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Singular Cotangent Bundle Reduction and Polar Actions

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Abstract
A conjecture of Lerman, Montgomery and Sjamaar states that two singular symplectic reductions $T^*M /\!\!/ G$ and $T^*N /\!\!/ H$ are isomorphic if M/G is diffeomorphic to N/H as stratified spaces. We confirm this conjecture under the assumptions that the action $G \times M \to M$ is polar with a section N and generalized Weyl group H.

Keywords Polar action · Singular symplectic reduction · Chevalley Restriction Theorem

Mathematics Subject Classification 53C20 · 53D20

1 Introduction

Let G be a compact Lie group acting isometrically on a complete Riemannian manifold (M, g). It is well known that the lifting action on the cotangent bundle T^*M with its canonical symplectic structure ω is a Hamiltonian action with a moment map given by $u : T^*M \to \mathfrak{g}^*$ with

$$u_X(x, \xi) = \langle \xi, X^*(x) \rangle, \tag{1.1}$$

where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$ and $u_X(x, \xi) = \langle u(x, \xi), X \rangle$, $X \in \mathfrak{g}$. Moreover, X^* is the vector field on M generated by X. The moment map satisfies the following equations:

$$\mathrm{d}u_X = i_{X^\#}\omega,$$

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$$u(g \cdot (x, \xi)) = Ad_g^* \cdot u(x, \xi), \ \forall g \in G,$$

where $X^\#$ is the vector field on T^*M generated by X.

The symplectic reduction $T^*M /\!\!/ G := u^{-1}(0)/G$ is not a smooth manifold in general. However, it is a stratified symplectic space defined in [20]. The reader is referred to [20] for the precise definition of stratified symplectic spaces. Singular symplectic reductions have played an important role in geometric quantization [10].

Following [20], we define a function $f : T^*M /\!\!/ G \to \mathbb{R}$ to be smooth if there exists a function $F \in C^\infty(T^*M)^G$ with $F|_{\mu^{-1}(0)} = \pi^* f$, where $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is the projection map. In other words, $C^\infty(T^*M /\!\!/ G)$ is isomorphic to $C^\infty(T^*M)^G/I^G$, where I^G is the ideal of G-invariant smooth functions on T^*M vanishing on $\mu^{-1}(0)$. The algebra $C^\infty(T^*M /\!\!/ G)$ inherits a Poisson algebra structure from $C^\infty(T^*M)$.

Let G and H be Lie groups and M, resp. N, be smooth manifolds on which G, resp. H act properly. The stratified symplectic spaces $T^*M /\!\!/ G$ and $T^*N /\!\!/ H$ are isomorphic if there exists a homeomorphism $\phi : T^*M /\!\!/ G \to T^*N /\!\!/ H$ and the pullback map

$$\begin{aligned} \phi^* : C^\infty(T^*N /\!\!/ H) &\to C^\infty(T^*M /\!\!/ G), \\ f &\mapsto f \circ \phi \end{aligned}$$

is an isomorphism of Poisson algebras.

In [9, p. 13, Conjecture 3.7], they made the following conjecture.

Conjecture 1.1 *Let G and H be Lie groups and M, resp. N be smooth manifolds on which G, resp. H act properly. Assume that the orbit spaces M/G and N/H are diffeomorphic in the sense that there exists a homeomorphism $\phi : M/G \to N/H$ such that the pullback map ϕ^* is an isomorphism from $C^\infty(N/H) := C^\infty(N)^H$ to $C^\infty(M/G) := C^\infty(M)^G$. Then $T^*M /\!\!/ G$ and $T^*N /\!\!/ H$ are isomorphic.*

Conjecture 1.1 has been verified for isotropic representation of symmetric spaces (p. 17 in [9] and [22]). In this paper, we confirm Conjecture 1.1 for a much general class of group actions. To start with, we recall that an isometric group action $G \times (M, g) \to (M, g)$ is polar if there exists a closed submanifold $\Sigma \subseteq M$ meeting all orbits orthogonally [13]. Then M is called a polar G-manifold and such a submanifold Σ is called a section and comes with a natural action by a discrete group of isometries $\Pi = \Pi(\Sigma)$, called its generalized Weyl group. Recall that by definition, $\Pi(\Sigma) := N(\Sigma)/Z(\Sigma)$, where

$$\begin{aligned} N(\Sigma) &= \{g \in G | g\Sigma = \Sigma\}, \\ Z(\Sigma) &= \{g \in G | gx = x, \ x \in \Sigma\}. \end{aligned}$$

Polar actions have nice properties and have been studied by many people, see for instance [3,6,12,13,15]. A basic example of polar action is given by the adjoint action of a compact Lie group on its Lie algebra. More generally, isotropy representations of symmetric spaces are also polar. It is a classical theorem of Dadok [3] which shows that a polar representation is (up to orbit equivalence) the isotropy representation of a symmetric space. It follows from classical Chevalley Restriction Theorem [13] that M/G is diffeomorphic to Σ/Π, i.e. the inclusion $\Sigma/\Pi \to M/G$ is a homeomorphism and the restriction $|_\Sigma : \ C^\infty(M)^G \to C^\infty(\Sigma)^\Pi$ is an isomorphism.

