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Perturbing Bézier coefficients for best constrained degree reduction in the L_2-norm

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Abstract

This paper first shows how the Bézier coefficients of a given degree n polynomial are perturbed so that it can be reduced to a degree m $(< n)$ polynomial with the constraint that continuity of a prescribed order is preserved at the two endpoints. The perturbation vector, which consists of the perturbation coefficients, is determined by minimizing a weighted Euclidean norm. The optimal degree $n-1$ approximation polynomial is explicitly given in Bézier form. Next the paper proves that the problem of finding a best L_2-approximation over the interval $[0, 1]$ for constrained degree reduction is equivalent to that of finding a minimum perturbation vector in a certain weighted Euclidean norm. The relevant weights are derived. This result is applied to computing the optimal constrained degree reduction of parametric Bézier curves in the L_2-norm.

Keywords: Degree reduction; Endpoint constraints; Bézier form; L_2-norm; Least squares perturbation

1. Introduction

Degree reduction of polynomial curves and surfaces is a common process in computer aided geometric design. It amounts to approximating a polynomial by a lower degree polynomial. This process is useful for many tasks in geometric modeling, such

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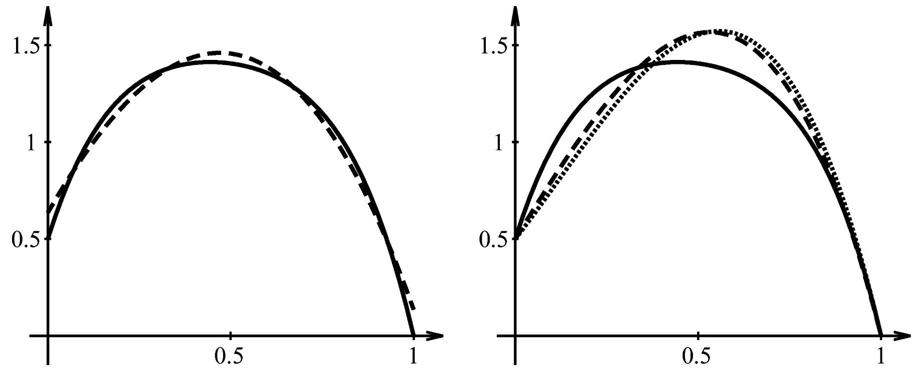
as data exchange, data compression, data comparison, Boolean operations and rendering [9,13]. For example, degree reduction is needed when data are transferred from one modeling system to another and these systems have different limitations on the maximum degree of polynomials. Degree reduction can also be used to generate a piecewise continuous lower degree approximation to a given curve or surface so as to simplify some geometric or graphical algorithms like intersection calculation or rendering.

There have been many methods developed for degree reduction. Forrest [6] and Farin [5] considered the inverse of degree elevation and obtained two sets of control points. A simple convex combination of these two sets of control points was used to generate the control points for the degree reduced curve. This approach is easy to compute, but not optimal in the usual L_p-norm for any $p = 1, 2, \ldots, +\infty$. Since degree reduction is a problem of approximation in nature, methods in the classical approximation theory can be employed. In particular, the optimal approximations with respect to the L_∞ or L_2 metric are of interest. Watkins and Worsey [13] used the Chebyshev economization to produce the best L_∞-approximation of degree $n-1$ to a given degree n polynomial. This best approximation, however, does not interpolate the given curve at the endpoints. The endpoint constraints that guarantee a prescribed order of continuity are frequently required in many applications and especially when degree reduction is combined with subdivision to generate continuous, piecewise approximations. A modified economization procedure was proposed by Bogacki et al. [1] which could achieve the best uniform approximation with endpoint interpolation. Lachance [8] and Eck [3] investigated in depth the Chebyshev economization for the best L_∞-approximation with continuity constraints at the boundaries. In general, as pointed out in [4], computing a C^k-constrained ($k \geqslant 1$) best degree reduction in the L_∞-norm needs a lot of implementation effort, and it seems that there is no explicit formula for the degree reduced curve. These deficiencies can be avoided by using the L_2-norm. The endpoint constrained degree reduction algorithm that minimized the L_2-norm was analyzed by Eck [4]. His method optimized Forrest's convex combination. The optimal degree reduction with respect to various norms was studied by Brunnett et al. [2] who also focused on separability of degree reduction into the different spatial components.

In CAGD the Bézier form of a polynomial is a popular representation for curves since the control polygons capture many geometric properties of curves. It is thus tempting to perform degree reduction based on just the control points, rather than the polynomials. Recently, Lutterkort, Peters and Reif [9] discovered a surprising coincidence: finding a best L_2-approximation over $[0, 1]$ from polynomials of degree m to a given polynomial p of degree n ($> m$) is equivalent to finding the best Euclidean approximation of the vector of Bézier coefficients of p from the vectors of Bézier coefficents of polynomials of degree m raised to degree n. This result can be extended to the multivariate case [10], but does not hold if degree reduction is subject to additional constraints like endpoint interpolation [9]. For example, Fig. 1(a) shows a degree four polynomial (solid):



(a) Unconstrained case (b) Constrained case

Fig. 1. Examples of degree reduction.

$$-\frac{19}{2}x^4 + 18x^3 - 15x^2 + 6x + \frac{1}{2},$$

and its best L_2-approximation of degree three (dashed):

$$-x^3 - \frac{39}{14}x^2 + \frac{23}{7}x + \frac{89}{140}.$$

This best degree three polynomial can be obtained by minimizing the Euclidean norm of the perturbation vector of the Bézier coefficients. However, if some end-point interpolations (C^0 at $t = 0$ and C^1 at $t = 1$) are enforced, the Bézier coefficients based Euclidean approximation method results in a non-optimal polynomial:

$$-\frac{87}{26}x^3 - \frac{3}{13}x^2 + \frac{40}{13}x + \frac{1}{2},$$

which is shown as the dotted line in Fig. 1(b), where the dashed line represents the best constrained L_2-approximation:

