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Curvature continuity between adjacent rational Bézier patches

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Abstract

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This paper discusses the curvature continuity between two adjacent rational Bézier surfaces which may be rectangular or triangular patches. The necessary and sufficient conditions are derived, and further, a series of simple sufficient conditions are developed. These conditions are either descriptive or constructive. Therefore with them one can both check the geometric continuity between two surfaces and construct a rational surface possessing curvature continuity with a given rational patch along a certain boundary. This is an important feature in CAGD applications.

Keywords. Rational surfaces; Bézier patches; curvature continuity; geometric continuity; connecting functions.

1. Introduction

The description of the surface shape is a fundamental task of surface modeling. Among various mathematical representations, the rational Bézier surface becomes one of the most popular models for free-form surface modeling. This is because it has a lot of powerful properties, such as the capability of describing both polynomial parametric surfaces and quadratic surfaces. However, it is usually impossible to represent a complex surface by a single patch. Thus the surface must be subdivided into several pieces, each of which is represented by a rational Bézier patch. To guarantee the smoothness of the resulting surface, continuity between adjacent surface patches becomes a crucial problem [Liang et al. '88, Watkins '88].

Much research has been devoted to this subject, and many approaches to the construction of Bézier surfaces that share tangent planes along their common boundary have been developed [Farin '82, Chiyokura & Kimura '83]. As most of them are based on some simplifying sufficient conditions, Liu and Hoschek made a study of necessary and sufficient conditions for tangent plane continuity (or GC^1) between rectangular or triangular Bézier patches [Liu & Hoschek '89]. Then Degen derived explicit representations of the first-order cross-boundary tangent vectors of two adjacent Bézier patches with GC^1 continuity [Degen '90]. Recently, GC^1 continuity for rational Bézier patches is further addressed by Liu and DeRose [Liu '90, DeRose '90], and a system of necessary and sufficient conditions is given.

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Vinacua and Brunet present a construction for GC^1 continuity of rational Bézier patches [Vinacua & Brunet '89].

In general, tangent plane continuity is sufficient. For some applications, however, curvature continuity (or GC^2) is required [Boehm '88]. The relevant work was made by Veron et al., Jones and Kahmann. Veron et al. described some sufficient conditions for ensuring GC^2 [Veron et al. '76]. Kahmann has presented a straightforward approach to the construction of the involved Bézier points of curvature continuously connected Bézier patches [Kahmann '83].

This paper studies the curvature continuous connection between adjacent rational Bézier patches. The necessary and sufficient conditions for general rational surfaces are deduced, and the constraints for rational Bézier patches are analysed. These results provide answers to the following problems:

- How to examine by a program if two given C^0 continuous rational Bézier patches are GC^2.
- How to construct a rational Bézier patch which meets a given rational Bézier patch curvature continuously.

Furthermore, some practical sufficient conditions are developed, and two simple constructions for smooth connection are given.

2. Curvature continuity for rational patches

Suppose that surfaces $r(u, v)$ and $\bar{r}(\bar{u}, \bar{v})$ share a common boundary curve:

$$CB: \quad r(0, v) = \bar{r}(0, \bar{v}), \quad 0 \leqslant v = \bar{v} \leqslant 1. \tag{2.1}$$

They are curvature continuous or GC^2 if they have coincident tangent planes and identical indicators of curvature at every point of CB.

By definition, Kahmann has obtained the general GC^2 conditions:

$$\bar{r}(0, \bar{v}) = r(0, v), \tag{2.2}$$

$$\bar{r}_{\bar{u}}(0, \bar{v}) = \alpha\, r_u(0, v) + \beta\, r_v(0, v), \tag{2.3}$$

$$\begin{aligned}\bar{r}_{\bar{u}\bar{u}}(0, \bar{v}) = {} & \alpha^2 r_{uu}(0, v) + 2\alpha\beta\, r_{uv}(0, v) + \beta^2 r_{vv}(0, v)\\ & + \nu\, r_u(0, v) + \eta\, r_v(0, v)\end{aligned} \tag{2.4}$$

where α, β, ν, η are functions of v.

Consider two regular rational surfaces $r(u, v)$ and $\bar{r}(\bar{u}, \bar{v})$ which are defined an an affine coordinate system of E^3 by

$$\begin{aligned} r(u, v) &= R(u, v)/\omega(u, v), \quad 0 \leqslant u, v \leqslant 1,\\ \bar{r}(\bar{u}, \bar{v}) &= \bar{R}(\bar{u}, \bar{v})/\bar{\omega}(\bar{u}, \bar{v}), \quad 0 \leqslant \bar{u}, \bar{v} \leqslant 1, \end{aligned} \tag{2.5}$$

and in a homogeneous coordinate system by

$$\begin{aligned} r: \quad & Q(u, v) = \{R(u, v), \omega(u, v)\},\\ \bar{r}: \quad & \bar{Q}(\bar{u}, \bar{v}) = \{\bar{R}(\bar{u}, \bar{v}), \bar{\omega}(\bar{u}, \bar{v})\} \end{aligned} \tag{2.6}$$

where $R(v, v)$ and $\bar{R}(\bar{u}, \bar{v})$ are surfaces of E^3, $\omega(u, v)$ and $\bar{\omega}(\bar{u}, \bar{v})$ are nonzero functions.

This section will deduce the GC^2 conditions for rational surface patches.

Theorem 1. *Let rational surface patches $r(u, v)$ and $\bar{r}(\bar{u}, \bar{v})$ share a common boundary curve CB. Then $r(u, v)$ and $\bar{r}(\bar{u}, \bar{v})$ are curvature continuous iff the following equations*

$$\bar{Q} = c_0 Q, \tag{2.7}$$

$$\bar{Q}_{\bar{u}} = c_1 Q + c_0 p_1 Q_u + c_0 q_1 Q_v, \tag{2.8}$$

$$\bar{Q}_{\bar{u}\bar{u}} = c_2 Q + p_2 Q_u + q_2 Q_v + c_0 p_1^2 Q_{uu} + 2 c_0 p_1 q_1 Q_{uv} + c_0 q_1^2 Q_{vv} \tag{2.9}$$

with some functions $c_0(v)$, $c_i(v)$, $p_i(v)$, $q_i(v)$ $(i = 1, 2)$ hold along CB.

