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An inverse source problem for distributed order time-fractional diffusion equation

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Abstract

We consider an inverse time-dependent source problem governed by a distributed time-fractional diffusion equation using interior measurement data. Such a problem arises in some ultra-slow diffusion phenomena in many applied areas. Based on the regularity result of the solution to the direct problem, we establish the solvability of this inverse problem as well as the conditional stability in suitable function space with a weak norm. By a variational identity connecting the unknown time-dependent source and the interior measurement data, the conjugate gradient method is also introduced to construct the inversion algorithm under the framework of regularizing scheme. We show the validity of the proposed scheme by several numerical examples.

Keywords: diffusion process, distributed order time-fractional derivative, uniqueness, conditional stability, numerics

(Some figures may appear in colour only in the online journal)

1. Introduction

It was found by physicists in recent decades that the ultraslow diffusion phenomena arise in many applied areas such as polymer physics and particle's motion in a quenched random force field, where the mean square displacement (MSD) has a logarithmic growth [1–5]. Mathematically, such phenomena should be described by a diffusion equation with distributed order time-fractional derivatives, instead of classical advection–diffusion or time-fractional derivatives. The reason is that the MSD of the diffusive particles described by classical advection–diffusion equation behaves like $O(t)$ as $t \to \infty$, and a typical behavior of MSD in the framework of the time-fractional derivative model is $O(t^{\alpha})$ with $\alpha \in (0, 1)$ as $t \to \infty$ [6]. Thus, to describe the ultraslow diffusion phenomena, a distributed-order time fractional derivative

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$$D^{(\mu)}z(t) \equiv D^{(\mu)}_{0+}z(t) := \int_0^1 \mu(\alpha)\partial^{\alpha}_{0+}z(t)\,\mathrm{d}\alpha \tag{1.1}$$

with some weight function $\mu(\alpha) \geqslant 0$ for $\alpha \in [0, 1]$ should be introduced in terms of the Caputo fractional left derivative

$$\partial^{\alpha}_{0+}z(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{z'(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau, \quad 0 < \alpha < 1, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

To describe ultraslow diffusion phenomena in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with piecewise smooth boundary $\partial\Omega$, we consider the following distributed time-fractional diffusion system

$$\begin{cases} D^{(\mu)}u(x,t) - \mathbb{L}u = F(x,t), & (x,t) \in \Omega \times (0,T] := \Omega_T, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T] =: \partial\Omega_T, \\ u(x,0) = a(x), & x \in \Omega \end{cases} \tag{1.3}$$

for internal source $F(x,t)$, where $\mathbb{L}\diamond := \nabla \cdot (\sigma(x)\nabla\diamond)$ is a known elliptic operator with $\sigma(\cdot) \in C^1(\overline{\Omega}), \sigma\big|_{\overline{\Omega}} > 0$.

The time-fractional derivatives which have been applied to describe some slow diffusion phenomena, for example see [7–9], can be considered as the special case of the derivative in (1.3). Indeed, if we replace the smooth weight function $\mu(\alpha)$ in (1.3) by an impulse function $\mu(\alpha) = \delta(\alpha - \alpha_0)$ for known $\alpha_0 \in (0, 1)$, we can immediately deduce the so-called time-fractional diffusion equation

$$\partial^{\alpha_0}_{0+}u(x,t) - \mathbb{L}u = F(x,t), \quad (x,t) \in \Omega_T. \tag{1.4}$$

The direct problem for the governed equation (1.4) and the related inverse problems have drawn extensive attentions of researchers during the recent years, see [10] for a tutorial review and also [11–15] for several concrete inverse problems. Especially for the inverse source problems aiming to the determination of the source $F(x,t)$ in some special form in (1.4), we refer the readers to [8, 9, 16–18] for the uniqueness and stability.

However, to our knowledge, for the diffusion system (1.3) with more general weight function such as $\mu(\cdot) \in C[0, 1]$, there are few literatures concerned with both the direct problems and the related inverse problems. In fact, the well-posedness of the direct problem (1.3) depends heavily on the properties of the weight function $\mu(\alpha)$ as well as the sources $F(x,t), a(x)$, see [6, 19–21] and the references therein.

For the inverse problems, the uniqueness for recovering $\mu(\alpha)$ from interior measurement data based on the well-posedness for (1.3) have been considered in [19, 20, 22, 23]. The essences of these problems is to detect the system properties represented by the weight function $\mu(\alpha)$, i.e., to what extent of this system diffuses slowly.

However, in many engineering configurations, the ultraslow diffusion system is given or known, people are often required to detect the unknown sources leading to the diffusion phenomena governed by this system, from some measurable data of the physical field. In this case, the source term $F(x,t)$ in (1.3) is unknown. Motivated by the above situations, we are interested in the identification of $F(x,t)$ in special form $\beta(t)f(x)$ with unknown time-dependent ingredient $\beta(t)$ and known $f(x)$ in (1.3), from the interior measurement data

$$h(x,t) := u(x,t), \quad (x,t) \in \Omega_0 \times [0,T], \tag{1.5}$$

where $\Omega_0 \subset\subset \Omega$ is the observation domain. Although this inverse problem is linear, such an inverse problem is novel and difficult due to the average effect of the distributed time-fractional derivative including the slow diffusion effects for all $\alpha \in (0,1)$, i.e., the local information (1.5) about t represents the average message of the slow diffusion described by $\partial_t^\alpha u$ for each $\alpha \in (0,1)$. We will prove the uniqueness and stability of the inverse problem in suitable functional spaces, with specified weight function $\mu(\alpha)$.

This paper is organized as follows. In section 2, we state some well-known equivalent expressions of fractional derivatives, and analyze the property of the ordinary differential equation with distributed fractional derivative. Then we establish the regularity of the solution to direct diffusion system, which is crucial to the solvability of our inverse problem (1.3)–(1.5) in a suitable function space in section 3. In section 4, we construct a variational identity specifying the relation between the unknown time-dependent source and interior measurements. Based on this identity, the uniqueness and conditional stability of the inverse problem are established. Then a conjugate gradient method is utilized to solve the optimization version of the inverse problem with several examples in section 5.

2. Preliminaries

For constructing the adjoint system in our inversion scheme, we need the right-hand side Caputo fractional derivative given by

$$\partial_{T-}^{\alpha} z(t) := -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{z'(s)}{(s-t)^\alpha}\, \mathrm{d}s, \quad \alpha \in (0,1) \tag{2.1}$$

and the corresponding right-hand side distributed-order fractional derivative

$$D_{T-}^{(\mu)} z(t) := \int_0^1 \mu(\alpha) \partial_{T-}^{\alpha} z(t)\, \mathrm{d}\alpha. \tag{2.2}$$

Moreover, for $\alpha \in (0,1)$, the Riemann-Liouville fractional left (right) integral operators

$$\begin{cases} I_{0+}^{R,\alpha} z(t) := \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{z(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, & 0 < t \leqslant T, \\ I_{T-}^{R,\alpha} z(t) := \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^T \dfrac{z(s)}{(s-t)^{1-\alpha}}\, \mathrm{d}s, & 0 \leqslant t < T \end{cases} \tag{2.3}$$

and the Riemann-Liouville fractional left (right) derivatives

$$\begin{cases} \partial_{0+}^{R,\alpha} z(t) := \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t \dfrac{z(s)}{(t-s)^{\alpha}}\, \mathrm{d}s, & 0 < t \leqslant T, \\ \partial_{T-}^{R,\alpha} z(t) := -\dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_t^T \dfrac{z(s)}{(s-t)^{\alpha}}\, \mathrm{d}s, & 0 \leqslant t < T, \end{cases} \tag{2.4}$$

as well as the Carputo derivatives have the relations

$$\begin{cases} \partial_{0+}^{\alpha} z(t) = (I_{0+}^{R,1-\alpha} z')(t), \quad \partial_{T-}^{\alpha} z(t) = -(I_{T-}^{R,1-\alpha} z')(t), \\ \partial_{0+}^{R,\alpha} z(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}(I_{0+}^{R,1-\alpha} z)(t), \quad \partial_{T-}^{R,\alpha} z(t) = -\dfrac{\mathrm{d}}{\mathrm{d}t}(I_{T-}^{R,1-\alpha} z)(t) \end{cases} \tag{2.5}$$

for $z \in AC[0,T]$ [24, 25].

Denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product and $\|\cdot\|_2$ the corresponding L^2 norm. To represent the solution of the direct problem in terms of the eigenfunctions, introduce $\{\lambda_n, \varphi_n\}_{n=1}^{\infty}$ the Dirichlet eigensystem of $-\mathbb{L}$ satisfying $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \to \infty$ as $n \to \infty$, while $\{\varphi_n : n \in \mathbb{N}\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ is L^2-unified orthogonal eigenfunction set forming the basis of $L^2(\Omega)$. Then the operator $(-\mathbb{L})^{\gamma}$ for $\gamma \geqslant 0$ is defined by

$$(-\mathbb{L})^{\gamma} v := \sum_{n=1}^{\infty} \lambda_n^{\gamma} (v, \varphi_n)\varphi_n$$

with the domain $\mathcal{D}((-\mathbb{L})^{\gamma}) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma}(v,\varphi_n)^2 < \infty \right\}$, which is a Hilbert space with the norm

$$\|v\|_{\mathcal{D}((-\mathbb{L})^{\gamma})} = \left(\sum_{n=1}^{\infty} |\lambda_n^{\gamma}(v,\varphi_n)|^2 \right)^{1/2}.$$

It is well-known that $\mathcal{D}((-\mathbb{L})^{\gamma}) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$ and especially there hold

$$\mathcal{D}((-\mathbb{L})^0) = L^2(\Omega), \quad \mathcal{D}((-\mathbb{L})^{1/2}) = H_0^1(\Omega), \quad \mathcal{D}((-\mathbb{L})^1) = H^2(\Omega) \cap H_0^1(\Omega).$$

Furthermore, we set $\mathcal{D}((-\mathbb{L})^{-\gamma}) = (\mathcal{D}((-\mathbb{L})^{\gamma}))'$ for $\gamma > 0$. For $1 \leqslant p \leqslant \infty$ and a Banach space X, we say that $f \in L^p(0,T;X)$ provided

$$\|f\|_{L^p(0,T;X)} := \begin{cases} \left(\int_0^T \|f(\cdot,t)\|_X^p \,\mathrm{d}t \right)^{1/p} < \infty, & \text{if } 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}_{0<t<T} \|f(\cdot,t)\|_X < \infty, & \text{if } p = \infty. \end{cases} \tag{2.6}$$

Similarly, for $0 \leqslant t_0 < T$, denote by $C([t_0,T];X)$ the space with the norm $\|f\|_{C([t_0,T];X)} := \max_{[t_0,T]} \|f(\cdot,t)\|_X$ and define

$$C((0,T];X) := \bigcap_{0<t_0<T} C([t_0,T];X), \ \ C([0,\infty);X) := \bigcap_{T>0} C([0,T];X). \tag{2.7}$$

By $W_t^1((0,T];X)$ and $W_t^2((0,T];X)$ we denote the space of functions $g \in C^1((0,T];X)$ such that $g' \in L^1(0,T;X)$ and $g' \in L^2(0,T;X)$, respectively.

To consider the property of operator $D^{(\mu)}$, we introduce the function

$$g_{\mu}(t) := \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{e}^{-rt} \int_0^1 \sin(\pi\alpha) r^{\alpha} \mu(\alpha)\,\mathrm{d}\alpha}{(\int_0^1 \cos(\pi\alpha) r^{\alpha}\mu(\alpha)\,\mathrm{d}\alpha)^2 + (\int_0^1 \sin(\pi\alpha) r^{\alpha}\mu(\alpha)\,\mathrm{d}\alpha)^2}\,\mathrm{d}r \tag{2.8}$$

and define a convolution operator $I^{(\mu)}[z](t) := g_{\mu}(t) * z(t)$ in terms of $g_{\mu}(t)$.

Lemma 2.1. *For $0 \leqslant \mu \in L^1(0,1)$, $\mu \not\equiv 0$, it follows that*

(*a*) *$g_\mu(\cdot) \in L^1_{\mathrm{loc}}[0,\infty)$ with the estimate $0 \leqslant g_\mu(t) \leqslant Ct^{-(1-\zeta)}$ for $t \in (0,T]$, where $C := C(\mu, T) > 0$ is a constant independent of t and the constant $\zeta \in \left(0, \frac{1}{2}\right)$ is determined from*

$$\int_\zeta^{1-\zeta} \mu(\alpha)\,\mathrm{d}\alpha = \frac{1-\zeta}{2}\int_0^1 \mu(\alpha)\,\mathrm{d}\alpha > 0. \tag{2.9}$$

(*b*) *The operator $I^{(\mu)}$ is the inverse of $D^{(\mu)}$ in the sense that*

$$D^{(\mu)}I^{(\mu)}[z](t) = z(t), \quad I^{(\mu)}D^{(\mu)}[z](t) = z(t) - z(0) \quad \text{for } z \in AC[0,T]. \tag{2.10}$$

(*c*) *If $\mu(\alpha)$ also meets $\int_{1/2}^1 \mu(\alpha)\,\mathrm{d}\alpha > 0$, then there exists $\zeta^* \in (1/2, 3/4)$ determined by*

$$\int_{\zeta^*}^{3/2-\zeta^*} \mu(\alpha)\,\mathrm{d}\alpha = \zeta^*\int_{1/2}^1 \mu(\alpha)\,\mathrm{d}\alpha > 0 \tag{2.11}$$

such that $0 \leqslant g_\mu(t) \leqslant Ct^{-(1-\zeta^*)}$ for $t \in (0,T]$, where $C := C(\mu,T)$ is a positive constant independent of t, i.e., $g_\mu(\cdot) \in L^2(0,T)$.

