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On the reconstruction of unknown time-dependent boundary sources for time fractional diffusion process by distributing measurement

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Abstract

We consider an inverse problem of recovering a time-dependent factor of an unknown source on some subboundary for a diffusion equation with time fractional derivative by nonlocal measurement data. Such fractional-order equations describe anomalous diffusion of some contaminants in heterogeneous media such as soil and model the contamination process from an unknown source located on a part of the boundary of the concerned domain. For this inverse problem, we firstly establish the well-posedness in some Sobolev space. Then we propose two regularizing schemes in order to reconstruct an unknown boundary source stably in terms of the noisy measurement data. The first regularizing scheme is based on an integral equation of the second kind which an unknown boundary source solves, and we prove a convergence rate of regularized solutions with a suitable choice strategy of the regularizing parameter. The second regularizing scheme relies directly on discretization by the radial basis function for the initial-boundary value problem for fractional diffusion equation, and we carry out numerical tests, which show the validity of our proposed regularizing scheme.

Keywords: anomalous diffusion, fractional order derivative, ill-posedness, regularization, stability, error estimate, numerics

(Some figures may appear in colour only in the online journal)

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1. Introduction

In this paper, we consider an inverse problem of determining a time-dependent factor of a boundary source for a fractional diffusion equation in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, with piecewise smooth boundary $\partial\Omega$. Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\Gamma_0 \subset \partial\Omega$ be a relatively open sub-boundary and set $\Gamma_1 \coloneqq \partial\Omega \backslash \overline{\Gamma_0}$. Here $\overline{D}$ denotes the closure of a set D under consideration.

Let $\alpha \in (1/2, 1)$. We consider the following diffusion problem:

$$\partial_t^\alpha u(x, t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u, \quad x \in \Omega,\ \ 0 < t < T \tag{1.1}$$

with

$$\begin{cases} u(x, t) = 0, & x \in \Gamma_0,\ 0 < t < T, \\ u(x, t) = F(t)g(x), & x \in \Gamma_1,\ 0 < t < T, \\ u(x, 0) = 0, & x \in \Omega. \end{cases} \tag{1.2}$$

Throughout this paper, we assume that $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $c \in C(\overline{\Omega})$, $c \leqslant 0$ in Ω, and there exists a constant $\gamma_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant \gamma_0 \sum_{i=1}^{d}\xi_i^2 \quad \text{for all} x \in \Omega \text{ and } \xi_1, \ldots, \xi_d \in \mathbb{R},$$

and we use the Caputo derivative for variable t defined by

$$\partial_t^\alpha u(x, t) \coloneqq \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha}\frac{\partial u}{\partial s}(x, s)\mathrm{d}s$$

and $\Gamma(\,\cdot\,)$ is the Gamma function.

Here we assume that $g \in H_0^{3/2}(\Gamma_1)$ is a spatial distribution amplitude factor of the source, while $F(t)$ describes a time amplitude factor of the source. Our inverse problem is to determine $F(t)$ by some extra data of the solution u, provided that g is known. Such a separation form of variables has also been assumed for example for a classical inverse heat conduction problem in Cannon *et al* [4]. In practice, the boundary source $F(t)g(x)$ in the form of variable separation is motivated by a point source $F(t)\delta_{x_0}(x)$ where δ_{x_0} is the Dirac delta function at $x_0 \in \Gamma_1$, i.e., the delta function is replaced by a suitable bell-shaped function centered at x_0, which is a reasonable approximation of the point source.

In (1.1) and (1.2), the function $u(x, t)$ denotes the density of e.g., a contaminant or temperature, and (1.1) is called a fractional diffusion equation. In modeling the diffusion in heterogeneous media such as soil, the classical diffusion equation like $\frac{\partial u}{\partial t} = \Delta u$ does not appropriately simulate profiles of diffusion as the time passes (e.g., Adams and Gelhar [1]). Thus several kinds of model equations have been proposed and the fractional diffusion equation in the form (1.1) is one of them (e.g., Suzuki *et al* [31]). We refer also for example to Klafter and Sokolov [16], Metzler and Klafter [22–24]) about the physical backgrounds. In (1.2), the boundary value $u(x, t) = F(t)g(x)$ for $x \in \Gamma_1$ and $0 < t < T$ is interpreted to describe a boundary input of contaminant or heat into the domain Ω and cause the anomalous diffusion in $\Omega \times (0, T)$.

For given F and g, as is proved later, there exists a unique solution to (1.1) and (1.2) in a suitable class (Gorenflo *et al* [9], and also theorem 2.2 in Sakamoto and Yamamoto [28]). By the linearity, we can similarly discuss non-zero boundary values on Γ_1 and non-zero initial values, but we do not treat here.