Our main result in this paper is the following theorem which gives a partial answer to Conjecture 1.1:

Theorem 1.1 *Let M be a polar G-manifold with a section Σ and generalized Weyl group Π. Then $T^*M /\!\!/ G$ and $T^*\Sigma /\!\!/ \Pi$ are isomorphic.*

Example 1.1 Let $M = (S^2 \times S^2)\#(S^2 \times S^2)$. Then M admits a polar action of $G = S^1 \times S^1$ with a section $\Sigma = T^2\#T^2$ and the generalized Weyl group $\Pi = \mathbb{Z}_2 \times \mathbb{Z}_2$ (see Example 2.1.1 in [11] and p. 309 in [6]). Applying Theorem 1.1, we get that $T^*M /\!\!/ G$ and $T^*\Sigma /\!\!/ \Pi$ are isomorphic.

Under a slightly different assumption, it was proved that $T^*M /\!\!/ G$ is homeomorphic to $T^*\Sigma /\!\!/ \Pi$ in [9, Proposition 3.8]. Also see [4,5,7] for related work. However, they considered cotangent bundle reduction under a strong assumption (p. 189 in [7]) which is not satisfied in general in our situation.

The proof of Theorem 1.1 consists of two parts. First we prove the inclusion $T^*\Sigma /\!\!/ \Pi \to T^*M /\!\!/ G$ is a homeomorphism. Secondly, we show that the restriction $|_{T^*\Sigma}$: $C^\infty(T^*M)^G \to C^\infty(T^*\Sigma)^\Pi$ is a surjective homomorphism of Poisson algebras. These will be proved in Sect. 3. A main ingredient of the proof is a characterization of symplectic slice representations of the lifting action G on T^*M, which is done by using the natural Sasaki metric on T^*M. Then combining the multivariable Chevalley restriction theorem proved by Tevelev [22] and other things, we are able to prove our results. For details, see Sect. 3.

2 Sasaki Metrics on *TM* and *T***M*

The proof of main theorem will use the geometry of Sasaki metrics on TM [16] which we describe here briefly. Given a Riemannian metric g on M, its Levi-Civita connection determines a splitting $TTM = \mathcal{H}M \oplus \mathcal{V}M$, where $\mathcal{V}M = \ker d\pi$, $\pi : TM \to M$ is the projection and $\mathcal{H}M$ is spanned by X^h, X is a smooth vector field on M. To describe X^h, let $(x, v) \in TM$ and $\gamma(t) : (-\epsilon, \epsilon) \to M$ be a smooth curve such that $\gamma(0) = x$, $\gamma'(0) = X(x)$. Let $Y(t) \in T_{\gamma(t)}M$ such that

$$\begin{cases} \nabla_{\gamma'} Y = 0 \\ Y(0) = v. \end{cases}$$

Then $X^h(x, v) =: \bar{\gamma}'(0)$, where $\bar{\gamma}(t) = (\gamma(t), Y(t))$. From the definition of X^h, we see that $d\pi(X^h(x, v)) = X(x)$. Let I_g be the natural isomorphism $T^*M \cong TM$ induced by the Riemannian metric g. Then using the splitting $TTM = \mathcal{H}M \oplus \mathcal{V}M \cong TM \oplus TM$, we define the Sasaki metric $\tilde{g}$ by

$$\tilde{g}\langle (X_1, X_2), (Y_1, Y_2)\rangle := g\langle X_1, Y_1\rangle + g\langle X_2, Y_2\rangle.$$

Define an almost complex structure J by setting $J(X, Y) = (-Y, X)$. Then $\tilde{g}\langle J\cdot, J\cdot\rangle = \tilde{g}\langle\cdot,\cdot\rangle$ and the symplectic form $\Omega := \tilde{g}(J\cdot,\cdot)$ is nothing but the pull-back of ω by the isomorphism $I_g^{-1} : TM \cong T^*M$, where ω is the standard symplectic form on T^*M.

The Sasaki metric on T^*M is the pullback of $\tilde{g}$ under the isomorphism $I_g : T^*M \to TM$. The following lemma will be important for us.

Lemma 2.1 *If Σ is a totally geodesic submanifold of (M, g), then $T\Sigma$ is a totally geodesic submanifold of $(TM, \tilde{g})$, where $\tilde{g}$ is the Sasaki metric on TM.*

Proof Let X be a smooth vector field on M such that $X(x) \in T_x\Sigma$, $\forall x \in \Sigma$. As Σ is totally geodesic, we see that $X^h|_{T\Sigma}$ is a smooth vector field on $T\Sigma$ from the construction of X^h.

The vector field X also induces a vertical vector field $X^\perp$ on TM. We choose a local coordinate to describe $X^\perp$. Let $(x^1, \ldots, x^n)$ be a local coordinate system at $x \in M$, where $n = \dim M$. Then any tangent vector $v \in T_xM$ can be decomposed as $v = v^i \frac{\partial}{\partial x_i}$. The set of parameters $\{x^1, \ldots, x^n, v^1, \ldots, v^n\}$ forms a natural coordinate system of TM. The natural frame in $T_{(x,v)}TM$ is given by $\tilde{\partial}_i = \frac{\partial}{\partial x_i}$ and $\tilde{\partial}_{n+i} = \frac{\partial}{\partial v_i}$. Now if $X = X^i \frac{\partial}{\partial x_i}$ is a vector field on M, then the vertical vector field $X^\perp$ on TM is given by $X^\perp = X^i \tilde{\partial}_{n+i}$. As $X(x) \in T_x\Sigma$, $\forall x \in \Sigma$, by definition we see that $X^\perp|_{T\Sigma}$ is a vector field on $T\Sigma$.

