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RECOVERING THE BOUNDARY CORROSION FROM ELECTRICAL POTENTIAL DISTRIBUTION USING PARTIAL BOUNDARY DATA

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Abstract. We study detecting a boundary corrosion damage in the inaccessible part of a rectangular shaped electrostatic conductor from a one set of Cauchy data specified on an accessible boundary part of conductor. For this nonlinear ill-posed problem, we prove the uniqueness in a very general framework. Then we establish the conditional stability of Hölder type based on some *a priori* assumptions on the unknown impedance and the electrical current input specified in the accessible part. Finally a regularizing scheme of double regularizing parameters, using the truncation of the series expansion of the solution, is proposed with the convergence analysis on the explicit regularizing solution in terms of a practical average norm for measurement data.

1. **Introduction.** Consider the following boundary value problem for $u(x)$ with $x=(x_1,x_2)$ in a rectangle domain $D:=(0,\pi)\times(0,1)\subset\mathbb{R}^2$:

$$\begin{cases}\Delta u:=u_{x_1x_1}+u_{x_2x_2}=0, & x=(x_1,x_2)\in D\\ u(0,x_2)=0, u(\pi,x_2)=0, & x_2\in[0,1]\\ u_{x_2}(x_1,0)=\phi(x_1), u_{x_2}(x_1,1)+\sigma(x_1)u(x_1,1)=0, & x_1\in[0,\pi].\end{cases} \tag{1.1}$$

Giving $\phi(x_1)$ taken as the input and knowing the impedance $\sigma(x_1)$, (1.1) defines a well-posed direct problem, i.e., $u(x)\in C^2(D)\bigcap C(\overline{D})$ can be uniquely determined, provided that $\phi(x_1)$ and $\sigma(x_1)$ satisfy certain conditions, see Lemma 2.1 in the sequel.

Physically, (1.1) can be applied to model either the electrostatic potential distribution in an electrostatic conductor D or the stationary temperature distribution in a heat conductor D. In the former configuration, $\sigma(x_1)$ describes the possible presence of boundary corrosion damage on the part $\{(x_1,1):x_1\in(0,\pi)\}$ with $\phi(x_1)$ the specified current density [12, 13]. In the latter case, $\sigma(x_1)$ represents the heat exchanging coefficient on the boundary in terms of the Fourier heat conduction's

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law, with $\phi(x_1)$ the boundary heat flux [6]. The boundary condition at $x_2 = 1$ is mathematically called the Robin boundary condition with impedance $\sigma(x_1)$. The boundary value problems with mixed boundary conditions such as (1.1) are of great importance, since such kinds of models allow the media boundary to be of different physical property in its different part, which are the practical situations.

On the other hand, in many branches of engineering and science, we are also faced with inverse problems to detect some unknowns such as the impedance $\sigma(x_1)$ in (1.1). This problem can be considered as an inverse problem to detect a boundary corrosion damage in the inaccessible part $\{(x_1, 1) : x_1 \in (0, \pi)\}$ of ∂D from some extra measurement data of $u(x)$ specified in the accessible part of ∂D. For this inverse problem, assume that we are given a noisy measurement of the exact boundary potential

$$u(x_1, 0) = u_0(x_1), \quad x_1 \in [0, \pi], \tag{1.2}$$

say, $u_0^\delta(x_1)$ satisfying

$$\left\| u_0^\delta - u_0 \right\|_{L^2(0,\pi)} \le \delta. \tag{1.3}$$

We need to reconstruct $\sigma(x_1)$ approximately using this data $u_0^\delta(\cdot)$ based on the system (1.1)-(1.3). It is well-known that this system is a nonlinear ill-posed problem corresponding to the Laplace equation, i.e., we try to identify $\sigma(x_1)$ from the partially noisy Cauchy data $(u(x_1, 0), u_{x_2}(x_1, 0)) \approx (u_0^\delta(x_1), \phi(x_1))$.

The ill-posedness of this inverse problem comes from two factors. Firstly, if we can determine the Cauchy data at $x_2 = 1$ from the Cauchy data at $x_2 = 0$ for harmonic function $u(x)$ with zero boundary conditions at $x_1 = 0, \pi$, then $\sigma(x_1)$ can be determined from the impedance boundary condition at $x_2 = 1$ by

$$\sigma(x_1) = -\frac{u_{x_2}(x_1, 1)}{u(x_1, 1)} \tag{1.4}$$

at the points x_1 satisfying $u(x_1, 1) \neq 0$. However it is well-known that the Cauchy problem for elliptic equation is ill-posed, see [14, 17] for its theoretical studies and [19, 20] for its numerical studies. Secondly, except for the ill-posedness of determining the Cauchy data at $x_2 = 1$, the determination of $\sigma(x_1)$ in terms of (1.4) is also unstable at points where the nonzero values of $u(x_1, 1)$ are very small, i.e., at these points, the error in the numerical solution of $u_{x_2}(x_1, 1)$ will be rapidly amplified.

Due to the above mathematical difficulties and the wide applications of boundary impedance detections in engineering, the determination of impedance has obtained much attention in recent years. As early as in 1999, the determination of Robin coefficient for Laplace equation in $\mathbb{R}^2$ with mixed boundary condition and specified Cauchy data in part of the boundary is studied in [6] for $\phi(x) \ge 0$. Then this problem is restudied in [1] in a general two-dimensional domain with smooth boundary, where the Neumann datum is allowed to change its sign in the accessible part of the boundary, considering that the nonnegative assumption on electrical current density $\phi(x_1)$ is not physically reasonable. The conditional stability of logarithm type with L^∞ norms for both the unknown impedance and the measurement data was established. For the determination of the inclusion shape under the Laplace equation model, the reconstruction scheme is researched in [15]. As for the reconstructions for both the impedance and the medium boundary, some optimization schemes based on the boundary integral equation techniques were also thoroughly studied, see for example [2, 3, 4, 5, 8, 15, 18].

The conditional stability for ill-posed problems means that the stabilities can be recovered, if the unknown parameters are restricted in some *a priori* admissible set. Such restrictions, besides the uniform boundedness requirements on the unknowns in some topology, are also often specified indirectly by some requirements on the solution of the direct problems, which of course depends on the unknowns. These requirements can also be realized by specifying some restrictions on the input data for direct problem which yield the inversion input for inverse problem, because the solution to direct problem depends also on the input data. The observation given here has been applied for some (especially nonlinear) ill-posed problems, for example, see [11], where the restriction on the input is specified to establish the stability for recovering the coefficients in the parabolic equations.

In this paper, we give a systematic study on the inverse problem (1.1)-(1.3), including the uniqueness, conditional stability and regularizing scheme for reconstruction. Since we consider the rectangle domain in $\mathbb{R}^2$, the explicit expansion of the solution in terms of the eigenfunctions enable us to give a deep insight on this problem quantitatively. The advantage of our inversion schemes using the series expansion in a rectangle is that this quantitative analysis can be generalized to deal with the three dimensional cubic model, while the techniques based on the analytic function theory in complex plane [1] does not work anymore in three dimensional case. On the other hand, most of studies available on mixed boundary value problems concern with very smooth boundaries, while the practical engineering problems are in general given in domains with simple but non smooth boundaries, such as polygons. The comprehensive studies on the direct elliptic problems with non-smooth boundaries were given in [9].

It is well-known that uniqueness coming from the conditional stability is quite weak, especially for nonlinear ill-posed inverse problems. Here we carry out our nonlinear ill-posed problem under several configurations for the unknown impedance $\sigma(x_1)$. Firstly, the uniqueness is established for $\sigma(x_1) \in C^1[0,\pi]$ for non-positive input $\phi(x_1)$ based on the maximum principle. Secondly as for the conditional stability, we establish the interior Hölder continuity under strong assumptions on $\sigma(x_1)$ for specially specified excitation function $\phi(x)$, with the error for exact inversion input data measured by H^2-norm. It is interesting to compare our Hölder type stability with the logarithm type estimate given in [1], where the L^∞ norm is applied for both the input data and the impedance. The reason we can get a Hölder type stability is that here we assume a strong condition both on $\phi(x_1)$ and on σ (see the definition $\mathcal{A}_3$ for Theorem 2.5), while only a L^2-bound is assumed for $\phi(x)$ in [1]. It should be mentioned that the Hölder type stability was already studied recently in [10] for 2-dimensional general geometric shape with piecewise constant or analytic impedance, where the Hölder index is some constant $\kappa \in (0,1)$. Finally, by measuring the input data error in terms of L^2 norm which is the practical situation, we propose a regularizing scheme with double regularizing parameters (N,α), to overcome the ill-posedness from both the Cauchy problem and the small value of $u(x_1,1)$. The convergence rate for our regularizing solution with explicit expression is proven to be $|\log\delta|^{-\beta/2}$ for $\beta \in (1/4,1/2)$.