Proof. If r and $\bar{r}$ are curvature continuous, (2.7) is obvious by setting $c_0(v) = \bar{\omega}/\omega \mid_{CB}$. Using the quotient rule

$$\frac{\partial}{\partial u}(r, 1) = \frac{\partial}{\partial u}\left(\frac{Q}{\omega}\right) = (Q_u \omega - Q \omega_u)/\omega^2,$$

yields

$$\bar{Q}_{\bar{u}} = c_0 \alpha Q_u + c_0 \beta Q_v + Q(\bar{\omega}_{\bar{u}} - c_0 \alpha \omega_u - c_0 \beta \omega_v)/\omega. \tag{2.10}$$

Let

$$p_1(v) = \alpha(v), \qquad q_1(v) = \beta(v),$$
$$c_1(v) = (\omega_{\bar{u}} - c_0 \alpha \omega_u - c_0 \beta \omega_v)/\omega. \tag{2.11}$$

One obtains (2.8). Similarly, (2.4) gives

$$\begin{aligned}\bar{Q}_{\bar{u}\bar{u}} = {} & c_0 p_1^2 Q_{uu} + 2 c_0 p_1 q_1 Q_{uv} + c_0 q_1^2 Q_{vv} + (c_0 \nu + 2 c_1 p_1) Q_u + (c_0 \eta + 2 c_1 q_1) Q_v \\ & + Q\big(\bar{\omega}_{\bar{u}\bar{u}} - \big(c_0 p_1^2 \omega_{uu} + 2 c_0 p_1 q_1 \omega_{uv} + c_0 q_1^2 \omega_{vv} \\ & \qquad + (c_0 \nu + 2 c_1 p_1) \omega_u + (c_0 \eta + 2 c_1 q_1) \omega_v\big)\big)/\omega. \end{aligned} \tag{2.12}$$

Set

$$p_2(v) = c_0 \nu + 2 c_1 p_1, \qquad q_2(v) = c_0 \eta + 2 c_1 q_1,$$
$$c_2 = \big(\bar{\omega}_{\bar{u}\bar{u}} - \big(c_0 p_1^2 \omega_{uu} + 2 c_0 p_1 q_1 \omega_{uv} + c_0 q_1^2 \omega_{vv} + p_2 \omega_u + q_2 \omega_v\big)/\omega.$$

Thus

$$\bar{Q}_{\bar{u}\bar{u}} = c_2 Q + p_2 Q_u + q_2 Q_v + c_0 p_1^2 Q_{uu} + 2 c_0 p_1 q_1 Q_{uv} + c_0 q_1^2 Q_{vv}.$$

Conversely, let (2.7)–(2.9) hold. (2.2) is immediately obtained from (2.7). By setting $\alpha = p_1$ and $\beta = q_1$, (2.10) holds. Therefore (2.3) holds too. Furthermore, assuming $\nu = (p_2 - 2 c_1 \alpha)/c_0$ and $\eta = (q_2 - 2 c_1 q_1)/c_0$, gives (2.12). This directly leads to (2.4). □

Remark. In the above definition, the condition to ensure that r and $\bar{r}$ share a common boundary curve accords with the definition of 'GC^0 continuity' presented in [Liu '90]. To simplify the discussion, in the remainder of this paper one takes 'C^0 continuity', which requires $Q(0, v) = \bar{Q}(0, \bar{v})$, instead of '$GC^0$ continuity'. In this case conditions (2.7)–(2.9) become

$$\bar{Q} = Q, \tag{2.13}$$

$$\bar{Q}_{\bar{u}} = c_1 Q + p_1 Q_u + q_1 Q_v, \tag{2.14}$$

$$\bar{Q}_{\bar{u}\bar{u}} = c_2 Q + p_2 Q_u + q_2 Q_v + p_1^2 Q_{uu} + 2 p_1 q_1 Q_{uv} + q_1^2 Q_{vv}. \tag{2.15}$$

3. GC^2 conditions for rectangular rational Bézier patches

The rational Bézier surfaces r of degree $m \times n$ and $\bar{r}$ of degree $\bar{m} \times n$ are defined in homogeneous coordinates by

$$r:\quad Q(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} Q_{ij} B_i^m(u) B_j^n(v), \quad 0 \leqslant u, v \leqslant 1 \tag{3.1}$$

and

$$\bar{r}:\quad \bar{Q}(\bar{u}, \bar{v}) = \sum_{i=0}^{\bar{m}} \sum_{j=0}^{n} \bar{Q}_{ij} B_i^{\bar{m}}(\bar{u}) B_j^n(\bar{v}), \quad 0 \leqslant \bar{u}, \bar{v} \leqslant 1 \tag{3.2}$$

where $Q_{ij} = (P_{ij}\omega_{ij}, \omega_{ij})$, $\bar{Q}_{ij} = (\bar{P}_{ij}\bar{\omega}_{ij}, \bar{\omega}_{ij})$; P_{ij}, $\bar{P}_{ij}$ are control points, ω_{ij}, $\bar{\omega}_{ij}$ are weights. $B_j^l(t) = \binom{l}{j}(1-t)^{l-j}t^j$, $j = 0, \ldots, l$, are the Bernstein polynomials of degree l.

Once Q_{ij} are found, then P_{ij} and ω_{i_j} are generally determined.

3.1. GC^2 necessary and sufficient condition

As shown in Section 2, the coefficients α, β, ν, η or c_i, p_i, q_i, which are called the connecting functions of $\bar{r}$ and r, are scalar functions of v. For rational Bézier patches $\bar{r}$ and r, the connecting functions have simple forms, i.e., they are rational polynomials.

Theorem 2. *A necessary and sufficient condition for curvature continuity between two adjacent 'C⁰ continuous' rational Bézier patches r and $\bar{r}$ along CB is that*

$$D\bar{Q}_{\bar{u}} = \bar{c}_1 Q + \bar{p}_1 Q_u + \bar{q}_1 Q_v, \tag{3.3}$$

$$D^3\bar{Q}_{\bar{u}\bar{u}} = \bar{c}_2 Q + \bar{p}_2 Q_u + \bar{q}_2 Q_v + \bar{p}_1^2 DQ_{uu} + 2\bar{p}_1\bar{q}_1 DQ_{uv} + \bar{q}_1^2 DQ_{vv} \tag{3.4}$$

hold at each point of CB, where $D\ (\neq 0)$, $\bar{c}_i$, $\bar{p}_i$, $\bar{q}_i$ $(i = 1, 2)$ are all polynomials of v, and their degrees are as follows:

	$D, \bar{c}_1, \bar{p}_1$	$\bar{q}_1$	$\bar{c}_2, \bar{p}_2$	$\bar{q}_2$
degree $\leqslant$	$3n-1$	$3n$	$9n-3$	$9n-2$

Proof. For convenience and simplicity, introduce the notation

$$\langle R, S, T\rangle_i = (-1)^{i+1}\det\begin{bmatrix} R_{i+1} & S_{i+1} & T_{i+1} \\ R_{i+2} & S_{i+2} & T_{i+2} \\ R_{i+3} & S_{i+3} & T_{i+3} \end{bmatrix}, \quad i = 1, 2, 3, 4$$

where R, S, T are arbitrary vectors in E^4. R_i denotes the ith component of R. If $i > 4$, $R_i = R_{\mathrm{mod}\,(i-1,\,4)+1}$.

As $r_u(u, v) \times r_v(u, v) \neq 0$ and

$$\begin{aligned} r_u \times r_v &= ((R_u\omega - R\omega_u) \times (R_v\omega - R\omega_v))/\omega^4 \\ &= \{\langle Q, Q_u, Q_v\rangle_1, \langle Q, Q_u, Q_v\rangle_2, \langle Q, Q_u, Q_v\rangle_3\}/\omega^3 \end{aligned}$$

there exists at least one $\langle Q, Q_u, Q_v\rangle_i$ at (u, v) satisfying

$$\langle Q, Q_u, Q_v\rangle_i \neq 0.$$

Therefore $Q(u, v)$, $Q_u(u, v)$, $Q_v(u, v)$ are independent.

