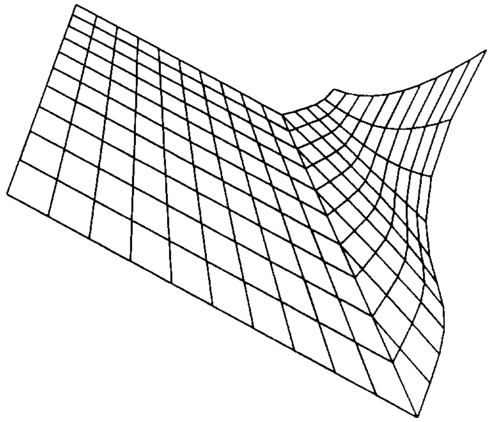

Fig. 2. Isoparametric lines of a surface consisting of patches $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$.

Since parametric continuity is a special case of geometric continuity which requires the transformation (2.3) being an identical mapping, we know $p_1 = 1$, $p_i = q_j = 0$, $i = 2, \ldots, n$; $j = 1, \ldots, n$ from the proof of Theorem 2. Thus we have

Corollary. *Regular rational surfaces* $\boldsymbol{r}(u, v)$ *and* $\bar{\boldsymbol{r}}(\bar{u}, \bar{v})$ *are* C^n *parametric continuous along* CB: $\bar{\boldsymbol{r}}(0, \bar{v}) = \boldsymbol{r}(1, v)$ $(\bar{v} = v)$ *iff there exist scalar functions* $c_i(v)$, $i = 0, \ldots, n$ *such that*

$$\left.\frac{\partial^s \overline{\boldsymbol{Q}}}{\partial \bar{u}^s}\right|_{CB} = c_s \boldsymbol{Q}\,|_{CB} + \sum_{j=1}^{s} \binom{s}{j} c_{s-j}(v) \left.\frac{\partial^j \boldsymbol{Q}}{\partial u^j}\right|_{CB}, \quad s = 0, \ldots, n. \tag{3.10}$$

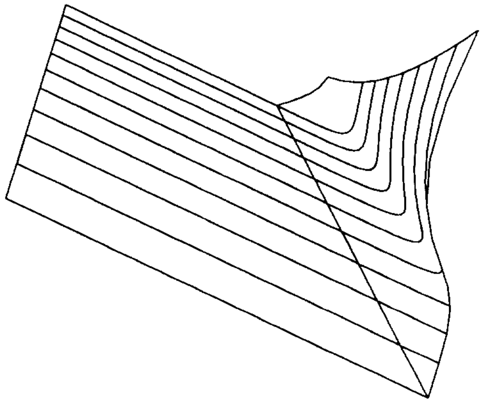

Fig. 3. Cross sections of the same surface.

Now let us give the first few geometric continuity conditions.

$$s=0:\ \bar{Q}|_{CB} = c_0(v)Q|_{CB}, \tag{3.11}$$

$$s=1:\ \bar{Q}_{\bar{u}}|_{CB} = c_1 Q|_{CB} + c_0 p_1 Q_u|_{CB} + c_0 q_1 Q_v|_{CB}, \tag{3.12}$$

$$s=2:\ \bar{Q}_{\bar{u}\bar{u}}|_{CB} = c_2 Q|_{CB} + (2c_1 p_1 + c_0 p_2) Q_u|_{CB} + (2c_1 q_1 + c_0 q_2) Q_v|_{CB} + c_0 p_1^2 Q_{uu}|_{CB} + 2c_0 p_1 q_1 Q_{uv}|_{CB} + c_0 q_1^2 Q_{vv}|_{CB}. \tag{3.13}$$

It can be shown that the above formulas include many analogous conditions occurring in the computer aided geometric design literature as their special cases. In (Liu, 1990), the conditions are (3.11) and (3.12) with $c_0(v) = 1$ which is due to the fact that its position continuity is defined as "C^0 continuity". In this paper, the position continuity is defined as "GC^0 continuity" which belongs to the second definition presented in the "Remark" of (Liu, 1990). Therefore we have obtained more general results.

In fact, if we let $c_0 = 1$ and $c_i = 0$ for $i \geqslant 1$ then rational geometric continuity reduces to simple geometric continuity. On the other hand, if $p_1 = 1$ and $p_i = q_j = 0$ for $i \geqslant 2$ and $j \geqslant 1$, then rational geometric continuity reduces to the rational parametric continuity. Finally, if c_i, p_i and q_i are all specified as above, the rational geometric continuity conditions reduce to simple parametric continuity conditions.