Proof. The results (a) and (b) can be found in [21]. Let us prove conclusion (c). Consider the function $H(s)$: $\left[\frac{1}{2}, \frac{3}{4}\right] \mapsto \mathbb{R}$ defined by

$$H(s) := \int_s^{3/2-s} \mu(\alpha)\,\mathrm{d}\alpha - s\int_{1/2}^1 \mu(\alpha)\,\mathrm{d}\alpha.$$

Since $H(1/2)H(3/4) < 0$, there exists $\zeta^* \in (1/2, 3/4)$ satisfying

$$\int_{\zeta^*}^{3/2-\zeta^*} \mu(\alpha)\,\mathrm{d}\alpha = \zeta^*\int_{1/2}^1 \mu(\alpha)\,\mathrm{d}\alpha =: \zeta^* c_\mu > 0,$$

where $0 < c_\mu < \infty$ is a constant due to $\int_{1/2}^1 \mu(\alpha)\,\mathrm{d}\alpha > 0$ and $\mu \in L^1(0,1)$. Now it follows from

$$\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\,\mathrm{d}\alpha \geqslant \int_{\zeta^*}^{3/2-\zeta^*} r^\alpha \sin(\pi\alpha)\mu(\alpha)\,\mathrm{d}\alpha \geqslant \begin{cases} \zeta^* c_\mu \sin(\pi\zeta^*) r^{3/2-\zeta^*}, & 0 < r \leqslant 1, \\ \zeta^* c_\mu \sin(\pi\zeta^*) r^{\zeta^*}, & r \geqslant 1 \end{cases}$$

that

$$g_\mu(t) \leqslant \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-rt}}{\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\,\mathrm{d}\alpha}\,\mathrm{d}r$$
$$\leqslant c_{\mu,\zeta^*}\left(\int_0^1 \mathrm{e}^{-rt}r^{\zeta^*-3/2}\,\mathrm{d}r + \int_1^\infty \mathrm{e}^{-rt}r^{-\zeta^*}\,\mathrm{d}r\right) =: I_1 + I_2,$$

where $c_{\mu,\zeta^*} := (\pi\zeta^* c_\mu \sin(\pi\zeta^*))^{-1} > 0$ is a constant. For I_1, we have for $t, r > 0$ that

$$I_1(t) := c_{\mu,\zeta^*}\int_0^1 \mathrm{e}^{-rt}r^{\zeta^*-3/2}\,\mathrm{d}r \leqslant c_{\mu,\zeta^*}\int_0^1 r^{\zeta^*-3/2}\,\mathrm{d}r = \frac{2c_{\mu,\zeta^*}}{2\zeta^*-1}. \tag{2.12}$$

As for $I_2(t)$, we have for $t > 0$ that

$$I_2(t) := c_{\mu,\zeta^*}\int_1^\infty \mathrm{e}^{-rt}r^{-\zeta^*}\,\mathrm{d}r \leqslant c_{\mu,\zeta^*}\int_0^\infty \mathrm{e}^{-rt}r^{-\zeta^*}\,\mathrm{d}r = c_{\mu,\zeta^*}\Gamma(1-\zeta^*)t^{-(1-\zeta^*)} \tag{2.13}$$

due to the Laplace transform $\mathcal{L}\{r^{-\zeta^*}\}(t) = \Gamma(1-\zeta^*)t^{\zeta^*-1}$. Noting $\zeta^* \in (1/2, 3/4)$, by (2.12) we can further obtain for $t \in (0, T]$ that

$$I_1(t) \leqslant \frac{2c_{\mu,\zeta^*}}{2\zeta^*-1} = \frac{2c_{\mu,\zeta^*}t^{1-\zeta^*}}{(2\zeta^*-1)t^{1-\zeta^*}} \leqslant \left(\frac{2c_{\mu,\zeta^*}T^{1-\zeta^*}}{2\zeta^*-1}\right)t^{-(1-\zeta^*)}, \quad 0 < t \leqslant T. \tag{2.14}$$

Combining (2.13) and (2.14), we immediately have

$$0 \leqslant g_\mu(t) \leqslant I_1 + I_2 \leqslant C_{\mu,\zeta^*,T}\,t^{-(1-\zeta^*)}, \quad 0 < t \leqslant T, \tag{2.15}$$

where $C_{\mu,\zeta^*,T} := \max\left\{\frac{2c_{\mu,\zeta^*}T^{1-\zeta^*}}{2\zeta^*-1}, c_{\mu,\zeta^*}\Gamma(1-\zeta^*)\right\} > 0$ is a constant independent of t. Noticing $\zeta^* \in (1/2, 3/4)$, it is clear that $2(1-\zeta^*) \in (1/2, 1)$. Thus, (2.15) implies that $g_\mu(\cdot) \in L^2(0, T)$. The proof is complete. □

To represent the solution to (1.3), we define $Q_n(t)$ for $n \in \mathbb{N}^+$ by the following system

$$\begin{cases} D^{(\mu)}Q_n(t) + \lambda_n Q_n(t) = 0, & t \in (0, T], \\ Q_n(0) = 1. \end{cases} \tag{2.16}$$

The unique existence of solution to (2.16) can be found in [20, 21]. Based on lemma 2.1, $Q_n(t)$ has the following properties.

Lemma 2.2. *For $0 \leqslant \mu \in C[0, 1]$ satisfying $\mu \not\equiv 0$, $\mu(0) \neq 0$, it follows that*

(a) $Q_n(\cdot) \in C[0, \infty) \cap C^\infty(0, \infty)$ *for any $n \in \mathbb{N}^+$ is completely monotone, which means*

$$(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m}Q_n(t) \geqslant 0 \tag{2.17}$$

for all $t > 0$ and $m = 0, 1, \ldots$.

(*b*) $0 \leqslant Q_n(t) \leqslant 1$ *for* $t \in [0, T]$; $t^{1-\zeta}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\cdot) \in C[0, T]$ *with the bound*

$$0 \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t) \leqslant C\lambda_n t^{-(1-\zeta)}, \quad t \in (0, T], \tag{2.18}$$

where $C > 0$ *is a constant and the constant* $\zeta \in (0, 1/2)$ *is determined by* (2.9).

(*c*) *If* $\mu(\alpha)$ *also meets* $\int_{1/2}^{1} \mu(\alpha)\,\mathrm{d}\alpha > 0$, *then* $t^{1-\zeta^*}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\cdot) \in C[0, T]$ *with the bound*

$$0 \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t) \leqslant C\lambda_n t^{-(1-\zeta^*)}, \quad t \in (0, T], \tag{2.19}$$

where $C > 0$ *is a constant and the constant* $\zeta^* \in (1/2, 3/4)$ *is determined by* (2.11).

Proof. The first conclusion can be proven based on the Laplace transform for (2.16), see (i) and (iii) in theorem 2.3 in [6], corresponding to the case $\lambda = -\lambda_n < 0$ there.

Now we prove conclusions (b) and (c). By the regularity of Q_n stated in conclusion (a), it follows from lemma 2.1 that the ODE problem (2.16) is equivalent to the following integral equation

$$Q_n(t) + \lambda_n \int_0^t g_\mu(t-\tau) Q_n(\tau)\,\mathrm{d}\tau = 1. \tag{2.20}$$

Noticing $Q_n(\cdot) \geqslant 0$ by conclusion (a), we then have $Q_n(\cdot) \leqslant 1$ immediately from $g_\mu \geqslant 0$.

Differentiating (2.20) with respect to t, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t) - \lambda_n \int_0^t g_\mu(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\tau)\,\mathrm{d}\tau = \lambda_n g_\mu(t).$$

Since $-\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\cdot) \geqslant 0$ from conclusion (a) and $g_\mu(\cdot) \geqslant 0$, the second term in the left hand side is nonnegative, therefore we have

$$0 \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t) \leqslant \lambda_n g_\mu(t), \quad t \in (0, T],$$

which leads to (2.18) and (2.19) by conclusions (a) and (b) in lemma 2.1 respectively, with the constant $\zeta \in (0, 1/2)$ determined by (2.9) and the constant $\zeta^* \in (1/2, 3/4)$ determined by (2.11), respectively. The regularity $t^{1-\zeta}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\cdot) \in C[0, T]$ can be found in [21]. The proof is complete. □

By similar arguments in [26], we give an inequality for right-hand side Caputo fractional derivative.

Lemma 2.3. *For any function* $v(t)$ *absolutely continuous on* $[0, T]$, *there holds*

$$v(t)\partial_{T-}^{\alpha} v(t) \geqslant \frac{1}{2}\partial_{T-}^{\alpha}(v^2), \quad \alpha \in (0, 1). \tag{2.21}$$

We also need the following standard result (lemma 2.7 in [24]).

Lemma 2.4. *Let* $\alpha > 0$, $p \geqslant 1$, $q \geqslant 1$, *and* $1/p + 1/q \leqslant 1 + \alpha$ ($p \neq 1$ *and* $q \neq 1$ *in the case when* $1/p + 1/q = 1 + \alpha$). *For* $u \in L^p(0, T)$ *and* $v \in L^q(0, T)$, *there holds*

$$\int_0^T u(t) I_{0+}^{R,\alpha} v(t)\,\mathrm{d}t = \int_0^T v(t) I_{T-}^{R,\alpha} u(t)\,\mathrm{d}t.$$

In the sequel, we always assume that the weight function satisfies

$$\mu \in C[0,1], \quad 0 \leqslant \mu \not\equiv 0,\ \ \mu(0) \neq 0, \tag{2.22}$$

and we also use $C > 0$ to represent a constant which may depends on f, a, Ω, T, μ, but does not depend on x, t. The values of C may be different in the sequel.

To show the regularity of the solution to the direct problem, we need the following result.

Lemma 2.5. *For $\lambda_n > 0$ and $Q_n(t)$ with $\mu(\alpha)$ satisfying (2.22), denote*

$$p_1^n(t) = \int_0^t \beta(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} Q_n(t-\tau)\ \mathrm{d}\tau, \quad t \in (0, +\infty), \tag{2.23}$$

$$p_2^n(t) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\beta(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} Q_n(\tau)\ \mathrm{d}\tau, \quad t \in (0, \infty), \tag{2.24}$$

and define $p_1^n(0) = 0$, $p_2^n(0) = 0$ for any $n \in \mathbb{N}^+$. Then $p_1^n \in C[0,+\infty)$ for $\beta(\cdot) \in C[0,+\infty)$ and $p_2^n \in C[0,+\infty)$ for $\beta(\cdot) \in C^1[0,+\infty)$.

Proof. For $t > 0$, $n \in \mathbb{N}^+$ and $\beta \in C[0,+\infty)$, by lemma 2.2 we have

$$|p_1^n(t)| = \left| \int_0^t \beta(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} Q_n(t-\tau)\,\mathrm{d}\tau \right| \leqslant \|\beta\|_\infty [1 - Q_n(t)] \to 0, \quad t \to 0^+, \tag{2.25}$$

hence p_1^n is continuous at $t = 0$. For $\beta \in C^1[0,T]$, we can prove $p_2^n(t)$ is continuous at $t = 0$ analogously. The continuity of $p_1^n(t)$ and $p_2^n(t)$ for any $t > 0$ follows from lemma 2.2.

The proof is complete. □

3. Strong solution to direct problem

To establish the conditional stability for our inverse problem in next section, we need high regularity of solution to (1.3), which is also essential to establish the integration formula by parts of distributed order fractional derivatives. To this end, we consider the strong solution to (1.3), which is defined as follows.

Definition 3.1. We call $u(x,t)$ a strong solution to initial-boundary value problem (1.3), if

- $u \in C(\overline{\Omega} \times [0,T]) \cap W_t^1((0,T]; C(\overline{\Omega}))$, $\quad D^{(\mu)}u \in C(\overline{\Omega} \times [0,T])$;
- The initial condition is satisfied in the sense $\lim_{t\to 0^+} \|u(\cdot,t) - a\|_{C(\overline{\Omega})} = 0$;
- The equation $D^{(\mu)}u - \mathbb{L}u = F$ holds in $C(\overline{\Omega} \times [0,T])$.

