Ergod. Th. & Dynam. Sys. (2021), **41**, 349–378
doi:10.1017/etds.2019.57

Distributional chaos in multifractal analysis, recurrence and transitivity

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(*Received* 8 *May* 2018 *and accepted in revised form* 15 *July* 2019)

Abstract. There is much research on the dynamical complexity on irregular sets and level sets of ergodic average from the perspective of density in base space, the Hausdorff dimension, Lebesgue positive measure, positive or full topological entropy (and topological pressure), etc. However, this is not the case from the viewpoint of chaos. There are many results on the relationship of positive topological entropy and various chaos. However, positive topological entropy does not imply a strong version of chaos, called DC1. Therefore, it is non-trivial to study DC1 on irregular sets and level sets. In this paper, we will show that, for dynamical systems with specification properties, there exist uncountable DC1-scrambled subsets in irregular sets and level sets. Meanwhile, we prove that several recurrent level sets of points with different recurrent frequency have uncountable DC1-scrambled subsets. The major argument in proving the above results is that there exists uncountable DC1-scrambled subsets in saturated sets.

Key words: irregular set and level set, recurrence and transitivity, specification, distributional chaos, scrambled set
2010 Mathematics Subject Classification: 37C50, 37B20, 37B05 (Primary); 37D45, 37C45 (Secondary)

1. *Introduction*

Throughout this paper, let (X, d) be a non-degenerate (i.e. with at least two points) compact metric space, and $f : X \to X$ be a continuous map. Such (X, f) is called a dynamical system.

1.1. *Multifractal analysis.* The theory of multifractal analysis is a subfield of the dimension theory of dynamical systems. Briefly, multifractal analysis studies the dynamical complexity of the level sets of the invariant local quantities obtained from a dynamical system. There is much research on dynamical complexity on irregular sets and level sets of ergodic average from the perspective of density in base space, positive or full Hausdorff dimension, topological entropy (and topological pressure) [**4**, **7**, **13**, **19**, **24**, **46**, **47**, **59–62**], Lebesgue positive measure [**32**, **58**] and references therein. However, this is not the case from the viewpoint of chaos. In the field of chaos theory, Li–Yorke chaos and distributional chaos are commonly used to describe the dynamical complexity. In this paper, we firstly study the dynamical complexity of irregular sets and level sets from the viewpoint of a strong chaotic property called DC1. Notice that Pikula showed in [**50**] that positive topological entropy does not imply DC1 so that it is not expected to show DC1 of irregular sets and level sets by using the results in [**6**, **7**, **47**, **59**] that irregular set and level sets carry positive (and full) topological entropy.

The notion of chaos was first introduced in mathematical language by Li and Yorke in [**37**] in 1975. For a dynamical system (X, f), they defined that (X, f) is Li–Yorke chaotic if there is an uncountable scrambled set $S \subseteq X$, where S is called a scrambled set if, for any pair of distinct two points x, y of S,

$$\liminf_{n\to+\infty} d(f^n x, f^n y) = 0, \quad \limsup_{n\to+\infty} d(f^n x, f^n y) > 0.$$

Since then, several refinements of chaos have been introduced and extensively studied. One of the most important extensions of the concept of chaos in sense of Li and Yorke is distributional chaos [**53**]. The stronger form of chaos has three variants: DC1 (distributional chaos of type 1), DC2 and DC3 (ordered from strongest to weakest). In this paper, we focus on DC1. Readers can refer to [**22**, **55**, **57**] for the definition of DC2 and DC3 and see [**1**, **8**, **9**, **11**, **18**, **31**, **42**, **43**] and references therein for related topics on chaos theory, if necessary. A pair $x, y \in X$ is DC1-scrambled if the following two conditions hold:

$$\text{for all } t > 0, \quad \limsup_{n\to\infty} \frac{1}{n}|\{i \in [0, n-1] : d(f^i(x), f^i(y)) < t\}| = 1,$$

$$\text{and there exists } t_0 > 0, \quad \liminf_{n\to\infty} \frac{1}{n}|\{i \in [0, n-1] : d(f^i(x), f^i(y)) < t_0\}| = 0.$$

In other words, the orbits of x and y are arbitrarily close with upper density one, but for some distances, with lower density zero.

Definition 1.1. A set S is called a DC1-scrambled set if any pair of distinct points in S is DC1-scrambled.

1.1.1. *DC1 in an irregular set.* For a continuous function φ on X, define the *φ-irregular set* as

$$I_\varphi(f) := \left\{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \text{ diverges}\right\}.$$

The φ-irregular set and the irregular set, the union of $I_\varphi(f)$ over all continuous functions of φ (denoted by $IR(f)$), arise in the context of multifractal analysis and have been studied a lot, for example, see [**7**, **13**, **19**, **46**, **47**, **60**]. The irregular points are also called points with historic behavior, see [**52**, **58**]. From Birkhoff's ergodic theorem, the irregular set is not detectable from the point of view of any invariant measure. However, the irregular set may have strong dynamical complexity in the sense of the Hausdorff dimension, the Lebesgue positive measure, topological entropy and topological pressure etc. Pesin and Pitskel [**47**] were the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. There are lots of advanced results to show that the irregular points can carry full entropy in symbolic systems, hyperbolic systems, non-uniformly expanding or hyperbolic systems and systems with specification-like or shadowing-like properties, for example, see [**7**, **13**, **19**, **38**, **46**, **60**, **64**]. For the topological pressure case see [**60**] and for the Lebesgue positive measure see [**32**, **58**]. Now let us state our first main theorem to study the dynamical complexity of an irregular set from the perspective of DC1.

THEOREM A. *Suppose that (X, f) is a dynamical system with the specification property, φ is a continuous function on X and $I_\varphi(f) \neq \emptyset$. Then, there is an uncountable* DC1-*scrambled subset in $I_\varphi(f)$.*

1.1.2. *DC1 in a level set.* A level set is a natural concept to slice points with a convergent Birkhoff's average operated by a continuous function, regarded as the multifractal decomposition [**14**, **25**]. For a dynamical system (X, f), let $\mathcal{M}(X)$, $\mathcal{M}_f(X)$, $\mathcal{M}_f^e(X)$ denote the space of probability measures, f-invariant, f-ergodic probability measures, respectively. (X, f) is called uniquely ergodic if $\mathcal{M}_f(X)$ is a singleton. Let $\varphi : X \to \mathbb{R}$ be a continuous function. Denote

$$L_\varphi = \left[\inf_{\mu\in\mathcal{M}_f(X)} \int \varphi \, d\mu, \sup_{\mu\in\mathcal{M}_f(X)} \int \varphi \, d\mu\right]$$

and

$$\mathrm{Int}(L_\varphi) = \left(\inf_{\mu\in\mathcal{M}_f(X)} \int \varphi \, d\mu, \sup_{\mu\in\mathcal{M}_f(X)} \int \varphi \, d\mu\right).$$

For any $a \in L_\varphi$, define the level set

$$R_\varphi(a) := \left\{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = a\right\}.$$

Denote $R_\varphi = \bigcup_{a\in L_\varphi} R_\varphi(a)$, called the regular points for φ. Many authors have considered the entropy of the $R_\varphi(a)$. For example, Barreira and Saussol proved in [**6**] that the following properties for a dynamical system (X, f) whose function of metric entropy is upper semi-continuous. Consider a Hölder continuous function φ (see [**4**, **5**] for almost additive functions with tempered variation) which has a unique equilibrium measure; then, for any constant $a \in \mathrm{Int}(L_\varphi)$,

$$h_{\mathrm{top}}(R_\varphi(a)) = t_a, \tag{1.1}$$

where

$$t_a = \sup_{\mu\in\mathcal{M}_f(X)} \left\{h_\mu : \int \varphi \, d\mu = a\right\},$$

$h_{\mathrm{top}}(R_\varphi(a))$ denotes the entropy of $R_\varphi(a)$ and h_μ denotes the measure entropy of μ. For φ being an arbitrary continuous function (hence there may exist more than one equilibrium measure), (1.1) was established by Takens and Verbitski [**59**] under the assumption that f has the specification property. This result was further generalized by Pfister and Sullivan [**49**] to dynamical systems with g-product property (see [**61**, **63**] for more related discussions). The method used in [**5**, **6**] mainly depends on thermodynamic formalism such as differentiability of the pressure function, while the method in [**49**, **59**] is a direct approach by constructing fractal sets. Here, we consider the distributional chaotic of $R_\varphi(a)$ and R_φ. Note that if $I_\varphi(f) \neq \emptyset$, then $\mathrm{Int}(L_\varphi) \neq \emptyset$. The inverse is also true if the system has the specification property, see [**60**] (see [**62**] for the case of the almost specification property), and it is easy to check that the continuous functions with $\mathrm{Int}(L_\varphi) \neq \emptyset$ form an open and dense subset in the space of continuous functions and that so do the functions with $I_\varphi(f) \neq \emptyset$ if the system has the specification property or almost specification property.

THEOREM B. *Suppose that (X, f) is a dynamical system with the specification property, φ is a continuous function on X and $\mathrm{Int}(L_\varphi) \neq \emptyset$. Then, for any $a \in \mathrm{Int}(L_\varphi)$, there is an uncountable* DC1*-scrambled subset in $R_\varphi(a)$.*

As a corollary, there are uncountable numbers of disjoint uncountable DC1-scrambled subsets.

COROLLARY A. *Suppose that (X, f) is a dynamical system with the specification property. Then, there exists a collection of subsets of X, $\{S_\alpha\}_{\alpha\in(0,1)}$, such that:*

(1) *for any $0 < \alpha_1 < \alpha_2 < 1$, $S_{\alpha_1} \cap S_{\alpha_2} = \emptyset$; and*

(2) *for any $\alpha \in (0, 1)$, S_α is an uncountable* DC1*-scrambled set.*

Let us explain why this result holds. By Proposition 2.4 there are two different invariant measures μ, ν or, equivalently, there exists a continuous function ϕ such that $\int \phi \, d\mu \neq \int \phi \, d\nu$. Thus, $\mathrm{Int}(L_\phi) \neq \emptyset$. Let $\varphi := (1/L)(\phi - \inf_{\mu\in\mathcal{M}_f(X)} \int \phi \, d\mu)$ where L denotes the length of interval L_ϕ. Then $\mathrm{Int}(L_\varphi) = (0, 1)$ and Theorem B implies this corollary since $R_\varphi(a) \cap R_\varphi(b) = \emptyset$ if $a \neq b$.

THEOREM 1.2. *Suppose that (X, f) is a dynamical system with the specification property and φ is a continuous function on X. Then, there is an uncountable* DC1*-scrambled subset in R_φ.*

Let us explain why Theorem 1.2 holds. If $\mathrm{Int}(L_\varphi) \neq \emptyset$, then this follows from Theorem B by taking one $a \in \mathrm{Int}(L_\varphi)$ since $R_\varphi(a) \subseteq R_\varphi$. On the other hand, $\mathrm{Int}(L_\varphi) = \emptyset$, so then $R_\varphi = X$ and this result follows from [**43**] (or see [**41**]).

1.2. *DC1 in recurrence.* In classical study of dynamical systems, an important concept is recurrence. Recurrent points such as periodic points and minimal points are typical objects to be studied. It is known that the whole recurrent points set has full measure for any invariant measure under f and that the minimal points set is not empty [**26**]. A fundamental question in dynamical systems is to search for the existence of periodic points. For systems with the Bowen specification property (such as topological mixing subshifts of finite type and topological mixing uniformly hyperbolic systems), the set of periodic points is dense in the whole space [**17**]. Further, many people pay attention to more refinements of recurrent points according to the 'recurrent frequency' such as almost periodic points (which naturally exist in any dynamical system since it is equivalent that they belong to a minimal set), weakly almost periodic points and quasi-weakly almost periodic points and measure them [**27**, **68**]. In [**28**, **63**] the authors considered various recurrences and showed that many different recurrent levels carry strong dynamical complexity from the perspective of topological entropy. In this paper, one of our aims is to consider these different recurrent levels from the perspective of chaos.

For any $x \in X$, the orbit of x is $\{f^n x\}_{n=0}^{\infty}$, denoted by $\mathrm{orb}(x, f)$. The ω-limit set of x is the set of all accumulation points of $\mathrm{orb}(x, f)$, denoted by $\omega(f, x)$.

Definition 1.3. A point $x \in X$ is *recurrent* if $x \in \omega(f, x)$. If $\omega(f, x) = X$, we say x is a transitive point of f. A point $x \in X$ is *almost periodic* if, for any open neighborhood U of x, there exists $N \in \mathbb{N}$ such that $f^k(x) \in U$ for some $k \in [n, n + N]$, for every $n \in \mathbb{N}$. It is well known that x is almost periodic $\Leftrightarrow$ x belongs to a minimal set. A point x is periodic if there exists a natural number n such that $f^n(x) = x$.

We denote the sets of all recurrent points, transitive points, almost periodic points and periodic points by Rec, Trans, AP and Per, respectively. Now we recall some notions of recurrence by using density. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, \ldots\}$. Let $S \subseteq \mathbb{N}$, and we denote

$$\overline{d}(S) := \limsup_{n\to\infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n}, \quad \underline{d}(S) := \liminf_{n\to\infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n},$$
$$B^*(S) := \limsup_{|I|\to\infty} \frac{|S \cap I|}{|I|}, \quad B_*(S) := \liminf_{|I|\to\infty} \frac{|S \cap I|}{|I|},$$

where $|A|$ denotes the cardinality of the set A. They are called the upper density and the lower density of S, and the Banach upper density and the Banach lower density of S, respectively. Let $U, V \subseteq X$ be two non-empty open sets and $x \in X$. Define sets of visiting time

$$N(U, V) := \{n \geq 1 | U \cap f^{-n}(V) \neq \emptyset\} \quad \text{and} \quad N(x, U) := \{n \geq 1 | f^n(x) \in U\}.$$

Definition 1.4. A point $x \in X$ is called Banach upper recurrent if $N(x, B(x, \varepsilon))$ has positive Banach upper density where $B(x, \varepsilon)$ denotes the ball centered at x with radius ε. Similarly, one can define the Banach lower recurrent, upper recurrent and lower recurrent.