This paper is organized as follows. In section 2, we establish the uniqueness and conditional stability for inverse problem, which constitute the fundamentals for our inversion scheme. Then we propose the regularizing scheme by the truncation of the series representation of the solution in section 3, with the number of truncated terms as the regularizing parameter. The choice strategy for this parameter together with the convergence rate is also analyzed in this section. Section 4 is devoted to the regularity and uniform lower boundedness of the solution to direct problem in the rectangle domain, which are required in the theoretical analysis for the inverse problem.

2. Uniqueness and conditional stability.

We will establish the uniqueness for our inverse problem under very general requirements on the input $\phi(x_1)$ and boundary impedance $\sigma(x_1)$. Firstly let

$$\mathcal{A}_1 := \{\sigma : 0 \leq \sigma \in C^1[0,\pi], \sigma'(0) = \sigma'(\pi) = 0, \sigma \not\equiv 0\}$$

and consider the regularity of solution of our direct problem, which is necessary to the uniqueness and convergence rate of our regularizing solution for the inverse problem.

Lemma 2.1. *For $\sigma \in \mathcal{A}_1$ and $\phi(x_1) \in H^{1/2}(0,\pi)$, there exists a unique solution $u[\sigma,\phi](x) \in C^2(D)\bigcap C(\overline{D})$ to direct problem (1.1).*

Proof. We will give the proof in section 4, which is quite technical due to the non-smooth boundary and mixed boundary condition. An explicit condition to guarantee $\phi(x_1) \in H^{1/2}(0,\pi)$ is (2.4) in the sequel together with $\phi(0) = \phi(\pi) = 0$. □

This lemma gives a sufficient condition for (σ,ϕ) such that the solution to (1.1) is in $C^2(D)\bigcap C(\overline{D})$. Now we can consider the uniqueness and conditional stability in a general framework.

Theorem 2.2. *For given nonzero function $\phi(x_1) \in (H^{1/2}(0,\pi))^*$, it follows $\sigma^1(x_1) = \sigma^2(x_1)$ for $\sigma^1,\sigma^2 \in \mathcal{A}_1$, if $u[\sigma^1,\phi](x_1,0) = u[\sigma^2,\phi](x_1,0)$ in $C[0,\pi]$.*

Proof. Define $u_i(x) = u[\sigma^i,\phi](x)$ for $i = 1,2$ and $w(x) := u_1(x) - u_2(x), \sigma(x_1) := \sigma_1(x_1) - \sigma_2(x_1)$, then $w(x) \in C^2(D)\bigcap C(\overline{D})$ due to Lemma 2.1. Moreover it satisfies

$$(2.1)\quad \begin{cases} \Delta w = 0, & x = (x_1,x_2) \in D \\ w(0,x_2) = 0, w(\pi,x_2) = 0, & x_2 \in [0,1] \\ w_{x_2}(x_1,0) = 0, & x_1 \in [0,\pi] \\ w_{x_2}(x_1,1) + \sigma_2(x_1)w(x_1,1) = -\sigma(x_1)u_1(x_1,1), & x_1 \in [0,\pi], \end{cases}$$

with extra boundary condition

$$(2.2)\qquad w(x_1,0) = 0, \quad x_1 \in [0,\pi].$$

So it follows that $w(x) \equiv 0$ on $\overline{D}$ from the uniqueness of the Cauchy problem and the regularity of $w(x)$. Then the impedance boundary condition for w yields

$$(2.3)\qquad \sigma(x_1)u_1(x_1,1) = 0, \quad x_1 \in (0,\pi).$$

We firstly claim that $u_1(x_1,1)$ cannot be identically zero in any closed nonempty interval $I \subset (0,\pi)$. Otherwise the impedance boundary condition in (1.1) will generate

$$\frac{\partial}{\partial x_2}u_1(x_1,1) \equiv u_1(x_1,1) \equiv 0, \quad x \in I.$$

Then the uniqueness of the Cauchy problem yields $u_1(x) \equiv 0$ on D which implies $\int_0^\pi \phi(x_1)v(x_1,0)\,dx_1 = 0$ for any $v \in H^1(D)$ and hence $\phi \equiv 0$ contradicting with $\phi(x_1) \not\equiv 0$.

For any $x_1^* \in (0,\pi)$, if $u_1(x_1^*,1) \neq 0$, then (2.3) yields $\sigma(x_1^*) = 0$. If $u_1(x_1^*,1) = 0$, since we have proven that the zero points of $u_1(x_1,1)$ in $(0,\pi)$ are discrete, there exists a sequence $\{\tilde{x}_n : n \in \mathbb{N}\} \subset [0,\pi]$ converging to x_1^* satisfying $u_1(\tilde{x}_n,1) \neq 0$. Then (2.3) yields $\sigma(\tilde{x}_n) = 0$ for $n \in \mathbb{N}$. Finally the continuity of $\sigma(x_1)$ in $(0,\pi)$ yields

$$\sigma(x_1^*) = \lim_{n\to\infty} \sigma(\tilde{x}_n) = 0.$$

Therefore we have proven $\sigma(x_1) \equiv 0$ in $(0,\pi)$. Finally $\sigma(0) = \sigma(\pi) = 0$ follows from the continuity of $\sigma \in C[0,\pi]$. The proof is complete. □

For the stability, we need several lemmas. To begin with we give the positivity of $u[\sigma,\phi](x_1,1)$. More precisely, we have the following

Lemma 2.3. *For $\phi \in H^{1/2}(0,\pi)$ and $\sigma(x_1) \in \mathcal{A}_1$, if $0 \not\equiv \phi(x_1) \leq 0$, then $u[\sigma,\phi](x_1,1) > 0$ in $x_1 \in (0,\pi)$.*

Proof. It follows from Lemma 2.1 that $u \in C^2(D) \bigcap C(\overline{D})$. Then the strong maximum principle for elliptic equation states that the minimum values of u in $\overline{D}$ can only be taken on ∂D, noticing that u is not constant in $\overline{D}$ from $\phi(x_1) \not\equiv 0$. On the other hand, the minimum value of u cannot be taken on $\{(x_1,0) : x_1 \in (0,\pi)\}$ also from $u_{x_2}(x_1,0) = \phi(x_1) \leq 0$ and the Hopf lemma.

1. If the minimum value is taken either on $\{(0,x_2) : x_2 \in [0,1]\}$ or on $\{(\pi,x_2) : x_2 \in [0,1]\}$, the minimum value on $\overline{D}$ is 0. Then if $u(x_1^*,1) = 0$ for some $x_1^* \in (0,\pi)$, u also takes the minimum value at point $(x_1^*,1)$. The Hopf lemma gives $u_{x_2}(x_1^*,1) < 0$. However, the impedance boundary condition yields $u_{x_2}(x_1^*,1) = -\sigma(x_1^*)u(x_1^*,1) = 0$, which is a contradiction. Therefore we have $u(x_1,1) > 0$ for $x_1 \in (0,\pi)$.

2. If the minimum value cannot be reached on neither $\{(0,x_2) : x_2 \in [0,1]\}$ nor $\{(\pi,x_2) : x_2 \in [0,1]\}$, then the minimum value in $\overline{D}$ is negative. Since the minimum value now can only be reached in $\{(x_1,1) : 0 < x_1 < \pi\}$, we must have some point $y_1^* \in (0,\pi)$ such that $u(y_1^*,1) = \min_{\overline{D}} u(x) < 0$. At this point, it follows $u_{x_2}(y_1^*,1) < 0$ from the Hopf Lemma. However we have from the impedance boundary condition that $u_{x_2}(y_1^*,1) = -\sigma(y_1^*)u(y_1^*,1) \geq 0$, which is a contradiction.

The proof is complete. □

Now we give a quantitative characterization of (ϕ,σ) such that we can represent the smooth solution to (1.1) in terms of the Cauchy data $(u(x_1,0), u_{x_2}(x_1,0))$ based on the eigenfunction expansions.

