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Different asymptotic behavior versus same dynamical complexity: Recurrence & (ir)regularity

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ABSTRACT

For any dynamical system $T : X \to X$ of a compact metric space X with g-almost product property and uniform separation property, under the assumptions that the periodic points are dense in X and the periodic measures are dense in the space of invariant measures, we distinguish various periodic-like recurrences and find that they all carry full topological entropy and so do their gap-sets. In particular, this implies that any two kind of periodic-like recurrences are essentially different. Moreover, we coordinate periodic-like recurrences with (ir)regularity and obtain lots of generalized multi-fractal analyses for all continuous observable functions. These results are suitable for all β-shifts ($\beta > 1$), topological mixing subshifts of finite type, topological mixing expanding maps or topological mixing hyperbolic diffeomorphisms, etc.
Roughly speaking, we combine many different "eyes" (i.e., observable functions and periodic-like recurrences) to observe the dynamical complexity and obtain a *Refined Dynamical Structure* for Recurrence Theory and Multi-fractal Analysis.

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1. Introduction

In the theory of dynamical systems, i.e., the study of the asymptotic behavior of orbits $\{T^n(x)\}_{n\in\mathbb{N}}$ (denoted by $Orb(x)$) when $T : X \to X$ is a continuous map of a compact metric space X and $x \in X$, one may say that two fundamental problems are to understand how to partition different asymptotic behavior and how the points with same asymptotic behavior control or determine the complexity of system T.

Topological entropy is a classical concept to describe the dynamical complexity. In this paper we are mainly to deal with a certain class of dynamical systems and show that various subsets characterized by distinct asymptotic behavior all carry full topological entropy. To make this more precise let us introduce the following terminology. $T : X \to X$ is a continuous map of a compact metric space X.

Definition 1.1. For a collection of subsets $Z_1, Z_2, \cdots, Z_k \subseteq X$ $(k \geq 2)$, we say $\{Z_i\}$ has *full entropy gaps* with respect to $Y \subseteq X$ if

$$h_{top}(T, (Z_{i+1} \setminus Z_i) \cap Y) = h_{top}(T, Y) \quad \text{for all } 1 \leq i < k,$$

where $h_{top}(T, Z)$ denotes the topological entropy of a set $Z \subseteq X$.

Often, but not always, the sets Z_i are nested $(Z_i \subseteq Z_{i+1})$. Remark that for any system with zero topological entropy, it is obvious that any collection $\{Z_i\}$ has full entropy gaps with respect to any $Y \subseteq X$. Notice that if X is a finite set, then any system on X is simple and carries zero entropy. Thus in present paper, we always assume that

X is a compact metric space with infinitely many points.

Given $x \in M$, let $\omega_T(x)$ denote the ω-limit set of x, let $M_x(T)$ be the limit set of the empirical measures for x and let $C_x := \overline{\bigcup_{\nu\in M_x(T)} S_\nu}$ where S_ν denotes the support of measure ν. In this paper, we consider the following subsets of X according to different asymptotic behavior:

$$\begin{aligned}
Per(T) &:= \{\text{periodic points of } T\},\\
A(T) &:= \{\text{almost periodic points of } T\} = \{\text{ points contained in minimal set}\},\\
Rec(T) &:= \{\text{recurrent points of } T\},\\
\Omega(T) &:= \{\text{non-wandering points of } T\},\\
W(T) &:= \{x \in Rec(T) \mid S_\mu = C_x \text{ for every } \mu \in M_x(T)\},\\
QW(T) &:= \{x \in Rec(T) \mid C_x = \omega_T(x)\},\\
V(T) &:= \{x \in QW(T) \mid \exists\, \mu \in M_x(T) \text{ such that } S_\mu = C_x\}.
\end{aligned}$$

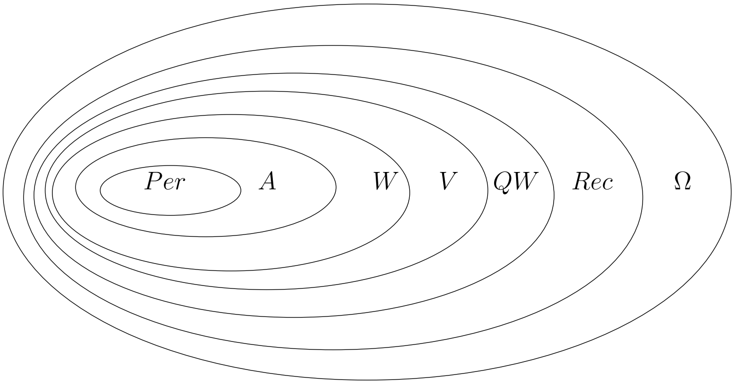

Fig. 1. $\Omega(T)$.

Most of notions in above considered sets are well-known. The notions of periodic, recurrent and non-wandering can be found in [59], the notion of almost periodic or minimal can be seen in [6,21,20,22,33] and others, for example, see [62,61,64,63]. We will recall their definitions, equivalent statements and relations in Section 3. Such sets are all T-invariant and they satisfy $Per(T) \subseteq A(T) \subseteq W(T) \subseteq V(T) \subseteq QW(T) \subseteq Rec(T) \subseteq \Omega(T)$. Fig. 1 is a simple Venn diagram illustrating the containment between various sets $\{Per(T), A(T), W(T), V(T), QW(T), Rec(T), \Omega(T)\}$ (simply, writing $\{Per, A, W, V, QW, Rec, \Omega\}$ in the figure).

A point $x \in X$ is generic for some invariant measure μ means that $M_x(T) = \{\mu\}$ (or equivalently, Birkhoff averages of all continuous functions converge to the integral of μ). Let G_μ denote the set of all generic points for μ. Let $M(T,X)$, $M_{erg}(T,X)$ and $M_p(T,X)$ denote the set of all T-invariant measures, T-ergodic measures and T-periodic measures respectively. We also consider

$$\begin{aligned} QR(T) &:= \{\text{quasiregular points of } T\} = \cup_{\mu \in M(T,X)} G_\mu, \\ I(T) &:= \{\text{irregular points of } T\} = X \setminus QR(T), \\ QR_{erg}(T) &:= \{\text{points generic for ergodic measures}\} = \cup_{\mu \in M_{erg}(T,X)} G_\mu, \\ QR_d(T) &:= \{\text{points of density in } QR(T)\} = \cup_{\mu \in M(T,X)} (G_\mu \cap S_\mu), \\ R(T) &:= \{\text{regular points of } T\} = QR_d(T) \cap QR_{erg}(T) = \cup_{\mu \in M_{erg}(T,X)} (G_\mu \cap S_\mu). \end{aligned}$$

Such sets are all T-invariant and remark that

$$R(T) \subseteq QR_d(T) \cup QR_{erg}(T) \subseteq QR(T) = X \setminus I(T).$$

Most notions except irregular point are from [39] (for quasiregular point, also see [15]) and the notion of irregular point can be found in [41,2,53,4] etc. We will recall them more precisely in Section 4.

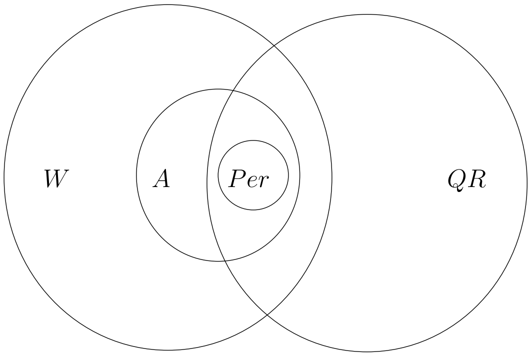

Fig. 2. $QR(T) \cup W(T)$.

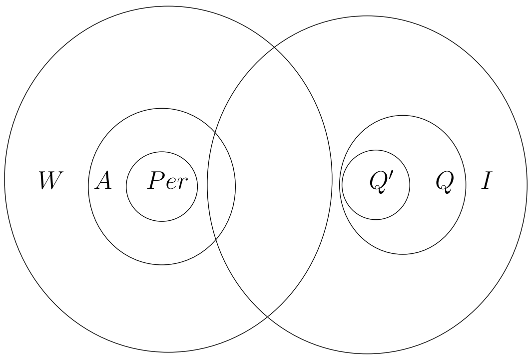

Fig. 3. $I(T) \cup W(T)$.

Fig. 2 is a simple Venn diagram illustrating the relations between various sets $\{Per(T), A(T), W(T), QR(T)\}$ (simply, writing $\{Per, A, W, QR\}$ in the figure). Fig. 3 is a simple Venn diagram to illustrate the relations between the following various sets $\{Per(T), A(T), W(T), V(T) \setminus W(T), QW(T) \setminus W(T), I(T)\}$ (simply, writing $\{Per, A, W, Q', Q, I\}$ in the figure where $Q' = V(T) \setminus W(T)$, $Q = QW(T) \setminus W(T)$. Remark that $Q \subseteq I(T)$, see Theorem 5.1). Precise discussions will appear later.

1.1. Main results

Now we start to state our main theorems. We need two conditions called *g-almost product property* and *uniform separation property* which are introduced in [43] and we will recall them later in Section 2.

Theorem 1.2. *Let T be a continuous map of a compact metric space X with g-almost product property and uniform separation property. If the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures* *(i.e., $\overline{M_p(T,X)} = M(T,X)$), then $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has full entropy gaps with respect to X.*

Theorem 1.2 seems to require too many very strong conditions, but they are satisfied by many examples, including all topological mixing subshifts of finite type and all β-shifts, etc. (see Section 1.2).

Given a continuous function $\phi : X \to \mathbb{R}$, let

$$R_\phi(T) := \{x \in X \mid \text{Birkhoff averages } \frac{1}{n}\sum_{i=0}^{n-1} \phi(T^i(x)) \text{ converge as } n \to +\infty\}.$$

For convenience, we call $R_\phi(T)$ to be regular set with respect to ϕ (simply, ϕ-regular set). Define the ϕ-irregular set $I_\phi(T) = X \setminus R_\phi(T)$. These two sets describe different asymptotic behavior under the observation of continuous functions.

Theorem 1.3. *Let T be a continuous map of a compact metric space X with g-almost product property and uniform separation property. If the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$), then for any continuous function $\phi : X \to \mathbb{R}$,*

(1) $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has full entropy gaps with respect to $R_\phi(T)$;

(2) if $I_\phi(T) \neq \emptyset$, then $\{QR(T), W(T), V(T), QW(T), I(T)\}$ has full entropy gaps with respect to $I_\phi(T)$.

Recall that for a certain class of dynamical systems (including mixing subshifts of finite type, mixing hyperbolic systems and β-shifts etc.), $I(T)$ (or nonempty $I_\phi(T)$) carries full topological entropy, for example, see [41,4,13,2,54,57]. Theorem 1.2 and Theorem 1.3 refine such prior results.

For any continuous function $\phi : X \to \mathbb{R}$ and any $a \in \mathbb{R}$, let

$$R_{\phi,a}(T) := \{x \in X \mid \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(T^i(x)) = a\}.$$

Remark that

$$R_\phi(T) = \bigsqcup_{a\in\mathbb{R}} R_{\phi,a}(T),$$

where $\sqcup$ denotes disjoint union. The sets of $R_{\phi,a}(T)$ refine the asymptotic behavior of $R_\phi(T)$. There are lots of classical results for $R_{\phi,a}(T)$, for example, see [43,54,38]. Define the domain of the multifractal spectrum for Birkhoff averages of ϕ,

$$L_\phi := [\inf\{\int \phi d\mu \mid \mu \in M(T,X)\}, \sup\{\int \phi d\mu \mid \mu \in M(T,X)\}].$$

Let $Int(L_\phi)$ denote the interior of L_ϕ. That is,

$$Int(L_\phi) = (\inf\{\int \phi d\mu |\, \mu \in M(T,X)\}, \sup\{\int \phi d\mu |\, \mu \in M(T,X)\}).$$

Remark that if $I_\phi(T) \neq \emptyset$, $Int(L_\phi)$ is a nonempty open interval, see (4.26) below.

Theorem 1.4. *Let T be a continuous map of a compact metric space X with g-almost product property and uniform separation property. If the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$), then for any continuous function $\phi : X \to \mathbb{R}$ satisfying $I_\phi(T) \neq \emptyset$ and for any $a \in Int(L_\phi)$,*

$$\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$$

has full entropy gaps with respect to $R_{\phi,a}(T)$.

Theorem 1.4 refines prior multifractal results on Birkhoff averages, for example, see [54,38,3,19] etc.

Remark that Theorem 1.3 (1) implies Theorem 1.2, since $R_\phi(T) = X$ if taking ϕ to be a constant function. We will prove Theorem 1.4 in Section 7.2 and then use it to prove Theorem 1.3 (1). We will prove Theorem 1.3 (2) in Section 7.1. The key tool for all the proofs is Theorem 1.1 from Pfister and Sullivan's paper [43], which is stated here as Lemma 2.8, and allows the entropy estimates to be reduced to the problem of describing the various gap sets in terms of $M_x(T)$.

Remark that the assumption of density of periodic points can be replaced by existence of an invariant measure with full support.

Theorem 1.5. *Let T be a continuous map of a compact metric space X with g-almost product property and uniform separation property. If the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$) and there exists an invariant measure with full support, then all results of Theorem 1.2, Theorem 1.3 and Theorem 1.4 hold.*

Let us explain why Theorem 1.5 holds. Under the assumption of density of periodic measures, $\overline{Per(T)} = X \Leftrightarrow \exists \mu \in M(T,X)$, $S_\mu = X$ (see Proposition 6.5 below). So the assumptions of Theorem 1.5 are equivalent to the ones of Theorem 1.2, Theorem 1.3 and Theorem 1.4 and thus Theorem 1.5 is valid.

Recall that from [43] g-almost product property is weaker than specification property and from [15], we know that for any dynamical system with Bowen's specification property, the periodic points are dense in X (i.e., $\overline{Per(f)} = X$, see Proposition 21.3 [15]) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$, see Proposition 21.8 [15] or see [49]). So we have a following result as a consequence of Theorem 1.2, Theorem 1.3 and Theorem 1.4.

Theorem 1.6. *Let T be a continuous map of a compact metric space X with Bowen's specification property and uniform separation property. Then all results of Theorem 1.2, Theorem 1.3 and Theorem 1.4 hold.*

Moreover, we have a following result when the system is expansive.

Theorem 1.7. *Let T be an expansive continuous map of a compact metric space X with Bowen's specification property. Then all results of Theorem 1.2, Theorem 1.3 and Theorem 1.4 hold. Moreover, Theorem 1.2 and Theorem 1.3 (1) can be stated for*

$$\{A(T), A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}.$$

Let us explain why Theorem 1.7 holds. It is known that expansiveness is stronger than uniform separation property, see [43]. By Theorem 1.6, Theorem 1.7 is valid except $R(T) \setminus A(T)$. Recall from [15] (see Chapter 22) that for any system with Bowen's specification property and expansiveness, it has unique maximal entropy measure and this measure has full support. So $R(T) \setminus A(T)$ having full entropy can be deduced from Theorem 5.4 below which shows that for any dynamical system, if there is an ergodic measure with maximal entropy and non-minimal support, then $R(T) \setminus A(T)$ has full entropy. Remark that $R(T) \setminus A(T) \subseteq R(T) \subseteq QR(T) \subseteq R_\phi(T)$ for any continuous function ϕ. So $\{A(T), R(T)\}$ has full entropy gaps with respect to X and $R_\phi(T)$.

1.2. Applications to standard examples

1.2.1. Mixing subshifts of finite type

Recall from [15] (Proposition 21.2) any topological mixing subshift of finite type satisfies Bowen's specification. As a subsystem of full shift, it is expansive. So by Theorem 1.7, we have

Theorem 1.8. *Let T be a topological mixing subshift of finite type. Then all results of Theorem 1.7 hold.*

1.2.2. β-shifts

Let us recall the definition of β-shift ($\beta > 1$) in [59] (Chapter 7.3). If $\beta \geq 2$ is an integer, β-shift is the full shift of β symbols. So we only need to recall the definition in the case that β is not an integer. Consider the expansion of 1 in powers of β^{-1}, i.e. $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$ where $a_1 = [\beta]$ and $a_n = [\beta^n - \sum_{i=1}^{n-1} a_i \beta^{n-i}]$. Here $[t]$ denotes the integral part of $t \in \mathbb{R}$. Let $k = [\beta] + 1$. Then $0 \leq a_n \leq k - 1$ for all n so we can consider $a = \{a_n\}_1^\infty$ as a point in the space $X = \prod_{n=1}^{+\infty} Y$ where $Y = \{0, 1, \cdots, k-1\}$. Consider the lexicographical ordering on X, i.e. $x = \{x_n\}_1^\infty < y = \{y_n\}_1^\infty$ if $x_j < y_j$ for the smallest j with $x_j \neq y_j$. Let $f : X \to X$ denote the one-sided shift transformation. Note that $f^n a \leq a$ for all $n \geq 0$. Let

$$\Sigma_\beta := \{x = \{x_n\}_1^\infty \,|\, x \in X \text{ and } f^n(x) \le a \text{ for all } n \ge 0\}.$$

Then Σ_β is a closed subset of X and $f(\Sigma_\beta) = \Sigma_\beta$. Let $\sigma_\beta := f|_{\Sigma_\beta}$. Then $(\Sigma_\beta, \sigma_\beta)$ is one-sided β-shift. One can obtain the two-sided β-shift by letting

$$\hat{\Sigma}_\beta := \{x = \{x_n\}_{-\infty}^\infty \,|\, x \in \prod_{n=-\infty}^{+\infty} Y \text{ and } (x_i, x_{i+1}, \cdots) \in \Sigma_\beta \text{ for all } i \in \mathbb{Z}\}.$$

Then $\hat{\Sigma}_\beta$ is a closed subspace of $\prod_{n=-\infty}^{+\infty} Y$ invariant under the two-sided shift

$$\hat{f} : \prod_{n=-\infty}^{+\infty} Y \to \prod_{n=-\infty}^{+\infty} Y.$$

The topological entropy of β-shift ($\beta > 1$) is $\log \beta$ (see Page 179 of Chapter 7.3 in [59]). Remark that by Variational Principle, there is an ergodic measure with positive entropy. Note that the Dirac measure supported on the fixed point $x = \{0\}_1^\infty \in \Sigma_\beta$ has zero entropy. So every β-shift is obviously not uniquely ergodic.

Theorem 1.9. *Every β-shift ($\beta > 1$) satisfies all results of Theorem 1.7.*

Let us explain why Theorem 1.9 holds. By definition every β-shift ($\beta > 1$) is a subsystem of full shift on $[\beta] + 1$ symbols and so every β-shift is expansive (which is stronger than uniform separation property, see [43]) and satisfies g-almost product property from [43] (see the Example on p. 934). It is known that the unique maximal entropy measure of β-shifts always carries full support (see [58], Theorem 13 (ii)). Furthermore, it was proved in [50] that the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T, X)} = M(T, X)$). So the hypotheses of Theorem 1.5 hold for all β-shifts and then Theorem 1.9 is obtained except $R(T) \setminus A(T)$. The set $R(T) \setminus A(T)$ can be deduced from Theorem 5.4 below which shows that for any dynamical system, if there is an ergodic measure with maximal entropy and non-minimal support, then $R(T) \setminus A(T)$ has full entropy. Remark that $R(T) \setminus A(T) \subseteq R(T) \subseteq QR(T) \subseteq R_\phi(T)$ for any continuous function ϕ. So $\{A(T), R(T)\}$ has full entropy gaps with respect to X and $R_\phi(T)$. So Theorem 1.9 is valid. In particular, we point out that one cannot use Theorem 1.7 to prove Theorem 1.9, since from [12] the set of parameters of β for which Bowen's specification holds, is dense in $(1, +\infty)$ but has Lebesgue zero measure.

In particular, for full shifts on finite symbols, we have a following result that contains more observations.

Theorem 1.10. *Let T be a full shift on k symbols ($k \ge 2$). Then Theorem 1.2 and Theorem 1.3 (1) can be stated for*

$$\{Per(T), A(T), A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}.$$

Let us explain why Theorem 1.10 holds. By Theorem 1.9, one only needs to consider $A(T) \setminus Per(T)$. This can be deduced from Theorem 5.2 below which shows that for full shifts of finite symbols, $R(T) \cap A(T) \setminus Per(T)$ has full entropy. Since $R(T) \cap A(T) \setminus Per(T) \subseteq QR(T) \cap A(T) \setminus Per(T) \subseteq R_\phi(T) \cap A(T) \setminus Per(T)$ for any continuous function ϕ. So $\{Per(T), A(T)\}$ has full entropy gaps with respect to X and $R_\phi(T)$.

1.2.3. Hyperbolic systems and Lyapunov exponents

From the classical uniform hyperbolicity theory, every subsystem restricted on a topological mixing locally maximal hyperbolic set (called basic set or elementary set) satisfies specification property (for example, see [49]) and satisfies expansiveness. So by Theorem 1.7, we have

Theorem 1.11. *Let $f : M \to M$ be a C^1 diffeomorphism of a compact Riemannian manifold M. Let T be a subsystem restricted on a topological mixing locally maximal hyperbolic set. Then all results of Theorem 1.7 hold.*

In particular, this result can be applicative to all topological mixing Anosov diffeomorphisms. Remark that similar results can be stated for topological mixing expanding maps (for example, $T : S^1 \to S^1$, $x \mapsto kx$ mod 1 for an integer $k \geq 2$).