We can find extensive references on the theoretical analysis and numerical algorithms on fractional diffusion equations and see Boris *et al* [3], Gorenflo *et al* [9], Lin and Xu [18], Podlubny [26], Wang and Liu [32], Yang and Liu [33], Ye and Xu [34], Zhang and Liu [35], for example.

The main problem in this paper is

Inverse problem of determining a t-factor of the boundary source: Given g in (1.1) and (1.2), determine $F(t)$ by

$$h(t) := \int_{\Omega} u(x, t)\mu(x)\mathrm{d}x, \quad 0 < t < T, \tag{1.3}$$

where $\mu \in C_0^{\infty}(\Omega), \not\equiv 0$ is a non-negative weight function.

In (1.3), the support of μ is presumed to be small and data mean that we are given the average spatial distribution information in a small sub-domain. Physically, the weight $\mu(x)$ can be considered as an internal tiny sensor measuring the average distribution in the small domain and see e.g., Prilepko *et al* [27, p 60].

As for inverse problems for the parabolic case ($\alpha = 1$) and the hyperbolic case ($\alpha = 2$), there are many works and we refer to Denisov [7], Isakov [14], Prilepko *et al* [27] as monographs. As for numerical methods for inverse problems, see also Duc and Tuan [8], Guo and Murio [12], Jonas and Louis [15], Lesnic and Elliott [17], Liu [19], and here we do not intend to create any comprehensive lists of the references. On the other hand, for inverse problems for fractional diffusion equations, the references are rapidly growing but we refer only to Bondaranko and Ivaschenko [2], Cheng *et al* [6], Liu and Yamamoto [20], Luchko *et al* [21], Miller and Yamamoto [25], Sakamoto and Yamamoto [28, 29]. Here the list is far from the complete. For fractional diffusion equations, the difficulty comes from the definition of the fractional-order derivatives, which is essentially an integral with the kernel of weak singularity. For such a non-classical derivative, some standard methods for treating the inverse problems such as the Carleman estimates cannot be applied. Such a difference implies that the inverse problems for fractional diffusion equations should be more difficult.

Moreover, for the reconstruction, available data $h^{\delta}(t)$ are often measured as noisy data of $h(t)$ satisfying

$$\left\|h^{\delta} - h\right\|_{L^2(0,T)} \leqslant \delta.$$

Then our inverse problem aims at the reconstruction of $F(t)$ and consequently the solution $u(x, t)$ in $\Omega \times [0, T]$ approximately from the available data $h^{\delta}(t)$ based on system (1.1) and (1.2).

This paper is composed of four sections. In section 2, we prove the well-posedness for our inverse problem provided that data h are in some Sobolev space, and the proof is based on the reduction of the inverse problem to an integral equation of the second kind. Section 3 is devoted to our first regularizing scheme which is based on the integral equation derived in section 2, and establish an *a priori* choice strategy of regularizing parameters which gives a convergence rate of regularized solutions. In section 4, we discuss the second regularizing scheme which is based on direct discretization by radial basis functions (RBFs) of the original initial-boundary value problem, and we give numerical examples which show the validity of our regularizing scheme.

2. Stability and uniqueness

Throughout this paper, the constant $C > 0$ denotes generic constants and may be different in the whole paper, and we denote the scalar product in $L^2(\Omega)$ by $(\cdot, \cdot)$. We assume

$$g \in H_0^{3/2}(\Gamma_1), \qquad \frac{1}{2} < \alpha < 1.$$

We can discuss the case $0 < \alpha \leqslant \frac{1}{2}$, but the treatments are more complicated and here we discuss only the case of $\frac{1}{2} < \alpha < 1$. The cases for $0 < \alpha \leqslant \frac{1}{2}$ require more work in the future.

Let $H^\alpha(0, T)$ denote the Sobolev space of the order α (e.g., Shimakura [30]). We set

$$_0H^\alpha(0, T) := \left\{ f \in H^\alpha(0, T) : f(0) = 0 \right\}.$$

Here *f*(0) is well-defined in the trace sense by the Sobolev embedding and $\alpha > \frac{1}{2}$.

Let G satisfy

$$\begin{cases} \sum_{i,j=1}^{d} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x) \dfrac{\partial G}{\partial x_j} \right) + c(x) G = 0, & x \in \Omega, \\ G \mid_{\Gamma_0} = 0, & G \mid_{\Gamma_1} = g \end{cases} \tag{2.1}$$

for $g \in H_0^{3/2}(\Gamma_1)$.