Theorem 3.2. *Let $F(x,t) := \beta(t) f(x)$ with $\beta \in C^1[0,T]$. Then for $a \in \mathcal{D}((-\mathbb{L})^{\gamma+1})$ and $f \in \mathcal{D}((-\mathbb{L})^{\gamma})$ with $\gamma > \frac{d}{2}$, the direct problem (1.3) admits a unique strong solution, which is represented by*

$$u(x,t)=\sum_{n=1}^{\infty}(a,\varphi_n)Q_n(t)\varphi_n(x)+\sum_{n=1}^{\infty}\frac{1}{\lambda_n}(f,\varphi_n)\int_0^t\beta(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}Q_n(t-\tau)\,\mathrm{d}\tau\varphi_n(x) \tag{3.1}$$

with the estimates

$$\|u\|_{C(\overline{\Omega}\times[0,T])}\leqslant C\left(\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}+\|\beta\|_{C[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma-1})}\right), \tag{3.2}$$

$$\|D^{(\mu)}u\|_{C(\overline{\Omega}\times[0,T])}\leqslant C\left(\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma+1})}+\|\beta\|_{C[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}\right). \tag{3.3}$$

Moreover, if $\int_{1/2}^1\mu(\alpha)\mathrm{d}\alpha>0$, *we also have* $u\in W_t^2((0,T];C(\overline{\Omega}))$.

Proof. By the separation of variables, we can obtain a formal solution for the direct problem (1.3) as (3.1). We will show that (3.1) is the defined strong solution to (1.3).

(a) We firstly verify $u\in C(\overline{\Omega}\times[0,T])$ and then give the estimate (3.2). Define

$$u_1(x,t):=\sum_{n=1}^{\infty}(a,\varphi_n)Q_n(t)\varphi_n(x)=\sum_{n=1}^{\infty}u_1^n(x,t), \tag{3.4}$$

$$u_2(x,t):=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}(f,\varphi_n)\int_0^t\beta(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}Q_n(t-\tau)\,\mathrm{d}\tau\varphi_n(x)=\sum_{n=1}^{\infty}u_2^n(x,t). \tag{3.5}$$

We note $\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\cdot)\leqslant 0$, $0\leqslant Q_n\leqslant 1$ from lemma 2.2 and

$$\|\varphi_n\|_{C(\overline{\Omega})}\leqslant\|\varphi_n\|_{H^{2k}(\Omega)}\leqslant\|(-\mathbb{L})^k\varphi_n\|_{L^2(\Omega)}\leqslant C\lambda_n^k,\quad n=1,2,\ldots \tag{3.6}$$

for $k>\frac{d}{4}$. For every $t\in[0,T]$, $x\in\overline{\Omega}$, we have for $k>\frac{d}{4}$ and $\beta\in C[0,T]$ that

$$|u_1^n(x,t)|\leqslant|(a,\varphi_n)|\|\varphi_n\|_\infty\leqslant C\lambda_n^k|(a,\varphi_n)|=:C\tilde{u}_1^n \tag{3.7}$$

and

$$\begin{aligned}|u_2^n(x,t)|&\leqslant\|\beta\|_\infty|1-Q_n(t)||(f,\varphi_n)|\frac{1}{\lambda_n}\|\varphi_n\|_\infty\\&\leqslant C\|\beta\|_\infty\lambda_n^{k-1}|(f,\varphi_n)|=:C\|\beta\|_\infty\tilde{u}_2^n,\end{aligned} \tag{3.8}$$

implying that

$$\sum_{n=1}^{\infty}|u_1^n(x,t)|\leqslant C\sum_{n=1}^{\infty}|\tilde{u}_1^n|\leqslant C\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2\,\mathrm{m}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{2(k+m)}(a,\varphi_n)^2\right)^{\frac{1}{2}} \tag{3.9}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}|u_2^n(x,t)| &\leqslant C\|\beta\|_\infty\sum_{n=1}^{\infty}|\tilde{u}_2^n| \\ &\leqslant C\|\beta\|_\infty\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2\,\mathrm{m}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{2(k+m-1)}(f,\varphi_n)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.10}$$

From $\lambda_n = O(n^{2/d})$ for $n\in\mathbb{N}^+$, we know $\frac{1}{\lambda_n^{2\,\mathrm{m}}}\leqslant C\frac{1}{n^{\frac{4\,\mathrm{m}}{d}}}$. Thus, by choosing $m>\frac{d}{4}$ which generates $\gamma := m+k > \frac{d}{2}$, $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2\,\mathrm{m}}}$ is convergent and the series in (3.4) and (3.5) are also convergent in $\overline{\Omega}\times[0,T]$ uniformly for $a\in\mathcal{D}((-\mathbb{L})^{\gamma})$ and $f\in\mathcal{D}((-\mathbb{L})^{\gamma-1})$ with $\gamma>\frac{d}{2}$. Noting $u_1^n, u_2^n\in C(\overline{\Omega}\times[0,T])$ for any $n\in\mathbb{N}^+$ by lemmas 2.2 and 2.5, we have $u\in C(\overline{\Omega}\times[0,T])$. Furthermore, by (3.9) and (3.10) we immediately have the estimate

$$\|u\|_{C(\overline{\Omega}\times[0,T])}\leqslant C\left(\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}+\|\beta\|_{C[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma-1})}\right). \tag{3.11}$$

(b) We further prove $\mathbb{L}u, D^{(\mu)}u\in C(\overline{\Omega}\times[0,T])$ and (3.3). By (3.1) we know

$$\begin{aligned}\mathbb{L}u(x,t) &= -\sum_{n=1}^{\infty}\lambda_n(a,\varphi_n)Q_n(t)\varphi_n(x)-\sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t\beta(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}Q_n(t-\tau)\,\mathrm{d}\tau\varphi_n(x)\\ &:= v_1(x,t)+v_2(x,t).\end{aligned} \tag{3.12}$$

Similarly, by lemma 2.2, $\lambda_n=O(n^{2/d})$ as well as (3.6), we can prove the series in $v_1(x,t)$ and $v_2(x,t)$ are uniformly convergent in $\overline{\Omega}\times[0,T]$ for every $x\in\overline{\Omega}$ and $t\in[0,T]$, if $a\in\mathcal{D}((-\mathbb{L})^{\gamma+1})$ and $f\in\mathcal{D}((-\mathbb{L})^{\gamma})$ with $\gamma>\frac{d}{2}$. This immediately leads to $\mathbb{L}u\in C(\overline{\Omega}\times[0,T])$ due to lemma 2.5 that the each term in the series in $v_1(x,t)$ and $v_2(x,t)$ are continuous in $\overline{\Omega}\times[0,T]$ for $\beta\in C[0,T]$. Also, it is easy to obtain the estimate

$$\|\mathbb{L}u\|_{C(\overline{\Omega}\times[0,T])}\leqslant C\left(\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma+1})}+\|\beta\|_{C[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}\right). \tag{3.13}$$

Henceforth, by the Sobolev embedding $\mathcal{D}((-\mathbb{L})^{\gamma})\subset C(\overline{\Omega})$ with $\gamma>\frac{d}{4}$, it follows by (2.16), (3.13) and

$$D^{(\mu)}u(x,t)=\mathbb{L}u(x,t)+\beta(t)f(x)$$

that $D^{(\mu)}u\in C(\overline{\Omega}\times[0,T])$ and

$$\|D^{(\mu)}u\|_{C(\overline{\Omega}\times[0,T])}\leqslant C\left(\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma+1})}+\|\beta\|_{C[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}\right).$$

Obviously the equation $D^{(\mu)}u-\mathbb{L}u=F$ holds in $C(\overline{\Omega}\times[0,T])$ under the given conditions.

(c) We next prove $u\in W_t^1((0,T];C(\overline{\Omega}))$. By the formal computation we have

$$\frac{\partial u_1}{\partial t}=\sum_{n=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)(a,\varphi_n)\varphi_n(x) \tag{3.14}$$

and

$$\frac{\partial u_2}{\partial t} = -\sum_{n=1}^{\infty}\frac{1}{\lambda_n}(f,\varphi_n)\varphi_n(x)\beta(0)\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)$$
$$-\sum_{n=1}^{\infty}\frac{1}{\lambda_n}(f,\varphi_n)\varphi_n(x)\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\beta(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}Q_n(\tau)\,\mathrm{d}\tau =: \tilde{I}_1(x,t)+\tilde{I}_2(x,t). \tag{3.15}$$

We note that each term in the series in (3.14) and (3.15) are continuous in $\overline{\Omega}\times(0,T]$ due to lemmas 2.2 and 2.5. Similar as before, we have for every $0<t\leqslant T$ and $x\in\overline{\Omega}$ that

$$\left|\frac{\partial u_1}{\partial t}(x,t)\right| \leqslant \sum_{n=1}^{\infty}\left|\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)\right||(a,\varphi_n)|\|\varphi_n\|_\infty \leqslant C\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma+1})}t^{-(1-\zeta)}, \tag{3.16}$$

$$\left|\tilde{I}_1(x,t)\right| \leqslant \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|(f,\varphi_n)\right|\|\varphi_n\|_\infty\left|\beta(0)\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)\right| \leqslant C|\beta(0)|\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}t^{-(1-\zeta)}, \tag{3.17}$$

and

$$\begin{aligned}\left|\tilde{I}_2(x,t)\right| &\leqslant \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|(f,\varphi_n)\right|\|\varphi_n\|_\infty\left|\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\beta(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}Q_n(\tau)\,\mathrm{d}\tau\right|\\ &\leqslant \|\beta\|_{C^1[0,T]}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}|(f,\varphi_n)|\|\varphi_n\|_\infty[1-Q_n(t)]\\ &\leqslant \|\beta\|_{C^1[0,T]}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}|(f,\varphi_n)|\|\varphi_n\|_\infty \leqslant C\|\beta\|_{C^1[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma-1})},\end{aligned}$$

where $\zeta\in(0,1/2)$ is determined by (2.9). This implies the series representations in (3.14) and (3.15) are uniformly convergent in $\overline{\Omega}\times[t_0,T]$ with any small $0<t_0<T$ and the above differentiations make sense in $C(\overline{\Omega})$ for $0<t\leqslant T$, if $\beta\in C^1[0,T]$, $a\in\mathcal{D}((-\mathbb{L})^{\gamma+1})$ and $f\in\mathcal{D}((-\mathbb{L})^{\gamma})$ with $\gamma>d/2$. Therefore, we have $u\in C^1((0,T];C(\overline{\Omega}))$ and estimate

$$\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{C(\overline{\Omega})} \leqslant C\left(\|a\|_{\mathcal{D}((-\mathbb{L})^{\gamma+1})}t^{-(1-\zeta)}+\|\beta\|_{C^1[0,T]}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma})}(1+t^{-(1-\zeta)})\right), \tag{3.18}$$

which implies $\partial_t u\in L^1(0,T;C(\overline{\Omega}))$.

If $\mu(\alpha)$ also meets $\int_{1/2}^1\mu(\alpha)\,\mathrm{d}\alpha>0$, then we have the estimate (2.19) in lemma 2.2. Therefore, replacing $\zeta\in(0,1/2)$ in (3.16) and (3.17) by $\zeta^*\in(1/2,3/4)$, we know $\partial_t u(x,\cdot)\in L^2(0,T)$ for every $x\in\overline{\Omega}$ and $\partial_t u\in L^2(0,T;C(\overline{\Omega}))$, which implies $u\in W_t^2((0,T];C(\overline{\Omega}))$.

(d) We finally prove the uniqueness of the strong solution to problem (1.3). It is enough to prove that the system (1.3) has only a trivial solution under the condition $F(x,t):=\beta(t)f(x)=0$, $a=0$. Denote $u_n(t)=(u(\cdot,t),\varphi_n)$, we obtain

$$\begin{cases} D^{(\mu)}u_n(t) + \lambda_n u_n(t) = 0, & t \in (0, T], \\ u_n(0) = 0. \end{cases}$$

Due to the existence and uniqueness of the ordinary distributed order fractional differential equation [20], we obtain that $u_n(t) \equiv 0$ for any $n \in \mathbb{N}^+$, which implies $u = 0$ in $\overline{\Omega} \times [0, T]$. The proof is complete. □

By analogous arguments to the proof of theorem 3.2, we immediately have

Remark 3.3. Let $F(x, t) := \beta(t) f(x)$ with $\beta \in C[0, T]$. For $f(x)$ and $a(x)$ satisfying the conditions in theorem 3.2, (3.1) admits a unique solution to problem (1.3) in the sense that

$$u \in C(\overline{\Omega} \times [0, T]), \quad D^{(\mu)}u \in C(\overline{\Omega} \times [0, T]), \quad \lim_{t \to 0+} \|u(\cdot, t) - a\|_{C(\overline{\Omega})} = 0, \tag{3.19}$$

and equation $D^{(\mu)}u - \mathbb{L}u = F$ holds in $L^2(\Omega_T)$. We also have the estimates (3.2) and (3.3).