Let BR denote the set of all Banach upper recurrent points and let QW, W denote the set of upper recurrent points and lower recurrent points, respectively (called quasi-weakly almost periodic and weakly almost periodic [**27**, **63**, **68**]). Note that AP coincides with the set of all Banach lower recurrent points and

$$\mathrm{AP} \subseteq W \subseteq \mathrm{QW} \subseteq \mathrm{BR} \subseteq \mathrm{Rec}.$$

So the recurrent set can be decomposed into several disjoint 'periodic-like' recurrent level sets which reflect different recurrent frequency:

$$\mathrm{Rec} = \mathrm{AP} \sqcup (W \setminus \mathrm{AP}) \sqcup (\mathrm{QW} \setminus W) \sqcup (\mathrm{BR} \setminus \mathrm{QW}) \sqcup (\mathrm{Rec} \setminus \mathrm{BR}).$$

To figure out exactly which level of recurrent point carries dynamical complexity and which level does not, a natural idea expressed in [**63**] is to study their 'gap' set (i.e. the disjoint part). In [**63**], the author uses topological entropy as index. It was shown that, except for $\mathrm{Rec} \setminus \mathrm{BR}$, these recurrent level sets all have full topological entropy ([**63**] for $\mathrm{QW} \setminus W$ and $W \setminus \mathrm{AP}$, [**28**] for $\mathrm{BR} \setminus \mathrm{QW}$, [**21**] for AP). From [**41**] Oprocha proved that there exists an uncountable DC1-scrambled subset in $\mathrm{Rec} \setminus \mathrm{AP}$. Recall that Pikula showed in [**50**] that positive topological entropy does not imply DC1. Thus, motivated by these results we can also ask the similar question from the perspective of chaos. That is, whether there is an uncountable DC1-scrambled set in every recurrent level set of $\mathrm{Rec} \setminus \mathrm{BR}$, $\mathrm{BR} \setminus \mathrm{QW}$, $\mathrm{QW} \setminus W$, $W \setminus \mathrm{AP}$ and AP. We will mainly show that there are uncountable DC1-scrambled subsets in $\mathrm{BR} \setminus \mathrm{QW}$ and $\mathrm{QW} \setminus W$ if the system has the specification property (and we also discuss an uncountable DC1-scrambled subset in $W \setminus \mathrm{AP}$ under more assumptions and an uncountable DC2-scrambled subset in AP in the last section).

THEOREM C. *Suppose that (X, f) is a dynamical system with the specification property. Then there exist uncountable* DC1*-scrambled subsets in* $\mathrm{QW} \setminus W$ *and* $\mathrm{BR} \setminus \mathrm{QW}$. *Moreover, the points in these subsets can be chosen to be transitive.*

1.3. *Combination of multifractal analysis and recurrence.* We give a DC1 result in combined sets of multifractal analysis and recurrence.

THEOREM D. *Suppose that (X, f) is a dynamical system with the specification property, φ is a continuous function on X and $\mathrm{Int}(L_\varphi) \neq \emptyset$. Then:*

(1) *there exist uncountable* DC1*-scrambled subsets in* $I_\varphi \cap (\mathrm{QW} \setminus W)$ *and* $I_\varphi \cap (\mathrm{BR} \setminus \mathrm{QW})$, *respectively;*

(2) *for any $a \in \mathrm{Int}(L_\varphi)$, there exist uncountable* DC1*-scrambled subsets in* $R_\varphi(a) \cap (\mathrm{QW} \setminus W)$ *and* $R_\varphi(a) \cap (\mathrm{BR} \setminus \mathrm{QW})$, *respectively.*

Moreover, the points in these subsets can be chosen to be transitive.

Obviously, Theorem D implies Theorems A and B. By Proposition 2.4, there are two different invariant measures μ, ν, or equivalently there exists a continuous function ϕ such that $\int \phi \, d\mu \neq \int \phi \, d\nu$. Thus $\mathrm{Int}(L_\phi) \neq \emptyset$. Therefore Theorems D(1) and (2) both imply Theorem C. So we only need to prove Theorem D in §4. As a corollary of Theorem D, we state the following result.

COROLLARY B. *Suppose that (X, f) is a dynamical system with specification property, φ is a continuous function on X and $\mathrm{Int}(L_\varphi) \neq \emptyset$. Then there exists an uncountable* DC1*-scrambled subset in* $\mathrm{Trans} \cap I_\varphi$. *And for any $a \in \mathrm{Int}(L_\varphi)$, there exists an uncountable* DC1*-scrambled subset in* $R_\varphi(a) \cap \mathrm{Trans}$.

1.4. *DC1 in recurrent level sets characterized by statistical ω-limit sets.* One problem in the study of dynamical systems is to consider the probability of finding one orbit entering in a set E: $(1/n) \sum_{i=0}^{n-1} \chi_E(f^i x)$ (for example, see [**2**, **3**, **40**]). Recently, several concepts of statistical ω-limit sets were introduced and studied in [**20**] (also see [**2**, **3**]) from the perspective of natural density and Banach density. They can describe different levels of recurrence and some cases coincide with above classifications of Banach recurrence.

Definition 1.5. For $x \in X$ and $\xi = \overline{d}, \underline{d}, B^*, B_*$, a point $y \in X$ is called $x - \xi$-accessible if, for any $\varepsilon > 0$, $N(x, B(y, \varepsilon))$ has positive density with respect to ξ. Let

$$\omega_\xi(x) := \{y \in X \mid y \text{ is } x - \xi\text{-accessible}\}.$$

For convenience, it is called the $\xi - \omega$*-limit set of* x. $\omega_{B_*}(x)$ is also called the *syndetic center* of x.

With these definitions, one can immediately note that

$$\omega_{B_*}(x) \subseteq \omega_{\underline{d}}(x) \subseteq \omega_{\overline{d}}(x) \subseteq \omega_{B^*}(x) \subseteq \omega(f, x). \tag{1.2}$$

For any $x \in X$, if $\omega_{B_*}(x) = \emptyset$, then we know that x satisfies one and only one of following 12 cases:
Case (1): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) = \omega_{B^*}(x) = \omega(f, x)$;
Case (1′): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) = \omega_{B^*}(x) \subsetneq \omega(f, x)$;
Case (2): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) = \omega(f, x)$;
Case (2′): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) \subsetneq \omega(f, x)$;
Case (3): $\omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) = \omega(f, x)$;
Case (3′): $\omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) \subsetneq \omega(f, x)$;
Case (4): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) = \omega(f, x)$;
Case (4′): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) \subsetneq \omega(f, x)$;
Case (5): $\omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) = \omega(f, x)$;
Case (5′): $\omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) \subsetneq \omega(f, x)$;
Case (6): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) = \omega(f, x)$;
Case (6′): $\omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) \subsetneq \omega(f, x)$.

Remark 1.6. There are 12 cases rather than 16 because $\omega_{\overline{d}}(x)$ must be a non-empty set (see Proposition 2.8).

THEOREM E. *Suppose that* (X, f) *is a dynamical system with the specification property. Then* $\{x \in \mathrm{Rec} \mid x$ *satisfies Case (i)*$\}$, $i = 2, 3, 4, 5, 6$ *contains an uncountable* DC1-*scrambled subset in* Trans. *Further, if* φ *is a continuous function on* X *and* $I_\varphi(f) \neq \emptyset$, *then for any* $a \in \mathrm{Int}(L_\varphi)$, *the recurrent level set of* $\{x \in \mathrm{Rec} \mid x$ *satisfies Case (i)*$\}$ *contains an uncountable* DC1-*scrambled subset in* $\mathrm{Trans} \cap I_\varphi(f)$, $\mathrm{Trans} \cap R_\varphi(a)$ *and* $\mathrm{Trans} \cap R_\varphi$, *respectively,* $i = 2, 3, 4, 5, 6$.

We will prove this theorem in §4. Case (1) is also known if the system has more assumptions, see the last section, but Cases (1′)–(6′) restricted on recurrent points all are still unknown, whether or not they have DC1 or weaker ones such as Li–Yorke chaos. Chaotic behavior in non-recurrent points and various non-recurrent levels by using the above statistical ω-limit sets will be discussed in another forthcoming paper.

1.5. *DC1 in saturated sets.*

To show the above results on irregular sets, level sets and different recurrence, one main proof idea follows from by Oprocha and Štefánková's results in [**43**] (or see [**42**]) that there is an uncountable DC1-scrambled subset in X when the dynamical system (X, f) has the specification property. One can construct corresponding uncountable DC1-scrambled subsets one by one but each one needs a long construction proof so it is not a good choice to do these constructions directly. Recall that in the case of an entropy estimate on recurrent levels, one main technique chosen in [**63**] is using the (transitively) saturated property, which can avoid a long construction proof for every object being considered. So, here we follow the way of [**63**] to give a DC1 result in saturated sets.

Given $x \in X$, denote $V_f(x) \subseteq \mathcal{M}_f(X)$ as the set of all accumulation points of the empirical measures

$$\mathcal{E}_n(x) := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)},$$

where δ_x is the Dirac measure concentrating on x. We say a dynamical system (X, f) has *saturated property* if, for any compact connected non-empty set $K \subseteq \mathcal{M}_f(X)$,

$$G_K \neq \emptyset \quad \text{and} \quad h_{\text{top}}(G_K) = \inf\{h_\mu(T) \mid \mu \in K\}, \tag{1.3}$$

where $G_K = \{x \in X \mid V_f(x) = K\}$ (called a saturated set). The existence of saturated sets is proved by Sigmund [**54**] for systems with uniform hyperbolicity or the specification property and generalized to non-uniformly hyperbolic systems in [**39**]. The property on entropy estimate was first established by Pfister and Sullivan in [**49**], provided that the system has g-product property (which is weaker than the specification property) and uniform separation property (which is weaker than expansiveness). In this subsection, we aim to establish DC1 in saturated sets. A point $x \in X$ which is generic for some invariant measure μ means that $V_f(x) = \mu$ (or equivalently, Birkhoff averages of all continuous maps converge to the integral of μ). Thus G_μ denotes the set of all generic points for μ.

For a dynamical system (X, f), we say a pair $p, q \in X$ is distal if $\liminf_{i\to\infty} d(f^i p, f^i q) > 0$. Otherwise, the pair p, q is proximal. Obviously, $\inf\{d(f^i p, f^i q) \mid i \in \mathbb{N}\} > 0$ if the pair p, q is distal. We say a subset $M \subseteq X$ has a distal pair if there are distinct $p, q \in M$ such that the pair p, q is distal.

THEOREM F. *Suppose that (X, f) is a dynamical system with the specification property and let K be a connected non-empty compact subset of $\mathcal{M}_f(X)$. If there is a $\mu \in K$ such that $\mu = \theta\mu_1 + (1-\theta)\mu_2$* ($\mu_1 = \mu_2$ could happen) *where $\theta \in [0, 1]$, and G_{μ_1}, G_{μ_2} both have distal a pair, then for any non-empty open set $U \subseteq X$, there exists an uncountable* DC1*-scrambled set $S_K \subseteq G_K \cap U \cap$* Trans.

We will prove this theorem in §3. Since an ergodic measure with non-degenerate minimal support has two generic points as a distal pair, see Proposition 4.2 below, one has the following result as a corollary of Theorem F.

COROLLARY C. *Suppose that (X, f) is a dynamical system with the specification property. For any ergodic measure μ, if its support is non-degenerate and minimal, then there exists an uncountable* DC1*-scrambled set $S \subseteq$* Trans *such that any point in S is generic for μ.*

Here μ admits to have zero metric entropy. If the system is not minimal, then the above set S has zero measure for μ, since $S \subseteq$ Trans, $S_\mu \neq X$ and by Birkhoff ergodic theorem $\mu(S_\mu \cap G_\mu) = 1$.

2. *Preliminaries*

2.1. *Specification Properties.* The specification property was first introduced by Bowen in [**10**]. However, we will use the definition used in [**60**] and [**65**] because, with this definition, the proofs of our main theorems will be much briefer. The differences between two kinds of definition have been elaborated in [**60**]. Before giving the definition, we suggest the notion that, for a dynamical system (X, f) and $x, y \in X$, $a, b \in \mathbb{N}$, we say x *ε-traces* y on $[a, b]$ if $d(f^i x, f^{i-a} y) < \varepsilon$ for all $i \in [a, b]$.

Definition 2.1. We say a dynamical system (X, f) has a *strong specification property* if, for any $\varepsilon > 0$, there is a positive integer K_ε such that, for any integer $s \geq 2$, any set $\{y_1, y_2, \ldots, y_s\}$ of s points of X, and any sequence

$$0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$$

of $2s$ integers with

$$a_{m+1} - b_m \geq K_\varepsilon$$

for $m = 1, 2, \ldots, s-1$, there is a point x in X such that the following two conditions hold:

(a) x *ε-traces* y_m on $[a_m, b_m]$ for all positive integers $m \leq s$;

(b) $f^n(x) = x$, where $n = b_s + K_\varepsilon$.

If the periodicity condition (b) is omitted, we say that f has the *specification property*.

PROPOSITION 2.2. [**23**] *Suppose that (X, f) is a dynamical system with the specification property. Then $\mathcal{M}^e_f(X)$ is dense in $\mathcal{M}_f(X)$.*

For a measure μ, define the support of μ by $S_\mu := \mathrm{supp}(\mu) = \{x \in X \mid \mu(U) > 0$ for any neighborhood U of $x\}$. Given $x \in X$, define the measure center of x by $C_x := \overline{\bigcup_{m \in V_f(x)} S_m}$. We say that a Borel set $U \subseteq X$ is universally null for f if $\mu(U) = 0$ for every $\mu \in \mathcal{M}_f(X)$. The *measure center* of a dynamical system (X, f) is the complement of the union of all universally null open sets.