4. GC^n condition for rational polynomial patches

In CAGD applications, rational Bézier surfaces belong to the most widely used surfaces. Other rational polynomial surfaces can be converted into rational Bézier form. In this section we only discuss the geometric continuity for rational Bézier surfaces.

Suppose that the rational Bézier surfaces $\boldsymbol{r}$ of degree $m \times l$ and $\bar{\boldsymbol{r}}$ of degree $\bar{m} \times l$ are expressed in homogeneous coordinates by

$$\boldsymbol{r}:\quad Q(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{l} Q_{ij} B_i^m(u) B_j^l(v), \quad 0 \leqslant u, v \leqslant 1$$

$$\bar{\boldsymbol{r}}:\quad \bar{Q}(\bar{u},\bar{v}) = \sum_{i=0}^{\bar{m}} \sum_{j=0}^{l} \bar{Q}_{ij} B_i^{\bar{m}}(\bar{u}) B_j^l(\bar{v}), \quad 0 \leqslant \bar{u}, \bar{v} \leqslant 1$$

where

$$Q_{ij} = (P_{ij}\omega_{ij}, \omega_{ij}), \qquad \bar{Q}_{ij} = \left(\bar{P}_{ij}\bar{\omega}_{ij}, \bar{\omega}_{ij}\right)$$

P_{ij}, $\bar{P}_{ij}$ are control points, ω_{ij}, $\bar{\omega}_{ij}$ are weights.

$B_i^l(t) = \binom{l}{i} t^i (1-t)^{l-i}$ are Bernstein polynomials of degree l.

In general, if Q_{ij} are found, then P_{ij}, ω_{ij} will be determined.

4.1. GC^n necessary and sufficient condition

The GC^n necessary and sufficient conditions for general rational surfaces have been given in Section 3. Here we concern ourselves with the rational Bézier surfaces and show that their connecting functions are rational polynomials.

Theorem 5. *A necessary and sufficient condition for GC^n continuity between two adjacent rational Bézier patches $\boldsymbol{r}$ of degree $m \times l$ and $\bar{\boldsymbol{r}}$ of degree $\bar{m} \times l$ along CB: $\boldsymbol{r}(1, v) = \bar{\boldsymbol{r}}(0, \bar{v})$ $(0 \leqslant v = \bar{v} \leqslant 1)$ is that*

$$G(v)\bar{\boldsymbol{Q}} = H(v)\boldsymbol{Q}, \qquad G(v)H(v) \neq 0,$$

$$D^{2s-1}H^{s-1}\frac{\partial^s \bar{\boldsymbol{Q}}}{\partial \bar{u}^s} = \underline{c}_s\boldsymbol{Q} + \sum_{k=1}^{s} \sum_{r_1+\cdots+r_k=s} \sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k} \cdot G^{k-1}D^{k-1}\underline{p}_{r_1}\cdots \underline{q}_{r_k} \cdot \frac{\partial^k Q}{\partial u^h \partial v^{k-h}} + \sum_{d=1}^{s-1}\binom{s}{d}\underline{c}_{s-d}(v)\sum_{k=1}^{d}\sum_{r_1+\cdots+r_k=d}\sum_{h=0}^{k} A^{dkh}_{r_1\ldots r_k} \cdot G^kD^k\underline{p}_{r_1}\cdots \underline{p}_{r_h}\underline{q}_{r_{h+1}}\cdots \underline{q}_{r_k} \cdot \frac{\partial^k Q}{\partial u^h \partial v^{k-h}}$$

$$s = 1, \ldots, n \tag{4.1}$$

hold at every point of CB, where G, H, D, $\underline{c}_i$, $\underline{p}_i$, $\underline{q}_i$ are all polynomials of v, $D(v) \neq 0$, $\underline{p}_1 \neq 0$, their degrees are as follows:

	G, H	D	$\underline{c}_i, \underline{p}_i$	$\underline{q}_i$
degrees $\leqslant$	l	$3l-1$	$(2i-1)(3l-1)$ $+(i-1)l$	$(2i-1)(3l-1)$ $+(i-1)l+1$

Proof. First, assume that $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are GC^n continuous. Then by Theorem 4, there exist $c_0(v)$, $c_i(v)$, $p_i(v)$ and $q_i(v)$, $i = 1, \ldots, n$, such that (3.9) holds.