In sections 2 and 3, we assume

$$(G, \mu) \neq 0. \tag{2.2}$$

Condition (2.2) is satisfied for example if $g \geqslant 0, \not\equiv 0$ on Γ_1 and $\mu \geqslant 0, \not\equiv 0$ in Ω. In fact, by (2.1) and $c \leqslant 0$, the maximum principle yields $G > 0$ in Ω. Therefore by $\mu \geqslant 0, \not\equiv 0$, it follows that $(G, \mu) \neq 0$.

Now we state our first main result on the well-posedness of our inverse problem in $_0H^\alpha(0, T)$:

Theorem 2.1. *For given $h \in {_0H^\alpha}(0, T)$, there exists a unique solution*

$$(u(F), F) \in ({_0H^\alpha}\left(0, T; L^2\left(\Omega\right)\right) \bigcap L^2\left(0, T; H^2(\Omega)\right)\right) \times {_0H^\alpha}\left(0, T\right)$$

to (1.1)–(1.3). Moreover there exists a constant $C > 0$ such that

$$C^{-1} \left\| h \right\|_{_0H^\alpha(0,T)} \leqslant \left\| F \right\|_{_0H^\alpha(0,T)} \leqslant C \left\| h \right\|_{_0H^\alpha(0,T)}$$

for each $h \in {_0H^\alpha}(0, T)$.

Theorem 2.1 is derived from theorem 2.2 stated below. For the statement of theorem 2.2, we need to introduce notations, functions and operators. Throughout this paper, by $E_{\alpha,\beta}(z)$ with $\alpha, \beta > 0$, we denote the Mittag–Leffler function which is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.$$

Here the series is absolutely convergent for all $z \in \mathbb{C}$ and $E_{\alpha,\beta}(z)$ is an entire function in $z \in \mathbb{C}$ (e.g, [26]).

We define an operator $-A$ in $L^2(\Omega)$ by

$$\begin{cases} -Au(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u, \quad x \in \Omega, \\ \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega). \end{cases}$$

Denote by $\{(\lambda_n, \varphi_n(x)) : n = 1, 2, \ldots\}$ an orthonormal eigensystem of A: $A\varphi_n = \lambda_n\varphi_n$ and $\|\varphi_n\|_{L^2(0,T)} = 1$. Henceforth we number λ_n by

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

and it is known that $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. Moreover, for $y \in L^2(0, T)$, we define

$$K(t-\tau) := \frac{\Gamma(\alpha)}{(G,\mu)} \sum_{n=1}^{\infty} \big(G, \varphi_n\big)\,(\mu, \varphi_n) E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^{\alpha}\big) - 1, \tag{2.3}$$

$$\begin{aligned}\big(B_\alpha y\big)(t) :&= \int_0^t \frac{1 + K(t-\tau)}{(t-\tau)^{1-\alpha}} y(\tau)\mathrm{d}\tau \\ &= \frac{\Gamma(\alpha)}{(G,\mu)} \sum_{n=1}^{\infty} \big(G, \varphi_n\big)\,(\mu, \varphi_n) \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^{\alpha}\big) y(\tau)\mathrm{d}\tau \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}(Ly)(t) :&= \frac{-1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ &\times \int_0^t \left(\int_\tau^t (s-\tau)^{-\alpha} \frac{\partial}{\partial s}\big((t-s)^{\alpha-1}K(t-s)\big)\mathrm{d}s\right) y(\tau)\mathrm{d}\tau, \quad 0 < t < T. \end{aligned} \tag{2.5}$$

The operator L is well-defined for $y \in L^2(0, T)$, which is proved in theorem 2.2 below. We are ready to state the key Volterra integral equation for the inverse problem.

Theorem 2.2. *(i)* $L : L^2(0, T) \longrightarrow {}_0H^{\alpha}(0, T)$ *is a linear bounded operator. (ii) If* $(u(F), F) \in ({}_0H^{\alpha}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))) \times {}_0H^{\alpha}(0, T)$ *satisfies (1.1) and (1.2), then* $F \in {}_0H^{\alpha}(0, T)$ *satisfies*

$$(\Gamma(\alpha) - 1)F(t) = -(LF)(t) + \frac{\Gamma(\alpha)}{(G,\mu)}h(t), \quad 0 < t < T. \tag{2.6}$$

In theorem 2.2, our inverse problem is reduced to the Volterra integral equation (2.6) of the second kind, which plays an important role also in the first regularization discussed in section 3.

2.1. Preliminaries

For the proof of the theorems, we show two lemmata.

Lemma 2.3. *For given* $F \in {}_0H^{\alpha}(0, T)$*, there exists a unique solution*

$$u = u(F) \in_0 H^{\alpha}(0, T; L^2(\Omega)) \bigcap L^2\big(0, T; H^2(\Omega)\big)$$

to (1.1) and (1.2). Moreover there exists a constant $C > 0$ *such that*

$$\|u(F)\|_{{}_0H^\alpha(0,T;L^2(\Omega))} + \|u(F)\|_{L^2(0,T;H^2(\Omega))} \leqslant \|F\|_{{}_0H^\alpha(0,T)}$$

for each $F \in {}_0H^\alpha(0, T)$.