PROPOSITION 2.3. *A dynamical system (X, f) with the specification property has measure with full support (i.e. $S_\mu = X$). Moreover, the set of such measures is dense in $\mathcal{M}_f(X)$.*

Proof. From [**16**], we know that, for any dynamical system with the specification property (not necessarily Bowen's strong version), the almost periodic points (AP) are dense in X. Take a sequence of points $\{x_i\} \in \mathrm{AP}$ dense in X. For any i, take μ_i to be an invariant measure on $\omega(f, x_i)$. Then $x_i \in \omega(f, x_i) = S_{\mu_i}$ and so $\overline{\bigcup_{i \geq 1} S_{\mu_i}} = X$. Let $\mu = \sum_{i \geq 1} \frac{1}{2^i} \mu_i$. Then $\mu \in \mathcal{M}_f(X)$ and $S_\mu = X$. By [**17**, Proposition 21.11], the proof is complete. □

PROPOSITION 2.4. *A dynamical system (X, f) with the specification property must not be uniquely ergodic.*

Proof. By [**36**], minimal points are dense in the measure center of map with the *almost specification property* (weaker than the specification property). So if we assume (X, f) is uniquely ergodic, then the measure center of (X, f) must be a minimal set.

By Proposition 2.3, the measure center of (X, f) is X, and thus X is a minimal set. Note that X is non-degenerate (stated at the beginning of the introduction). So by [**36**, Theorem 5.3], X contains a horseshoe (definition referring to [**36**]), which contradicts the minimality. □

2.2. *Levels of recurrence and statistical ω-limit sets.* Let us recall some equivalent statements of recurrence referring to [**20**, **27**, **66**, **68**] whose proofs are fundamental and standard. These statements reveal the close connection between points with different recurrent frequency and the support of measures 'generated' by the points.

PROPOSITION 2.5. [**27**] *For a dynamical system (X, f), let $x \in \mathrm{Rec}$. Then the following conditions are equivalent:*
(a) $x \in W$;
(b) $x \in C_x = S_\mu$ *for any* $\mu \in V_f(x)$;
(c) $S_\mu = \omega(f, x)$ *for any* $\mu \in V_f(x)$.

PROPOSITION 2.6. [**27**] *For a dynamical system (X, f), let $x \in \mathrm{Rec}$. Then the following conditions are equivalent:*
(a) $x \in \mathrm{QW}$;
(b) $x \in C_x$;
(c) $C_x = \omega(f, x)$.

A point x is called *quasi-generic* for some measure μ if there are two sequences of positive integers $\{a_k\}$, $\{b_k\}$ with $b_k > a_k$ and $b_k - a_k \to \infty$ such that

$$\lim_{k\to\infty} \frac{1}{b_k - a_k} \sum_{j=a_k}^{b_k-1} \delta_{f^j(x)} = \mu$$

in weak* topology. Let $V_f^*(x) = \{\mu \in \mathcal{M}_f(X) : x \text{ is quasi-generic for } \mu\}$. This concept is from [**26**, p. 65] and from there it is known that $V_f^*(x)$ is always non-empty, compact and connected. Obviously, $V_f(x) \subseteq V_f^*(x)$. Let $C_x^* := \overline{\bigcup_{m\in V_f^*(x)} S_m}$.

PROPOSITION 2.7. [**29**] *For a dynamical system (X, f), let $x \in \mathrm{Rec}$. Then the following conditions are equivalent:*
(a) $x \in \mathrm{BR}$;
(b) $x \in C_x^*$;
(c) $x \in \omega(f, x) = C_x^*$.

PROPOSITION 2.8. *Suppose (X, f) is a dynamical system.*
(a) *For any $x \in X$, $\omega_{\underline{d}}(x) = \bigcap_{\mu\in V_f(x)} S_\mu$.*
(b) *For any $x \in X$, $\omega_{\overline{d}}(x) = C_x \neq \emptyset$.*
(c) *For any $x \in X$, $\omega_{B_*}(x) = \bigcap_{\mu\in V_f^*(x)} S_\mu$. If $\omega_{B_*}(x) \neq \emptyset$, then $\omega_{B_*}(x)$ is minimal.*
(d) *For any $x \in X$, $\omega_{B^*}(x) = C_x^* \neq \emptyset$.*

Proof. The proofs of the four items in Proposition 2.8 are similar and ordinary. So, we only prove item (a). On the one hand, consider an arbitrary $y \in \omega_{\underline{d}}(x)$. For any $\mu \in V_f(x)$, there is a positive integer sequence $m_k \to \infty$ such that $\lim_{k\to\infty} \mathcal{E}_{m_k}(x) = \mu$. Then, for any $\varepsilon > 0$, one has

$$\begin{aligned}\mu(B(y, 2\varepsilon)) \geq \mu(\overline{B(y, \varepsilon)}) &\geq \limsup_{k\to\infty} \mathcal{E}_{m_k}(\overline{B(y, \varepsilon)}) \\ &= \limsup_{k\to\infty} \frac{1}{m_k} \sum_{i=0}^{m_k-1} \delta_{f^i x}(\overline{B(y, \varepsilon)}) \\ &\geq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(\overline{B(y, \varepsilon)}) > 0,\end{aligned}$$

which implies that $y \in S_\mu$. Thus, $\omega_{\underline{d}}(x) \subseteq \bigcap_{\mu\in V_f(x)} S_\mu$.

On the other hand, consider an arbitrary $y \in \bigcap_{\mu\in V_f(x)} S_\mu$. For any $\varepsilon > 0$, let $n_k \to \infty$ be a sequence such that

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i x}(B(y, \varepsilon)) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(B(y, \varepsilon)).$$

Choose a subsequence n_{k_l} of n_k such that $\lim_{l\to\infty} \mathcal{E}_{n_{k_l}}(x) = \tau$ for some $\tau \in V_f(x)$. Note that $y \in S_\tau$. Then

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}(B(y, \varepsilon)) &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i x}(B(y, \varepsilon)) \\ &= \lim_{l\to\infty} \frac{1}{n_{k_l}} \sum_{i=0}^{n_{k_l}-1} \delta_{f^i x}(B(y, \varepsilon)) \geq \tau(B(y, \varepsilon)) > 0,\end{aligned}$$

which implies $y \in \omega_{\underline{d}}(x)$. Thus, $\omega_{\underline{d}}(x) \supseteq \bigcap_{\mu\in V_f(x)} S_\mu$. □

3. *Proof of Theorem F*

One of our major ideas is motivated by Oprocha and Štefánková's results in [**43**] that there is an uncountable DC1-scrambled subset in X when the dynamical system (X, f) has the specification property. Before the proof, we introduce some basic facts and lemmas.

3.1. *Ergodic average.* If $r, s \in \mathbb{N}$, $r \leq s$, we set $[r, s] := \{j \in \mathbb{N} \mid r \leq j \leq s\}$, and the cardinality of a finite set Λ is denoted by $|\Lambda|$. We set

$$\langle f, \mu\rangle := \int_X f \, d\mu.$$

There exists a countable and separating set of continuous functions $\{f_1, f_2, \ldots\}$ with $0 \leq f_k(x) \leq 1$, and such that

$$d(\mu, \nu) := \sum_{k\geq 1} 2^{-k} |\langle f_k, \mu\rangle - \langle f_k, \nu\rangle|$$

defines a metric for the weak*-topology on $\mathcal{M}_f(X)$. We refer to [**49**] and use the metric on X as follows defined by Pfister and Sullivan:

$$d(x, y) := d(\delta_x, \delta_y),$$

which is equivalent to the original metric on X. Readers will find the benefits of using this metric in our proof later.

LEMMA 3.1. *For any $\varepsilon > 0$, $\delta > 0$, and any two sequences $\{x_i\}_{i=0}^{n-1}$, $\{y_i\}_{i=0}^{n-1}$ of X, if $d(x_i, y_i) < \varepsilon$ holds for any $i \in [0, n-1]$, then for any $J \subseteq \{0, 1, \ldots, n-1\}$ with $(n - |J|)/n < \delta$:*

(a) $d((1/n)\sum_{i=0}^{n-1}\delta_{x_i}, (1/n)\sum_{i=0}^{n-1}\delta_{y_i}) < \varepsilon.$;

(b) $d((1/n)\sum_{i=0}^{n-1}\delta_{x_i}, (1/|J|)\sum_{i\in J}\delta_{y_i}) < \varepsilon + 2\delta.$

Lemma 3.1 is easily verified and shows us that if any two orbits of x and y in finite steps are mostly close, then the two empirical measures induced by x, y are also close.

LEMMA 3.2. *Suppose that (X, f) is a dynamical system with the specification property. Let K be a connected non-empty compact subset of $\mathcal{M}_f(X)$ and $\mu \in K$. Then for any $\varepsilon > 0$ there exists a $N_\varepsilon^\mu \in \mathbb{N}$ such that, for any $\alpha \in K$, any $N > N_\varepsilon^\mu$ and any $M > N$, there is an $x \in X$ and $N^* > M$ such that:*

(a) $\mathcal{E}_n(x) \in B(\mu, \varepsilon)$, *for all $n \in [N_\varepsilon^\mu, N]$;*

(b) $\mathcal{E}_n(x) \in B(K, \varepsilon)$, *for all $n \in [N, N^*]$;*

(c) $\mathcal{E}_{N^*}(x) \in B(\alpha, \varepsilon)$.

Proof. For any fixed $\varepsilon > 0$, by Proposition 2.2, there exists $p^\mu \in X$ and $n^\mu \in \mathbb{N}$ such that $\mathcal{E}_n(p^\mu) \in B(\mu, \varepsilon/6)$ holds for any $n \geq n^\mu$. Set $N_\varepsilon^\mu := n^\mu$. We will prove that such N_ε^μ makes this lemma true. Note that K is connected, so for any $\alpha \in K$ we can find a sequence $\{\beta_1, \beta_2, \ldots, \beta_{m_\varepsilon}\} \subseteq K$ such that $d(\beta_{i+1}, \beta_i) < \varepsilon$, for all $i \in \{1, 2, \ldots, m_\varepsilon - 1\}$ and $\beta_1 = \mu$, $\beta_{m_\varepsilon} = \alpha$. By Proposition 2.2, for any $i \in \{2, \ldots, m_\varepsilon\}$, there exists $p^{\beta_i} \in X$ and $n^{\beta_i} \in \mathbb{N}$ such that $\mathcal{E}_n(p^{\beta_i}) \in B(\beta_i, \varepsilon/6)$ holds for any $n \geq n^{\beta_i}$. For any $N > N_\varepsilon^\mu$ and $M > N$, we choose $\{T_i\}_{i=1}^{2m_\varepsilon}$ with $T_i \in \mathbb{N}$ such that, for $i \in \{1, \ldots, m_\varepsilon - 1\}$,

$$T_1 = 0, \quad T_2 = N. \tag{3.1}$$

$$T_{2i+1} = T_{2i} + K_{\varepsilon/6} \quad \text{where } K_{\varepsilon/6} \text{ defined in } \textit{Definiton } 2.1. \tag{3.2}$$

$$\frac{\varepsilon}{12}(T_{2i} - T_{2i-1}) > n^{\beta_{i+1}}. \tag{3.3}$$

$$\frac{K_{\varepsilon/6} + T_{2i-1}}{T_{2i} - T_{2i-1}} < \frac{\varepsilon}{12}. \tag{3.4}$$

So far, we have fixed $\{T_i\}_{i=1}^{2m_\varepsilon - 1}$. We choose T_{2m_ε} large enough such that

$$T_{2m_\varepsilon} \geq \max\{M, T_{2m_\varepsilon - 1} + n^{\beta_{m_\varepsilon}}\}. \tag{3.5}$$

$$\frac{T_{2m_\varepsilon - 1}}{T_{2m_\varepsilon}} < \frac{\varepsilon}{12}. \tag{3.6}$$

By (3.2), we can use the specification property. So there is an $x \in X$ where x $\varepsilon/6$-*traces* x^* on $[T_1, T_2]$ and $\varepsilon/6$-*traces* p^{β_i} on $[T_{2i-1}, T_{2i}]$, for all $i \in \{2, \ldots, m_\varepsilon\}$. Now, we claim that such x and $N^* = T_{2m_\varepsilon}$ satisfy items (a)–(c). (a) and (c) are easy to check by (3.1), (3.5), (3.6) and Lemma 3.1. Here we check (b). If $n \in (T_{2i}, T_{2i+1})$ for some $i \in \{1, \ldots, m_\varepsilon - 1\}$, we have

$$\frac{n - T_{2i} + T_{2i-1}}{T_{2i} - T_{2i-1}} < \frac{\varepsilon}{12},$$

by (3.2), (3.4). So, by Lemma 3.1, we have

$$\begin{aligned} d(\mathcal{E}_n(x), \beta_i) &< d(\mathcal{E}_n(x), \mathcal{E}_{T_{2i}-T_{2i-1}}(p^{\beta_i})) + d(\mathcal{E}_{T_{2i}-T_{2i-1}}(p^{\beta_i}), \beta_i) \\ &< \frac{\varepsilon}{6} + 2 \cdot \frac{\varepsilon}{12} + \frac{\varepsilon}{6} \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{3.7}$$

If $n \in [T_{2i-1}, T_{2i}]$ for some $i \in \{2, 3, \ldots, m_\varepsilon\}$, we split this situation into the following two cases.

Case 1. $(n - T_{2i-1})/(T_{2i-2} - T_{2i-3}) < \varepsilon/12$. Then

$$d(\mathcal{E}_n(x), \beta_{i-1}) < \frac{\varepsilon}{6} + 2 \cdot \left(\frac{\varepsilon}{12} + \frac{\varepsilon}{12}\right) + \frac{\varepsilon}{6} = \frac{2\varepsilon}{3}, \tag{3.8}$$

by Lemma 3.1 and (3.4).

