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Morse-Novikov cohomology of almost nonnegatively curved manifolds

Xiaoyang Chen

School of Mathematical Sciences, Institute for Advanced Study, Tongji University, Shanghai, China

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ABSTRACT

Let M^n be a closed manifold of almost nonnegative sectional curvature and nonzero first de Rham cohomology group. Using a topological argument, we show that the Morse-Novikov cohomology group $H^p(M^n, \theta)$ vanishes for any p and $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$. Based on a new integral formula, we also show that a similar result holds for a closed manifold of almost nonnegative Ricci curvature under the additional assumption that its curvature operator is uniformly bounded from below.

1. Introduction

Let M^n be a smooth manifold and θ a real valued closed one form on M^n. Set $\Omega^p(M^n)$ the space of real smooth p-forms and define $d_\theta : \Omega^p(M^n) \to \Omega^{p+1}(M^n)$ as $d_\theta\alpha = d\alpha + \theta \wedge \alpha$ for $\alpha \in \Omega^p(M^n)$. Then we have a complex

$$\cdots \to \Omega^{p-1}(M^n) \xrightarrow{d_\theta} \Omega^p(M^n) \xrightarrow{d_\theta} \Omega^{p+1}(M^n) \to \cdots$$

E-mail address: xychen100@tongji.edu.cn.

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whose cohomology $H^p(M,\theta) = H^p(\Omega^*(M^n), d_\theta)$ is called the p-th Morse-Novikov cohomology group of M^n with respect to θ. If θ_1, θ_2 are two representatives in the cohomology class $[\theta]$, then $H^p(M,\theta_1) \simeq H^p(M,\theta_2)$. Hence $H^p(M,\theta)$ only depends on the de Rham cohomology class of θ. This cohomology shares many properties with the ordinary de Rham cohomology. See [11,18,19] and section 2 for details.

If $[\theta] = 0$, the Novikov cohomology group $H^p(M,\theta)$ is isomorphic to the de Rham cohomology group $H^p_{dR}(M^n)$. There are lots of work relating de Rham cohomology to curvature properties of Riemannian manifolds. See for example [20]. In particular, a celebrated theorem of Gromov says that the Betti number of a closed manifold with almost nonnegative sectional curvature is bounded above by a constant depending only the dimension of the manifold [10]. Here we say that a Riemannian manifold M^n has almost nonnegative sectional curvature if it admits a sequence of Riemannian metrics g_i such that

$$sec(g_i) \geq -\frac{1}{i}$$

$$D(g_i) \leq 1,$$

where $sec(g_i)$ is the sectional curvature of g_i and $D(g_i)$ is the diameter of g_i.

However, there are quite few work discussing the relationship between Morse-Novikov cohomology $H^p(M,\theta)$ and curvature when $[\theta] \neq 0$. This paper is trying to make an attempt towards this direction. Our first result is the following theorem.

Theorem 1.1. *Let M^n be a closed Riemannian manifold of almost nonnegative sectional curvature and nonzero first de Rham cohomology group, then the Morse-Novikov cohomology $H^p(M,\theta) = 0$ for any p (including $p = 0$) and any $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$.*

From the work in [8,15], we know that a closed Riemannian manifold M^n of almost nonnegative sectional curvature is an almost nilpotent space. Namely, there is a finite cover of M^n, denoted by $\hat{M}^n$, such that $\pi_1(\hat{M}^n)$ is a nilpotent group that operates nilpotently on $\pi_k(\hat{M}^n)$ for every $k \geq 2$. Recall that an action by automorphisms of a group G on an abelian group V is called nilpotent if V admits a finite sequence of G-invariant subgroups

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_k = 0$$

such that the induced action of G on V_j/V_{j+1} is trivial for any j. Now Theorem 1.1 is a consequence of the following topological result.

Theorem 1.2. *Let M^n be a smooth manifold with nonzero first de Rham cohomology group. If M^n is an almost nilpotent space, then the Morse-Novikov cohomology $H^p(M,\theta) = 0$ for any p and any $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$.*

By Theorem 2.1 in section 2, we see that $\sum_{p=0}^{n}(-1)^p dim H^p(M^n,\theta)$ is equal to the Euler characteristic number of M^n. Hence we get the following

Corollary 1.3. *Let M^n be a smooth manifold with nonzero first de Rham cohomology group. If M^n is an almost nilpotent space, then its Euler characteristic number vanishes.*

Corollary 1.3 implies that a closed Riemannian manifold of almost nonnegative sectional curvature and nonzero first de Rham cohomology group has vanishing Euler characteristic number. This result has previously been proved by Yamaguchi in [23] using collapsing theory.

Theorem 1.1 fails for closed manifolds of almost nonnegative Ricci curvature. Recall that a Riemannian manifold has almost nonnegative Ricci curvature if it admits a sequence of Riemannian metrics g_i such that

$$Ric(g_i) \geq -\frac{n-1}{i}$$
$$D(g_i) \leq 1,$$

where $Ric(g_i)$ is the Ricci curvature of g_i and $D(g_i)$ is the diameter of g_i. Let M^4 be the manifold performing surgery along a meridian curve in T^4, i.e., removing a tubular neighborhood of the curve and attaching a copy of $D^2 \times S^2$. In [1], Anderson showed that M^4 admits a sequence of Riemannian metrics g_i such that

$$|Ric(g_i)| \leq \frac{n-1}{i}$$
$$D(g_i) \leq 1.$$

Moreover, its fundamental group is isomorphic to $\mathbb{Z}^3$ and its Euler characteristic number is nonzero. For any $[\theta] \in H^1_{dR}(M^4)$, $[\theta] \neq 0$, by Theorem 2.1 and Theorem 2.3 in section 2, we get $H^p(M^4,\theta) = 0$ for $p \neq 2$ and $H^2(M^4,\theta) \neq 0$. However, the sectional curvature of g_i constructed by Anderson can *not* have a uniform lower bound. Otherwise, there will be also an upper bound of the sectional curvature and by Theorem 1 in [22], M^4 will fiber over S^1 which is impossible by the construction. In particular, the curvature operator of g_i can *not* have a uniform lower bound. By the following Theorem 1.4 and its Corollary 1.5, M^4 in fact can *not* admit a sequence of Riemannian metrics g_i of almost nonnegative Ricci curvature with curvature operator uniformly bounded from below.

Theorem 1.4. *Let M^n be a closed Riemannian manifold with nonzero first de Rham cohomology group and admits a sequence of Riemannian metrics g_i such that*

$$Ric(g_i) \geq -\frac{n-1}{i}$$
$$D(g_i) \leq 1.$$

If the curvature operator of g_i is uniformly bounded from below by $-Id$, then for any $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$, there exists some $t \in \mathbb{R}, t \neq 0$ such that $H^p(M, t\theta) = 0$ for any p, where $H^p(M, t\theta)$ is the Morse-Novikov cohomology group with respect to $t\theta$.

Corollary 1.5. *Let M^n be a closed Riemannian manifold with nonzero first de Rham cohomology group. If M^n admits a sequence of Riemannian metrics of almost nonnegative Ricci curvature with curvature operator uniformly bounded from below, then the Euler characteristic number of M^n vanishes.*

For a closed Riemannian manifold (M^n, g_i) with almost nonnegative Ricci curvature and nonzero first de Rham cohomology group, Theorem 1 in [22] also implies that M^n has vanishing Euler number if the sectional curvature of g_i has a uniform upper bound. Theorem 1 in [22] was proved by collapsing theory and is quite different from our method in this paper.