$$-\frac{73}{16}x^3 + \frac{13}{8}x^2 + \frac{39}{16}x + \frac{1}{2}.$$

This paper extends Lutterkort et al.'s discovery and analysis to the constrained degree reduction case. We prove that finding the best L_2-approximation for degree reduction with endpoint constraints (the continuity order could be different at the two endpoints) is equivalent to finding the best constrained approximation of Bézier coefficients under a certain weighted Euclidean norm. For instance, if the weighted Euclidean norm with the weights {1, 20/3, 9, 1, 1} is used in the above example, we can get the polynomial displayed as the dashed line in Fig. 1(b). These weights are derived from Eq. (24) later in the paper by using $k = 1$, $l = 2$ and $n = 4$. Furthermore, in computing the best degree reduced polynomial, unlike the approach in [9] that took use of the degree-raising matrices, we directly perturb the Bézier coefficients of the given polynomial so as to impose an exact degree reduction on the perturbed polynomial. In this way, we can easily get the explicit formula for the degree reduced polynomial. Since it has been shown in [2] that for any L_p-norm, the optimal degree reduction where the norm is applied to the Euclidean distance function of two curves is identical to the optimal component-wise degree reduction, the result obtained in this paper can be applied to computing the degree reduction of parametric Bézier curves.

The paper is organized as follows. Section 2 gives a review on how the best constrained L_2-approximation can be solved by the classical method in approximation theory, which will be used in Section 4. Section 3 proposes a new degree reduction algorithm. This algorithm is based on Bézier coefficients. The explicit formulas for the new Bézier coefficients are derived. Section 4 proves that under a certain weighted Euclidean norm, the method developed in Section 3 produces the same output as the method based on the L_2-norm does. The relevant weights are derived. Finally in Section 5 some practical issues are discussed and a few examples are presented.

2. Constrained L_2-approximation

We begin with the constrained approximation problem: Given a degree n polynomial $f(t)$, find a degree $m(< n)$ polynomial $g(t)$ such that

- $g(t)$ and $f(t)$ have the same first $k-1$ derivatives at $t=0$ and the same first $l-1$ derivatives at $t=1$, i.e.

$$\begin{aligned} g^{(i)}(0) &= f^{(i)}(0), \quad i=0,\dots,k-1; \\ & \qquad\qquad\qquad\qquad l+k \leqslant m \\ g^{(j)}(1) &= f^{(j)}(1), \quad j=0,\dots,l-1; \end{aligned} \tag{1}$$

- $g(t)$ minimizes the L_2-error $E = \left(\int_0^1 (f(t)-g(t))^2\,\mathrm{d}t\right)^{1/2}$ for all such possible polynomials of degree $\leqslant m$ that satisfy the endpoint constraints (1).

This problem can be solved through two stages. In the first stage, we construct a degree $k+l-1$ polynomial $F(t)$ interpolating $f(t)$ at $t=0$ up to the $(k-1)$th order continuity and at $t=1$ up to the $(l-1)$th order continuity. This polynomial can be written as

$$F(t) = \sum_{i=0}^{k-1} f^{(i)}(0) H_i^{k,l}(t) + \sum_{j=0}^{l-1} f^{(j)}(1) G_j^{l,k}(t), \tag{2}$$

where $H_i^{k,l}(t)$ and $G_j^{l,k}(t)$ are the degree $k+l-1$ polynomials satisfying

$$\begin{aligned} &\left.\frac{\mathrm{d}^j H_i^{k,l}(t)}{\mathrm{d}t^j}\right|_{t=0} = \begin{cases} 1, & j=i \\ 0, & \text{otherwise} \end{cases}, \quad \left.\frac{\mathrm{d}^h H_i^{k,l}(t)}{\mathrm{d}t^h}\right|_{t=1} = 0, \\ &i,j=0,\dots,k-1, \quad h=0,\dots,l-1, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\left.\frac{\mathrm{d}^j G_i^{l,k}(t)}{\mathrm{d}t^j}\right|_{t=1} = \begin{cases} 1, & j=i \\ 0, & \text{otherwise} \end{cases}, \quad \left.\frac{\mathrm{d}^h G_i^{l,k}(t)}{\mathrm{d}t^h}\right|_{t=0} = 0, \\ &i,j=0,\dots,l-1, \quad h=0,\dots,k-1, \end{aligned} \tag{4}$$

Both $H_i^{k,l}(t)$ and $G_j^{l,k}(t)$ have $k+l$ degrees of freedom, and $k+l$ constraints as well. It is easy to show that they are uniquely determined (see Fig. 2 for the case of $k=2$ and $l=3$). Usually these functions are called Hermite basis functions.

The second stage is then to determine $g(t)-F(t)$ (or equivalently $g(t)$). Note that the polynomials $f(t)-F(t)$ and $g(t)-F(t)$ have k-fold zeros at $t=0$ and l-fold zeros at $t=1$. A common factor $t^k(1-t)^l$ can be factored out from $f(t)-F(t)$ and $g(t)-F(t)$. Thus, the settings of $f(t)-F(t)=t^k(1-t)^l F_{n-k-l}(t)$ and $g(t)-F(t)=t^k(1-t)^l G_{m-k-l}(t)$ with a degree $n-k-l$ polynomial $F_{n-k-l}(t)$ and a degree $m-k-l$ polynomial $G_{m-k-l}(t)$ are appropriate. Next we use the least squares method to evaluate $G_{m-k-l}(t)$.