If r and $\bar{r}$ are curvature continuous, then (2.14) holds. As $r(u, v)$ is regular, without loss of generality, one can choose i such that $\langle Q, Q_u, Q_v \rangle_i \neq 0$ at $(u, v) = (0, 0)$.

Taking

$$D(v) = \langle Q, Q_u, Q_v \rangle_i, \quad \bar{c}_1(v) = \langle \bar{Q}_{\bar{u}}, Q_u, Q_v \rangle_i, \quad \bar{p}_1(v) = \langle Q, \bar{Q}_{\bar{u}}, Q_v \rangle_i, \quad \bar{q}_1(v) = \langle Q, Q_u, \bar{Q}_{\bar{u}} \rangle_i \tag{3.5}$$

gives

$$D\bar{Q}_{\bar{u}} = \bar{c}_1 Q + \bar{p}_1 Q_u + \bar{q}_1 Q_v, \quad 0 \leqslant v \leqslant 1$$

and

$$\bar{c}_1 = D(v)c(v), \qquad \bar{p}_1 = D(v)p_1(v), \qquad \bar{q}_1 = D(v)q_1(v).$$

The degrees of D, $\bar{c}_1$, $\bar{p}_1$ are not larger than $3n - 1$ and that of $\bar{q}_1$ is not larger than $3n$. Let

$$DC = D^2\left(\bar{Q}_{\bar{u}\bar{u}} - p_1^2 Q_{uu} - 2p_1 q_1 Q_{uv} - q_1^2 Q_{vv}\right). \tag{3.6}$$

DC is a polynomial curve with degree not larger than $7n - 2$. Rewriting (2.15), yields

$$DC = D^2 c_2 Q + D^2 p_2 Q_u + D^2 q_2 Q_v.$$

If

$$\bar{c}_2 = \langle DC, Q_u, Q_v \rangle_i, \qquad \bar{p}_2 = \langle Q, DC, Q_v \rangle_i, \qquad \bar{q}_2 = \langle Q, Q_u, DC \rangle_i \tag{3.7}$$

are set, then $\bar{c}_2$, $\bar{p}_2$, $\bar{q}_2$ are respectively of degree not larger than $9n - 3$, $9n - 3$, $9n - 2$ in v, and

$$D\,DC = \bar{c}_2 Q + \bar{p}_2 Q_u + \bar{q}_2 Q_v,$$

i.e.,

$$D^3 Q_{\bar{u}\bar{u}} = \bar{c}_2 Q + \bar{p}_2 Q_u + \bar{q}_2 Q_v + D\left(\bar{p}_1^2 Q_{uu} + 2\bar{p}_1 \bar{q}_1 Q_{uv} + \bar{q}_1^2 Q_{vv}\right).$$

On the other hand, assume that (3.3) and (3.4) hold, and $D(v)$ has m zeropoints $v_1, \ldots, v_m$ in [0,1]. They are also the zeropoints of $\bar{c}_1$,$\bar{p}_1$, $\bar{q}_1$ and triple ones of $\bar{c}_2$, $\bar{p}_2$, $\bar{q}_2$ because of the independence of Q, Q_u, Q_v. These zero factors can be eliminated by

$$D' = D/vm, \qquad c_1' = \bar{c}_1/vm, \qquad p_1' = \bar{p}_1/vm, \qquad q_1' = \bar{q}_1/vm,$$
$$c_2' = \bar{c}_2/vm^3, \qquad p_2' = \bar{p}_2/vm^3, \qquad q_2' = \bar{q}_2/vm^3 \tag{3.8}$$

where $vm = (v - v_1) \cdots (v - v_m)$.

It is clearly shown that $D'(v) \neq 0$, $v \epsilon [0,1]$, and (3.3), (3.4) still hold if the connecting functions D, $\bar{c}_i$, $\bar{p}_i$, $\bar{q}_i$ are replaced by D', c_i', p_i', q_i' respectively. Thus the sufficiency is fulfilled by taking

$$c_1 = c_1'/D', \qquad p_1 = p_1'/D', \qquad q_1 = q_1'/D',$$
$$c_2 = c_2'/D'^3, \qquad p_2 = p_2'/D'^3, \qquad q_2 = q_2'/D'^3. \qquad \square$$

Theorem 2 not only gives a necessary and sufficient condition for curvature continuity, but also presents a means to construct the connecting functions. From the theorem one can easily develop an algorithm to check the continuity of two given surface patches.

3.2. Connecting rectangular rational Bézier patches

The preceding results are more descriptive than constructive. To further reveal the continuity condition and to construct patches meeting a given patch along a certain boundary

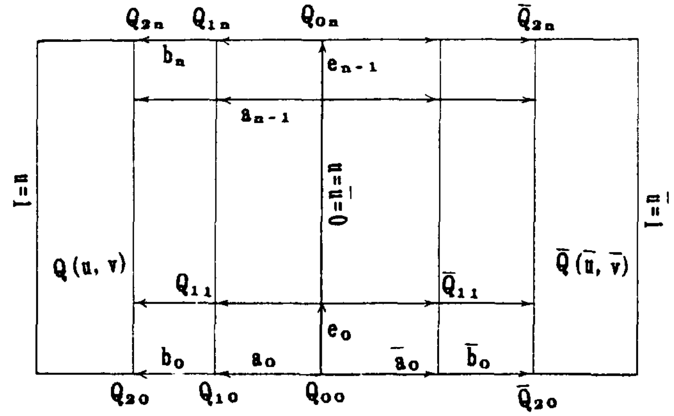

Fig. 1. Notations for the directional derivatives of rectangular patches.

with curvature continuity more conveniently, the foregoing conditions must be converted into conditions on the control points and weights of the patches.

First express $D(v)$, $\bar{c}_i$, $\bar{p}_i$, $\bar{q}_i$ as Bernstein polynomials

$$D(v)=\sum_{j=0}^{3n-1} d_j B_j^{3n-1}(v),\qquad \bar{c}_1=\sum_{j=0}^{3n-1} c_j^{(1)} B_j^{3n-1}(v),$$

$$\bar{p}_1=\sum_{j=0}^{3n-1} p_j^{(1)} B_j^{3n-1}(v),\qquad \bar{q}_1=\sum_{j=0}^{3n} q_j^{(1)} B_j^{3n}(v),\tag{3.9}$$

$$\bar{c}_2=\sum_{j=0}^{9n-3} c_j^{(2)} B_j^{9n-3}(v),\qquad \bar{p}_2=\sum_{j=0}^{9n-3} p_j^{(2)} B_j^{9n-3}(v),\qquad \bar{q}_2=\sum_{j=0}^{9n-2} q_j^{(2)} B_j^{9n-2}(v).$$

Then let

$$a_i=Q_{1i}-Q_{0i},\qquad b_i=Q_{2i}-Q_{1i},\qquad e_i=Q_{0i+1}-Q_{0i},$$

and similarly for $\bar{r}$ (see Fig. 1).