Case 2. $(n - T_{2i-1})/(T_{2i-2} - T_{2i-3}) \geq \varepsilon/12$. If so, we have $n - T_{2i-1} > n^{\beta_i}$ by (3.3), which implies $\mathcal{E}_{n-T_{2i-1}}(p^{\beta_i}) \in B(\beta_i, \varepsilon/6)$. We consider $d(\mathcal{E}_n(x), \beta_i)$ and $d(\mathcal{E}_n(x), \beta_{i-1})$:

$$\begin{aligned} d(\mathcal{E}_n(x), \beta_i) &= d\left(\frac{T_{2i-1}}{n}\mathcal{E}_{T_{2i-1}}(x) + \frac{n - T_{2i-1}}{n}\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i\right) \\ &\leq \frac{T_{2i-1}}{n} d(\mathcal{E}_{T_{2i-1}}(x), \beta_i) + \frac{n - T_{2i-1}}{n} d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i) \\ &\leq \frac{T_{2i-1}}{n} d(\mathcal{E}_{T_{2i-1}}(x), \beta_{i-1}) + \frac{T_{2i-1}}{n} d(\beta_{i-1}, \beta_i) \\ &\quad + \frac{n - T_{2i-1}}{n} d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i) \\ &< \frac{T_{2i-1}}{n}\left(\frac{\varepsilon}{6} + 2 \cdot \frac{\varepsilon}{12} + \frac{\varepsilon}{6}\right) + \frac{T_{2i-1}}{n}\varepsilon + \frac{n - T_{2i-1}}{n}\left(\frac{\varepsilon}{6} + \frac{\varepsilon}{6}\right) \\ &< \frac{\varepsilon}{2} + \frac{T_{2i-1}}{n}\varepsilon, \\ d(\mathcal{E}_n(x), \beta_{i-1}) &= d\left(\frac{T_{2i-1}}{n}\mathcal{E}_{T_{2i-1}}(x) + \frac{n - T_{2i-1}}{n}\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_{i-1}\right) \\ &\leq \frac{T_{2i-1}}{n} d(\mathcal{E}_{T_{2i-1}}(x), \beta_{i-1}) + \frac{n - T_{2i-1}}{n} d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_{i-1}) \\ &\leq \frac{T_{2i-1}}{n} d(\mathcal{E}_{T_{2i-1}}(x), \beta_{i-1}) + \frac{n - T_{2i-1}}{n} d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i) \\ &\quad + \frac{n - T_{2i-1}}{n} d(\beta_i, \beta_{i-1}) \\ &< \frac{T_{2i-1}}{n}\left(\frac{\varepsilon}{6} + 2 \cdot \frac{\varepsilon}{12} + \frac{\varepsilon}{6}\right) + \frac{n - T_{2i-1}}{n}\left(\frac{\varepsilon}{6} + \frac{\varepsilon}{6}\right) + \frac{n - T_{2i-1}}{n}\varepsilon \\ &< \frac{\varepsilon}{2} + \frac{n - T_{2i-1}}{n}\varepsilon. \end{aligned}$$

So,

$$\min\{d(\mathcal{E}_n(x), \beta_i), d(\mathcal{E}_n(x), \beta_{i-1})\} < \varepsilon. \tag{3.9}$$

With the combination of (3.7), (3.8) and (3.9), one has (b). □

LEMMA 3.3. *Suppose that (X, f) is a dynamical system with the specification property. Let K be a connected non-empty compact subset of $\mathcal{M}_f(X)$ and $\mu \in K$. Then for any $\varepsilon > 0$ there exists an $M_\varepsilon^\mu \in \mathbb{N}$ such that, for any $\alpha \in K$ and any $M > M_\varepsilon^\mu$, there exist $t_2 > t_1 > M$ and $x \in X$ such that:*

(a) $\mathcal{E}_n(x) \in B(\mu, \varepsilon)$, *for all* $n \in [M_\varepsilon^\mu, M]$;

(b) $\mathcal{E}_n(x) \in B(K, \varepsilon)$, *for all* $n \in [M, t_1]$;

(c) $\mathcal{E}_{t_1}(x) \in B(\alpha, \varepsilon)$;

(d) $\mathcal{E}_n(x) \in B(K, \varepsilon)$, *for all* $n \in [t_1, t_2]$;

(e) $\mathcal{E}_{t_2}(x) \in B(\mu, \varepsilon)$.

Proof. By Lemma 3.2, for $\varepsilon/3$, we obtain $N_{\varepsilon/3}^\mu$ and $N_{\varepsilon/3}^\alpha$ such that, for any $N_1 > N_{\varepsilon/3}^\mu$, there is an x_1 and N^* such that

$$N^* > \max\left\{N_1, \frac{K_{\varepsilon/3} + N_{\varepsilon/3}^\alpha}{\varepsilon/6}\right\}, \tag{3.10}$$
$$\mathcal{E}_n(x_1) \in B(\mu, \varepsilon/3) \quad \text{for all } n \in [N_{\varepsilon/3}^\mu, N_1];$$
$$\mathcal{E}_n(x_1) \in B(K, \varepsilon/3) \quad \text{for all } n \in [N_1, N^*];$$
$$\mathcal{E}_{N^*}(x_1) \in B(\alpha, \varepsilon/3),$$

and for

$$N_2 > \max\left\{N_{\varepsilon/3}^\alpha, \frac{N^* + K_{\varepsilon/3}}{\varepsilon/6}\right\}, \tag{3.11}$$

there exists $N^{**} > N_2$ and x_2 such that

$$\mathcal{E}_n(x_2) \in B(\alpha, \varepsilon/3) \quad \text{for all } n \in [N_{\varepsilon/3}^\alpha, N_2]; \tag{3.12}$$
$$\mathcal{E}_n(x_2) \in B(K, \varepsilon/3) \quad \text{for all } n \in [N_2, N^{**}];$$
$$\mathcal{E}_{N^{**}}(x_2) \in B(\mu, \varepsilon/3).$$

By the specification property, we can obtain an $x \in X$ such that x $\varepsilon/3$-*traces* x_1 on $[0, N^*]$ and $\varepsilon/3$-*traces* x_2 on $[N^* + K_{\varepsilon/3}, N^* + K_{\varepsilon/3} + N^{**}]$. Now we consider $\mathcal{E}_n(x)$, $n \in [N_{\varepsilon/3}^\mu, N^* + K_{\varepsilon/3} + N^{**}]$ and split it into the following cases.

Case 1. When $n \in [N_{\varepsilon/3}^\mu, N^*]$, we have $d(\mathcal{E}_n(x), \mathcal{E}_n(x_1)) < \varepsilon/3$. So

$$\mathcal{E}_n(x) \in B(\mu, \varepsilon) \quad \text{for all } n \in [N_{\varepsilon/3}^\mu, N_1];$$
$$\mathcal{E}_n(x) \in B(K, \varepsilon) \quad \text{for all } n \in [N_1, N^*];$$
$$\mathcal{E}_{N^*}(x) \in B(\alpha, \varepsilon).$$

Case 2. When $n \in [N^*, N^* + K_{\varepsilon/3} + N_{\varepsilon/3}^\alpha]$, we have $d(\mathcal{E}_n(x), \mathcal{E}_{N^*}(x_1)) < 2\varepsilon/3$ by (3.10) and Lemma 3.1. So $d(\mathcal{E}_n(x), \alpha) < \varepsilon$.

Case 3. When $n \in [N^* + K_{\varepsilon/3} + N_{\varepsilon/3}^\alpha, N_2]$,

$$\begin{aligned}
&d(\mathcal{E}_n(x), \alpha)\\
&= d\left(\frac{N^* + K_{\varepsilon/3}}{n}\mathcal{E}_{N^*+K_{\varepsilon/3}}(x) + \frac{n - N^* - K_{\varepsilon/3}}{n}\mathcal{E}_{n-N^*-K_{\varepsilon/3}}(f^{N^*+K_{\varepsilon/3}}x), \alpha\right)\\
&\leq \frac{N^* + K_{\varepsilon/3}}{n}d(\mathcal{E}_{N^*+K_{\varepsilon/3}}(x), \alpha) + \frac{n - N^* - K_{\varepsilon/3}}{n}d(\mathcal{E}_{n-N^*-K_{\varepsilon/3}}(f^{N^*+K_{\varepsilon/3}}x), \alpha).
\end{aligned}$$

Note that $n - N^* - K_{\varepsilon/3} \geq N^{\alpha}_{\varepsilon/3}$ and $n \leq N_2$, and then we have $d(\mathcal{E}_{n-N^*-K_{\varepsilon/3}}(f^{N^*+K_{\varepsilon/3}}x), \alpha) < \varepsilon$ by (3.12). So

$$d(\mathcal{E}_n(x), \alpha) < \frac{N^* + K_{\varepsilon/3}}{n}\varepsilon + \frac{n - N^* - K_{\varepsilon/3}}{n}\varepsilon = \varepsilon.$$

Case 4. When $n \in [N_2, N^{**}]$, note that $N^{**} > N_2 > (N^* + K_{\varepsilon/3})/\varepsilon/6$, so by Lemma 3.1 we have

$$d(\mathcal{E}_n(x), \mathcal{E}_{n-N^*-K_{\varepsilon/3}}(x_2)) < 2\varepsilon/3.$$

Thus

$$\mathcal{E}_n(x) \in B(K, \varepsilon) \quad \text{for all } n \in [N_2, N^{**}];$$
$$\mathcal{E}_{N^{**}}(x_2) \in B(\mu, \varepsilon).$$

Set $M^{\mu}_{\varepsilon} = N^{\mu}_{\varepsilon/3}$, $M = N_1$ $t_1 = N^*$ $t_2 = N^{**}$, and we finish the proof. □

LEMMA 3.4. *Suppose that (X, f) is a dynamical system with the specification property. Suppose there are $\mu_1, \mu_2 \in \mathcal{M}_f(X)$ such that G_{μ_1}, G_{μ_2} have distal pair (p_1, q_1), (p_2, q_2), respectively. Let*

$$\zeta = \min\{\inf\{d(f^i p_1, f^i q_1) | \ i \in \mathbb{N}\}, \inf\{d(f^i p_2, f^i q_2) | \ i \in \mathbb{N}\}\}.$$

Then, for any $\delta > 0$, any $0 < \varepsilon < \zeta$ and any $\theta \in [0, 1]$, there exist $x_1, x_2 \in X$ and $N \in \mathbb{N}$ such that, for any $n > N$,

(a) $\mathcal{E}_n(x_1) \in B(\theta\mu_1 + (1 - \theta)\mu_2, \varepsilon + \delta)$ *and* $\mathcal{E}_n(x_2) \in B(\theta\mu_1 + (1 - \theta)\mu_2, \varepsilon + \delta)$*;*

(b) $(|\{0 \leq i \leq n - 1 | d(f^i x_1, f^i x_2) < \zeta - \varepsilon\}|)/n < \delta$.

Proof. We will prove this lemma for the case when θ is rational. Then, the lemma naturally holds for any $\theta \in [0, 1]$ by the denseness of rational numbers. For any fixed $\delta > 0$, $0 < \varepsilon < \zeta$ and $\theta/(1 - \theta) = s/t$, where $s, t \in \mathbb{N}^+$, we can obtain an M_1 such that $\mathcal{E}_n(p_i) \in B(\mu_i, \varepsilon/2)$ and $\mathcal{E}_n(q_i) \in B(\mu_i, \varepsilon/2)$, $i = \{1, 2\}$, hold for any $n \geq M_1$. We choose M, $r \in \mathbb{N}^+$ such that

$$M > \max\left\{M_1, \frac{4K_{\varepsilon/2}}{\delta}\right\}, \tag{3.13}$$

$$r > \frac{4}{\delta}. \tag{3.14}$$

For any $k \geq 1$, by the specification property, we can obtain an x^k_1 such that, for any $j \in [0, k - 1]$, $i \in [0, s - 1]$, x^k_1 $\varepsilon/2$-*traces* p_1 on

$$[j(s + t)(M + K_{\varepsilon/2}) + i(M + K_{\varepsilon/2}), j(s + t)(M + K_{\varepsilon/2}) + (i + 1)M + iK_{\varepsilon/2}]$$

and, for any $j \in [0, k - 1]$, $i \in [s, s + t - 1]$, x^k_1 $\varepsilon/2$-*traces* p_2 on

$$[j(s + t)(M + K_{\varepsilon/2}) + i(M + K_{\varepsilon/2}), j(s + t)(M + K_{\varepsilon/2}) + (i + 1)M + iK_{\varepsilon/2}].$$

Also, we can obtain an x^k_2 such that, for any $j \in [0, k - 1]$, $i \in [0, s - 1]$, x^k_2 $\varepsilon/2$-*traces* q_1 on

$$[j(s + t)(M + K_{\varepsilon/2}) + i(M + K_{\varepsilon/2}), j(s + t)(M + K_{\varepsilon/2}) + (i + 1)M + iK_{\varepsilon/2}]$$

and for any $j \in [0, k-1]$, $i \in [s, s+t-1]$, x_2^k $\varepsilon/2$-*traces* q_2 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}].$$

We can assume that (take a subsequence if necessary) $x_1 = \lim_{k\to\infty} x_1^k$, $x_2 = \lim_{k\to\infty} x_2^k$. By the continuity of f, we have, for any $j \in \mathbb{N}$, $i \in [0, s-1]$, x_1 $\varepsilon/2$-*traces* p_1 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}]$$

and, for any $j \in \mathbb{N}$, $i \in [s, s+t-1]$, x_1 $\varepsilon/2$-*traces* p_2 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}].$$

Similarly, for any $j \in \mathbb{N}$, $i \in [0, s-1]$, x_2 $\varepsilon/2$-*traces* q_1 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}]$$

and, for any $j \in \mathbb{N}$, $i \in [s, s+t-1]$, x_2 $\varepsilon/2$-*traces* q_2 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}].$$

Set $N := r(s+t)(M+K_{\varepsilon/2})$. We will show that such N and x_1, x_2 satisfy (a) and (b). For any $n > N$, n lies in $[k(s+t)(M+K_{\varepsilon/2}), (k+1)(s+t)(M+K_{\varepsilon/2})]$ for some $k \ge r$. By (3.14) and Lemma 3.1, we have

$$d(\mathcal{E}_n(x_1), \mathcal{E}_{k(s+t)(M+K_{\varepsilon/2})}(x_1)) < \frac{\delta}{2};\quad d(\mathcal{E}_n(x_2), \mathcal{E}_{k(s+t)(M+K_{\varepsilon/2})}(x_2)) < \frac{\delta}{2}. \tag{3.15}$$