To see that $T\Sigma$ is totally geodesic in TM, choose two vector fields X, Y on M such that $X(x)$, $Y(x) \in T_x\Sigma$, $\forall x \in \Sigma$, then we have the following formula [8]:

$$\tilde{\nabla}_{X^\perp} Y^\perp = 0, \tag{2.1}$$

$$(\tilde{\nabla}_{X^h} Y^\perp)(x, v) = (\nabla_X Y)^\perp(x, v) + \frac{1}{2}(R_x(v, Y_x, X_x))^h(x, v), \tag{2.2}$$

$$(\tilde{\nabla}_{X^\perp} Y^h)(x, v) = \frac{1}{2}(R_x(v, X_x, Y_x))^h(x, v), \tag{2.3}$$

$$(\tilde{\nabla}_{X^h} Y^h)(x, v) = (\nabla_X Y)^h(x, v) - \frac{1}{2}(R_x(X_x, Y_x, v))^\perp(x, v), \tag{2.4}$$

where $(x, v) \in T\Sigma$ and ∇, resp. $\tilde{\nabla}$ are Levi-Civita connections of g, resp. $\tilde{g}$ and R is the Riemann curvature tensor of g.

Since Σ is totally geodesic, then $\nabla_X Y(x)$, $R_x(v, X_x, Y_x) \in T_x\Sigma$. From (2.1) to (2.4), it follows that $T\Sigma$ is totally geodesic. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The first key observation is the following proposition.

Proposition 3.1 *Let (M, g) be a polar G-manifold with a section Σ. Then $T^*\Sigma$ meets all G-orbits of the action $G \times u^{-1}(0) \to u^{-1}(0)$.*

Here $T^*\Sigma$ is seen as a submanifold of T^*M under the natural isomorphism $T^*M \cong TM$ induced by the Riemannian metric g. Note that in general $T^*\Sigma$ cannot meet all orbits of the lifting action of G on T^*M as it is easy to see that $T^*\Sigma \subseteq u^{-1}(0)$ from (1.1).

The crucial properties of polar actions we will use in the proof are the following results [13]:

Proposition 3.2 *Let M be a polar G-manifold with a section Σ. Then:*

(1) Σ *is totally geodesic.*
(2) $G \cdot x \cap \Sigma = \Pi \cdot x, \ \forall x \in \Sigma.$
(3) *The slice representation at x is polar with a section $T_x\Sigma$, $\forall x \in \Sigma$.*

Given Proposition 3.2, we can now give a proof of Proposition 3.1.

Recall that $u : T^*M \to \mathfrak{g}^*$ is given by

$$u_X(x, \xi) = \langle \xi, X^*(x) \rangle.$$

Then for any $(x, \xi) \in u^{-1}(0)$, $\langle \xi, X^*(x) \rangle = 0$, $\forall X \in \mathfrak{g}$. Under the isomorphism $I_g : T^*M \cong TM$ induced by the Riemannian metric g, the vector $\xi^\# := I_g(\xi)$ is orthogonal to $T_x(G \cdot x)$, i.e. $\xi^\# \in T_x(G \cdot x)^\perp$.

As the isometric action $G \times M \to M$ is polar with a section Σ, there exists $h_1 \in G$ such that $h_1 x \in \Sigma$. Then $h_1\xi^\# \in T_{h_1 x}(G \cdot x)^\perp$.

By Proposition 3.2, the slice representation

$$G_{h_1 x} \times T_{h_1 x}(G \cdot x)^\perp \to T_{h_1 x}(G \cdot x)^\perp$$

is polar. Hence there exists $h_2 \in G_{h_1 x}$ such that $h_2(h_1\xi^\#) \in T_{h_1 x}\Sigma$.

Let $h = h_2 h_1$, then

$$h(x, \xi^\#) = (hx, h\xi^\#) = (h_1 x, h_2 h_1 \xi^\#) \in T\Sigma.$$

So $T^*\Sigma$ meets all orbits of the action $G \times u^{-1}(0) \to u^{-1}(0)$.

Theorem 1.1 will follow from the following two theorems:

Theorem 3.1 *Let (M, g) be a polar G-manifold with a section Σ and generalized Weyl group Π. The inclusion $T^*\Sigma /\!\!/ \Pi \to T^*M /\!\!/ G$ is a homeomorphism.*

Theorem 3.2 *Let (M, g) be a polar G-manifold with a section Σ and generalized Weyl group Π. Then the following restriction to $T^*\Sigma$ is a surjective homomorphism of Poisson algebras:*

$$|_{T^*\Sigma} : C^\infty(T^*M)^G \to C^\infty(T^*\Sigma)^\Pi.$$

First, we give the Proof of Theorem 3.1. By Proposition 3.1, it suffices to show

$$G \cdot (x, \xi) \cap T^*\Sigma = \Pi \cdot (x, \xi), \ \forall (x, \xi) \in T^*\Sigma.$$