When $n = 0$, $\bar{\boldsymbol{Q}}|_{CB} = c_0\boldsymbol{Q}|_{CB}$ implies

$$\bar{\boldsymbol{R}}|_{CB} = c_0\boldsymbol{R}|_{CB} \quad \text{and} \quad \bar{\omega}|_{CB} = c_0\omega|_{CB}.$$

If $H = \bar{\omega}|_{CB}$, $G = \omega|_{CB}$ are set, then $G\bar{\boldsymbol{Q}} = H\boldsymbol{Q}$, and $c_0 = H/G$.

When $n = 1$,

$$\bar{\boldsymbol{Q}}_{\bar{u}}|_{CB} = c_1\boldsymbol{Q}|_{CB} + c_0p_1\boldsymbol{Q}_u|_{CB} + c_0q_1\boldsymbol{Q}_v|_{CB}.$$

In terms of Lemma 2, there exists i $(1 \leqslant i \leqslant 3)$ such that $\langle \boldsymbol{Q}, \boldsymbol{Q}_u, \boldsymbol{Q}_v\rangle_i|_{u=1,v=1} \neq 0$. Taking

$$D(v) = \langle \boldsymbol{Q}, \boldsymbol{Q}_u, \boldsymbol{Q}_v\rangle_i|_{CB}, \qquad \underline{c}_1 = \langle \bar{\boldsymbol{Q}}_{\bar{u}}, \boldsymbol{Q}_u, \boldsymbol{Q}_v\rangle_i|_{CB},$$

$$\underline{p}_1(v) = \langle \boldsymbol{Q}, \bar{\boldsymbol{Q}}_{\bar{u}}, \boldsymbol{Q}_v\rangle_i|_{CB}, \qquad \underline{q}_1(v) = \langle \boldsymbol{Q}, \boldsymbol{Q}_u, \bar{\boldsymbol{Q}}_{\bar{u}}\rangle_i|_{CB} \tag{4.2}$$

gives

$$D\bar{\boldsymbol{Q}}_{\bar{u}}|_{CB} = \underline{c}_1\boldsymbol{Q}|_{CB} + \underline{p}_1\boldsymbol{Q}_u|_{CB} + \underline{q}_1\boldsymbol{Q}_v|_{CB}$$

and

$$\underline{c}_1 = Dc_1, \qquad \underline{p}_1 = Dc_0 p_1, \qquad \underline{q}_1 = Dc_0 q_1.$$

Obviously, the degrees of D, $\underline{c}_1$, $\underline{p}_1$, $\underline{q}_1$ are not larger than $3l-1$, $3l-1$, $3l-1$ and $3l$ respectively. Now suppose that the necessity has been proved for $n \leqslant s-1$. Then when $n = s$,

$$\left.\frac{\partial^s \overline{Q}}{\partial \overline{u}^s}\right|_{CB} = c_s Q \mid_{CB} + c_0 p_s Q_u \mid_{CB} + c_0 q_s Q_v \mid_{CB} + \delta \tag{4.3}$$

where

$$\delta = \sum_{k=2}^{s} \sum_{r_1+\cdots+r_k=s} \sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k} c_0 \cdot p_{r_1} \cdots q_{r_k} \cdot \left.\frac{\partial^k Q}{\partial u^h \partial v^{k-h}}\right|_{CB}$$
$$+ \sum_{d=1}^{s-1} \binom{s}{d} c_{s-d}(v) \sum_{k=1}^{d} \sum_{r_1+\cdots+r_k=d} \sum_{h=0}^{k} A^{dkh}_{r_1\ldots r_k} \cdot p_{r_1} \cdots p_{r_h} q_{r_{h+1}} \cdots q_{r_k}$$
$$\cdot \left.\frac{\partial^k Q}{\partial u^h \partial v^{k-h}}\right|_{CB}.$$