Assume that $(u_0(x_1), \phi(x_1), \sigma(x_1))$ has the following Fourier expansion

$$(u_0(x_1), \phi(x_1), \sigma(x_1)) = \sum_{n=1}^{\infty} (c_n, \phi_n, \sigma_n) \sin(nx_1),$$

where $u_0(x_1) = u[\sigma,\phi](x_1,0)$ is the Dirichlet data from the direct problem (1.1), taking as our inversion input data. Moreover, assume that the nonzero excitation $\phi(x_1)$ meets

$$\sum_{n=1}^{\infty} \phi_n \cosh(n), \quad \sum_{n=1}^{\infty} \phi_n^2 \cosh^2(n) < +\infty. \tag{2.4}$$

For three positive constants B_0, L_0, ϵ_0, introduce the admissible set

$$\mathcal{A}_2 := \left\{ \sigma : \ \sigma \in \mathcal{A}_1, \int_0^{\pi} \sigma(x_1)dx_1 \geq B_0, 0 \leq \sigma(x_1) \leq L_0, \|\sigma'\|_{L^2} \leq \epsilon_0 \right\}.$$

Lemma 2.4. *If $\sigma(x_1) \in \mathcal{A}_2$ for large B_0 and small ϵ_0, whereas $\phi(x_1)$ meets (2.4), then the unique solution $u \in C^2(D) \bigcap C(\overline{D})$ to (1.1) has the representation*

$$u(x) = \sum_{n=1}^{\infty} \left(c_n \cosh(nx_2) + \frac{1}{n}\phi_n \sinh(nx_2) \right) \sin(nx_1), \quad x \in \overline{D} \tag{2.5}$$

with the estimate

$$\sum_{n=1}^{\infty} c_n^2 n^3 e^{2n} \leq C(B_0, L_0, \epsilon_0, \phi). \tag{2.6}$$

Proof. We construct a function $v(x)$ in terms of the Cauchy data (u_0, ϕ) by

$$v(x) = \sum_{n=1}^{\infty} \left(c_n \cosh(nx_2) + \frac{1}{n}\phi_n \sinh(nx_2) \right) \sin(nx_1), \quad x \in \overline{D}. \tag{2.7}$$

We will prove, for (ϕ, σ) specified in this lemma, the following points:

1. the series (2.7) and its termwise derivative converge compactly in $\overline{D}$;
2. the second order termwise derivative of the series (2.7) converges local uniformly in D;
3. the function v also meets the impedance boundary condition on $x_2 = 1$.

Once we have these, then by the uniqueness of direct problem (1.1) in $C^2(D) \bigcap C(\overline{D})$, we have $u(x) = v(x)$ on $\overline{D}$ and hence we have (2.6). The similar arguments were already applied in §4, Chapter 3 in [16] for treating an ill-posed problem in the two dimensional rectangle by the Fourier series expansion of a solution.

To begin with, we firstly assume that point 1 and point 2 are true. Then we have

c1: $v \in C^1(\overline{D})$ which meets the Dirichlet boundary conditions on $x_1 = 0, \pi$ and Cauchy data on $x_2 = 0$, where the first order partial derivatives at four corners of the rectangle are understood as the limitation of corresponding partial derivatives from the boundaries.

c2: $v \in C^2(D)$ which satisfies the Laplace equation.

Then the uniqueness of the Cauchy problem for Laplace equation implies that $v(x)$ must also meet the impedance boundary condition, noticing that $v(x)$ and $u(x)$ have the same Cauchy data in $x_2 = 0$ and u satisfies the impedance boundary condition. Therefore, from point 1, we can make termwise differentiation for the series v from which the boundary impedance condition yields

$$\begin{aligned} &\sum_{n=1}^{\infty} n(c_n \sinh(n) + \frac{1}{n}\phi_n \cosh(n)) \sin(nx_1) + \\ &\sigma(x_1) \sum_{n=1}^{\infty} (c_n \cosh(n) + \frac{1}{n}\phi_n \sinh(n)) \sin(nx_1) = 0. \end{aligned} \tag{2.8}$$

Considering the second term as a function of x_1, then it follows that

$$\begin{aligned} & n(c_n \sinh(n) + \frac{1}{n}\phi_n \cosh(n)) \\ = & -\frac{2}{\pi} \int_0^{\pi} \sigma(x_1) \sum_{m=1}^{\infty} (c_m \cosh(m) + \frac{1}{m}\phi_m \sinh(m)) \sin(mx_1) \ \sin(nx_1) dx_1 \\ = & -\sum_{m=1}^{\infty} (c_m \cosh(m) + \frac{1}{m}\phi_m \sinh(m)) \sigma_{n,m}, \end{aligned} \tag{2.9}$$

where

$$\sigma_{n,m} := \frac{2}{\pi}\int_0^\pi \sigma(x_1)\sin(mx_1)\sin(nx_1)dx_1 = \frac{1}{2}\left(\frac{1}{m+n}\sigma'_{m+n}+s_{n,m}\right)$$

with $\{\sigma'_k : k\in\mathbb{N}\}$ the Fourier sin coefficients of $\sigma'(x_1)$ and

$$s_{n,m} = \begin{cases} -\frac{1}{|m-n|}\sigma'_{|m-n|}, & m\neq n,\\ \frac{2}{\pi}\int_0^\pi \sigma(x_1)dx_1, & m=n. \end{cases} \tag{2.10}$$

Define $C_n^s := nc_n\sinh(n)$, $\Phi_n^c := -\phi_n\cosh(n)$ for $n=1,2,\cdots$, then we have

$$C_n^s + \sum_{m=1}^\infty C_m^s \frac{\cosh(m)}{m\sinh(m)}\sigma_{n,m} = \Phi_n^c + \sum_{m=1}^\infty \Phi_m^c\frac{\sinh(m)}{m\cosh(m)}\sigma_{n,m},\quad n=1,2,\cdots,$$

which can be written as for $n=1,2,\cdots$ that

$$\begin{aligned} &\left[1+\frac{1}{2n}\frac{\cosh(n)}{\sinh(n)}\left(\frac{1}{2n}\sigma'_{2n}+\frac{2}{\pi}\int_0^\pi\sigma(x_1)dx_1\right)\right]C_n^s + \\ &\sum_{m=1,m\neq n}^\infty \frac{\cosh(m)}{2m\sinh(m)}\left(\frac{1}{m+n}\sigma'_{m+n}-\frac{1}{|m-n|}\sigma'_{|m-n|}\right)C_m^s \\ = &\left[1+\frac{1}{2n}\frac{\sinh(n)}{\cosh(n)}\left(\frac{1}{2n}\sigma'_{2n}+\frac{2}{\pi}\int_0^\pi\sigma(x_1)dx_1\right)\right]\Phi_n^c + \\ &\sum_{m=1,m\neq n}^\infty \frac{\sinh(m)}{2m\cosh(m)}\left(\frac{1}{m+n}\sigma'_{m+n}-\frac{1}{|m-n|}\sigma'_{|m-n|}\right)\Phi_m^c \\ =&: \Phi_n^c[\sigma]. \end{aligned}\tag{2.11}$$

For all $\sigma\in\mathcal{A}_2$, it follows

$$\frac{1}{2n}\sigma'_{2n}+\frac{2}{\pi}\int_0^\pi\sigma(x_1)dx_1 \geq \frac{2}{\pi}B_0-\frac{1}{2}\epsilon_0\geq 0,\quad n=1,2,\cdots$$

for large B_0 and small ϵ_0. So the existence of solution $\{C_n^s : n\in\mathbb{N}\}$, which is defined as the solution to the $N-$terms truncation of the series in the left-hand side of (2.11) for any $N\in\mathbb{N}$, is obvious, since this $N-$dimensional linear system with respect to $\{C_n^s : n=1,2,\cdots,N\}$ is diagonal dominant for small ϵ_0.