In least squares problems, choosing proper basis functions often simplifies the computation. It allows the coefficients of the approximant to be determined directly, without solving a linear system. In our case, the appropriate basis functions should be orthogonal over the interval $[0,1]$ with respect to the weighting function $t^{2k}(1-t)^{2l}$. The Jacobi polynomials that have been thoroughly studied in the past [11,12] are just such a set of orthogonal polynomials. Denote by $J_i(t)$ the Jacobi polynomials of degree i with respect to the weighting function $t^{2k}(1-t)^{2l}$, $(k,l \geqslant 0)$ on the interval $[0,1]$. They are defined by Rodrigues' formula

$$J_i(t)=\frac{(-1)^i}{i!}t^{-2k}(1-t)^{-2l}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[t^{2k+i}(1-t)^{2l+i}\right] \quad \text{for} \quad i=1,2,\ldots \tag{5}$$

and $J_0(t)=1$. The first few polynomials are $J_0(t)=1$, $J_1(t)=2(k+l+1)t-(2k+1)$ and $J_2(t)=(k+l+2)(2k+2l+3)t^2-2(k+1)(2k+2l+3)t+(k+1)(2k+1)$. These Jacobi polynomials satisfy the orthogonality relation

$$\int_0^1 t^{2k}(1-t)^{2l}J_i(t)J_j(t)\,\mathrm{d}t=\begin{cases}\frac{1}{2i+2k+2l+1}\dfrac{\binom{i+2k}{2k}}{\binom{i+2k+2l}{2k}} & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases} \tag{6}$$

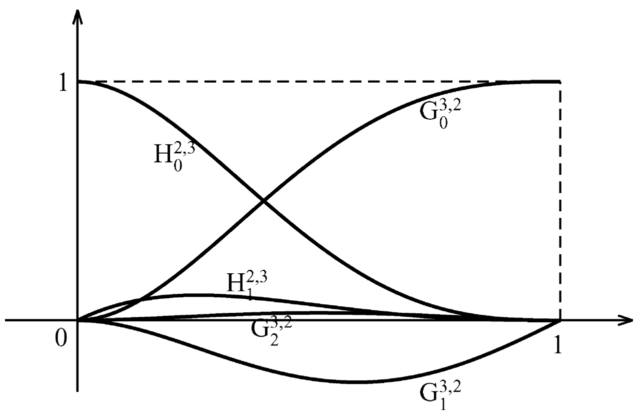


Fig. 2. The graphs of $H_i^{2,3}$ and $G_i^{3,2}$.

Now coming back to our problem of determining $G_{m-k-l}(t)$, we assume $F_{n-k-l}(t)$ is expressed:

$$F_{n-k-l}(t) = a_0 J_0(t) + \cdots + a_{n-k-l} J_{n-k-l}(t), \tag{7}$$

where the coefficients a_i can actually be evaluated by

$$a_i = \frac{(2i+2k+2l+1)\binom{i+2k+2l}{2k}}{\binom{i+2k}{2k}} \int_0^1 t^k (1-t)^l (f(t) - F(t)) J_i(t)\, \mathrm{d}t.$$

We also write $G_{m-k-l}(t)$ in terms of Jacobi polynomials symbolically: $G_{m-k-l}(t) = b_0 J_0(t) + \cdots + b_{m-k-l} J_{m-k-l}(t)$ where the b_i are the coefficients to be determined.

Consider the square of the L_2-error

$$\begin{aligned} E^2 &= \int_0^1 ([f(t) - F(t)] - [g(t) - F(t)])^2\, \mathrm{d}t \\ &= \int_0^1 t^{2k}(1-t)^{2l} \left(\sum_{i=0}^{m-k-l} (a_i - b_i) J_i(t) + \sum_{i=m-k-l+1}^{n-k-l} a_i J_i(t) \right)^2 \mathrm{d}t. \end{aligned}$$

Differentiating it with respect to the coefficient b_j gives

$$\frac{\partial E^2}{\partial b_j} = -2 \int_0^1 t^{2k}(1-t)^{2l} \left(\sum_{i=0}^{m-k-l} (a_i - b_i) J_i(t) J_j(t) + \sum_{i=m-k-l+1}^{n-k-l} a_i J_i(t) J_j(t) \right) \mathrm{d}t.$$

To minimize E, we equate this derivative to zero. Upon invoking the orthogonality relation, we obtain $b_i = a_i$ for $i = 0, \ldots, m-k-l$. Thus, the best degree m constrained L_2-approximant is

$$g(t) = F(t) + t^k (1-t)^l [a_0 J_0(t) + \cdots + a_{m-k-l} J_{m-k-l}(t)]. \tag{8}$$

Eq. (8) shows that if the decomposition

$$f(t) = F(t) + t^k (1-t)^l [a_0 J_0(t) + \cdots + a_{n-k-l} J_{n-k-l}(t)] \tag{9}$$

for a given polynomial $f(t)$ is available, the constrained L_2-approximation of degree m can be immediately obtained by just removing the last $n-m$ terms in the square bracket of (9). This implies that a single multidegree reduction is equivalent to step-by-step reductions of one degree at a time.

3. Weighted least squares perturbation

Instead of pursuing the decomposition (9) for degree reduction, in this section we perform degree reduction directly based on the Bézier coefficients of the given polynomial. A similar idea was also used in [7].

A degree $n-1$ Bézier polynomial $g(t) = \sum_{i=0}^{n-1} q_i B_i^{n-1}(t)$ can always be degree elevated to a degree n Bézier polynomial

$$f(t) = \sum_{i=0}^{n} p_i B_i^n(t) \tag{10}$$

with the new Bézier coefficients $p_0 = q_0; p_i = (i/n)q_{i-1} + ((n-i)/n)q_i, i = 1, \cdots, n-1;$ $p_n = q_{n-1}$. However, the converse is generally not true unless the coefficients p_i satisfy

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} p_i = 0. \tag{11}$$

This is because

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} p_i = \frac{1}{n!} \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$$

is the coefficient of the nth degree term of $f(t)$ or the nth forward difference of the Bézier coefficients. It is also easy to prove by induction that under the condition (11), we have (see [3])

$$q_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} \binom{n}{j} p_j, \quad i = 0, \ldots, n-1. \tag{12}$$