Note that, for any $j \in \mathbb{N}$, $i \in [0, s-1]$, x_1 $\varepsilon/2$-*traces* p_1 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}]$$

and, for any $j \in \mathbb{N}$, $i \in [s, s+t-1]$, x_1 $\varepsilon/2$-*traces* p_2 on

$$[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}),\ j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}].$$

We have

$$\begin{aligned}
&d(\mathcal{E}_{k(s+t)(M+K_{\varepsilon/2})}(x_1), \theta\mathcal{E}_M(p_1) + (1-\theta)\mathcal{E}_M(p_2))\\
&\quad\le d\bigg(\sum_{i=1}^{k} \frac{1}{k}\mathcal{E}_{(s+t)(M+K_{\varepsilon/2})}(f^{(i-1)(s+t)(M+K_{\varepsilon/2})}x_1), \theta\mathcal{E}_M(p_1) + (1-\theta)\mathcal{E}_M(p_2)\bigg)\\
&\quad\le \frac{1}{k}\sum_{i=1}^{k} d(\mathcal{E}_{(s+t)(M+K_{\varepsilon/2})}(f^{(i-1)(s+t)(M+K_{\varepsilon/2})}x_1), \theta\mathcal{E}_M(p_1) + (1-\theta)\mathcal{E}_M(p_2))\\
&\quad\le \frac{1}{k}\sum_{i=1}^{k}\bigg[d\bigg(\frac{s}{s+t}\mathcal{E}_{s(M+K_{\varepsilon/2})}(f^{(i-1)(s+t)(M+K_{\varepsilon/2})}x_1), \theta\mathcal{E}_M(p_1)\bigg)\\
&\qquad + d\bigg(\frac{t}{s+t}\mathcal{E}_{t(M+K_{\varepsilon/2})}(f^{[(i-1)(s+t)+s](M+K_{\varepsilon/2})}x_1), (1-\theta)\mathcal{E}_M(p_2)\bigg)\bigg]\\
&\quad< \frac{1}{k}\sum_{i=1}^{k}[\theta(\varepsilon/2+\delta/2) + (1-\theta)(\varepsilon/2+\delta/2)]\\
&\quad= \varepsilon/2 + \delta/2.
\end{aligned}$$

Combining with (3.15) and $\mathcal{E}_M(p_i) \in B(\mu_i, \varepsilon/2)$, we have $d(\mathcal{E}_n(x_1), \theta\mu_1 + (1-\theta\mu_2)) < \varepsilon + \delta$. Similarly, we can prove $d(\mathcal{E}_n(x_2), \theta\mu_1 + (1-\theta\mu_2)) < \varepsilon + \delta$. Hence (a) holds. Note that $\zeta = \min\{\inf\{d(f^i p_1, f^i q_1) \mid i \in \mathbb{N}\}, \inf\{d(f^i p_2, f^i q_2) \mid i \in \mathbb{N}\}\}$, so then we have

$$\frac{|\{i | d(f^i x_1, f^i x_2) < \zeta - \varepsilon\}|}{n} < \frac{1}{k} + \frac{K_{\varepsilon/2}}{M} < \delta.$$

Hence (b) holds. □

3.2. *Proof of Theorem F.* We assume that (p_1, q_1), (p_2, q_2) are the distal pairs of G_{μ_1}, G_{μ_2}, respectively, and $\min\{\inf\{d(f^i p_1, f^i q_1) \mid i \in \mathbb{N}\}, \inf\{d(f^i p_2, f^i q_2) \mid i \in \mathbb{N}\}\} = \zeta > 0$. For any non-empty open set U, we can fix an $\varepsilon > 0$ and a transitive point $z \in U$ such that $\overline{B(z, \varepsilon)} \subseteq U$, since transitive points are dense for systems with the specification property. Let $\varepsilon_i = \varepsilon/2^i$, $K_i = K_{\varepsilon_i}$ (cf. the definition of the specification property). Let $\delta_1 < 1$, $\delta_i = \delta_{i-1}/2$. By [**49**, p. 944], there exists a sequence $\{\alpha_1, \alpha_2, \ldots\} \subseteq K$ such that

$$\overline{\{\alpha_j : j \in \mathbb{N}^+, j > n\}} = K \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 3.4, for any $s \in \mathbb{N}^+$, we can obtain $x_1^{\varepsilon_s,\delta_s}$, $x_2^{\varepsilon_s,\delta_s}$ and $N^{\varepsilon_s,\delta_s}$ such that, for any $n \geq N^{\varepsilon_s,\delta_s}$,

$$\mathcal{E}_n(x_1^{\varepsilon_s,\delta_s}) \in B(\mu, \varepsilon_s + \delta_s), \quad \mathcal{E}_n(x_2^{\varepsilon_s,\delta_s}) \in B(\mu, \varepsilon_s + \delta_s), \tag{3.16}$$

$$\frac{|\{i \in [0, n-1] | d(f^i x_1^{\varepsilon_s,\delta_s}, f^i x_2^{\varepsilon_s,\delta_s}) < \zeta - \varepsilon\}|}{n} < \delta_s. \tag{3.17}$$

Also, for any $s \in \mathbb{N}^+$, we can obtain an $M_{\varepsilon_s}^{\mu}$ such that the result of Lemma 3.3 holds. Now, given an $\xi = (\xi_1, \xi_2, \ldots) \in \{1, 2\}^\infty$, we construct the x_ξ inductively.

Step 1. Construct x_{ξ_1}. We fix $T_1 = 2K_1$. By Lemma 3.3, for a large enough $M_1 > M_{\varepsilon_1}^{\mu}$ satisfying

$$\delta_1 M_1 > \max\{T_1 + 2K_1, N^{\varepsilon_1,\delta_1}\} \tag{3.18}$$

we can obtain an $x_{\varepsilon_1}^{\alpha_1}$ and $t_2^{\varepsilon_1,\alpha_1} > t_1^{\varepsilon_1,\alpha_1} > M_1$ such that

$$\begin{cases} \mathcal{E}_n(x_{\varepsilon_1}^{\alpha_1}) \in B(\mu, \varepsilon_1) & \text{for all } n \in [M_{\varepsilon_1}^{\mu}, M_1]; \\ \mathcal{E}_n(x_{\varepsilon_1}^{\alpha_1}) \in B(K, \varepsilon_1) & \text{for all } n \in [M_1, t_1^{\varepsilon_1,\alpha_1}]; \\ \mathcal{E}_{t_1^{\varepsilon_1,\alpha_1}}(x_{\varepsilon_1}^{\alpha_1}) \in B(\alpha_1, \varepsilon_1); \\ \mathcal{E}_n(x_{\varepsilon_1}^{\alpha_1}) \in B(K, \varepsilon_1) & \text{for all } n \in [t_1^{\varepsilon_1,\alpha_1}, t_2^{\varepsilon_1,\alpha_1}]; \\ \mathcal{E}_{t_2^{\varepsilon_1,\alpha_1}}(x_{\varepsilon_1}^{\alpha_1}) \in B(\mu, \varepsilon_1). \end{cases} \tag{3.19}$$

Set $T_{1\to 2} = T_1 + t_1^{\varepsilon_1,\alpha_1}$, $T_2 = T_1 + t_2^{\varepsilon_1,\alpha_1}$, $T_3 = T_2 + 2K_1$, T_4 large enough such that

$$\delta_1 T_4 > \max\{T_3 + 2K_2, M_{\varepsilon_2}^{\mu}\}, \quad T_4 - T_3 > N^{\varepsilon_1,\delta_1}. \tag{3.20}$$

By the specification property, we can obtain an x_{ξ_1} ε_1-*traces* $z, x_{\varepsilon_1}^{\alpha_1}, x_{\xi_1}^{\varepsilon_1,\delta_1}$ on $[0, 0]$, $[T_1, T_2]$, $[T_3, T_4]$, respectively.

Step k. Construct $x_{\xi_1\cdots\xi_k}$. If $x_{\xi_1\cdots\xi_{k-1}}$, $\{T_i\}_{i=1}^{2k(k-1)}$ and $\{T_{4i-3\to 4i-2}\}_{i=1}^{k(k-1)/2}$ have been defined, we construct $x_{\xi_1\cdots\xi_k}$ in the following way. For any $i\in\{1,2,\ldots,k\}$, let $T_{2k(k-1)+4i-2}$ and $T_{2k(k-1)+4i}$ be indefinite; $T_{2k(k-1)+4i-3}=T_{2k(k-1)+4i-4}+2K_k$ and $T_{2k(k-1)+4i-1}=T_{2k(k-1)+4i-2}+2K_k$. By Lemma 3.3, for a large enough $M_{(k(k-1)/2)+i}>M^{\mu}_{\varepsilon_k}$ satisfying

$$\delta_k M_{(k(k-1)/2)+i}>\max\{T_{2k(k-1)+4i-3}+2K_k,\ N^{\varepsilon_k,\delta_k}\}\tag{3.21}$$

we can obtain an $x^{\alpha_i}_{\varepsilon_k}$ and $t_2^{\varepsilon_k,\alpha_i}>t_1^{\varepsilon_k,\alpha_i}>M_{(k(k-1)/2)+i}$ such that

$$\begin{cases}\mathcal{E}_n(x^{\alpha_i}_{\varepsilon_k})\in B(\mu,\varepsilon_k) & \text{for all } n\in[M^{\mu}_{\varepsilon_k},M_{(k(k-1)/2)+i}];\\ \mathcal{E}_n(x^{\alpha_i}_{\varepsilon_k})\in B(K,\varepsilon_k) & \text{for all } n\in[M_{(k(k-1)/2)+i},t_1^{\varepsilon_k,\alpha_i}];\\ \mathcal{E}_{t_1^{\varepsilon_k,\alpha_i}}(x^{\alpha_i}_{\varepsilon_k})\in B(\alpha_i,\varepsilon_k);\\ \mathcal{E}_n(x^{\alpha_i}_{\varepsilon_k})\in B(K,\varepsilon_k) & \text{for all } n\in[t_1^{\varepsilon_k,\alpha_i},t_2^{\varepsilon_k,\alpha_i}];\\ \mathcal{E}_{t_2^{\varepsilon_k,\alpha_i}}(x^{\alpha_i}_{\varepsilon_k})\in B(\mu,\varepsilon_k).\end{cases}\tag{3.22}$$

Set $T_{2k(k-1)+4i-3\to 2k(k-1)+4i-2}=T_{2k(k-1)+4i-3}+t_1^{\varepsilon_k,\alpha_i}$, $T_{2k(k-1)+4i-2}=T_{2k(k-1)+4i-3}+t_2^{\varepsilon_k,\alpha_i}$. If $i<k$, we select $T_{2k(k-1)+4i}$ large enough such that

$$\delta_k T_{2k(k-1)+4i}>\max\{T_{2k(k-1)+4i-1}+2K_k,\ M^{\mu}_{\varepsilon_k}\},\tag{3.23}$$

$$T_{2k(k-1)+4i}-T_{2k(k-1)+4i-1}>N^{\varepsilon_k,\delta_k}.\tag{3.24}$$

If $i=k$, $T_{2k(k-1)+4i}$ is large enough such that

$$\delta_k T_{2k(k-1)+4i}>\max\{T_{2k(k-1)+4i-1}+2K_{k+1},\ M^{\mu}_{\varepsilon_{k+1}}\},\tag{3.25}$$

$$T_{2k(k-1)+4i}-T_{2k(k-1)+4i-1}>N^{\varepsilon_k,\delta_k}.\tag{3.26}$$

Hence, we have defined the $T_{2(k-1)k+1},\ldots,T_{2k(k+1)}$ and $T_{2k(k-1)+4i-3\to 2k(k-1)+4i-2}$ for all $i\in[1,k]$. By the specification property, we can obtain an $x_{\xi_1\cdots\xi_k}$ ε_k*-traces* $x_{\xi_1\cdots\xi_{k-1}}$, $f^{k-1}z$, $x^{\alpha_1}_{\varepsilon_k}$, $x^{\varepsilon_k,\delta_k}_{\xi_1}$, $x^{\alpha_2}_{\varepsilon_k}$, $x^{\varepsilon_k,\delta_k}_{\xi_2}$, $\ldots$, $x^{\alpha_k}_{\varepsilon_k}$, $x^{\varepsilon_k,\delta_k}_{\xi_k}$ on

$$\begin{gathered}[0,T_{2k(k-1)}],\\ [T_{2k(k-1)}+K_k,T_{2k(k-1)}+K_k],\\ [T_{2k(k-1)+1},T_{2k(k-1)+2}],\\ \cdots,\\ [T_{2k(k-1)+4k-1},T_{2k(k-1)+4k}],\end{gathered}$$

respectively. Obviously, $d(x_{\xi_1\cdots\xi_{k-1}},x_{\xi_1\cdots\xi_k})<\varepsilon_k$, so $\{x_{\xi_1\cdots\xi_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\overline{B(z,\varepsilon)}$ since $\sum_{i=k}^{+\infty}\varepsilon_i\le 2\varepsilon_k$. Denote the accumulation point of $\{x_{\xi_1\cdots\xi_k}\}_{k=1}^{\infty}$ by x_ξ, and it is easy to verify that x_ξ $2\varepsilon_k$*-traces* $f^{k-1}z$, $x^{\alpha_1}_{\varepsilon_k}$, $x^{\varepsilon_k,\delta_k}_{\xi_1}$, $x^{\alpha_2}_{\varepsilon_k}$, $x^{\varepsilon_k,\delta_k}_{\xi_2}$, $\ldots$, $x^{\alpha_k}_{\varepsilon_k}$, $x^{\varepsilon_k,\delta_k}_{\xi_k}$ on

$$\begin{gathered}[T_{2k(k-1)}+K_k,T_{2k(k-1)}+K_k],\\ [T_{2k(k-1)+1},T_{2k(k-1)+2}],\\ \cdots,\\ [T_{2k(k-1)+4k-1},T_{2k(k-1)+4k}],\end{gathered}$$

respectively, since $\sum_{i=k}^{+\infty}\varepsilon_i\le 2\varepsilon_k$. Note that $\mathrm{orb}(x_\xi,f)$ has a subsequence which shadows the orbit of the transitive point z more and more closely so we can conclude that x_ξ