Now we estimate $\{C_n^s : n\in\mathbb{N}\}$. The above equation for any fixed n yields

$$\begin{aligned} |C_n^s| &\leq \left[1+\frac{1}{2n}\frac{\cosh(n)}{\sinh(n)}\left(\frac{1}{2n}\sigma'_{2n}+\frac{2}{\pi}\int_0^\pi\sigma(x_1)dx_1\right)\right]|C_n^s| \\ &\leq \frac{1}{1-e^{-2}}\sum_{m=1,m\neq n}^\infty\left(\frac{1}{m+n}|\sigma'_{m+n}|+\frac{1}{|m-n|}|\sigma'_{|m-n|}|\right)|C_m^s| + |\Phi_n^c[\sigma]| \\ &\leq \frac{1}{1-e^{-2}}\sum_{m=1}^\infty\left(\frac{1}{m+n}|\sigma'_{m+n}|+\tilde\sigma'_{m,n}\right)|C_m^s|+|\Phi_n^c[\sigma]|, \end{aligned}\tag{2.12}$$

where we define the constants

$$\tilde\sigma'_{m,n} = \begin{cases} \frac{1}{n-m}|\sigma'_{n-m}|, & m=1,\cdots,n-1\\ 0, & m=n\\ \frac{1}{m-n}|\sigma'_{m-n}|, & m=n+1,\cdots \end{cases}$$

By straightforward computations, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m+n}|\sigma'_{m+n}|+\tilde{\sigma}'_{m,n}\right)|C^s_m|\\ =&\sum_{m=1}^{\infty}\left(\sum_{k=m+1}^{\infty}\frac{1}{k}|\sigma'_k|+\sum_{n=1}^{m-1}\frac{1}{m-n}|\sigma'_{m-n}|+\sum_{n=m+1}^{\infty}\frac{1}{n-m}|\sigma'_{n-m}|\right)|C^s_m|\\ =&\sum_{m=1}^{\infty}\left(\sum_{k=m+1}^{\infty}\frac{1}{k}|\sigma'_k|+\sum_{k=1}^{m-1}\frac{1}{k}|\sigma'_k|+\sum_{k=1}^{\infty}\frac{1}{k}|\sigma'_k|\right)|C^s_m|\le\sum_{k=1}^{\infty}\frac{2}{k}|\sigma'_k|\sum_{m=1}^{\infty}|C^s_m|.\end{aligned}$$

Taking summation for $n=1,2,\cdots$ in (2.12) and using the above bounds, it follows

$$\sum_{n=1}^{\infty}|C^s_n|\le\frac{2}{1-e^{-2}}\sum_{k=1}^{\infty}\frac{1}{k}|\sigma'_k|\sum_{m=1}^{\infty}|C^s_m|+\sum_{n=1}^{\infty}|\Phi^c_n[\sigma]|. \tag{2.13}$$

Then for $\epsilon_0>0$ small enough such that

$$\frac{2}{1-e^{-2}}\sum_{k=1}^{\infty}\frac{1}{k}|\sigma'_k|\le\frac{2}{1-e^{-2}}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{1/2}\epsilon_0:=h<1$$

for all $\sigma\in\mathcal{A}_2$, we finally have

$$\sum_{n=1}^{\infty}|C^s_n|\le\frac{1}{1-h}\sum_{n=1}^{\infty}|\Phi^c_n[\sigma]|\le\frac{1}{1-h}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{1/2}C(\epsilon_0,L_0)\sum_{n=1}^{\infty}|\Phi^c_n|<+\infty$$

by using the similar arguments to estimate $\sum_{n=1}^{\infty}|\Phi^c_n[\sigma]|$ and (2.4). Therefore, we have proven that

$$\sum_{n=1}^{\infty}|c_n|\sinh(n)<\sum_{n=1}^{\infty}n|c_n|\sinh(n)<+\infty, \tag{2.14}$$

which yields $v,\nabla v\in C(\overline{D})$, noticing (2.4).

In order to prove $v\in C^2(D)$, we only need to prove that v is of second continuous derivatives at interior points. For any $x_0=(x_1^0,x_2^0)\in D$, consider

$$G:=\{x=(x_1,x_2):x_1\in(0,\pi),x_2\in(x_2^0-\epsilon,x_2^0+\epsilon)\}$$

with small $\epsilon>0$ satisfying $x_2^0+\epsilon<1$. In G all the twofold termwise differentiations of the series in (2.7) essentially yield

$$\sum_{n=1}^{\infty}n^2\left(c_n\cosh(nx_2)+\frac{1}{n}\phi_n\sinh(nx_2)\right)\sin(nx_1).$$

For all $x\in G$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^2|c_n\cosh(nx_2)\sin(nx_1)|\le C\sum_{n=1}^{\infty}n^2|c_n|e^{n(x_2^0+\epsilon)}\\ \le\ & C\sum_{n=1}^{\infty}n|c_n|e^n ne^{-n(1-(x_2^0+\epsilon))}\le C\sum_{n=1}^{\infty}ne^{-n(1-(x_2^0+\epsilon))}\le C.\end{aligned} \tag{2.15}$$

Finally the estimate (2.14) shows that the series for $v(x)$ and the series from termwise differentiation are uniformly absolute convergent in $\overline{D}$ and therefore point 1 is true. Also point 2 is ensured from (2.15) and (2.4). Thus we have proven (2.5).

It is easy to see for given $\phi \in (H^{1/2}(0,\pi))^*$ that

$$\begin{aligned}\int_D |\nabla u|^2 dx &\leq -\int_0^\pi \phi(x_1)u_0(x_1)dx_1 - \int_0^\pi \sigma(x_1)u^2(x_1,1)dx_1 \\ &\leq \|\phi\|_{(H^{1/2}(0,\pi))^*}\|u_0\|_{H^{1/2}(0,\pi)}.\end{aligned} \tag{2.16}$$

However, we have from the uniform convergence of the series in (2.5) in $\overline{D}$ and (2.14) that

$$\|u_0\|^2_{H^{1/2}(0,\pi)} = 2\sum_{n=1}^\infty nc_n^2 \leq \sum_{n=1}^\infty ne^{-2n} \leq C,$$

which generates $u \in H^1(D)$ due to the zero data on $x_1 = 0, \pi$. Then we have $u(x_1, 1) \in H^{1/2}(0, \pi)$, which says $u_{x_2}(x_1, 1) = -\sigma(x_1)u(x_1, 1) \in H^{1/2}(0, \pi)$, i.e.,

$$\sum_{n=1}^\infty n^3c_n^2e^{2n} = \sum_{n=1}^\infty n(nc_ne^n)^2 \leq C\,\|u_{x_2}(\cdot,1)\|^2_{H^{1/2}(0,\pi)} < C$$

from the $\overline{D}$ uniform convergence of the series obtained from the termwise differentiation of (2.5).

The proof is complete. □

Now we can establish the conditional stability. To get the Hölder type estimate, we need to give σ more restrictions instead of the admissible set $\mathcal{A}_2$. Noticing that the expansion coefficients $\{c_n : n \in \mathbb{N}\}$ in Lemma 2.4 depend both on σ and ϕ, we fix the excitation function $\phi(x)$ satisfying $0 \not\equiv \phi(x) \leq 0$ and (2.4), and then define

$$\mathcal{A}_3 := \left\{\sigma \in \mathcal{A}_2 : \sum_{n=1}^\infty |c_n|^{2(1-\beta)}n^2e^{2n} < C^2(\phi, \epsilon_0, L_0, B_0)\right\}$$

for given $\beta \in (1/4, 1/2)$ as the admissible set of σ, which is nonempty for $\phi(x)$ satisfying (2.4). In fact, we have (2.6) for all $\sigma \in \mathcal{A}_2$. Then the identity

$$\sum_{n=1}^\infty c_n^2n^3e^{2n} \equiv \sum_{n=1}^\infty |c_n|^{2(1-\beta)}n^2e^{2n}\; n|c_n|^{2\beta}$$

together with (2.6) ensures that $\mathcal{A}_3$ is nonempty and well-defined at least

1. if $c_n \neq 0$ holds only for finite number of n, or,
2. if $n|c_n|^{2\beta} \geq C_0 > 0$ for some $2\beta < 1$ uniformly (notice, this condition is impossible for $2\beta \geq 1$), which means that $|c_n|^{2\beta} \to 0$ as $n \to \infty$ should not be too fast,

since in both cases, we can always give upper bound on $\sum_{n=1}^\infty |c_n|^{2(1-\beta)}n^2e^{2n}$ by $\sum_{n=1}^\infty c_n^2n^3e^{2n}$. The above assumptions on $\{c_n : n \in \mathbb{N}\}$ can be realized for some special σ, see Remark 2.6 below. Thus (2.6) ensures that the set $\mathcal{A}_3$ is well-defined, which is a direct characterization on σ for our specified $\phi(x)$.

For $i = 1, 2$, denote by $u^i(x)$ the corresponding solution to (1.1) with boundary impedance $\sigma^i(x_1) \in \mathcal{A}_3$. We have the following conditional stability.