Moreover, we can use Lyapunov exponents to observe the "periodic-like" recurrence. Let $f : M \to M$ be a C^1 diffeomorphism of a compact Riemannian manifold M. Let $E \subset TM$ be a Df-invariant subbundle. Define the (maximal) Lyapunov exponent of E at a point $x \in M$ by

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n|_{E_x}\|$$

if the limit exists. Such points are called *Lyapunov-regular*. Otherwise, the points are called *Lyapunov-irregular*. Denote the sets of all Lyapunov-regular and Lyapunov-irregular points by $R_{Lya}(f)$ and $I_{Lya}(f)$ respectively. Let $\Lambda \subseteq M$ be a compact invariant set. The subbundle E is called *conformal* on Λ, if for any $x \in \Lambda$, $n \geq 1$

$$\|Df^n|_{E_x}\| = \prod_{j=0}^{n-1} \|Df|_{E_{f^j(x)}}\|.$$

Let $\phi(x) = \log \|Df|_{E_x}\|$, if E is a continuous subbundle then it is a continuous function. Thus, for subsystem $T = f|_\Lambda$, using this ϕ in Theorem 1.3 and Theorem 1.4 we have

Theorem 1.12. *Let $f : M \to M$ be a C^1 diffeomorphism of a compact Riemannian manifold M. Let $T : \Lambda \to \Lambda$ be a subsystem restricted on a topological mixing locally maximal hyperbolic set Λ and let $E \subset T_\Lambda M$ be a continuous conformal Df-invariant subbundle* *on* Λ. *Then Theorem 1.3 and Theorem 1.4 hold for the function* $\phi(x) = \log \|Df|_{E(x)}\|$ *(replacing* $R_\phi(T)$ *and* $I_\phi(T)$ *by* $R_{Lya}(T)$ *and* $I_{Lya}(T)$ *respectively).*

In other words, we can use the "eyes" of Lyapunov exponents to distinguish different "periodic-like" recurrence. Remark that similar results can be stated for topological mixing conformal expanding maps.

1.3. An answer for Zhou and Feng's question

There is an open problem in [63] by Zhou and Feng that whether the set

$$\{QW(T) \setminus W(T) | \exists\, \mu \in M_x(T)\, \text{s.t.}\, S_\mu = C_x\} \neq \emptyset?$$

This is the set $V \setminus W(T)$ according to the definition of V at the beginning of the paper. It has been solved positively by constructing examples, see [37,25,60] etc. From Theorem 1.2, for a certain class of dynamical systems (including topological mixing subshifts of finite type, all β-shifts, systems restricted on mixing locally maximal hyperbolic sets), $V \setminus W(T)$ is not only nonempty but also has full topological entropy (and so does its complementary set in $QW(T) \setminus W(T)$). In other words, $V \setminus W(T)$ has very strong dynamical complexity which reaches the complexity of dynamical system itself. In particular, we know that positive topological entropy implies $V \setminus W(T)$ has uncountable elements. So our Theorem 1.2 can be as a strong answer for Zhou and Feng's open problem, provided that the given system has positive entropy.

1.4. Layout of the paper

The remainder of this paper is organized as follows. In Section 2 we will recall the notions of entropy, g-almost product property, uniform separation and recall some classical results including saturated property and entropy-dense property. In Section 3 we will recall the notions of various 'periodic-like' recurrence and introduce some simple observation. In Section 4 we will recall the notions of regularity and irregularity and introduce some simple facts. In Section 5 we will coordinate 'periodic-like' recurrence and (ir)regularity and give some basic discussion. In Section 6 we recall and introduce some useful facts and lemmas. In Section 7 we divide our main theorems into several propositions to prove and in this process we state every proposition for possibly applicative to more general dynamical systems, in particular some results can be applied for time-t maps of mixing hyperbolic flows. In Section 8 we give some similar results involving the set of transitive points. Finally, in Section 9 for generic systems or systems with Bowen's specification, we give some topological or geometric characterization of various subsets with distinct asymptotic behavior.

2. Entropy, g-almost product property & uniform separation

2.1. Entropy

Let $T : X \to X$ be a continuous map of a compact metric space X. Now let us to recall the definition of topological entropy in [10] by Bowen.

Let $x \in X$. The dynamical ball $B_n(x, \varepsilon)$ is the set

$$B_n(x,\varepsilon) := \{y \in X| \max\{d(T^j(x), T^j(y))| 0 \le j \le n-1\} \le \varepsilon\}.$$

Let $E \subseteq X$, and $\mathfrak{F}_n(E,\epsilon)$ be the collection of all finite or countable covers of E by sets of the form $B_m(x,\epsilon)$ with $m \ge n$. We set

$$C(E;t,n,\epsilon,T) := \inf\{\sum_{B_m(x,\epsilon)\in\mathcal{C}} 2^{-tm} : \mathcal{C} \in \mathfrak{F}_n(E,\epsilon)\},$$

and

$$C(E;t,\epsilon,T) := \lim_{n\to\infty} C(E;t,n,\epsilon,T).$$

Then

$$h_{top}(E,\epsilon,T) := \inf\{t : C(E;t,\epsilon,T) = 0\} = \sup\{t : C(E;t,\epsilon,T) = \infty\}$$

and the *topological entropy* of E is defined as

$$h_{top}(T,E); = \lim_{\epsilon\to 0} h_{top}(E,\epsilon,T).$$

In particular, if $E = X$, we also denote $h_{top}(T,X)$ by $h_{top}(T)$. It is known from [10] that if E is an invariant compact subset, then the topological entropy $h_{top}(T,E)$ is same as the classical definition (for classical definition of topological entropy, see Chapter 7 in [59]).

Let us recall some basic facts about topological entropy. From [10] for any subsets $Y_1 \subseteq Y_2 \subseteq X$,

$$h_{top}(T,Y_1) \le h_{top}(T,Y_2). \tag{2.1}$$

If one considers a collection of subsets of X: $\{Y_i\}_{i=1}^{+\infty}$, from [10] we know that the topological entropy satisfies

$$h_{top}(T, \bigcup_{i=1}^{+\infty} Y_i) = \sup_{i\ge 1} h_{top}(T,Y_i). \tag{2.2}$$

Let $M(X)$ denote the space of all Borel probability measures supported on X. Let $\xi = \{V_i | i = 1, 2, \cdots, k\}$, be a finite partition of measurable sets of X. The entropy of $\nu \in M(X)$ with respect to ξ is

$$H(\nu, \xi) := -\sum_{V_i \in \xi} \nu(V_i) \log \nu(V_i).$$

We write $T^{\vee n}\xi := \vee_{k\in\Lambda} T^{-k}\xi$. The entropy of $\nu \in M(T, X)$ with respect to ξ is

$$h(T, \nu, \xi) := \lim_{n\to\infty} \frac{1}{n} H(\nu, T^{\vee n}\xi),$$

and the *metric entropy* of ν is

$$h_\nu(T) := \sup_\xi h(T, \nu, \xi).$$

More information of metric entropy, see Chapter 4 of [59].

Let us recall some relations on metric entropy and topological entropy. By classical Variational Principle (Theorem 8.6 and Corollary 8.6.1 in [59]), we know that

$$h_{top}(T) = \sup_{\mu\in M(T,X)} h_\mu(T) = \sup_{\mu\in M_{erg}(T,X)} h_\mu(T). \tag{2.3}$$

From Theorem 1 in [10] for any ergodic measure μ and any subset $Z \subseteq X$, if $\mu(Z) = 1$, then

$$h_\mu(T) \le h_{top}(T, Z). \tag{2.4}$$

Moreover, every set with totally full measure (i.e., being full measure for all invariant measures) carries full topological entropy, that is,

Theorem 2.1. *Let T be a continuous map of a compact metric space X. If $Y \subseteq X$ is a set with totally full measure, then*

$$h_{top}(T, Y) = h_{top}(T). \tag{2.5}$$

Proof. Let $Y \subseteq X$ be a set with totally full measure. By (2.1), (2.3) and (2.4),

$$\begin{aligned} h_{top}(T, Y) \le h_{top}(T, X) = h_{top}(T) &= \sup_{\mu\in M(T,X)} h_\mu(T) \\ &= \sup_{\mu\in M_{erg}(T,X)} h_\mu(T) \le h_{top}(T, Y). \end{aligned}$$

This means that Y carries full entropy. □

2.2. g-Almost product property

Firstly we recall the definition of specification property which is stronger than g-almost product property, see [15,49,9,11,8,56]. Let T be a continuous map of a compact metric space X.

Definition 2.2. We say that the dynamical system T satisfies *specification property*, if the following holds: for any $\epsilon > 0$ there exists an integer $M(\epsilon)$ such that for any $k \geq 2$, any k points $x_1, \cdots, x_k$, any integers

$$a_1 \leq b_1 < a_2 \leq b_2 \cdots < a_k \leq b_k$$

with $a_{i+1} - b_i \geq M(\epsilon)$ $(2 \leq i \leq k)$, there exists a point $x \in X$ such that

$$d(T^j(x), T^j(x_i)) < \epsilon, \quad \text{for } a_i \leq j \leq b_i,\ 1 \leq i \leq k. \tag{2.6}$$

The original definition of specification, due to Bowen, was stronger.

Definition 2.3. We say that the dynamical system T satisfies *Bowen's specification property*, if under the assumptions of Definition 2.2 and for any integer $p \geq M(\epsilon) + b_k - a_1$, there exists a point $x \in X$ with $T^p(x) = x$ satisfying (2.6).

Now we start to recall the concept g-almost product property in [43] (there is a slightly weaker variant, called almost specification, see [57]). It is weaker than specification property (see Proposition 2.1 in [43]). A striking and typical example of g-almost product property (and almost specification) is that it applies to every β-shift [43,57]. In sharp contrast, the set of β for which the β-shift has specification property has zero Lebesgue measure [12,48].

Let $\Lambda_n = \{0, 1, 2, \cdots, n-1\}$. The cardinality of a finite set Λ is denoted by $\#\Lambda$. Let $x \in X$. The dynamical ball $B_n(x, \varepsilon)$ is the set

$$B_n(x, \varepsilon) := \{y \in X |\ \max\{d(T^j(x), T^j(y)) |\, j \in \Lambda_n\} \leq \varepsilon\}.$$

Definition 2.4. Let $g : \mathbb{N} \to \mathbb{N}$ be a given nondecreasing unbounded map with the properties

$$g(n) < n \ \text{ and } \lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

The function g is called *blowup function.* Let $x \in X$ and $\varepsilon > 0$. The g-blowup of $B_n(x, \varepsilon)$ is the closed set

$$\begin{aligned} &B_n(g; x, \varepsilon) \\ &\quad := \{y \in X |\, \exists \Lambda \subseteq \Lambda_n, \#(\Lambda_n \setminus \Lambda) \leq g(n) \ \text{and} \ \max\{d(T^j(x), T^j(y)) |\, j \in \Lambda\} \leq \varepsilon\}. \end{aligned}$$

Definition 2.5. We say that the dynamical system T satisfies *g-almost product property* with blowup function g, if there is a nonincreasing function $m : \mathbb{R}^+ \to \mathbb{N}$, such that for any $k \geq 2$, any k points $x_1, \cdots, x_k \in X$, any positive $\varepsilon_1, \cdots, \varepsilon_k$ and any integers $n_1 \geq m(\varepsilon_1), \cdots, n_k \geq m(\varepsilon_k)$,

$$\bigcap_{j=1}^{k} T^{-M_{j-1}} B_{n_j}(g; x_j, \varepsilon_j) \neq \emptyset,$$

where $M_0 := 0$, $M_i := n_1 + \cdots + n_i$, $i = 1, 2, \cdots, k-1$.

2.3. Uniform separation

Now we recall the definition of uniform separation property [43]. For $x \in X$, define

$$\Upsilon_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$$

where δ_y is the Dirac probability measure supported at $y \in X$. For $\delta > 0$ and $\varepsilon > 0$, two points x and y are (δ, n, ε)-separated if

$$\#\{j : d(T^j x, T^j y) > \varepsilon,\ j \in \Lambda_n\} \geq \delta n.$$

A subset E is (δ, n, ε)-separated if any pair of different points of E are (δ, n, ε)-separated. Let $F \subseteq M(X)$ be a neighborhood of $\nu \in M(T, X)$. Define

$$X_{n,F} := \{x \in X | \, \Upsilon_n(x) \in F\},$$

and define

$$N(F; \delta, n, \varepsilon) := \text{maximal cardinality of a } (\delta, n, \varepsilon) - \text{separated subset of } X_{n,F}.$$

Definition 2.6. We say that the dynamical system T satisfies *uniform separation property*, if the following holds. For any $\eta > 0$, there exist $\delta^* > 0$, $\epsilon^* > 0$ such that for μ ergodic and any neighborhood $F \subseteq M(X)$ of μ, there exists $n^*_{F,\mu,\eta}$, such that for $n \geq n^*_{F,\mu,\eta}$,

$$N(F; \delta^*, n, \epsilon^*) \geq 2^{n(h_\mu(f)-\eta)}.$$

Now let us recall a basic relation between expansiveness and uniform separation in [43].

Theorem 2.7. *(See Theorem 3.1 in [43].) Let T be a continuous map of a compact metric space X. If T is expansive (or asymptotically h-expansive), T satisfies uniform separation.*

2.4. Variational Principle for saturated sets

Let T be a continuous map of a compact metric space X. Recall the definition that $\Upsilon_n(x) := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{T^j(x)}$. Let $M_x(T)$ be the limit set of the empirical measures for x, i.e., the set of all limits of $\Upsilon_n(x)$ in weak* topology.

Now we recall a result from [43]. The system T is said to be *saturated* (or T has saturated property), if for any compact connected nonempty set $K \subseteq M(T,X)$,

$$h_{top}(T, G_K) = \inf\{h_\mu(T) \,|\, \mu \in K\},$$

where $G_K = \{x \in X |\, M_x(T) = K\}$.

Lemma 2.8 *(Variational Principle). (See Theorem 1.1 in [43].) Let T be a continuous map of a compact metric space X with g-almost product property and uniform separation property. Then T is saturated.*

Remark that Lemma 2.8 is the key tool for all the proofs of our main results in present article. It allows the entropy estimates to be reduced to the problem of describing the various gap sets in terms of $M_x(T)$ (precise proof details will appear in Section 7).

On the other hand, from [43] if one does not have uniform separation property, then the saturated property just holds for any singleton K. For convenience to compare saturated property, we give a following notion called single-saturated property. We say T is *single-saturated*, if $h_{top}(T, G_\mu) = h_\mu(T)$ holds for any $\mu \in M(T,X)$, where $G_\mu = \{x \in X |\, M_x(T) = \{\mu\}\}$.

Lemma 2.9 *(Variational Principle). (See Theorem 1.2 in [43].) Let T be a continuous map of a compact metric space X with g-almost product property. Then T is single-saturated.*

Remark that for any continuous map T of a compact metric space X, there is a general fact (see Theorem 4.1 (3) in [43]): for any compact connected nonempty set $K \subseteq M(X,T)$,

$$h_{top}(T, G_K) \le \inf\{h_\mu(T) \,|\, \mu \in K\}, \quad \text{where} \quad G_K = \{x \in X |\, M_x(T) = K\}. \tag{2.7}$$

In particular, for any $\mu \in M(X,T)$, we have

$$h_{top}(T, G_\mu) \le h_\mu(T), \quad \text{where} \quad G_\mu = \{x \in X |\, M_x(T) = \{\mu\}\}. \tag{2.8}$$

2.5. Entropy-dense property

Now let's recall the entropy-dense property of Theorem 2.1 in [42] (or see [43], also see [17] for similar discussion). Roughly speaking, any invariant probability measure μ is the limit of a sequence of ergodic measures $\{\mu_n\}_{n=1}^{\infty}$ in weak* topology such that the entropy of μ is the limit of the entropies of μ_n. Recall $M(X)$ and $M(T, X)$ denote the space of all Borel probability measures and the space of invariant measures, respectively. Here we further require $S_{\mu_n} \neq X$ in this property (whose statement is a little stronger than usual entropy-dense property). More precisely, we say T has *entropy-dense* property, if for any $\nu \in M(T, X)$, any neighborhood $G \subseteq M(X)$ of ν and any $h_* < h_\nu(T)$, there exists an ergodic measure $\mu \in G \cap M(T, X)$ such that $S_\mu \neq X$ and $h_\mu(T) > h_*$.

Lemma 2.10. *Let T be a continuous map of a compact metric space X with g-almost product property. Suppose that $M(T, X)$ is not a singleton. Then T has entropy-dense property.*

Proof. Here this lemma is slightly stronger than Theorem 2.1 in [42], since we further add $S_\mu \neq X$. Now let us explain this more precisely. Let $\nu \in M(T, X)$ and $G \subseteq M(X)$ be a neighborhood of ν. Since $M(T, X)$ is not a singleton, then we can take an open ball $G' \subseteq M(X)$ such that $\nu \in G' \subseteq \overline{G'} \subset G$ and $M(T, X) \setminus \overline{G'} \neq \emptyset$. Note that $M(T, X) \setminus \overline{G'}$ is open in $M(T, X)$.

From the proof of Proposition 2.3 (1) of [42], one construct a closed invariant set Y and there exists $n_{G'} \in \mathbb{N}$ such that $h_{top}(T, Y) > h_*$ and for any $y \in Y$ and any $n \geq n_{G'}$, $\Upsilon_n(y) \in G'$. So for any $m \in M_{erg}(T, Y)$, by Birkhoff ergodic theorem there is $y \in Y$ such that $\Upsilon_n(y)$ converge to m in weak* topology and thus $m \in \overline{G'}$. In other words, $M_{erg}(T, Y) \subseteq \overline{G'}$. By convex property of the ball G' and Ergodic Decomposition theorem, $M(T, Y) \subseteq \overline{G'}$. Then $Y \neq X$, since $M(T, X) \setminus \overline{G'} \neq \emptyset$. By Variational Principle, for $0 < \varepsilon < h_{top}(T, Y) - h_*$, take a $\mu \in M_{erg}(T, Y)$ such that $h_\mu(T) > h_{top}(T, Y) - \varepsilon > h_*$. Then μ is the measure we need. For more details, see [42]. □

3. Periodic and periodic-like recurrence

One important way to partition points with different asymptotic behavior is according to the recurrence property.

In the classical study of dynamical systems, an important concept is non-wandering point. A point $x \in X$ is called *wandering*, if there is a neighborhood U of x such that the sets $T^{-n}U$, $n \geq 0$, are mutually disjoint. Otherwise, x is called non-wandering. Let $\Omega(T)$ denote the set of all non-wandering points, called *non-wandering set*. The interesting action of T takes place in $\Omega(T)$ and recall that from Theorem 5.6, Theorem 6.15 and Corollary 8.6.1 in [59] $\Omega(T)$ is always invariant, compact, carries totally full measure and owns the whole complexity of the system

$$h_{top}(T) = h_{top}(T, \Omega(T)).$$

So in general one can always consider the subsystem $T : \Omega(T) \to \Omega(T)$ to replace the original system $T : X \to X$. It is interesting to ask for general dynamical systems, how about the dynamical complexity of $X \setminus \Omega(T)$? Unfortunately, in this paper the systems studied is the case $X = \Omega(T)$, since we require that the periodic points are dense in X in our main theorems (Theorems 1.2–1.4).

The set $\Omega(T)$ consists of those points with a weak recurrence property. Now let us recall the concept of recurrent point. It is known that recurrent points play important roles in the ergodic theory of dynamical systems. Given $x \in X$, let $\omega_T(x)$ denote the ω-limit set. We call $x \in X$ to be *recurrent*, if

$$x \in \omega_T(x).$$

Let $Rec(T)$ denote the set of all recurrent points. By definition, it is obvious that $Rec(T) \subseteq \Omega(T)$.

Theorem 3.1. *For any continuous map $T : X \to X$ of a compact metric space X, the recurrent set $Rec(T)$ has full topological entropy.*

Proof. It is known that from Poincaré recurrent theorem recurrent set $Rec(T)$ has totally full measure (see Remark on Page 157 of [59]) so that by Theorem 2.1 we complete the proof of Theorem 3.1. □

In the study of (smooth or topological) dynamical systems, many people pay attention to refine recurrent set according to the 'recurrent frequency'. A standard and important kind of recurrent point with same asymptotic behavior is periodic point, which returns itself through finite iterates. Let $Per(T)$ denote the set of periodic points. Then

$$Per(T) \subseteq Rec(T).$$

A fundamental question in dynamical systems is to search the existence of periodic points. For system with Bowen's specification (such as topological mixing subshifts of finite type, topological mixing uniformly hyperbolic and topological mixing expanding systems), it is well known that the set of periodic points is dense in the whole space (for example, see Proposition 21.3 in [15]) and moreover, if the system is expansive, then the periodic set is countable and thus has zero entropy. Here we give a basic observation on the topological entropy of periodic set for general dynamical systems, admitting existence of uncountable periodic points.