Proof. First we prove the uniqueness of $u(F)$. Let $F = 0$. Then we have

$$\begin{cases} \partial_t^\alpha u = -Au, & x \in \Omega,\ 0 < t < T, \\ u(x, t) = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x, 0) = 0, & x \in \Omega. \end{cases}$$

Therefore the uniqueness to the initial-boundary value problem (e.g., theorem 4.2 in [9]) yields $u \equiv 0$ in $\Omega \times (0, T)$, that is, the uniqueness is proved.

Next let

$$v(x, t) = u(x, t) - F(t)G(x), \quad x \in \Omega,\ 0 < t < T, \tag{2.7}$$

where $G \in H^2(\Omega)$ is a unique solution to (2.1). Then, in terms of (2.1) and $F(0) = 0$, we can rewrite (1.1) and (1.2) as

$$\begin{cases} \partial_t^\alpha v = -Av - G(x)\partial_t^\alpha F(t), & x \in \Omega,\ 0 < t < T, \\ v(x, t) = 0, & x \in \partial\Omega,\ 0 < t < T. \\ v(x, 0) = 0, & x \in \Omega. \end{cases} \tag{2.8}$$

By $F \in {}_0H^\alpha(0, T)$, applying theorem 4.2 in [9], we see that there exists a unique solution $v \in_0 H^\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ to (2.8). Therefore $u = v + GF \in_0 H^\alpha(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ satisfies (1.1) and (1.2). □

Henceforth we set $K'(t) := \frac{\mathrm{d}K}{\mathrm{d}t}(t)$. Next we prove

Lemma 2.4. *With some constants* d_0, d_1, a_1, a_2 *(which may be 0) and* $p_0, p_1, r \in C[0, T]$, *we have*

(i)

$$K(t) = d_0 t^\alpha + t^{2\alpha} p_0(t), \quad 0 \leqslant t \leqslant T.$$

(ii)

$$K'(t) = d_1 t^{\alpha-1} + t^{2\alpha-1} p_1(t), \quad 0 < t \leqslant T.$$

(iii)

$$\frac{\partial}{\partial s}\big((t-s)^{\alpha-1} K(t-s)\big) = a_1 (t-s)^{2\alpha-2} + a_2 (t-s)^{3\alpha-2} + r(t-s)(t-s)^{4\alpha-2} \tag{2.9}$$

for $0 < t - s < T$ *and*

$$|r'(t)| \leqslant C t^{-1} \tag{2.10}$$

for $0 < t < T$.

Proof. Henceforth by $\mu \in C_0^\infty(\Omega) \subset \mathcal{D}(A^m)$ with $m \in \mathbb{N}$, we note that $(\mu, A^m \varphi_n) = (A^m \mu, \varphi_n)$.

By the definition of K and

$$E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}, \quad (G, \mu) = \sum_{n=1}^{\infty}(G, \varphi_n)(\mu, \varphi_n),$$

we have

$$K(t) = \frac{\Gamma(\alpha)}{(G, \mu)}\sum_{n=1}^{\infty}(G, \varphi_n)(\mu, \varphi_n)\left(E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) - E_{\alpha,\alpha}(0)\right).$$

On the other hand, we have

$$\begin{aligned}
E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) - E_{\alpha,\alpha}(0) &= \sum_{k=1}^{\infty}\frac{(-\lambda_n t^{\alpha})^k}{\Gamma(\alpha k + \alpha)} = -\lambda_n t^{\alpha}\sum_{k=1}^{\infty}\frac{(-\lambda_n t^{\alpha})^{k-1}}{\Gamma(\alpha k + \alpha)}\\
&= -\lambda_n t^{\alpha}\left(\frac{1}{\Gamma(2\alpha)} - \lambda_n t^{\alpha}\sum_{k=2}^{\infty}\frac{(-\lambda_n t^{\alpha})^{k-2}}{\Gamma(\alpha k + \alpha)}\right)\\
&= -\lambda_n t^{\alpha}\left(\frac{1}{\Gamma(2\alpha)} - \lambda_n t^{\alpha}\sum_{j=0}^{\infty}\frac{(-\lambda_n t^{\alpha})^{j}}{\Gamma(\alpha j + 3\alpha)}\right)\\
&= -\lambda_n t^{\alpha}\frac{1}{\Gamma(2\alpha)} + \lambda_n^2 t^{2\alpha}E_{\alpha,3\alpha}(-\lambda_n t^{\alpha}).
\end{aligned}$$