is also a transitive point. Fix $\xi, \eta \in \{1,2\}^\infty$; we claim that $x_\xi \neq x_\eta$ and x_ξ, x_η is a DC1-scrambled pair if $\xi \neq \eta$. Suppose $\xi_s \neq \eta_s$ (implied by $\xi \neq \eta$), so then, for any $k \geq s$ x_ξ $2\varepsilon_k$-*traces* $x_{\xi_s}^{\varepsilon_k,\delta_k}$ on $[T_{2(k-1)k+4s-1}, T_{2(k-1)k+4s}]$ and x_η $2\varepsilon_k$-*traces* $x_{\eta_s}^{\varepsilon_k,\delta_k}$ on $[T_{2(k-1)k+4s-1}, T_{2(k-1)k+4s}]$. For any fixed $\kappa < \zeta$, we can get an $I_\kappa > s$ such that $\zeta - \kappa > 5\varepsilon_{I_\kappa}$. Note that, from (3.17),

$$\frac{|\{i \in [T_{2k(k-1)+4s-1}, T_{2k(k-1)+4s}]|\, d(f^i x_{\xi_s}^{\varepsilon_k,\delta_k}, f^i x_{\eta_s}^{\varepsilon_k,\delta_k}) < \zeta - \varepsilon_k\}|}{T_{2k(k-1)+4s} - T_{2k(k-1)+4s-1} + 1} < \delta_k < 1$$

holds for any $k \geq I_\kappa$. So

$$\frac{|\{i \in [T_{2k(k-1)+4s-1}, T_{2k(k-1)+4s}]|d(f^i x_\xi, f^i x_\eta) < \zeta - 5\varepsilon_k\}|}{T_{2k(k-1)+4s} - T_{2k(k-1)+4s-1} + 1} < \delta_k < 1$$

holds for any $k \geq I_\kappa$, which implies, for any $k \geq I_\kappa$, that there exists $t \in [T_{2(k-1)k+4s-1}, T_{2(k-1)k+4s}]$ such that $d(f^t x_\xi, f^t x_\eta) \geq \zeta - 5\varepsilon_k > \kappa$. Therefore, $x_\xi \neq x_\eta$ and $\{x_\xi\}_{\xi\in\{1,2\}^\infty}$ (denote by S) is an uncountable set. Meanwhile,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} |\{j \in [0, n-1] : d(f^j x_\xi, f^j x_\eta) < \kappa\}| \\
&\quad \leq \liminf_{k\geq I_\kappa, k\to\infty} \frac{1}{T_{2(k-1)k+4s}} |\{j \in [0, T_{2(k-1)k+4s} - 1] : d(f^j x_\xi, f^j x_\eta) < \kappa\}| \\
&\quad \leq \liminf_{k\geq I_\kappa, k\to\infty} \frac{T_{2(k-1)k+4s-1}}{T_{2(k-1)k+4s}} + \delta_k \\
&\quad \leq \liminf_{k\geq I_\kappa, k\to\infty} 2\delta_k = 0.
\end{aligned}$$

On the other hand, for any fixed $t > 0$, we can choose $k_t \in \mathbb{N}$ large enough such that $4\varepsilon_k < t$ holds for any $k \geq k_t$. Note that x_ξ and x_η are both $2\varepsilon_k$-*traces* $x_{\varepsilon_k}^{\alpha_1}$ on $[T_{2(k-1)k+1}, T_{2(k-1)k+2}]$. So

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} |\{j \in [0, n-1] : d(f^i x_\xi, f^i x_\eta) < t\}| \\
&\quad \geq \limsup_{n\to\infty} \frac{1}{n} |\{j \in [0, n-1] : d(f^j x_\xi, f^j x_\eta) < 4\varepsilon_{k_t}\}| \\
&\quad \geq \limsup_{k\geq k_t, k\to\infty} \frac{1}{T_{2(k-1)k+2}} |\{j \in [0, T_{2(k-1)k+2} - 1] : d(f^j x_\xi, f^j x_\eta) < 4\varepsilon_k\}| \\
&\quad \geq \limsup_{k\geq k_t, k\to\infty} \left(1 - \frac{T_{2(k-1)k+1}}{T_{2(k-1)k+2}}\right) \\
&\quad \geq \limsup_{k\geq k_t, k\to\infty} (1 - \delta_k) \\
&\quad = 1.
\end{aligned}$$

So far we have proved that $S = \{x_\xi\}_{\xi\in\{1,2\}^\infty} \subseteq \overline{B(z, \varepsilon)} \subseteq U$ is an uncountable DC1-scrambled set. To complete this proof, we need to check that $V_f(x_\xi) = K$ for any $\xi \in \{1,2\}^\infty$. On the one hand, for any fixed $s \in \mathbb{N}^+$, when $k \geq s$, note (3.21), $T_{2(k-1)k+4s-3\to 2(k-1)k+4s-2} - T_{2(k-1)k+4s-3} > M_{(k(k-1)/2)+s}$, and x_ξ $2\varepsilon_k$-*traces* $x_{\varepsilon_k}^{\alpha_s}$

on $[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-3\to 2(k-1)k+4s-2}]$, so we have

$$\begin{aligned}
&d(\mathcal{E}_{T_{2(k-1)k+4s-3\to 2(k-1)k+4s-2}}(x_\xi), \alpha_s)\\
&\quad\le d(\mathcal{E}_{T_{2(k-1)k+4s-3\to 2(k-1)k+4s-2}-T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}}x_\xi), \alpha_s) + 2\delta_k\\
&\quad\le d(\mathcal{E}_{T_{2(k-1)k+4s-2}-T_{2(k-1)k+4s-3}}(x_{\varepsilon_k}^{\alpha_s}), \alpha_s) + 2\varepsilon_k + 2\delta_k\\
&\quad\le \varepsilon_k + 2\varepsilon_k + 2\delta_k\\
&\quad= 3\varepsilon_k + 2\delta_k
\end{aligned}$$

by Lemma 3.1. Let $k\to\infty$; we have $\alpha_s \in V_f(x_\xi)$ for any $s\in\mathbb{N}^+$, which implies $K\subseteq V_f(x_\xi)$.

On the other hand, for any fixed $n\in\mathbb{N}^*$, we consider $\mathcal{E}_n(x_\xi)$. Obviously, there is a $k\in\mathbb{N}$ such that $n\in[T_{2(k-1)k+1}, T_{2k(k+1)}+2K_{k+1}]$. If n lies in $[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-2}+2K_k]$ for certain $s\in\{2,3,\ldots,k\}$,

$$\begin{aligned}
\mathcal{E}_n(x_\xi) &= \frac{T_{2(k-1)k+4s-3}}{n}\mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi)\\
&\quad+\frac{n-T_{2(k-1)k+4s-3}}{n}\mathcal{E}_{n-T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}}x_\xi).
\end{aligned}$$

Notice that $T_{2(k-1)k+4s-3} = T_{2(k-1)k+4(s-1)} + 2K_k$, x_ξ $2\varepsilon_k$*-traces* $x_{\xi_s}^{\varepsilon_k,\delta_k}$ on $[T_{2(k-1)k+4(s-1)-1}, T_{2(k-1)k+4(s-1)}]$ and (3.16), (3.23), so by Lemma 3.1, we have

$$\begin{aligned}
d(\mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi), \mu) &< d(\mathcal{E}_{T_{2(k-1)k+4(s-1)}-T_{2(k-1)k+4(s-1)-1}}(f^{T_{2(k-1)k+4(s-1)-1}}x_\xi), \mu) + 2\delta_k\\
&< d(\mathcal{E}_{T_{2(k-1)k+4(s-1)}-T_{2(k-1)k+4(s-1)-1}}(x_{\xi_s}^{\varepsilon_k,\delta_k}), \mu) + 2\varepsilon_k + 2\delta_k\\
&< \varepsilon_k + \delta_k + 2\varepsilon_k + 2\delta_k,
\end{aligned}$$

i.e.,

$$d(\mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi), \mu) < 3\varepsilon_k + 3\delta_k. \tag{3.27}$$

If $n\in[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-3}+M_{\varepsilon_k}^{\mu}]$, note that (3.21) and $M_{(2k(k-1)/2)+s} > M_{\varepsilon_k}^{\mu}$, then we have $d(\mathcal{E}_n(x_\xi), \mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi)) < 2\delta_k$ by Lemma 3.1. So,

$$d(\mathcal{E}_n(x_\xi), \mu) < 2\delta_k + 3\varepsilon_k + 3\delta_k = 3\varepsilon_k + 5\delta_k. \tag{3.28}$$

If $n\in[T_{2(k-1)k+4s-3}+M_{\varepsilon_k}^{\mu}, T_{2(k-1)k+4s-3}+M_{(2k(k-1)/2)+s}]$, by (3.22), one has

$$\begin{aligned}
d(\mathcal{E}_{n-T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}}x_\xi), \mu) &< d(\mathcal{E}_{n-T_{2(k-1)k+4s-3}}(x_{\varepsilon_k}^{\alpha_s}), \mu) + 2\varepsilon_k\\
&< \varepsilon_k + 2\varepsilon_k\\
&= 3\varepsilon_k.
\end{aligned}$$

Combining with (3.27), gives

$$d(\mathcal{E}_n(x_\xi), \mu) < 3\varepsilon_k + 3\delta_k. \tag{3.29}$$

If $n\in[T_{2(k-1)k+4s-3}+M_{(2k(k-1)/2)+s}, T_{2(k-1)k+4s-2}+2K_k]$, by (3.21) and Lemma 3.1, we have

$$d(\mathcal{E}_n(x_\xi), \mathcal{E}_{n-T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}}x_\xi)) < 2\delta_k. \tag{3.30}$$

Then $\mathcal{E}_n(x_\xi)\in B(K, \varepsilon_k+2\delta_k)$ by (3.22). So, when $n\in[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-2}+2K_k]$, $\mathcal{E}_n(x_\xi)\subseteq B(K, 3\varepsilon_k+5\delta_k)$. In other situations of the interval where n lies, we can also prove $\mathcal{E}_n(x_\xi)\subseteq B(K, 3\varepsilon_k+5\delta_k)$ with a little modification of the above method. When $n\to\infty$, forcing $k\to\infty$, $B(K, 3\varepsilon_k+5\delta_k)\to K$, and hence we have $\mathcal{E}_n(x_\xi) = K$. □

Remark 3.5. Theorem F only states the situation where K contains a measure μ which is the convex combination of two measures. Actually, with little modification, Theorem F also holds for any $K \subseteq \mathcal{M}_f(X)$ if K contains a measure μ which is the convex combination of finite measures. Here we omit it.

4. *Proof of Theorems D and E*

4.1. *Distal pair in minimal sets.*

LEMMA 4.1. *Given a dynamical system (X, f), suppose that $\mu \in \mathcal{M}_f^e(X)$, S_μ is non-degenerate and minimal. Then, G_μ has a distal pair.*

Proof. $S_\mu \cap G_\mu \neq \emptyset$ since $\mu \in \mathcal{M}_f^e(X)$. Let $p \in S_\mu \cap G_\mu$, so $f(p) \in S_\mu \cap G_\mu$. Assume that p, $f(p)$ are proximal; then $\omega_f(p)$ contains a fixed point, which implies $\omega_f(p)$ is either degenerate or non-minimal. Then S_μ is either degenerate or non-minimal since $\omega_f(p) \subseteq S_\mu$. □

PROPOSITION 4.2. *Suppose that X has at least 2 elements and (X, f) is a dynamical system with the specification property. Then*

$$\{\mu \in \mathcal{M}_f(X) | \mu \text{ is ergodic}, S_\mu \text{ is non-degenerate and minimal}\}$$

is dense in $\mathcal{M}_f(X)$ and, for any μ in such a set, G_μ has a distal pair.

Proof. By [**16**, Theorem 3], $G_\nu \neq \emptyset$ for any $\nu \in \mathcal{M}_f(X)$. Take $y \in G_\nu$. For any $\varepsilon > 0$, let m, x_1 and ε_1 in [**36**, Theorem 5,2] equal 1, y, $\varepsilon/3$, respectively. One can construct a closed and non-empty set Z which contains a minimal point q by [**36**, Theorem 5,2]. So $\omega(f, q) \subseteq Z$ and $\omega(f, q)$ is a minimal set. By Lemma 3.1,

$$V_f(z) \subseteq B(\nu, \varepsilon) \quad \text{for any } z \in Z. \tag{4.1}$$

Fix a $\mu \in \mathcal{M}_f^e(\omega(f, q))$, so then $S_\mu = \omega(f, q)$ and $S_\mu \cap G_\mu \neq \emptyset$. So, by (4.1), $\mu \in B(\nu, \varepsilon)$. Thus,

$$\{\mu \in \mathcal{M}_f(X) | \mu \text{ is ergodic}, S_\mu \text{ is minimal}\}$$

is dense in $\mathcal{M}_f(X)$. Here we claim that

$$\{\mu \in \mathcal{M}_f(X) | \mu \text{ is ergodic}, S_\mu \text{ is non-degenerate and minimal}\}$$

is also dense in $\mathcal{M}_f(X)$. If not, there will be an open set $U \subseteq \mathcal{M}_f(X)$ such that

$$\{\mu \in \mathcal{M}_f(X) | \mu \text{ is ergodic}, S_\mu \text{ is degenerate and minimal}\}$$

is dense in U, which implies that any measure in U can be approximated by the Dirac measure concentrating on a fixed point, i.e. for any $\mu \in U$, there is a sequence $\{x_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} \delta_{x_i} = \mu$. Without loss of generality, we can assume that $\lim_{i\to\infty} x_i = x$. Then for any continuous function f on X,

$$\int f\, d\mu = \lim_{i\to\infty} \int f\, d\delta_{x_i} = \lim_{i\to\infty} f(x_i) = f(x) = \int f\, d\delta_x.$$

So, $\mu = \delta_x$, which means measures in U are all Dirac measures, which contradicts Proposition 2.3. Thus, the contradiction and Lemma 4.1 complete this proof. □