Theorem 2.5. *For fixed $\phi(x)$ satisfying $0 \not\equiv \phi(x) \leq 0$ and (2.4), denote by $u_0^i(x_1) := u^i(x_1, 0)$ the exact inversion input data corresponding to $\sigma^i(x_1) \in \mathcal{A}_3$ for given constant $\beta \in (1/4, 1/2)$ for $i = 1, 2$. Then it follows for any given small $\epsilon > 0$ that*

$$\left\|\sigma^1 - \sigma^2\right\|_{C[\epsilon,\pi-\epsilon]} \leq C(\epsilon, \beta, B_0, L_0, \epsilon_0)\|u_0^1 - u_0^2\|^\beta_{H^2(0,\pi)}. \tag{2.17}$$

Proof. Define $(w(x), \sigma(x_1))$ as that in the proof of Theorem 2.2. Then $w(x)$ satisfies

$$\begin{cases} \Delta w = 0, & x = (x_1, x_2) \in D \\ w(0, x_2) = 0, w(\pi, x_2) = 0, & x_2 \in [0, 1] \\ w_{x_2}(x_1, 0) = 0, & x_1 \in [0, \pi] \\ w_{x_2}(x_1, 1) + \sigma_2(x_1) w(x_1, 1) = -\sigma(x_1) u^1(x_1, 1), & x_1 \in [0, \pi], \end{cases} \tag{2.18}$$

with the extra boundary condition

$$w(x_1, 0) = u_0^1(x_1) - u_0^2(x_1), \quad x_1 \in [0, \pi]. \tag{2.19}$$

Due to Lemma 2.4, we can express $w(x) \in C^2(D) \bigcap C(\overline{D})$ as

$$w(x) = \sum_{n=1}^{\infty} C_n \cosh(n x_2) \sin(n x_1)$$

with the coefficients

$$C_n = \frac{2}{\pi} \int_0^{\pi} [u_0^1 - u_0^2](x_1) \sin(n x_1) dx_1 = \frac{2}{n^2 \pi} \int_0^{\pi} [u_0^1 - u_0^2]''(x_1) \sin(n x_1) dx_1$$

from the zero Dirichlet data on $x_1 = 0, \pi$ and Cauchy data on $x_2 = 0$. We finally get

$$\sigma(x_1) u_1(x_1, 1) = -\sum_{n=1}^{\infty} C_n [n \sinh(n) + \sigma_2(x_1) \cosh(n)] \sin(n x_1). \tag{2.20}$$

From Lemma 2.3, we know that $u[\sigma^1](x_1, 1) > 0$ for $x_1 \in (0, \pi)$. We claim that, for any $\sigma^1(x_1) \in \mathcal{A}_3 \subset \mathcal{A}_2$, there exists a constant $c(\epsilon) > 0$ depending also on B_0, L_0, ϵ_0 but independent of σ^1 such that $u[\sigma^1](x_1, 1) > c^{-1}(\epsilon)$ in the closed interval $[\epsilon, \pi - \epsilon] \subset (0, \pi)$. The verification of this claim will be given in section 4. Therefore we have from (2.20) that

$$|\sigma(x_1)| \leq C(\epsilon) \left(\left| \sum_{n=1}^{\infty} C_n n \sinh(n) \sin(n x_1) \right| + \left| \sum_{n=1}^{\infty} C_n \cosh(n) \sin(n x_1) \right| \right)$$

for $x_1 \in [\epsilon, \pi - \epsilon]$. So we have for the parameter $\beta \in (1/4, 1/2)$ that

$$\begin{aligned} \|\sigma\|_{C[\epsilon, \pi-\epsilon]} &\leq C(\epsilon) \sum_{n=1}^{\infty} |C_n| n e^n \\ &\leq C(\epsilon) \left(\sum_{n=1}^{\infty} |C_n|^{2\beta} \right)^{1/2} \left(\sum_{n=1}^{\infty} |C_n|^{2(1-\beta)} n^2 e^{2n} \right)^{1/2} \end{aligned} \tag{2.21}$$

from (2.20). Using the expression of C_n, we have $C_n = c_n^1 - c_n^2$, where c_n^i are the Fourier coefficients of $u_0^i(x_1)$ corresponding to $\sigma^i(x_1) \in \mathcal{A}_3$. Therefore we have

$$\begin{aligned} \left\|\sigma^1 - \sigma^2\right\|_{C[\epsilon, \pi-\epsilon]} &\leq C(\epsilon, B_0, L_0, \epsilon_0) \left(\sum_{n=1}^{\infty} \frac{1}{n^{4\beta}} \right)^{1/2} \left\|(u_0^1 - u_0^2)''\right\|_{L^1(0,\pi)}^{\beta} \\ &\leq C(\epsilon, \beta, B_0, L_0, \epsilon_0) \|u_0^1 - u_0^2\|_{H^2(0,\pi)}^{\beta} \end{aligned} \tag{2.22}$$

for $\sigma^i \in \mathcal{A}_3$ with $\beta \in (1/4, 1/2)$. The proof is complete. □

This result gives the conditional stability for reconstructing $\sigma(x_1)$. Since we establish the Hölder type pointwise estimate on σ, here we must add restrictive requirements on $\sigma(x_1)$ indirectly by $\mathcal{A}_3$. The other important observation is that we can only establish the pointwise estimate (2.17) in the interior interval $[\varepsilon, \pi-\varepsilon] \subset (0,\pi)$. This is natural, since for our model, the consistent conditions at points $(0,1)$ and $(\pi,1)$ require $u(0,1) = u(\pi,1) = 0$, therefore there is no information about $\sigma(0)$ and $\sigma(\pi)$ in the impedance boundary condition of our model. This phenomenon has also been clarified for the general model in [1] with Lipschitz smooth boundary, where the conditional stability for impedance with L^∞ norm was also established only in $\Gamma_d \subset \Gamma$, where Γ is the part of impedance boundary.

Remark 1. The admissible set $\mathcal{A}_3$ for σ can be explicitly realized for special excitation flux ϕ. A simple example is for any nonnegative constant impedance $\sigma \in \mathcal{A}_3$, the function

$$u(x) = (c_1 \cosh(x_2) + \phi_1 \sinh(x_2)) \sin(x_1)$$

with

$$\phi_1 = -1 < 0, c_1 = \frac{(e+e^{-1}) + \sigma(e-e^{-1})}{(e-e^{-1}) + \sigma(e+e^{-1})} > 0$$

solves (1.1), and meets the definition of $\mathcal{A}_3$ for Theorem 2.5. This solution in special form of the series (2.5) motivates us to construct the regularization strategy in terms of the truncations of series (2.5) in finite terms, which will be the topic of the next section.

3. Regularization scheme by series truncations.

For given Cauchy data

$$(u(x_1,0), u_{x_2}(x_1,0)) = (u_0(x_1), \phi(x_1)), \tag{3.1}$$

where $0 \not\equiv \phi(x_1) \le 0$ is the specified excitation data, and $u_0(x_1) := u[\sigma](x_1,0)$ is the corresponding response generated from (1.1) for $\sigma \in \mathcal{A}_3$, our uniqueness result ensures that there exists a unique $\sigma(x_1) \in \mathcal{A}_3$ satisfying

$$u_{x_2}(x_1,1) + \sigma(x_1) u(x_1,1) = 0, \quad x_1 \in [0,\pi]. \tag{3.2}$$

Since we already give the expression of the solution u in (2.5) by the Fourier coefficients (c_n, ϕ_n) of $(u_0(x_1), \phi(x_1))$, we can theoretically solve $u_{x_2}(x_1,1)$ and $u(x_1,1)$ for exact data using this series and then determine $\sigma(x_1)$ in $(0,\pi)$ under the condition $u(x_1,1) > 0$.

However, in case that $u_0(x_1)$ is obtained by observations, some noise $\delta > 0$ should be introduced. Then the series (2.5) with c_n replaced by c_n^δ, the Fourier coefficients of $u_0^\delta(x_1)$, need not be convergent, which reveals the ill-posedness of the Cauchy problem of the Laplace equation, and also the reconstruction of $\sigma(x_1)$ from (3.2). The other factor causing the instability of the reconstruction of $\sigma(x_1)$ is that $u^\delta(x_1,1)$ may be very small for $x_1 \approx 0, \pi$ and hence contaminates the reconstruction performance of $\sigma(x_1)$ at points near to $x_1 \approx 0, \pi$.