Let

$$\overline{\delta} = D^{2s-2} H^{s-1} \left(\left.\frac{\partial^s \overline{Q}}{\partial \overline{u}^s}\right|_{CB} - \delta \right)$$
$$= D^{2s-2} H^{s-1} \left(c_s Q \mid_{CB} + c_0 p_s Q_u \mid_{CB} + c_0 q_s Q_v \mid_{CB} \right).$$

We have

$$\overline{\delta} = D^{2s-2} H^{s-1} \frac{\partial^s \overline{Q}}{\partial \overline{u}^s} - \sum_{k=2}^{s} \sum_{r_1+\cdots+r_k=s} \sum_{h=0}^{k} A^{skh}_{r_1\ldots r_k} \cdot G^{k-1} D^{k-2} \underline{p}_{r_1} \cdots \underline{q}_{r_k}$$
$$\cdot \left.\frac{\partial^k Q}{\partial u^h \partial v^{k-h}}\right|_{CB} - \sum_{d=1}^{s-1} \binom{s}{d} \underline{c}_{s-d}(v) \sum_{k=1}^{d} \sum_{r_1+\cdots+r_k=d} \sum_{h=0}^{k} A^{dkh}_{r_1\ldots r_k}$$
$$\cdot G^k D^{k-1} \underline{p}_{r_1} \cdots \underline{p}_{r_h} \underline{q}_{r_{h+1}} \cdots \underline{q}_{r_k} \cdot \left.\frac{\partial^k Q}{\partial u^h \partial v^{k-h}}\right|_{CB}.$$

It is easy to show that $\overline{\delta}$ is a curve of degree not larger than $s(7l-2)-6l+2$. If we let

$$\underline{c}_s = \langle \overline{\delta}, Q_u, Q_v \rangle_i \mid_{CB}, \qquad \underline{p}_s(v) = \langle Q, \overline{\delta}, Q_v \rangle_i \mid_{CB},$$
$$\underline{q}_s(v) = \langle Q, Q_u, \overline{\delta} \rangle_i \mid_{CB} \tag{4.4}$$

then

$$\underline{c}_s = c_s D^{2s-1} H^{s-1}, \qquad \underline{p}_s = c_0 p_s D^{2s-1} H^{s-1}, \qquad \underline{q}_s = c_0 \hat{q}_s D^{2s-1} H^{s-1}, \tag{4.5}$$

and

$$D \cdot \bar{\delta} = \underline{c}_s \boldsymbol{Q}|_{CB} + \underline{p}_s \boldsymbol{Q}_u|_{CB} + \underline{q}_s \boldsymbol{Q}_v|_{CB}.$$

The degrees of $\underline{c}_s$, $\underline{p}_s$, $\underline{q}_s$ are not larger than $(2s-1)(3l-1)+(s-1)l$, $(2s-1)(3l-1)+(s-1)l$ and $(2s-1)(3l-1)+(s-1)l+1$ respectively, and the second equation of (4.1) holds for $n=s$. Thus the necessity is fulfilled by induction.

On the other hand, take $c_0(v)=H(v)/G(v)$, and assume that $D(v)$ has j zero points $v_1,\dots,v_j$ in [0, 1]. From Eq. (4.5), we know that $v_1,\dots,v_j$ are also the roots of $\underline{c}_s$, $\underline{p}_s$, $\underline{q}_s$ with multiplicity $2s-1$. These zero factors can be eliminated by

$$\bar{D}=D/VJ, \qquad \bar{c}_i=\underline{c}_i/VJ^{2i-1}, \qquad \bar{p}_i=\underline{p}_i/VJ^{2i-1},$$

$$\bar{q}_i=\underline{q}_i/VJ^{2i-1}, \quad i=1,\dots,n,$$

where $VJ=(v-v_1)\cdots(v-v_j)$.

It is clear that $\bar{D}\neq 0$, $v\in[0,1]$ and (4.1) still holds if D, $\underline{c}_i$, $\underline{p}_i$, $\underline{q}_i$ are replaced by $\bar{D}$, $\bar{c}_i$, $\bar{p}_i$ and $\bar{q}_i$ respectively. Thus the sufficiency follows immediately if we take

$$c_i=\bar{c}_i/(\bar{D}^{2i-1}H^{i-1}), \qquad p_i=G\bar{p}_i/(\bar{D}^{2i-1}H^i),$$

$$q_i=G\bar{q}_i/(\bar{D}^{2i-1}H^i). \qquad \square$$

Note that there is no loss of generality by assuming the degree of $\boldsymbol{r}$ with respect to v to be equal to that of $\bar{\boldsymbol{r}}$ with respect to $\bar{v}$ in Theorem 5 because of the elevation formula.