Theorem 3.2. *For any continuous map $T : X \to X$ of a compact metric space X, the periodic set $Per(T)$ either is empty or has zero topological entropy.*

We emphasize that the topological entropy we used is Bowen's version of entropy [10], as a critical point a la Hausdorff dimension, rather than the capacity version of entropy as a growth rate (see Chapter 7.2 of [59]).

Proof of Theorem 3.2. Notice that $Per(T) = \bigcup_{n\geq 1} P_n(T)$, where $P_n(T) = \{x \in X \,|\, T^n(x) = x\}$. By (2.2), we only need to show that for any $n \geq 1$, $P_n(T)$ carries zero entropy. Note that for any fixed $n \geq 1$, $P_n(T)$ is an invariant closed (compact) set, every ergodic measure μ supported on $P_n(T)$ is periodic and so μ has zero metric entropy. Applying classical Variational Principle (Corollary 8.6.1 in [59]) for subsystem $T|_{P_n(T)}$, we have

$$h_{top}(T, P_n(T)) = \sup_{\mu \in M_{erg}(T, P_n(f))} h_\mu(f) = 0.$$

This ends the proof of Theorem 3.2. $\square$

From Theorem 3.2, the periodic set has no dynamical complexity in the sense of topological entropy. However, a classical interesting result states that for expansive systems with Bowen's specification (which implies topological mixing), the topological entropy can be characterized by the exponential growth of periodic points with same period (for example, see Proposition 22.7 in [15]). More precisely,

$$h_{top}(T) = \lim_{n \to \infty} \frac{1}{n} \log \# P_n(T)$$

where $P_n(T) = \{x |\, T^n(x) = x\}$ and $\#A$ denotes the cardinality of the set A (more better characterization of entropy and the growth of periodic points, see Proposition 22.6 in [15]). Roughly speaking, periodic points have same complexity as the system itself from the viewpoint of two different ways to interpret complexity. This result holds for Axiom A diffeomorphisms in any dimension (see [11]), but this is a somewhat special situation. It is well known that Axiom A diffeomorphisms are not dense in $\mathrm{Diff}^1(M)$ (the space of all diffeomorphisms on a compact Riemannian manifold M). Kaloshin [28] showed that in general $\#P_n(T)$ can grow much faster than entropy. Moreover, it is well known that for C^1 generic diffeomorphisms, all periodic points are hyperbolic so that countable and they form a dense subset of the non-wandering set (by classical Kupka–Smale theorem, Pugh's or Mañé's Ergodic Closing lemma from Smooth Ergodic Theory, for example, see [31,51,45,44,34]). For $C^{1+\alpha}$ nonuniformly hyperbolic systems, many people studied the existence of periodic point by showing closing lemma, for example, see [29]. In particular, an interesting result from [29] is that any $C^{1+\alpha}$ surface diffeomorphism with positive entropy always carries a lot of periodic points. All in all, periodic points have been studied more and more in the research of modern dynamical systems.

In general, it is well known that there are lots of topological dynamical systems without periodic points. The standard example is irrational rotation. So many generalizations of periodic points are introduced. One such kind 'periodic-like' point is almost periodic point. A point $x \in X$ is *almost periodic*, if for every open neighborhood U of x, there exists $N > 0$ such that $f^k(x) \in U$ for some $k \in [n, n + N]$ and every integer $n \geq 1$. Let $A(T)$ denote the set of all almost periodic points, called almost periodic set for convenience. It is well-known from [6] that a point x is almost periodic if and only if x is minimal, i.e., $x \in \omega_T(x)$ and $\omega_T(x)$ is minimal (see [21,20,22,33] for more related discussions in the sense that the space X is more general, not necessarily being compact metric space). Here an invariant set $E \subseteq X$ is called minimal, if for every point $y \in E$, $\omega_T(y) = E$. In particular, if X is minimal, we say the system T to be minimal. Remark that the almost periodic set $A(T)$ can be written as the union of all minimal sets.

Remark that for any uniquely ergodic system, the support of the unique invariant measure must be minimal. However, there are minimal invariant sets which are not uniquely ergodic [39]. Note that $E \subseteq X$ is a minimal invariant subset if and only if the support of any invariant measure supported on E coincides with E. By Zorn's lemma, one can show that any dynamical system contains at least one minimal invariant subset. So different from periodic points, almost periodic points naturally exist and thus played important roles in the study of all topological dynamical systems. Moreover, constructions of minimal examples are studied a lot by many researchers. For homeomorphisms, there are many examples of subshifts which are strictly ergodic and has positive entropy [23,24]. For systems on manifolds, from [46] there are minimal homeomorphisms on 2-torus with positive entropy and it was proved in [5] that any compact manifold of dimension $d \geq 2$ which carries a minimal uniquely ergodic homeomorphism also carries a minimal uniquely ergodic homeomorphism with positive topological entropy. M. Herman asked whether, for diffeomorphisms, positive topological entropy was compatible with minimality or strict ergodicity. It was constructed in [26] a 4-dimensional example of a minimal (but not strictly ergodic) diffeomorphism with positive topological entropy. An interesting fact is that any $C^{1+\alpha}$ surface diffeomorphism with positive entropy is not minimal, since there are lots of periodic points, see Corollary 4.4 in [29]. For general non-minimal systems, we have a following observation.

Theorem 3.3. *For any non-minimal continuous map $T : X \to X$ of a compact metric space X, if there is an ergodic measure μ with full support, then $\mu(A(T)) = 0$.*

Proof. By contradiction, $\mu(A(T))>0$. By ergodicity and invariance of $A(T)$, $\mu(A(T)) = 1$. Let S_μ be the support of μ and by assumption $X = S_\mu$. By ergodicity and full support of μ, it is known that the set $D := \{x \in X | \overline{Orb(x)} = X\}$ has μ full measure (for example, see Theorem 1.7 or Theorem 5.16 in [59]). So $\mu(A(T) \cap D) = 1$ and thus we can take $z \in A(T)$ such that the orbit of z is dense in X. $z \in A(T)$ implies that $z \in \omega_T(z)$ and the closed invariant set $\omega_T(z)$ is minimal. $z \in \omega_T(z)$ implies that $\omega_T(z) = \overline{Orb(z)}$. Then $X = \overline{Orb(z)} = \omega_T(z)$ is minimal, contradicting that T is not minimal. □

Notice that for systems such as topological mixing Anosov systems, mixing subshifts of finite type, the unique measure with maximal entropy is ergodic and its support is not minimal. Thus, by Theorem 3.3 if one wants to find some kind of "periodic-like" points with totally full measure, we need to generalize almost periodic point to be more general. Zhou etc. (see [61,64,63]) introduced such two more general concepts of "periodic-like" points which have totally full measure for all dynamical systems.

One is called *weakly almost periodic* and another is more weaker called *quasi-weakly almost periodic*. Different 'recurrent frequency' determines different asymptotic behavior. If $E \subseteq X$ is nonempty and $x \in X$, define

$$\underline{P}_x(E) := \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)) \quad \text{and} \quad \overline{P}_x(E) := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)).$$

In other words, recalling the definition of $\Upsilon_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)}$,

$$\underline{P}_x(E) = \liminf_{n\to\infty} \Upsilon_n(x)(E), \quad \overline{P}_x(E) = \limsup_{n\to\infty} \Upsilon_n(x)(E).$$

If $\underline{P}_x(E) = \overline{P}_x(E)$, we denote by $P_x(E)$, which means the probability of the orbit of x enters in E. Let $V_\varepsilon(x)$ denote ε-neighborhood of x, i.e., $V_\varepsilon(x) = \{y \in M \mid d(x,y) < \varepsilon\}$.

Definition 3.4 *((Quasi-)weakly almost periodic).* We call x to be a weakly almost periodic point, if for any $\varepsilon > 0$,

$$\underline{P}_x(V_\varepsilon(x)) > 0.$$

x is called to be a quasi-weakly almost periodic point, if for any $\varepsilon > 0$,

$$\overline{P}_x(V_\varepsilon(x)) > 0.$$

Let $QW(T)$ and $W(T)$ denote the sets of all weakly almost periodic points and all quasi-weakly almost periodic points, respectively. Remark that

$$\Omega(T) \supseteq Rec(T) \supseteq QW(T) \supseteq W(T) \supseteq A(T) \supseteq Per(T). \tag{3.9}$$

We have known $\Omega(T)$, $Rec(T)$ both have full measure for any invariant measure. Now we show that $QW(T)$ and $W(T)$ also both have full measure for any invariant measure.

Theorem 3.5. *For any continuous map $T : X \to X$ of a compact metric space X, $QW(T)$ and $W(T)$ both have full measure for any invariant measure. In particular, each one of $QW(T)$ and $W(T)$ carries full topological entropy.*

Proof. For any ergodic measure μ, let $G(\mu)$ denote the set of all points satisfying that $\Upsilon_n(x)$ converges to μ in weak* topology. By Birkhoff ergodic theorem $G(\mu)$ is of μ full measure. Let S_μ denote the support of μ, meaning that S_μ is the smallest closed invariant set with μ full measure. So $S_\mu \cap G(\mu)$ is of μ full measure and every $x \in S_\mu \cap G(\mu)$ satisfies $\Upsilon_n(x) \to \mu$ in weak* topology. By weak* topology for open sets (see Remarks (3) (iii) on Page 149 of [59]), we have for any $x \in S_\mu \cap G(\mu)$, $\varepsilon > 0$,

$$\underline{P}_x(V_\varepsilon(x)) = \liminf_{n\to\infty} \Upsilon_n(x)(V_\varepsilon(x)) \geq \mu(V_\varepsilon(x)) > 0.$$

That is, for any ergodic measure μ, μ a.e. x belongs to $W(T)$. Thus by Ergodic Decomposition theorem, $W(T)$ has full measure for any invariant measure. By (3.9) $QW(T)$ also has full measure for any invariant measure. By Theorem 2.1 we complete the proof. □

Remark that by (3.9) the proof of Theorem 3.5 also can be as an alternative proof to show that $\Omega(T)$, $Rec(T)$ both have full measure for any invariant measure.

There are some equivalent statements of weakly almost periodic point and quasi-weakly almost periodic point. Firstly we need to recall some notions (see [62], for the case of flow see [36]). A set E is said to be a *center of attraction* of T with respect to $X_0 \subseteq X$, if E is a closed invariant set, and for any $x \in X_0$ and $\varepsilon > 0$, the limit

$$P_x(V(E,\varepsilon)) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{V(E,\varepsilon)}(T^i(x))$$

exists and equals to 1, where $V(E,\varepsilon)$ denotes the ε-neighborhood of E, i.e., $V(E,\varepsilon) = \{y \in X | d(E,y) < \varepsilon\}$. A set E is called *minimal center of attraction* relative to X_0 if E is a center of attraction relative to X_0 but no proper subset of E also is. The minimal center of attraction always exists and is unique (this is a basic fact and it follows from Lemma 3.6 which comes soon). Denote by $C(X_0)$ the minimal center of attraction relative to X_0. In particular, let

$$C_x := C(\{x\}),\ x \in X.$$

Let $M_x(T)$ be the set of all limits of $\Upsilon_n(x)$ in weak* topology. Recall that S_μ denotes the support of a measure μ (that is, the smallest closed subset of X with μ full measure). Let $M_{X_0}(T) = \cup_{x\in X_0} M_x(T)$. From [62,61,64,63] we know some basic facts:

Lemma 3.6. *Let $T : X \to X$ be a continuous map of a compact metric space X. For any $X_0 \subseteq X$, $x \in X$,*

$$C(X_0) = \overline{\cup_{m\in M_{X_0}(T)} S_m},\ \textit{in particular}\ C_x = \overline{\cup_{m\in M_x(T)} S_m}; \tag{3.10}$$

$$\forall\ x \in X,\ C_x \subseteq \omega_T(x); \tag{3.11}$$

For any recurrent point $x \in Rec(T)$,

$$x \in W(T) \Leftrightarrow C_x = S_\mu,\ \forall \mu \in M_x(T) \Leftrightarrow \omega_T(x) = S_\mu,\ \forall \mu \in M_x(T); \tag{3.12}$$

$$x \in QW(T) \Leftrightarrow x \in C_x \Leftrightarrow \omega_T(x) = C_x \Leftrightarrow \omega_T(x) = \overline{\cup_{\mu\in M_x(T)} S_\mu}. \tag{3.13}$$

If $C_x = X$, then by (3.11) $\omega_T(x) = X \ni x$ so that $x \in Rec(T) \cap C_x$. By (3.13), $x \in QW(T)$. That is, we have the following corollary.

Corollary 3.7. *Let $T : X \to X$ be a continuous map of a compact metric space X. For any $x \in X$,*

$$C_x = X \Rightarrow x \in QW(T). \tag{3.14}$$

For convenience of readers, we prefer to introduce some classical equivalent definitions. Let $S \subseteq \mathbb{N}$, define

$$\bar{d}(S) := \limsup_{n\to\infty} \#(S \cap \{0, 1, \cdots, n-1\})$$

and

$$\underline{d}(S) := \liminf_{n\to\infty} \#(S \cap \{0, 1, \cdots, n-1\}).$$

These two concepts are called *lower density* and *upper density* of S, respectively. If $\bar{d}(S) = \underline{d} = d$, we call S to have density of d. A set $S \subseteq \mathbb{N}$ is called *syndetic*, if there is $N > 0$ such that for any $n \geq 1$,

$$S \cap \{n, n+1, \cdots, n+N\} \neq \emptyset.$$

Let $U, V \subseteq X$ be two nonempty open subsets and $x \in X$. Define sets of recurrent time

$$N(U, V) := \{n \geq 1 |\, U \cap f^{-n}(V) \neq \emptyset\}$$

and

$$N(x, U) := \{n \geq 1 |\, f^n(x) \in U\}.$$

Then it is easy to check that for any $x \in X$,

$$\begin{aligned}
x \text{ is almost periodic } &\Leftrightarrow \forall\, \epsilon > 0,\ N(x, V_\epsilon(x)) \text{ is syndetic},\\
x \text{ is weakly almost periodic } &\Leftrightarrow \forall\, \epsilon > 0,\ N(x, V_\epsilon(x)) \text{ has positive lower}\\
&\qquad \text{density},\\
x \text{ is quasi-weakly almost periodic } &\Leftrightarrow \forall\, \epsilon > 0,\ N(x, V_\epsilon(x)) \text{ has positive upper}\\
&\qquad \text{density},\\
x \text{ is recurrent } &\Leftrightarrow \forall\, \epsilon > 0,\ N(x, V_\epsilon(x)) \neq \emptyset,\\
x \text{ is non-wandering } &\Leftrightarrow \forall\, \epsilon > 0,\ N(V_\epsilon(x), V_\epsilon(x)) \neq \emptyset.
\end{aligned}$$

Remark that from these equivalent statements, for recurrent and non-wandering points it is not required positive upper or lower density and for almost periodic points it is required not only lower density but also some uniformly good order. Thus, these "periodic-like" recurrences essentially reflect different "recurrent frequency". A natural question arises:

How much difference are there between these "periodic-like" recurrences?

One fundamental way is to consider non-emptiness of gap-sets. Recall that $A(T)$ maybe not have totally full measure (for example, mixing subshifts of finite type and β-shifts) but $W(T)$ has totally full measure, thus the concepts of weakly and quasi-weakly almost periodic is more general than almost periodic from the probabilistic perspective. Moreover, many people pay attention to which system has *nonempty* gap between two "periodic-like" sets (see [63,37,25,60] etc.), $QW(T) \setminus W(T)$. For generic diffeomorphisms and systems with Bowen's specification such as topological mixing subshifts of finite type and topological mixing hyperbolic systems, we will prove that $QW(T) \setminus W(T)$ always contains a dense G_δ subset (see Proposition 9.6 and Theorem 9.9 below). This is a characterization from topological or geometric perspective so that the concept of quasi-weakly almost periodic is more general than weakly almost periodic point. However, note that the sets $QW(T) \setminus W(T)$ and $Rec(T) \setminus QW(T)$ have zero measure for any invariant measure. So we need to find another way to characterize the difference.

As said in the beginning of the paper, it is known that topological entropy is a better and deeper tool to study dynamical complexity than non-emptiness. Now let us firstly consider some simple observation for general dynamical systems.

Theorem 3.8. *For any continuous map $T : X \to X$ of a compact metric space X and a set $Y \subseteq X$ with totally full measure,*

$$Y \setminus Per(T) \text{ either is empty or carries full topological entropy.} \tag{3.15}$$

In particular, $W(T) \setminus Per(T)$ either is empty or carries full topological entropy and by (3.9) so does $QW(T) \setminus Per(T)$ and $Rec(T) \setminus Per(T)$.

Proof. Suppose $Y \setminus Per(T) \neq \emptyset$. For $Y_1 = Per(T)$ and $Y_2 = Y \setminus Per(T)$, by (2.2) and Theorem 3.2 $Y \setminus Per(T)$ should carry full topological entropy. In particular, $W(T) \setminus Per(T)$ carries full topological entropy and by (3.9) so do $QW(T) \setminus Per(T)$, $Rec(T) \setminus Per(T)$ and $\Omega(T) \setminus Per(T)$. □

Note that

$$W(T) \setminus Per(T) = (W(T) \setminus A(T)) \sqcup (A(T) \setminus Per(T)),$$

where $\sqcup$ denotes the disjoint union. By the discussion of $W(T) \setminus Per(T)$ in Theorem 3.8 and (2.2),

Theorem 3.9. *For any continuous map $T : X \to X$ of a compact metric space X, at least one of $W(T) \setminus A(T)$ and $A(T) \setminus Per(T)$ carries full topological entropy.*

4. Regularity, quasiregularity and irregularity

Another important way to observe points is from Birkhoff Ergodic theorem, called *quasiregular points* (for example, see [15,39]), *regular points* (see [39]) and *irregular points* (for example, see Chapter 8 in [2]).

Firstly let us recall the definition of generic point and quasiregular point (see Chapter 4 in [15]). A point $x \in X$ is said to be *generic* for a measure $\mu \in M(T, X)$, if for any continuous function $\phi : X \to \mathbb{R}$, the limit

$$\phi^*(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x))$$

exists and equals to $\int \phi d\mu$. Let G_μ (or $G(\mu)$) denote the set of all generic points for μ, called generic set of μ for convenience. By weak* topology,

$$x \in G_\mu \;\Leftrightarrow\; \lim_{n\to\infty} \Upsilon_n(x) = \mu \Leftrightarrow M_x(T) = \{\mu\}. \tag{4.16}$$

Remark that for different $\mu \neq \nu \in M(T, X)$, $G_\mu \cap G_\nu = \emptyset$. Recall a classical result (see Theorem 3 in [10]) that for any continuous map $T : X \to X$, every ergodic measure $\mu \in M_{erg}(T, X)$ satisfies that

$$h_\mu(f) = h_\mu(G_\mu). \tag{4.17}$$

By Birkhoff ergodic theorem, for any ergodic measure $\mu \in M_{erg}(T, X)$, G_μ is of μ full measure. Thus, by Ergodic Decomposition theorem, for any invariant measure $\mu \in M(T, X)$,

$$\mu(\cup_{\nu \in M_{erg}(T,X)} G_\nu) = 1.$$

This implies for any invariant measure $\mu \in M(T, X) \setminus M_{erg}(T, X)$, G_μ either is empty or satisfies that

$$\mu(G_\mu) = 0.$$

A point $x \in X$ is called *quasiregular* with respect to T, if it is generic with respect to some invariant measure. Denote by $QR(T)$ the set of all quasiregular points with respect to T, called quasiregular set for convenience. By definition,

$$x \in QR(T) \Leftrightarrow\; \exists\, \mu \in M(T, X),\; x \in G_\mu \Leftrightarrow M_x(T) \text{ is a singleton.} \tag{4.18}$$

For convenience, for any $x \in QR(T)$, denote by μ_x the invariant measure for which x is generic. Now let us introduce another concept, called ϕ-regular, and then we use it to give more equivalent statements of $QR(T)$. Let $\phi : X \to \mathbb{R}$ be a continuous observable function. A point $x \in X$ is called to be *quasiregular for* ϕ with respect to T (simply,

ϕ-regular), if the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi(T^i(x))$ exists. Define the ϕ-regular set to be the set of all ϕ-regular points, that is,

$$R_\phi(T):=\{x\in X|x \text{ is } \phi\text{-regular}\}.$$

Let $C^0(X)$ denote the space of all continuous functions on X. Then by definition and weak* topology,

Theorem 4.1. *For any continuous map $T:X\to X$ of a compact metric space X,*

$$QR(T)=\bigcup_{\mu\in M(T,X)}G_\mu=\bigcap_{\phi\in C^0(X)}R_\phi(T).$$

Proof. The first equality is from definition. For the second equality, by definition, the relation "$\subseteq$" is obvious and so we only need to prove the relation "$\supseteq$". By contradiction, there is some $x\in X$ such that x is ϕ-regular point for any continuous function $\phi:X\to\mathbb{R}$, but x is not in $QR(T)$. From (4.18) $\Upsilon_n(x)$ do not converge to a unique measure, then this sequence has at least two different limit points μ and ν. By weak* topology and the definition of ϕ-regular point, for any continuous function $\phi:X\to\mathbb{R}$, the limit of