Clearly $\Pi \cdot (x, \xi) \subseteq G \cdot (x, \xi) \cap T^*\Sigma$. On the other hand, $\forall h_1(x, \xi) \in G \cdot (x, \xi) \cap T^*\Sigma$, we have $h_1 x \in G \cdot x \cap \Sigma$ and $h_1 \xi^\# \in T_{h_1 x}\Sigma$. By Proposition 3.2, we get

$$G \cdot x \cap \Sigma = \Pi \cdot x, \ \forall x \in \Sigma.$$

Hence

$$h_1 x = h_2 x, \ h_2 \in \Pi. \tag{3.1}$$

Then $(h_2^{-1} h_1)x = x$ and so $h_2^{-1} h_1 \in G_x$. Since $(x, \xi) \in T^*\Sigma$, we get $\xi^\# \in T_x(G \cdot x)^\perp$. By Proposition 3.2, the slice representation: $G_x \times T_x(G \cdot x)^\perp \to T_x(G \cdot x)^\perp$ is polar with a section $T_x\Sigma$ and generalized Weyl group Π_x. By Proposition 3.2 again,

$$G_x \cdot \xi^\# \cap T_x\Sigma = \Pi_x \cdot \xi^\#.$$

As $h_2^{-1} h_1 \in G_x$, $h_2 \in \Pi$, $h_1 \xi^\# \in T_{h_1 x}\Sigma$, we get $h_2^{-1} h_1 \xi^\# \in G_x \cdot \xi^\# \cap T_x\Sigma$. Then there exists $h_3 \in \Pi_x$ such that

$$h_2^{-1} h_1 \xi^\# = h_3 \xi^\#.$$

Hence $h_1 \xi^\# = h_2 h_3 \xi^\# \in \Pi \cdot \xi^\#$. Combined with (3.1), we obtain $h_1(x, \xi^\#) = (h_2 x, h_2 h_3 \xi^\#) = h_2 h_3 (x, \xi^\#) \in \Pi \cdot (x, \xi^\#)$. So $G \cdot (x, \xi) \cap T^*\Sigma \subseteq \Pi \cdot (x, \xi)$. □

Then we prove Theorem 3.2. Recall that we have a splitting $TT^*M \cong \mathcal{H}M \oplus \mathcal{V}M$, which induces an isomorphism $TT^*M \overset{(d\pi, I_g)}{\cong} TM \oplus TM$, where $d\pi$ is the differential of the projection $T^*M \to M$ and I_g is the natural isomorphism $T^*M \cong TM$ induced by the Riemannian metric g.

Let $\{x^1, \ldots, x^n, \xi_1, \ldots \xi_n\}$ be a local coordinate of T^*M at (x, ξ) and Γ_{ij}^k be the Christoffel symbols of the Levi-Civita connection ∇ induced by g. Then the horizontal lift of $\frac{\partial}{\partial x_i}$ at (x, ξ) is given by

$$\frac{\tilde{\partial}}{\partial x_i} = \frac{\partial}{\partial x_i} + \Gamma_{il}^k \xi_k \frac{\partial}{\partial \xi^l}.$$

Here a horizontal lift of a vector X at (x, ξ) is defined to be the unique vector $\tilde{X} \in \mathcal{H}M$ such that $d\pi(x, \xi)(\tilde{X}) = X$.

In terms of local coordinate system $\{x^1, \ldots, x^n, \xi_1, \ldots \xi_n\}$, the almost complex structure J defined in Sect. 2 can be rephrased as

$$J\left(\frac{\tilde{\partial}}{\partial x_i}\right) = g_{ij} \frac{\partial}{\partial \xi^j},$$

$$J\left(\frac{\partial}{\partial \xi^i}\right) = -g^{ij} \frac{\tilde{\partial}}{\partial x_j},$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and (g^{ij}) is the inverse matrix of (g_{ij}).

Let $X^* = X^i \frac{\partial}{\partial x_i}$ be a vector field on M generated by $X \in \mathfrak{g}$. Then the corresponding vector field on T^*M generated by X is

$$X^\#(x,\xi) = X^i \frac{\partial}{\partial x_i} - \sum_{i,j} \frac{\partial X^j}{\partial x_i} \xi_j \frac{\partial}{\partial \xi^i},$$

see [1, p. 16, Lemma 11].

The Sasaki metric $\tilde{g}$ on T^*M satisfies

$$\tilde{g}\left\langle \frac{\tilde{\partial}}{\partial x_i}, \frac{\tilde{\partial}}{\partial x_j} \right\rangle = g_{ij},$$
$$\tilde{g}\left\langle \frac{\tilde{\partial}}{\partial x_i}, \frac{\partial}{\partial \xi^j} \right\rangle = 0,$$
$$\tilde{g}\left\langle \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right\rangle = g^{ij}.$$

Lemma 3.1 *$\forall (x,\xi) \in T^*\Sigma$, the Sasaki metric $\tilde{g}$ on T^*M induces an orthogonal splitting*

$$T_{(x,\xi)}T^*M = T_{(x,\xi)}(G\cdot(x,\xi)) \oplus JT_{(x,\xi)}(G\cdot(x,\xi)) \oplus V$$

with $T_{(x,\xi)}T^*\Sigma \subseteq V$ and V is the orthogonal complement of $T_{(x,\xi)}(G\cdot(x,\xi)) \oplus JT_{(x,\xi)}(G\cdot(x,\xi))$.