It has been known that the fundamental group of a closed manifold M of almost nonnegative Ricci curvature is almost nilpotent [3,16]. By Theorem 2.3, $H^1(M, \theta) = 0$ for any $[\theta] \neq 0$ without any additional assumption. See [14] for related work on noncollapsed almost Ricci flat manifolds.

Finally, we point out that for a closed Riemannian manifold M of nonnegative Ricci curvature and nonzero first de Rham cohomology group, then the Morse-Novikov cohomology $H^p(M, \theta) = 0$ for any p and $[\theta] \in H^1_{dR}(M), [\theta] \neq 0$. In fact, by Cheeger-Gromoll splitting theorem [5], a finite cover of M is diffeomorphic to a product of a torus and a simply connected manifold. By Theorem 2.1 and Example 1, we see that the Morse-Novikov cohomology $H^p(M, \theta) = 0$ for any p and $[\theta] \in H^1_{dR}(M), [\theta] \neq 0$.

The proof of Theorem 1.2 is based on Cartan-Leray spectral sequence on equivalent homology [4]. By passing to a finite cover, we can assume that M^n is a nilpotent space. The closed one form θ on M^n defines a linear representation of the fundamental group of M^n:

$$\rho : \pi_1(M^n) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{\int_\gamma \theta}.$$

The representation ρ defines a complex rank one local system $\mathbb{C}_\rho$ over M^n [6]. We denote by $H^p(M^n, \mathbb{C}_\rho)$ the p-th cohomology group of M^n with coefficients in this local system. By Theorem 2.2 in section 2, for any p, we have

$$H^p(M^n, \theta) \simeq H^p(M^n, \mathbb{C}_\rho).$$

By duality, it suffices to show that $H_p(M^n, \mathbb{C}_\rho) = 0$, where $H_p(M^n, \mathbb{C}_\rho)$ is the p-th homology group of M^n with coefficients in this local system. Let $\pi : \widetilde{M^n} \to M^n$ be the universal cover of M^n. By the Cartan-Leray spectral sequence [4], we have

$$E^2_{kl} = H_k(\pi_1(M^n), H_l(\widetilde{M^n}, \mathbb{C})) \Rightarrow H_{k+l}(M^n, \mathbb{C}_\rho), \tag{1.1}$$

where $H_k(\pi_1(M^n), H_l(\widetilde{M^n}, \mathbb{C}))$ is the k-th homology group of $\pi_1(M^n)$ with coefficients in the $\pi_1(M^n)$-module $H_l(\widetilde{M^n}, \mathbb{C})$. Then we prove by induction to get the vanishing of $H_p(M^n, \mathbb{C}_\rho)$.

The proof of Theorem 1.4 is based on Hodge theory of Morse-Novikov cohomology. Let d^* be the formal L^2 adjoint of d with respect to the Riemannian metric g_i. We can also define an operator d_θ^* as the formal L^2 adjoint of d_θ with respect to g_i. Further, $\Delta_\theta = d_\theta d_\theta^* + d_\theta^* d_\theta$ is the corresponding Laplacian. These operators are lower-order perturbations of the corresponding operators in the usual Hodge-de Rham theory and therefore have much the same analytic properties. For example, the usual proof of the Hodge decomposition theorem goes through, and one obtains an orthogonal decomposition

$$\Omega^p(M^n) = \mathcal{H}^p(M^n) \oplus d_\theta(\Omega^{p-1}(M^n)) \oplus d_\theta^*(\Omega^{p+1}(M^n)),$$

where $\mathcal{H}^p(M^n)$ is the space of Δ_θ harmonic forms, which is isomorphic to $H^p(M^n, \theta)$.

By Hodge theory, for each i we can choose a harmonic form θ_i in the cohomology class $[\theta]$. Let $V(g_i)$ be the volume of (M^n, g_i), dV_i the volume form of g_i and X_i the dual vector field of θ_i defined by $g_i(X_i, Y) = \theta(Y)$. Set $t_i = (\frac{V(g_i)}{\int_{M^n} |X_i|^2 dV_i})^{1/2} > 0$. Choose a $\Delta_{t_i\theta_i}$ harmonic form α_i in $H^p(M^n, t_i\theta_i)$. The idea is to show that $\alpha_i \equiv 0$ for sufficiently large i, which relies on the following crucial integral inequality proved in Corollary 4.3.

$$\int\limits_{M^n} t_i^2 |X_i|^2 |\alpha_i|^2 dV_i \leq C_n \int\limits_{M^n} (t_i|\nabla X_i| + t_i|div(X_i)|)|\alpha_i|^2 dV_i \tag{1.2}$$

for some constant C_n depending only on n.

As $Ric(g_i) \geq -\frac{n-1}{i}$, applying Bochner formula to X_i, we get

$$\int\limits_{M^n} |\nabla X_i|^2 dV_i \leq \frac{n-1}{i} \int\limits_{M^n} |X_i|^2 dV_i. \tag{1.3}$$

Combining (1.2) and (1.3), for sufficiently large i we will show

$$\int\limits_{M^n} |\alpha_i|^2 dV_i \leq \frac{1}{2} \int\limits_{M^n} |\alpha_i|^2 dV_i.$$

Hence $\alpha_i \equiv 0$. See section 5 for details.

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2. Basic properties of Morse-Novikov cohomology

In this section we collect some basic properties of Morse-Novikov cohomology.

Theorem 2.1. *Let M^n be a compact n-dimensional manifold and θ a closed one form on M^n. Then:*
(1) If $\theta' = \theta + df, f \in C^\infty(M^n, \mathbb{R})$, then for any p, we have $H^p(M^n, \theta') \simeq H^p(M^n, \theta)$ and the isomorphism is given by the map $[\alpha] \mapsto [e^f \alpha]$;
(2) If $[\theta] \neq 0$ and M^n is connected and orientable, then $H^0(M^n, \theta)$ and $H^n(M^n, \theta)$ vanish. Moreover, the integration $\int : H^p(M^n, \theta) \times H^{n-p}(M^n, -\theta), (\alpha, \beta) \mapsto \int_{M^n} \alpha \wedge \beta$ induces an isomorphism $H^p(M^n, \theta) \simeq (H^{n-p}(M^n, -\theta))^*$.
(3) $\sum_{p=0}^n (-1)^p dim H^p(M^n, \theta)$ is equal to the Euler characteristic number of M^n;
(4) If N^d be a d-dimensional manifold and γ be a closed one form on N^d, then we have $H^k(M^n \times N^d, \pi_1^*\theta + \pi_2^*\gamma) \simeq \bigoplus_{p+q=k} H^p(M^n, \theta) \bigotimes H^q(N^d, \gamma)$, where $\pi_1 : M^n \times N^d \to M^n, \pi_2 : M^n \times N^d \to N^d$ are the projection maps.
(5) If $\pi : \widehat{M}^n \to M^n$ is a covering space with finite sheet, then $\pi^* : H^p(M^n, \theta) \to H^p(\widehat{M}^n, \pi^*\theta)$ is injective for any p.