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(T^i(x))$$

equals to $\int\phi(x)d\mu$ and $\int\phi(x)d\nu$. In other words, $\int\phi(x)d\mu=\int\phi(x)d\nu$ holds for any continuous functions. By weak* topology this implies $\mu=\nu$, which is a contradiction. □

Now we start to recall the concept of regular point (see [39]). A point $x\in QR(T)$ is called a *point of density*, if $\mu_x(U)>0$ for every open set $U\subseteq X$ containing x. Let $QR_d(T)$ denote the set of all points of density in $QR(T)$ and for convenience in present paper $QR_d(T)$ is called density set. It is easy to check that for any $x\in QR(T)$,

$$x\in QR_d(T)\Leftrightarrow x\in S_{\mu_x}. \tag{4.19}$$

Thus

$$QR_d(T)=\bigcup_{\mu\in M(T,X)}(G_\mu\cap S_\mu). \tag{4.20}$$

Let $QR_{erg}(T):=\cup_{\nu\in M_{erg}(T,X)}G_\nu$. In [39] the point in $QR_{erg}(T)$ is called transitive, but in present paper transitive point means that its orbit is dense in the whole space X. To avoid confusion, in this paper points in $QR_{erg}(T)$ are called *ergodic-transitive* and the set $QR_{erg}(T)$ is called ergodic-transitive set. A point $x\in X$ is called *regular*, if it belongs to the set $R(T)=QR_d(T)\cap QR_{erg}(T)$ (called regular set). Remark that

$$R(T)=\bigcup_{\mu\in M_{erg}(T,X)}(G_\mu\cap S_\mu)\subseteq QR_d(T)\cup QR_{erg}(T)\subseteq QR(T). \tag{4.21}$$

By Birkhoff Ergodic theorem and Ergodic Decomposition theorem, $R(T)$ has totally full measure (see [39] for a proof) and so does $QR(T)$ and every ϕ-regular set $R_\phi(T)$. Thus, by Theorem 2.1 regular points (resp., quasiregular points or ϕ-regular points) essentially determine the dynamical complexity of any system $T : X \to X$. That is,

Theorem 4.2. *For any continuous map $T : X \to X$ of a compact metric space X, $R(T)$ carries full topological entropy and so does $QR_d(T)$, $QR_{erg}(T)$, $QR(T)$ and every $R_\phi(T)$ $(\forall \phi \in C^0(X))$.*

Let $I(T)$ denote the complementary set of quasiregular set, that is $I(T) = X \setminus QR(T)$. This set is called *irregular set* and its element is called *irregular point* (for example, see Chapter 8 in [2], also called *point with historic behavior* in [47,53] and called 'non-typical' point in [4]). In other words, a point $x \in X$ is irregular if and only if there is some continuous function $\phi : X \to \mathbb{R}$ such that the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(T^i(x))$ does not exist. Note that every irregular point is observed by at least one continuous function and so let us recall a concept called irregular point for the Birkhoff averages of a function (for example, see [4,57,56]). Let $\phi : X \to \mathbb{R}$ be a continuous function. A point $x \in X$ is called to be *irregular for the Birkhoff averages of* ϕ (simply, *ϕ-irregular*), if the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(T^i(x))$ does not exist. Remark that $I_\phi(T) = X \setminus R_\phi(T)$. By Theorem 4.1,

$$I(T) = X \setminus QR(T) = \bigcap_{\mu \in M(T,X)} (X \setminus G_\mu) = \bigcup_{\phi \in C^0(X)} X \setminus R_\phi(T) = \bigcup_{\phi \in C^0(X)} I_\phi(T). \tag{4.22}$$

Thus by (4.18) we have

$$x \in I(T) \Leftrightarrow M_x(T) \text{ is not a singleton} \Leftrightarrow \Upsilon_n(x) \text{ does not converge.} \tag{4.23}$$

From Theorem 4.2, $QR(T)$ has totally full measure. Thus by (4.22) $I(T)$ has zero measure for any invariant measure and so does every $I_\phi(T)$.

However, in recent several years many people have focused on studying the dynamical complexity of irregular set from different sights, for example, in the sense of dimension theory and topological entropy (or pressure) etc. Pesin and Pitskel [41] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. Barreira, Schmeling, etc. studied the irregular set in the setting of subshifts of finite type and beyond, see [4,2,54] etc. Recently, Thompson shows in [56,57] that every ϕ-irregular set $I_\phi(T)$ either is empty or carries full topological entropy (or pressure) when the system satisfies (almost) specification, which is inspired from [43] by Pfister and Sullivan and [54] by Takens and Verbitskiy. For convenience of readers, let us recall a result from [57].

Theorem 4.3. *(See [57].) Let T be a continuous map of a compact metric space X with almost specification (slightly weaker than g-almost product property). Then for any continuous function $\phi : X \to \mathbb{R}$, the ϕ-irregular set $I_\phi(T)$ either is empty or carries full topological entropy.*

$I(T)$ contains any ϕ-irregular set so that it has similar results in a certain class of dynamical systems such as [4,2,54,56,57] etc. For example, by Theorem 4.3 we have

Theorem 4.4. *Let T be a continuous map of a compact metric space X with almost specification. Then the irregular set $I(T)$ either is empty or carries full topological entropy.*

For the case of specification, a separate proof of this result can be found in [13] (irregular point is called *divergence point* there). Moreover, if further assuming that the system is not uniquely ergodic, we have

Theorem 4.5. *Let T be a continuous map of a compact metric space X with almost specification. If T is not uniquely ergodic (equivalently, $M(T,X)$ is not a singleton), then the irregular set $I(T)$ is not empty and carries full topological entropy.*

To prove Theorem 4.5, we need a following fact from [57] that

Lemma 4.6. *Let T be a continuous map of a compact metric space X with almost specification (which is a little weaker than g-almost product property). Let $\psi : X \to \mathbb{R}$ be a continuous function. Then*

$$\inf_{\mu\in M(T,X)} \int \psi(x)d\mu < \sup_{\mu\in M(T,X)} \int \psi(x)d\mu \Leftrightarrow I_\psi(T) \neq \emptyset.$$

Proof. For the case of '$\Leftarrow$', it is the fact (4.27). For the case of '$\Rightarrow$', see the paragraph behind of Lemma 2.1 in [57], as a corollary of Lemma 2.1 and Theorem 4.1 there on Page 5397 (for the case of specification, see Lemma 1.6 of [56]). □

Proof of Theorem 4.5. By assumption, there are two different invariant measures μ_1, μ_2. By weak* topology, there is a continuous function $\phi : X \to \mathbb{R}$ such that

$$\int \phi d\mu_1 \neq \int \phi d\mu_2.$$

Then

$$\inf_{\mu\in M(T,X)} \int \phi(x)d\mu < \sup_{\mu\in M(T,X)} \int \phi(x)d\mu.$$

By Lemma 4.6 this implies $I_\phi(T) \neq \emptyset$ and so does $I(T)$. Thus by Theorem 4.4 we complete the proof. □

Some classical results are known on multi-fractal analysis of Birkhoff averages, for example see [43,54,38]. More precisely, firstly let's recall $R_{\phi,a}(T)$ which denote the set of points whose Birkhoff average by ϕ equal to a, that is,

$$R_{\phi,a}(T) := \{x \in X| \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x)) = a\}.$$

Remark that

$$R_\phi(T) = \bigsqcup_{a\in\mathbb{R}} R_{\phi,a}(T),$$

where $\sqcup$ denotes the disjoint union. Let us state some basic facts as follows. For $\phi \in C^0(X)$ and $a \in \mathbb{R}$, by the continuity of ϕ and weak* topology,

$$x \in R_{\phi,a}(T) \Leftrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = a \Leftrightarrow M_x(T) \subseteq \{\rho| \int \phi d\rho = a\}. \tag{4.24}$$

So by (4.24)

$$x \in R_\phi(T) \Leftrightarrow \exists\ a \in \mathbb{R},\ \ M_x(T) \subseteq \{\rho| \int \phi d\rho = a\}. \tag{4.25}$$

Thus one has

$$x \in I_\phi(T) \Leftrightarrow \exists\ \mu_1, \mu_2 \in M_x(T) \text{ such that } \int \phi d\mu_1 \neq \int \phi d\mu_2. \tag{4.26}$$

In particular,

$$I_\phi(T) \neq \emptyset \Rightarrow \inf\{\int \phi d\mu|\, \mu \in M(T,X)\} < \sup\{\int \phi d\mu|\, \mu \in M(T,X)\}. \tag{4.27}$$

Recall $L_\phi := [\inf\{\int \phi d\mu|\, \mu \in M(T,X)\}, \sup\{\int \phi d\mu|\, \mu \in M(T,X)\}]$, and let $Int(L_\phi)$ denote the interior of L_ϕ. That is,

$$Int(L_\phi) = (\inf\{\int \phi d\mu|\, \mu \in M(T,X)\}, \sup\{\int \phi d\mu|\, \mu \in M(T,X)\}).$$

Remark that if $I_\phi(T) \neq \emptyset$, by (4.26) $Int(L_\phi)$ is a nonempty and open interval.

Now let us recall a result of [43] about variational principle on $R_{\phi,a}(T)$.

Proposition 4.7. *(See Proposition 7.1 of [43].) Let T be a continuous map of a compact metric space X. Suppose that T is single-saturated. Let $\phi : X \to \mathbb{R}$ be a continuous function. Then for any real number $a \in L_\phi$,*

$$h_{top}(T, R_{\phi,a}(T)) = \sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}.$$

For uniformly hyperbolic maps and Hölder continuous functions, Barreira and Saussol [3] established similar result as Proposition 4.7. The study of multifractal analysis for arbitrary (that is, non-Hölder) continuous functions was initiated in the symbolic dynamics setting by Olivier [38], Fan and Feng [18,19]. Similar results for maps with specification can be found in [54] (for pressure, see [55]).

Let us consider the entropy map $\Psi : L_\phi \to \mathbb{R}$:

$$a \mapsto \sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}.$$

Proposition 4.8. *Let T be a continuous map of a compact metric space X with positive entropy. Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then Ψ is a positive function on the interval $Int(L_\phi)$.*

Proof. Take a $\omega \in M(T,X)$ with positive entropy. If $\int \phi d\omega = a$, then

$$\sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\} \geq h_\omega(f) > 0.$$

Otherwise, without loss of generality, we assume $\int \phi d\omega > a$. By definition of L_ϕ and connectedness of $M(T,X)$, we can take a measure $\sigma \in M(T,X)$ such that $\int \phi d\sigma < a$. Then we can choose suitable $\xi \in (0,1)$ such that $\nu = \xi\omega + (1-\xi)\sigma$ satisfies $\int \phi d\nu = a$. Remark that $h_\nu(T) \geq \xi h_\omega(T) > 0$. So $\sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\} \geq h_\nu(T) > 0$. □

Moreover, we point out a result on the continuity and concave property of entropy function.

Proposition 4.9. *Let T be a continuous map of a compact metric space X. Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then the entropy map $\Psi : L_\phi \to \mathbb{R}$ is a concave function and thus continuous on $Int(L_\phi)$. In particular, $h_{top}(T) = \sup_{a \in Int(L_\phi)} \Psi(a)$.*

Proof. For any $a_1, a_2 \in L_\phi$, $\theta \in [0,1]$,

$$\begin{aligned}
&\theta \sup\{h_\mu(f) : \int \phi(x) d\mu = a_1\} + (1-\theta) \sup\{h_\mu(f) : \int \phi(x) d\mu = a_2\} \\
&= \sup\{\theta h_{\mu_1}(f) + (1-\theta) h_{\mu_2}(f) : \int \phi(x) d\mu_i = a_i,\ i = 1,2\} \\
&= \sup\{h_{\theta\mu_1 + (1-\theta)\mu_2}(f) : \int \phi(x) d\mu_i = a_i,\ i = 1,2\} \\
&\leq \sup\{h_\mu(f) : \int \phi(x) d\mu = \theta a_1 + (1-\theta) a_2\}.
\end{aligned}$$

By classical convex analysis theory, convex function is always (locally Lipshitz) continuous over interior subset of the domain.

Fix $\varepsilon > 0$. By Variational Principle, we can take one $\mu \in M(T, X)$ such that

$$h_\mu(T) > h_{top}(T) - \varepsilon.$$

Then by (4.27) there is another $\nu \in M(T, X)$ such that

$$\int \phi d\mu \neq \int \phi d\nu.$$

Take $\theta \in (0, 1)$ close to 1 such that

$$h_\omega(T) > h_{top}(T) - \varepsilon,$$

where $\omega = \theta\mu + (1-\theta)\nu$. Remark that the value of $\int \phi d\omega$ is between $\int \phi d\mu$ and $\int \phi d\nu$ so that if $a = \int \phi d\omega$, then $a \in Int(L_\phi)$ and

$$\Psi(a) = \sup\{h_\rho(T) | \ \rho \in M(T, X) \text{ and } \int \phi d\rho = a\} \geq h_\omega(T) > h_{top}(T) - \varepsilon.$$

Now we complete the proof. □

For higher smoothness of the concave function Ψ, it may be worth mentioning here that regularity of Ψ depends on the form of the pressure function $t \to P(t\phi)$, of which Ψ is the Legendre transform. Although the fact that the two are related by a Legendre transform is most well-known when the system is uniformly hyperbolic or has specification or some such other hypothesis, in fact it holds for any continuous map on a compact metric space; for example, see Proposition 2.9 (specifically (2.20)) of [14]. The notation there is different and the proof is for rather more general multifractal objects, but in particular it implies the first part of Proposition 4.9, concavity of Ψ, and shows that as one expects, non-differentiability of Ψ corresponds to intervals of t on which $P(t\phi)$ is linear. Thus one can easily construct examples where Ψ is non-differentiable.

5. Overlap of regularity and periodic-like recurrence

Regularity and recurrence are two different "eyes" to study asymptotic behavior. Inspired from above analysis, a natural idea is to consider the recurrence and (ir)regularity simultaneously. Roughly speaking, under the observation of two "eyes", we aim to obtain more deeper results. In this section let us first deal with some simple relations.

Theorem 5.1. *For any continuous map $T : X \to X$ of a compact metric space X,*

$$R(T) \subseteq W(T) \cap QR(T), \quad Rec(T) \cap QR(T) \subseteq W(T), \quad QW(T) \setminus W(T) \subseteq I(T).$$

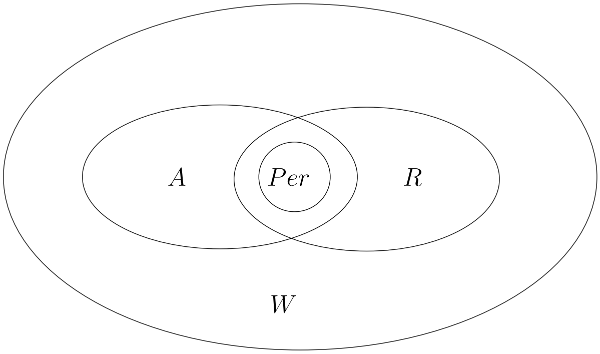

Fig. 4. $W(T)$.

Proof. We prove the three relations respectively as follows:

(1) If $x \in R(T)$, by definition $x \in QR(T)$. Now we start to prove $x \in W(T)$. Recall μ_x denotes the ergodic measure for which x is generic. Then by definition of regular point, it is a point of density so that for any $\varepsilon > 0$, $\mu(V_\varepsilon(x)) > 0$. By weak* topology for open sets (see Remarks (3) (iii) on Page 149 of [59]),

$$\underline{P}_x(V_\varepsilon(x)) = \liminf_{n\to\infty} \Upsilon_n(x)(V_\varepsilon(x)) \geq \mu(V_\varepsilon(x)) > 0.$$

So $x \in W(T)$.

(2) If $x \in Rec(T) \cap QR(T)$, then there is $\mu \in M(T, X)$ such that $x \in G_\mu$ (the set of all generic points of μ). So $M_x(T) = \{\mu\}$ and by (3.10) $C_x = \overline{\cup_{m\in M_x(T)} S_m} = S_\mu$. Then by (3.12) $x \in Rec(T)$ and $C_x = S_\nu$, $\forall \nu \in M_x(T)$ imply $x \in W(T)$.

(3) Let $x \in QW(T) \setminus W(T)$. Then by (3.9) $x \in Rec(T)$. If $x \in QR(T)$, by the second statement $x \in W(T)$, which is a contradiction to $x \in QW(T) \setminus W(T)$. $\square$

Fig. 4 is to illustrate the relations between $R(T)$, $W(T)$, $Per(T)$, $A(T)$ (simply, writing R, W, Per, A respectively in the figure).

Note that obviously one has $Per(f) \subseteq R(T)$. From (3.15) for any dynamical system, $R(T) \setminus Per(T)$ always carries full topological entropy, since $R(T)$ always has totally measure. Now we deal with $R(T)$ and $A(T)$. For full shifts on finite symbols, $R(T) \cap A(T)$ has full topological entropy.

Theorem 5.2. *For full shift on k symbols ($k \geq 2$), $R(T) \cap A(T) \setminus Per(T)$ carries full topological entropy and so does $A(T) \setminus Per(T)$. In particular, the almost periodic set $A(T)$ carries full topological entropy.*

Proof. It is not difficult to prove. Denote the set of all ergodic measures of all uniquely ergodic minimal subshifts with *positive* entropy by $M^*_{erg}(T, X)$. Then

$$\bigcup_{\mu\in M^*_{erg}(T,X)} S_\mu = \bigcup_{\mu\in M^*_{erg}(T,X)} G_\mu \cap S_\mu \subseteq R(T) \cap A(T) \setminus Per(T)$$

and for any $\mu \in M^*_{erg}(T,X)$, by classical Variational Principle (Theorem 8.6 and Corollary 8.6.1 in [59]) for $T|_{S_\mu}$,

$$h(T,S_\mu) = \sup_{\nu \in M_{erg}(T,S_\mu)} h_\nu(f) = h_\mu(T).$$

Recall a result from [23,24] that for any full shift on finite symbols, there exist uniquely ergodic minimal subshifts with any given entropy. This implies that

$$h_{top}(T) = \sup_{\mu \in M^*_{erg}(T,X)} h_\mu(T) = \sup_{\mu \in M^*_{erg}(T,X)} h(T,S_\mu).$$

Thus by (2.1),

$$\begin{aligned} &h_{top}(T, R(T) \cap A(T) \setminus Per(T)) \le h_{top}(T,X) = h_{top}(T) \\ &= \sup_{\mu \in M^*_{erg}(T,X)} h(T,S_\mu) \le h_{top}(T, R(T) \cap A(T) \setminus Per(T)). \end{aligned} \tag{5.28}$$

We complete the proof. □

Remark that even for full shifts on finite symbols, it is still unknown the non-emptiness of $A(T) \setminus R(T)$ and its entropy estimate (to the best of my knowledge).

Now we consider $R(T) \setminus A(T)$. Firstly let us give some simple observation as follows.

Lemma 5.3. *For any continuous map $T : X \to X$ of a compact metric space X, if there is an invariant measure μ with non-minimal support, then $G_\mu \cap A(T) = \emptyset$.*

Proof. By contradiction, there is $x \in G_\mu \cap A(T)$. Then $\omega_T(x)$ is a minimal compact set and $x \in \omega_T(x)$. Thus $\omega_T(x) = \overline{Orb(x)}$. We claim that $\mu(\omega_T(x)) = 1$. More precisely, by weak* topology for closed sets (see Remarks (3) (ii) on Page 149 of [59]), $\Upsilon_n(x) \to \mu$ implies that

$$1 = \limsup_{n\to\infty} \Upsilon_n(x)(Orb(x)) \le \limsup_{n\to\infty} \Upsilon_n(x)(\overline{Orb(x)}) \le \mu(\overline{Orb(x)}) = \mu(\omega_T(x)).$$

So $\mu(\omega_T(x)) = 1$ implies $S_\mu \subseteq \omega_T(x)$, contradicting that S_μ is not minimal. □

Theorem 5.4. *For any continuous map $T : X \to X$ of a compact metric space X, if there is an ergodic measure μ with maximal entropy and non-minimal support, then $R(T) \setminus A(T)$ carries full topological entropy.*

Proof. Since μ is ergodic, by Birkhoff ergodic theorem G_μ is of μ full measure. Then $G_\mu \cap S_\mu$ also has full measure, since S_μ has full measure. Since μ has maximal entropy, then by (2.4) $h_{top}(G_\mu \cap S_\mu) \ge h_\mu(f) = h_{top}(T)$. By non-minimal assumption of S_μ

and Lemma 8.1 $A(T) \cap G_\mu = \emptyset$. By ergodicity, $G_\mu \cap S_\mu \subseteq R(T)$ and thus $G_\mu \cap S_\mu \subseteq R(T) \cap G_\mu \subseteq R(T) \setminus A(T)$. Then by (2.1) $h_{top}(R(T) \setminus A(T)) \geq h_{top}(G_\mu \cap S_\mu) \geq h_{top}(T)$. We complete the proof. □

From Theorem 5.1, $R(T) \subseteq QR(T) \cap W(T)$ and thus $R(T) \setminus A(T) \subseteq QR(T) \cap W(T) \setminus A(T)$. So by Theorem 5.4 we have a following consequence.