Hence

$$\begin{aligned}
K(t) &= \frac{\Gamma(\alpha)}{\Gamma(2\alpha)(G, \mu)}\sum_{n=1}^{\infty}(G, \varphi_n)(\mu, \varphi_n)(-\lambda_n t^{\alpha})\\
&\quad + \frac{\Gamma(\alpha)}{(G, \mu)}\sum_{n=1}^{\infty}(G, \varphi_n)(\mu, \varphi_n)\lambda_n^2 t^{2\alpha}E_{\alpha,3\alpha}(-\lambda_n t^{\alpha})\\
&= \frac{\Gamma(\alpha)}{\Gamma(2\alpha)(G, \mu)}(G, -A\mu)t^{\alpha} + \frac{\Gamma(\alpha)}{(G, \mu)}t^{2\alpha}\sum_{n=1}^{\infty}(G, \varphi_n)(A^2\mu, \varphi_n)\\
&\quad \times E_{\alpha,3\alpha}(-\lambda_n t^{\alpha}) =: d_0 t^{\alpha} + p_0(t)t^{2\alpha}.
\end{aligned}$$

Here, by $\mu \in C_0^{\infty}(\Omega) \subset \mathcal{D}(A^2)$, we used $(\mu, A\varphi_n) = (A\mu, \varphi_n)$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(G, \varphi_n)(\mu, \varphi_n)(-\lambda_n t^{\alpha}) = \sum_{n=1}^{\infty}(G, \varphi_n)(\mu, -\lambda_n\varphi_n)t^{\alpha}\\
&= \sum_{n=1}^{\infty}(G, \varphi_n)(\mu, -A\varphi_n)t^{\alpha} = \sum_{n=1}^{\infty}(G, \varphi_n)(-A\mu, \varphi_n)t^{\alpha} = (G, -A\mu)t^{\alpha}
\end{aligned} \tag{2.11}$$

and

$$(G, \varphi_n)(\mu, \varphi_n)\lambda_n^2 = (G, \varphi_n)(\mu, \lambda_n^2\varphi_n) = (G, \varphi_n)(\mu, A^2\varphi_n) = (G, \varphi_n)(A^2\mu, \varphi_n).$$

By theorem 1.6 (p 35) in [26], for $\alpha, \beta > 0$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|E_{\alpha,\beta}(-\lambda_n t^{\alpha})| \leqslant C_{\alpha,\beta}, \quad t > 0, n \in \mathbb{N}. \tag{2.12}$$

Therefore it follows from $G, A^2\mu \in L^2(\Omega)$ that $p_0 \in C[0, T]$ and the proof of (i) is completed.

Next since $E_{\alpha,\gamma}(z)$ is an entire function, the termwise differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)\right) = t^{\gamma-2}E_{\alpha,\gamma-1}\left(-\lambda_n t^{\alpha}\right), \quad \alpha, \gamma > 0. \tag{2.13}$$

We apply (2.13) to have

$$\begin{aligned}\left(E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)\right)' &= \left(t^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)t^{1-\gamma}\right)' \\ &= \left(t^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)\right)' t^{1-\gamma} + t^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)(1-\gamma)t^{-\gamma},\end{aligned}$$

that is

$$\left(E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)\right)' = t^{-1}\left(E_{\alpha,\gamma-1}\left(-\lambda_n t^{\alpha}\right) - (\gamma-1)E_{\alpha,\gamma}\left(-\lambda_n t^{\alpha}\right)\right). \tag{2.14}$$

Therefore we have

$$\begin{aligned}K'(t) &= \frac{\Gamma(\alpha)}{(G, \mu)}\sum_{n=1}^{\infty}\left(G, \varphi_n\right)\left(\mu, \varphi_n\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\alpha,\alpha}\left(-\lambda_n t^{\alpha}\right)\right) \\ &= \frac{\Gamma(\alpha)}{(G, \mu)}t^{-1}\sum_{n=1}^{\infty}\left(G, \varphi_n\right)\left(\mu, \varphi_n\right)\left(E_{\alpha,\alpha-1}\left(-\lambda_n t^{\alpha}\right) - (\alpha-1)E_{\alpha,\alpha}\left(-\lambda_n t^{\alpha}\right)\right).\end{aligned}$$