4. Uniqueness and stability of inverse problem

Without loss of generality, we assume the null initial status, i.e., we consider the identification of the time-dependent source $\beta(t)$ in the system

$$\begin{cases} D^{(\mu)}u(x, t) - \mathbb{L}u = \beta(t) f(x), & (x, t) \in \Omega_T, \\ u(x, t) = 0, & (x, t) \in \partial\Omega_T, \\ u(x, 0) = 0, & x \in \Omega, \end{cases} \tag{4.1}$$

from the interior measurement given by (1.5).

We focus on the uniqueness and stability of this inverse problem.

4.1. Uniqueness of inverse problem

We will show the uniqueness of inverse problem based on the explicit expression of solution and its regularity $D^{(\mu)}u \in C(\overline{\Omega} \times [0, T])$. The similar arguments have been applied for the inverse source problem in the case of $\mu(\alpha) = \delta(\alpha - \alpha_0)$ with given $\alpha_0 \in (0, 1)$ in [7, 9].

Theorem 4.1. *Suppose $\beta \in C[0, T]$, $f \in \mathcal{D}\left((-\mathbb{L})^{\gamma+1}\right)$ with $\gamma > \frac{d}{2}$ and $f(x_0) \neq 0$ for the observation point $x_0 \in \Omega_0$. Then there exists a constant $C := C(\Omega, T, f, x_0, \zeta) > 0$ such that*

$$\|\beta_1 - \beta_2\|_{C[0,T]} \leqslant C \|D^{(\mu)} (u[\beta_1](x_0, \cdot) - u[\beta_2](x_0, \cdot))\|_{C[0,T]}. \tag{4.2}$$

Proof. For given $f \in \mathcal{D}((-\mathbb{L})^{\gamma})$ with $\gamma > \frac{d}{2}$, denote $U = u[\beta_1] - u[\beta_2]$, $\tilde{\beta} = \beta_1 - \beta_2 \in C[0, T]$. Then $U(x, t)$ solves

$$\begin{cases} D^{(\mu)}U - \mathbb{L}U = \tilde{\beta}(t)f(x), & (x,t) \in \Omega_T, \\ U(x,t) = 0, & (x,t) \in \partial\Omega_T, \\ U(x,0) = 0, & x \in \Omega. \end{cases} \tag{4.3}$$

By remark 3.3, we obtain

$$D^{(\mu)}U(x,t) = \tilde{\beta}(t)f(x) + \sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t \tilde{\beta}(\tau)\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t-\tau)\,\mathrm{d}\tau\varphi_n(x) \tag{4.4}$$

in $C(\overline{\Omega}\times[0,T])$. Henceforth, we have

$$D^{(\mu)}U(x_0,t) = \tilde{\beta}(t)f(x_0) + \sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t \tilde{\beta}(\tau)\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t-\tau)\,\mathrm{d}\tau\varphi_n(x_0) \tag{4.5}$$

for $t \in [0,T]$. Setting

$$P(x_0,t) := \sum_{n=1}^{\infty}(f,\varphi_n)t^{1-\zeta}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)\varphi_n(x_0) \tag{4.6}$$

with ζ determined by (2.9), we have

$$D^{(\mu)}U(x_0,t) = \tilde{\beta}(t)f(x_0) + \int_0^t \tilde{\beta}(\tau)(t-\tau)^{\zeta-1}P(x_0,t-\tau)\,\mathrm{d}\tau \tag{4.7}$$

in $C[0,T]$. By lemma 2.2 we know $t^{1-\zeta}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(\cdot) \in C[0,T]$ and $\left|t^{1-\zeta}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)\right| \leqslant C\lambda_n$ for each $n \in \mathbb{N}$. Then it follows by the asymptotic $\lambda_n = O(n^{2/d})$ and the regularity of φ_n as shown in (3.6) that

$$\sum_{n=1}^{\infty}\left\|(f,\varphi_n)t^{1-\zeta}\frac{\mathrm{d}}{\mathrm{d}t}Q_n(t)\varphi_n(x_0)\right\|_{C[0,T]} \leqslant C\sum_{n=1}^{\infty}|\lambda_n(f,\varphi_n)|\|\varphi_n(x)\|_{C(\overline{\Omega})} \leqslant \tilde{C}\|f\|_{\mathcal{D}((-\mathbb{L})^{\gamma+1})} \tag{4.8}$$

for $t \in [0,T]$ uniformly. Therefore the series in (4.6) is uniformly convergent on $[0,T]$ for $f \in \mathcal{D}((-\mathbb{L})^{\gamma+1})$ with $\gamma > \frac{d}{2}$, which says $P(x_0,\cdot) \in C[0,T]$. So we have

$$\tilde{\beta}(t) = \frac{D^{(\mu)}U(x_0,t)}{f(x_0)} - \frac{1}{f(x_0)}\int_0^t \tilde{\beta}(\tau)(t-\tau)^{\zeta-1}P(x_0,t-\tau)\,\mathrm{d}\tau, \quad t \in [0,T] \tag{4.9}$$

from $f(x_0) \neq 0$, which leads to

$$|\tilde{\beta}(t)| \leqslant \frac{1}{|f(x_0)|}\left(\left\|D^{(\mu)}U(x_0,t)\right\|_{C[0,T]} + \|P(x_0,\cdot)\|_{C[0,T]}\int_0^t (t-\tau)^{\zeta-1}|\tilde{\beta}(\tau)|\,\mathrm{d}\tau\right), \quad t \in [0,T].$$

Applying the inequality of Gronwall type with weakly singular kernel $(t-\tau)^{\zeta-1}$ (e.g. lemma 7.1.1 in [27] or theorem 1 in [28]), the above inequality leads to

$$|\tilde{\beta}(t)| \leqslant C(t)\left\|D^{(\mu)}U(x_0,t)\right\|_{C[0,T]}, \quad t\in[0,T], \tag{4.10}$$

where the constant $C(t) := \left(1+\zeta^{-1}t^{\zeta}\right)\ \max\left\{|f(x_0)|^{-1}, \|P(x_0,\cdot)\|_{C[0,T]}\right\} > 0$.

The proof is complete. □

As a direct result of this estimate and the regularity $u[\beta_i] \in C(\overline{\Omega}\times[0,T])$, we have proven the following uniqueness result for our inverse problem.

Theorem 4.2. *For known spatial function $f(x)$ satisfying the condition in theorem 4.1 and $\beta_i \in C[0,T]$, it holds $\beta_1(t) = \beta_2(t)$ in $C[0,T]$, if $u[\beta_1](x_0,\cdot) = u[\beta_2](x_0,\cdot)$ in $C[0,T]$.*

Remark 4.3. The *a priori* requirement $f(x_0)\neq 0$ is assumed for the observation location for our uniqueness result. Such a restriction can be relaxed, by specifying the observation data in a domain $\Omega_0 \subset\subset \Omega$ satisfying $\int_{\Omega_0} f(x)\,\mathrm{d}x \neq 0$, rather than a fixed point $x_0\in\Omega$, due to the following identity

$$D^{(\mu)}\int_{\Omega_0} U(x,t)\mathrm{d}x = \tilde{\beta}(t)\int_{\Omega_0} f(x)\mathrm{d}x + \int_0^t \tilde{\beta}(\tau)(t-\tau)^{\zeta-1}\int_{\Omega_0} P(x,t-\tau)\mathrm{d}x\,\mathrm{d}\tau \tag{4.11}$$

in $C[0,T]$ by (4.7) from which we can also show the uniqueness for the inverse problem. On the other hand, we should also point out that, from the theoretical point of view, the condition $\int_{\Omega_0} f(x)\,\mathrm{d}x \neq 0$ leads to $f(x_0)\neq 0$ for some $x_0\in\Omega_0$ immediately. However $\int_{\Omega_0} f(x)\mathrm{d}x \neq 0$ is easy to be guaranteed by the average measurement in Ω_0 satisfying $\operatorname{supp} f(x) \subset\subset \Omega_0$.

4.2. Variational identity

To establish a variational identity which is necessary for establishing the stability for our inverse problem, we firstly introduce the adjoint problem

$$\begin{cases} \mathcal{S}^{(\mu)}_{T-}\phi - \mathbb{L}\phi = F(x,t), & (x,t)\in\Omega\times(0,T), \\ \phi(x,t) = 0, & (x,t)\in\partial\Omega\times[0,T), \\ I^{(\mu),R}_{T-}\phi(x,T) = 0, & x\in\Omega, \end{cases} \tag{4.12}$$

where the operator $\mathcal{S}^{(\mu)}_{T-}\phi := -\,\partial_t I^{(\mu),R}_{T-}\phi$, and the distributed order right integral operator $I^{(\mu),R}_{T-}$ is defined by

$$I^{(\mu),R}_{T-}\phi := \int_0^1 \mu(\alpha) I^{R,1-\alpha}_{T-}\phi\ \mathrm{d}\alpha, \tag{4.13}$$

where $I^{R,1-\alpha}_{T-}$ defined by (2.3) denotes the $(1-\alpha)$-th order Riemann-Liouville fractional right integral operator.

Now we consider the weak solution to adjoint problem (4.12).

Definition 4.4. For $F\in L^2(\Omega_T)$, we call $\phi(x,t)$ a weak solution to (4.12), if

- $\phi\in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$, $\mathcal{S}^{(\mu)}_{T-}\phi\in L^2(\Omega_T)$;
- The final condition holds in the sense $\lim_{t\to T^-}\left\|I^{(\mu),R}_{T-}\phi(\cdot,t)\right\|_{L^2(\Omega)} = 0$;
- The equation $\mathcal{S}^{(\mu)}_{T-}\phi - \mathbb{L}\phi = F(x,t)$ holds in the sense

$$(\mathcal{S}_{T-}^{(\mu)}\phi - \mathbb{L}\phi, \eta) = (F(\cdot\,, t), \eta), \quad \forall \eta \in L^2(\Omega),\ t > 0.$$

Then we show the well-posedness of adjoint problem (4.12) in the sense of definition 4.4.

Theorem 4.5. *Let $F(x,t) \in L^2(\Omega_T)$. Then there exists a unique weak solution to (4.12), which meets the estimate*

$$\|\phi\|_{L^2(0,T;H^2(\Omega))} + \|\mathcal{S}_{T-}^{(\mu)}\phi\|_{L^2(\Omega_T)} \leqslant 3\|F\|_{L^2(\Omega_T)}. \tag{4.14}$$

Moreover, if $\int_{1/2}^{1}\mu(\alpha)\,\mathrm{d}\alpha > 0$, we also have $\phi \in C([0,T]; L^2(\Omega))$.

Proof. The proof is under the same framework as that for theorem 4 in [21], which was basically based on the formal representation of the solution in terms of the eigenvalue function expansions for the existence of the solution and then the energy estimate for the regularity of the solution. Here we only give the outline of the proof.

Firstly we extend F by odd reflection to the interval $(0,T)$ and set zero elsewhere. For the standard smoothing kernel $\xi_\epsilon(t)$ with the support in $[-\epsilon, \epsilon]$, denote by $\overline{*}$ the convolution on real line and set

$$F_{\frac{1}{N}}(x,t) := \xi_{\frac{1}{N}}(\cdot)\overline{*}F(x,\cdot)(t). \tag{4.15}$$

We then construct

$$\phi^N(x,t) = \sum_{n=1}^{N} \phi_n^N(t)\varphi_n(x), \quad N \in \mathbb{N}^+, \tag{4.16}$$

which will be proven to generate the weak solution to (4.12) as $N \to \infty$ in some way, where $\phi_n^N(t)$ is determined by

$$\begin{cases} \mathcal{S}_{T-}^{(\mu)}\phi_n^N(t) + \lambda_n\phi_n^N(t) = F_n^N(t) := \left(F_{\frac{1}{N}}(\cdot, t), \varphi_n\right)_{L^2(\Omega)}, & t \in [0,T), \\ I_{T-}^{(\mu),R}\phi_n^N(T) = 0 \end{cases} \tag{4.17}$$

with the operator $I_{T-}^{(\mu),R}$ defined in (4.13). To ensure the well-posedness of (4.17), we firstly notice that, it was proven in lemma 11 in [21] that, for $F \in L^2(\Omega_T)$, there exists a unique solution to

$$\begin{cases} D^{(\mu)}\psi_n^N(t) + \lambda_n\psi_n^N(t) = F_n^N(T-t), & t \in (0,T], \\ \psi_n^N(0) = 0, \end{cases}$$

in $AC[0,T]$. Therefore $\tilde{\psi}_n^N(t) := \psi_n^N(T-t) \in AC[0,T]$ meets

$$\begin{cases} D_{T-}^{(\mu)}\tilde{\psi}_n^N(t) + \lambda_n\tilde{\psi}_n^N(t) = F_n^N(t), & t \in [0,T), \\ \tilde{\psi}_n^N(T) = 0. \end{cases}$$

On the other hand, for $\tilde{\psi}_n^N(t) \in AC[0,T]$, it follows

$$\mathcal{S}_{T-}^{(\mu)}\tilde{\psi}_n^N := -\int_0^1 \mu(\alpha)\partial_t I_{T-}^{R,1-\alpha}\tilde{\psi}_n^N \,\mathrm{d}\alpha = \int_0^1 \mu(\alpha)\partial_{T-}^{R,\alpha}\tilde{\psi}_n^N \,\mathrm{d}\alpha$$
$$= \int_0^1 \mu(\alpha)\partial_{T-}^{\alpha}\tilde{\psi}_n^N \,\mathrm{d}\alpha =: D_{T-}^{(\mu)}\tilde{\psi}_n^N$$

due to $\tilde{\psi}_n^N(T) = 0$. So there exists a unique solution $\phi_n^N(t) := \tilde{\psi}_n^N(t) \in AC[0,T]$ to (4.17).