Therefore we hope to perturb the coefficients of the given polynomial $f(t)$ so that the perturbed polynomial can be degree reduced and meanwhile the perturbation is as small as possible in some measure. More specifically, given $f(t)$ as in (10) we want to find a perturbation vector $(\epsilon_0, \epsilon_1, \ldots, \epsilon_n)$ such that the perturbed polynomial

$$f_\epsilon(t) = \sum_{i=0}^{n} (p_i + \epsilon_i) B_i^n(t) \tag{13}$$

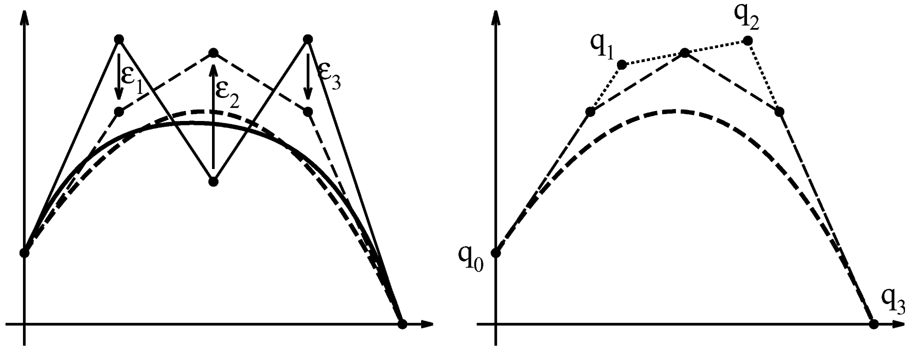
satisfies $\sum_{i=0}^{n} (-1)^i \binom{n}{i} (p_i + \epsilon_i) = 0$ and the sum $\sum_{i=0}^{n} w_i \epsilon_i^2$ is minimized for a given set of positive weights w_i. This is demonstrated by Fig. 3 where $\epsilon_0 = \epsilon_4 = 0$ and the polylines stand for the control polygons.

This formulation is very convenient for imposing the continuity conditions on the endpoints. If $f_\epsilon(t)$ is required to match $f(t)$ up to the $(k-1)$th order continuity at $t = 0$, just set $\epsilon_0 = \cdots = \epsilon_{k-1} = 0$. Similarly, $\epsilon_{n-l+1} = \cdots = \epsilon_n = 0$ guarantees C^{l-1} continuity between f_ϵ and $f(t)$ at $t = 1$. In the following we derive the explicit formula for ϵ_i under such constraints.

Introducing a Lagrange's multiplier λ and including the constraints into the objective function, we obtain a single unconstrained objective function:

$$L(\epsilon_k, \ldots, \epsilon_{n-l}; \lambda) = \sum_{i=k}^{n-l} w_i \epsilon_i^2 - \lambda \sum_{i=0}^{n} (-1)^i \binom{n}{i} p_i - \lambda \sum_{i=k}^{n-l} (-1)^i \binom{n}{i} \epsilon_i.$$

Taking the partial derivative of L with respect to ϵ_i and setting the derivative equal to zero lead to



(a) $f(t)$ (solid) and $f_\epsilon(t)$ (dashed) (b) Reduce $f_\epsilon(t)$ to degree 3

Fig. 3. Coefficient perturbation for C^0-continuous degree reduction.

$$\frac{1}{2}\frac{\partial L}{\partial \epsilon_i} = w_i\epsilon_i - \frac{1}{2}\lambda(-1)^i\binom{n}{i} = 0 \quad \text{for } i = k, \ldots, n-l$$

which gives

$$\epsilon_i = \frac{(-1)^i\binom{n}{i}}{w_i}\frac{\lambda}{2}, \quad i = k, \ldots, n-l. \tag{14}$$

In addition,

$$\frac{\partial L}{\partial \lambda} = -\sum_{i=0}^{n}(-1)^i\binom{n}{i}p_i - \sum_{i=k}^{n-l}(-1)^i\binom{n}{i}\epsilon_i = 0.$$

Combining this with (14), we have

$$\lambda = -2\sum_{i=0}^{n}(-1)^i\binom{n}{i}p_i \bigg/ \sum_{i=k}^{n-l}\frac{\binom{n}{i}^2}{w_i}. \tag{15}$$

Thus, we obtain the perturbed Bézier coefficients

$$p_i + \epsilon_i = p_i - \frac{(-1)^i\binom{n}{i}\sum_{j=0}^{n}(-1)^j\binom{n}{j}p_j}{w_i\sum_{j=k}^{n-l}\binom{n}{j}^2/w_j}, \quad i = k, \ldots, n-l. \tag{16}$$

With such perturbed coefficients, the degree of $f_\epsilon(t)$ is less than n. Substituting (16) into (12), we have the polynomial in degree $n-1$ Bézier form: $f_\epsilon(t) = \sum_{i=0}^{n-1} q_i B_i^{n-1}(t)$, with the Bézier coefficients q_i explicitly given by

$$\begin{cases} q_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} (-1)^j \binom{n}{j} p_j, \quad i = 0, \dots, k-1, \\ q_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} (-1)^j \binom{n}{j} p_j + \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=k}^{i} \frac{\binom{n}{j}^2}{w_j} \frac{\lambda}{2}, \quad i = k, \dots, n-l, \\ q_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=i}^{n-1} (-1)^j \binom{n}{j+1} p_{j+1}, \quad i = n-l+1, \dots, n-1, \end{cases} \tag{17}$$

where λ is given by (15).

Remark. (1) When the polynomial $f(t)$ is of degree $< n$, λ and thus all ϵ_i are equal to zero, which implies $f_\epsilon(t) = f(t)$. That is, the accurate solution can be obtained using the perturbation method whenever the exact degree reduction exists. (2) Note that the above approach reduces the degree only by one. If we would like $f_\epsilon(t)$ to be of degree m ($\leqslant n-1$), $n-m$ constraints

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i} p_i + \sum_{i=k}^{\min(n-l,j)} (-1)^i \binom{j}{i} \epsilon_i = 0$$

for $j = m+1, \dots, n$ should be put in. Construct the objective function by introducing $n-m$ Lagrange multipliers $\lambda_1, \dots, \lambda_{n-m}$:

$$L(\epsilon_k, \dots, \epsilon_{n-l}; \lambda_1, \dots, \lambda_{n-m}) = \sum_{i=k}^{n-l} w_i \epsilon_i^2 - \sum_{j=m+1}^{n} \lambda_{j-m} \left[\sum_{i=0}^{j} (-1)^i \binom{j}{i} p_i + \sum_{i=k}^{\min(n-l,j)} (-1)^i \binom{j}{i} \epsilon_i \right].$$

Then solving the minimization problem:

$$\text{minimize}\, L(\epsilon_k, \dots, \epsilon_{n-l}; \lambda_1, \dots, \lambda_{n-m})$$

gives the perturbation coefficients ϵ_i and thus the degree reduced polynomial $f_\epsilon(t)$. Compared to the case of $m = n-1$, ϵ_i are unlikely to have simple explicit formulas for $m < n-1$. Nevertheless, with the special choice of the weights, the multidegree reduction can be decomposed into several reductions of one degree at a time that have explicit formulas. This will be discussed in Section 5.1.