Here the termwise differentiation can be justified by the convergence of (2.15) in $C[0, T]$ which can be seen below. Moreover

$$\begin{aligned}&E_{\alpha,\alpha-1}\left(-\lambda_n t^{\alpha}\right) - (\alpha-1)E_{\alpha,\alpha}\left(-\lambda_n t^{\alpha}\right) \\ &= -\lambda_n t^{\alpha}\left(\sum_{k=1}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{k-1}}{\Gamma(\alpha k + \alpha - 1)} - \sum_{k=1}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{k-1}(\alpha-1)}{\Gamma(\alpha k + \alpha)}\right) \\ &= -\lambda_n t^{\alpha}\left(\frac{1}{\Gamma(2\alpha-1)} - \frac{\alpha-1}{\Gamma(2\alpha)}\right) \\ &\quad + \lambda_n^2 t^{2\alpha}\left(\sum_{k=2}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{k-2}}{\Gamma(\alpha k + \alpha - 1)} - \sum_{k=2}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{k-2}(\alpha-1)}{\Gamma(\alpha k + \alpha)}\right) \\ &= -\lambda_n t^{\alpha}\left(\frac{1}{\Gamma(2\alpha-1)} - \frac{\alpha-1}{\Gamma(2\alpha)}\right) \\ &\quad + \lambda_n^2 t^{2\alpha}\left(\sum_{j=0}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{j}}{\Gamma(\alpha j + 3\alpha - 1)} - (\alpha-1)\sum_{j=0}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{j}}{\Gamma(\alpha j + 3\alpha)}\right) \\ &= -\lambda_n t^{\alpha}\left(\frac{1}{\Gamma(2\alpha-1)} - \frac{\alpha-1}{\Gamma(2\alpha)}\right) \\ &\quad + \lambda_n^2 t^{2\alpha}\left(E_{\alpha,3\alpha-1}\left(-\lambda_n t^{\alpha}\right) - (\alpha-1)E_{\alpha,3\alpha}\left(-\lambda_n t^{\alpha}\right)\right).\end{aligned}$$

Therefore, again similarly to (2.11), we obtain

$$\begin{aligned}K'(t) = &-\frac{\Gamma(\alpha)}{(G, \mu)}t^{\alpha-1}\left(\frac{1}{\Gamma(2\alpha-1)} - \frac{\alpha-1}{\Gamma(2\alpha)}\right)\sum_{n=1}^{\infty}\left(G, \varphi_n\right)\lambda_n\left(\mu, \varphi_n\right) \\ &+ \frac{\Gamma(\alpha)}{(G, \mu)}t^{-1}t^{2\alpha}\sum_{n=1}^{\infty}\left(G, \varphi_n\right)\lambda_n^2\left(\mu, \varphi_n\right)\end{aligned}$$

$$
\begin{aligned}
&\times \left(E_{\alpha,3\alpha-1}\left(-\lambda_n t^{\alpha}\right) - (\alpha-1)E_{\alpha,3\alpha}\left(-\lambda_n t^{\alpha}\right)\right)\\
&= \frac{\Gamma(\alpha)}{(G,\,\mu)}\left(\frac{\alpha-1}{\Gamma(2\alpha)} - \frac{1}{\Gamma(2\alpha-1)}\right)(G,\,A\mu)t^{\alpha-1}\\
&\quad + \frac{\Gamma(\alpha)}{(G,\,\mu)}t^{2\alpha-1}\sum_{n=1}^{\infty}\left(G,\,\varphi_n\right)\left(A^2\mu,\,\varphi_n\right)\\
&\quad \times \left(E_{\alpha,3\alpha-1}\left(-\lambda_n t^{\alpha}\right) - (\alpha-1)E_{\alpha,3\alpha}\left(-\lambda_n t^{\alpha}\right)\right)\\
&:= d_1 t^{\alpha-1} + t^{2\alpha-1}p_1(t) \quad \text{in } C(0,\,T].
\end{aligned}
\tag{2.15}
$$

In terms of (2.12), we verify that $p_1 \in C[0,\,T]$, which implies (ii). Next we will prove (iii). By (i) and (ii), we have

$$
\begin{aligned}
&\frac{\partial}{\partial s}\left((t-s)^{\alpha-1}K(t-s)\right) = (1-\alpha)(t-s)^{\alpha-2}K(t-s) - (t-s)^{\alpha-1}K'(t-s)\\
&\quad = (1-\alpha)(t-s)^{\alpha-2}\left(d_0(t-s)^{\alpha} + p_0(t-s)(t-s)^{2\alpha}\right)\\
&\qquad - (t-s)^{\alpha-1}\left(d_1(t-s)^{\alpha-1} + p_1(t-s)(t-s)^{2\alpha-1}\right)\\
&\quad = \left((1-\alpha)d_0 - d_1\right)(t-s)^{2\alpha-2} + \left((1-\alpha)p_0(t-s) - p_1(t-s)\right)(t-s)^{3\alpha-2}\\
&\quad =: a_1 t^{2\alpha-2} + (t-s)^{3\alpha-2}R(t-s).
\end{aligned}
$$