Proof. See page 476-480 in [11] and Proposition 1.2 in [18] for the proof of parts 1-4. For part 5, by Theorem 2.2, we have

$$H^p(M^n, \theta) \simeq H^p(M^n, \mathbb{C}_\rho),$$

where $\mathbb{C}_\rho$ is the complex rank one local system defined by the linear representation

$$\rho : \pi_1(M^n) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{\int_\gamma \theta}$$

and $H^p(M^n, \mathbb{C}_\rho)$ is the p-th cohomology group of M^n with coefficients in this local system.

As $\pi : \widehat{M}^n \to M^n$ is a covering space with finite sheet, one can construct a transfer map (see e.g. [9,12]) $h : H^p(\widehat{M}^n, \pi^*\mathbb{C}_\rho) \to H^p(M^n, \mathbb{C}_\rho)$ such that $h\pi^* = kId$, where k is the degree of π. It follows that $\pi^* : H^p(M^n, \theta) \simeq H^p(M^n, \mathbb{C}_\rho) \to H^p(\widehat{M}^n, \pi^*\mathbb{C}_\rho) \simeq H^p(\widehat{M}^n, \pi^*\theta)$ is injective. □

As a corollary of Theorem 2.1, we get

Example 1. Let M^n be n-dimensional torus, then $H^p(M^n, \theta) = 0$ for any p and $[\theta] \neq 0$ by Theorem 2.1.

Let θ be a closed one form on M^n. Consider the following linear representation of the fundamental group of M^n:

$$\rho : \pi_1(M^n) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{\int_\gamma \theta}.$$

The representation ρ defines a complex rank one local system $\mathbb{C}_\rho$ over M^n [6]. We denote by $H^p(M^n, \mathbb{C}_\rho)$ the p-th cohomology group of M^n with coefficients in this local system.

Theorem 2.2. *$H^p(M^n, \theta) \simeq H^p(M^n, \mathbb{C}_\rho)$ for any p.*

Proof. The proof is contained in [19]. For the convenience of the reader, we provide the details here. Let $\pi : \widetilde{M}^n \to M^n$ be the universal cover of M^n. The cohomology groups $H^p(M^n, \mathbb{C}_\rho)$ are isomorphic to $H^p_\rho(\widetilde{M}^n)$, the cohomology groups of the complex $\Omega(\widetilde{M}^n, \rho)$, consisting of the ρ-equivariant differential forms on $\widetilde{M}^n$ relative to the usual differential (the proof is analogous to the sheaf-theoretic proof of de Rham's theorem). Let h be a function on $\widetilde{M}^n$ such that $dh = \pi^*\theta$. We give a mapping $F : \Omega^*(M^n) \to \Omega^*(\widetilde{M}^n, \rho)$ by the formula $F(w) = e^h \pi^* w$. It is easy to see that F is one-to-one and commutes with the differentials. Hence

$$H^p(M^n, \theta) \simeq H^p_\rho(\widetilde{M}^n) \simeq H^p(M^n, \mathbb{C}_\rho). \quad \square$$

Theorem 2.3. *Let M^n be a n-dimensional manifold and θ a closed one form on M^n. If the fundamental group of M^n has a finitely generated nilpotent subgroup of finite index, then $H^1(M^n, \theta) = H^{n-1}(M^n, \theta) = 0$ for any $[\theta] \neq 0$.*

Proof. Let $G \subseteq \pi_1(M^n)$ be a finitely generated nilpotent subgroup of finite index and $\pi : \widehat{M}^n \to M^n$ the covering space of M^n with $\pi_1(\widehat{M}^n) \simeq G$. The closed one form $\pi^*\theta$ defines a linear representation of G:

$$\rho : G = \pi_1(\widehat{M}^n) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{\int_\gamma \pi^*\theta}.$$

The representation ρ defines a complex rank one local system $\mathbb{C}_\rho$ over $\widehat{M}^n$. We denote by $H^p(\widehat{M}^n, \mathbb{C}_\rho)$ the p-th cohomology group of $\widehat{M}^n$ with coefficients in the local system $\mathbb{C}_\rho$. Let $K(G, 1)$ be the topological space such that $\pi_1(K(G, 1)) = G, \pi_i(K(G, 1)) = 0, i \geq 2$ and $\mathbb{L}_\rho$ the complex rank one local system over $K(G, 1)$ defined by ρ. Since the classifying map $\widehat{M}^n \to K(G, 1)$ induces over $\mathbb{Q}$ a cohomology isomorphism in degree one, we get

$$H^1(\widehat{M}^n, \mathbb{C}_\rho) \simeq H^1(K(G, 1), \mathbb{L}_\rho).$$

As $\pi : \widehat{M}^n \to M^n$ is a finite cover, $[\theta] \neq 0$ implies that $[\pi^*\theta] \neq 0$. Then $\mathbb{L}_\rho$ is a nontrivial local system over $K(G, 1)$. As G is a finitely generated nilpotent group, by Theorem 2.2 in [17], for any p, we have

$$H^p(K(G, 1), \mathbb{L}_\rho) = 0.$$

In particular,

$$H^1(\widehat{M}^n, \mathbb{C}_\rho) \simeq H^1(K(G, 1), \mathbb{L}_\rho) = 0.$$

By Theorem 2.1 and Theorem 2.2, we have

$$H^1(\widehat{M}^n, \pi^*\theta) = 0$$
$$H^1(M^n, \theta) = 0$$
$$H^{n-1}(M^n, \theta) \simeq H^1(M^n, -\theta) = 0. \quad \square$$

For a smooth manifold which is *not* an almost nilpotent space, its Morse-Novikov cohomology does not necessarily vanish as the following example shows.

Example 2. [15] Let $h : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ be defined by

$$h : (x, y) \to (xy, yxy).$$

This map is a diffeomorphism with inverse given by

$$h^{-1} : (u, v) \to (u^2v^{-1}, vu^{-1}).$$

Let M be the mapping torus of h. Then M has the structure of a fiber bundle:

$$\mathbb{S}^3 \times \mathbb{S}^3 \to M \to \mathbb{S}^1.$$

The induced map $h^{*,3}$ on $H^3_{dR}(\mathbb{S}^3 \times \mathbb{S}^3)$ is given by the matrix

$$A_h = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{2.1}$$

Notice that the eigenvalues of A_h are different from 1 in absolute value. Hence M^n is *not* an almost nilpotent space. Let λ be a eigenvalue of A_h with $\lambda = e^{-t}, t \neq 0, t \in \mathbb{R}$ and θ a generator of $H^1_{dR}(M)$. We claim that $H^3(M, t\theta) \neq 0$. To see this, observe that $t\theta$ defines a linear representation of the fundamental group of M:

$$\rho_t : \pi_1(M) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{t\int_\gamma \theta}.$$

The representation ρ_t defines a complex rank one local system $\mathbb{C}_{\rho_t}$ over M^n [6]. We denote by $H^p(M^n, \mathbb{C}_{\rho_t})$ the p-th cohomology group of M^n with coefficients in this local system. By Theorem 2.2 in section 2, for any p, we have

$$H^p(M, t\theta) \simeq H^p(M^n, \mathbb{C}_{\rho_t}).$$

On the other hand, by Wang's exact sequence in Proposition 6.4.8 in [6] page 212, we have

$$\dim_{\mathbb{C}} H^p(M^n, \mathbb{C}_{\rho_t}) = \dim_{\mathbb{C}} \ker(h^{*,p} - e^{-t}Id) + \dim_{\mathbb{C}} coker(h^{*,p-1} - e^{-t}Id),$$

where $h^{*,p} : H^p(\mathbb{S}^3 \times \mathbb{S}^3, \mathbb{C}) \to H^p(\mathbb{S}^3 \times \mathbb{S}^3, \mathbb{C})$ is the linear map induced by h. As e^{-t} is an eigenvalue of $h^{*,3}$, we see that $\dim_{\mathbb{C}} \ker(h^{*,3} - e^{-t} Id) > 0$ and $H^3(M, t\theta) \neq 0$.