4. Equivalence of two methods

The L_2-norm of the polynomials and the weighted Euclidean norm of the coefficients are in general different. It is not surprising that minimizing these two norms in degree reduction leads to different results. However, under some circumstances, we can expect the same solution. Lutterkort et al. [9] have studied the case of unconstrained degree reduction. In this section we analyze the constrained case.

Let $\mathbb{P}_n$ denote the linear space of polynomials in t of degree less than or equal to n. We also introduce four notations: $\langle,\rangle_L$, $\langle,\rangle_{L+}$, $\langle,\rangle_E$ and $\langle,\rangle_{E+}$. $\langle,\rangle_L$ stands for the L_2-inner product on $\mathbb{P}_n$:

$$\langle f,g\rangle_L := \int_0^1 f(t)g(t)\,\mathrm{d}t \tag{18}$$

and $\langle,\rangle_{L+}$ stands for the weighted L_2-inner product with respect to the weighting function $t^{2k}(1-t)^{2l}$:

$$\langle f,g\rangle_{L+} := \int_0^1 t^{2k}(1-t)^{2l} f(t)g(t)\,\mathrm{d}t. \tag{19}$$

$\langle,\rangle_E$ is a map from $\mathbb{P}_n \times \mathbb{P}_n$ to the real number field $\mathbb{R}$:

$$\langle p,q\rangle_E := \sum_{i=0}^{n} w_i p_i q_i, \quad p(t),q(t) \in \mathbb{P}_n, \tag{20}$$

where p_i and q_i are the Bézier coefficients of the polynomials $p(t)$ and $q(t)$ when they are expressed in the degree n Bézier form, and the weights w_i are given positive numbers. With the same weights, $\langle,\rangle_{E+}$ is a map from $\mathbb{P}_{n-k-l} \times \mathbb{P}_{n-k-l}$ to $\mathbb{R}$:

$$\langle p,q\rangle_{E+} := \sum_{i=k}^{n-l} w_i \frac{\binom{n-k-l}{i-k}^2}{\binom{n}{i}^2} p_{i-k} q_{i-k}, \quad p(t),q(t) \in \mathbb{P}_{n-k-l}, \tag{21}$$

where p_{i-k} and q_{i-k} are the Bézier coefficients of the polynomials $p(t)$ and $q(t)$ expressed in the degree $n-k-l$ Bézier form. Obviously, $\langle,\rangle_E$ and $\langle,\rangle_{E+}$ are inner products on $\mathbb{P}_n$ and $\mathbb{P}_{n-k-l}$, respectively.

Now we consider a polynomial $f(t) \in \mathbb{P}_n$, It has been shown in Section 2 that $f(t)$ can be expressed as

$$f(t) = F(t) + t^k(1-t)^l f_{n-k-l}(t), \quad f_{n-k-l}(t) \in \mathbb{P}_{n-k-l}, \tag{22}$$

where $F(t)$ is a degree $k+l-1$ polynomial defined by (2). Furthermore, any degree m polynomial $g(t)$ that matches $f(t)$ up to the $(k-1)$th order derivative at $t=0$ and up to the $(l-1)$th order derivative at $t=1$ must have the decomposition

$$g(t) = F(t) + t^k(1-t)^l g_{m-k-l}(t), \quad g_{m-k-l}(t) \in \mathbb{P}_{m-k-l}. \tag{23}$$

The perturbation method is actually to find the polynomial $g(t)$ with minimum $\langle f-g, f-g\rangle_E$ among all possible polynomials having the form of (23). The following lemma shows that this is equivalent to finding $g_{m-k-l}(t)$ through an unconstrained minimization procedure.

Lemma 4.1. *Given a polynomial* $f(t) \in \mathbb{P}_n$ *expressed by* (22), *the equality* $\langle f-g, f-g\rangle_E = \langle f_{n-k-l} - g_{m-k-l}, f_{n-k-l} - g_{m-k-l}\rangle_{E+}$ *holds for any polynomial* $g(t)$ *having the form of* (23).

Proof. Assume $f_{n-k-l}(t) = \sum_{i=0}^{n-k-l} a_i B_i^{n-k-l}(t)$. Degree elevate $g_{m-k-l}(t)$ to $n-k-l$ and let $g_{m-k-l}(t) = \sum_{i=0}^{n-k-l} b_i B_i^{n-k-l}(t)$. Then

$$\begin{aligned} t^k(1-t)^l(f_{n-k-l}(t) - g_{m-k-l}(t)) &= \sum_{i=0}^{n-k-l}(a_i - b_i)\binom{n-k-l}{i}(1-t)^{n-k-i}t^{i+k} \\ &= \sum_{j=k}^{n-l}(a_{j-k} - b_{j-k})\frac{\binom{n-k-l}{j-k}}{\binom{n}{j}}b_j^n(t), \end{aligned}$$

which implies

$$(a_{j-k} - b_{j-k})\binom{n-k-l}{j-k}\Big/\binom{n}{j}$$

is the jth Bézier coefficient of polynomial $t^k(1-t)^l(f_{n-k-l}(t) - g_{m-k-l}(t))$ in the degree n Bézier form. So if $g(t) = F + t^k(1-t)^l g_{m-k-l}(t)$, we have

$$\begin{aligned} \langle f-g, f-g\rangle_E &= \langle t^k(1-t)^l(f_{n-k-l} - g_{n-k-l}), t^k(1-t)^l(f_{n-k-l} - g_{n-k-l})\rangle_E \\ &= \sum_{j=k}^{n-l} w_j \frac{\binom{n-k-l}{j-k}^2}{\binom{n}{j}^2}(a_{j-k} - b_{j-k})^2 \\ &= \langle f_{n-k-l} - g_{m-k-l}, f_{n-k-l} - g_{m-k-l}\rangle_{E+}. \qquad \square \end{aligned}$$

The constrained L_2-approximation method is to find the polynomial $g(t)$ with minimum $\langle f-g, f-g\rangle_L$ among all possible polynomials in the form of (23). In a similar way, we can easily prove.