Thus the 'C^0 condition' is simply forced by

$$\bar{Q}_{0i}=Q_{0i}.\tag{3.10}$$

If one substitutes (3.9) into (3.3) and regroups the equation, then (3.3) becomes

$$\sum_{\substack{i=0\\ i+j=s}}^{n}\sum_{j=0}^{3n-1}\binom{3n-1}{j}\binom{n}{i}\bar{m}d_j\bar{a}_i=\sum_{\substack{i=0\\ i+j=s}}^{n}\sum_{j=0}^{3n-1}\binom{3n-1}{j}\binom{n}{i}\left(c_j^{(1)}Q_{0i}+mp_j^{(1)}a_i\right)$$
$$+\sum_{\substack{i=0\\ i+j=s}}^{n-1}\sum_{j=0}^{3n}\binom{3n}{j}\binom{n-1}{i}nq_j^{(1)}e_i,\quad s=0,\ldots,4n-1.\tag{3.11}$$

Note that $\binom{k}{j}=0$ when $j<0$ or $j>k$.

Making the additional definitions

$$[C\bar{A}] = \begin{pmatrix} \binom{n}{0}\bar{a}_0 \\ \vdots \\ \binom{n}{n}\bar{a}_n \end{pmatrix}, \qquad [CQ] = \begin{pmatrix} \binom{n}{0}Q_{00} \\ \vdots \\ \binom{n}{n}Q_{0n} \end{pmatrix}, \qquad [CA] = \begin{pmatrix} \binom{n}{0}a_0 \\ \vdots \\ \binom{n}{n}a_n \end{pmatrix},$$

$$\begin{pmatrix} D^{11}_{(n+1)\times(n+1)} \\ D^{12}_{(3n-1)\times(n+1)} \end{pmatrix} = \begin{pmatrix} \bar{d}_0 & & \\ \vdots & \ddots & \bar{d}_0 \\ \bar{d}_{3n-1} & & \vdots \\ & \ddots & \bar{d}_{3n-1} \end{pmatrix}, \qquad [CE] = \begin{pmatrix} \binom{n-1}{0}e_0 \\ \vdots \\ \binom{n-1}{n-1}e_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} C^{11}_{(n+1)\times(n+1)} \\ C^{12}_{(3n-1)\times(n+1)} \end{pmatrix} = \begin{pmatrix} \bar{c}^{(1)}_0 & & \\ \vdots & \ddots & \bar{c}^{(1)}_0 \\ \bar{c}^{(1)}_{3n-1} & & \vdots \\ & \ddots & \bar{c}^{(1)}_{3n-1} \end{pmatrix} \tag{3.12}$$

and similarly for

$$\begin{pmatrix} p^{11}_{(n+1)\times(n+1)} \\ p^{12}_{(3n-1)\times(n+1)} \end{pmatrix}, \qquad \begin{pmatrix} q^{11}_{(n+1)\times n} \\ q^{12}_{(3n-1)\times n} \end{pmatrix}$$

where

$$\begin{aligned} &\bar{d}_i = \bar{m}\binom{3n-1}{i}d_i, \qquad \bar{c}^{(1)}_i = \binom{3n-1}{i}c^{(1)}_i, \\ &\bar{p}^{(1)}_i = m\binom{3n-1}{i}p^{(1)}_i, \quad \bar{q}^{(1)}_i = n\binom{3n}{i}q^{(1)}_i, \end{aligned} \tag{3.13}$$

equation (3.11) can be rewritten in matrix form:

$$\begin{pmatrix} D^{11} \\ D^{12} \end{pmatrix}[C\bar{A}] = \begin{pmatrix} C^{11} \\ C^{12} \end{pmatrix}[CQ] + \begin{pmatrix} p^{11} \\ p^{12} \end{pmatrix}[CA] + \begin{pmatrix} q^{11} \\ q^{12} \end{pmatrix}[CE]. \tag{3.14}$$

By the choice made in (3.5), it is easy to show that $d_0 \neq 0$. Thus $\bar{d}_0 \neq 0$. Let $[H^1] = [D^{11}]^{-1}$, then

$$[H^1] = \begin{pmatrix} h^{(1)}_0 & & \\ \vdots & \ddots & \\ h^{(1)}_n & \cdots & h^{(1)}_0 \end{pmatrix} \tag{3.15}$$

where

$$h^{(1)}_0 = \frac{1}{\bar{d}_0}, \qquad h^{(1)}_i = -h^{(1)}_0 \sum_{j=0}^{i-1} \bar{d}_{i-j}h^{(1)}_j. \tag{3.16}$$

Therefore solving equation (3.14), gives

$$[C\bar{A}] = [H^1][C^{11}][CQ] + [H^1][p^{11}][CA] + [H^1][q^{11}][CE],$$

i.e.,

$$\binom{n}{i}\bar{a}_i = \sum_{j=0}^{i}\sum_{k=j}^{i} h_{i-k}^{(1)}\left(\binom{n}{j}\bar{c}_{k-j}^{(1)}Q_{0j} + \binom{n}{j}\bar{p}_{k-j}^{(1)}a_j + \binom{n-1}{j}\bar{q}_{k-j}^{(1)}e_j\right)$$

$$\text{for } i = 0, \ldots, n. \tag{3.17}$$

Meanwhile, coefficients d_i, $c_i^{(1)}$, $p_i^{(1)}$, $q_i^{(1)}$ are not in general independent. They should satisfy the constraint equations

$$\begin{aligned}&([D^{12}][H^1][C^{11}] - [C^{12}])[CQ] + ([D^{12}][H^1][p^{11}] - [p^{12}])[CA]\\&\quad + ([D^{12}][H^1][q^{11}] - [q^{12}])[CE] = 0.\end{aligned} \tag{3.18}$$

Similarly, one can obtain the following equation by substituting (3.9) into (3.4):

$$\begin{aligned}\begin{pmatrix} D^{21} \\ D^{22}\end{pmatrix}[C\overline{BA}] &= \begin{pmatrix} C^{21} \\ C^{22}\end{pmatrix}[CQ] + \begin{pmatrix} p^{21} \\ p^{22}\end{pmatrix}[CA] + \begin{pmatrix} q^{21} \\ q^{22}\end{pmatrix}[CE]\\ &\quad + \begin{pmatrix} F^{1} \\ F^{2}\end{pmatrix}[CBA] + \begin{pmatrix} G^{1} \\ G^{2}\end{pmatrix}[CAA] + \begin{pmatrix} T^{1} \\ T^{2}\end{pmatrix}[CEE]\end{aligned} \tag{3.19}$$

where

$$[C\overline{BA}] = \begin{pmatrix} \binom{n}{0}(\bar{b}_0 - \bar{a}_0) \\ \vdots \\ \binom{n}{n}(\bar{b}_n - \bar{a}_n)\end{pmatrix}, \qquad [CBA] = \begin{pmatrix} \binom{n}{0}(b_0 - a_0) \\ \vdots \\ \binom{n}{n}(b_n - a_n)\end{pmatrix},$$