To overcome these difficulties, we propose to solve this nonlinear ill-posed problem by considering the following cost functional with double regularizing parameters N, α:

$$J_{N,\alpha}(\sigma) := \left\| \partial_{x_2} u_N^\delta(\cdot,1) + \sigma(\cdot) u_N^\delta(\cdot,1) \right\|^2_{L^2(0,\pi)} + \alpha \left\| \sigma \right\|^2_{L^2(0,\pi)}, \tag{3.3}$$

where $u_N^\delta(x_1,x_2)$ is the regularizing solution to Cauchy problem of Laplace equation defined by

$$u_N^\delta(x_1,x_2) := \sum_{n=1}^{N}\left(c_n^\delta\cosh(nx_2)+\frac{1}{n}\phi_n\sinh(nx_2)\right)\sin(nx_1), \tag{3.4}$$

with $\{c_n^\delta : n\in\mathbb{N}\}$ the Fourier coefficients of given noisy data $u_0^\delta(x_1)\in L^2(0,\pi)$ satisfying (1.3). Our final aim is to estimate $\left\|\sigma_{N,\alpha}^\delta-\sigma\right\|$, where $\sigma_{N,\alpha}^\delta(x)$ is the unique minimizer of (3.3), which is easily verified to satisfy

$$[\alpha+(u_N^\delta(x_1,1))^2]\sigma_{N,\alpha}^\delta(x_1)+u_N^\delta(x_1,1)\,\partial_{x_2}u_N^\delta(x_1,1)=0. \tag{3.5}$$

Define

$$\partial_{x_2}e(x_1,1):=\partial_{x_2}(u_N^\delta(x_1,1)-u(x_1,1)),\quad e(x_1,1):=u_N^\delta(x_1,1)-u(x_1,1), \tag{3.6}$$

then it follows from (3.2) and $u(x_1,1)>0$ in $(0,\pi)$ that

$$\begin{aligned} & [\sigma_{N,\alpha}^\delta(x_1)-\sigma(x_1)][\alpha+(u_N^\delta(x_1,1))^2] \\ = & -u_N^\delta(x_1,1)[\sigma(x_1)e(x_1,1)+\partial_{x_2}e(x_1,1)]-\alpha\sigma(x_1). \end{aligned} \tag{3.7}$$

By straightforward computations using (3.4), (2.7) and (3.6), we have

$$\begin{aligned} \begin{pmatrix}\partial_{x_2}e(x_1,1)\\ e(x_1,1)\end{pmatrix} = & \sum_{n=1}^{N}(c_n^\delta-c_n)\begin{pmatrix}n\sinh(n)\\ \cosh(n)\end{pmatrix}\sin(nx_1)- \\ & \sum_{n=N+1}^{\infty}\left[\begin{pmatrix}nc_n\\ \frac{1}{n}\phi_n\end{pmatrix}\sinh(n)+\begin{pmatrix}\phi_n\\ c_n\end{pmatrix}\cosh(n)\right]\sin(nx_1), \end{aligned}$$

which yields the estimates for $j=0,1$ that

$$\left\|\partial_{x_2}^j e(\cdot,1)\right\|_{C([0,\pi])}\le C\left[\sum_{n=1}^{N}|c_n^\delta-c_n|n^je^n+\sum_{n=N+1}^{\infty}\left|c_n+\frac{1}{n}\phi_n\right|n^je^n\right]. \tag{3.8}$$

Now let us estimate $|u_N^\delta(x_1,1)|$. It is easy to see for $x_1\in(0,\pi)$ that

$$\begin{aligned} |u_N^\delta(x_1,1)|^2 \ge\ & \frac{1}{2}|u(x_1,1)|^2-|u_N^\delta(x_1,1)-u(x_1,1)|^2 \\ \ge\ & \frac{1}{2}|u(x_1,1)|^2-2\left(\sum_{n=1}^{N}|c_n^\delta-c_n|\cosh(n)\right)^2- \\ & 2\left(\sum_{n=N+1}^{\infty}\left|c_n\cosh(n)+\frac{1}{n}\phi_n\sinh(n)\right|\right)^2 \\ \ge\ & \frac{1}{2}|u(x_1,1)|^2-2\left(\sum_{n=1}^{N}|c_n^\delta-c_n|^2\right)\left(\sum_{n=1}^{N}\cosh^2(n)\right)- \\ & C\left(\sum_{n=N+1}^{\infty}(c_n^2+\frac{1}{n^2}\phi_n^2)e^{2n}\right). \end{aligned}$$

From Lemma 2.4 and (2.4), we know $c_n^2e^{2n}, \frac{1}{n^2}\phi_n^2e^{2n}\sim\frac{1}{n^2}$, so the above estimate yields

$$|u_N^\delta(x_1,1)|^2\ge\frac{1}{2}|u(x_1,1)|^2-Ce^{2N}\left\|u_0^\delta-u_0\right\|_{L^2(0,\pi)}^2-C\frac{1}{N}.$$

Therefore if we choose the number of truncation terms $N = N(\delta)$ such that

$$N(\delta) \to +\infty, \quad e^{N(\delta)}\delta \to 0 \tag{3.9}$$

as $\delta \to 0$, then we have for fixed $\varepsilon > 0$ that

$$|u_N^\delta(x_1, 1)| > \frac{1}{2}|u(x_1, 1)| \geq c(\epsilon) > 0, \quad x \in [\varepsilon, \pi - \varepsilon] \tag{3.10}$$

for $\delta > 0$ small enough from $u(x_1, 1) > 0$ in $(0, \pi)$. Finally it follows from (3.7), (3.8) and (3.10) that

$$\begin{aligned} \left\|\sigma_{N,\alpha}^\delta - \sigma\right\|_{C[\varepsilon,\pi-\varepsilon]} &\leq C(\epsilon)\left[\sum_{n=1}^{N}|c_n^\delta - c_n|ne^n + \sum_{n=N+1}^{\infty}\left|c_n + \frac{1}{n}\phi_n\right|ne^n + \sqrt{\alpha}\right] \\ &\leq C(\epsilon)\left[Ne^N\delta + \sum_{n=N+1}^{\infty}\left|c_n + \frac{1}{n}\phi_n\right|ne^n + \sqrt{\alpha}\right] \end{aligned} \tag{3.11}$$

by using the Cauchy-Schwartz inequality with $C(\epsilon)$ depending also on B_0, L_0, ϵ_0, provided that (3.9) hold. For $\sigma \in \mathcal{A}_3$, we have

$$\begin{aligned} \sum_{n=N+1}^{\infty}|c_n|ne^n &\leq \left(\sum_{n=N+1}^{\infty}|c_n|^{2(1-\beta)}n^2e^{2n}\right)^{1/2}\left(\sum_{n=N+1}^{\infty}|c_n|^{2\beta}\right)^{1/2} \\ &\leq C\left(\sum_{n=N+1}^{\infty}\left(\frac{e^{-2n}}{n^3}\right)^\beta\right)^{1/2} \leq C(\beta)e^{-\beta N} \end{aligned} \tag{3.12}$$

for $\beta \in (1/4, 1/2)$. Assume that the excitation function $\phi(x)$ also meets

$$\sum_{n=1}^{\infty}|\phi_n|^{2(1-\beta)}e^{2n} < +\infty. \tag{3.13}$$

Then it follows from $\sum_{n=1}^{\infty}\phi_n^2\cosh^2(n) < +\infty$ that

$$\begin{aligned} \sum_{n=N+1}^{\infty}|\phi_n|e^n &\leq \left(\sum_{n=N+1}^{\infty}|\phi_n|^{2(1-\beta)}e^{2n}\right)^{1/2}\left(\sum_{n=N+1}^{\infty}|\phi_n|^{2\beta}\right)^{1/2} \\ &\leq C\left(\sum_{n=N+1}^{\infty}(e^{-2n})^\beta\right)^{1/2} \leq C(\beta)e^{-\beta N}. \end{aligned} \tag{3.14}$$

So we have from (3.12) and (3.14) that

$$\sum_{n=N+1}^{\infty}\left|c_n + \frac{1}{n}\phi_n\right|ne^n \leq C(\phi, \beta)e^{-\beta N}.$$

Finally (3.11) leads to the estimate for $\beta \in (1/4, 1/2)$ that

$$\left\|\sigma_{N,\alpha}^\delta - \sigma\right\|_{C[\varepsilon,\pi-\varepsilon]} \leq C(\epsilon, \beta, B_0, L_0, \epsilon_0)\left[Ne^N\delta + \frac{1}{e^{\beta N}} + \sqrt{\alpha}\right]. \tag{3.15}$$

Now we can give the choice strategy for the regularizing parameters (N, α) in terms of the noise level $\delta > 0$ based on this estimate, which is stated by

Theorem 3.1. *For given input $\phi(x)$ satisfying (2.4) and (3.13), denote by $u_0(x_1) := u[\sigma,\phi](x_1,0)$ the inversion input data for $\sigma \in \mathcal{A}_3$. For the noisy measurement data $u_0^\delta(x_1)$ satisfying (1.3), if we take the regularizing parameters (N,α) in the cost functional (3.3) as*

$$N = N(\delta) = \frac{1}{2}\eta_0|\log\delta|, \quad \alpha = \alpha(\delta) = (\eta_0|\log\delta|)^{-\beta} \tag{3.16}$$

for any fixed $\eta_0 \in (0,1)$, then we have the convergence estimate for the regularizing solution

$$\left\|\sigma^\delta_{N(\delta),\alpha(\delta)} - \sigma\right\|_{C[\varepsilon,\pi-\varepsilon]} \le C(\epsilon, B_0, L_0, \epsilon_0, \beta, \eta_0)|\log\delta|^{-\beta/2} \tag{3.17}$$

for $\delta > 0$ small enough with $\beta \in (1/4, 1/2)$.