Proof Let $X_i^\#$ be two vector fields on T^*M generated by $X_i \in \mathfrak{g}, i = 1, 2$, respectively, and $Y \in T_{(x,\xi)}T^*\Sigma$. Then $\tilde{g}\langle JX_1^\#, X_2^\#\rangle = \omega(X_1^\#, X_2^\#) = (i_{X_1^\#}\omega)(X_2^\#)$. Let u be the moment map defined in (1.1), as $(x,\xi) \in T^*\Sigma \subseteq u^{-1}(0)$, by the G-equivalence of u, we get $G\cdot(x,\xi) \subseteq u^{-1}(0)$. Hence

$$\tilde{g}\langle JX_1^\#, X_2^\#\rangle = (i_{X_1^\#}\omega)(X_2^\#) = du_{X_1}(X_2^\#) = 0.$$

By the definition of J, we get $JT_{(x,\xi)}T^*\Sigma \subseteq T_{(x,\xi)}T^*\Sigma$. As $T^*\Sigma \subseteq u^{-1}(0)$, we get

$$\tilde{g}\langle X_1^\#, Y\rangle = \tilde{g}\langle JX_1^\#, JY\rangle = \omega(X_1^\#, JY) = (i_{X_1^\#}\omega)(JY) = du_{X_1}(JY) = 0.$$

Similarly, $\tilde{g}\langle JX_1^\#, Y\rangle = 0$. Hence $T_{(x,\xi)}T^*\Sigma \subseteq V$. □

Similar statements as Lemma 3.1 were also obtained in [14,18].

The representation

$$G_{(x,\xi)} \times V \to V$$

is called the symplectic slice representation at (x, ξ). Note that $G_{(x,\xi)} = (G_x)_\xi =: \{h \in G_x |\, h\xi = \xi\}$.

The following lemma will be crucial for us.

Lemma 3.2 *Let M be a polar G-manifold with a section Σ. Then the symplectic slice representation at $(x, \xi) \in T^*\Sigma$ is the diagonal action (up to identification)*

$$(G_x)_{\xi^\#} \times (W \oplus W) \to W \oplus W,$$

where $W := (G_x \cdot \xi^\#)^\perp$ is the orthogonal complement of $G_x \cdot \xi^\#$ in the slice $(G \cdot x)^\perp$, i.e. we have

$$T_x M = T_x(G \cdot x) \oplus (T_x(G \cdot x)^\perp),$$
$$T_x(G \cdot x)^\perp = G_x \cdot \xi^\# \oplus (G_x \cdot \xi^\#)^\perp.$$

Proof Let $G_{(x,\xi)} \times V \to V$ be the symplectic slice representation at (x, ξ). Under the isomorphism $\Phi : \mathcal{H}M \oplus \mathcal{V}M \overset{(d\pi, I_g)}{\longrightarrow} TM \oplus TM$, we first claim that

$$\Phi(V) = W \oplus W.$$

Choose a local coordinate system $\{x^1, \ldots, x^n, \xi_1 \ldots, \xi_n\}$ of T^*M at (x, ξ). Then we have

$$d\pi \left(\frac{\tilde{\partial}}{\partial x_i} \right) = \frac{\partial}{\partial x_i},$$
$$I_g \left(\frac{\partial}{\partial \xi^i} \right) = g^{ij} \frac{\partial}{\partial x_j}.$$

Let $Z = a^i \frac{\tilde{\partial}}{\partial x_i} + b_i \frac{\partial}{\partial \xi^i} \in TT^*M$. Then $\Phi(Z) = (d\pi, I_g)(Z) = (a^i \frac{\partial}{\partial x_i}, g^{ij} b_i \frac{\partial}{\partial x_j}) =: (Y_1, Y_2)$.

Let $X^* = X^i \frac{\partial}{\partial x_i}$ be the vector field on M generated by $X \in \mathfrak{g}$, then the corresponding vector field on T^*M is

$$\begin{aligned} X^\#(x, \xi) &= X^i \frac{\partial}{\partial x_i} - \sum_{i,j} \frac{\partial X^j}{\partial x_i} \xi_j \frac{\partial}{\partial \xi^i} \\ &= X^i \frac{\tilde{\partial}}{\partial x_i} - X^i \Gamma^k_{il} \xi_k \frac{\partial}{\partial \xi^l} - \sum_{ij} \frac{\partial X^j}{\partial x_i} \xi_j \frac{\partial}{\partial \xi^i} \\ &= X^i \frac{\tilde{\partial}}{\partial x_i} - g\langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^\# \rangle \frac{\partial}{\partial \xi^i}. \end{aligned}$$

Then we have

$$\begin{aligned}\tilde{g}(X^{\#}, Z) &= \tilde{g}\left\langle X^i \frac{\tilde{\partial}}{\partial x_i} - g\big\langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^{\#}\big\rangle \frac{\partial}{\partial \xi^i}, a^j \frac{\tilde{\partial}}{\partial x_j} + b_j \frac{\partial}{\partial \xi^j}\right\rangle \\ &= g_{ij} X^i a^j - g^{ij} b_j g\big\langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^{\#}\big\rangle \\ &= g\big\langle X^*, Y_1\big\rangle - g\big\langle \nabla_{Y_2} X^*, \xi^{\#}\big\rangle. \end{aligned} \tag{3.2}$$

We also have

$$\begin{aligned}\tilde{g}\langle X^{\#}, JZ\rangle &= \tilde{g}\left\langle X^i \frac{\tilde{\partial}}{\partial x_i} - g\big\langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^{\#}\big\rangle \frac{\partial}{\partial \xi^i}, J\left(a^j \frac{\tilde{\partial}}{\partial x_j} + b_j \frac{\partial}{\partial \xi^j}\right)\right\rangle \\ &= \tilde{g}\left\langle X^i \frac{\tilde{\partial}}{\partial x_i} - g\big\langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^{\#}\big\rangle \frac{\partial}{\partial \xi^i}, a^j g_{jk} \frac{\partial}{\partial \xi^k} - b_j g^{jk} \frac{\tilde{\partial}}{\partial x_k}\right\rangle \\ &= -X^i b_i - a^i g\big\langle \nabla_{\frac{\partial}{\partial x_i}} X^*, \xi^{\#}\big\rangle \\ &= -g\langle X^*, Y_2\rangle - g\big\langle \nabla_{Y_1} X^*, \xi^{\#}\big\rangle. \end{aligned} \tag{3.3}$$