$$[CAA] = \begin{pmatrix} \binom{n-1}{0}(a_1 - a_0) \\ \vdots \\ \binom{n-1}{n-1}(a_n - a_{n-1})\end{pmatrix}, \qquad [CEE] = \begin{pmatrix} \binom{n-2}{0}(e_1 - e_0) \\ \vdots \\ \binom{n-2}{n-2}(e_{n-1} - e_{n-2})\end{pmatrix},$$

$$\begin{pmatrix} D^{21}_{(n+1)\times(n+1)} \\ D^{22}_{(9n-3)\times(n+1)}\end{pmatrix} = \begin{pmatrix} \bar{d}_0^{(2)} & & \\ \vdots & \ddots & \bar{d}_0^{(2)} \\ \bar{d}_{9n-3}^{(2)} & & \vdots \\ & \ddots & \bar{d}_{9n-3}^{(2)}\end{pmatrix}, \qquad \begin{pmatrix} F^{1}_{(n+1)\times(n+1)} \\ F^{2}_{(9n-3)\times(n+1)}\end{pmatrix} = \begin{pmatrix} \bar{f}_0 & & \\ \vdots & \ddots & \bar{f}_0 \\ \bar{f}_{9n-3} & & \vdots \\ & \ddots & \bar{f}_{9n-3}\end{pmatrix}$$

and

$$\begin{pmatrix} C^{21}_{(n+1)\times(n+1)} \\ C^{22}_{(9n-3)\times(n+1)}\end{pmatrix}, \quad \begin{pmatrix} p^{21}_{(n+1)\times(n+1)} \\ p^{22}_{(9n-3)\times(n+1)}\end{pmatrix}, \quad \begin{pmatrix} q^{21}_{(n+1)\times n} \\ q^{22}_{(9n-3)\times n}\end{pmatrix}, \quad \begin{pmatrix} G^{1}_{(n+1)\times n} \\ G^{2}_{(9n-3)\times n}\end{pmatrix},$$

$$\begin{pmatrix} T^{1}_{(n+1)\times(n-1)} \\ T^{2}_{(9n-3)\times(n-1)}\end{pmatrix}$$

are defined in the same way.

$$\bar{c}_i^{(2)} = \binom{9n-3}{i} c_i^{(2)}, \qquad \bar{p}_i^{(2)} = m\binom{9n-3}{i} p_i^{(2)}, \qquad \bar{q}_i^{(2)} = n\binom{9n-2}{i} q_i^{(2)},$$

$$\bar{d}_i^{(2)} = \sum_{\substack{j_1,j_2,j_3=0\\ j_1+j_2+j_3=i}}^{3n-1} \bar{d}_{j_1}\bar{d}_{j_2}\bar{d}_{j_3}(\bar{m}-1)/\bar{m}^2, \qquad \bar{f}_i = \sum_{\substack{j_1,j_2,j_3=0\\ j_1+j_2+j_3=i}}^{3n-1} \bar{d}_{j_1}\bar{p}_{j_2}^{(1)}\bar{p}_{j_3}^{(1)}(m-1)/m\bar{m},$$

$$\bar{g}_i = 2\sum_{\substack{j_1,j_2=0\\ j_1+j_2+j_3=i}}^{3n-1}\sum_{j_3=0}^{3n} \bar{d}_{j_1}\bar{p}_{j_2}^{(1)}\bar{q}_{j_3}^{(1)}/\bar{m}, \qquad \bar{t}_i = \sum_{\substack{j_1=0\\ j_1+j_2+j_3=i}}^{3n-1}\sum_{j_2,j_3=0}^{3n} \bar{d}_{j_1}\bar{q}_{j_2}^{(1)}\bar{q}_{j_3}^{(1)}(n-1)/\bar{m}n. \tag{3.20}$$

It is clear that $\bar{d}_0^{(2)}$ is nonzero. Let $[H^2]=[D^{21}]^{-1}$, then equation (3.19) is equivalent to

$$\begin{aligned}\binom{n}{i}(\bar{b}_i-\bar{a}_i) = \sum_{j=0}^{i}\sum_{k=j}^{i} h_{i-k}^{(2)}\Bigg(&\binom{n}{j}\bar{c}_{k-j}^{(2)}Q_{0_j} + \binom{n}{j}\bar{p}_{k-j}^{(2)}a_j + \binom{n-1}{j}\bar{q}_{k-j}^{(2)}e_j\\ &+\binom{n}{j}\bar{f}_{k-j}(b_j-a_j) + \binom{n-1}{j}\bar{g}_{k-j}(a_{j+1}-a_j) + \binom{n-2}{j}\bar{t}_{k-j}(e_{j+1}-e_j)\Bigg)\end{aligned}$$

$$\text{for } i=0,\ldots,n \tag{3.21}$$

with the constraint equation

$$\begin{aligned}&([D^{22}][H^2][C^{21}]-[C^{22}])[CQ] + ([D^{22}][H^2][p^{21}]-[p^{22}])[CA]\\ &+([D^{22}][H^2][q^{21}]-[q^{22}])[CE] + ([D^{22}][H^2][F^1]-[F^2])[CBA]\\ &+([D^{22}][H^2][G^1]-[G^2])[CAA] + ([D^{22}][H^2][T^1]-[T^2])[CEE]=0\end{aligned} \tag{3.22}$$

where

$$h_0^{(2)} = \frac{1}{\bar{d}_0^{(2)}}, \qquad h_i^{(2)} = -h_0^{(2)}\sum_{j=0}^{i-1}\bar{d}_{i-j}^{(2)}h_j^{(2)}.$$

Combining the above results, one can conclude:

Theorem 3. *A necessary and sufficient condition for curvature continuity between adjacent 'C^0 continuous' rectangular rational Bézier patches r and $\bar{r}$ along the common boundary CB is that there exist real numbers* $\{d_i\}_0^{3n-1}$, $\{c_i^{(1)}\}_0^{3n-1}$, $\{p_i^{(1)}\}_0^{3n-1}$, $\{q_i^{(1)}\}_0^{3n}$, $\{c_i^{(2)}\}_0^{9n-3}$, $\{p_i^{(2)}\}_0^{9n-3}$, $\{q_i^{(2)}\}_0^{9n-2}$ *satisfying* (3.14) *and* (3.19).

Theorem 4. *Given a regular rational Bézier patch r and a set of real numbers* $\{d_i\}$, $\{c_i^{(j)}\}$, $\{p_i^{(j)}\}$, $\{q_i^{(j)}\}$ *which satisfy the constraint equations* (3.18) *and* (3.22). *Let the first three columns of the control points and the weights of rational Bézier patch $\bar{r}$ be determined by* (3.10), (3.17) *and* (3.21). *Then $\bar{r}$ meets r curvature continuously.*

3.3. Some GC^2 sufficient conditions

The GC^2 condition in Theorem 3 contains $39n-4$ coefficients $\{d_i\}$, $c_i^{(j)}\}$, $\{p_i^{(j)}\}$, $\{q_i^{(j)}\}$ and many constraints which those coefficients have to satisfy. However, so many free coefficients are not necessary, and to solve the complicated constraint equations is very difficult. It is not convenient in practice. For this reason one reduces the degrees of the connecting functions. Thereby a series of GC^2 sufficient conditions is obtained.