Proof. It follows from (3.15) that

$$\left\|\sigma^\delta_{N,\alpha} - \sigma\right\|_{C[\varepsilon,\pi-\varepsilon]} \le C(\epsilon, B_0, L_0, \epsilon_0)\left[e^{2N}\delta + \frac{1}{e^{\beta N}} + \sqrt{\alpha}\right]. \tag{3.18}$$

Obviously, for any fixed $\eta_0 \in (0,1)$, we have

$$e^{2N(\delta)}\delta = \delta^{1-\eta_0} \to 0, \quad \frac{1}{e^{\beta N}} = \frac{1}{[\eta_0 \log\frac{1}{\delta}]^{\beta/2}} \to 0$$

as $\delta \to 0$. Noticing that $\delta^{1-\eta_0}[\eta_0\log\frac{1}{\delta}]^{\beta/2} \to 0$ as $\delta \to 0$.

The proof is complete. □

Remark 2. The optimal choice strategy for $(N(\delta), \alpha(\delta))$ is to determine these two parameters from

$$e^{(1-\beta)N}(N+1)\delta = \beta, \quad \alpha(\delta) = \frac{1}{e^{2\beta N}}$$

for small $\delta > 0$. Since it is very hard to give an explicit solution to the above equation with respect to $N(\delta)$, we propose the suboptimal strategy (3.16).

We already propose a strategy for choosing our regularizing parameters N and α. The numerical implementations of this scheme for concrete models will be given elsewhere. Noticing that we also establish the conditional stability in section 2, it has been known that such kinds of result can be applied to give some choice strategy for regularizing parameter in Tikhonov setting to get some approximate solution with error estimate [7]. The application of the conditional stability in section 2 for the choice $N = N(\delta), \alpha = \alpha(\delta)$ will be the other interesting topic for further researches.

4. **The regularity and bounds for direct problem.** Since we deal with our nonlinear inverse problems under the regularity $u[\sigma,\phi] \in C^2(D)\bigcap C(\overline{D})$ for direct problem (1.1), we need to give some characterizations on (σ,ϕ) to ensure this regularity. Here we give the proof for Lemma 2.1.

Proof. Rewrite (1.1) as

$$\begin{cases} \Delta u = 0, & x \in D \\ u = 0, & x \in \overline{\Gamma_0}\bigcup\overline{\Gamma_+} \\ \partial_\nu u = -\phi, & x \in N_+, \\ \partial_\nu u + \sigma u = 0, & x \in I_+, \end{cases} \tag{4.1}$$

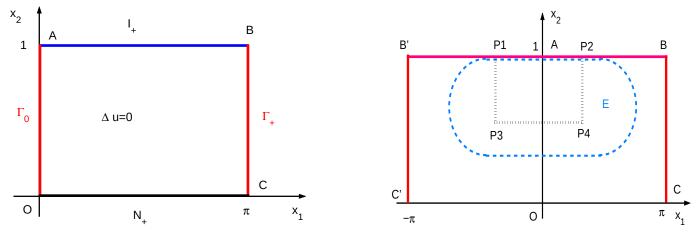

FIGURE 1. Geometric configuration for our direct problem (left) and the extension for dealing with the corner point A (right).

where ν is the outward normal direction of ∂D, see Figure 1 (left) for the geometric configuration.

We separate the proof as the following steps.

Step 1. There exists a unique solution in $u \in H^1(D)$ to (4.1). That is, we need to prove $\exists^1 u \in W := \{v \in H^1(D), v = 0 \text{ in } \Gamma_0 \bigcup \Gamma_+\}$, which is a closed subspace of $H^1(D)$ due to the continuity of the trace $H^1(D) \to H^{1/2}(\Gamma_0 \bigcup \Gamma_+)$ (Theorem 1.5.2.1 in [9]), such that

$$E(u,v) := \int_D \nabla u \cdot \nabla v dx + \int_{I_+} \sigma u v ds(x) = -\int_{N_+} \varphi v ds(x), \quad \forall v \in W \tag{4.2}$$

for $\phi \in H^{1/2}(N_+)^*$ and $v \in H^{1/2}(N_+)$. This is a direct result from the Lax-Milgram theorem for Hilbert space W, noticing for $\sigma \in \mathcal{A}_1$ that

$$\begin{aligned} |E(u,v)| &\leq C\|u\|_{H^1(D)}\|v\|_{H^1(D)} + C\|u\|_{L^2(I_+)}\|v\|_{L^2(I_+)} \\ &\leq C\|u\|_{H^1(D)}\|v\|_{H^1(D)} + C\|u\|_{H^1(D)}\|v\|_{H^1(D)} = C\|u\|_W\|v\|_W \end{aligned}$$

from the trace theorem and $\|v\|_W^2 \leq C\|\nabla v\|_{L^2(D)}^2 \leq CE(v,v)$ from the Poincare inequality.

Step 2. The solution in Step 1 also has the regularity $u \in H^2(D)$. The key ingredient is to prove that the solution can have the boundary H^2-regularity near the four corner points O, A, B, C, see Figure 1 (left).

We firstly analyze the boundary regularity of u near point A. Introduce the cut-off functions $\chi(x) := \chi_1(x_1)\chi_2(x_2)$ with $\chi_i(x_i) \in C^\infty(\mathbb{R}^1)$ defined as

$$\chi_1(x_1) = \begin{cases} 1, & x_1 \leq \pi/3 \\ 0, & x_1 \geq 2\pi/3, \end{cases} \quad \chi_2(x_2) = \begin{cases} 1, & x_1 \geq 2/3 \\ 0, & x_1 \leq 1/3. \end{cases}$$

Define $v = \chi\, u$. Then the boundary value problem (4.1) yields

$$\Delta v = \Delta\chi u + 2\nabla\chi \cdot \nabla u + \Delta u \chi := f \in C^\infty(D) \tag{4.3}$$

from the interior regularity of u satisfying $\Delta u = 0$ in D. Moreover, it also has the boundary conditions

$$v = 0 \text{ on } \Gamma_0 \bigcup \Gamma_+, \quad \partial_\nu v = 0 \text{ on } N_+, \quad \partial_\nu v + \sigma v = 0 \text{ on } I_+. \tag{4.4}$$

Now we make the odd extensions for (v, ϕ) and even extension for (σ, f) with respect to x_1. Denote by $\tilde{v}, \tilde{\phi}, \tilde{\sigma}, \tilde{f}$ the extended function. Then we consider the boundary value problem for $\tilde{v}$ in the rectangle domain cycled by $B'BCC'B'$(see

Figure 1 (right)). It is easy to verify that

$$\partial_\nu \tilde{v} + \tilde{\sigma}\tilde{v} = 0 \text{ on } \overline{B'B}, \quad \tilde{v} = 0 \text{ on } \overline{BC}\bigcup\overline{B'C'}, \quad \partial_\nu \tilde{v} = -\tilde{\phi} \text{ on } \overline{C'C}.$$

On the other hand, for $x_1 < 0$, it follows from

$$\partial_{x_1}\tilde{v}(x_1, x_2) = -\partial_{x_1} v(-x_1, x_2) = \partial_z v(z, x_2)|_{z=-x_2}$$

that $\lim_{x_1\to 0-}\partial_{x_1}\tilde{v}(x_1, x_2) = \partial_{x_1} v(0, x_2)$, i.e., $\partial_{x_1}\tilde{v}$ is also continuous at $x_1 = 0$. Therefore we conclude $\tilde{v} \in H^1(\tilde{D})$ from $\Delta\tilde{v} = \tilde{f}$ in

$$\tilde{D} := \{x : (x_1, x_2) \in D\}\bigcup\{x : (-x_1, x_2) \in D\}.$$

Now let us restrict the solution $\tilde{v}$ in E (see Figure 1, right), which is of smooth boundary $\partial E \in C^2$ containing the impedance part for $\tilde{v}$ in $x_2 = 1$. Since $\tilde{v}$ is of the support $P1P2P3P4P1$ and we assume $\sigma'(0) = 0$, we can extend σ to the whole boundary ∂E, denoted by $\hat{\sigma}(x)$ such that $\hat{\sigma} \in C^1(\partial E)$. Noticing the compact support of $\tilde{v}$ is $P1P2P3P4P1$, $\tilde{v}$ satisfies

$$\begin{cases} \Delta\tilde{v} = \tilde{f}, & x \in E \\ \partial_\nu\tilde{v} + \hat{\sigma}\tilde{v} = 0, & x \in \partial E. \end{cases} \tag{4.5}$$

Then it follows from Theorem 2.20 in [21] that $\tilde{v} \in H^2(E)$. Then we know that v is of the boundary regularity H^2 near the corner point A.