3. Cartan-Leray spectral sequence

In this section we apply Cartan-Leray spectral sequence to prove Theorem 1.2. By passing to a finite cover, we can assume that M^n is a nilpotent space. The closed one form θ induces a linear representation of $G = \pi_1(M^n)$:

$$\rho : \pi_1(M^n) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{\int_\gamma \theta}.$$

By Theorem 2.2, for any p, we have

$$H^p(M^n, \theta) \simeq H^p(M^n, \mathbb{C}_\rho),$$

where $\mathbb{C}_\rho$ is the complex rank one local system over M^n defined by ρ. By duality, it suffices to prove the vanishing of $H_p(M^n, \mathbb{C}_\rho)$, which is the homology group of M^n with coefficients in the local system $\mathbb{C}_\rho$. Let $\widetilde{M}^n$ be the universal cover of M^n. The representation ρ together with the G action on $\widetilde{M}^n$ by deck transformation induces the diagonal action on $H_l(\widetilde{M}^n, \mathbb{C}) \simeq H_l(\widetilde{M}^n, \mathbb{Z}) \otimes \mathbb{C}$. By the Cartan-Leray spectral sequence (Theorem 7.9, page 173 in [4]), we have

$$E^2_{kl} = H_k(G, H_l(\widetilde{M}^n, \mathbb{C})) \Rightarrow H_{k+l}(M^n, \mathbb{C}_\rho),$$

where $H_k(G, H_l(\widetilde{M}^n, \mathbb{C}))$ is the k-th homology group of G with coefficients in the G-module $H_l(\widetilde{M}^n, \mathbb{C})$. See [4] for more details of homology of groups. For us, we only need the following long exact sequence (Proposition 6.1, page 71 in [4]).

Lemma 3.1. *For any short exact sequence* $0 \to M' \to M \to M'' \to 0$ *of* G*-modules, there is the following long exact sequence:*

$$\begin{aligned} \cdots \to H_i(G, M') \to H_i(G, M) \to H_i(G, M'') \to H_{i-1}(G, M') \to H_{i-1}(G, M) \to \cdots \\ \to H_1(G, M') \to H_1(G, M) \to H_1(G, M'') \to H_0(G, M') \to H_0(G, M) \\ \to H_0(G, M'') \to 0. \end{aligned}$$

As M^n is a nilpotent space, then $G = \pi_1(M^n)$ is a nilpotent group that operates nilpotently on $\pi_m(M^n)$ for every $m \geq 2$. By Lemma 2.18 in [13], G operates nilpotently on $H_l(\widetilde{M}^n, \mathbb{Z})$ for every l, that is $V = H_l(\widetilde{M}^n, \mathbb{Z})$ admits a finite sequence of G-invariant subgroups

$$V = V_0 \supseteq V_1 \supseteq \ldots V_k = 0$$

such that the induced action of G on V_j/V_{j+1} is trivial for any j. The representation ρ of G induces a diagonal action on $V_j \otimes \mathbb{C}$ and we have the following short exact sequence of G modules:

$$0 \to V_{j+1} \otimes \mathbb{C} \to V_j \otimes \mathbb{C} \to V_j/V_{j+1} \otimes \mathbb{C} \to 0.$$

We now prove $H_k(G, V_j \otimes \mathbb{C}) = 0$ for any j by induction. It is clear that $H_k(G, V_k \otimes \mathbb{C}) = H_k(G, 0) = 0$. As $[\theta] \neq 0$, we see that ρ is a nontrivial representation of G. By assumption, the induced action of G on V_j/V_{j+1} is trivial for any j. Then the diagonal action of G on $V_j/V_{j+1} \otimes \mathbb{C}$ is nontrivial. As G is a finitely generated nilpotent group, by Theorem 2.2 in [17], we get

$$H_k(G, V_j/V_{j+1} \otimes \mathbb{C}) = 0.$$

By Lemma 3.1 and induction, for any j, we get

$$H_k(G, V_j \otimes \mathbb{C}) = 0.$$

In particular,

$$H_k(G, H_l(\widetilde{M}^n, \mathbb{C})) = H_k(G, V_0 \otimes \mathbb{C}) = 0.$$

By the Cartan-Leray spectral sequence [4], we have

$$E^2_{kl} = H_k(G, H_l(\widetilde{M}^n, \mathbb{C})) \Rightarrow H_{k+l}(M^n, \mathbb{C}_\rho).$$

Hence for any $k, l \geq 0$, we have

$$H_{k+l}(M^n, \mathbb{C}_\rho) = 0.$$

Then we get $H^p(M^n, \theta) = 0$ for any p and $[\theta] \neq 0$.

4. An integral formula of Δ_θ harmonic forms

In section we derive an integral formula of Δ_θ harmonic forms which will be crucial in the proof of Theorem 1.4.

Let (M^n, g) be a closed Riemannian manifold and θ a closed real one form on M^n. Define $d_\theta : \Omega^p(M^n) \to \Omega^{p+1}(M^n)$ as $d_\theta \alpha = d\alpha + \theta \wedge \alpha$ for $\alpha \in \Omega^p(M^n)$. Let d^* be the formal L^2 adjoint of d with respect to g. We can also define an operator d^*_θ as the formal L^2 adjoint of d_θ with respect to g. Further, $\Delta_\theta = d_\theta d^*_\theta + d^*_\theta d_\theta$ is the corresponding Laplacian. These operators are lower-order perturbations of the corresponding operators in the usual Hodge-de Rham theory and therefore have much the same analytic properties.

For example, the usual proof of the Hodge decomposition theorem goes through, and one obtains an orthogonal decomposition

$$\Omega^p(M^n) = \mathcal{H}^p(M^n) \oplus d_\theta(\Omega^{p-1}(M^n)) \oplus d_\theta^*(\Omega^{p+1}(M^n)),$$

where $\mathcal{H}^p(M^n)$ is the space of Δ_θ harmonic forms, which is isomorphic to $H^p(M^n, \theta)$.

Let dV be the volume form of g and X the dual vector field of θ defined by $g(X, Y) = \theta(Y)$. Choose a Δ_θ harmonic form α in $H^p(M^n, \theta)$. Then

$$d_\theta \alpha = d\alpha + \theta \wedge \alpha = 0$$
$$d_\theta^* \alpha = d^* \alpha + i_X \alpha = 0.$$

The following integral formula and its Corollary 4.3 will be crucial in the proof of Theorem 1.4.

Theorem 4.1.

$$\int_{M^n} |X|^2 |\alpha|^2 dV = \frac{1}{2} \int_{M^n} \alpha \wedge [L_X, *]\alpha,$$

where $[L_X, *]\alpha = L_X * \alpha - *L_X \alpha$ and $L_X \alpha$ is the Lie derivative of α in the direction X.