Multiplying both sides of the equation in (4.17) by $\phi_n^N(t) \in AC[0,T]$ and integrating for $t \in (0,T)$, we have

$$\int_0^1 \mu(\alpha)\int_0^T \phi_n^N(t)\partial_{T-}^{\alpha}\phi_n^N(t)\,\mathrm{d}t\,\mathrm{d}\alpha + \lambda_n\|\phi_n^N\|_{L^2(0,T)}^2 = \left(F_n^N,\phi_n^N\right)_{L^2(0,T)}. \tag{4.18}$$

Using (2.5) and lemma 2.3 for $\phi_n^N(t) \in AC[0,T]$, we have

$$\int_0^T \phi_n^N(t)\partial_{T-}^{\alpha}\phi_n^N(t)\mathrm{d}t \geqslant \int_0^T \frac{1}{2}\partial_{T-}^{\alpha}(\phi_n^N)^2\,\mathrm{d}t = -\int_0^T \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{\left(\phi_n^N(\tau)\right)^2}{(\tau-t)^{\alpha}}\,\mathrm{d}\tau\right)\mathrm{d}t$$
$$= \frac{1}{2\Gamma(1-\alpha)}\int_0^T \frac{\left(\phi_n^N(\tau)\right)^2}{\tau^{\alpha}}\mathrm{d}\tau \geqslant 0.$$

So (4.18) leads to $\lambda_n\left\|\phi_n^N\right\|_{L^2(0,T)} \leqslant \left\|F_n^N\right\|_{L^2(0,T)}$, which also generates

$$\begin{aligned}\left\|\mathcal{S}_{T-}^{(\mu)}\phi_n^N\right\|_{L^2(0,T)} &= \left\|D_{T-}^{(\mu)}\phi_n^N\right\|_{L^2(0,T)}\\ &\leqslant \lambda_n\left\|\phi_n^N\right\|_{L^2(0,T)} + \left\|F_n^N\right\|_{L^2(0,T)} \leqslant 2\left\|F_n^N\right\|_{L^2(0,T)}\end{aligned} \tag{4.19}$$

from the equation in (4.17). Based on these estimates on ϕ_n^N, for ϕ^N constructed in (4.16), similarly to the arguments in section 3.3 in [21], there exists a subsequence of $\{\phi^N : n \in \mathbb{N}^+\}$, denoted by $\{\phi^{N_j} : j = 1,2,\ldots\}$, weakly converging to $\phi \in L^2(0,T;H^2(\Omega))$ in $L^2(0,T;H^2(\Omega))$, and $I_{T-}^{(\mu),R}\phi^{N_j} \rightharpoonup I_{T-}^{(\mu),R}\phi$ in ${}_0H^1(0,T;L^2(\Omega))$ which means $\mathcal{S}_{T-}^{(\mu)}\phi^{N_j} \rightharpoonup \mathcal{S}_{T-}^{(\mu)}\phi$ in $L^2(\Omega_T)$. Therefore the first two-point in the definition 4.4 is verified for ϕ. On the other hand, it is easy to see that

$$\mathcal{S}_{T-}^{(\mu)}\phi^N - \mathbb{L}\phi^N = \sum_{n=1}^N F_n^N(t)\varphi_n(x) = \sum_{n=1}^N (F(\cdot,t),\varphi_n)\varphi_n(x) - \sum_{n=1}^N (F(\cdot,t) - F_{\frac{1}{N}}(\cdot,t),\varphi_n)\varphi_n(x)$$

and

$$\left\|\sum_{n=1}^N (F(\cdot,t) - F_{\frac{1}{N}}(\cdot,t),\varphi_n)\varphi_n(x)\right\|_{L^2(\Omega)}^2 = \sum_{n=1}^N (F(\cdot,t) - F_{\frac{1}{N}}(\cdot,t),\varphi_n)^2$$
$$\leqslant \sum_{n=1}^\infty (F(\cdot,t) - F_{\frac{1}{N}}(\cdot,t),\varphi_n)^2 = \|F(\cdot,t) - F_{\frac{1}{N}}(\cdot,t)\|_{L^2(\Omega)}^2,$$

which verify the third point of definition 4.4 by taking $N \to \infty$. So we have the existence of the weak solution.

By the above convergence property, we also have

$$
\begin{aligned}
\|\phi\|^2_{L^2(0,T;H^2(\Omega))} &\leqslant \lim_{N_j\to\infty} \left\|\phi^{N_j}\right\|^2_{L^2(0,T;H^2(\Omega))} \\
&= \lim_{N_j\to\infty} \sum_{n=1}^{N_j} \lambda_n^2 \left\|\phi_n^{N_j}\right\|^2_{L^2(0,T)} \leqslant \lim_{N_j\to\infty} \sum_{n=1}^{N_j} \left\|F_n^{N_j}\right\|^2_{L^2(0,T)} \\
&\leqslant \lim_{N_j\to\infty} \sum_{n=1}^{\infty} \left\|F_n^{N_j}\right\|^2_{L^2(0,T)} = \lim_{N_j\to\infty} \left\|F_{\frac{1}{N_j}}\right\|^2_{L^2(\Omega_T)} \leqslant \|F\|^2_{L^2(\Omega_T)}
\end{aligned}
$$

and $\|\mathcal{S}^{(\mu)}_{T-}\phi\|_{L^2(\Omega_T)} \leqslant 2\|F\|_{L^2(\Omega_T)}$ analogously by (4.19). So we prove the estimate (4.14).

As for the higher regularity $\phi \in C([0,T], L^2(\Omega))$ for the weight function satisfying $\int_{1/2}^1 \mu(\alpha)\,\mathrm{d}\alpha > 0$, the readers are referred to the proof of theorem 4 in [21].

The proof is complete. □

In the sequel, we always assume

$$
\int_{1/2}^{1} \mu(\alpha)\,\mathrm{d}\alpha > 0, \tag{4.20}
$$

which is essential to ensure the regularity of solution $u \in W_t^2((0,T]; C(\overline{\Omega}))$ to (4.1). Then we can establish the integration relation between (4.1) and its adjoint system (4.12).

Lemma 4.6. *Let $u(x,t)$ be the strong solution to problem (4.1) for $\beta \in C^1[0,T]$, $f \in \mathcal{D}((-\mathbb{L})^\gamma)$ with $\gamma > \frac{d}{2}$ and $\phi(x,t)$ be the weak solution to problem (4.12) for $F \in L^2(\Omega_T)$. Then $u(x,t)$ and $\phi(x,t)$ have the relation*

$$
\int_0^T \int_\Omega D^{(\mu)} u(x,t)\,\phi(x,t)\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega u(x,t)\,\mathcal{S}^{(\mu)}_{T-}\phi(x,t)\,\mathrm{d}x\mathrm{d}t. \tag{4.21}
$$

Proof. Noting the assumption (4.20) for $\mu(\alpha)$, by theorems 4.5 and 3.2, we know $\phi[\omega](\cdot,\cdot) \in L^2(0,T; H^2(\Omega)\cap H_0^1(\Omega))$ and $u \in C(\overline{\Omega}\times[0,T]) \cap W_t^2((0,T]; C(\overline{\Omega}))$, which implies $\phi(x,\cdot) \in L^2(0,T)$ and $\partial_t u(x,\cdot) \in L^2(0,T)$, respectively. Thus, for every fixed $x \in \Omega$ and all $\alpha \in (0,1)$, lemma 2.4 is applicable for $\phi(x,\cdot)$ and $\partial_t u(x,\cdot)$. Then, by applying $u(x,0) = 0$, (2.4) and (2.5) as well as lemma 2.4, we have for any fixed $x \in \Omega$ that

$$
\begin{aligned}
\int_0^T (\partial^\alpha_{0+} u)\phi\,\mathrm{d}t &= \int_0^T \left(I^{R,1-\alpha}_{0+}\partial_t u\right)\cdot\phi\,\mathrm{d}t = \int_0^T \partial_t u \cdot \left(I^{R,1-\alpha}_{T-}\phi\right)\,\mathrm{d}t \\
&= u\cdot (I^{R,1-\alpha}_{T-}\phi)|_{t=0}^{t=T} - \int_0^T u\cdot\partial_t\left(I^{R,1-\alpha}_{T-}\phi\right)\,\mathrm{d}t \\
&= u(x,T)\cdot I^{R,1-\alpha}_{T-}\phi(x,T) - \int_0^T u\cdot\partial_t\left(I^{R,1-\alpha}_{T-}\phi\right)\,\mathrm{d}t,
\end{aligned} \tag{4.22}
$$

where we should understand $I^{R,1-\alpha}_{T-}\phi(x,T)$ in the sense $\lim_{t\to T^-} I^{R,1-\alpha}_{T-}\phi(x,t)$.

Multiplying both sides of (4.22) by $\mu(\alpha)$ and integrating for $\alpha \in (0,1)$ and $x \in \Omega$ yield

$$\begin{aligned}\int_0^1 \mu(\alpha)\int_0^T \int_\Omega (\partial_{0+}^\alpha u)\phi\,\mathrm{d}x\mathrm{d}t\mathrm{d}\alpha &= u(x,T)\cdot \int_0^1 \mu(\alpha) I_{T-}^{R,1-\alpha}\phi(x,T)\,\mathrm{d}\alpha \\ &\quad - \int_0^1 \mu(\alpha)\int_0^T\int_\Omega u\cdot \partial_t\left(I_{T-}^{R,1-\alpha}\phi\right)\,\mathrm{d}x\mathrm{d}t\mathrm{d}\alpha \\ &= u(x,T)\cdot I_{T-}^{(\mu),R}\phi(x,T) + \int_0^T\int_\Omega u\cdot \mathcal{S}_{T-}^{(\mu)}\phi\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.23}$$

Noting the definition of $D^{(\mu)}u$ and $I_{T-}^{(\mu),R}\phi(x,T)=0$ which should be understood in the sense $\lim_{t\to T^-}\|I_{T-}^{(\mu),R}\phi(\cdot,t)\|_{L^2(\Omega)}=0$, we immediately have (4.21). The proof is complete. □

Based on lemma 4.6, we can establish the following variational identity.

Theorem 4.7. *For $f\in\mathcal{D}((-\mathbb{L})^\gamma)$ with $\gamma>\frac{d}{2}$ and $\beta_i\in C^1[0,T]$ for $i=1,2$, $u[\beta_i](x,t)$ and β_i have the relation*

$$\int_{\Omega_T}(\beta_1-\beta_2)(t)f(x)\phi[\omega](x,t)\,\mathrm{d}x\mathrm{d}t \equiv \int_0^T\int_{\Omega_0}(u[\beta_1]-u[\beta_2])(x,t)\omega(x,t)\,\mathrm{d}x\mathrm{d}t, \tag{4.24}$$

where $\phi[\omega](x,t)$ is the weak solution to (4.12) with the specified source $F\in L^2(\Omega_T)$ given by

$$F(x,t):=\begin{cases}\omega(x,t), & (x,t)\in\Omega_0\times[0,T),\\ 0, & (x,t)\in\Omega\backslash\overline{\Omega}_0\times[0,T)\end{cases} \tag{4.25}$$

for $\omega\in L^2(\Omega_0\times(0,T))$.