Lemma 4.2. *If polynomials* $f(t)$ *and* $g(t)$ *are expressed by* (22) *and* (23), *respectively, then the following equality holds*:

$$\langle f-g, f-g\rangle_L = \langle f_{n-k-l} - g_{m-k-l}, f_{n-k-l} - g_{m-k-l}\rangle_{L+}.$$

Before developing our main result, let us recall one nice property of Bézier coefficients. That is [9].

Lemma 4.3. *A polynomial* $p(t) = \sum_{i=0}^n p_i B_i^n(t)$ *is of degree* $\leqslant k$ *if and only if there exists a polynomial* $q(t)$ *of degree* $\leqslant k$ *such that the Bézier coefficients* $p_i = q(i)$.

We are now ready to derive the particular weights w_i that enable the inner products $\langle,\rangle_{E+}$ and $\langle,\rangle_{L+}$ to define the same orthogonal complements.

Theorem 4.1. *The orthogonal complements of $\mathbb{P}_{m-k-l}$ in $\mathbb{P}_{n-k-l}$ with respect to the weighted L_2-inner product $\langle,\rangle_{L+}$ and the weighted Euclidean inner product of the Bézier coefficients $\langle,\rangle_{E+}$ are equal if the weights w_i are chosen to be*

$$w_i = \frac{(i+k)\cdots(i+1)(n-i+l)\cdots(n-i+1)}{i\cdots(i-k+1)(n-i)\cdots(n-i-l+1)}$$

for $k \leqslant i \leqslant n-l$.

Proof. Let $\mathbb{P}^{\perp}_{m-k-l}$ denote the orthogonal complement of $\mathbb{P}_{m-k-l}$ in $\mathbb{P}_{n-k-l}$ with respect to the weighted Euclidean inner product, and let $E_1, \ldots, E_{n-m}$ be some basis of this complement space. By equality of dimensions, it suffices to show that $\mathbb{P}^{\perp}_{m-k-l}$ is contained in the orthogonal complement of $\mathbb{P}_{m-k-l}$ in $\mathbb{P}_{n-k-l}$ with respect to the weighted L_2-inner product $\langle,\rangle_{L+}$. That is, the polynomials E_h, $(1 \leqslant h \leqslant n-m)$, have to be orthogonal to all polynomials in $\mathbb{P}_{m-k-l}$ with respect to $\langle,\rangle_{L+}$:

$$\langle E_h, t^i\rangle_{L+} = \int_0^1 t^{2k}(1-t)^{2l} E_h t^i \, dt = 0, \quad 0 \leqslant i \leqslant m-k-l.$$

Let $E_h = \sum_{j=0}^{n-k-l} a_j B_j^{n-k-l}(t)$. Then

$$\begin{aligned}
\langle E_h, t^i\rangle_{2+} &= \sum_{j=0}^{n-k-l} \int_0^1 t^{2k}(1-t)^{2l} a_j B_j^{n-k-l}(t) t^i \, dt \\
&= \sum_{j=0}^{n-k-l} a_j \int_0^1 \binom{n-k-l}{j} (1-t)^{n-k+l-j} t^{j+2k+i} \, dt \\
&= \sum_{j=0}^{n-k-l} \frac{a_j}{n+k+l+i+1} \binom{n-k-l}{j} \Big/ \binom{n+k+l+i}{2k+i+j}.
\end{aligned}$$

The last equality is due to the identity $\int_0^1 B_i^n(t)\,dt = 1/(n+1)$. Comparing with the inner product $\langle,\rangle_{E+}$, we further rewrite

$$\begin{aligned}
\langle E_h, t^i\rangle_{L+} &= \sum_{j=k}^{n-l} w_j \frac{\binom{n-k-l}{j-k}^2}{\binom{n}{j}^2} \frac{\binom{n-k-l}{j-k}}{\binom{n+k+l+i}{k+i+j}} \frac{a_{j-k}}{n+k+l+i+1} \\
&\quad \times \frac{1}{w_j} \frac{\binom{n}{j}^2}{\binom{n-k-l}{j-k}^2} \\
&= \sum_{j=k}^{n-l} w_j \frac{\binom{n-k-l}{j-k}^2}{\binom{n}{j}^2} a_{j-k} b_{j-k},
\end{aligned}$$

where

$$b_{j-k} = \frac{1}{n+k+l+i+1}\frac{1}{w_j}\frac{\binom{n}{j}^2}{\binom{n+k+l+i}{k+i+j}\binom{n-k-l}{j-k}},$$
$$0 \leqslant i \leqslant m-k-l; \quad k \leqslant j \leqslant n-1$$

or for $i = 0,\ldots,m-k-l$; $j = 0,\ldots,n-k-l$,

$$\begin{aligned} b_j &= \frac{1}{n+k+l+i+1}\frac{1}{w_{j+k}}\frac{\binom{n}{j+k}^2}{\binom{n+k+l+i}{2k+i+j}\binom{n-k-l}{j}} \\ &= \frac{(n-k-j+l)\cdots(n-k-j+1)(j+2k+i)\cdots(j+k+1)}{(j+k)\cdots(j+1)(n-j-k)\cdots(n-j-k-l+1)} \\ &\quad \times n!/(n+k+l+i+1)!(n-k-l)!w_{j+k}. \end{aligned}$$

So if

$$w_i = \frac{(i+k)\cdots(i+1)(n-i+l)\cdots(n-i+1)}{i\cdots(i-k+1)(n-i)\cdots(n-i-l+1)},$$

then

$$b_j = \frac{n!}{(n+k+l+1)!(n-k-l)!}(j+2k+i)\cdots(j+2k+1)$$

is of degree i in j. Thus, by Lemma 4.3, $p(t) = \sum_{i=0}^{n-k-l} b_i B_i^{n-k-l}(t)$ is a polynomial of degree $\leqslant i$. By assumption, we have

$$\langle E_h, t^i\rangle_{L+} = \langle E_h, p(t)\rangle_{E+} = 0. \qquad \square$$