Remark 4.2. When θ is exact and $X = \nabla f$ for some smooth function f on M^n, we believe that the integral formula in Theorem 4.1 is the same as [7]. It is also possible to adapt the method in [7] to prove Theorem 4.1. However, we present a different proof here.

Corollary 4.3.

$$\int_{M^n} |X|^2 |\alpha|^2 dV \le C_n \int_{M^n} (|\nabla X| + |div(X)|) |\alpha|^2 dV$$

for some constant C_n depending only on n.

Proof. The Riemannian metric g on M^n induces a linear map between TM^n and T^*M^n defined by

$$g : TM^n \to T^*M^n$$
$$< g(X), Y >= g(X, Y), \forall X, Y \in TM^n.$$

Let g^{-1} be the inverse of the above map g and h the endomorphism of the bundle $T^*M^n \to M^n$ by

$$h = L_X g \circ g^{-1}.$$

The derivation of the Grassmann algebra $\Lambda T^* M^n$ induced by h is denoted by $i(h)$. This is a linear map such that, if $\gamma \in T^* M^n$, then $i(h)(\gamma) = h(\gamma)$, and

$$i(h)(\omega_1 \wedge \omega_2) = (i(h)\omega_1) \wedge \omega_2 + \omega_1 \wedge (i(h)\omega_2) \tag{4.1}$$

for any $\omega_1, \omega_2 \in \Lambda T^* M^n$. The following formula is proved in [21].

$$[L_X, *]\omega = (i(h) - \frac{1}{2}Trh) * \omega \tag{4.2}$$

for any $\omega \in \Lambda T^* M^n$.

Let $div(X)$ be the divergence of X with respect to g. As

$$(L_X g)(Y, Z) = \ g(\nabla_Y X, Z) + \ g(Y, \nabla_Z X)$$

for all $Y, Z \in TM^n$, we see that $Trh = 2div(X)$. Then by Theorem 4.1, we get

$$\int_{M^n} |X|^2 |\alpha|^2 dV \leq C_n \int_{M^n} (|\nabla X| + |divX|)|\alpha|^2 dV$$

for some constant C_n depending only on n. $\square$

Now we prove Theorem 4.1. We firstly need the following lemmas.

Lemma 4.4. *For any p form ω, we have*

$$*i_X \omega = (-1)^{p-1} \theta \wedge *\omega, \tag{4.3}$$

where $*$ is the Hodge star operator with respect to g.

Proof. For any $p-1$ form ξ, we have

$$\begin{aligned} \int_{M^n} \xi \wedge *i_X \omega &= \int_{M^n} g(\xi, i_X \omega) dV \\ = \int_{M^n} g(\theta \wedge \xi, \omega) dV &= \int_{M^n} \theta \wedge \xi \wedge *\omega \\ &= (-1)^{p-1} \int_{M^n} \xi \wedge \theta \wedge *\omega. \end{aligned}$$

Hence

$$*i_X \omega = (-1)^{p-1} \theta \wedge *\omega. \quad \square$$

Lemma 4.5. *Let $\beta = *\alpha$, then*

$$d\beta - \theta \wedge \beta = 0.$$

Proof. As $d^*\alpha = (-1)^{n(p+1)+1} * d * \alpha$ and $d^*\alpha + i_X\alpha = 0$, we get

$$(-1)^{n(p+1)+1} * d * \alpha + i_X\alpha = 0.$$

Hence

$$(-1)^{n(p+1)+1} * *d * \alpha + *i_X\alpha = 0.$$

By Lemma 4.4, we have

$$*i_X\alpha = (-1)^{p-1}\theta \wedge *\alpha.$$

It follows that

$$(-1)^p d * \alpha + (-1)^{p-1}\theta \wedge *\alpha = 0$$

So

$$d\beta - \theta \wedge \beta = 0. \quad \square$$

Now we proceed to prove Theorem 4.1. As $d\alpha + \theta \wedge \alpha = 0$, we get

$$i_X d\alpha + i_X(\theta \wedge \alpha) = 0.$$

So

$$i_X d\alpha \wedge \beta + |X|^2\alpha \wedge \beta - \theta \wedge i_X\alpha \wedge \beta = 0. \tag{4.4}$$

On the other hand, as $d\beta - \theta \wedge \beta = 0$, we get

$$i_X d\beta - i_X(\theta \wedge \beta) = 0.$$

So

$$i_X d\beta \wedge \alpha - |X|^2\beta \wedge \alpha + \theta \wedge i_X\beta \wedge \alpha = 0.$$

Then

$$\alpha \wedge i_X d\beta - |X|^2\alpha \wedge \beta + (-1)^p\theta \wedge \alpha \wedge i_X\beta = 0. \tag{4.5}$$

By (4.4), (4.5), we get

$$-i_X d\alpha \wedge \beta + \alpha \wedge i_X d\beta - 2|X|^2 \alpha \wedge \beta + \theta \wedge i_X \alpha \wedge \beta + (-1)^p \theta \wedge \alpha \wedge i_X \beta = 0. \tag{4.6}$$

Combined with

$$\begin{aligned}\theta \wedge i_X \alpha \wedge \beta + (-1)^p \theta \wedge \alpha \wedge i_X \beta &= \theta \wedge i_X(\alpha \wedge \beta)\\ = |X|^2 \alpha \wedge \beta - i_X(\theta \wedge \alpha \wedge \beta) &= |X|^2 \alpha \wedge \beta,\end{aligned}$$

we get

$$-i_X d\alpha \wedge \beta + \alpha \wedge i_X d\beta = |X|^2 \alpha \wedge \beta. \tag{4.7}$$

Since

$$d(i_X \alpha \wedge \beta) = d i_X \alpha \wedge \beta + (-1)^{p-1} i_X \alpha \wedge d\beta,$$

we get

$$\int_{M^n} i_X \alpha \wedge d\beta = (-1)^p \int_{M^n} d i_X \alpha \wedge \beta. \tag{4.8}$$

On the other hand, we have

$$0 = i_X(\alpha \wedge d\beta) = i_X \alpha \wedge d\beta + (-1)^p \alpha \wedge i_X d\beta. \tag{4.9}$$

Combining (4.8), (4.9), we get

$$\int_{M^n} \alpha \wedge i_X d\beta = - \int_{M^n} d i_X \alpha \wedge \beta. \tag{4.10}$$

From (4.7), (4.10), we get

$$\begin{aligned}\int_{M^n} |X|^2 \alpha \wedge \beta &= - \int_{M^n} i_X d\alpha \wedge \beta - \int_{M^n} d i_X \alpha \wedge \beta = - \int_{M^n} L_X \alpha \wedge \beta\\ &= - \int_{M^n} L_X(\alpha \wedge \beta) + \int_{M^n} \alpha \wedge L_X \beta = \int_{M^n} \alpha \wedge L_X \beta.\end{aligned} \tag{4.11}$$

As $\beta = *\alpha$, we get

$$\int_{M^n} \alpha \wedge L_X \beta = \int_{M^n} \alpha \wedge L_X * \alpha = \int_{M^n} \alpha \wedge * L_X \alpha + \int_{M^n} \alpha \wedge [L_X, *]\alpha. \tag{4.12}$$

Moreover,

$$\int_{M^n} \alpha \wedge *L_X\alpha = \int_{M^n} L_X\alpha \wedge *\alpha$$
$$= \int_{M^n} L_X(\alpha \wedge *\alpha) - \int_{M^n} \alpha \wedge L_X * \alpha = - \int_{M^n} \alpha \wedge L_X * \alpha$$
$$= - \int_{M^n} \alpha \wedge *L_X\alpha - \int_{M^n} \alpha \wedge [L_X, *]\alpha.$$

Hence

$$\int_{M^n} \alpha \wedge *L_X\alpha = -\frac{1}{2} \int_{M^n} \alpha \wedge [L_X, *]\alpha. \tag{4.13}$$

By (4.11), (4.12), (4.13), we get

$$\int_{M^n} |X|^2|\alpha|^2 dV = \frac{1}{2} \int_{M^n} \alpha \wedge [L_X, *]\alpha.$$

5. Proof of Theorem 1.4

In this section we give a proof of Theorem 1.4. The proof is based on Corollary 4.3. Another crucial tool is the following Poincaré-Sobolev inequality ([2], page 397).