Theorem 5.5. *For any continuous map $T : X \to X$ of a compact metric space X, if there is an ergodic measure μ with maximal entropy and non-minimal support, then $QR(T) \cap W(T) \setminus A(T)$ carries full topological entropy and so does $QR(T) \setminus A(T)$ and $W(T) \setminus A(T)$.*

Remark that Theorem 5.4 and Theorem 5.5 are applicative to all the examples in Section 1.2, since each example is non-minimal and the unique maximal entropy measure is ergodic and has full support.

6. Some useful facts and lemmas

Firstly let us recall two basic properties of invariant measures with full support from [15]. Let $M_{full}(T, X)$ denote the set of all invariant measures with full support.

Lemma 6.1. *(See Proposition 21.11 in [15].) Let $T : X \to X$ be a continuous map on a compact metric space X. Then the set $M_{full}(T, X)$ is either empty or a dense G_δ subset in $M(T, X)$.*

Lemma 6.2. *(See Proposition 21.12 in [15].) Let $T : X \to X$ be a continuous map on a compact metric space X. If the periodic points are dense in X, then the set $M_{full}(T, X)$ is a dense G_δ subset in $M(T, X)$.*

A direct corollary of Lemma 6.1 is the following.

Lemma 6.3. *Let $T : X \to X$ be a continuous map on a compact metric space X. If there is an invariant measure with full support, then the set $M_{full}(T, X)$ is a dense G_δ subset in $M(T, X)$.*

Moreover, we discuss the relation between periodic points, periodic measures and measures with full support.

Proposition 6.4. *Let $T : X \to X$ be a continuous map on a compact metric space X. If there is an invariant measure μ with full support and a sequence of periodic measures μ_n such that μ_n converges to μ in weak* topology. Then the periodic points are dense in X.*

Proof. By weak* topology for closed sets (see Remarks (3) (ii) on Page 149 of [59]),

$$1 = \limsup_{n\to\infty} \mu_n(\overline{Per(T)}) \leq \mu(\overline{Per(T)}).$$

So $S_\mu \subseteq \overline{Per(T)}$ and then $X = \overline{Per(T)}$, since μ has full support. □

By Proposition 6.4 and Lemma 6.2, we have a following consequence.

Proposition 6.5. *Let $T : X \to X$ be a continuous map on a compact metric space X. Suppose that the periodic measures are dense in the space of invariant measures. Then*

$$\overline{Per(T)} = X \Leftrightarrow \exists \mu \in M(T,X),\ S_\mu = X.$$

For non-ergodic measures, we have a following result.

Lemma 6.6. *Let $T : X \to X$ be a not uniquely ergodic continuous map on a compact metric space X. Then the set of non-ergodic measures $M(T,X) \setminus M_{erg}(T,X)$ is a dense subset in $M(T,X)$.*

Proof. One can use Ergodic Decomposition theorem to prove, here we give a constructed proof. In fact, we only need to prove for any given $\mu \in M_{erg}(T,X)$, μ can be approximated by non-ergodic measures. More precisely, by the assumption of not uniquely ergodic, take another ergodic measure $\nu \neq \mu$. Let $\nu_n = \frac{1}{n}\nu + (1 - \frac{1}{n})\mu$. Then ν_n converges to μ in weak* topology. Remark that obviously ν_n are all not ergodic. □

Now we start to consider the number of periodic points in open sets.

Proposition 6.7. *Let $T : X \to X$ be a continuous map on a compact metric space X. If the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$), then for any nonempty open set $U \subseteq X$, there are infinite periodic points in U. In particular, every nonempty set U is not finite.*

Proof. By contradiction, there is some nonempty open set $U \subseteq X$ such that U contains at most finite periodic points. This implies that

$$\Delta := \{x \in Per(T) |\ Orb(x) \cap U \neq \emptyset\}$$

is an invariant set with at most finite elements and thus closed. By assumption of density of periodic points, the open set U contains some periodic point. Thus Δ is a nonempty finite, closed invariant set. Take $\mu := \frac{1}{\#\Delta}\sum_{x\in\Delta}\delta_x$. It is a finite convex sum of all periodic measures supported on Δ and thus $\mu(U) > 0$.

Since in present paper X is assumed infinite, then by assumption, the number of periodic points are infinite. Then we can take a periodic measure ν whose orbit is contained in $X \setminus U$. So $\nu(\Delta) = 0$. Let $\omega = \frac{1}{2}\mu + \frac{1}{2}\nu$. Then by assumption, there is a sequence of periodic measures ω_n such that ω_n converges to ω in weak* topology. By weak* topology for open sets (see Remarks (3) (iii) on Page 149 of [59]),

$$\liminf_{n\to\infty} \omega_n(U) \geq \omega(U) = \frac{1}{2}\mu(U) > 0.$$

Then we can take a large N_1 such that for any $n \geq N_1$, $\omega_n(U) > 0$. So the periodic orbit S_{ω_n} is contained in Δ and thus $\omega_n(\Delta) = 1$. On the other hand, using weak* topology for the open set $X \setminus \Delta$,

$$\liminf_{n\to\infty} \omega_n(X \setminus \Delta) \geq \omega(X \setminus \Delta) \geq \frac{1}{2}\nu(X \setminus \Delta) = \frac{1}{2} > 0.$$

Then we can take a large integer $N > N_1$ such that $\omega_N(X \setminus \Delta) > 0$. This contradicts $\omega_N(\Delta) = 1$. □

Given $\phi \in C^0(X)$ with $Int(L_\phi) \neq \emptyset$ and $\mathcal{R} \subseteq M(T, X)$, we say that $\mathcal{R}$ has *Property* (P) w.r.t. ϕ, if for every $\nu \in M(T, X)$, there is $\varrho \in \mathcal{R}$ such that $\int \phi d\varrho \neq \int \phi d\nu$, and if moreover, for any $n \in \mathbb{Z}^+$ and $a \in Int(L_\phi)$, there is $\mu_1, \mu_2, \cdots, \mu_n \in \mathcal{R}$ and $\nu_1, \nu_2, \cdots, \nu_n \in \mathcal{R}$ such that

$$\int \phi(x)d\mu_n < \cdots < \int \phi(x)d\mu_1 < a < \int \phi(x)d\nu_1 < \cdots < \int \phi(x)d\nu_n.$$

By (4.27) $I_\phi(T) \neq \emptyset \Rightarrow Int(L_\phi) \neq \emptyset$. By weak* topology and the continuity of $\phi \in C^0(X)$, it is easy to see that if $\mathcal{R}$ is a dense subset of $M(T, X)$, then

$$\{\int \phi(x)d\nu |\ \nu \in \mathcal{R}\}$$

is dense in L_ϕ. This implies that

Lemma 6.8. *Let $T : X \to X$ be a continuous map on a compact metric space X and let $\mathcal{R}$ be a dense subset of $M(T, X)$. Let $\phi \in C^0(X)$ with $I_\phi(T) \neq \emptyset$. Then $\mathcal{R}$ has Property (P) w.r.t. ϕ.*

In particular, by Lemma 6.8 we have a following consequence.

Lemma 6.9. *Let $T : X \to X$ be a continuous map on a compact metric space X. Let $\phi \in C^0(X)$ with $I_\phi(T) \neq \emptyset$. Then*

(1) if $M_p(T, X)$ is dense in $M(T, X)$, then $M_p(T, X)$ has Property (P) w.r.t. ϕ. Moreover,

$$Int(L_\phi) \subseteq \{\int \phi d\mu \,|\, \mu \in M(T,X),\ S_\mu \textit{ is composed of at most two periodic orbits}\};$$

(2) if there is some invariant measure with full support, then $M_{full}(T,X)$ has Property (P) w.r.t. ϕ. Moreover,

$$Int(L_\phi) \subseteq \{\int \phi d\mu \,|\, \mu \in M(T,X),\ S_\mu = X\};$$

(3) if $M_{erg}(T,X)$ is dense in $M(T,X)$, then $M_{erg}(T,X)$ has Property (P) w.r.t. ϕ;

(4) if there is some invariant measure with full support and $M_{erg}(T,X)$ is dense in $M(T,X)$, then $M_{erg}(T,X) \cap M_{full}(T,X)$ has Property (P) w.r.t. ϕ;

(5) if there is some invariant measure with full support and $M_p(T,X)$ is dense in $M(T,X)$, then $M_{erg}(T,X) \cap M_{full}(T,X)$ has Property (P) w.r.t. ϕ;

(6) $M(T,X) \setminus M_{erg}(T,X)$ has Property (P) w.r.t. ϕ. Moreover,

$$Int(L_\phi) \subseteq \{\int \phi d\mu \,|\, \mu \in M(T,X) \setminus M_{erg}(T,X)\}.$$

Proof. (1) Letting $\mathcal{R} = M_p(T,X)$, by Lemma 6.8 we only need to prove

$$Int(L_\phi) \subseteq \{\int \phi d\mu \,|\, \mu \in M(T,X),\ S_\mu \text{ is composed of at most two periodic orbits}\}.$$

Take two periodic measures μ_1, ν_1 such that $\int \phi(x)d\mu_1 < a < \int \phi(x)d\nu_1$. Then we can choose some suitable $\theta \in (0,1)$ such that $\int \phi d\omega = a$, where $\omega = \theta\mu_1 + (1-\theta)\nu_1$. Remark that S_ω is composed of at most two periodic orbits.

(2) By Lemma 6.3 $M_{full}(T,X)$ is dense in $M(T,X)$. So by Lemma 6.8 we only need to prove

$$Int(L_\phi) \subseteq \{\int \phi d\mu \,|\, \mu \in M(T,X),\ S_\mu = X\}.$$

Take two invariant measures μ_1, ν_1 with full support such that $\int \phi(x)d\mu_1 < a < \int \phi(x)d\nu_1$. Then we can choose some suitable $\theta \in (0,1)$ such that $\int \phi d\omega = a$, where $\omega = \theta\mu_1 + (1-\theta)\nu_1$. Remark that S_ω also has full support.

(3) It is obvious just letting $\mathcal{R} = M_{full}(T,X)$ in Lemma 6.8.

(4) It is well-known that $M_{erg}(T,X)$ is a non-empty G_δ subset of $M(T,X)$ (see Proposition 5.7 in [15]). Thus, by assumption and Lemma 6.3, $M_{erg}(T,X)$ and $M_{full}(T,X)$ both are dense G_δ subsets of $M(T,X)$. So $M_{erg}(T,X) \cap M_{full}(T,X)$ is also dense G_δ in $M(T,X)$. Then Lemma 6.8 implies that (4) is true.

(5) Note that $M_p(T,X) \subseteq M_{erg}(T,X)$. So by assumption, $M_{erg}(T,X)$ is dense in $M(T,X)$ and thus (4) implies (5).

(6) From (4.27) $I_\phi(T) \neq \emptyset$ implies that the system is naturally not uniquely ergodic. By Lemma 6.6, $M(T,X) \setminus M_{erg}(T,X)$ is dense in $M(T,X)$. Let $\mathcal{R} = M(T,X) \setminus M_{erg}(T,X)$ and then by Lemma 6.3, we only need to show

$$Int(L_\phi) \subseteq \{\int \phi d\mu \,|\, \mu \in M(T,X) \setminus M_{erg}(T,X)\}.$$

Take two non-ergodic measures μ_1, ν_1 such that $\int \phi(x)d\mu_1 < a < \int \phi(x)d\nu_1$. Then we can choose some suitable $\theta \in (0,1)$ such that $\int \phi d\omega = a$, where $\omega = \theta\mu_1 + (1-\theta)\nu_1$. Remark that ω is not ergodic, since it is known that every ergodic measure is extremal point in $M(T,X)$ (for example, see Proposition 5.6 in [15]). □

7. Proofs of main results

7.1. Proof of Theorem 1.3 (2)

In this section we divide the proof of Theorem 1.3 (2) into several propositions, for which the system is just assumed saturated. Recall from Section 2.4 that a system T is called saturated, if for any compact connected nonempty set $K \subseteq M(T,X)$,

$$h_{top}(T, G_K) = \inf\{h_\mu(T) \,|\, \mu \in K\},$$

where $G_K = \{x \in X |\, M_x(T) = K\}$. By Theorem 2.8 we know that g-almost product property + uniform separation gives that the system should be saturated. So Theorem 1.3 (2) can be deduced from the following several propositions. They are stated for possibly applicable to more general systems (for example, it is possible to apply partial results in some partially hyperbolic systems, see Section 7.3).

Proposition 7.1. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and there is some invariant measure μ with full support (i.e., $S_\mu = X$). Then for any $\phi \in C^0(X)$, either $I_\phi(T) = \emptyset$ or*

$$h_{top}(T, (W(T) \setminus QR(T)) \cap I_\phi(T)) = h_{top}(T, I_\phi(T)) = h_{top}(T).$$

Proof. Suppose $I_\phi(T) \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [59]), we can take $\mu \in M(T,X)$ such that

$$h_\mu(T) > h_{top}(T) - \varepsilon.$$

By Lemma 6.9 (2), we can take $\nu \in M(T,X)$ such that $S_\nu = X$ and $\int \phi d\nu \neq \int \phi d\mu$. Then we can choose two different numbers $0 < \theta_1 < \theta_2 < 1$ close to 1 enough such that for $\omega_i = \theta_i\mu + (1-\theta_i)\nu$, $i = 1,2$, one has

$$h_{\omega_i}(T) = \theta_i h_\mu(T) + (1-\theta_i)h_\nu(T) \geq \theta_i h_\mu(T) > h_{top}(T) - \varepsilon, \, i = 1,2. \tag{7.29}$$

Remark that $\theta_1 \neq \theta_2$ and $\int \phi d\mu \neq \int \phi d\nu$ imply

$$\int \phi d\omega_1 \neq \int \phi d\omega_2; \tag{7.30}$$

and $S_\nu = X$ implies

$$S_{\omega_i} = S_\mu \cup S_\nu = X,\ i = 1, 2. \tag{7.31}$$

Let

$$K = \{\tau\omega_1 + (1-\tau)\omega_2 |\, \tau \in [0,1]\}.$$

Then by (7.31) and (7.29) for any $m = \tau\omega_1 + (1-\tau)\omega_2 \in K$,

$$S_m = X,\ \ h_m(T) \geq \min\{h_{\omega_1}(T), h_{\omega_2}(T)\} > h_{top}(T) - \varepsilon.$$

Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_m(T) \,|\, m \in K\} \geq h_{top}(T) - \varepsilon,$$

where $G_K = \{x \in X |\, M_x(T) = K\}$.

To complete the proof, we only need to prove $G_K \subseteq I_\phi(T) \cap W(T)$. On one hand, for any $x \in G_K$, notice that $\omega_1, \omega_2 \in K = M_x(T)$ and thus by (4.26), (7.30) implies $x \in I_\phi(T)$. On the other hand, by (3.10) the equality

$$C_x = \overline{\cup_{\kappa \in M_x(T)} S_\kappa} = \overline{\cup_{\kappa \in K} S_\kappa} = X = S_m$$

hold for all $m \in K = M_x(T)$. By (3.11) $x \in X = C_x \subseteq \omega_T(x)$ and thus $x \in Rec(T)$. So from (3.12) one has $x \in W(T)$. □

Proposition 7.2. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated, has entropy-dense property and there is some invariant measure with full support. Then for any $\phi \in C^0(X)$, either $I_\phi(T) = \emptyset$ or*

$$h_{top}(T, I_\phi(T) \cap V(T) \setminus W(T)) = h_{top}(T, I_\phi(T)) = h_{top}(T).$$

Proof. Suppose $I_\phi(T) \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [59]), we can take $\mu_0 \in M(T, X)$ such that

$$h_{\mu_0}(T) > h_{top}(T) - \varepsilon.$$

By entropy-dense property we can choose $\mu \in M_{erg}(T, X)$ (close to μ_0) such that $S_\mu \subsetneqq X$ and

$$h_\mu(T) > h_{top}(T) - \varepsilon.$$

By Lemma 6.9 (2), $I_\phi(T) \neq \emptyset$ implies that we can take $\nu \in M(T,X)$ such that $S_\nu = X$ and $\int \phi d\nu \neq \int \phi d\mu$. Then we can take $\theta \in (0,1)$ close to 1 such that $h_\omega(T) \geq \theta h_\mu(T) > h_{top}(T) - \varepsilon$ where $\omega = \theta\mu + (1-\theta)\nu$. Remark that $S_\omega = S_\mu \cup S_\nu = X$ and

$$\int \phi d\omega \neq \int \phi d\mu. \tag{7.32}$$

Let

$$K = \{\tau\omega + (1-\tau)\mu |\, \tau \in [0,1]\}.$$

Then for any $m = \tau\omega + (1-\tau)\mu \in K \setminus \{\mu\}$, $S_m = X$ and for any $m = \tau\omega + (1-\tau)\mu \in K$

$$h_m(T) \geq \min\{h_\omega(T), h_\mu(T)\} > h_{top}(T) - \varepsilon.$$

Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_m(T) \,|\, m \in K\} > h_{top}(T) - \varepsilon,$$

where $G_K = \{x \in X |\, M_x(T) = K\}$.

To complete the proof, we only need to prove $G_K \subseteq I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists\, \omega \in M_x(T) \,\text{s.t.}\, S_\omega = C_x\}$. In fact, fix $x \in G_K$. On one hand, notice that $\omega, \mu \in K = M_x(T)$ and thus by (4.26), (7.32) implies $x \in I_\phi(T)$. On the other hand, by (3.10) one has the following equality

$$C_x = \overline{\cup_{m \in M_x(T)} S_m} = \overline{\cup_{m \in K} S_m} = X.$$

By (3.14) $x \in QW(T)$. Thus by (3.9) $x \in Rec(T)$. Notice that $\mu \in M_x(T)$ and $C_x = X \neq S_\mu$ so that from (3.12) $x \in X \setminus W(T)$. Recall that $\omega \in K = M_x(T)$ and $S_\omega = X$. So $x \in I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists\, \omega \in M_x(T) \,\text{s.t.}\, S_\omega = C_x\}$. □

Proposition 7.3. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and has entropy-dense property, the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Then for any $\phi \in C^0(X)$, either $I_\phi(T) = \emptyset$ or*

$$h_{top}(T, I_\phi(T) \cap QW(T) \setminus V(T)) = h_{top}(T, I_\phi(T)) = h_{top}(T).$$

Proof. Suppose $I_\phi(T) \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [59]), we can take $\mu_0 \in M(T,X)$ such that

$$h_{\mu_0}(T) > h_{top}(T) - \varepsilon.$$

By entropy-dense property, we can choose $\mu \in M_{erg}(T,X)$ (close to μ_0) such that $S_\mu \subsetneqq X$ and

$$h_\mu(T) > h_{top}(T) - \varepsilon.$$

Since X and $M(T,X)$ are compact metric spaces, by assumption we can take a countable dense subset $P_1 \subseteq Per(T)$ and a countable dense subset $M_1 \subseteq M_p(T,X)$. Then $\cup_{\mu \in M_1} S_\mu \cup P_1$ is still a countable dense subset of $Per(T)$, denoted by $\{x_i\}_{i=1}^\infty$. Moreover, $\bigcup_{x\in P_1} m_x \cup M_1$ is countable and dense in $M(T,X)$, where m_x denote the T-invariant measure supported on the periodic orbit of x. So $\overline{\{m_i\}_{i=1}^\infty} = M(T,X)$, where m_i denotes the T-invariant measure supported on the periodic orbit of x_i.

Take a strictly increasing sequence of $\{\theta_i |\ \theta_i \in (0,1)\}_{i=1}^\infty$ such that

$$\lim_{i\to+\infty} \theta_i = 1$$

and

$$h_{\nu_i}(T) \geq \theta_i h_\mu(T) > h_{top}(T) - \varepsilon$$

where $\nu_i = \theta_i\mu + (1-\theta_i)m_i$, $i = 1,2,3,\cdots$. Remark that for any i, $S_{\nu_i} = S_\mu \cup S_{m_i}$.

By Lemma 6.8, if let $\mathcal{R} = \{m_i\}_{i=1}^\infty$, then $I_\phi(T) \neq \emptyset$ implies that we can take one periodic measure m_{i_0} such that $\int \phi dm_{i_0} \neq \int \phi d\mu$. Without loss of generality, we can assume $m_1 = m_{i_0}$. Then

$$\int \phi d\nu_1 \neq \int \phi d\mu. \tag{7.33}$$

Now we consider

$$K = \bigcup_{i\geq 1} \{\tau\nu_i + (1-\tau)\nu_{i+1} |\, \tau \in [0,1]\} \ \bigcup\ \{\mu\}.$$

Remark that K is a nonempty connected compact subset of $M(T,X)$ because $\nu_i \to \mu$ in weak* topology. Since T is saturated, then

$$h_{top}(T,G_K) = \inf\{h_m(T) \,|\, m \in K\} = \min\{\inf_{i\geq 1}\{h_{\nu_i}(T)\}, h_\mu(T)\} \geq h_{top}(T) - \varepsilon,$$

where $G_K = \{x \in X |\, M_x(T) = K\}$.