Now we proceed to prove $\Phi(V) = W \oplus W$. Let $Z \in TT^*M$ such that $\Phi(Z) = (Y_1, Y_2) \in W \oplus W$. We claim that $Z \in V$ and it follows that $W \oplus W \subseteq \Phi(V)$. In fact, as $(Y_1, Y_2) \in W \oplus W$, we get

$$g\langle X^*, Y_1\rangle = 0, \tag{3.4}$$

$$g\langle X^*, Y_2\rangle = 0. \tag{3.5}$$

As X^* is a Killing vector field, we get

$$g\langle \nabla_{Y_2} X^*, \xi^{\#}\rangle = -g\langle \nabla_{\xi^{\#}} X^*, Y_2\rangle = g\langle \nabla_{\xi^{\#}} Y_2, X^*\rangle, \tag{3.6}$$

and

$$g\langle \nabla_{Y_1} X^*, \xi^{\#}\rangle = g\langle \nabla_{\xi^{\#}} Y_1, X^*\rangle. \tag{3.7}$$

As M is a polar G-manifold with a section Σ, By Proposition 3.2, the slice representation $G_x \times T_x(G\cdot x)^{\perp} \to T_x(G\cdot x)^{\perp}$ is polar with a section $T_x\Sigma$. Then by Proposition 3.2 again, the slice representation $(G_x)_{\xi^{\#}} \times W \to W$ is polar with a section $T_{\xi^{\#}}T_x\Sigma$. As $Y_1 \in W$, there exists $h \in (G_x)_{\xi^{\#}}$ such that $hY_1 \in T_{\xi^{\#}}T_x\Sigma$. Hence $Y_1 \in h^{-1}(T_{\xi^{\#}}(T_x\Sigma)) = T_{\xi^{\#}}T_x(h^{-1}\Sigma) \cong T_x(h^{-1}\Sigma)$. We also have $\xi^{\#} = h^{-1}\xi^{\#} \in T_x(h^{-1}\Sigma)$, as Σ is totally geodesic by Proposition 3.2, so is $h^{-1}\Sigma$.

Then

$$g\langle \nabla_{\xi^{\#}} Y_1, X^*\rangle = g\langle B(\xi^{\#}, Y_1), X^*\rangle = 0, \tag{3.8}$$

$$g\langle \nabla_{\xi^{\#}} Y_2, X^*\rangle = g\langle B(\xi^{\#}, Y_2), X^*\rangle = 0 \tag{3.9}$$

where $B(\ ,\ )$ is the second fundamental form of $h^{-1}\Sigma$.

By (3.2), (3.4), (3.6) and (3.9), we get

$$\tilde{g}\langle X^{\#}, Z\rangle = g\langle X^{*}, Y_1\rangle - g\langle \nabla_{Y_2} X^{*}, \xi^{\#}\rangle = 0.$$

Similarly we get $\tilde{g}\langle X^{\#}, JZ\rangle = 0$. Hence $\tilde{g}\langle JX^{\#}, Z\rangle = -g\langle X^{\#}, JZ\rangle = 0$. It follows that $Z \in V$, which implies that $W \oplus W \subseteq \Phi(V)$.

On the other hand, we claim that $\dim (W \oplus W) = \dim \Phi(V)$. In fact,

$$\begin{aligned}
\dim(W \oplus W) &= 2 \dim W \\
&= 2(\dim(T_x G \cdot x)^{\perp} - \dim(G_x \cdot \xi^{\#})) \\
&= 2(\dim M - \dim G \cdot x - (\dim G_x - \dim(G_x)_{\xi^{\#}})) \\
&= 2 \dim M - 2(\dim G - \dim G_{(x,\xi)}),
\end{aligned}$$

and

$$\begin{aligned}
\dim \Phi(V) &= \dim V \\
&= \dim T^{*}M - 2 \dim G \cdot (x, \xi) \\
&= 2(\dim M - \dim G \cdot (x, \xi)) \\
&= 2(\dim M - (\dim G - \dim G_{(x,\xi)})).
\end{aligned}$$

Hence $\dim \Phi(V) = \dim (W \oplus W)$ and we have $\Phi(V) = W \oplus W$.

Now Lemma 3.2 follows from the following commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\Phi} & W \oplus W \\
{\scriptstyle G_{(x,\xi)}}\downarrow & & \downarrow{\scriptstyle (G_x)_{\xi^{\#}}} \\
V & \xrightarrow{\Phi} & W \oplus W
\end{array}$$

□

Given Lemma 3.2, we can now give a proof of Theorem 3.2. We first show that the following restriction map is surjective:

$$|_{T^{*}\Sigma} :\ C^{\infty}(T^{*}M)^{G} \to C^{\infty}(T^{*}\Sigma)^{\Pi}.$$

For any $(x, \xi) \in T^{*}\Sigma$, the Sasaki metric $\tilde{g}$ on $T^{*}M$ induces an orthogonal splitting

$$T_{(x,\xi)}T^{*}M = T_{(x,\xi)}G \cdot (x, \xi) \oplus JT_{(x,\xi)}G \cdot (x, \xi) \oplus V,$$

where $\Phi(V) \cong W \oplus W$ by Lemma 3.2.