Theorem 5.1. *Let (M^n, g) be a closed smooth Riemannian manifold such that for some constant $b > 0$,*

$$r_{min}(g)D^2(g) \geq -(n-1)b^2,$$

where $D(g)$ is the diameter of g, $Ric(g)$ is the Ricci curvature of g and

$$r_{min}(g) = inf\{Ric(g)(u,u) : u \in TM, g(u,u) = 1\}.$$

Let $R = \frac{D(g)}{bC(b)}$, where $C(b)$ is the unique positive root of the equation

$$x \int_0^b (cht + xsht)^{n-1} dt = \int_0^\pi sin^{n-1} t dt.$$

Then for each $1 \leq p \leq \frac{nq}{n-q}, p < \infty$ and $f \in W^{1,q}(M^n)$, we have

$$\|f - \frac{1}{V(g)} \int_{M^n} f dV\|_p \leq S_{p,q} \|df\|_q$$

$$\|f\|_p \leq S_{p,q}\|df\|_q + V(g)^{1/p-1/q}\|f\|_q,$$

where $V(g)$ is the volume of (M^n, g), $S(p,q) = (V(g)/vol(S^n(1))^{1/p-1/q}R\Sigma(n,p,q)$ and $\Sigma(n,p,q)$ is the Sobolev constant of the canonical unit sphere S^n defined by

$$\Sigma(n,p,q) = sup\{\|f\|_p/\|df\|_q : f \in W^{1,q}(S^n), f \neq 0, \int_{S^n} f = 0\}.$$

Let $p = \frac{2n}{n-2}, q = 2$ in Theorem 5.1 and apply Theorem 3 and Proposition 6 in [2] pages 395-396, then we get the following mean value inequality.

Theorem 5.2. *Let $n \geq 3$ and (M^n, g) be a closed n-dimensional smooth Riemannian manifold such that for some constant $b > 0$,*

$$r_{min}(g)D^2(g) \geq -(n-1)b^2.$$

If $f \in W^{1,2}(M^n)$ is a nonnegative continuous function such that $f\Delta f \geq -cf^2$ (here Δ is a negative operator) in the sense of distribution for some positive number c, then

$$max_{x\in M^n}|f|^2(x) \leq B_n(\sigma_n Rc^{1/2})\frac{\int_{M^n} f^2 dV}{V(g)},$$

where $\sigma_n = vol(S^n)^{1/n}\Sigma(n, \frac{2n}{n-2}, 2)$ and $B_n : \mathbb{R}_+ \to \mathbb{R}_+$ is a function defined by

$$B_n(x) = \prod_{i=0}^{\infty}(x\nu^i(2\nu^i - 1)^{-1/2} + 1)^{2\nu^{-i}}, \nu = \frac{n}{n-2}.$$

The function B_n satisfies the inequalities

$$B_n(x) \leq exp(2x\sqrt{\nu}/(\sqrt{\nu}-1)), 0 \leq x \leq 1$$
$$B_n(x) \leq B_n(1)x^{2\nu/(\nu-1)}, x \geq 1.$$

In particular, $\lim_{x\to 0_+ B_n(x)} = 1$ and $B_n(x) \leq B_n(1)x^n$ for $x \geq 1$.

Let M^n be a closed Riemannian manifold with nonzero first de Rham cohomology group and admits a sequence of Riemannian metrics g_i such that

$$Ric(g_i) \geq -\frac{n-1}{i}$$
$$D(g_i) \leq 1.$$

Moreover, the curvature operator of g_i is uniformly bounded from below by $-Id$. For any $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$, we are going to prove that there exists some $t \in \mathbb{R}, t \neq 0$ such that $H^p(M^n, t\theta) = 0$ for any p. If $n = 2$, since the first Betti number of M^2 is bounded by 2 (see e.g. [2]), the genus of M^2 is at most 1 and $H^p(M^2, t\theta) = 0$ by Example 1. Now we assume that $n \geq 3$. Let d^* be the formal L^2 adjoint of d with respect to g_i. By Hodge theory, we can choose a harmonic one form θ_i in the cohomology class $[\theta]$. Then

$$d\theta_i = 0$$
$$d^*\theta_i = 0$$
$$\theta_i \neq 0.$$

Let $t_i = (\frac{V(g_i)}{\int_{M^n} |X_i|^2 dV_i})^{1/2} > 0$, where $V(g_i)$ is the volume of (M^n, g_i), dV_i is the volume form of g_i, $|X_i|^2 = g_i(X_i, X_i)$ and X_i is the dual vector field of θ_i defined by $g_i(X_i, Y) = \theta(Y)$. We claim that for sufficiently large i, $H^p(M^n, t_i\theta_i) = 0$ for any p. Choose a $\Delta_{t_i\theta_i}$ harmonic form α_i in $H^p(M^n, t_i\theta_i)$. Then

$$d\alpha_i + t_i\theta_i \wedge \alpha_i = 0$$
$$d^*\alpha_i + i_{t_i X_i}\alpha_i = 0.$$

The goal is to prove that $\alpha_i = 0$. By Theorem 2.1, we can assume that $1 \leq deg(\alpha_i) \leq n-1$. As $Ric(g_i) \geq -\frac{n-1}{i}$, applying Bochner formula to X_i [20], we get

$$\frac{1}{2}\Delta|X_i|^2 = |\nabla X_i|^2 + Ric(g_i)(X_i, X_i) \geq |\nabla X_i|^2 - \frac{n-1}{i}|X_i|^2, \tag{5.1}$$

where Δ is the Laplacian acting on functions which is a negative operator. Then

$$\int_{M^n} |\nabla X_i|^2 dV_i \leq \frac{n-1}{i} \int_{M^n} |X_i|^2 dV_i. \tag{5.2}$$

Let $div(X_i)$ be the divergence of X_i with respect to g_i. As θ_i is a harmonic one form, we see $div(X_i) = 0$ (see e.g. Proposition 31 in [20] page 206). By Corollary 4.3, we have

$$\int_{M^n} t_i^2 |X_i|^2 |\alpha_i|^2 dV_i \leq C_n \int_{M^n} t_i |\nabla X_i| |\alpha_i|^2 dV_i, \tag{5.3}$$

for some constant C_n depending only on n. Applying Hölder's inequality on (5.3) and using (5.2), we get

$$\begin{aligned}\int_{M^n} t_i^2 |X_i|^2 |\alpha_i|^2 dV_i &\leq C_n \int_{M^n} t_i |\nabla X_i| |\alpha_i|^2 dV_i \\ &\leq C_n (\int_{M^n} t_i^2 |\nabla X_i|^2 dV_i)^{\frac{1}{2}} (\int_{M^n} |\alpha_i|^4 dV_i)^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{C_n}{\sqrt{i}}|\alpha_i|_\infty(\int\limits_{M^n} t_i^2|X_i|^2 dV_i)^{\frac{1}{2}}(\int\limits_{M^n} |\alpha_i|^2 dV_i)^{\frac{1}{2}}, \tag{5.4}$$

where $|\alpha_i|_\infty = max_{x\in M^n}|\alpha_i|(x)$.