Proof. Let $u[\beta_i](x,t)$ be the strong solution to direct problem (4.1) corresponding to β_i for $i=1,2$, then $U:=u[\beta_1]-u[\beta_2]$ satisfies

$$\begin{cases}D^{(\mu)}U(x,t)-\mathbb{L}U=f(x)(\beta_1(t)-\beta_2(t)), & (x,t)\in\Omega_T,\\ U(x,t)=0, & (x,t)\in\partial\Omega_T,\\ U(x,0)=0, & x\in\Omega.\end{cases} \tag{4.26}$$

Let $\phi(x,t):=\phi[\omega](x,t)$ be the weak solution to (4.12) with the source term given by (4.25). Then, multiplying two sides of the equation in (4.26) by $\phi(x,t)$ and integrating on Ω_T, we get

$$\int_0^T\int_\Omega\left(D^{(\mu)}U-\mathbb{L}U\right)\phi(x,t)\,\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega(\beta_1-\beta_2)(t)f(x)\phi(x,t)\,\mathrm{d}x\mathrm{d}t.$$

Using lemma 4.6 and Green formula, we know

$$\int_0^T \int_\Omega (D^{(\mu)}U - \mathbb{L}U)\phi \, \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega U(\mathcal{S}^{(\mu)}_{T-}\phi - \mathbb{L}\phi) \, \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Omega_0} U \cdot \omega(x,t) \, \mathrm{d}x\mathrm{d}t.$$

The proof is complete. □

4.3. Lipschitz stability of inverse problem

Now we apply theorem 4.7 to establish the Lipschitz stability by some weak norm to our inverse problem, as done in [16]. For $f \in \mathcal{D}((-\mathbb{L})^{\gamma+1})$ with $\gamma > \frac{d}{2}$ and $f \not\equiv 0$ in Ω_0, we define a bilinear functional with respect to $\beta(t)$ and $\omega(x,t)$ in terms of $f(x)$ by

$$\mathcal{B}(\beta, \omega) := \int_{\Omega_T} \beta(t) f(x) \phi[\omega](x,t) \, \mathrm{d}x\mathrm{d}t, \tag{4.27}$$

where $\phi[\omega](x,t)$ is a linear functional of $\omega \in L^2(\Omega_0 \times (0,T))$ defined by (4.12) with source term given by (4.25). It is easy to see from theorem 4.5 that $\mathcal{B}(f, \omega)$ is well defined for $\beta \in C^1[0,T]$ and $\omega \in L^2(\Omega_0 \times (0,T))$. Define

$$\|\beta\|_{\mathcal{B}} := \frac{1}{\|f\|_{L^2(\Omega)}} \sup_{\omega \in W} \frac{|\mathcal{B}(\beta,\omega)|}{\|\omega\|_{L^2(\Omega_0 \times (0,T))}}, \tag{4.28}$$

where $W := \{\omega : \omega \in L^2(\Omega_0 \times (0,T)), \omega \not\equiv 0\}$.

Lemma 4.8. *For $\beta \in C^1[0,T]$, $\|\beta\|_{\mathcal{B}}$ is a norm of β and $\|\beta\|_{\mathcal{B}} \leqslant C\|\beta\|_{L^2(0,T)}$.*

Proof. Firstly, it is easy to see that $\|f\|_{\mathcal{B}}$ is well-defined. In fact, it follows that

$$\begin{aligned} |\mathcal{B}(\beta,\omega)| &\leqslant C\|\beta\|_{L^2(0,T)}\|f\|_{L^2(\Omega)}\|\phi[\omega]\|_{L^2(\Omega_T)} \\ &\leqslant C\|\beta\|_{L^2(0,T)}\|f\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega_0 \times (0,T))} \end{aligned} \tag{4.29}$$

due to theorem 4.5, which says $\frac{|\mathcal{B}(\beta,\omega)|}{\|f\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega_0 \times (0,T))}} \leqslant C\|\beta\|_{L^2(0,T)}$ uniformly for all $\omega \in W$. So $\|\beta\|_{\mathcal{B}}$ is well-defined satisfying $\|\beta\|_{\mathcal{B}} \leqslant C\|\beta\|_{L^2(0,T)}$. Now we prove that this quantity is also a norm for $\beta \in C^1[0,T]$. It is enough to prove that $\|\beta\|_{\mathcal{B}} = 0$ leads to $\beta = 0$.

In fact, if $\|\beta\|_{\mathcal{B}} = 0$, we have

$$\frac{|\mathcal{B}(\beta,\omega)|}{\|f\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega_0 \times (0,T))}} \leqslant \|\beta\|_{\mathcal{B}} = 0, \quad \forall \omega \in W \tag{4.30}$$

and $\mathcal{B}(\beta,\omega) = 0$ for $\omega = 0$ due to $\phi[\omega]|_{\omega=0} = 0$ by theorem 4.5. Therefore we have $\mathcal{B}(\beta,\omega) = 0$ for any $\omega \in L^2(\Omega_0 \times (0,T))$, which leads to

$$\int_0^T \int_{\Omega_0} u[\beta](x,t)\omega(x,t) \, \mathrm{d}x\mathrm{d}t = 0, \quad \forall \omega \in L^2(\Omega_0 \times (0,T)) \tag{4.31}$$

by variational identity (4.24), so $u[\beta](x,t) = 0$ in $L^2(\Omega_0 \times (0,T))$. By theorem 3.2 we see that $u[\beta] \in C(\Omega_0 \times [0,T])$ under the given conditions for $\beta(t)$ and $f(x)$, so we have $u[\beta](x,t) = 0$

in $C(\Omega_0 \times [0,T])$. Finally by theorem 4.1, we have $\beta(t) = 0$, i.e., $\|\beta\|_{\mathcal{B}}$ is a norm of β. The proof is complete. □

Now we can establish the conditional stability of Lipschitz type for the inverse problem by the weighted norm $\|\beta\|_{\mathcal{B}}$.

Theorem 4.9. *For the direct problem (4.1) with $f \in \mathcal{D}((-\mathbb{L})^{\gamma+1}), \gamma > \frac{d}{2}$ and $f \not\equiv 0$ in Ω_0, denote by $u[\beta_i](x,t)$ the solution corresponding to $\beta_i \in C^1[0,T]$ for $i = 1, 2$. Then it follows that*

$$\|\beta_1 - \beta_2\|_{\mathcal{B}} \leqslant \frac{1}{\|f\|_{L^2(\Omega)}} \|u[\beta_1] - u[\beta_2]\|_{L^2(\Omega_0 \times (0,T))}. \tag{4.32}$$

Proof. By (4.24) and the definition (4.28), we have

$$\begin{aligned} \|\beta_1 - \beta_2\|_{\mathcal{B}} &= \sup_{\omega \in W} \frac{|\mathcal{B}(\beta_1 - \beta_2, \omega)|}{\|f\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega_0 \times (0,T))}} \\ &= \sup_{\omega \in W} \frac{|\int_0^T \int_{\Omega_0} (u[\beta_1] - u[\beta_2]) \omega(x,t) \, \mathrm{d}x \mathrm{d}t|}{\|f\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega_0 \times (0,T))}}. \end{aligned} \tag{4.33}$$

The proof is complete from the Cauchy inequality. □

Remark 4.10. In this theorem, for the unknown β restricted in $C^1[0,T]$, we only established its Lipschitz continuous dependance by our introduced $\mathcal{B}$-norm, which is even weaker than the $L^2(0,T)$-norm in terms of our lemma 4.8. Therefore, theorem 4.9 is essentially the conditional stability for our inverse problem in the sense that we use the weak norm for a very smooth function.

5. Inversion algorithm and implementations

In this section, we first introduce the conjugate gradient method (CGM) based on the theoretical results in above. Then we present the numerical inversions for three examples in one-dimensional and two-dimensional cases to show the effectiveness of the proposed CG algorithm.

5.1. The conjugate gradient method

For known $f \in \mathcal{D}((-\mathbb{L})^{\gamma})$ with $\gamma > \frac{d}{2}$, (4.1) defines an observation operator $G : \beta \in C^1[0,T] \mapsto u[\beta] \in C(\overline{\Omega} \times [0,T]) \cap W_t^2((0,T); C(\overline{\Omega}))$, where $u[\beta]$ is a strong solution to (4.1) for given $\beta(t)$. Then the inverse problem is to solve the operator equation

$$G[\beta](x,t) = h(x,t), \quad (x,t) \in \Omega_0 \times [0,T]. \tag{5.1}$$

By theorem 3.2, G is a well-defined linear bounded operator. For exact observation data, this equation has a unique solution from theorem 4.2. For given noisy data $h^\delta(x,t)$ satisfying

$$\|h^\delta - h\|_{L^2(\Omega_0 \times (0,T))} \leqslant \delta, \tag{5.2}$$

we consider the approximate solution of (5.1) by the minimizer of the cost functional

$$\mathcal{J}_{\delta,\eta}(\beta) := \frac{1}{2}\|G[\beta] - h^{\delta}\|^2_{L^2(\Omega_0\times(0,T))} + \frac{\eta}{2}\|\beta\|^2_{L^2(0,T)} \tag{5.3}$$

for specified $\eta > 0$.

Based on the regularity of direct problem shown in theorems 3.2 and 4.5, we will show the differentiability of $\mathcal{J}_{\delta,\eta}(\beta)$ and compute its gradient for our inversion algorithm by the variational identity in theorem 4.7.

Theorem 5.1. *The functional $\mathcal{J}_{\delta,\eta}$ is Fréchet differentiable. For given $f(x)$ and any fixed $\eta > 0$, the Fréchet derivative of $\mathcal{J}_{\delta,\eta}$ defined in (5.3) is*

$$\mathcal{J}'_{\delta,\eta}(\beta)(t) = \int_{\Omega} f(x)\phi[\omega_{\beta,\delta}](x,t)\,\mathrm{d}x + \eta\beta(t), \tag{5.4}$$

where $\phi := \phi[\omega_{\beta,\delta}](x,t)$ satisfies the adjoint problem (4.12) corresponding to the input

$$F(x,t) = \begin{cases} \omega_{\beta,\delta}(x,t) := G[\beta](x,t) - h^{\delta}(x,t), & (x,t)\in\Omega_0\times[0,T),\\ 0, & (x,t)\in\Omega\backslash\overline{\Omega}_0\times[0,T). \end{cases} \tag{5.5}$$

Proof. For $\beta, \beta+\delta\beta \in C^1[0,T]$, it follows from (5.3) that

$$\begin{aligned} &\mathcal{J}_{\delta,\eta}(\beta+\delta\beta) - \mathcal{J}_{\delta,\eta}(\beta)\\ &\quad = \int_0^T\int_{\Omega_0}(G[\beta]-h^{\delta})(G[\beta+\delta\beta]-G[\beta])\,\mathrm{d}x\mathrm{d}t + \eta\int_0^T \beta\,\delta\beta\mathrm{d}t + o(\|\delta\beta\|_2), \end{aligned} \tag{5.6}$$

Let $\beta_1 = \beta + \delta\beta$, $\beta_2 = \beta$ and take the input for the adjoint problem (4.12) as (5.5). Since $\beta_1, \beta_2 \in C^1[0,T]$ and $F \in L^2(\Omega_T)$, the variational identity (4.24) is applicable for $G[\beta_1], G[\beta_2]$ and $\phi[\omega_{\beta,\delta}](x,t)$. Then combining (4.24) with (5.6) yields that

$$\begin{aligned} \mathcal{J}'_{\delta,\eta}(\beta)\bullet\delta\beta &= \int_0^T\int_{\Omega_0}\omega_{\beta,\delta}(x,t)(G[\beta_1]-G[\beta_2])\,\mathrm{d}x\mathrm{d}t + \eta\int_0^T\beta\,\delta\beta\mathrm{d}t\\ &= \int_0^T\int_{\Omega} f(x)\,\delta\beta(t)\cdot\phi[\omega_{\beta,\delta}](x,t)\,\mathrm{d}x\mathrm{d}t + \eta\int_0^T\beta\,\delta\beta\mathrm{d}t, \end{aligned} \tag{5.7}$$

i.e., the Fréchet derivative of $\mathcal{J}_{\delta,\eta}$ is (5.4). The proof is complete. □

By the gradient $\mathcal{J}'_{\delta,\eta}(\beta)$, we propose an iteration scheme applying the conjugate gradient (CG) method for generating the minimizer of $\mathcal{J}_{\delta,\eta}$ approximately. We approximate $\beta(t)$ by the following iterative process

$$\beta_{k+1} = \beta_k + r_k d_k, \quad k = 0, 1, \ldots \tag{5.8}$$

for suitably chosen step size $r_k > 0$ and initial guess β_0, where d_k is the iterative direction defined by

$$d_k = \begin{cases} -\mathcal{J}'_{\delta,\eta}(\beta_0), & \text{if } k = 0, \\ -\mathcal{J}'_{\delta,\eta}(\beta_k) + s_k d_{k-1}, & \text{if } k > 0 \end{cases} \tag{5.9}$$

with

$$s_k = \frac{\|\mathcal{J}'_{\delta,\eta}(\beta_k)\|_2^2}{\|\mathcal{J}'_{\delta,\eta}(\beta_{k-1})\|_2^2}, \quad r_k = \arg\min_{r\geqslant 0} \mathcal{J}_{\delta,\eta}(\beta_k + r d_k). \tag{5.10}$$