To complete the proof, we only need to prove $G_K \subseteq I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \forall\, m \in M_x(T) \text{ s.t. } S_m \neq C_x\}$. Fix $x \in G_K$. Recall that $\nu_1, \mu \in K = M_x(T)$ and thus by (4.26), (7.33) implies $x \in I_\phi(T)$. Clearly by (3.10)

$$C_x = \overline{\cup_{m\in M_x(T)} S_m} = \overline{\cup_{m\in K} S_m} \supseteq \overline{\cup_{i\geq 1} S_{\nu_i}} = \overline{\cup_{i\geq 1}(S_{m_i} \cup S_\mu)} \supseteq \overline{\cup_{i\geq 1}\{x_i\}} = X.$$

So $C_x = X$. By (3.14) $x \in QW(T)$. By (3.9) $x \in Rec(T)$. Notice that $C_x = X \neq S_\mu$ and $\mu \in M_x(T)$ so that from (3.12) $x \in X \setminus W(T)$. For any $m \in M_x(T) = K$, by definition of

K there is some i such that $S_m \subseteq S_\mu \cup S_{m_i} \cup S_{m_{i+1}}$. Note that $X \setminus S_\mu$ is a nonempty open set so that by Proposition 6.7 $X \setminus S_\mu$ is not a finite set. But $S_{m_i} \cup S_{m_{i+1}}$ is just composed of two periodic orbits and thus is a finite set. Hence $S_m \subsetneqq X = C_x$. I.e., $S_m \neq C_x$. □

Proposition 7.4. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and has entropy-dense property, the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Then for any $\phi \in C^0(X)$, either $I_\phi(T) = \emptyset$ or*

$$h_{top}(T, I_\phi(T) \cap I(T) \setminus QW(T)) = h_{top}(T, I_\phi(T)) = h_{top}(T).$$

Proof. Suppose $I_\phi(T) \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [59]), we can take $\mu_0 \in M(T,X)$ such that

$$h_{\mu_0}(T) > h_{top}(T) - \varepsilon.$$

By entropy-dense property, we can choose $\mu \in M_{erg}(T,X)$ (close to μ_0) such that $S_\mu \subsetneqq X$ and

$$h_\mu(T) > h_{top}(T) - \varepsilon.$$

Since $X \setminus S_\mu$ is nonempty and open, by density of periodic points

$$B := \{\nu \in M_p(T,X) |\ S_\nu \setminus S_\mu \neq \emptyset\} \neq \emptyset.$$

Now we will construct an invariant measure κ such that the set $S_\kappa \setminus S_\mu$ is composed of one periodic orbit and

$$\int \phi d\kappa \neq \int \phi d\mu.$$

More precisely, if there is a periodic measure $\nu \in B$ such that $\int \phi d\nu \neq \int \phi d\mu$, then take $\kappa = \nu$ and remark that $S_\kappa \setminus S_\mu = S_\nu$. Otherwise, for any $\nu \in B$, $\int \phi d\nu = \int \phi d\mu$. Take such a measure ν. By Lemma 6.9 (1), $\overline{M_p(T,X)} = M(T,X)$ and $I_\phi(T) \neq \emptyset$ imply

$$Y := \{\tau |\ \int \phi d\tau \neq \int \phi d\mu,\ \tau \in M_p(T,X)\} \neq \emptyset.$$

Then we can take $\nu' \in Y$ such that $\int \phi d\nu' \neq \int \phi d\mu$. Remark that in this case $Y \cap B = \emptyset$ so that $S_{\nu'} \setminus S_\mu = \emptyset$. Then $S_{\nu'} \subseteq S_\mu$. So if we take $\kappa = \frac{1}{2}(\nu + \nu')$, then $\int \phi d\kappa \neq \int \phi d\mu$. Note that $S_\kappa = S_\nu \cup S_{\nu'}$ and $S_\kappa \setminus S_\mu = S_\nu$.

Take $\theta \in (0,1)$ close to 1 such that $\omega = \theta\mu + (1-\theta)\kappa$ satisfies $h_\omega(T) \geq \theta h_\mu(T) > h_{top}(T) - \varepsilon$. Then ω also satisfies that $\int \phi d\omega \neq \int \phi d\mu$. Remark that $S_\omega = S_\mu \cup S_\nu \cup S_{\nu'} = S_\mu \sqcup S_\nu$. Let $K = \{\tau\mu + (1-\tau)\omega |\ \tau \in [0,1]\}$. Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_m(T) \,|\, m \in K\} \geq \min\{h_\mu(T), h_\omega(T)\} > h_{top}(T) - \varepsilon,$$

where $G_K = \{x \in X | \, M_x(T) = K\}$.

To complete the proof, we only need to prove

$$G_K \subseteq I_\phi(T) \setminus QW(T).$$

For any $x \in G_K$, recall that $\omega, \mu \in K = M_x(T)$ and $\int \phi d\omega \neq \int \phi d\mu$. Thus by (4.26) $x \in I_\phi(T)$. If $x \in QW(T)$, then by (3.9) $x \in Rec(T)$ so that by (3.13) $x \in C_x = \omega_T(x)$. Then by (3.10)

$$x \in \omega_T(x) = C_x = \overline{\cup_{m \in M_x(T)} S_m} = \overline{\cup_{m \in K} S_m} = S_\mu \sqcup S_\nu,$$

which implies $x \in S_\mu$ or S_ν. So by invariance of S_μ and S_ν, one has $Orb(x) \subseteq S_\mu$ or $Orb(x) \subseteq S_\nu$ and thus by compactness of S_μ and S_ν, we have $\omega_T(x) \subseteq S_\mu \subsetneqq S_\mu \sqcup S_\nu = C_x = \omega_T(x)$ or $\omega_T(x) \subseteq S_\nu \subsetneqq S_\mu \sqcup S_\nu = C_x = \omega_T(x)$. That $\omega_T(x) \subsetneqq \omega_T(x)$ is a contradiction. Hence, x is not in $QW(T)$. □

Proof of Theorem 1.3 (2). By Lemma 6.2, there is some invariant measure with full support. By Lemma 2.8, T is saturated. By Lemma 2.10, T has entropy-dense property. So Theorem 1.3 (2) can be deduced from Propositions 7.1–7.4. □

7.2. Proofs of Theorem 1.3 (1) and Theorem 1.4

We divide Theorem 1.4 into several propositions and then use it to prove Theorem 1.3 (1). Here we will state every proposition for possible more general systems.

Proposition 7.5. *Let T be a continuous map of a compact metric space X. Suppose that T is single-saturated. Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,*

(1) $\{QR_{erg}(T), QR(T)\}$ has full entropy gaps with respect to $R_{\phi,a}(T)$.

(1′) $\{R(T), QR(T)\}$ has full entropy gaps with respect to $R_{\phi,a}(T)$.

(2) If there is some invariant measure such that its support is not minimal, then $\{A(T) \cup QR_{erg}(T), QR(T)\}$ has full entropy gaps with respect to $R_{\phi,a}(T)$.

(2′) If there is some invariant measure such that its support is not minimal, then $\{A(T) \cup R(T), QR(T)\}$ has full entropy gaps with respect to $R_{\phi,a}(T)$.

Proof. Since $R(T) \subseteq QR_{erg}(T)$, (1) implies (1′) and (2) implies (2′) so that we only need to prove (1) and (2). Remark that from (4.27) $I_\phi(T) \neq \emptyset$ implies that the system is naturally not uniquely ergodic.

Fix $a \in Int(L_\phi)$ and let $t = \sup\{h_\rho(T) | \; \rho \in M(T, X)$ and $\int \phi d\rho = a\}$. For any $\epsilon > 0$, we can take $\rho \in M(T, X)$ such that $\int \phi d\rho = a$, $h_\rho(f) > t - \epsilon$.

(1) By Proposition 4.7, we only need to show that

$$h_{top}(T, R_{\phi,a}(T) \cap QR(T) \setminus QR_{erg}(T)) \geq t.$$

If ρ is non-ergodic, then take $m := \rho$. Otherwise, ρ is ergodic, by Lemma 6.9 (6), take a non-ergodic measure σ such that $\int \phi d\sigma = a$. Take $\theta' \in (0,1)$ close to 1 such that $m := \theta'\rho + (1-\theta')\sigma$ satisfies $h_m(f) \geq \theta' h_\rho(f) > t - \epsilon$. Remark that $\int \phi dm = a$ and m is not ergodic, since σ is not ergodic. Since T is single-saturated, then $h(T, G_m) = h_m(f) > t - \epsilon$.

We only need to prove that $G_m \subseteq R_{\phi,a}(T) \cap QR(T) \setminus QR_{erg}(T)$. By construction of m, it is easy to see that $G_m \subseteq QR(T) \setminus QR_{erg}(T)$. For any $x \in G_m$, $M_x(T) = \{m\}$ so that $M_x(T) \subseteq \{\vartheta | \int \phi d\vartheta = a\}$. By (4.24) $x \in R_{\phi,a}(T)$. Now we complete the proof of (1).

(2) Take same m as in (1). If S_m is not minimal, then we claim that (2) can be deduced from $G_m \subseteq R_{\phi,a}(T) \cap QR(T) \setminus (QR_{erg}(T) \cup A(T))$. Otherwise, there is $x \in G_m \cap A(T)$. Then $\omega_T(x)$ is a minimal compact set and $x \in \omega_T(x)$. By (3.10) and (3.11) $S_m = C_x \subseteq \omega_T(x)$, contradicting that S_m is not minimal.

Now we face the case that S_m is minimal. By assumption there is some invariant measure μ with non-minimal support. Then $m \neq \mu$. Let $b := \int \phi d\mu$. If $b = a$, take $\omega := \mu$. If $b \neq a$, without loss of generality, we can assume $b > a$. By definition of L_ϕ and connectedness of $M(T, X)$, we can choose some ν such that $c := \int \phi d\nu < a$. In this case we can take suitable $\theta \in (0,1)$ such that $\omega = \theta\mu + (1-\theta)\nu$ satisfies that $\int \phi d\omega = a$. So in any case, ω satisfies that $\int \phi d\omega = a$ and $S_\omega \supseteq S_\mu$ is not minimal. Take $\theta' \in (0,1)$ close to 1 such that $\tau := \theta' m + (1-\theta')\omega$ satisfies $h_\tau(f) \geq \theta' h_m(f) > t - \epsilon$. Remark that $\int \phi d\tau = a$ and $S_\tau = S_\omega \cup S_m \supseteq S_\omega$ is not minimal, and S_m is minimal implies that $m \neq \omega$ and so τ is non-ergodic. Since T is single-saturated, then $h(T, G_\tau) = h_\tau(f) > t - \epsilon$.

We need to prove $G_\tau \subseteq R_{\phi,a}(T) \cap QR(T) \setminus (QR_{erg}(T) \cup A(T))$. Similar as the proof of (1), it is easy to check that $G_\tau \subseteq R_{\phi,a}(T) \cap QR(T) \setminus QR_{erg}(T)$. Now we start to prove $G_\tau \cap A(T) = \emptyset$. Otherwise, there is $x \in G_\tau \cap A(T)$. Then $\omega_T(x)$ is a minimal compact set and $x \in \omega_T(x)$. By (3.10) and (3.11) $S_\tau = C_x \subseteq \omega_T(x)$, contradicting that S_τ is not minimal. Now we complete the proof of (2). □

Proposition 7.6. *Let T be a continuous map of a compact metric space X. Suppose that T is single-saturated and there is some invariant measure with full support. Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,*

(1) $\{QR_{erg}(T), QR_d(T)\}$ has full entropy gaps with respect to $W(T) \cap R_{\phi,a}(T)$.

(1′) $\{R(T), QR_d(T)\}$ has full entropy gaps with respect to $W(T) \cap R_{\phi,a}(T)$.

(2) If further the system T is not minimal, then $\{A(T) \cup QR_{erg}(T), QR_d(T)\}$ has full entropy gaps with respect to $W(T) \cap R_{\phi,a}(T)$.

(2′) $\{A(T) \cup R(T), QR_d(T)\}$ has full entropy gaps with respect to $W(T) \cap R_{\phi,a}(T)$.

Proof. Since $R(T) \subseteq QR_{erg}(T)$, (1) implies (1′) and (2) implies (2′) so that we only need to prove (1) and (2).

Remark that from (4.27) $I_\phi(T) \neq \emptyset$ implies that the system is naturally not uniquely ergodic. By Lemma 6.9 (2) there is some invariant measure ξ with full support such that $\int \phi d\xi = a$. Fix $a \in Int(L_\phi)$ and let $t = \sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}$. Fix $\epsilon > 0$,

(1) Take m same as in the proof of Proposition 7.5 (1) such that m is not ergodic, $\int \phi dm = a$, $h_m(f) > t - \epsilon$. Take $\alpha \in (0,1)$ close to 1 enough such that the measure $m' = \alpha m + (1-\alpha)\xi$ such that $h_{m'}(f) \geq \alpha h_m(f) > t - \epsilon$. Remark that m' is not ergodic, has full support and $\int \phi dm' = a$. Since T is single-saturated, then $h(T, G_{m'}) = h_{m'}(f) > t - \epsilon$.

We only need to prove that $G_{m'} \subseteq W(T) \cap R_{\phi,a}(T) \cap QR_d(T) \setminus QR_{erg}(T)$. By construction of m', it is easy to see that $G_{m'} \subseteq QR(T) \setminus QR_{erg}(T)$. Now we start to show that $G_{m'} \subseteq QR_d(T)$. In fact, $S_{m'} = X$ implies $G_{m'} = G_{m'} \cap S_{m'}$ and thus by (4.20) $G_{m'} \subseteq QR_d(T)$. For any $x \in G_{m'}$, $M_x(T) = \{m'\}$ so that $M_x(T) \subseteq \{\rho|\ \int \phi d\rho = a\}$. Thus by (4.24) $x \in R_{\phi,a}(T)$. Moreover, for $x \in G_{m'}$, $M_x(T) = \{m'\}$ implies that $C_x = S_{m'} = X \ni x$. By (3.11) $x \in Rec(T)$ and then by (3.12) $x \in W(T)$. Now we complete the proof of (1).

(2) By non-minimal assumption the measure m' in (1) satisfies that $S_{m'} = X$ is not minimal. Similar as the proof of Proposition 7.5 (2) it is easy to check that $G_{m'} \cap A(T) = \emptyset$. Thus by (1) $G_{m'} \subseteq W(T) \cap R_{\phi,a}(T) \cap QR_d(T) \setminus (A(T) \cup QR_{erg}(T))$. Then we can follow the proof of (1) to complete the proof of (2). □

Proposition 7.7. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated, there is some invariant measure with full support and $M_{erg}(T,X)$ is dense in $M(T,X)$. Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,*

$$h_{top}(T, R_{\phi,a}(T) \cap W(T) \setminus QR(T)) = h_{top}(T, R_{\phi,a}(T)).$$

Proof. Fix $a \in Int(L_\phi)$ and let $t = \sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}$. By Proposition 4.7, we only need to show that

$$h_{top}(T, R_{\phi,a}(T) \cap W(T) \setminus QR(T)) \geq t.$$

Fix $\varepsilon > 0$. We need to construct two measures as follows, which are also useful to prove other propositions.

Lemma 7.8. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated, there is some invariant measure with full support and $M_{erg}(T,X)$ is dense in $M(T,X)$. Then there are two different measures $\omega, \omega' \in M(T,X)$ $(\omega \neq \omega')$ such that*

$$\min\{h_\omega(T),\ h_{\omega'}(T)\} > t - \varepsilon$$

and $\int \phi d\omega = \int \phi d\omega' = a$, $S_\omega = S_{\omega'} = X$.

Proof. By Lemma 6.9 (4) we can take three different ergodic measures of μ_i $(i = 0, 1, 2)$ with support X such that

$$\int \phi d\mu_0 < a < \int \phi d\mu_1 < \int \phi d\mu_2.$$

Then we can choose suitable $\theta_i \in (0, 1)$ $(i = 1, 2)$ such that $\nu_i = \theta_i\mu_0 + (1 - \theta_i)\mu_i$ satisfy

$$\int \phi d\nu_i = a, i = 1, 2.$$

Remark that by ergodicity of μ_i, $\nu_1 \neq \nu_2$ and $S_{\nu_i} = S_{\mu_0} \cup S_{\mu_i} = X$, $i = 1, 2$.

By definition of t, we can take $\mu \in M(T, X)$ such that $\int \phi d\mu = a$ and $h_\mu(T) > t - \varepsilon$. Then we can choose $0 < \theta < 1$ close to 1 such that $\omega = \theta\mu + (1 - \theta)\nu_1$, $\omega' = \theta\mu + (1 - \theta)\nu_2$ satisfy

$$h_\omega(T) = \theta h_\mu(T) + (1 - \theta)h_{\nu_1}(T) \geq \theta h_\mu(T) > t - \varepsilon,$$
$$h_{\omega'}(T) = \theta h_\mu(T) + (1 - \theta)h_{\nu_2}(T) \geq \theta h_\mu(T) > t - \varepsilon.$$

Remark that $\int \phi d\omega = \int \phi d\omega' = a$, $S_\omega = S_{\omega'} = X$ and $\nu_1 \neq \nu_2$ implies $\omega \neq \omega'$. □

Now we continue the proof of Proposition 7.7. Let $K = \{\tau\omega + (1 - \tau)\omega' | \tau \in [0, 1]\}$, then for any $m = \tau\omega + (1 - \tau)\omega' \in K$,

$$S_m = X, \;\; h_m(T) \geq \min\{h_\omega(T), h_{\omega'}(T)\} > t - \varepsilon, \int \phi dm = a.$$

Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_m(T) \,|\, m \in K\} > t - \varepsilon,$$

where $G_K = \{x \in X | \, M_x(T) = K\}$. We only need to prove $G_K \subseteq R_{\phi,a}(T) \cap W(T) \setminus QR(T)$.

Fix $x \in G_K$. Then $M_x(T) = K$ so that for any $m \in M_x(T)$, $\int \phi dm = a$. Thus $M_x(T) \subseteq \{\rho | \; \int \phi d\rho = a\}$ and so by (4.24) $x \in R_{\phi,a}(T)$. Notice that for any $m \in M_x(T)$, by (3.10)

$$C_x = \overline{\cup_{m \in M_x(T)} S_m} = \overline{\cup_{m \in K} S_m} = X = S_m.$$

By (3.11) $x \in X = C_x \subseteq \omega_T(x)$ and thus $x \in Rec(T)$. So from (3.12) one has $x \in W(T)$. Since $M_x(T) = K$ is not a singleton, by (4.23) $x \in I(T)$. □

Proposition 7.9. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and has entropy-dense property, the periodic points are dense in X (i.e.,*

$\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,

$$h_{top}(T, R_{\phi,a}(T) \cap V(T) \setminus W(T)) = h_{top}(T, R_{\phi,a}(T)).$$

Proof. Fix $a \in Int(L_\phi(T))$ and let $t = \sup\{h_\rho(T) | \ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}$. By Proposition 4.7, we only need to show that

$$h_{top}(T, R_{\phi,a}(T) \cap V(T) \setminus W(T)) \geq t.$$

Fix $\varepsilon \in (0,t)$. We need to construct a measure as follows, which is also useful to prove other propositions.

Lemma 7.10. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and has entropy-dense property, the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Then there is a measure $\omega \in M(T,X)$ such that*

$$h_\omega(T) > t - \varepsilon \ \text{ and } \ \int \phi d\omega = a, \ S_\omega \subsetneqq X.$$

Proof. Take a $\nu \in M(T,X)$ such that $h_\nu(f) > t - \frac{\varepsilon}{3}$ and $\int \phi d\nu = a$. By Lemma 6.9 (1), take two periodic measures ν_i $(i = 1,2)$ such that $b_1 := \int \phi d\nu_1 > a > \int \phi d\nu_2 =: b_2$. Let $\delta > 0$ small enough such that

$$\min\{\frac{b_1 - a}{b_1 - a + \delta}, \frac{a - b_2}{a - b_2 + \delta}\} > \frac{t - \varepsilon}{t - \frac{2\varepsilon}{3}}.$$

Then by entropy-dense property, we can take one ergodic measure μ close to ν enough (in weak* topology) such that $S_\mu \subsetneqq X$ and

$$|\int \phi d\mu - a| = |\int \phi d\mu - \int \phi d\nu| < \delta, \ h_\mu(f) > t - \frac{2\varepsilon}{3}.$$

If $\int \phi d\mu = a$, then take $\omega = \mu$. Otherwise, $\int \phi d\mu \neq a$. Without loss of generality, we assume $\int \phi d\mu < a$. Take

$$\omega = \frac{b_1 - a}{b_1 - \int \phi d\mu}\mu + (1 - \frac{b_1 - a}{b_1 - \int \phi d\mu})\nu_1.$$

Then $\int \phi d\omega = a$, $h_\omega(f) \geq \frac{b_1 - a}{b_1 - \int \phi d\mu} h_\mu(f) > \frac{b_1 - a}{b_1 - a + \delta} h_\mu(f) > t - \varepsilon$. Recall that S_{ν_1} is a finite closed set but $X \setminus S_\mu$ is nonempty, open and thus by Proposition 6.7 it is not a finite set. So $S_\omega = S_\mu \cup S_{\nu_1} \neq X$. □

Now we continue the proof of Proposition 7.9. By density of periodic points and Lemma 6.2, there is some invariant measure with full support and by density of periodic measures, $M_{erg}(T,X)$ is dense in $M(T,X)$. Then one can construct ω' same as in Lemma 7.8 such that

$$h_{\omega'}(T) > t-\varepsilon, \int \phi d\omega' = a \text{ and } S_{\omega'} = X.$$

Clearly $S_\omega \neq S_{\omega'}$ and thus $\omega \neq \omega'$.