Lemma 5.3.

$$|X_i|^2_\infty =: max_{x\in M^n}|X_i|^2(x) \leq B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})\frac{\int_{M^n}|X_i|^2 dV_i}{V(g_i)}, \tag{5.5}$$

$$|\alpha_i|^2_\infty =: max_{x\in M^n}|\alpha_i|^2(x) \leq B_n(\sigma_n R_i(t_i^2|X_i|^2_\infty + C_n)^{\frac{1}{2}})\frac{\int_{M^n}|\alpha_i|^2 dV_i}{V(g_i)}, \tag{5.6}$$

where $R_i = \frac{D(g_i)}{\frac{1}{\sqrt{i}}C(\frac{1}{\sqrt{i}})}$, $C(\frac{1}{\sqrt{i}}), \sigma_n, B_n(x)$ *are defined in Theorem 5.1 and Theorem 5.2 and* C_n *is a positive constant depending only on* n.

Proof. Since θ_i is a harmonic one form, $divX_i = 0$. As $Ric(g_i) \geq -\frac{n-1}{i}$, applying Bochner formula to X_i, we get

$$\frac{1}{2}\Delta|X_i|^2 = |\nabla X_i|^2 + Ric(g_i)(X_i, X_i) \geq |\nabla X_i|^2 - \frac{n-1}{i}|X_i|^2, \tag{5.7}$$

where Δ is the Laplacian acting on functions which is a negative operator. On the other hand, by Kato's inequality [2], we have $|\nabla X_i| \geq |\nabla|X_i||$. It follows that

$$|X_i|\Delta|X_i| \geq -\frac{n-1}{i}|X_i|^2. \tag{5.8}$$

Since $Ric(g_i) \geq -\frac{n-1}{i}$, $D(g_i) \leq 1$, we have

$$r_{min}(g_i)D^2(g_i) \geq -\frac{n-1}{i}.$$

Apply Theorem 5.2 to $|X_i|$, we get

$$|X_i|^2_\infty =: max_{x\in M^n}|X_i|^2(x) \leq B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})\frac{\int_{M^n}|X_i|^2 dV_i}{V(g_i)}, \tag{5.9}$$

where $R_i = \frac{D(g_i)}{\frac{1}{\sqrt{i}}C(\frac{1}{\sqrt{i}})}$. As $1 \leq deg(\alpha_i) \leq n-1$, applying Bochner formula to α_i (Theorem 51 in page 221 in [20]), we get

$$\frac{1}{2}\Delta|\alpha_i|^2 \geq |\nabla\alpha_i|^2 - |d\alpha_i|^2 - |d^*\alpha_i|^2 + \frac{1}{4}\lambda_k|[\Theta_k, \alpha_i]|^2, \tag{5.10}$$

where λ_k are the eigenvalues of the curvature operator of g_i and Θ_k the dual of eigenvectors for the curvature operator. Since the curvature operator of g_i is bounded from below by $-Id$, we have

$$\frac{1}{4}\lambda_k|[\Theta_k, \alpha_i]|^2 \geq -C_n|\alpha_i|^2$$

for some positive constant C_n depending only on n.

Lemma 5.4.

$$t_i^2|X_i|^2|\alpha_i|^2 = |d\alpha_i|^2 + |d^*\alpha_i|^2.$$

Proof. Firstly, we have

$$\begin{aligned} t_i^2|X_i|^2|\alpha_i|^2 dV_i &= t_i\theta_i \wedge i_{t_iX_i}(\alpha_i \wedge *\alpha_i) \\ &= t_i^2\theta_i \wedge i_{X_i}\alpha_i \wedge *\alpha_i + (-1)^p t_i^2\theta_i \wedge \alpha_i \wedge i_{X_i}(*\alpha_i) \\ &= (-1)^{p-1}t_i^2 i_{X_i}\alpha_i \wedge \theta_i \wedge *\alpha_i + (-1)^p t_i^2\theta_i \wedge \alpha_i \wedge i_{X_i}(*\alpha_i). \end{aligned} \tag{5.11}$$

By Lemma 4.4, we get

$$*i_{X_i}\alpha_i = (-1)^{p-1}\theta_i \wedge *\alpha_i; \tag{5.12}$$

$$*i_{X_i}(*\alpha_i) = (-1)^{n-p-1}\theta_i \wedge * * \alpha_i = (-1)^{n-p-1}(-1)^{np+p}\theta_i \wedge \alpha_i. \tag{5.13}$$

Hence

$$\theta_i \wedge *\alpha_i = (-1)^{p-1} * i_{X_i}\alpha_i \tag{5.14}$$

$$i_{X_i}(*\alpha_i) = (-1)^{n(n-p-1)+n-p-1} * *i_{X_i}(*\alpha_i) = (-1)^p * (\theta_i \wedge \alpha_i). \tag{5.15}$$

By (5.11), (5.14), (5.15), we get

$$\begin{aligned} t_i^2|X_i|^2|\alpha_i|^2 dV_i =& t_i^2 i_{X_i}\alpha_i \wedge *(i_{X_i}\alpha_i) + t_i^2\theta_i \wedge \alpha_i \wedge *(\theta_i \wedge \alpha_i) \\ =& \left(t_i^2\, |i_{X_i}\alpha_i|^2 + t_i^2|\theta_i \wedge \alpha_i|^2\right) dV_i. \end{aligned} \tag{5.16}$$

Since $d\alpha_i + t_i\theta_i \wedge \alpha_i = 0, d^*\alpha_i + i_{t_iX_i}\alpha_i = 0$, we get

$$t_i^2|X_i|^2|\alpha_i|^2 = |d\alpha_i|^2 + |d^*\alpha_i|^2. \quad \square$$

Given Lemma 5.4, we have

$$\frac{1}{2}\Delta|\alpha_i|^2 \geq |\nabla\alpha_i|^2 - t_i^2|X_i|^2|\alpha_i|^2 - C_n|\alpha_i|^2. \tag{5.17}$$

By Kato's inequality, we have $|\nabla\alpha_i| \geq |\nabla|\alpha_i||$. It follows that

$$|\alpha_i|\Delta|\alpha_i| \geq -(t_i^2|X_i|^2 + C_n)|\alpha_i|^2 \geq -(t_i^2|X_i|^2_\infty + C_n)|\alpha_i|^2. \tag{5.18}$$

Applying Theorem 5.2 to $|\alpha_i|$, we get

$$|\alpha_i|_\infty^2 =: max_{x\in M^n}|\alpha_i|^2(x) \le B_n(\sigma_n R_i(t_i^2|X_i|_\infty^2 + C_n)^{\frac{1}{2}})\frac{\int_{M^n}|\alpha_i|^2 dV_i}{V(g_i)}. \quad \square$$

Lemma 5.5.

$$\frac{\int_{M^n}|X_i|^2 dV_i}{V(g_i)}\int_{M^n}|\alpha_i|^2 dV_i \le \int_{M^n}|X_i|^2|\alpha_i|^2 dV_i + \frac{2C_n|\alpha_i|_\infty^2}{\sqrt{i}}R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}\int_{M^n}|X_i|^2 dV_i \quad (5.19)$$

for some constant C_n depending only n.