Since the best approximation problem is closely related to the orthogonal decomposition of the space, combining the previous results, we have

Theorem 4.2. *If the weights satisfy*

$$w_i = \begin{cases} 1, & i = 0,\ldots,k-1 \text{ or } \\ & \quad n-l+1,\ldots,n, \\ \frac{(i+k)\cdots(i+1)(n-i+l)\cdots(n-i+1)}{i\cdots(i-k+1)(n-i)\cdots(n-i-l+1)}, & \text{otherwise,} \end{cases} \tag{24}$$

then for a given polynomial $f(t) \in \mathbb{P}_n$, the constrained approximation problem $\min_{g\in\mathbb{P}_m}\|f(t)-g(t)\|$ subject to $g^{(i)}(0) = f^{(i)}(0)$ for $i = 0,\ldots,k-1$ and $g^{(j)}(1) = f^{(j)}(1)$ for $j = 0,\ldots,l-1$ has the same minimizer for the norm $\|\cdot\|$ induced either by the L_2-inner product $\langle,\rangle_L$ or the weighted Euclidean inner product $\langle,\rangle_E$.

Proof. Let $f(t) = F(t) + t^k(1-t)^l f_{n-k-l}(t)$. From Theorem 4.1, we know that the polynomial $f_{n-k-l}(t) \in \mathbb{P}_{n-k-l}$ can be decomposed uniquely according to

$$f_{n-k-l}(t) = g_{m-k-l}(t) + g^{\perp}_{m-k-l}(t), \quad g_{m-k-l} \in \mathbb{P}_{m-k-l}, \quad g^{\perp}_{m-k-l} \in \mathbb{P}^{\perp}_{m-k-l}.$$

Thus, by orthogonality, Lemma 4.1 and 4.2, $g(t) = F(t) + t^k(1-t)^l g_{m-k-l}(t)$ is the best approximation for both norms induced by $\langle , \rangle_L$ and $\langle , \rangle_E$. $\square$

Let us look at two simple cases. The C^0 continuity at the two endpoints corresponds to $k = l = 1$. The weights for achieving the best L_2-approximation in this case are

$$C^0 : \begin{cases} w_i = \frac{(i+1)(n+1-i)}{i(n-i)}, & i \neq 0, n \\ w_0 = w_n = 1 \end{cases}$$

The C^1 continuity at the two endpoints corresponds to $k = l = 2$ and in this case the weights are

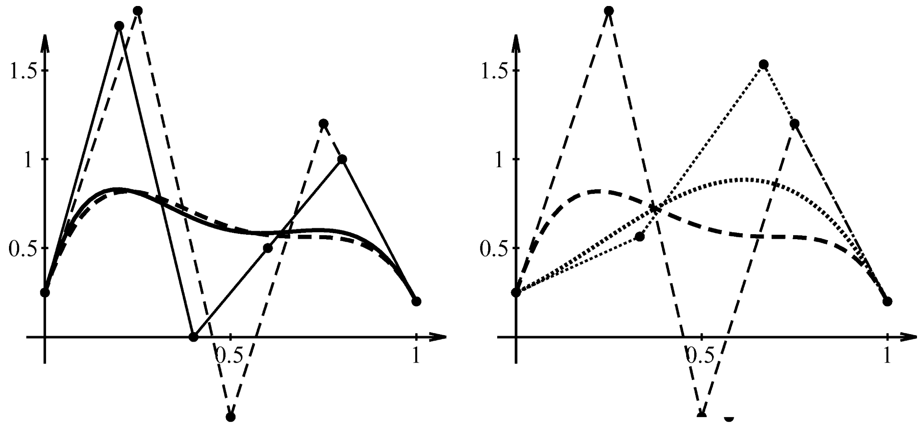
$$C^1 : \begin{cases} w_i = \frac{(i+2)(i+1)(n+2-i)(n+1-i)}{i(i-1)(n-i)(n-1-i)}, & i \neq 0, 1, n-1, n \\ w_0 = w_1 = w_{n-1} = w_n = 1 \end{cases}$$

5. Discussion

In this section we discuss some practical issues related to the L_2-degree reduction, such as the stepwise degree reduction, approximation error and the combination of degree reduction and subdivision.

5.1. Stepwise degree reduction

In order to reduce the degree of a polynomial from n to any $m < n$, Section 3 includes $n - m$ constraints on the objective function and solves a complex minimization problem. An alternative approach is to recursively apply the procedure of reducing the degree by one at each time. If in each step the weights are chosen according to (24), then each degree reduction based on the perturbation method is a constrained L_2-degree reduction. By the property of the constrained L_2-approximation mentioned at the end of Section 2, the best approximation property with respect to the L_2-norm still holds for this stepwise approach. Note that this is non-trivial to prove directly in the discrete weighted Euclidean norm. Besides the best approximation property, the continuity at the endpoints also remains the same after performing this stepwise degree reduction. Fig. 4 illustrates an L_2-degree reduction of a quintic to degree 4 and degree 3 polynomials with C^0 continuity at $t = 0$ and C^1 continuity at $t = 1$. The quintic polynomial is with Bézier coefficients $[1/4, 7/4, 0, 1/2, 1, 1/5]$. The best approximating quartic and cubic polynomials are defined by the Bézier coefficients $[1/4, 367/200, -9/10, 6/5, 1/5]$ and $[1/4, 271/480, 23/15, 1/5]$. The weights used to derive these polynomials are {1, 5, 5, 8, 1, 1} and {1, 20/3, 9, 1, 1}.



(a) Degree 5 (solid) to degree 4 (dashed) (b) Degree 4 (dashed) to degree 3 (dotted)

Fig. 4. Stepwise L_2 degree reduction with C^0 at $t = 0$ and C^1 at $t = 1$.