Theorem 5. *Two adjacent rational Bézier patches r and $\bar{r}$ expressed by* (3.1) *and* (3.2) *are curvature continuous if* (2.13), (3.3) *and* (3.4) *hold, where $D(v)$ ($\not\equiv 0$), $\bar{c}_i$, $\bar{p}_i$, $\bar{q}_i$ ($i = 1, 2$) are all polynomials of v, and their degrees are as follows*:

	$D, \bar{c}_1, \bar{p}_1$	$\bar{q}_1$	$\bar{c}_2, \bar{p}_2$	$\bar{q}_2$
degree $\leqslant$	σ	$\sigma + 1$	3σ	$3\sigma + 1$

with a fixed integer σ ranging from 0 *to* $3n - 1$.

After some analogous discussions, one can easily obtain the representations of connecting patches similar to (3.10), (3.17) and (3.21), which are omitted here. In general, the sufficient conditions reduce the complexity of the constraint equations, but the equations are still difficult to solve except in some special cases. In the case of $\sigma = 0$, D, $\bar{c}_i$, $\bar{p}_i$ are constants, $\bar{q}_i$ is linear, and the constraint equations are trivial. Assuming

$$\begin{aligned} &D(v) = 1, \qquad \bar{c}_i = \gamma_i \bar{m}!/(\bar{m} - i)!, \\ &\bar{p}_i = \alpha_i \bar{m}!/(\bar{m} - i)!m, \qquad \bar{q}_i = \left(\beta_i^0(1 - v) + \beta_i^1 v\right)\bar{m}!/(\bar{m} - i)!, \end{aligned} \tag{3.23}$$

the GC^2 constructions of the rational Bézier patch $\bar{r}$ are given as follows:

(a) $\quad \bar{Q}_{0i} = Q_{0i}$ or $\bar{\omega}_{0i} = \omega_{0i}$, $\quad \bar{P}_{0i} = P_{0i}$,

(b) $\quad \bar{a}_i = \gamma_1 Q_{0i} + \alpha_1 a_i + \beta_1^1 i e_{i-1} + \beta_1^0 (n - i) e_i$,

(c)
$$\begin{aligned} \bar{b}_i = {} & \bar{a}_i + \gamma_2 Q_{0i} + \alpha_2 a_i + \beta_2^1 i e_{i-1} + \beta_2^0 (n - i) e_i \\ & + \frac{\bar{m}}{\bar{m} - 1}\Bigg[\frac{m - 1}{m}\alpha_1^2(b_i - a_i) + 2\alpha_1\left(\beta_1^1 i(a_i - a_{i-1}) + \beta_1^0(n - i)(a_{i+1} - a_i)\right) \\ & + (n - i)(n - i - 1)\left(\beta_1^0\right)^2(e_{i+1} - e_i) + 2i(n - i)\beta_1^0\beta_1^1(e_i - e_{i-1}) \\ & + i(i - 1)\left(\beta_1^1\right)^2(e_{i-1} - e_{i-2})\Bigg]. \end{aligned}$$

It is well known that the real numbers γ_i, α_i, β_i^0, β_i^1 are coefficients which can be chosen freely. One can generate various complex shapes for surface $\bar{r}$ by choosing them flexibly. So these coefficients are called the shape parameters in the literature.

4. GC^2 conditions for triangular rational Bézier patches

In addition to the rectangular Bézier surfaces, the Bernstein–Bézier surface over a triangular domain is also a popular model in CAGD application, especially for modeling surfaces of arbitrary topology. This section focuses on the connecting conditions for curvature continuity of triangular rational Bézier patches, and rectangular patches as well.

Theorem 6. *Suppose that $r(u, v)$ and $\bar{r}(\bar{u}, \bar{v})$ are rectangular or triangular rational Bézier patches which are 'C^0 continuous' along CB: $u = \bar{u} = 0$, $0 \leqslant v = \bar{v} \leqslant 1$. Then r and $\bar{r}$ are curvature continuous iff* (3.3), (3.4) *hold, where D ($\not\equiv 0$), $\bar{c}_i$, $\bar{p}_i$, $\bar{q}_i$ ($i = 1, 2$) are all polynomials of v, and their degrees are determined according to Table* 1.

Table 1

degree $\leqslant$	$\bar{r}: n$ $r: n$	$\bar{r}: \bar{m} \times n$ $r: n$	$\bar{r}: n$ $r: m \times n$
D	$3n-2$	$3n-2$	$3n-1$
$\bar{c}_1$	$3n-3$	$3n-2$	$3n-2$
$\bar{p}_1$	$3n-2$	$3n-1$	$3n-2$
$\bar{q}_1$	$3n-2$	$3n-1$	$3n-1$
$\bar{c}_2$	$9n-8$	$9n-6$	$9n-5$
$\bar{p}_2$	$9n-7$	$9n-5$	$9n-5$
$\bar{q}_2$	$9n-7$	$9n-5$	$9n-4$

$s: n$ means surface s is a triangular patch of degree n
$s: m \times n$ means surface s is a rectangular patch of degree $m \times n$

The proof is completely analogous to that of Theorem 2, and the method for GC^2 constructions of rectangular patches can be directly applied into this section. In the following only two triangular rational Bézier patches r and $\bar{r}$ are considered as an illustration. Suppose that r and $\bar{r}$ have the representations

$$r: \quad Q(u, v) = \sum_{i+j+k=n} (P_{ijk}\omega_{ijk}, \omega_{ijk}) \frac{n!}{i!j!k!} u^i v^j (1-u-v)^k,$$

$$\bar{r}: \quad \bar{Q}(\bar{u}, \bar{v}) = \sum_{i+j+k=n} \left(\bar{P}_{ijk}\bar{\omega}_{ijk}, \bar{\omega}_{ijk}\right) \frac{n!}{i!j!k!} \bar{u}^i \bar{v}^j (1-\bar{u}-\bar{v})^k. \tag{4.1}$$