We now deal with two special cases.

Case 1. If the position continuity of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ along CB is defined as "C^0 continuity" of (Liu, 1990), i.e. $\bar{\boldsymbol{Q}}|_{CB}=\boldsymbol{Q}|_{CB}$, then $c_0=1$. Without loss of generality, taking $H=G=1$, conditions (4.1) become

$$\bar{\boldsymbol{Q}}|_{CB}=\boldsymbol{Q}|_{CB},$$

$$\begin{aligned} D^{2s-1}\left.\frac{\partial^s\bar{\boldsymbol{Q}}}{\partial\bar{u}^s}\right|_{CB} &= \underline{c}_s\boldsymbol{Q}|_{CB} + \sum_{k=1}^{s}\sum_{r_1+\cdots+r_k=s}\sum_{h=0}^{k}A^{skh}_{r_1\dots r_k} \cdot D^{k-1}\underline{p}_{r_1}\cdots\underline{q}_{r_k}\cdot\left.\frac{\partial^k\boldsymbol{Q}}{\partial u^h\partial v^{k-h}}\right|_{CB} \\ &\quad + \sum_{d=1}^{s-1}\binom{s}{d}\underline{c}_{s-d}(v)\sum_{k=1}^{d}\sum_{r_1+\cdots+r_k=d}\sum_{h=0}^{k}A^{dkh}_{r_1\dots r_k} \\ &\quad \cdot D^k\underline{p}_{r_1}\cdots\underline{p}_{r_h}\underline{q}_{r_{h+1}}\cdots\underline{q}_{r_k}\cdot\left.\frac{\partial^k\boldsymbol{Q}}{\partial u^h\partial v^{k-h}}\right|_{CB}, \quad s=1,\dots,n. \end{aligned} \tag{4.6}$$

After an analogous discussion, we obtain

Corollary 1. *If regular rational Bézier surfaces* $\boldsymbol{r}$ *of degree* $m\times l$ *and* $\bar{\boldsymbol{r}}$ *of degree* $\bar{m}\times l$ *satisfy "*C^0 *continuity" along their common boundary curve CB, then they meet with* GC^n *iff there exist some polynomials* $D(v)$, $\underline{c}_i(v)$, $\underline{p}_i(v)$, $\underline{q}_i(v)$, *whose* *degrees are respectively not larger than* $3l-1$, $(2i-1)(3l-1)$, $(2i-1)(3l-1)$, $(2i-1)(3l-1)+1$, $D(v)\not\equiv 0$, $p_1(v)\not\equiv 0$ *such that* (4.6) *hold.*

Case 2. If surfaces $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are integral Bézier surfaces, then it is a natural choice that $H=G=1$. For any $s\geqslant 1$, $\partial^s\omega/\partial u^h\partial v^{s-h}=0$, and (4.1) imply $0=\underline{c}_s\omega+0=\underline{c}_s$. Thus conditions (4.1) become

$$\bar{\boldsymbol{r}}\,|_{CB}=\boldsymbol{r}\,|_{CB},$$
$$D^{2s-1}\frac{\partial^s\bar{\boldsymbol{r}}}{\partial\bar{u}^s}\bigg|_{CB}=\sum_{k=1}^{s}\sum_{r_1+\cdots+r_k=s}\sum_{h=0}^{k}A^{skh}_{r_1\ldots r_k}\cdot D^{k-1}\underline{p}_{r_1}\cdots\underline{q}_{r_k}\cdot\frac{\partial^k\boldsymbol{r}}{\partial u^h\partial v^{k-h}}\bigg|_{CB}. \tag{4.7}$$

Similarly, the degrees of D, $\underline{p}_i$, $\underline{q}_i$ must be properly adjusted.