Proof. Let $h_i = |X_i|^2$ and $\overline{h_i} = \frac{\int_{M^n}|X_i|^2 dV_i}{V(g_i)}$. By Theorem 5.1 in the case $p = q = 2$, we get

$$\begin{aligned}
\int_{M^n}|h_i - \overline{h_i}||\alpha_i|^2 dV_i &\le |\alpha_i|_\infty^2(\int_{M^n}|h_i - \overline{h_i}|^2 dV_i)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}}\\
&\le C_n|\alpha_i|_\infty^2 R_i(\int_{M^n}|\nabla h_i|^2 dV_i)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}}\\
&= 2C_n|\alpha_i|_\infty^2 R_i(\int_{M^n}|X_i|^2|\nabla|X_i||^2|dV_i)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}}\\
&\le 2C_n|\alpha_i|_\infty^2 R_i(\int_{M^n}|X_i|^2|\nabla X_i|^2 dV_i)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}}\\
&\le 2C_n|\alpha_i|_\infty^2 R_i|X_i|_\infty(V(g_i))^{\frac{1}{2}}(\int_{M^n}|\nabla X_i|^2 dV_i)^{\frac{1}{2}}\\
&\le 2C_n|\alpha_i|_\infty^2 R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}(\int_{M^n}|X_i|^2 dV_i)^{\frac{1}{2}}(\int_{M^n}|\nabla X_i|^2 dV_i)^{\frac{1}{2}}\\
&\le \frac{2C_n|\alpha_i|_\infty^2}{\sqrt{i}}R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}\int_{M^n}|X_i|^2 dV_i.
\end{aligned}$$

It follows that

$$\frac{\int_{M^n}|X_i|^2 dV_i}{V(g_i)}\int_{M^n}|\alpha_i|^2 dV_i \le \int_{M^n}|X_i|^2|\alpha_i|^2 dV_i + \frac{2C_n|\alpha_i|_\infty^2}{\sqrt{i}}R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}\int_{M^n}|X_i|^2 dV_i. \quad \square$$

Lemma 5.6. *Let $C(b)$ be the function defined in Theorem 5.1. Namely, $C(b)$ is the unique positive root of the equation*

$$x\int_0^b (cht + xsht)^{n-1}dt = \int_0^\pi sin^{n-1}tdt.$$

Then

$$\liminf_{b\to 0} bC(b) \geq a_n > 0 \tag{5.20}$$

for some constant a_n depending only on n.

Proof. Let $\omega_n = \int_0^\pi sin^{n-1}tdt$. Then

$$\omega_n = C(b)\int_0^b (cht + C(b)sht)^{n-1}dt = C(b)\int_0^b \left(\frac{e^t + e^{-t}}{2} + C(b)\frac{e^t - e^{-t}}{2}\right)^{n-1}dt \geq C(b)b.$$

On the other hand, for any sequence $b_i \to 0$, we have

$$\begin{aligned}\omega_n &= C(b_i)\int_0^{b_i} \left(\frac{e^t + e^{-t}}{2} + C(b_i)\frac{e^t - e^{-t}}{2}\right)^{n-1}dt \\ &\leq C(b_i)\int_0^{b_i} \left(\frac{e + e^{-1}}{2} + C(b_i)\frac{e^t - e^{-t}}{2}\right)^{n-1}dt \\ &\leq C(b_i)b_i\left(\frac{e + e^{-1}}{2} + 2b_iC(b_i)\right)^{n-1} \\ &\leq C(b_i)b_i\left(\frac{e + e^{-1}}{2} + 2\omega_n\right)^{n-1}\end{aligned}$$

Hence for some constant a_n depending only on n, we have

$$\liminf_{b\to 0} bC(b) \geq a_n > 0 \quad \square$$

By (5.4), (5.5), (5.6) and (5.19), we get

$$\begin{aligned}&\frac{\int_{M^n} t_i^2|X_i|^2dV_i}{V(g_i)}\int_{M^n} |\alpha_i|^2dV_i \\ &\leq \int_{M^n} t_i^2|X_i|^2|\alpha_i|^2dV_i\end{aligned}$$

$$
\begin{aligned}
&+\frac{2C_n|\alpha_i|_\infty^2}{\sqrt{i}}R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}\int\limits_{M^n} t_i^2|X_i|^2dV_i\\
&\le \frac{C_n}{\sqrt{i}}|\alpha_i|_\infty(\int\limits_{M^n} t_i^2|X_i|^2dV_i)^{\frac{1}{2}}(\int\limits_{M^n}|\alpha_i|^2dV_i)^{\frac{1}{2}}\\
&\quad+\frac{2C_n|\alpha_i|_\infty^2}{\sqrt{i}}R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}\int\limits_{M^n} t_i^2|X_i|^2dV_i\\
&\le \frac{C_n\sqrt{B_n(\sigma_n R_i(t_i^2|X_i|_\infty^2+C_n)^{\frac{1}{2}})}}{\sqrt{i}}\sqrt{\frac{\int_{M^n}t_i^2|X_i|^2dV_i}{V(g_i)}}\int\limits_{M^n}|\alpha_i|^2dV_i\\
&+\frac{2C_nB_n(\sigma_n R_i(t_i^2|X_i|_\infty^2+C_n)^{\frac{1}{2}})}{\sqrt{i}}R_i\sqrt{B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})}\frac{\int_{M^n}t_i^2|X_i|^2dV_i}{V(g_i)}\int\limits_{M^n}|\alpha_i|^2dV_i,
\end{aligned}
\tag{5.21}
$$

where

$$
|X_i|_\infty^2 =: max_{x\in M^n}|X_i|^2(x)\le B_n(\sigma_n R_i\sqrt{\frac{n-1}{i}})\frac{\int_{M^n}|X_i|^2dV_i}{V(g_i)}.
$$

As $t_i=(\frac{V(g_i)}{\int_{M^n}|X_i|^2dV_i})^{1/2}$, we see

$$
\frac{\int_{M^n}t_i^2|X_i|^2dV_i}{V(g_i)}=1. \tag{5.22}
$$

Recall that $R_i=\frac{D(g_i)}{\frac{1}{\sqrt{i}}C(\frac{1}{\sqrt{i}})}$ and $D(g_i)\le 1$. By (5.20), (5.21) and (5.22), using the properties of $B_n(x)$ in Theorem 5.2, we see that for sufficiently large i,

$$
\int\limits_{M^n}|\alpha_i|^2dV_i\le\frac{1}{2}\int\limits_{M^n}|\alpha_i|^2dV_i.
$$

Hence $\alpha_i\equiv 0$ and $H^p(M^n,t_i\theta_i)=0$ when $n\ge 3$.

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