Let the connecting functions be written as

$$D(v) = \sum_{j=0}^{3n-2} d_j B_j^{3n-2}(v),$$

$$\bar{c}_1 = \sum_{j=0}^{3n-3} c_j^{(1)} B_j^{3n-3}(v), \qquad \bar{c}_2 = \sum_{j=0}^{9n-8} c_j^{(2)} B_j^{9n-8}(v),$$

$$\bar{p}_1 = \sum_{j=0}^{3n-2} p_j^{(1)} B_j^{3n-2}(v), \qquad \bar{p}_2 = \sum_{j=0}^{9n-7} p_j^{(2)} B_j^{9n-7}(v),$$

$$\bar{q}_1 = \sum_{j=0}^{3n-2} q_j^{(1)} B_j^{3n-2}(v), \qquad \bar{q}_2 = \sum_{j=0}^{9n-7} q_j^{(2)} B_j^{9n-7}(v)$$

and introduce the following notations (see Fig. 2)

$$Q_u = n \sum_{j=0}^{n-1} a_j B_j^{n-1}(v), \qquad Q_v = n \sum_{j=0}^{n-1} e_j B_j^{n-1}(v),$$

$$Q = \sum_{i=0}^{n} Q_{0i} B_i^n(v), \qquad Q_{uu} = n(n-1) \sum_{j=0}^{n-2} (b_j - a_j) B_j^{n-2}(v),$$

$$Q_{uv} = n(n-1) \sum_{j=0}^{n-2} (a_{j+1} - a_j) B_j^{n-2}(v),$$

$$Q_{vv} = n(n-1) \sum_{j=0}^{n-2} (e_{j+1} - e_j) B_j^{n-2}(v),$$

$$\bar{Q}_{\bar{u}} = n \sum_{j=0}^{n-1} \bar{a}_j B_j^{n-1}(v), \qquad \bar{Q}_{\bar{u}\bar{u}} = n(n-1) \sum_{j=0}^{n-2} \left(\bar{b}_j - \bar{a}_j\right) B_j^{n-2}(v).$$

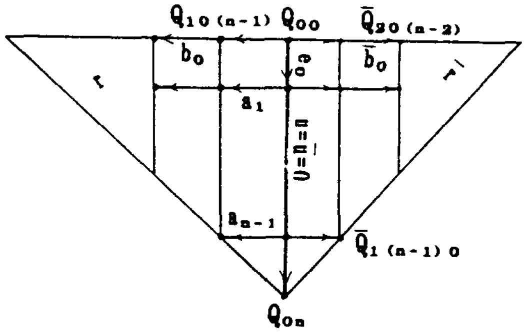

Fig. 2. Notations for the directional derivatives of triangular patches

After analogous deduction to the rectangular case, one obtains equations (3.10), (3.14) and (3.19) with some changes in variable assignment. For instance,

$$[C\bar{A}] = \begin{pmatrix} \binom{n-1}{0}\bar{a}_0 \\ \vdots \\ \binom{n-1}{n-1}\bar{a}_{n-1} \end{pmatrix}, \qquad [CA] = \begin{pmatrix} \binom{n-1}{0}a_0 \\ \vdots \\ \binom{n-1}{n-1}a_{n-1} \end{pmatrix},$$

$$[C\bar{B}\bar{A}] = \begin{pmatrix} \binom{n-2}{0}(\bar{b}_0 - \bar{a}_0) \\ \vdots \\ \binom{n-2}{n-2}(\bar{b}_{n-2} - \bar{a}_{n-2}) \end{pmatrix},$$

$$[CBA]\begin{pmatrix} \binom{n-2}{0}(b_0 - a_0) \\ \vdots \\ \binom{n-2}{n-2}(b_{n-2} - a_{n-2}) \end{pmatrix}, \qquad [CAA] = \begin{pmatrix} \binom{n-2}{0}(a_1 - a_0) \\ \vdots \\ \binom{n-2}{n-2}(a_{n-1} - a_{n-2}) \end{pmatrix},$$

$$\begin{pmatrix} D^{11}_{n\times n} \\ D^{12}_{(3n-2)\times n} \end{pmatrix} = \begin{pmatrix} \bar{d}_0 & & \\ \vdots & \ddots & \bar{d}_0 \\ \bar{d}_{3n-2} & & \vdots \\ & \ddots & \bar{d}_{3n-2} \end{pmatrix}, \qquad \begin{pmatrix} C^{11}_{n\times(n+1)} \\ C^{12}_{(3n-2)\times(n+1)} \end{pmatrix} = \begin{pmatrix} \bar{c}^{(1)}_0 & & \\ \vdots & \ddots & \bar{c}^{(1)}_0 \\ \bar{c}^{(1)}_{3n-3} & & \vdots \\ & \ddots & \bar{c}^{(1)}_{3n-3} \end{pmatrix},$$

$$\begin{pmatrix} D^{21}_{(n-1)\times(n-1)} \\ D^{22}_{(9n-6)\times(n-1)} \end{pmatrix} = \begin{pmatrix} \bar{d}^{(2)}_0 & & \\ \vdots & \ddots & \bar{d}^{(2)}_0 \\ \bar{d}^{(2)}_{9n-6} & & \vdots \\ & \ddots & \bar{d}^{(2)}_{9n-6} \end{pmatrix}, \qquad \begin{pmatrix} C^{21}_{(n-1)\times(n+1)} \\ C^{22}_{(9n-6)\times(n+1)} \end{pmatrix} = \begin{pmatrix} \bar{c}^{(2)}_0 & & \\ \vdots & \ddots & \bar{c}^{(2)}_0 \\ \bar{c}^{(2)}_{9n-8} & & \vdots \\ & \ddots & \bar{c}^{(2)}_{9n-8} \end{pmatrix}$$

where

$$\bar{d}_i = \binom{3n-2}{i} d_i, \qquad \bar{c}_i^{(1)} = \binom{3n-3}{i} c_i^{(1)},$$

$$\bar{p}_i^{(1)} = \binom{3n-2}{i} p_i^{(1)}, \qquad \bar{q}_i^{(1)} = \binom{3n-2}{i} q_i^{(1)},$$

$$\bar{c}_i^{(2)} = \binom{9n-8}{i} c_i^{(2)}/n(n-1), \qquad \bar{p}_i^{(2)} = \binom{9n-7}{i} p_i^{(2)}/(n-1),$$

$$\bar{q}_i^{(2)} = \binom{9n-7}{i} q_i^{(2)}/(n-1),$$

$$\bar{d}_i^{(2)} = \sum_{\substack{j_1, j_2, j_3 = 0 \\ j_1 + j_2 + j_3 = i}}^{3n-2} \bar{d}_{j_1} \bar{d}_{j_2} \bar{d}_{j_3}, \qquad \bar{f}_i = \sum_{\substack{j_1, j_2, j_3 = 0 \\ j_1 + j_2 + j_3 = i}}^{3n-2} \bar{d}_{j_1} \bar{p}_{j_2}^{(1)} \bar{p}_{j_3}^{(1)},$$

$$\bar{g}_i = 2 \sum_{\substack{j_1, j_2, j_3 = 0 \\ j_1 + j_2 + j_3 = i}}^{3n-2} \bar{d}_{j_1} \bar{p}_{j_2}^{(1)} \bar{q}_{j_3}^{(1)}, \qquad \bar{t}_i = \sum_{\substack{j_1, j_2, j_3 = 0 \\ j_2 + j_2 + j_3 = i}}^{3n-2} \bar{d}_{j_1} \bar{q}_{j_2}^{(1)} \bar{q}_{j_3}^{(1)}.$$