Corollary 2. *Bézier surfaces* $\boldsymbol{r}$ *of degree* $m\times l$ *and* $\bar{\boldsymbol{r}}$ *of degree* $\bar{m}\times l$ *are* GC^n *continuous along* CB: $\bar{\boldsymbol{r}}(0,\bar{v})=\boldsymbol{r}(1,v)$ $(0\leqslant\bar{v}=v\leqslant 1)$ *iff conditions* (4.7) *hold, where* $D(v)$ $(\not\equiv 0)$, $p_i(v)$ $(p_1(v)\not\equiv 0)$, $q_i(v)$, $i=1,\ldots,n$, *are all polynomials of* v *with degrees not larger than* $2l-1$, $(2i-1)(2l-1)$ *and* $(2i-1)(2l-1)+1$.

4.2. Some practical sufficient conditions

One application of geometric continuity is the construction of smoothly connecting surfaces by using the continuity conditions. In general, as shown in (Zheng et al., 1992), conditions (4.1) provide the designer numerous coefficients while they include a system of constraint equations which these coefficients have to satisfy. In practice, too many free coefficients will confuse the designer and the complicated constraints will give rise to difficulties for CAGD application. For this reason, we reduce the degrees of the connecting functions, and let the position continuity be "C^0 continuity". Thus some GC^n sufficient conditions are obtained.

Theorem 6. *Regular rational Bézier surfaces* $\boldsymbol{r}$ *of degree* $m\times l$ *and* $\boldsymbol{r}$ *of degree* $\bar{m}\times l$ *are* GC^n *along* CB: $\bar{\boldsymbol{r}}(0,\bar{v})=\boldsymbol{r}(1,v)$ $(0\leqslant\bar{v}=v\leqslant 1)$ *if conditions* (4.6) *hold, where* D $(\not\equiv 0)$, $\underline{c}_i$, $\underline{p}_i$ $(\underline{p}_1\not\equiv 0)$, $\underline{q}_i$ *are all polynomials of* v *respectively with degrees not larger than* σ, $(2i-1)\sigma$, $(2i-1)\sigma$, $(2i-1)\sigma+1$. *The number* σ *is an integer ranging from* 0 *to* $3l-1$. *If* $\sigma=3l-1$, *the condition is also necessary.*

As an example, consider the case of $\sigma=0$. Then D, $\underline{c}_i$, $\underline{p}_i$ are constants, and $\underline{q}_i$ are linear polynomials. Assume that $D(v)=1$, $\underline{c}_i=\gamma_i$, $\underline{p}_i=\alpha_i$, $\underline{q}_i=\beta_i^0(1-v)+\beta_i^1 v$. Then conditions (4.6) become

$$\bar{\boldsymbol{Q}}\,|_{CB}=\boldsymbol{Q}\,|_{CB},$$
$$\frac{\partial^s\bar{\boldsymbol{Q}}}{\partial\bar{u}^s}\bigg|_{CB}=\gamma_s\boldsymbol{Q}_s\,|_{CB}+\sum_{d=1}^{s}\binom{s}{d}\gamma_{s-d}(v)\sum_{k=1}^{d}\sum_{r_1+\cdots+r_k=d}\sum_{h=0}^{k}A^{dkh}_{r_1\ldots r_k}$$
$$\times\left\{\alpha_{r_1}\cdots\alpha_{r_h}\cdot\left(\beta^0_{r_{h+1}}(1-v)+\beta^1_{r_{h+1}}v\right)\cdots\left(\beta^0_{r_k}(1-v)+\beta^1_{r_k}v\right)\right\}$$
$$\times\frac{\partial^k\boldsymbol{Q}}{\partial u^h\partial v^{k-h}}\bigg|_{CB},$$
$$s=1,\ldots,n.$$

After some calculations, it is possible to obtain the first $(n+1)$-columns of the control points of surface $\bar{\boldsymbol{r}}$ which meets $\boldsymbol{r}$ with GC^n along CB:

$$\bar{\boldsymbol{Q}}_{0i} = \boldsymbol{Q}_{mi}, \quad i = 0, \ldots, l,$$

$$\begin{aligned} \bar{\boldsymbol{Q}}_{si} = & -\sum_{j=0}^{s-1} \binom{s}{j} (-1)^{s-j} \bar{\boldsymbol{Q}}_{ji} + \frac{(\bar{m}-s)!}{\bar{m}!} \\ & \times \left\{ \gamma_s \boldsymbol{Q}_{mi} + \sum_{d=1}^{s} \sum_{k=1}^{d} \sum_{r_1+\cdots+r_k=d} \sum_{h=0}^{k} \sum_{\substack{j=0 \\ j+i_{h+1}+\cdots+i_k=i}}^{l-k+h} \sum_{i_{h+1}=0}^{1} \cdots \sum_{i_k=0}^{1} \right. \\ & \times \binom{s}{d} A^{dkh}_{r_1 \ldots r_k} \frac{m!}{(m-h)!} \frac{l!}{(l-k+h)!} \\ & \left. \cdot \left[\left(\binom{l-k+h}{j} \Big/ \binom{l}{i} \right) \gamma_{s-d} \alpha_{r_1} \cdots \alpha_{r_h} \beta^{i}_{r_{h+1}} \cdots \beta^{i}_{r_k} \nabla_1^h \nabla_2^{k-h} Q_{m-h\,j} \right] \right\}, \end{aligned}$$

$$s = 1, \ldots, n \tag{4.8}$$

where

$$\nabla_1 Q_{ij} = Q_{i+1\,j} - Q_{ij}, \qquad \nabla_1^h Q_{ij} = \nabla_1\left(\nabla_1^{h-1} Q_{ij}\right),$$

$$\nabla_2 Q_{ij} = Q_{ij+1} - Q_{ij}, \qquad \nabla_2^h Q_{ij} = \nabla_2\left(\nabla_2^{h-1} Q_{ij}\right).$$

and γ_i, α_i, β_i^0, β_i^1 ($\alpha_1 \neq 0$) are free coefficients which are sometimes called shape parameters in the literature.

Further, suppose that $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ are bicubic, that is $m = l = \bar{m} = 3$. If we choose the first two columns of the control points of $\bar{\boldsymbol{r}}$ as follows, then $\bar{\boldsymbol{r}}$ meets $\boldsymbol{r}$ with GC^1 continuity.

$$\bar{\boldsymbol{Q}}_{0i} = \boldsymbol{Q}_{3i}, \quad i = 0, \ldots, 3,$$

$$\bar{\boldsymbol{Q}}_{10} = \gamma_1 \boldsymbol{Q}_{30} + \alpha_1(\boldsymbol{Q}_{30} - \boldsymbol{Q}_{20}) + \beta_1^0(\boldsymbol{Q}_{31} - \boldsymbol{Q}_{30}),$$

$$\bar{\boldsymbol{Q}}_{11} = \gamma_1 \boldsymbol{Q}_{31} + 3\alpha_1(\boldsymbol{Q}_{31} - \boldsymbol{Q}_{21}) + \tfrac{2}{3}\beta_1^0(\boldsymbol{Q}_{32} - \boldsymbol{Q}_{31}) + \tfrac{1}{3}\beta_1^1(\boldsymbol{Q}_{31} - \boldsymbol{Q}_{30}),$$

$$\bar{\boldsymbol{Q}}_{12} = \gamma_1 \boldsymbol{Q}_{32} + 3\alpha_1(\boldsymbol{Q}_{32} - \boldsymbol{Q}_{22}) + \tfrac{1}{3}\beta_1^0(\boldsymbol{Q}_{33} - \boldsymbol{Q}_{32}) + \tfrac{2}{3}\beta_1^1(\boldsymbol{Q}_{32} - \boldsymbol{Q}_{31}),$$

$$\boldsymbol{Q}_{13} = \gamma_1 \boldsymbol{Q}_{33} + \alpha_1(\boldsymbol{Q}_{33} - \boldsymbol{Q}_{23}) + \beta_1^1(\boldsymbol{Q}_{33} - \boldsymbol{Q}_{32}).$$

5. Conclusion

This paper has presented a set of necessary and sufficient conditions of GC^n continuity for two adjacent rational surfaces along a general intersection curve, which do not require the homogeneous surfaces to be smooth. These conditions are represented by the associated homogeneous surfaces and a set of connecting functions. Thus they can be further treated like integral surfaces. Specifically, for rational Bézier surfaces, it can be shown that these conditions can be converted into a series of constraints represented by the control points and weights as in (Zheng et al., 1992). This is convenient for constructing the connecting surfaces and examining by a program if two given rational Bézier patches are GC^n.

This paper has also shown that geometric continuity is, in essence, the existence of a reparameterization. If two given surface patches are GC^n, then transformations (3.3) and (3.6) present a method to reparameterize the two surfaces so that they are C^n. This is an important feature in some CAGD applications.

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