The Slice Theorem says that for an open G-invariant tubular neighborhood $U_{(x,\xi)}$ of the orbit $G \cdot (x, \xi)$, there is a G-equivalent diffeomorphism

$$\exp^{\perp} :\ G \times_{G_{(x,\xi)}} S^{\perp}_{(x,\xi)}(\epsilon) \to U_{(x,\xi)},$$

where $S^{\perp}_{(x,\xi)} := JT_{(x,\xi)}G\cdot(x,\xi)\oplus V$, $S^{\perp}_{(x,\xi)}(\epsilon)$ is the ϵ-ball in $S^{\perp}_{(x,\xi)}$ and $\exp^{\perp}$ is the normal exponential map of $G\cdot(x,\xi)$.

Let $U = \bigcup_{(x,\xi)\in T^*\Sigma} U_{(x,\xi)}$. As $T^*\Sigma$ intersects all orbits in $u^{-1}(0)$ by Proposition 3.1, we see that U is a G-invariant open neighborhood of $u^{-1}(0)$. $\forall f\in C^{\infty}(T^*\Sigma)^{\Pi}$, we first show that there exists $F_\epsilon \in C^{\infty}(U_{(x,\xi)})^G$ such that

$$F_\epsilon|_{T^*\Sigma\cap U_{(x,\xi)}} = f|_{T^*\Sigma\cap U_{(x,\xi)}} \tag{3.10}$$

By the existence of G-invariant partition of unity subject to the cover $U = \bigcup_{(x,\xi)\in T^*\Sigma} U_{(x,\xi)}$, then there exists $F\in C^{\infty}(U)^G$ such that $F|_{T^*\Sigma} = f$. Extending F to $\tilde{F}\in C^{\infty}(T^*M)^G$, we then prove our desired result.

To prove (3.10), we first recall some facts on polar representations which we will use. Let (G,K) be a symmetric pair and consider the isotropy representation of K on $\mathfrak{p} = T_K(G/K)$. It is a polar action and any maximal abelian sub-algebra Σ is a section. Its generalized Weyl group Π is also called the ”baby” Weyl group. Consider the diagonal action of K on $\mathfrak{p}^m$ (respectively Π on Σ^m) and the corresponding algebra of invariant (m-variable) polynomials $\mathbb{R}[\mathfrak{p}^m]^K$ (respectively $\mathbb{R}[\Sigma^m]^{\Pi}$). Then we have the following result due to Tevelev [22].

Theorem 3.3 *The restriction map* $|_\Sigma : \mathbb{R}[\mathfrak{p}^m]^K \to \mathbb{R}[\Sigma^m]^{\Pi}$ *is surjective.*

As a polar representation is (up to orbit equivalence) the isotropy representation of a symmetric space [3]. Theorem 3.3 generalizes to the class of polar representations [12, Corollary 2].

Corollary 3.1 *Let* $K\subseteq O(\mathfrak{p})$ *be a linear representation which is also polar with a section* Σ *and generalized Weyl group* Π. *Then the restriction is surjective:*

$$|_\Sigma : \mathbb{R}[\mathfrak{p}^m]^K \to \mathbb{R}[\Sigma^m]^{\Pi}.$$

Corollary 3.2 *Let* $\mathfrak{p}$ *be a polar representation of a compact Lie group* K *with a section* Σ *and generalized Weyl group* Π. *Then the restriction to* Σ *is surjective:*

$$|_\Sigma : C^{\infty}(\mathfrak{p}^m)^K \to C^{\infty}(\Sigma^m)^{\Pi}.$$

Proof It is a classical result of Hilbert [21, Proposition 2.4.14] that $\mathbb{R}[\mathfrak{p}^m]^K$ is finitely generated. Let $\rho_1,\ldots,\rho_n$ be generators. By Corollary 3.1, $\rho_1|_\Sigma,\ldots,\rho_n|_\Sigma$ generate $\mathbb{R}[\Sigma^m]^{\Pi}$.

For any $f\in C^{\infty}(\Sigma^m)^{\Pi}$, apply Schwarz's Theorem [19] to the action of Π on Σ^m, we get $F\in C^{\infty}(\mathbb{R}^n)$ such that $f = F\circ\rho|_\Sigma$, where $\rho|_\Sigma : \Sigma^m\to\mathbb{R}^n$ be the map whose coordinates are $\rho_1|_\Sigma,\ldots,\rho_n|_\Sigma$. Then $\tilde{f} = F\circ\rho\in C^{\infty}(\mathfrak{p}^m)^K$ such that $\tilde{f}|_\Sigma = f$. □