Let $K = \{\tau\omega + (1-\tau)\omega' | \tau \in [0,1]\}$, then for any $m = \tau\omega + (1-\tau)\omega' \in K \setminus \{\omega\}$, $S_m = X$ and for any $m = \tau\omega + (1-\tau)\omega' \in K$,

$$h_m(T) \geq \min\{h_\omega(T), h_{\omega'}(T)\} > t-\varepsilon, \int \phi dm = a.$$

Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_m(T) \,|\, m \in K\} > t-\varepsilon,$$

where $G_K = \{x \in X |\, M_x(T) = K\}$. We only need to prove

$$G_K \subseteq R_{\phi,a}(T) \cap V(T) \setminus W(T).$$

Fix $x \in G_K$. Then $M_x(T) = K$ so that for any $m \in M_x(T)$, $\int \phi dm = a$. Thus

$$M_x(T) \subseteq \{\rho| \int \phi d\rho = a\}$$

and so by (4.24) $x \in R_{\phi,a}(T)$. Notice that by (3.10)

$$C_x = \overline{\cup_{m\in M_x(T)} S_m} = \overline{\cup_{m\in K} S_m} = X.$$

So by (3.14) $C_x = X$ implies $x \in QW(T)$. By (3.9) $x \in Rec(T)$. Then from (3.12) $C_x = X \neq S_\omega$ and $\omega \in M_x(T)$ imply $x \in X \setminus W(T)$. Recall $S_{\omega'} = X$ and $\omega' \in K = M_x(T)$. So $x \in R_{\phi,a}(T) \cap V(T) \setminus W(T)$. □

Proposition 7.11. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and has entropy-dense property, the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,*

$$h_{top}(T, R_{\phi,a}(T) \cap QW(T) \setminus V(T)) = h_{top}(T, R_{\phi,a}(T)).$$

Proof. Fix $a \in L_\phi(T)$ and let $t = \sup\{h_\rho(T)|\ \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}$. By Proposition 4.7, we only need to show that

$$h_{top}(T, R_{\phi,a}(T) \cap QW(T) \setminus V(T)) \geq t.$$

Fix $\varepsilon > 0$. Firstly, by assumption we can take same ω as in Lemma 7.10 such that $\int \phi d\omega = a$, $h_\omega(f) > t - \varepsilon$ and $S_\omega \neq X$.

By density of periodic points and density of periodic measures, we can choose $\{x_i\}_{n=1}^\infty$ and $\{m_i\}_{n=1}^\infty$ same as in the proof Proposition 7.3. They are composed of periodic points and periodic measures and $\overline{\{x_i\}_{i=1}^\infty} = X$, $\overline{\{m_i\}_{i=1}^\infty} = M(T,X)$, $\cup_{i\geq 1} S_{m_i} = \{x_i\}_{i=1}^\infty$.

Let

$$K_1 := \{m| \int \phi dm > a,\, m \in \{m_i\}_{n=1}^\infty\},$$

$$K_2 := \{m| \int \phi dm < a,\, m \in \{m_i\}_{n=1}^\infty\}$$

and

$$K_3 := \{m| \int \phi dm = a,\, m \in \{m_i\}_{n=1}^\infty\}.$$

By Lemma 6.9 if let $\mathcal{R} = \{m_i\}_{n=1}^\infty$, then it is easy to see that K_1 and K_2 are countable. Remark that K_3 may be empty, finite or countable. Without loss of generality, we can assume $K_i = \{m_j^{(i)}\}_{j=1}^\infty$, $i = 1,2,3$. Then we can choose suitable $\theta_{j,k} \in (0,1)$ such that $m_{j,k} = \theta_{j,k} m_j^{(1)} + (1-\theta_{j,k}) m_k^{(2)}$ satisfies $\int \phi dm_{j,k} = a$. For any $n \geq 1$, let

$$l_n = \frac{\sum_{j+k=n} m_{j,k} + m_n^{(3)}}{n},$$

then $\int \phi dl_n = a$. Remark that every S_{l_n} is composed of finite periodic orbits and $\bigcup_{n\geq 1} S_{l_n} = \{x_i\}_{n=1}^\infty$ is dense in X.

Take an increasing sequence of $\{\theta_i|\ \theta_i \in (0,1)\}_{i=1}^\infty$ convergent to 1 such that $h_{\omega_i}(T) > t - \varepsilon$ where $\omega_i = \theta_i \omega + (1-\theta_i) l_i$. Remark that $S_{\omega_i} = S_\omega \cup S_{l_i}$. In particular, for all i, $\int \phi d\omega_i = \int \phi d\omega = a$.

Now we consider

$$K = \{\omega\} \cup \bigcup_{i\geq 1} \{\tau\omega_i + (1-\tau)\omega_{i+1} | \tau \in [0,1]\}.$$

Then K is nonempty connected compact subset of $M(T,X)$ because $\omega_i \to \omega$ in weak* topology. Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_\nu(T)\,|\,\nu \in K\} = \min\{\inf_{i\geq 1}\{h_{\omega_i}(T)\}, h_\omega(T)\} \geq t - \varepsilon,$$

where $G_K = \{x \in X | \, M_x(T) = K\}$. We only need to prove

$$G_K \subseteq \{x \in R_{\phi,a}(T) \cap QW(T) \setminus W(T) | \forall\, m \in M_x(T) \,\text{s.t.}\, S_m \neq C_x\}.$$

Fix $x \in G_K$. Note that for any $m \in M_x(T) = K$, $\int \phi dm = a$. Thus $M_x(T) \subseteq \{\rho| \int \phi d\rho = a\}$ and so by (4.25) $x \in R_{\phi,a}(T)$. Notice that by (3.10)

$$C_x = \overline{\cup_{m \in M_x(T)} S_m} = \overline{\cup_{m \in K} S_m} \supseteq \overline{\cup_{i \geq 1} S_{\omega_i}} = \overline{\cup_{i \geq 1} (S_{l_i} \cup S_\omega)} \supseteq \overline{\cup_{i \geq 1} \{x_i\}} = X.$$

So $X = C_x$. From (3.14) one has $x \in QW(T)$. By (3.9) $x \in Rec(T)$. Notice that from (3.12) $C_x = X \neq S_\omega$ and $\omega \in M_x(T)$ imply $x \in X \setminus W(T)$. For any $m \in M_x(T) = K$, there is some i such that $S_m \subseteq S_\omega \cup S_{l_i} \cup S_{l_{i+1}} \subsetneqq X = C_x$. This is because $S_{l_i} \cup S_{l_{i+1}}$ is a finite set but $X \setminus S_\omega$ is nonempty, open and thus by Proposition 6.7 it is an infinite set. Hence $S_m \neq C_x$. □

Proposition 7.12. *Let T be a continuous map of a compact metric space X. Suppose that T is saturated and has entropy-dense property, the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Let $\phi : X \to \mathbb{R}$ be a continuous function with $I_\phi(T) \neq \emptyset$. Then for any real number $a \in Int(L_\phi)$,*

$$h_{top}(T, R_{\phi,a}(T) \cap I(T) \setminus QW(T)) = h_{top}(T, R_{\phi,a}(T)).$$

Proof. Fix $a \in L_\phi(T)$ and let $t = \sup\{h_\rho(T)| \, \rho \in M(T,X) \text{ and } \int \phi d\rho = a\}$. By Proposition 4.7, we only need to show that

$$h_{top}(T, R_{\phi,a}(T) \cap I(T) \setminus QW(T)) \geq t.$$

Fix $\varepsilon > 0$. Firstly, by assumption we can take same ω as in Lemma 7.10 such that $\int \phi d\omega = a$, $h_\omega(f) > t - \varepsilon$ and $S_\omega \neq X$.

Let $B := \{m \in M_p(T,X)| \, S_m \setminus S_\omega \neq \emptyset\}$. Since $X \setminus S_\omega$ is nonempty, open and invariant, by density of periodic points, $B \neq \emptyset$. If there is $m \in B$ such that $\int \phi dm = a$, take $\mu = m$. Otherwise, for any $m \in B$, $\int \phi dm \neq a$. Take one $\mu_1 \in B$. Then $\int \phi d\mu_1 \neq a$. Without loss of generality, we assume $\int \phi d\mu_1 < a$. By Lemma 6.9 (1) we can take a periodic measure μ_2 such that $\int \phi d\mu_2 > a$. Then we can choose suitable $\theta \in (0,1)$ such that $\mu = \theta \mu_1 + (1-\theta)\mu_2$ satisfies $\int \phi d\mu = a$. Remark that $S_\mu \setminus S_\omega$ is composed of one periodic orbit or two periodic orbits containing S_{μ_1}.

Take $\theta' \in (0,1)$ close to 1 such that $\omega' = \theta' \omega + (1-\theta')\mu$ satisfies $h_{\omega'}(T) > t - \varepsilon$. Remark that $\int \phi d\omega' = a$ and $S_{\omega'} \setminus S_\omega = S_\mu \setminus S_\omega$ is composed of one periodic orbit or two periodic orbits. So $S_{\omega'} \setminus S_\omega$ is nonempty, finite, invariant and compact and thus $\omega \neq \omega'$.

Let $K = \{\tau\omega + (1-\tau)\omega' | \, \tau \in [0,1]\}$, then for any $m \in K$,

$$h_m(T) \geq \min\{h_\omega(T), h_{\omega'}(T)\} > t - \varepsilon, \; \int \phi dm = a.$$

Since T is saturated, then

$$h_{top}(T, G_K) = \inf\{h_m(T) \,|\, m \in K\} > t - \varepsilon,$$

where $G_K = \{x \in X | \, M_x(T) = K\}$. We only need to prove $G_K \subseteq R_{\phi,a}(T) \cap I(T) \setminus QW(T)$.

Fix $x \in G_K$. Since K is not singleton, by (4.23) $x \in I(T)$. Note that for any $m \in M_x(T) = K$, $\int \phi dm = a$. Thus $M_x(T) \subseteq \{\rho | \int \phi d\rho = a\}$ and so by (4.25) $x \in R_{\phi,a}(T)$. If $x \in QW(T)$, by (3.9) $x \in Rec(T)$. By (3.13) $x \in C_x = \omega_T(x)$. Then by (3.10)

$$x \in \omega_T(x) = C_x = \overline{\cup_{m \in M_x(T)} S_m} = \overline{\cup_{m \in K} S_m} = S_\omega \sqcup (S_{\omega'} \setminus S_\omega).$$

So $x \in S_\omega$ or $S_{\omega'} \setminus S_\omega$. Recall that $S_\omega, S_{\omega'} \setminus S_\omega$ are both compact and invariant. One has $Orb(x) \subseteq S_\omega$ or $Orb(x) \subseteq S_{\omega'} \setminus S_\omega$ and thus $\omega_T(x) \subseteq S_\omega \subsetneqq S_\omega \sqcup (S_{\omega'} \setminus S_\omega) = C_x = \omega_T(x)$ or $\omega_T(x) \subseteq (S_{\omega'} \setminus S_\omega) \subsetneqq S_\omega \sqcup (S_{\omega'} \setminus S_\omega) = C_x = \omega_T(x)$. That $\omega_T(x) \subsetneqq \omega_T(x)$ is a contradiction. Hence, x is not in $QW(T)$. □

Proof of Theorem 1.4. Since every periodic measure is ergodic, by density of periodic measures, ergodic measures are dense in the space of invariant measures. Since X is assumed infinite in this paper, by density of periodic points, X is not a minimal set (in other words, T is not minimal). By density of periodic measures, ergodic measures are dense in the space of invariant measures. By Lemma 6.2, there is some invariant measure such that it has full support and in particular, the support is not minimal. By Lemma 2.8, T is saturated. By Lemma 2.10, T has entropy-dense property (which also implies density of ergodic measures). So Theorem 1.4 can be deduced from Propositions 7.6, 7.7, 7.9, 7.11, 7.12. □

Remark 7.13. In Theorem 1.4, we require a to satisfy

$$\inf\{\int \phi d\mu | \, \mu \in M(T,X)\} < a < \sup\{\int \phi d\mu | \, \mu \in M(T,X)\}.$$

For full shifts of finite type, let us give an example why we do not choose a from extreme points. More precisely, firstly recall a result from [23,24] that for any full shift on finite symbols, there exist uniquely ergodic minimal subshifts with any given entropy. Then we can take an ergodic measure μ with positive entropy such that S_μ is minimal, $S_\mu \neq X$ and μ is the unique invariant measure supported on S_μ. By density of periodic points, take a periodic measure ν such that $S_\nu \subseteq X \setminus S_\mu$. Since S_μ and S_ν are two disjoint closed sets, we can take a continuous function ϕ which restricts on the set S_μ with value 0 and on the set S_ν with value 1 respectively, and the values of other points are in the open interval of $(0, 1)$. Take $a = 0$, let us show that $R_{\phi,a}(T)$ is nonempty and has positive entropy but $R_{\phi,a}(T) \cap I(T) = \emptyset$. More precisely, on one hand, by (4.24), $R_{\phi,a}(T) \supseteq G_\mu$. By ergodicity of μ and (4.17), $h_{top}(T, G_\mu) = h_\mu(f) > 0$. So $h_{top}(T, R_{\phi,a}(T)) \geq h_{top}(T, G_\mu) > 0$. On the other hand, observe that μ is the unique invariant measure with integral by ϕ equal to a. So by weak* topology, $R_{\phi,a}(T) = G_\mu$ and thus $R_{\phi,a}(T)$ does not contain irregular point.

Proof of Theorem 1.3 (1). Step 1. Firstly we consider the case of $I_\phi(T) \neq \emptyset$. That is, we need to prove that

$$\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$$

has full entropy gaps with respect to $R_\phi(T)$.

Fix $\varepsilon > 0$. By Proposition 4.9, we can take a number $a \in Int(L_\phi)$ such that

$$h_{top}(T, R_{\phi,a}(T)) > h_{top}(T) - \varepsilon.$$

Recall that

$$R_\phi(T) = \bigsqcup_{b \in \mathbb{R}} R_{\phi,b}(T).$$

So by Theorem 1.4,

$$h_{top}(T, R_\phi(T) \cap \xi) \geq h_{top}(T, R_{\phi,a}(T) \cap \xi) = h_{top}(T, R_{\phi,a}(T)) > h_{top}(T) - \varepsilon,$$

where

$$\begin{aligned} \xi = {} & QR(T) \setminus (A(T) \cup R(T)), W(T) \setminus QR(T), V(T) \setminus W(T), \\ & QW(T) \setminus V(T), I(T) \setminus QW(T). \end{aligned}$$

By arbitrariness of ε, we complete the proof of Step 1.

Step 2. The case that $I_\phi(T)$ is empty. That is, $R_\phi(T) = X$.

Since X is assumed infinite in this paper, by density of periodic measures, $M(T, X)$ is not a singleton. Take two invariant measures $\mu_1 \neq \mu_2$. By weak* topology, there is some continuous function $\varphi : X \to \mathbb{R}$ such that $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. Since T has g-almost product property (which is a little stronger than almost specification, see [57]), by Lemma 4.6, we have $I_\varphi(T) \neq \emptyset$. By Step 1 for the function φ,

$$\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$$

has full entropy gaps with respect to $R_\varphi(T)$. Since $R_\varphi(T) \subseteq X = R_\phi(T)$, then

$$\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$$

also has full entropy gaps with respect to $R_\phi(T)$.

Now we complete the proof of Theorem 1.3 (1). □

Remark that Step 2 also can be as a proof of Theorem 1.2.

7.3. Time-t map of hyperbolic flows

Let $f : X \to X$ be the time-t ($t \neq 0$) map of a topologically mixing Anosov flow of a compact Riemannian manifold X. In this case, f is partially hyperbolic with one-dimension central bundle. Then f is far from tangency so that f is entropy-expansive (see [32] or see [16,40]). Recall that from [35] entropy-expansive implies asymptotically h-expansive and from Theorem 3.1 of [43] any expansive or asymptotically h-expansive system satisfies uniform separation property. Recall that the unique maximal entropy measure of the flow has full support and note that the invariant measure of the flow is also invariant for the time-t map. On the other hand, as said in Section 4.3 of [56], the time-t map f satisfies specification property (even though possible not Bowen's specification) which is stronger than g-almost product property. So f satisfies entropy-dense property which implies density of ergodic measures in the space of invariant measures. Thus, f satisfies the assumptions of Propositions 7.1, 7.2, 7.6 (2) and 7.7 and so we have a following result.

Theorem 7.14. *Let $f : X \to X$ be a time-t map ($t \neq 0$) of a topological mixing Anosov flow of a compact Riemannian manifold X (in this case, f is partially hyperbolic whose central bundle only has zero Lyapunov exponents). Then*

(A) $\{A(T) \cup R(T), QR(T), W(T), V(T)\}$ has full entropy gaps with respect to X;

(B) for any continuous function $\phi : X \to \mathbb{R}$, $\{A(T) \cup R(T), QR(T), W(T)\}$ has full entropy gaps with respect to $R_\phi(T)$;

(C) for any continuous function $\phi : X \to \mathbb{R}$ satisfying $I_\phi(T) \neq \emptyset$, $\{QR(T), W(T), V(T)\}$ has full entropy gaps with respect to $I_\phi(T)$;

(D) for any continuous function $\phi : X \to \mathbb{R}$ satisfying $I_\phi(T) \neq \emptyset$ and for any $a \in Int(L_\phi)$, $\{A(T) \cup R(T), QR(T), W(T)\}$ has full entropy gaps with respect to $R_{\phi,a}(T)$.

Proof. Recall from above analysis, f satisfies specification and uniform separation, and there is an invariant measure with full support. Now we start to prove.

(C) Since f satisfies the assumptions of Propositions 7.1, 7.2, then (C) is valid.

(D) Since f satisfies the assumptions of 7.6 (2) and 7.7, then (D) is valid.

(B) One can follow same idea as the proof of Theorem 1.3 (1) to show that (D) implies (B).

(A) If take $\phi \equiv 1$, then $R_\phi(T) = X$ and thus (B) implies (A) except $V(T) \setminus W(T)$. Notice that Anosov flow carries periodic orbit. Take a periodic measure for flow, then it is also invariant (even though not ergodic) measure for time-t map f. This measure is just supported on one periodic orbit of flow and thus is different from the maximal entropy measure which has full support. In other words, f is not uniquely ergodic. By weak* topology, there is a continuous function $\phi : X \to \mathbb{R}$ such that

$$\inf_{\mu \in M(T,X)} \int \phi(x) d\mu < \sup_{\mu \in M(T,X)} \int \phi(x) d\mu.$$

So by Lemma 4.6 $I_\phi(T) \neq \emptyset$. By (C), $\{W(T), V(T)\}$ has full entropy gaps with respect to $I_\phi(T)$. From Theorem 4.4 we know that $I_\phi(T)$ carries full topological entropy. So $V(T)\backslash W(T)$ also carries full topological entropy and thus the proof of (A) is finished. □

Theorems 17.6.2 and 18.3.6 in [30] (originally due to Anosov) ensure that the geodesic flow of any compact connected Riemannian manifold of negative sectional curvature is topologically mixing and Anosov. So Theorem 7.14 can be applicative to the time-t map of the geodesic flow of any compact connected Riemannian manifold of negative sectional curvature.

8. Transitive points

Let $D(T)$ denote the set of all transitive points, i.e. the points whose orbit is dense in the whole space (see Chapter 5.4 of [59]). In other words, $D(T) = \{x \in X | \, \overline{Orb(x)} = X\}$. If T is a minimal system (for example, irrational rotation on the circle), it is obvious that $D(T) = A(T) = X$. Now we consider non-minimal systems.

Firstly we state some simple observation. For any dynamical system which is not minimal, $D(T)$ does not contain any periodic points and almost periodic points. Here we admit $D(T)$ is empty set (for example, the system of rational rotation on the circle).

Lemma 8.1. *For any non-minimal continuous map $T : X \to X$ of a compact metric space X, $A(T) \cap D(T) = \emptyset$.*

Proof. By contradiction, there is some $x \in A(T)$ such that the orbit of x is dense in X. $x \in A(T)$ implies that $x \in \omega_T(z)$ and the closed invariant set $\omega_T(x)$ is minimal. $x \in \omega_T(x)$ implies that $\omega_T(x) = \overline{Orb(x)}$. Then $X = \overline{Orb(x)} = \omega_T(x)$ is minimal, contradicting that T is not minimal. □

We will coordinate $D(T)$ with $R(T) \setminus A(T)$ together.