Since the operator G is linear, we have $G[\beta_k + r_k d_k] = G[\beta_k] + r_k G[d_k]$. Then there holds

$$\begin{aligned} \mathcal{J}_{\delta,\eta}(\beta_k + r_k d_k) = & \frac{1}{2}\left[\|G[\beta_k] - h^\delta\|_2^2 + {r_k}^2\|G[d_k]\|_2^2\right] + r_k\left(G[\beta_k] - h^\delta, G[d_k]\right) \\ & + \frac{\eta}{2}\left(\|\beta_k\|_2^2 + {r_k}^2\|d_k\|_2^2 + 2r_k(\beta_k, d_k)\right). \end{aligned} \tag{5.11}$$

To determine the step size r_k, by $\frac{\mathrm{d}\mathcal{J}_{\delta,\eta}(\beta_k + r d_k)}{\mathrm{d}r} = 0$, it is easy to obtain

$$r_k = -\frac{\left(G[\beta_k] - h^\delta, G[d_k]\right)_2 + \eta(\beta_k, d_k)_2}{\|G[d_k]\|_2^2 + \eta\|d_k\|_2^2}. \tag{5.12}$$

We summarize the CGM for reconstructing the unknown $\beta(t)$ as follows:

- Step 1: set $k = 0$, the initial guess β_0.
- Step 2: compute $d_0(t) = -\mathcal{J}'_{\delta,\eta}(\beta_0)$ from (5.4).
- Step 3: compute the step size $r_0 > 0$ from (5.12) and update $\beta_1(t) = \beta_0(t) + r_0 d_0(t)$.
- Step 4: for $k = 1, \ldots,$ compute s_k, $d_k(t)$ and r_k by (5.9), (5.10) and (5.12), respectively.
- Step 5: update $\beta_{k+1}(t) = \beta_k(t) + r_k d_k(t)$. If a stopping criterion is satisfied, output $\beta_{k+1}(t)$ and stop. Otherwise, set $k+1 \Rightarrow k$ and go to step 4.

5.2. The finite difference scheme to direct problem

Without the loss of generality, we consider the following one-dimensional problem

$$\begin{cases} D^{(\mu)}u(x,t) = u_{xx}(x,t) + F(x,t), & (x,t) \in (0,l) \times (0,T], \\ u(0,t) = u(l,t) = 0, & t \in (0,T], \\ u(x,0) = a(x), & x \in [0,l]. \end{cases} \tag{5.13}$$

Firstly we discrete the space domain by $x_i = i\Delta h (i = 0, 1, \ldots, M)$, and the time domain by $t_n = n\tau (n = 0, 1, \ldots, N)$, here $\Delta h = l/M$ is the space mesh step and $\tau = T/N$ is the time mesh step. For the weight function $\mu(\alpha), \alpha \in [0,1]$, we discrete its variable by $\alpha_s = s\Delta\alpha (s = 0, 1, \ldots, 2S)$ with the mesh step size $\Delta\alpha = 1/(2S)$. By the composite trapezoid formula we have for the distributed order fractional derivative that

$$D^{(\mu)}u(x_i, t_{n+1}) = \sum_{s=0}^{2S} r_s \partial_{0+}^{\alpha_s} u(x_i, t_{n+1}) + O(\Delta\alpha^2), \tag{5.14}$$

where the coefficients $r_s := \Delta\alpha c_s \mu(\alpha_s)$ with $c_1 = c_{2S} = 1/2$ and $c_s = 1$ for $2 \leqslant s \leqslant 2S - 1$.

For the fractional derivative $\partial_{0+}^{\alpha} u(x,t)$ with different order, we define

$$\partial_{0+}^{\alpha} u(x,t) := \begin{cases} u(x,t) - u(x,0), & \alpha = 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{1}{(t-\tau)^{\alpha}} \frac{\partial u}{\partial \tau}(x,\tau)\,\mathrm{d}\tau, & 0 < \alpha < 1, \\ \dfrac{\partial u}{\partial t}(x,t), & \alpha = 1. \end{cases} \tag{5.15}$$

Then, by the well-known L^1 approximation we compute (5.15) for $\alpha_s \in [0,1]$ that

$$\partial_{0+}^{\alpha_s} u(x_i, t_{n+1}) = \frac{\tau^{-\alpha_s}}{\Gamma(2-\alpha_s)} \sum_{k=0}^{n} \left[u(x_i, t_{n+1-k}) - u(x_i, t_{n-k})\right] e_k^{\alpha_s} + O(\tau), \tag{5.16}$$

where we defined

$$e_k^{\alpha} := \begin{cases} (k+1)^{1-\alpha} - k^{1-\alpha}, & 0 < \alpha < 1,\ k = 0,1,\ldots,n, \\ 1, & \alpha = 0,\ k = 0,1,\ldots,n, \\ 1, & \alpha = 1,\ k = 0, \\ 0, & \alpha = 1,\ k = 1,\ldots,n. \end{cases} \tag{5.17}$$

As for the integer-order derivative u_{xx} in (1.1), we apply the central difference scheme given as

$$u_{xx}(x_i, t_{n+1}) = \frac{u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})}{\Delta h^2} + O(\Delta h^2). \tag{5.18}$$

Denote by $u_i^n = u(x_i, t_n)$, $F_i^n = F(x_i, t_n)$, $a_i = a(x_i)$, $\mu_s = \mu(\alpha_s)$ and substituting (5.14)–(5.18) into (1.1), and ignoring the remainder terms, we get

$$\begin{cases} -p u_{i-1}^{n+1} + (1+2p) u_i^{n+1} - p u_{i+1}^{n+1} = u_i^n - \sum_{k=1}^{n} q_k (u_i^{n+1-k} - u_i^{n-k}) + \frac{1}{R} F_i^{n+1}, \\ u_i^0 = a_i, \quad u_0^n = u_M^n = 0 \end{cases} \tag{5.19}$$

together with the initial and boundary value conditions, where the coefficients are defined by

$$\begin{cases} R := \sum_{s=0}^{2S} r_s \frac{\tau^{-\alpha_s}}{\Gamma(2-\alpha_s)}, \quad p := \frac{1}{R\Delta h^2}, \\ q_k := \dfrac{1}{R} \sum_{s=0}^{2S} r_s \frac{\tau^{-\alpha_s}}{\Gamma(2-\alpha_s)} e_k^{\alpha_s}, \quad k = 0,1,\ldots,n. \end{cases} \tag{5.20}$$

By (2.22) we know $r_s \geqslant 0$ for $s = 0,1,\ldots,2S$ and there exists at least one $s \in \{1,\ldots,2S-1\}$ such that $r_s > 0$ and $R > 0$.

For $n = 1,2,\ldots,N$, we define the vectors

$$U^n = (u_1^n, u_2^n, \ldots, u_{M-1}^n)^T, \quad \mathbb{F}^n = (F_1^n, F_2^n, \ldots, F_{M-1}^n)^T, \quad a = (a_1, a_2, \ldots, a_{M-1})^T$$

and the matrix $B = (b_{ij})_{(M-1)\times(M-1)}$, with the elements for $i, j = 1, \ldots, M-1$ that

$$b_{ij} = \begin{cases} -p, & j = i \pm 1, \\ 1 + 2p, & j = i, \\ 0, & \text{else}. \end{cases} \tag{5.21}$$

Then, with the help of the coefficients $\{z_k : k = 1, \ldots, n\}$ defined by

$$z_1 := 1 - q_1, \quad z_k := q_{k-1} - q_k \quad \text{for } k = 2, \ldots, n, \tag{5.22}$$

we get the implicit finite difference scheme in matrix form

$$\begin{cases} BU^1 = U^0 + \dfrac{\mathbb{F}^1}{R}, \quad U^0 = a; \\ BU^{n+1} = z_1 U^n + z_2 U^{n-1} + \cdots + z_n U^1 + q_n U^0 + \dfrac{\mathbb{F}^{n+1}}{R} \end{cases} \tag{5.23}$$

for solving U^n iteratively.

Proposition 5.2. *The matrix B is strictly diagonally dominant, so the iterative scheme (5.23) can be uniquely solved. For z_k and q_k given by (5.22) and (5.20), respectively, there hold $z_k > 0$ ($k = 1, 2, \ldots, n$), and*

$$\sum_{k=1}^{n} z_k + q_n = 1. \tag{5.24}$$

Proof. By (5.20), (5.22) and the definition (5.17), we have

$$\begin{aligned} z_k = q_{k-1} - q_k &= \frac{1}{R}\sum_{s=0}^{2S} r_s \frac{\tau^{-\alpha_s}}{\Gamma(2-\alpha_s)}[e_{k-1}^{\alpha_s} - e_k^{\alpha_s}] \\ &= \frac{1}{R}\sum_{s=1}^{2S-1} r_s \frac{\tau^{-\alpha_s}}{\Gamma(2-\alpha_s)}[e_{k-1}^{\alpha_s} - e_k^{\alpha_s}] + \frac{1}{R} r_{2S}\tau^{-1}[e_{k-1}^{1} - e_k^{1}] \\ &\geqslant \frac{1}{R}\sum_{s=1}^{2S-1} r_s \frac{\tau^{-\alpha_s}}{\Gamma(2-\alpha_s)}[2k^{1-\alpha_s} - (k-1)^{1-\alpha_s} - (k+1)^{1-\alpha_s}]. \end{aligned}$$

Since $2k^{1-\alpha_s} - (k-1)^{1-\alpha_s} - (k+1)^{1-\alpha_s} > 0$ holds for $k = 1, \ldots, n$ and all $\alpha_s \in (0, 1)$, we immediately have $z_k > 0$ for $k = 1, 2, \ldots, n$. Noting $q_0 = 1$, (5.24) is clear. □

Proposition 5.3. *If the coefficients $r_s \geqslant 0$ for $s = 0, 1, \ldots, 2S$, and there exists at least one $s \in \{1, \ldots, 2S-1\}$ such that $r_s > 0$ and $R > 0$, then $p > 0$ and*

$$\begin{aligned} &b_{ii} \geqslant 1 + \sum_{j=1, j\neq i}^{M-1} |b_{ij}|, \quad 1 \leqslant i \leqslant M-1 \\ &\sum_{j=1, j\neq i}^{M-1} b_{ij} < 0, \quad 1 \leqslant i \leqslant M-1. \end{aligned} \tag{5.25}$$

Based on propositions 5.2 and 5.3, we can prove the implicit scheme (5.23) along time direction is unconditionally stable and convergent with analogous method to that used in [29] for multi-term time fractional equation, i.e., the case of $\mu(\alpha) = \sum_{k=1}^{K} r_k \delta(\alpha - \alpha_k)$, where $\delta(\alpha - \alpha_k)$ is the Dirac function with $\alpha_k \in (0, 1)$.

5.3. Numerical inversions

Since the uniqueness of the inverse problem from input data at one fixed observation $x_0 \in \Omega$ satisfying $f(x_0) \neq 0$ is ensured by theorem 4.1, instead of using the noisy data $h^\delta(\cdot, t)$ in the domain Ω_0 satisfying (5.2), here we consider the numerical implementations using the data at one fixed point $x_0 \in \Omega_0$. Such data can also reduce the computational costs of inversion algorithm from practical points of view. In fact, from our numerical performances, the reconstructions are indeed satisfactory.

Let β_k be the iterative source function at kth iteration step. We stop the iteration by using the well-known Morozov's discrepancy principle. In the sequel, we denote by β_{true}, β_0, $\beta^\delta_{\text{rec}}$ the exact source, initial guess and the reconstructed source function, respectively. To show the accuracy of reconstruction, we define the relative error in L^2 norm by

$$\text{Err} := \frac{\|\beta^\delta_{\text{rec}} - \beta_{\text{true}}\|_2}{\|\beta_{\text{true}}\|_2}. \tag{5.26}$$

In our numerical implementations for the observation data with one fixed point $x_0 \in \Omega_0$, similarly to the derivations in theorem 5.1, the gradient expression (5.4) still holds, but $\phi[\omega_{\beta,\delta}](x, t)$ satisfies (4.12) corresponding to the input data

$$F(x, t) = \omega_{\beta,\delta}(x, t) := \delta(x - x_0)(G[\beta](x_0, t) - h^\delta(x_0, t)), \quad (x, t) \in \Omega \times [0, T]. \tag{5.27}$$

In this case, (4.12) is an initial boundary value problem for $\phi[\omega_{\beta,\delta}](x, t)$ with spatial impulse source $\delta(x - x_0)$, which can be solved by approximating singular function $\delta(x - x_0)$ with a smooth function numerically, or expanding the solution in terms of the eigenfunctions and then determining the expansion coefficients as done in [30]. In our computations, we take finite difference method to solve the direct problem and adjoint problem where $\delta(x - x_0)$ is replaced by a smooth function approximately at each iteration step. The noisy data are generated by

$$h^\delta(x_0, t_j) = G[\beta_{\text{true}}](x_0, t_j) + \delta \times G[\beta_{\text{true}}](x_0, t_j) \times (2\,\text{rand}\,(j) - 1) \tag{5.28}$$

for observation instants $t_j \in [0, T]$, where $\delta > 0$ is the noise level, and rand $(j) \in [0, 1]$ is the random number. We divide the interval $[0, 1]$ for α into 100 equidistant meshes, and test our inversion algorithm for two cases:

Case 1: fixed observation point x_0 and different noise levels $\delta > 0$;

Case 2: fixed noise level $\delta > 0$ and different observation points x_0.