LEMMA 4.3. *Suppose that a subset $B' \subseteq \mathcal{M}_f(X)$ is dense in $\mathcal{M}_f(X)$. If there is an invariant measure μ with full support, then $\overline{\bigcup_{\omega \in B'} S_\omega} = X$.*

Proof. By assumption, there is a sequence of invariant measures $\mu_i \in B'$ converging to μ. Then $1 = \limsup_{i\to\infty} \mu_i(\overline{\bigcup_{\omega\in B'} S_\omega}) \le \mu(\overline{\bigcup_{\omega\in B'} S_\omega})$. It follows that $X = S_\mu \subseteq \overline{\bigcup_{\omega\in B'} S_\omega}$. □

PROPOSITION 4.4. *Suppose that (X, f) is a dynamical system with the specification property. Then $x \in$ Trans implies $x \in$ BR.*

Proof. From [**26**, Proposition 3.9] we know that for a point x_0 and an ergodic measure $\mu_0 \in \mathcal{M}_f(\omega(f, x_0))$, x_0 is quasi-generic for μ_0. So if $x \in$ Trans, $\mathcal{M}_f(\omega(f, x)) = \mathcal{M}_f(X)$. By Propositions 2.2, 2.3 and Lemma 4.3,

$$C_x^* = X. \tag{4.2}$$

By Proposition 2.7, the proof is completed. □

4.2. *Proof of Theorem D.* For any $\mu_1, \mu_2 \in \mathcal{M}_f(X)$, we define

$$\mathrm{conv}\{\mu_1, \mu_2\} = \{\theta\mu_1 + (1-\theta)\mu_2 \mid \theta \in [0, 1]\}.$$

Proof of Item (1). If $\mathrm{Int}(L_\varphi) \ne \emptyset$, then there exist $\lambda_1, \lambda_2 \in \mathcal{M}_f(X)$ such that $\int \varphi\, d\lambda_1 \ne \int \varphi\, d\lambda_2$. Note that the measures satisfying Proposition 4.2 and measures with full support are both dense in $\mathcal{M}_f(X)$. Then we can choose μ_1, μ_2 satisfying Proposition 4.2 and μ with full support such that $\int \varphi\, d\mu_1 \ne \int \varphi\, d\mu_2 \ne \int \varphi\, d\mu$. Obviously, $S_{\mu_1} \cup S_{\mu_2} \ne X$ since S_{μ_1}, S_{μ_2} are minimal. Let

$$K_1 := \mathrm{conv}\{\mu_1, \mu_2\};$$
$$K_2 := \mathrm{conv}\{\mu_1, \mu\}.$$

One can observe that $G_{K_i} \subseteq I_\varphi(f)$, $i \in \{1, 2\}$. Applying Theorem F to K_i, $i \in \{1, 2\}$, for any open set U, there is an uncountable scrambled set $S_i \subseteq G_{K_i} \cap U \cap \mathrm{Trans}$. By Propositions 4.4 and 2.6(c), we have $G_{K_1} \cap \mathrm{Trans} \subseteq I_\varphi(f) \cap (\mathrm{BR} \setminus \mathrm{QW})$. By Propositions 2.5(c) and 2.6(c), we have $G_{K_2} \cap \mathrm{Trans} \subseteq I_\varphi(f) \cap (\mathrm{QW} \setminus W)$. □

Proof of Item (2). If $\mathrm{Int}(L_\varphi) \ne \emptyset$, then, for any $a \in \mathrm{Int}(L_\varphi)$, there exist $\lambda_1, \lambda_2 \in \mathcal{M}_f(X)$ such that $\int \varphi\, d\lambda_1 < a < \int \varphi\, d\lambda_2$. Then we can take μ_1, μ_2, μ_3 satisfying Proposition 4.2 with $\int \varphi\, d\mu_1 < \int \varphi\, d\mu_2 < a < \int \varphi\, d\mu_3$. By Proposition 2.3, we can take ν_1, ν_2 with full support and $\int \varphi\, d\nu_1 < a < \int \varphi\, d\nu_2$. Now, we can choose proper $\theta_1, \theta_2, \theta_3 \in (0, 1)$ such that

$$\begin{aligned}\theta_1 \int \varphi\, d\mu_1 + (1-\theta_1) \int \varphi\, d\mu_3 &= \theta_2 \int \varphi\, d\mu_2 + (1-\theta_2) \int \varphi\, d\mu_3 \\ &= \theta_3 \int \varphi\, d\nu_1 + (1-\theta_3) \int \varphi\, d\nu_2 = a.\end{aligned}$$

Set $\rho_1 = \theta_1\mu_1 + (1-\theta_1)\mu_3$, $\rho_2 = \theta_2\mu_2 + (1-\theta_2)\mu_3$, $\rho_3 = \theta_3\nu_1 + (1-\theta_3)\nu_2$. Let

$$K_1 := \mathrm{conv}\{\rho_1, \rho_2\};$$
$$K_2 := \mathrm{conv}\{\rho_1, \rho_3\}.$$

One can observe that $G_{K_i} \subseteq R_\varphi(a)$, $i \in \{1, 2\}$. Based on the discussion in the proof of item (1), the proof is complete. □

Remark 4.5. If $a \in L_\varphi \setminus \mathrm{Int}(L_\varphi)$, Theorem D may not be true even for Li–Yorke chaos. For example, if the dynamical system (X, f) is a full shift of two symbols (which satisfies the specification property), taking $\mathrm{orb}(p, f)$, $\mathrm{orb}(q, f)$ to be two different periodic orbits with period ≥ 2 and letting φ be a continuous function such that $\varphi|_{\mathrm{orb}(p,f)} = 0$, $\varphi|_{\mathrm{orb}(q,f)} = 1$ and for any $x \in X \setminus (\mathrm{orb}(p, f) \cup \mathrm{orb}(q, f))$, $0 < \varphi(x) < 1$. In this case, $L_\phi = [0, 1]$. Let μ_p, μ_q denote the periodic measures supported on the orbit of p, q, respectively. It is not difficult to check that $G_{\mu_p} \cap \mathrm{Trans} \subseteq R_\phi(0) \cap \mathrm{Trans} \subseteq \mathrm{BR} \setminus \mathrm{QW}$ and $G_{\mu_q} \cap \mathrm{Trans} \subseteq R_\phi(1) \cap \mathrm{Trans} \subseteq \mathrm{BR} \setminus \mathrm{QW}$. So $R_\varphi(0) \cap \mathrm{Trans} \cap (\mathrm{QW} \setminus W) = \emptyset$ and $R_\varphi(1) \cap \mathrm{Trans} \cap (\mathrm{QW} \setminus W) = \emptyset$. So most cases cannot have any kind of chaotic behavior with respect to $R_\phi(0) \cap \mathrm{Trans}$ and $R_\phi(1) \cap \mathrm{Trans}$. By Theorem F, G_{μ_p}, G_{μ_q} all contain uncountable DC1-scrambled subsets and so do $R_\phi(0) \cap \mathrm{Trans}$ and $R_\phi(1) \cap \mathrm{Trans}$. However, $R_\phi(0)$ and $R_\phi(1)$ has zero topological entropy by (1.1). In particular, this implies that there exists an uncountable DC1-scrambled set with zero topological entropy.

4.3. *Proof of Theorem E.* Take μ_1, μ_2, satisfying Proposition 4.2. Let μ be a measure with full support and take $\nu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Let

$$\begin{aligned} K_2 &:= \{\mu_1\}; \\ K_3 &:= \mathrm{conv}\{\mu_1, \mu\} \cup \mathrm{conv}\{\mu_1, \mu_2\}; \\ K_4 &:= \mathrm{conv}\{\mu_1, \mu\}; \\ K_5 &:= \mathrm{conv}\{\mu_1, \mu_2\}; \\ K_6 &:= \mathrm{conv}\{\mu_1, \nu\}. \end{aligned}$$

Applying Theorem F to K_i, $i \in \{2, 3, 4, 5, 6\}$, for any open set U, there is an uncountable scrambled set $S_i \subseteq G_{K_i} \cap U \cap \mathrm{Trans}$. By Propositions 4.4, 2.8(d) and (4.2), $\omega_{B^*}(x) = X$. Since the dynamical systems with the specification property are not minimal but minimal points are dense, for any $x \in \mathrm{Trans}$, $\omega_{B_*}(x) = \emptyset$. Then, one can verify that $\{x \in \mathrm{Rec} |\ x \text{ satisfies Case (i)}\}$, $i = 2, 3, 4, 5, 6$, contains an uncountable DC1-scrambled subset S_i in Trans. Here, we omit the proof of the left part of Theorem E since it is similar to the proof of Theorem D. The major argument is that the density of measures satisfy Proposition 4.2 and the measures with full support. □

5. *Applications*

5.1. *Examples with specification.* It is known from [**12**] that any topologically mixing interval map satisfies the specification property. For example, [**30**] showed that there exists a set of parameter values $\Lambda \subseteq [0, 4]$ of positive Lebesgue measure such that if $\lambda \in \Lambda$, then the logistic map $f_\lambda(x) = \lambda x(1 - x)$ is topological mixing.

Moreover, maps satisfying the specification property include the mixing subshift of finite type, mixing sofic subshift, topological mixing uniformly hyperbolic systems and the time-1 map of the geodesic flow of compact connected negative curvature manifolds; for example, see [**54**, **60**]. So, all the results of Theorems A–F are all suitable for such systems.

5.2. *Examples without specification.* Now, we use our theorem on a type of subshift which may not have the specification property. Before proceeding, we need some preparation.

For any finite alphabet A, the *full symbolic space* is the set $A^{\mathbb{Z}} = \{\cdots x_{-1}x_0x_1\cdots : x_i \in A\}$, which is viewed as a compact topological space with the discrete product topology. The set $A^{\mathbb{N}_+} = \{x_1x_2\cdots : x_i \in A\}$ is called the *one-side full symbolic space*. The *shift action* on the *one-side full symbolic space* is defined by

$$\sigma : A^{\mathbb{N}_+} \to A^{\mathbb{N}_+}, \quad x_1x_2\cdots \mapsto x_2x_3\cdots .$$

$(A^{\mathbb{N}_+}, \sigma)$ forms a dynamical system under the discrete product topology which we called a shift. A closed subset $X \subseteq A^{\mathbb{N}_+}$ is called a *subshift* if it is invariant under the shift action σ. $\mathbf{w} \in A^n \triangleq \{x_1x_2\cdots x_n : x_i \in A\}$ is a *word* of *subshift* X if there is an $x \in X$ and $k \in \mathbb{N}$ such that $\mathbf{w} = x_kx_{k+1}\cdots x_{k+n-1}$. Here we call n the length of $\mathbf{w}$, denoted by $|\mathbf{w}|$. The *language* of a subshift X, denoted by $\mathcal{L}(X)$, is the set of all words of X. Denote $\mathcal{L}_n(X) \triangleq \mathcal{L}(X) \bigcap A^n$, i.e., the set of all the words of X with length n.

Here we present one type of *subshift*, β-shift, basically referring to [**48**, **51**, **56**]. It is worth mentioning that from [**12**] the set of parameters of β for which the specification property holds is dense in $(1, +\infty)$ but has Lebesgue zero measure.

Let $\beta > 1$ be a real number. We denote by $[x]$ and $\{x\}$ the integer and fractional parts of the real number x. Consider the β-*transformation* $f_\beta : [0, 1) \to [0, 1)$ given by

$$f_\beta(x) = \beta x \pmod 1.$$

For $\beta \notin \mathbb{N}$, let $b = [\beta]$ and for $\beta \in \mathbb{N}$, let $b = \beta - 1$. Then, we split the interval $[0, 1)$ into a $b + 1$ partition as below:

$$J_0 = \left[0, \frac{1}{\beta}\right), \quad J_1 = \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \ldots, J_1 = \left[\frac{b}{\beta}, 1\right).$$

For $x \in [0, 1)$, let $i(x, \beta) = (i_n(x, \beta))_1^\infty$ be the sequence given by $i_n(x, \beta) = j$ when $f^{n-1}x \in J_j$. We call $i(x, \beta)$ the greedy β-*expansion* of x and we have

$$x = \sum_{n=1}^{\infty} i_n(x, \beta)\beta^{-n}.$$

We call (Σ_β, σ) the β-shift, where σ is the shift map and Σ_β is the closure of $\{i(x, \beta)\}_{x\in[0,1)}$ in $\prod_{i=1}^{\infty}\{0, 1, \ldots, b\}$.

From the discussion above, we can define the greedy β-*expansion* of 1, denoted by $i(1, \beta)$. Parry showed that the set of sequences which belong to Σ_β can be characterized as

$$\omega \in \Sigma_\beta \Leftrightarrow f^k(\omega) \leq i(1, \beta) \quad \text{for all } k \geq 1,$$

where $\leq$ is taken in the lexicographic ordering [**45**]. By the definition of Σ_β above, $\Sigma_{\beta_1} \subsetneq \Sigma_{\beta_2}$ for $\beta_1 < \beta_2$ [**45**].

LEMMA 5.1. *For the β-shift, there exists an increasing sequence $\{\Sigma_\beta^n\}$ of compact σ-invariant subsets of Σ_β with the following properties:*

(a) *each $\{\Sigma_\beta^n\}$ is a sofic shift and has the specification property;*

(b) *for any $\mu \in \mathcal{M}_f(\Sigma_\beta)$, and any neighborhood U of μ in $\mathcal{M}_f(\Sigma_\beta)$, there exists an $n \geq 1$ and $\mu' \in \mathcal{M}_f^e(\Sigma_\beta^n) \cap U$.*

Lemma 5.1 is a particular case of [**15**, Proposition 3.6]. The reader can refer to [**15**] for the details of the proof. The lemma above shows us that to figure out the irregular set for the whole space(Σ_β), it is sufficient to study the irregular set for certain asymptotic 'horseshoe-like' (Σ_β^n) of the whole space.