For corner point B, we can also do the similar extension and yield the boundary regularity H^2 near the corner point B.

As for the corner point O, we have the Neumann condition $\partial_\nu u = \phi$ in N_+ instead of the impedance boundary condition $\partial_\nu u + \sigma u = 0$ in I_+. In this case, we can consider the condition as $\sigma \equiv 0$ and make the same extension on ϕ. The above arguments also work. Therefore we get the boundary regularity H^2 near the corner point O. The same result holds near the corner point C. Combing the above arguments together, we know that $u \in H^2(D)$.

Step 3. Prove $u \in C^2(D)\bigcap C(\overline{D})$.

Now the Sobolev embedding theorem $C^{0,h}(\overline{D}) \hookrightarrow H^2(D)$ for $h \in (0, 1)$ says $u \in C(\overline{D})$. $u \in C^\infty(D)$ is a direct result of interior regularity of the Poisson equation for $\tilde{v}$ with $\tilde{f} \in C^\infty(\tilde{D})$. Finally we get that $u \in C^2(D)\bigcap C(\overline{D})$.

The proof is complete. □

Next we give the proof of $u[\sigma](x_1, 1) > c^{-1}(\epsilon)$ uniformly for all $\sigma \in \mathcal{A}_2$, which is used in the proof of Theorem 2.5.

For any $\sigma \in \mathcal{A}_2$, denote by $u[\sigma](x_1, x_2)$ the solution to (1.1) with $0 \not\equiv \phi(x_1) \le 0$. Introduce the solution $z(x_1, x_2)$ to

$$\begin{cases} \Delta z := z_{x_1x_1} + z_{x_2x_2} = 0, & x = (x_1, x_2) \in D \\ z(0, x_2) = 0, z(\pi, x_2) = 0, & x_2 \in [0, 1] \\ z_{x_2}(x_1, 0) = \phi(x_1), z_{x_2}(x_1, 1) + L_0 z(x_1, 1) = 0, & x_1 \in [0, \pi]. \end{cases} \tag{4.6}$$

Using the same arguments as those in the proof of Lemma 2.3, it follows that $z(x_1, 1) > 0$ for $x_1 \in (0, \pi)$. Now consider $w(x_1, x_2) := u[\sigma](x_1, x_2) - z(x_1, x_2)$, which satisfies

$$(4.7)\quad \begin{cases} \Delta w := w_{x_1x_1} + w_{x_2x_2} = 0, & x = (x_1, x_2) \in D \\ w(0, x_2) = 0, w(\pi, x_2) = 0, & x_2 \in [0, 1] \\ w_{x_2}(x_1, 0) = 0, & x_1 \in [0, \pi] \\ w_{x_2}(x_1, 1) + \sigma(x_1)w(x_1, 1) = (L_0 - \sigma(x_1))z(x_1, 1), & x_1 \in [0, \pi]. \end{cases}$$

We assert that $d := \min_{\overline{D}} w(x_1, x_2) \geq 0$. In fact, the maximal principle says that there exists some point $(x_1^*, x_2^*) \in \partial D$ such that $w(x_1^*, x_2^*) = d$. If $d < 0$, then either $(x_1^*, x_2^*) \in \{(x_1, 0) : x_1 \in (0, \pi)\}$ or $(x_1^*, x_2^*) \in \{(x_1, 1) : x_1 \in (0, \pi)\}$. The strong maximal principle says that $(x_1^*, x_2^*) \in \{(x_1, 0) : x_1 \in (0, \pi)\}$ is impossible from $w_{x_2}(x_1, 0) = 0$. So the only possibility is $(x_1^*, x_2^*) \in \{(x_1, 1) : x_1 \in (0, \pi)\}$. In this case, we have

$$w_{x_2}(x_1^*, 1) + \sigma(x_1^*)w(x_1^*, 1) < 0$$

from $w_{x_2}(x_1^*, 1) < 0$ and $\sigma(x_1^*) \geq 0, w(x_1^*, 1) < 0$, which contradicts with the boundary condition in (4.7) since $(L_0 - \sigma(x_1^*))z(x_1^*, 1) \geq 0$ for all $\sigma \in \mathcal{A}_2$.

Now we have $w(x_1, 1) \geq 0$, i.e., $u[\sigma](x_1, 1) \geq z(x_1, 1) > 0$ in $x_1 \in (0, \pi)$ for all $\sigma \in \mathcal{A}_2$. Since $z(x_1, x_2)$ depends only on $(\phi(x_1), L_0)$, there exists a constant $c^{-1}(\epsilon)$ independent of $\sigma \in \mathcal{A}_2$ such that

$$u[\sigma](x_1, 1) \geq z(x_1, 1) > c^{-1}(\epsilon) > 0, \quad x_1 \in [\epsilon, \pi - \epsilon].$$

The proof is complete.

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REFERENCES

[1] G. Alessandrini, L. D. Piero and L. Rondi, Stable determination of corrosion by a single electrostatic boundary measurement, *Inverse Problems*, **19** (2003), 973–984.

[2] F. Cakoni and R. Kress, Integral equations for inverse problems in corrosion detection from partial Cauchy data, *Inverse Problems and Imaging*, **1** (2007), 229–245.

[3] F. Cakoni, Y. Q. Hu and R. Kress, Simultaneous reconstruction of shape and generalized impedance functions in electrostatic imaging, *Inverse Problems*, **30** (2014), 105009(19pp).

[4] F. Cakoni, R. Kress and C. Schuft, Simultaneous reconstruction of shape and impedance in corrosion detection, *Methods and Applications of Analysis*, **17** (2010), 357–377.

[5] F. Cakoni, R. Kress and C. Schuft, Integral equations for shape and impedance reconstruction in corrosion detection, *Inverse Problems*, **26** (2010), 095012(24pp).

[6] S. Chaabane and M. Jaoua, Identification of Robin coefficients by themeans of boundary measurements, *Inverse Problems*, **15** (1999), 1425–1438.

[7] J. Cheng and M. Yamamoto, One new strategy for a-priori choice of regularizing parameters in Tikhonov regularization, *Inverse Problems*, **16** (2000), L31–L38.

[8] H. Eckel and R. Kress, Nonlinear integral equations for the inverse electrical impedance problem, *Inverse Problems*, **23** (2007), 475–491.

[9] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, SIAM Classics in Applied Mathematics Series, 2011.

[10] G. H. Hu and M. Yamamoto, Hölder stability estimate of Robin coeffcient in corrosion detection with a single boundary measurement, *Inverse Problems*, **31** (2015), 115009, 20pp.

[11] V. Isakov, *Inverse Problems for Partial Differential Equations*, 2^{nd} edition, Springer, 2006.

[12] P. G. Kaup and F. Santosa, Nondestructive evaluation of corrosion damage using electrostatic measurements, *J. Nondestruct. Eval.*, **14** (1995), 127–136.

[13] P. G. Kaup, F. Santosa and M. Vogelius, Method for imaging corrosion damage in thin plates from electrostatic data, *Inverse Problems*, **12** (1996), 279–293.

[14] M. V. Klibanov and F. Santosa, A computational quasi-reversibility method for Cauchy problems for Laplace's equation, *SIAM J. Appl. Math.*, **51** (1991), 1653–1675.

[15] R. Kress and W. Rundell, Nonlinear integral equations and the iterative solution for an inverse boundary value problem, *Inverse Problems*, **21** (2005), 1207–1223.

[16] M. M. Lavrentev, V. G. Romanov and S. P. Shishatskii, *Ill-posed Problems of Mathematical Physics and Analysis*, AMS, Trans. Math. Monographs, Vol.64, Providance, Rhode Island, 1986.

[17] L. E. Payne, Bounds in the Cauchy problem for the Laplace equation, *Arch. Ration Mech. Anal.*, **5** (1960), 35–45.

[18] W. Rundell, Recovering an obstacle and its impedance from Cauchy data, *Inverse Problems*, **24** (2008), 045003(22p).

[19] T. Shigeta and D. L. Young, Method of fundamental solutions with optimal regularization techniques for the Cauchy problem of the Laplace equation with singular points, *J. Comput. Phys.*, **228** (2009), 1903–1915.

[20] T. Wei, Y. C. Hon and L. V. Ling, Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators, *Engineering Analysis with Boundary Elements*, **31** (2007), 373–385.

[21] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1987.

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