That is, $a_1 = (1-\alpha)d_0 - d_1$ and

$$
\begin{aligned}
R(t) &:= (1-\alpha)p_0(t) - p_1(t)\\
&= \frac{\Gamma(\alpha)}{(G,\,\mu)}\sum_{n=1}^{\infty}\left(G,\,\varphi_n\right)\left(A^2\mu,\,\varphi_n\right)\{(1-\alpha)E_{\alpha,3\alpha}\left(-\lambda_n t^{\alpha}\right)\\
&\quad - (E_{\alpha,3\alpha-1}\left(-\lambda_n t^{\alpha}\right) - (\alpha-1)E_{\alpha,3\alpha}\left(-\lambda_n t^{\alpha}\right))\}\\
&= -\frac{\Gamma(\alpha)}{(G,\,\mu)}\sum_{n=1}^{\infty}\left(G,\,\varphi_n\right)\left(A^2\mu,\,\varphi_n\right)E_{\alpha,3\alpha-1}\left(-\lambda_n t^{\alpha}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
E_{\alpha,3\alpha-1}\left(-\lambda_n t^{\alpha}\right) &= \frac{1}{\Gamma(3\alpha-1)} + \sum_{k=1}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^k}{\Gamma(\alpha k + 3\alpha - 1)}\\
&= \frac{1}{\Gamma(3\alpha-1)} - \lambda_n t^{\alpha}\sum_{k=1}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{k-1}}{\Gamma(\alpha k + 3\alpha - 1)}\\
&= \frac{1}{\Gamma(3\alpha-1)} - \lambda_n t^{\alpha}\sum_{j=0}^{\infty}\frac{\left(-\lambda_n t^{\alpha}\right)^{j}}{\Gamma(\alpha j + 4\alpha - 1)}\\
&= \frac{1}{\Gamma(3\alpha-1)} - \lambda_n t^{\alpha}E_{\alpha,4\alpha-1}\left(-\lambda_n t^{\alpha}\right),
\end{aligned}
$$

and so

$$R(t) = -\frac{\Gamma(\alpha)}{(G,\ \mu)}\sum_{n=1}^{\infty}\big(G,\ \varphi_n\big)\big(A^2\mu,\ \varphi_n\big)\frac{1}{\Gamma(3\alpha-1)} - \frac{\Gamma(\alpha)}{(G,\ \mu)}\sum_{n=1}^{\infty}\big(G,\ \varphi_n\big)\big(A^2\mu,\ \varphi_n\big)\big(-\lambda_n t^{\alpha}\big)E_{\alpha,4\alpha-1}\big(-\lambda_n t^{\alpha}\big)$$
$$= -\frac{\Gamma(\alpha)}{(G,\ \mu)\Gamma(3\alpha-1)}\big(A^2\mu,\ G\big) + \frac{\Gamma(\alpha)}{(G,\ \mu)}t^{\alpha}\sum_{n=1}^{\infty}\big(G,\ \varphi_n\big)\big(A^3\mu,\ \varphi_n\big)E_{\alpha,4\alpha-1}\big(-\lambda_n t^{\alpha}\big).$$

Setting

$$a_2 = -\frac{\Gamma(\alpha)}{(G,\ \mu)\Gamma(3\alpha-1)}\big(A^2\mu,\ G\big)$$

and

$$r(t) = \frac{\Gamma(\alpha)}{(G,\ \mu)}\sum_{n=1}^{\infty}\big(G,\ \varphi_n\big)\big(A^3\mu,\ \varphi_n\big)E_{\alpha,4\alpha-1}\big(-\lambda_n t^{\alpha}\big),$$

we have

$$R(t-s)(t-s)^{3\alpha-2} = a_2(t-s)^{3\alpha-2} + r(t-s)(t-s)^{4\alpha-2}.$$

Therefore we obtain (2.9). Finally we have to prove (2.10). By (2.14), we have

$$r'(t) = \frac{\Gamma(\alpha)}{(G,\ \mu)}t^{-1}\sum_{n=1}^{\infty}\big(G,\ \varphi_n\big)\big(A^3\mu,\ \varphi_n\big) \times \Big(E_{\alpha,4\alpha-2}\big(-\lambda_n t^{\alpha}\big) - (4\alpha-2)E_{\alpha,4\alpha-1}\big(-\lambda_n t^{\alpha}\big)\Big).$$

Again, by (2.12), we have proved (2.10). Thus the proof of lemma 2.4 is completed. □

2.2. Proof of theorem 2.2

We define the Riemann–Liouville fractional integral operator $J^{\alpha} : L^2(0,\ T) \longrightarrow L^2(0,\ T)$ by

$$\big(J^{\alpha}y\big)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s \tag{2.16}$$

(e.g., Gorenflo and Vessella [10]).