THEOREM 5.2. *For any $\beta > 1$ and (Σ_β, σ), suppose φ is a continuous function on Σ_β. Then:*

(a) *there exist uncountable* DC1*-scrambled subsets in* $\mathrm{QW} \setminus W$ *and* $\mathrm{BR} \setminus \mathrm{QW}$*;*

(b) *if $I_\varphi(\sigma) \neq \emptyset$, then there exist uncountable* DC1*-scrambled subsets in $I_\varphi(\sigma) \cap (\mathrm{QW} \setminus W)$ and $I_\varphi(\sigma) \cap (\mathrm{BR} \setminus \mathrm{QW})$;*

(c) *if $\mathrm{Int}(L_\varphi) \neq \emptyset$, then for any $a \in \mathrm{Int}(L_\varphi)$, there exist uncountable* DC1*-scrambled subsets in $R_\varphi(a) \cap (\mathrm{QW} \setminus W)$ and $R_\varphi(a) \cap (\mathrm{BR} \setminus \mathrm{QW})$;*

(d) *there exist uncountable* DC1*-scrambled subsets in $R_\varphi \cap (\mathrm{QW} \setminus W)$ and $R_\varphi \cap (\mathrm{BR} \setminus \mathrm{QW})$.*

Proof. (a) Referring to [**56**], we have that $\{\beta \in (1, +\infty) \mid (\Sigma_\beta, \sigma)$ has the specification property$\}$ is dense in $(1, +\infty)$. Then, for any $\beta > 1$, we can find an $\alpha < \beta$ such that (Σ_α, σ) has the specification property. By Theorem C, for (Σ_α, σ), $\mathrm{QW}' \setminus W'$ and $\mathrm{BR}' \setminus \mathrm{QW}'$ of Σ_α both have an uncountable DC1-scrambled subset. It is easy to see that $\mathrm{QW}' \setminus W'$ and $\mathrm{BR}' \setminus \mathrm{QW}'$ of Σ_α are the subsets of $\mathrm{QW} \setminus W$ and $\mathrm{BR} \setminus \mathrm{QW}$ of Σ_β, respectively, since Σ_α is σ-*invariant* as a subset of Σ_β. Then item (a) has been proved.

(b) If $I_\varphi(\sigma) \neq \emptyset$, there exist $\lambda_1, \lambda_2 \in \mathcal{M}_\sigma(\Sigma_\beta)$ such that $\int \varphi \, d\lambda_1 \neq \int \varphi \, d\lambda_2$. By Lemma 5.1, we have (Σ_β^n, σ) which has the specification property and $\mu_1, \mu_2 \in \mathcal{M}_\sigma(\Sigma_\beta^n)$ such that $\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2$. By Theorem D, for (Σ_β^n, σ), $I_\varphi(\sigma) \cap (\mathrm{QW}' \setminus W')$ and $I_\varphi(\sigma) \cap (\mathrm{BR}' \setminus \mathrm{QW}')$ of Σ_β^n both have an uncountable DC1-scrambled subset. Like the analysis in the proof of item (a), we complete the proof.

(c) If $\mathrm{Int}(L_\varphi) \neq \emptyset$, then for any $a \in \mathrm{Int}(L_\varphi)$, there exist λ_1, λ_2 such that $\int \varphi \, d\lambda_1 < a < \int \varphi \, d\mu_2$. By Lemma 5.1, we have (Σ_β^n, σ) which has the specification property and $\mu_1, \mu_2 \in \mathcal{M}_f(\Sigma_\beta^n)$ such that $\int \varphi \, d\mu_1 < a < \int \varphi \, d\mu_2$. By Theorem D, for (Σ_β^n, σ), $R_\varphi(a) \cap (\mathrm{QW}' \setminus W')$ and $R_\varphi(a) \cap (\mathrm{BR}' \setminus \mathrm{QW}')$ of Σ_β^n both have an uncountable DC1-scrambled subset. Like the analysis in the proof of item (a), we complete the proof.

(d) If $\mathrm{Int}(L_\varphi) \neq \emptyset$, item (d) is from item (c). Otherwise, $R_\varphi = X$ so that item (d) is from item (a). □

6. *Comments and questions*

6.1. *Weakly almost periodic points.* The reason why we cannot analyse whether there is an uncountable DC1-scrambled set in W by our method is that we did not find a measure μ with full support and G_μ has a distal pair. For a point $x \in W \cap \mathrm{Trans}$, we can observe that x must be an element of the generic point of a measure with full support. But Theorem F does not cover this situation.

THEOREM 6.1. *Suppose that (X, f) is a dynamical system with the specification property. If, for any invariant measure μ with full support, G_μ has a distal pair, then:*

(1) *there is an uncountable* DC1*-scrambled set $S \subseteq W \cap \mathrm{Trans}$;*

(2) *if φ is a continuous function on X and $I_\varphi(f) \neq \emptyset$, there is an uncountable* DC1*-scrambled set $S \subseteq W \cap \mathrm{Trans} \cap I_\varphi(f)$;*

(3) *if φ is a continuous function on X and* $\mathrm{Int}(L_\varphi) \neq \emptyset$*, then, for any $a \in L_\varphi$, there is an uncountable* DC1*-scrambled set $S \subseteq W \cap \mathrm{Trans} \cap R_\varphi(a)$;*
(4) *for any continuous function φ on X, there is an uncountable* DC1*-scrambled set $S \subseteq W \cap \mathrm{Trans} \cap R_\varphi$.*

Remark 6.2. The set of points with Case (1) restricted on a recurrent set coincides with the set of $W \setminus \mathrm{AP}$. For systems with the specification property, note that $W \cap \mathrm{Trans} \subseteq W \setminus \mathrm{AP}$ so that the above result can be stated for the set of points with Case (1) restricted on the recurrent set or $W \setminus \mathrm{AP}$.

Remark 6.3. For a transitive dynamical system (X, f) without periodic points with period m, it is easy to check that, for any $x \in \mathrm{Trans}$, $(x, f^m x)$ must be a distal pair. This implies that, for any invariant measure μ (not necessarily with full support), $G_\mu \cap \mathrm{Trans}$ has a distal pair. So Theorem 6.1 is suitable for systems with the specification property but without periodic points with period m for some m. In particular, it applies to mixing subshifts of finite type without periodic points with period m for some m. For example, it can be a subshift of finite type defined by a graph with two distinct cycles of length $m+1$ and $m+2$, starting from the same vertex. For such dynamical systems, Theorem F holds for any non-empty compact connected set K, since G_μ has a distal pair for any μ in K.

Proof. Let μ be an invariant measure with full support.
(1) Take $K = \{\mu\}$. Then one can use Proposition 2.5 and Theorem F to give the proof.
(2) By Proposition 2.3, one can choose an invariant measure μ' with full support such that $\int \varphi\, d\mu \neq \int \varphi\, d\mu'$. Take $K = \mathrm{conv}\{\mu, \mu'\}$. Then one can use Proposition 2.5 and Theorem F to give the proof.
(3) If $\int \varphi\, d\mu = a$, take $\omega = \mu$. Otherwise, by Proposition 2.3, one can choose an invariant measure μ' with full support such that $\int \varphi\, d\mu' < a < \int \varphi\, d\mu$ or $\int \varphi\, d\mu < a < \int \varphi\, d\mu'$. Take suitable $\theta \in (0, 1)$ such that $\omega = \theta\mu + (1-\theta)\mu'$ and $\int \varphi\, d\omega = a$. In this case take $K = \{\omega\}$. One can use Proposition 2.5 and Theorem F to give the proof.
(4) If $\mathrm{Int}(L_\varphi) \neq \emptyset$, item (4) is from item (3). Otherwise, $R_\varphi = X$ so that item (4) is from item (1). □

6.2. *Minimal points.* For minimal points, it is still unknown whether DC1 appears but here we point out that DC2 appears. In fact, by [**36**, Theorem 5.3], a dynamical system (X, f) with the specification property contains a horseshoe, and therefore also contains a minimal subsystem with positive entropy. So DC2 appears by [**22**].

From [**12**], the set of parameters of β for which the specification property holds is dense in $(1, +\infty)$ but has Lebesgue zero measure. However, every β shift has almost the specification property by [**49**]. Thus DC2 appears in the minimal set for all β shifts.

Let $C(M)$ be the set of continuous maps on a compact manifold M and $H(M)$ be the set of homeomorphisms on M. Recall that C^0, generic $f \in H(M)$ (or $f \in C(M)$) has the shadowing property and infinite topological entropy (see [**35**] and [**33**, **34**], respectively). Thus, DC2 appears in the minimal set for C^0 generic dynamical systems.

6.3. *Zhou and Feng's question.* There is an open problem in [**67**] by Zhou and Feng concerning the set V:

$$V := \{x \in \mathrm{QW} \setminus W | \exists\, \mu \in V_f(x) \text{ s.t. } S_\mu = C_x\} \neq \emptyset?$$

It has been solved positively by constructing examples, see [**27**, **43**], etc. From [**63**], for a certain class of dynamical systems (including topological mixing subshifts of finite type, all β-shifts, systems restricted on mixing locally maximal hyperbolic sets), V is not only non-empty but also is a dense G_δ subset and has full topological entropy. Here, we give an answer from the perspective of distributional chaos.

THEOREM 6.4. *Suppose that (X, f) is a dynamical system with the specification property. Then, there is an uncountable* DC1*-scrambled subset in the set $\{x \in \mathrm{QW} \setminus W | \exists\, \mu \in V_f(x)$ s.t. $S_\mu = C_x\}$.*

Proof. Let μ satisfy Proposition 4.2 and ν be a measure with full support. Let $K = \mathrm{conv}\{\mu, \nu\}$. Applying Theorem F to K, there is an uncountable DC1-scrambled subset $S \subseteq G_K \cap \mathrm{Trans}$. By Propositions 2.5(c) and 2.6(c), $S \subseteq \{x \in \mathrm{QW} \setminus W | \exists\, \mu \in V_f(x)$ s.t. $S_\mu = C_x\}$. □

6.4. *Regular points.* Recall that $\mathrm{QR} = \bigcup_{\mu \in \mathcal{M}_f(X)} G_\mu$ and the points in QR are called *quasiregular* points of f in [**17**]. Now, we start to recall the concept of regular point (see [**44**]). A point $x \in \mathrm{QR}$ is called a *point of density* if $\mu_x(U) > 0$ for every open set $U \subseteq X$ containing x where μ_x is the single measure in $V_f(x)$. Let $\mathrm{QR}_d(f)$ (QR_d briefly) denote the set of all points of density in QR and, for convenience in the present paper, QR_d is called the density set. It is easy to check that, for any $x \in \mathrm{QR}$,

$$x \in \mathrm{QR}_d \Leftrightarrow x \in S_{\mu_x}. \tag{6.1}$$

Thus

$$\mathrm{QR}_d = \bigcup_{\mu \in \mathcal{M}_f(X)} (G_\mu \cap S_\mu). \tag{6.2}$$

Let $\mathrm{QR}_{\mathrm{erg}} := \cup_{\nu \in \mathcal{M}_f^e(X)} G_\nu$. In [**44**], the point in $\mathrm{QR}_{\mathrm{erg}}$ is called transitive, but, in the present paper, 'transitive point' means that its orbit is dense in the whole space X. To avoid confusion, in this paper, points in $\mathrm{QR}_{\mathrm{erg}}$ are called *ergodic-transitive* and the set $\mathrm{QR}_{\mathrm{erg}}$ is called the ergodic-transitive set. A point $x \in X$ is called *regular* if it belongs to the set $R(f) = \mathrm{QR}_d \cap \mathrm{QR}_{\mathrm{erg}}$ (called the regular set). We note that

$$R(f) = \bigcup_{\mu \in \mathcal{M}_f^e(X)} (G_\mu \cap S_\mu) \subseteq \mathrm{QR}_d \cup \mathrm{QR}_{\mathrm{erg}} \subseteq \mathrm{QR}. \tag{6.3}$$

By the Birkhoff ergodic theorem and the ergodic decomposition theorem, $R(f)$ has totally full measure (see [**44**] for a proof) and so does $\mathrm{QR}_{\mathrm{erg}}$, QR_d and QR.

THEOREM 6.5. *Suppose that (X, f) is a dynamical system with the specification property. Then $\mathrm{QR} \setminus (\mathrm{QR}_d \cup \mathrm{QR}_{\mathrm{erg}})$ and $\mathrm{QR}_{\mathrm{erg}} \setminus R(f)$ both have an uncountable* DC1*-scrambled subset.*

Proof. Let $\mu_1, \mu_2 \in \mathcal{M}_f(X)$ satisfy Proposition 4.2. Let $\nu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Then $\nu \in \mathcal{M}_f(X) \setminus \mathcal{M}_f^e(X)$. Let

$$K_1 := \{\nu\};$$
$$K_2 := \{\mu_1\}.$$

Let $U_1 \subseteq X \setminus (S_{\mu_1} \cup S_{\mu_2})$ be an open set. Applying Theorem F to U_1, K_1, $\mathrm{QR} \setminus (\mathrm{QR}_d \cup \mathrm{QR}_{\mathrm{erg}})$ has an uncountable DC1-scrambled subset. Let $U_2 \subseteq X \setminus S_{\mu_1}$ be an open set. Applying Theorem F to U_2, K_2, $\mathrm{QR}_{\mathrm{erg}} \setminus R(f)$ has an uncountable DC1-scrambled subset. □

Remark 6.6. Like the analysis in Remark 6.3, if we assume that the dynamical system (X, f) does not contain periodic points with period m, then there are $\nu_1 \in \mathcal{M}_f(X) \setminus \mathcal{M}_f^e(X)$ and $\nu_2 \in \mathcal{M}_f^e(X)$ such that G_{ν_1}, G_{ν_2} both have a distal pair and $S_{\nu_1} = S_{\nu_2} = X$. Let $K_1 = \{\nu_1\}$; $K_2 = \{\nu_2\}$. Then by Theorem F, $\mathrm{QR}_d \setminus R(f)$ and $R(f)$ both have an uncountable DC1-scrambled subset.

Acknowledgements. The authors would like to thank the anonymous referees for their useful suggestions and comments to improve our paper. X. Tian is the corresponding author and supported by the National Natural Science Foundation of China (grant no. 11671093).

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