Theorem 8.2. *For any non-minimal continuous map $T : X \to X$ of a compact metric space X, if there is an ergodic measure μ with maximal entropy and full support, then $D(T) \cap R(T) \backslash A(T)$ carries full topological entropy. In particular, each one of $R(T) \backslash A(T)$, $D(T) = D(T) \setminus A(T)$ and $R(T) \cap D(T)$ carries full topological entropy.*

Before proof let us first give some simple observation as follows.

Lemma 8.3. *For any continuous map $T : X \to X$ of a compact metric space X,*

$$\{x \in X \,|\, C_x = X\} \subseteq D(T).$$

In particular,

(1) for any nonempty connected compact set $K \subseteq M(T,X)$, if $\overline{\cup_{\mu\in K} S_\mu} = X$, then $G_K \subseteq D(T)$, where $G_K = \{x \in X \mid M_x(T) = K\}$.

(2) for any invariant measure $\mu \in M(T,X)$, if μ has full support, then $G_\mu \subseteq D(T)$.

Proof. Fix a point $x \in X$ such that $C_x = X$. By (3.11), $x \in X = C_x \subseteq \omega_T(x)$. So $\overline{Orb(x)} = X$, i.e., $x \in D(T)$.

In particular, for any nonempty connected compact set $K \subseteq M(T,X)$, if $\overline{\cup_{\mu\in K} S_\mu} = X$, by (3.10) $G_K \subseteq \{x \mid C_x = X\}$ and thus $G_K \subseteq D(T)$. This implies (1). For the case (2), it is obvious from (1) by taking singleton K. $\square$

Proof of Theorem 8.2. Since μ is ergodic, by Birkhoff ergodic theorem G_μ is of μ full measure. Since $S_\mu = X$, from Lemma 8.3 we know that $G_\mu \cap S_\mu = G_\mu \subseteq D(T)$.

Remark that $G_\mu = G_\mu \cap S_\mu \cap D(T) \subseteq R(T) \cap D(T) = R(T) \cap D(T) \setminus A(T)$, since by non-minimal assumption and Lemma 8.1 $A(T) \cap D(T) = \emptyset$. Then by (2.1) we obtain that $D(T) \cap R(T) \setminus A(T)$ carries full topological entropy. We complete the proof. $\square$

From Theorem 5.1, $R(T) \subseteq W(T)$ and thus

$$D(T) \cap R(T) \setminus A(T) \subseteq D(T) \cap W(T) \setminus A(T).$$

So by Theorem 8.2 we have a following consequence.

Theorem 8.4. *For any non-minimal continuous map $T : X \to X$ of a compact metric space X, if there is an ergodic measure μ with maximal entropy and full support, then $D(T) \cap (W(T) \setminus A(T))$ carries full topological entropy and so does $W(T) \setminus A(T)$.*

Remark 8.5. Remark that Theorem 8.2 and Theorem 8.4 are applicative to all the examples in Section 1.2, since each example is non-minimal and the unique maximal entropy measure is ergodic and has full support.

Now we consider $D(T)$ under same assumption of our main theorems in the first section.

Theorem 8.6. *Let T be a continuous map of a compact metric space X with g-almost product property and uniform separation property. If the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$), then*

(A) $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T)\}$ has full entropy gaps with respect to $D(T)$;

(B) for any continuous function $\phi : X \to \mathbb{R}$, $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T)\}$ has full entropy gaps with respect to $D(T) \cap R_\phi(T)$;

(C) for any continuous function $\phi : X \to \mathbb{R}$ satisfying $I_\phi(T) \neq \emptyset$, $\{QR(T), W(T), V(T), QW(T)\}$ has full entropy gaps with respect to $D(T) \cap I_\phi(T)$;

(D) for any continuous function $\phi : X \to \mathbb{R}$ satisfying $I_\phi(T) \neq \emptyset$ and for any $a \in Int(L_\phi)$, $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T)\}$ has full entropy gaps with respect to $D(T) \cap R_{\phi,a}(T)$.

Remark 8.7. Remark that Theorem 8.6 holds for all the examples in Section 1.2. However, for the results of $QW(T) \setminus V(T)$ and $I(T) \setminus QW(T)$, it is still unknown.

Proof. Since X is assumed infinite in this paper, by density of periodic measures, X is not a minimal set (in other words, T is not minimal). By Lemma 6.2, there is some invariant measure such that it has full support and in particular, the support is not minimal. By Lemma 2.8, T is saturated. By Lemma 2.10, T has entropy-dense property.

Case (C). Observe that from the proofs of Propositions 7.1, 7.2 and 7.3, each constructed K satisfies $\overline{\cup_{\mu \in K} S_\mu} = X$, and thus by Lemma 8.3, G_K is contained in D(T). Then we can follow the proof of Propositions 7.1, 7.2 and 7.3 to complete the proof of (C).

Case (D). Observe that from the proofs of Propositions 7.6 (2), 7.7, 7.9 and 7.11, each constructed K (or a single measure) satisfies $\overline{\cup_{\mu \in K} S_\mu} = X$, and thus by Lemma 8.3, G_K is contained in D(T). Then we can follow the proof of Propositions 7.6 (2), 7.7, 7.9 and 7.11 to complete the proof of (D).

Case (B). One can follow same idea as the proof of Theorem 1.3 (1) to show that (D) implies (B).

Case (A). Take $\phi \equiv 1$ in (B), then $R_\phi(T) = X$ and thus (B) implies (A). □

In particular, for full shifts of finite symbols, we also have some more observations.

Theorem 8.8. *Let T be a full shift on k symbols ($k \geq 2$). Then $\{Per(T), A(T), A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has full entropy gaps with respect to $X \setminus D(T)$.*

Proof. By Theorem 1.10, $A(T) \setminus Per(T)$ has full entropy. Since T is not minimal, by Lemma 8.1 $A(T) \cap D(T) = \emptyset$. Then $A(T) \setminus Per(T)$ also has full entropy with respect to $X \setminus D(T)$. Now we start to consider other cases.

Recall a classical result in §7.3 of [59] that for full shift T of k symbols, T has proper subshifts (that is, subshifts not equal to the full shift) with topological entropy equal to any given positive real number less than the topological entropy of the shift itself. The constructed subshift is in fact the well-known β-shift ($\beta > 1$). Recall that any β-shift is not minimal (which contains a fixed point) so that it is not uniquely ergodic. For convenience to explain, denote the subshift by T_β and the subspace by $\Sigma_\beta \subsetneqq X$.

Note that for $\beta < k$, $\xi = I(T) \setminus QW(T), QW(T) \setminus V(T), V(T) \setminus W(T), W(T) \setminus QR(T), QR(T) \setminus (R(T) \cup A(T)), R(T) \setminus A(T)$,

$$\xi \cap \Sigma_\beta \subseteq \xi \cap (X \setminus D(T)).$$

By Theorem 1.9, we have $h_{top}(T, \xi \cap \Sigma_\beta) = \log \beta$ and thus $h_{top}(T, \xi \cap (X \setminus D(T))) \geq \log \beta$. Let $\beta \uparrow k$, then every set $\xi \cap (X \setminus D(T))$ carries full topological entropy of $\log k$. □

Roughly speaking, with the help of various "periodic-like" recurrence and (ir)regularity, for a certain class of dynamical systems (including mixing subshifts of finite type, mixing hyperbolic systems and β-shifts etc.), we obtain a refined classification of transitive points and each one carries full topological entropy. In particular, for full shifts of finite symbols, we have a similar classification for non-transitive points.

9. Geometric characterization of gap-sets

Under the assumption of Bowen's specification property, we will show that the gap-sets of Theorems 1.2, 1.3 and 1.4 are all dense in the whole space. For convenience of making these more precise, we introduce a concept as follows. Let $T : X \to X$ be a continuous map of a compact metric space X.

Definition 9.1. For a collection of subsets $Z_1, Z_2, \cdots, Z_k \subseteq X$ $(k \geq 2)$, we say $\{Z_i\}$ has *dense gaps* with respect to $Y \subseteq X$ if

$$\overline{(Z_{i+1} \setminus Z_i) \cap Y} = \overline{Y} \quad \text{for all } 1 \leq i < k.$$

Often, but not always, the sets Z_i are nested $(Z_i \subseteq Z_{i+1})$.

Now we state the geometric characterization as follows.

Proposition 9.2. *Let T be a continuous map of a compact metric space X with Bowen's specification property. Then*

(A) $\{A(T), A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to X;

(B) for any $\phi \in C^0(X)$, $\{A(T), A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to $R_\phi(T)$;

(C) for any $\phi \in C^0(X)$, if $I_\phi(T) \neq \emptyset$, then $\{QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to $I_\phi(T)$;

(D) for any $\phi \in C^0(X)$, if $I_\phi(T) \neq \emptyset$, then for any $a \in Int(L_\phi)$, $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to $R_{\phi,a}(T)$.

Remark that Theorem 9.2 can be applicative to all topological mixing subshifts of finite type, systems restricted on topological mixing locally maximal hyperbolic sets.

We need some classical properties of Bowen's specification property, see [15] (also see [49,9,11]).

Lemma 9.3. *(See Propositions 21.3 and 21.8 in [15].) Let T be a continuous map of a compact metric space X with Bowen's specification property. Then*

(1) the set of periodic points is dense in the whole space X;
(2) the set of periodic measures is dense in the set of T-invariant measures.

Lemma 9.4. *(See Propositions 21.9 and 21.12 in [15].) Let T be a continuous map of a compact metric space X with Bowen's specification property. Then there is a dense G_δ subset $\mathcal{R}$ of T-invariant measures such that for any $\mu \in \mathcal{R}$, μ is ergodic and $S_\mu = X$ (also saying μ has full support).*

Lemma 9.5. *(See Proposition 21.14 in [15].) Let T be a continuous map of a compact metric space X with Bowen's specification property. Then for any compact connected nonempty set $K \subseteq M(X,T)$,*

$$G_K := \{x \in X |\, M_x(T) = K\}$$

(called saturated set of K) is nonempty and dense in X. In particular,

$$G_{max} := \{x \in X |\, M_x(T) = M(X,T)\}$$

is nonempty and contains a dense G_δ subset of X.

Now let us start to prove Proposition 9.2.

Proof of Proposition 9.2. Notice that under the assumption of Bowen's specification, by Lemma 9.3 the periodic points are dense in X (i.e., $\overline{Per(f)} = X$) and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T,X)} = M(T,X)$). Moreover, by Lemma 2.10, T has entropy-dense property, since specification is stronger than g-almost product property.

(C). Observe that the constructions of K in Propositions 7.1–7.4 all did not use the saturated property. So for such K, by Lemma 9.5, we complete the proof of Proposition 9.2 (C).

(D). Observe that the constructions of K in Propositions 7.6, 7.7, 7.9, 7.11, 7.12 all did not use the saturated property. So for such K, by Lemma 9.5, we complete the proof of Proposition 9.2 (D).

(B). Firstly we prove that $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to $R_\phi(T)$. We need two steps, since (D) just holds for functions with $I_\phi(T) \neq \emptyset$.

Step 1. If $I_\phi(T) \neq \emptyset$, then $Int(L_\phi)$ is nonempty and we can take $a \in Int(L_\phi)$. Observe that $R_{\phi,a}(T) \subseteq R_\phi(T)$ and thus (D) implies that $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to $R_\phi(T)$.

Step 2. If $I_\phi(T) = \emptyset$, then $R_\phi(T) = X$. Similar as Step 2 in the proof of Theorem 1.3 (1), there is some continuous function $\varphi : X \to \mathbb{R}$ such that $I_\varphi(T) \neq \emptyset$. Then by Step 1 we have $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ has dense gaps with respect to $R_\varphi(T)$. Notice that $R_\varphi(T) \subseteq X = R_\phi(T)$, then $\{A(T) \cup R(T), QR(T), W(T), V(T), QW(T), I(T)\}$ also has dense gaps with respect to $R_\phi(T)$.

Secondly let us consider the gap-set $R(T) \setminus A(T)$. By Lemma 9.4, we can take an ergodic measure μ with full support. Remark that $G_\mu = G_\mu \cap X = G_\mu \cap S_\mu \subseteq R(T)$. Notice that T is not minimal so that by Lemma 5.3 $S_\mu = X$ implies that $G_\mu \cap A(T) = \emptyset$. So $G_\mu \subseteq R(T) \setminus A(T)$. By Lemma 9.5, G_μ is dense in X and thus $R(T) \setminus A(T)$ is also dense in X. Now we complete the proof of Proposition 9.2 (B).

(A). Take $\phi \equiv 1$ in (B), then $R_\phi(T) = X$ and thus (B) implies (A). Now we complete the proof of Proposition 9.2. □

In particular, we have a better characterization for the set of $V(T) \setminus W(T)$, that is,

$$I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists\, \omega \in M_x(T) \,\text{s.t.}\, S_\omega = C_x\},$$

which is to answer the open problem in Section 1.3 from different sight.

Proposition 9.6. *Let T be a continuous map of a compact metric space X with Bowen's specification property. Let*

$$IC^0(X) := \{\phi \in C^0(X) |\ I_\phi(T) \neq \emptyset\}.$$

Then the set

$$\bigcap_{\phi \in IC^0(X)} I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists\, \omega \in M_x(T)\ s.t.\ S_\omega = C_x\}$$

contains a dense G_δ subset of X (called residual in X).

Proof. By Lemma 9.5, $G_{max} = \{x \in X |\, M_x(T) = M(X,T)\}$ is residual in X. So we only need to prove for any $\phi \in IC^0(X)$,

$$G_{max} \subseteq I_\phi(T) \cap \{x \in QW(T) \setminus W(T) | \exists\, \omega \in M_x(T) \,\text{s.t.}\, S_\omega = C_x\}.$$

Since $\phi \in IC^0(X)$, then $I_\phi(T) \neq \emptyset$ and thus by (4.27) there are two invariant measures $\omega, \mu \in M(X,T)$ such that they have different integrals for ϕ. Fix $x \in G_{max}$. Remark that $M(T,X) = M_x(T)$ and so $\mu, \omega \in M_x(T)$. By (4.26) $x \in I_\phi(T)$. By density of periodic points (Lemma 9.3),

$$C_x = \overline{\bigcup_{m \in M_x(T)} S_m} = \overline{\bigcup_{m \in M(T,X)} S_m} \supseteq \overline{\bigcup_{m \in M(T,X) \text{ is periodic measure}} S_m} = X$$

(remark that $C_x = X$ can be also obtained from the existence of invariant measure with full support by Lemma 9.4). By (3.14) $X = C_x$ implies $x \in QW(T)$. By (3.9) $x \in Rec(T)$.

By Lemma 9.3, one can take a periodic measure $\nu \in M(T,X) = M_x(T)$ whose support $S_\mu \subsetneqq X = C_x$, then by (3.12) $x \in X \setminus W(T)$. Take an invariant measure μ with full support X by Lemma 9.4, then $\mu \in M(T,X) = M_x(T)$ and $S_\mu = X = C_x$. We complete the proof. □

Remark that G_{max} has zero topological entropy from (2.7), since $K := M(T,X)$ contains periodic measures which have zero entropy. By (2.2) and Theorem 1.6 we know that

Theorem 9.7. *Let T be a continuous map of a compact metric space X with Bowen's specification property and uniform separation. The complementary set of G_{max},*

$$\{x \in I_\phi(T) \cap QW(T) \setminus W(T) | \exists\, \omega \in M_x(T)\, s.t.\, S_\omega = C_x\} \setminus G_{max},$$

has full topological entropy. This complementary set is just dense but not residual in X.

Proof. Full entropy can be deduced from (2.2) and Theorem 1.6. Density can be seen from Lemma 9.5, since the choice of K in the proof of Proposition 7.2 is a proper subset of $M(T,X)$ and so $G_K \cap G_{max} = \emptyset$. □

Recall from [57] that for any system with almost specification (not necessarily satisfying uniform separation), every ϕ-irregular set either is empty or carries full topological entropy. Together with the observation of Proposition 9.2, a natural question arises for systems with Bowen's specification property (not necessarily satisfying uniform separation):

Question 9.8. *Let f be a continuous map of a compact metric space X with Bowen's specification (or almost specification). Whether all the results in Theorems 1.2, 1.3 and Theorem 1.4 hold? If not, which results hold, which results do not hold and how to construct such a counterexample?*

Let us give some simple observation for Question 9.8 in the case of Bowen's specification. Since Bowen's specification is stronger than g-almost product property, then by Theorem 2.9 f is single-saturated. By Lemma 6.2, there is an invariant measure with full support. So we can use Proposition 7.6 (2) to obtain $\{A(T) \cup R(T), QR(T)\}$ has full entropy gaps with respect to $R_{\phi,a}(T)$ (resp., $R_\phi(T)$ and X). For other results of Theorems 1.2, 1.3 and Theorem 1.4, from the proofs of such results the considered $K \subseteq M(T,X)$ is not singleton. So single-saturated property is not enough. Moreover, let us recall another possible idea by Thompson [57], one needs to take two needed ergodic measures and then use these two measures to construct a set $F \subseteq I_\phi(T)$ such that the topological entropy of F is larger than $h_{top}(f) - \epsilon$. In this process, Entropy Distribution Principle plays an important role. One can see [56,57] for more details. The constructed measures are required ergodic. However, the constructed measures in present paper are not all ergodic so that we are not sure the idea of [56,57] works. So Question 9.8 is still open except the case for $\{A(T) \cup R(T), QR(T)\}$.

Remark that if the answer of Question 9.8 is positive, then it can be as a generalization of Theorem 1.6. In particular, if Question 9.8 is true, it would be applicative to all topological mixing interval maps, since it is known from [7,12] that any topologically mixing interval map satisfies Bowen's specification. For example, Jakobson [27] showed that there exists a set of parameter values $\Lambda \subseteq [0,4]$ of positive Lebesgue measure such that if $\lambda \in \Lambda$, then the logistic map $f_\lambda(x) = \lambda x(1-x)$ is topologically mixing. From Proposition 21.4 of [15] we also know that the factor of a system with Bowen's specification has Bowen's specification.

At the end of this section, we show that Proposition 9.6 holds for a larger class of systems, such as C^1 generic (volume-preserving) diffeomorphisms, which means the open problem of [63] by Zhou and Feng is solved for generic systems. Let M be a compact Riemannian manifold and m be a volume measure on M. Let $\mathrm{Diff}^1(M)$ and $\mathrm{Diff}^1_m(M)$ denote the space of all C^1 diffeomorphisms on M and all volume-preserving C^1 diffeomorphisms on M respectively. If $f : M \to M$ and Λ is a compact subset of M, let

$$IC^0_f(\Lambda) := \{\phi \in C^0(\Lambda) |\ I_\phi(f) \neq \emptyset\}.$$

Theorem 9.9. *(1) Let Λ be an isolated non-trivial transitive set of a C^1 generic diffeomorphism $f \in \mathrm{Diff}^1(M)$. Then the set*

$$\bigcap_{\phi \in IC^0_f(\Lambda)} I_\phi(f) \cap \{x \in \Lambda \cap QW(f) \setminus W(f) | \exists\, \omega \in M_x(f)\ s.t.\ S_\omega = C_x\}$$

contains a dense G_δ subset of Λ (called residual in Λ).

(2) Let $f \in \mathrm{Diff}^1_m(M)$ be a C^1 generic volume-preserving diffeomorphism. Then the set

$$\bigcap_{\phi \in IC^0_f(M)} I_\phi(f) \cap \{x \in QW(f) \setminus W(f) | \exists\, \omega \in M_x(f)\ s.t.\ S_\omega = C_x\}$$

contains a dense G_δ subset of M.

Proof. For the first case, by the main result of [52]

$$G_{max} = \{x \in \Lambda |\ M_x(f) = M(f, \Lambda)\}$$

is residual in X. Recall Theorem 3.5 of [1] that generic invariant measures have full support. Thus, forward the proof of Proposition 9.6, one can replace Lemma 9.4 by Theorem 3.5 of [1] to prove.

For the second case, it is known that generic $f \in \mathrm{Diff}^1_m(M)$ is transitive so that we can take $\Lambda = M$. Notice that Theorem 3.5 of [1] and the main result of [52] also can be stated as the volume-preserving case. Then the proof is similar as above. Here we omit the details. □

Inspired by Theorem 9.9, it is possible to ask

Question 9.10. *Let Λ be an isolated non-trivial transitive set of a C^1 generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ or let $f \in \mathrm{Diff}^1_m(M)$ be a C^1 generic volume-preserving diffeomorphism. Then whether $\{W(f), V(f)\}$ (resp., $\{W(f), QW(f)\}$) has full entropy gaps with respect to Λ or M?*

Recall that uniformly hyperbolic systems are not dense in the space of all diffeomorphisms. So we cannot use Theorem 1.11 to answer this question.

Moreover, it is unknown whether we can use our main theorems (Theorems 1.2–1.4) to answer this question. From [1] periodic points are all hyperbolic and dense in Λ and periodic measures are dense in the space of all invariant measures supported on Λ. However, we do not know whether g-almost product property and uniform separation hold for generic diffeomorphisms. So Question 9.10 seems to be nontrivial.

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