Let $[H^1] = [D^{11}]^{-1}$, $[H^2] = [D^{21}]^{-1}$, then

$$[H^1] = \begin{pmatrix} h_0^{(1)} & & \\ \vdots & \ddots & \\ h_{n-1}^{(1)} & \cdots & h_0^{(1)} \end{pmatrix}, \qquad [H^2] = \begin{pmatrix} h_0^{(2)} & & \\ \vdots & \ddots & \\ h_{n-2}^{(2)} & \cdots & h_0^{(2)} \end{pmatrix}.$$

Thus the construction of the first three columns of the control net is easily obtained by

$$\bar{Q}_{0i(n-i)} = Q_{0i(n-i)}, \quad i = 0, \ldots, n,$$

$$\binom{n-1}{i} \bar{a}_i = \sum_{j=0}^{i} \sum_{k=j}^{i} h_{i-k}^{(1)} \left(\binom{n}{j} \bar{c}_{k-j}^{(1)} Q_j + \binom{n-1}{j} \bar{p}_{k-j}^{(1)} a_j + \binom{n-1}{j} \bar{q}_{k-j}^{(1)} e_j \right),$$

$$i = 0, \ldots, n-1,$$

$$\binom{n-2}{i} (\bar{b}_i - \bar{a}_i) = \sum_{j=0}^{i} \sum_{k=j}^{i} h_{i-k}^{(2)} \left[\binom{n}{j} \bar{c}_{k-j}^{(2)} Q_j + \binom{n-1}{j} \left(\bar{p}_{k-j}^{(2)} a_j + \bar{q}_{k-j}^{(2)} e_j \right) \right.$$

$$\left. + \binom{n-2}{j} \left(\bar{f}_{k-j}(b_j - a_j) + \bar{g}_{k-j}(a_{j+1} - a_j) + \bar{t}_{k-j}(e_{j+1} - e_j) \right) \right],$$

$$i = 0, \ldots, n-2 \tag{4.2}$$

with the constraint equations (3.18) and (3.22).

Clearly, this construction provides many free coefficients, and many restrictions too. For convenience in practice, some simple sufficient conditions are more useful.

Theorem 7. *Triangular rational Bézier patches $\bar{r}$ and r of degree n meet curvature continuously if* (2.13), (3.3) *and* (3.4) *hold, where D ($\not\equiv 0$),$\bar{c}_i$, $\bar{p}_i$, $\bar{q}_i$ ($i = 1, 2$) are all polynomials of v, and their degrees are as follows*:

	$D, \bar{p}_1, \bar{q}_1$	$\bar{c}_1$	$\bar{p}_2, \bar{q}_2$	$\bar{c}_2$
degree $\leqslant$	σ	$\sigma - 1$	$3\sigma - 1$	$3\sigma - 2$

with a fixed integer σ ranging from 0 *to* $3n - 2$.

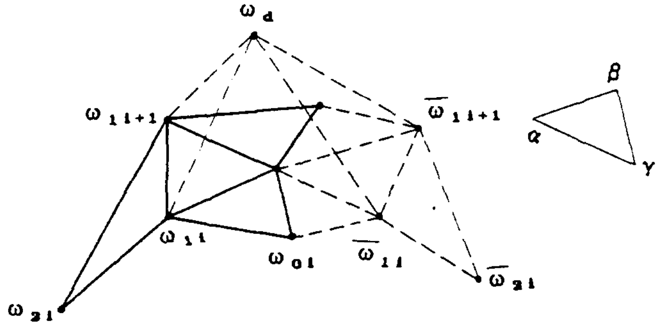

Fig. 3. GC^2 construction for weights.

In the above theorem, polynomials vanish by convention if their degrees are less than 0. As an example, take $\sigma = 0$ and let $D(v) = 1$. Then

$$\bar{c}_1 = \bar{c}_2 = \bar{p}_2 = \bar{q}_2 = 0, \qquad \bar{p}_1 = \alpha, \qquad q_1 = \beta$$

where α, β are constants which can be freely chosen. For simplicity, the convention is adopted that $\omega_{ij} = \omega_{ij(n-i-j)}$, $p_{ij} = p_{ij(n-i-j)}$ and so on. Thus the construction of the patch $\bar{r}$ is as follows (see Fig. 3 and Fig. 4), which is similar to that in [Kahmann '83] or [Farin '86].

$$\bar\omega_{0i} = \omega_{0i}, \qquad \bar{P}_{0i} = P_{0i}, \tag{4.3}$$

$$\begin{cases} \bar\omega_{1i} = \alpha\omega_{1i} + \beta\omega_{0i+1} + \gamma\omega_{0i}, \\ \bar{P}_{1i} = (\alpha\omega_{1i}P_{1i} + \beta\omega_{0i+1}P_{0i+1} + \gamma\omega_{0i}P_{0i})/\bar\omega_{1i}, \end{cases} \tag{4.4}$$

$$\begin{cases} \bar\omega_{2i} = \alpha\omega_d + \beta\bar\omega_{1i+1} + \gamma\bar\omega_{1i}, \\ \bar{P}_{2i} = \left(\alpha\omega_d P_d + \beta\bar\omega_{1i+1}\bar{P}_{1i+1} + \gamma\bar\omega_{1i}\bar{P}_{1i}\right)/\bar\omega_{2i} \end{cases} \tag{4.5}$$

where

$$\gamma = 1 - \alpha - \beta, \qquad \omega_d = \alpha\omega_{2i} + \beta\omega_{1i+1} + \gamma\omega_{1i},$$
$$P_d = (\alpha\omega_{2i}P_{2i} + \beta\omega_{1i+1}P_{1i+1} + \gamma\omega_{1i}P_{1i})/\omega_d. \tag{4.6}$$

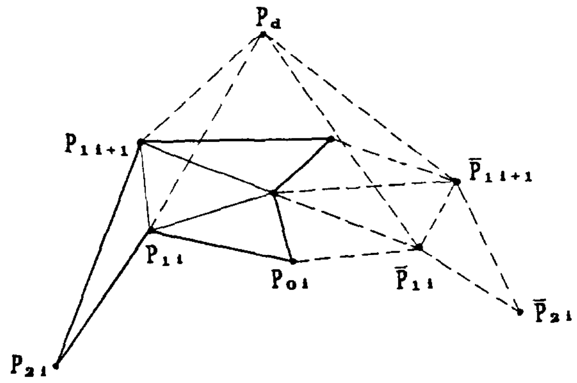

Fig. 4. GC^2 construction for control points.

5. Conclusion

The conditions for curvature continuity of rational Bézier patches have been presented, some of which can be expressed by the control points and the connecting functions, so they are easy to carry out on a computer. In general, the connecting functions, which are in polynomial or rational polynomial form, are not freely chosen. They are subject to some constraint conditions. In the case of adjoining two rectangular rational Bézier patches, the constraint conditions vanish if one makes the choice that $D(v)$, $\bar{c}_i(v)$, $\bar{p}_i(v)$ are constants and $\bar{q}_i(v)$ are linear functions. This provides great convenience and flexibility for constructing a curvature continuous surface and controlling the shape of patches. However, adjoining two triangular rational Bézier patches appears to have more restrictions on the connecting functions. Therefore, it is an interesting problem if one could obtain more free coefficients to control the shape while keeping the constraint equations as simple as possible.

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