We can now give a proof of (3.10). By Lemma 2.1, as Σ is totally geodesic in M, then $T^*\Sigma$ is totally geodesic in T^*M. Hence the normal exponential map $\exp^{\perp}$ of the orbit $G \cdot (x, \xi)$ maps the ϵ-ball B_ϵ in $T_{(x,\xi)}T^*\Sigma \cong T_x\Sigma \oplus T_x\Sigma$ diffeomorphically onto $T^*\Sigma \cap U_{(x,\xi)}$. $\forall f \in C^\infty(T^*\Sigma)^\Pi$, $f \circ \exp^\perp : B_\epsilon \to \mathbb{R}$ is a $\Pi_{(x,\xi)}$-invariant smooth function, where $\Pi_{(x,\xi)} = \{h \in \Pi | \ h(x, \xi) = (x, \xi)\}$. Let W be a polar representation of $K := G_{(x,\xi)}$ with a section $T_{\xi^\#}T_x\Sigma \cong T_x\Sigma$ defined in Lemma 3.2. By Corollary 3.2, we see that there exists $f_\epsilon \in C^\infty(W \oplus W)^K$ such that

$$f_\epsilon|_{B_\epsilon} = f \circ \exp^\perp.$$

Hence $f_\epsilon \circ (\exp^\perp)^{-1} = f$ on $T^*\Sigma \cap U_{(x,\xi)}$. Combined with Lemma 3.2 and the Slice Theorem, then f_ϵ is pulled back to be a smooth function on $G \times S^\perp_{(x,\xi)}(\epsilon)$ which descends to $F_\epsilon \in C^\infty(U_{(x,\xi)})^G$ such that $F_\epsilon = f$ on $T^*\Sigma \cap U_{(x,\xi)}$. We finish the proof of the surjectivity part of Theorem 3.2.

Let ω be the standard symplectic form on T^*M. We show that the restriction to $T^*\Sigma$ preserves Poisson brackets $(C^\infty(T^*M)^G, \{\ ,\ \}_1)$ and $(C^\infty(T^*\Sigma)^\Pi, \{\ ,\ \}_2)$, where $\{\ ,\ \}_{i=1,2}$ are Poisson brackets induced by ω and $\omega|_{T^*\Sigma}$, respectively.

Let $\mathring{M} \subseteq M$ be the union of principal orbits and $\mathring{\Sigma} = \Sigma \cap \mathring{M}$. Then $\mathring{\Sigma}$ is open and dense in Σ [6, Proposition 1.3]. It follows that $T^*\mathring{\Sigma} \subseteq T^*\Sigma$ is also open and dense. $\forall (x, \xi) \in T^*\mathring{\Sigma}$, we have the following orthogonal splitting with respect to the Sasaki metric $\tilde{g}$ on T^*M:

$$T_{(x,\xi)}T^*M \cong T_{(x,\xi)}G \cdot (x, \xi) \oplus JT_{(x,\xi)}G \cdot (x, \xi) \oplus T_{(x,\xi)}T^*\mathring{\Sigma}. \tag{3.11}$$

To see this, as $\mathring{\Sigma}$ consists of principal orbits, the slice representation at $x \in \mathring{\Sigma}$ is trivial. Hence $G_{(x,\xi)} = G_x$, $\forall (x, \xi) \in T^*\mathring{\Sigma}$. By [13], we also have $\dim G \cdot x + \dim \mathring{\Sigma} = \dim M$. Then the dimension of the vector space on the right-hand side of (3.11) is equal to

$$2 \dim G \cdot (x, \xi) + 2 \dim \mathring{\Sigma} = 2(\dim G \cdot x + \dim \mathring{\Sigma}) = 2 \dim M$$

which finishes the proof of (3.11).

$\forall f \in C^\infty(M)^G$, at $(x, \xi) \in T^*\mathring{\Sigma}$, we can write

$$X_f = X + JY + Z,$$

where $X, Y \in T_{(x,\xi)}G \cdot (x, \xi)$, $Z \in T_{(x,\xi)}T^*\mathring{\Sigma}$.

Recall that $i_{X_f}\omega = df$, ω is the standard symplectic form on T^*M. Since f is G-invariant, we get $(i_{X_f}\omega)(Y) = df(Y) = 0$. Then

$$\omega(X_f, Y) = \tilde{g}(JX_f, Y) = \tilde{g}(JX - Y + JZ, Y) = -\tilde{g}(Y, Y).$$

It follows that $Y = 0$.

Now let $f_1, f_2 \in C^\infty(T^*M)^G$, at $(x, \xi) \in T^*\mathring{\Sigma}$, we have

$$X_{f_i} = X_i + Z_i, \ i = 1, 2$$

where $X_i \in T_{(x,\xi)} G \cdot (x,\xi)$, $Z_i \in T_{(x,\xi)} T^* \mathring{\Sigma}$.

Let $\hat{f}_1 = f_1|_{T^*\mathring{\Sigma}}$, then we claim that $X_{\hat{f}_1} = Z_1$. In fact $i_{X_{\hat{f}_1}} \omega|_{T^*\Sigma} = d\hat{f}_1$. $\forall Y \in T_{(x,\xi)} T^* \mathring{\Sigma}$, we have

$$\begin{aligned}\tilde{g}\langle X_{\hat{f}_1}, Y\rangle &= \omega(X_{\hat{f}_1}, JY)\\ &= d\hat{f}_1(JY)\\ &= df_1(JY)\\ &= \omega(X_{f_1}, JY)\\ &= \tilde{g}\langle Z_1, Y\rangle.\end{aligned}$$

Then at (x,ξ),

$$\begin{aligned}\{f_1, f_2\}_1 &= \omega(X_{f_1}, X_{f_2})\\ &= \tilde{g}(JX_1 + JZ_1, X_2 + Z_2)\\ &= \tilde{g}(JZ_1, Z_2)\\ &= \omega|_{T^*\Sigma}(Z_1, Z_2)\\ &= \{\hat{f}_1, \hat{f}_2\}_2.\end{aligned}$$

By continuity, $\{f_1, f_2\}_1 = \{\hat{f}_1, \hat{f}_2\}_2$ on $T^*\Sigma$ everywhere.

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