5.2. Error estimation

From the weighted least squares perturbation approach, we can also estimate the L_∞-error. In fact, consider the error between $f(t)$ and $f_\epsilon(t)$, where $f(t)$ and $f_\epsilon(t)$ are defined by (10) and (13):

$$|f(t) - f_\epsilon(t)| = \left|\sum_{i=k}^{n-l} \epsilon_i B_i^n(t)\right|$$
$$= \left|\sum_{i=k}^{n-l} \frac{(-1)^i\binom{n}{i}}{w_i} B_i^n(t)\right| \left|\sum_{j=0}^{n}(-1)^j\binom{n}{j}p_j\right| \Big/ \sum_{j=k}^{n-l}\frac{\binom{n}{j}^2}{w_j}.$$

In practical applications, the continuity orders at the two endpoints are often the same, i.e., $k = l$. In the following we estimate the maximal approximation error for the L_2-degree reduction with such C^k endpoint constraints. Since the weights w_i for the L_2-degree reduction are given by (24), thus in this case

$$\binom{n}{j}^2 \Big/ w_j = \binom{n}{j+k}\binom{n}{j-k}.$$

Using the binomial identity [4]:

$$\sum_{j=k}^{n-k}\binom{n}{j+k}\binom{n}{j-k} = \binom{2n}{n-2k},$$

we have

$$|f(t) - f_\epsilon(t)| = \frac{\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j}p_j\right|}{\binom{2n}{n-2k}}\left|\sum_{i=k}^{n-k}(-1)^i\frac{\binom{n}{i-k}\binom{n}{i+k}}{\binom{n}{i}}B_i^n(t)\right|. \tag{25}$$

Thus, the maximal error depends on the estimation of

$$M_{n,k} = \max_{t\in[0,1]}\left|\sum_{i=k}^{n-k}(-1)^i\frac{\binom{n}{i-k}\binom{n}{i+k}}{\binom{n}{i}}B_i^n(t)\right|.$$

Eck [4] used numerical methods to obtain several tight upper bounds for $M_{n,1}$, $M_{n,2}$, and $M_{n,3}$ with $n \leqslant 30$. If a precise bound is not required, the convex-hull property of Bézier form can provide a simple upper bound:

$$\max_{t\in[0,1]}|f(t) - f_\epsilon(t)| \leqslant \frac{\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j}p_j\right|}{\binom{2n}{n-2k}}\max_{k\leqslant i\leqslant n-k}\frac{\binom{n}{i-k}\binom{n}{i+k}}{\binom{n}{i}}$$

$$= \frac{\binom{n}{[\frac{n}{2}]-k}\binom{n}{[\frac{n}{2}]+k}}{\binom{n}{[\frac{n}{2}]}\binom{2n}{n-2k}}\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j}p_j\right|.$$

In the case of the stepwise degree reduction, i.e., the degree of a polynomial is reduced more than one, the bound on the approximation error can be estimated by adding up all the maximal errors appearing in each step. But this bound is usually excessive.

5.3. Degree reduction with subdivision

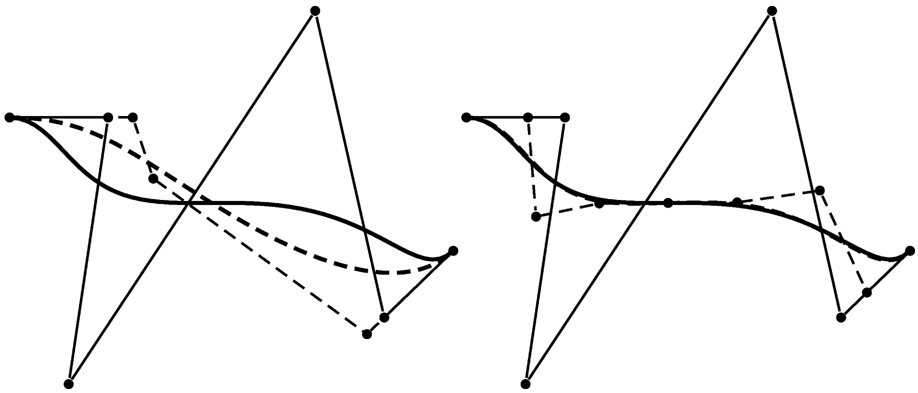
When the approximation error between $f(t)$ and $f_\epsilon(t)$ is larger than the prescribed tolerance, we can subdivide the interval $[0, 1]$ and perform constrained degree reduction on each subinterval. Since C^k continuity is preserved at the two endpoints, we finally get a continuous, piecewise approximation to $f(t)$. Observe that in (25) the maximal error consists of two parts:

$$M_{n,k}\bigg/\binom{2n}{n-2k}$$

depends only on degree n and integer k; and

$$\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j}p_j\right|$$

depends on the polynomial. So the process of subdivision only changes the value of the second part. For example, if we subdivide $f(t)$ at $h-1$ equidistant parameter



(a) Degree 5 (solid) to degree 4 (dashed) (b) Degree 5 (solid) to two degree 4 (dashed)

Fig. 5. C^1 degree reduction with subdivision.

values $t_i = i/h$ $(i = 1, \ldots, h-1)$, then the second part corresponding to each segment is decreased by a factor of $1/h^n$. This can be used to determine h—the number of segments such that the approximation error between each segment and the original polynomial is within the given tolerance.

It is worth mentioning that the techniques developed in the paper are also valid for degree reduction of parametric Bézier curves if the Euclidean distance function between curves is considered. Refer to Fig. 5, for example. Fig. 5(a) shows a degree 5 Bézier curve and its degree 4 approximation They are C^1 continuous at both endpoints. The maximal approximation error is 0.33829. If we split the degree 5 curve at $t = 1/2$ and perform the C^1-degree reduction on each segment, we obtain two degree 4 Bézier curves. The maximal approximation error between these degree 4 curves and the original degree 5 curve is 0.01057. The result is illustrated in Fig. 5(b).

So far we have shown two different methods for approximating a polynomial by a lower degree one. These different approaches yield the same L_2-approximation and thus provide more insights to the constrained L_2-degree reduction problem. Moreover, the perturbation method proposed in Section 3 not only produces the constrained L_2-approximation, but also presents a family of degree reduction methods that depend on the choice of the weights. Therefore how to choose appropriate weights so as to achieve various approximation efforts is an interesting problem that warrants further study.

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