We start with the one-dimensional case, with the space-time region $[0, 1] \times [0, 1]$ into 100×100 equidistant meshes.

Example 1. Let $\mu(\alpha) = \Gamma(2 - \alpha)$. Consider the following one-dimensional system

Table 1. The reconstructions for (5.30).

$\delta(x_0 = 0.5)$	0.1%	1%	3%	5%	10%
Err	0.0066	0.0082	0.0185	0.0304	0.0608
$x_0(\delta = 5\%)$	0.05	0.1	0.2	0.6	0.8
Err	0.0259	0.0211	0.0283	0.0302	0.0282

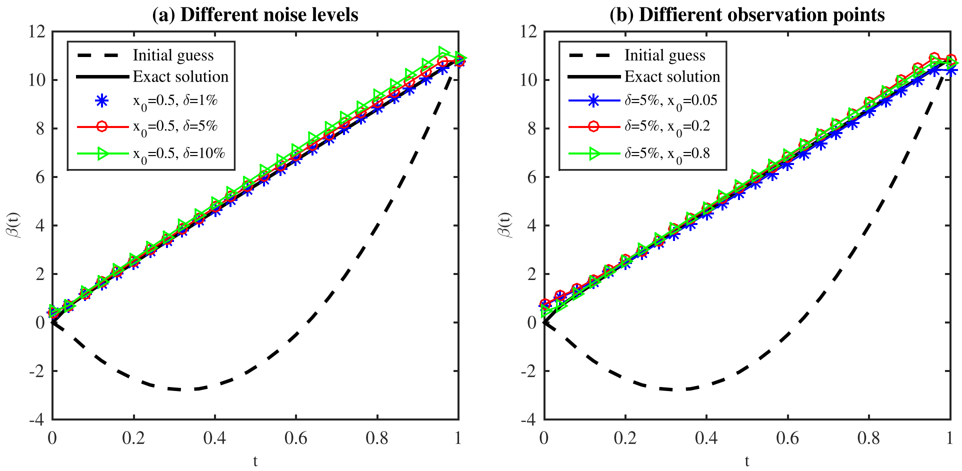

Figure 1. The reconstructions for example 1.

$$\begin{cases} D^{(\mu)}u(x,t) = u_{xx}(x,t) + \sin(\pi x)\beta(t), & (x,t) \in (0,1) \times (0,1], \\ u(0,t) = 0, u(1,t) = 0, & t \in (0,1], \\ u(x,0) = 0, & x \in (0,1), \end{cases} \tag{5.29}$$

which is a particular case of (4.1) with $\sigma(x) = 1$ and

$$\beta_{\text{true}}(t) = \frac{t-1}{\ln t} + \pi^2 t. \tag{5.30}$$

(5.29) has the exact solution $u(x,t) = t\sin(\pi x)$. For this example, we take the initial guess $\beta_0(t) := \beta_{\text{true}}(t) - 30t(1-t)$.

The quantitative descriptions for the reconstructions are shown in table 1. We also give the reconstructions in figure 1(a) for case 1 and figure 1(b) for case 2, respectively.

Form table 1 and figure 1, it can be observed that the reconstructions are satisfactory for exact solution both for case 1 and case 2. By table 1 and figure 1(a), we can see that the relative error Err decreases as the noise level in the data decreases and the numerical results are quite accurate up to 5% noise added in the exact data $G[\beta_{\text{true}}](x_0, t)$, implying our proposed CGM scheme is very stable against the measurement noise, which is consistent with the theoretical estimate in theorem 4.1. On the other hand, it can be seen from table 1 and figure 1(b) that the reconstructions by our proposed CGM scheme are not sensitive to the observation position x_0. In other words, our proposed algorithm is really robust against the choice of observation point. Here we should mention that the initial choice $\beta_0(t)$ has some influence on the accuracy of

Table 2. The reconstructions for (5.32).

$\delta(x_0 = 0.8)$	1%	5%	10%	15%
Err	0.0166	0.0413	0.0803	0.0858
$x_0(\delta = 5\%)$	0.02	0.1	0.4	0.95
Err	0.0366	0.0414	0.0418	0.0366

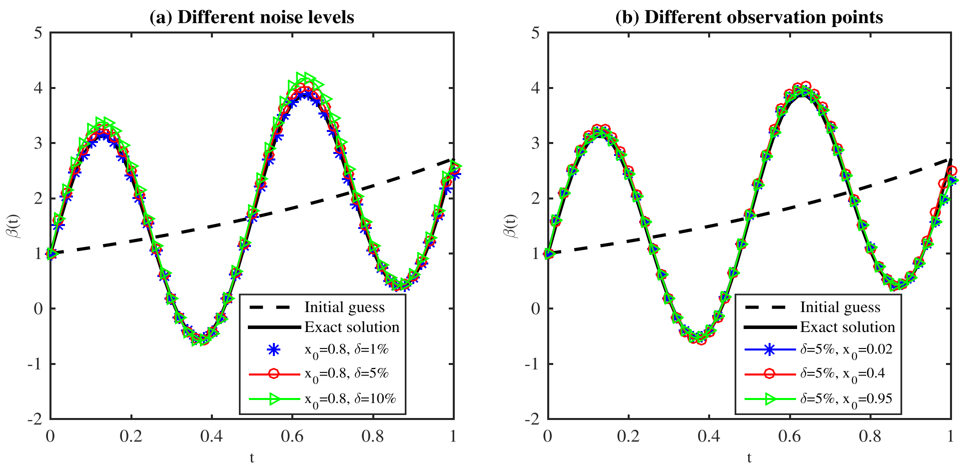

Figure 2. The reconstructions for example 2.

our reconstruction at $t = T$. Generally we need to choose an initial guess $\beta_0(t)$ such that $\beta_0(t)$ could provide exact information at $t = T$. This is because we cannot update $\beta_k(T)$ if $\eta = 0$ in (5.4). Such fact has been verified in [16, 30].

Example 2. Let $\mu(\alpha) = \Gamma(4 - \alpha)$. Consider the one-dimensional system

$$\begin{cases} \partial_t^\alpha u(x,t) = \dfrac{\partial}{\partial x}\left(\sigma(x)\dfrac{\partial u}{\partial x}\right) + \sin(\pi x)\beta(t), & (x,t) \in (0,1) \times (0,1), \\ u(0,t) = 0, u(1,t) = 0, & t \in (0,1), \\ u(x,0) = 0, & x \in (0,1) \end{cases} \tag{5.31}$$

with $\sigma(x) = 2 + \cos(2\pi x)$ and exact time-dependent source

$$\beta_{\text{true}}(t) = 2\,\sin(4\pi t) + \exp(t). \tag{5.32}$$

For this example, the analytic expression of exact solution is unknown. We simulate the interior measurement data by solving the direct problem using finite difference method.

We set the initial guess $\beta_0(t) = \exp(t)$. The numerical reconstructions for case 1 and case 2 are shown in table 2 and figure 2. We can see that the numerical results are very accurate up to 5% nose added in the exact measurements. By figure 2(a) we can see that the reconstructions on the corner become difficult as the noise level increases. On the other hand, it can be readily observed from table 2 and figure 2(b) that the numerical result is satisfactory even if the observation point is very close to the boundary, which reflects the robustness of the proposed algorithm again.

Table 3. The reconstructions for (5.35).

$\delta((x_0, y_0) = (0.3, 0.7))$	0.1%	1%	5%	10%
Err	0.0175	0.0182	0.0346	0.0607
$(x_0, y_0)(\delta = 5\%)$	$\left(\frac{1}{10}, \frac{1}{10}\right)$	$\left(\frac{3}{10}, \frac{3}{10}\right)$	$\left(\frac{3}{5}, \frac{3}{5}\right)$	$\left(\frac{4}{5}, \frac{4}{5}\right)$
Err	0.0356	0.0344	0.0344	0.0359

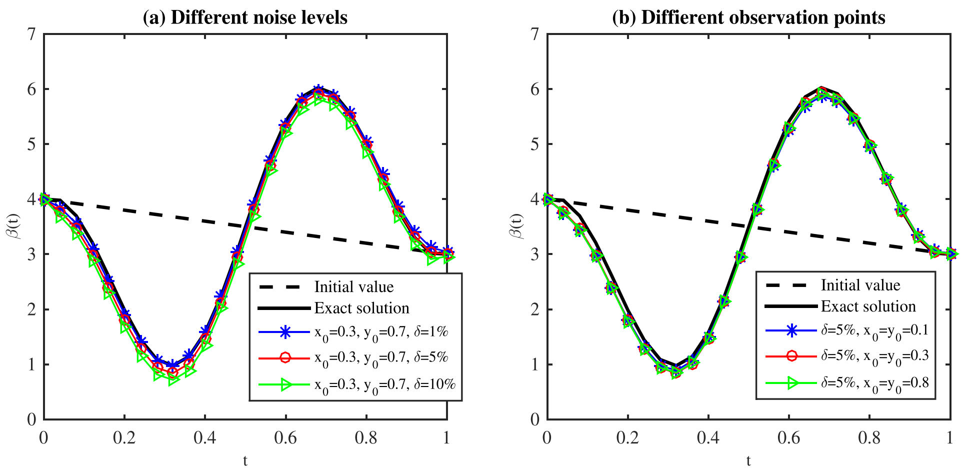

Figure 3. The reconstructions for example 3.

Now we proceed to the two-dimensional case, where we divide the space-time region $\overline{\Omega} \times [0, T] = [0, 1]^2 \times [0, 1]$ into a $40^2 \times 100$ equidistant meshes. Similarly to the one-dimensional cases, we will test the numerical performances of proposed CGM scheme for case 1 and case 2. We notice that all the tested observation points in examples 1 and 2 satisfy $f(x_0) \neq 0$, which means that the observation point should locate in the inside of the spatial source. However, such a restriction on the observation location is in general not realistic, because for some inverse source problems such as the nuclear radiative sources, people cannot have access to the source, the measurement can only be implemented at points away from the source distribution. Thus, we will also consider the case of observation point that $f(x_0) = 0$ in the following example.

Example 3. Let $\mu(\alpha) = \Gamma(3 - \alpha)$. Consider two-dimensional diffusion system

$$\begin{cases} D^{(\mu)}u(x, y, t) = u_{xx} + u_{yy} + f(x, y)\beta(t), & (x, y) \in (0, 1)^2, t \in (0, 1], \\ u|_{\partial\Omega} = 0, & t \in (0, 1], \\ u(x, y, 0) = 0, & (x, y) \in (0, 1)^2 \end{cases} \tag{5.33}$$

with the space-dependent source function

$$f(x, y) = \begin{cases} 1, & (x, y) \in [1/4, 3/4]^2, \\ 0, & (x, y) \notin [1/4, 3/4]^2, \end{cases} \tag{5.34}$$

and the time-dependent source

$$\beta_{\mathrm{true}}(t) = 2\cos(3\pi t) + 3t + 2 \tag{5.35}$$

to be recovered. The reconstructions from one observation point (x_0, y_0) and initial guess $\beta_0(t) = -t + 4$ for case 1 and case 2 are shown in table 3 and figure 3.

It can be observed by the numerical results shown in table 3 and figure 3 that the reconstructions are in good agreement with the exact shape even for the two-dimensional case. By table 3 we can see that the relative errors for the cases of $(x_0, y_0) = (1/10, 1/10), (4/5, 4/5)$ satisfying $f(x_0, y_0) = 0$ are Err $= 0.0356, 0.0359$, respectively, while the the errors are almost always 0.0344 for the cases of $(x_0, y_0) = (3/10, 3/10), (3/5, 3/5)$ satisfying $f(x_0, y_0) \neq 0$. Since we applied the same noisy data set for different observation points, the reconstructions from the cases of $f(x_0, y_0) = 0$ are a little bit worse than the ones from the cases of $f(x_0, y_0) \neq 0$, which is consistent with the expected results (e.g., [31, 32] for the parabolic case). However, we can also see that the improvement of reconstruction by choosing observation inside the source is not excellent. Indeed, the reconstructions from the cases of $f(x_0, y_0) = 0$ are also satisfactory.

Summarizing the numerical results obtained from examples 1–3, we have observed that our proposed CGM scheme is very stable against the noise in measurements and robust against the choice of observation points.

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