Lemma 2.4 (iii) yields

$$\int_{\tau}^{t}(s-\tau)^{-\alpha}\frac{\partial}{\partial s}\big((t-s)^{\alpha-1}K(t-s)\big)\mathrm{d}s$$
$$= \int_{\tau}^{t}(s-\tau)^{-\alpha}\big(a_1(t-s)^{2\alpha-2} + a_2(t-s)^{3\alpha-2} + r(t-s)(t-s)^{4\alpha-2}\big)\mathrm{d}s$$
$$= \frac{a_1\Gamma(1-\alpha)\Gamma(2\alpha-1)}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{a_2\Gamma(1-\alpha)\Gamma(3\alpha-1)}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} + \int_{\tau}^{t}(s-\tau)^{-\alpha}r(t-s)(t-s)^{4\alpha-2}\mathrm{d}s.$$

Hence

$$(Ly)(t) = \frac{-a_1\Gamma(2\alpha - 1)}{\Gamma(\alpha)}\left(J^{\alpha}y\right)(t) - \frac{a_2\Gamma(3\alpha - 1)}{\Gamma(\alpha)}\left(J^{2\alpha}y\right)(t)$$
$$- \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)}\int_0^t \left(\int_\tau^t (s - \tau)^{-\alpha} r(t - s)(t - s)^{4\alpha-2}\mathrm{d}s\right) y(\tau)\mathrm{d}\tau$$
$$=: \frac{-a_1\Gamma(2\alpha - 1)}{\Gamma(\alpha)}\left(J^{\alpha}y\right)(t) - \frac{a_2\Gamma(3\alpha - 1)}{\Gamma(\alpha)}\left(J^{2\alpha}y\right)(t) + I(t). \tag{2.17}$$

We calculate $I'(t)$. By $4\alpha - 2 > 0$, $0 < \alpha < 1$ and lemma 2.4 (iii), we have

$$I'(t) = \frac{-1}{\Gamma(\alpha)\Gamma(1 - \alpha)}\int_0^t \partial_t\left(\int_\tau^t (s - \tau)^{-\alpha} r(t - s)(t - s)^{4\alpha-2}\mathrm{d}s\right) y(\tau)\mathrm{d}\tau$$
$$= \frac{-1}{\Gamma(\alpha)\Gamma(1 - \alpha)}\int_0^t \left(\int_\tau^t (s - \tau)^{-\alpha}\left(r'(t - s)(t - s)^{4\alpha-2}\right.\right.$$
$$\left.\left. + r(t - s)(4\alpha - 2)(t - s)^{4\alpha-3}\right)\mathrm{d}s\right) y(\tau)\mathrm{d}\tau.$$

By lemma 2.4, we obtain $|r'(t - s)| \leqslant C(t - s)^{-1}$ for $0 < t < T$, so that

$$|(s - \tau)^{-\alpha}\left(r'(t - s)(t - s)^{4\alpha-2} + r(t - s)(4\alpha - 2)(t - s)^{4\alpha-3}\right)|$$
$$\leqslant C(s - \tau)^{-\alpha}(t - s)^{4\alpha-3}.$$

Since $4\alpha - 3 > -1$ by $\alpha > \frac{1}{2}$, we obtain

$$\left|\int_0^t \left(\int_\tau^t (s - \tau)^{-\alpha}\left(r'(t - s)(t - s)^{4\alpha-2} + r(t - s)(4\alpha - 2)(t - s)^{4\alpha-3}\right)\mathrm{d}s\right) y(\tau)\mathrm{d}\tau\right|$$
$$\leqslant C\int_0^t \left(\int_\tau^t (s - \tau)^{-\alpha}(t - s)^{4\alpha-3}\mathrm{d}s\right)|y(\tau)|\mathrm{d}\tau = C\int_0^t (t - \tau)^{3\alpha-2}|y(\tau)|\mathrm{d}\tau,$$

and

$$\int_0^T |\int_0^t \left(\int_\tau^t (s - \tau)^{-\alpha}\left(r'(t - s)(t - s)^{4\alpha-2}\right.\right.$$
$$\left.\left. + r(t - s)(4\alpha - 2)(t - s)^{4\alpha-3}\right)\mathrm{d}s\right)|y(\tau)|\mathrm{d}\tau\,|^2\mathrm{d}t$$
$$\leqslant C\int_0^T \left(\int_0^t (t - \tau)^{3\alpha-2}|y(\tau)|\,\mathrm{d}\tau\right)^2 \mathrm{d}t = C\,\|t^{3\alpha-2}*|y(t)|\|^2_{L^2(0,T)}$$
$$\leqslant C\left(\int_0^T t^{3\alpha-2}\mathrm{d}t\right)^2\left(\int_0^T |y(\tau)|^2\mathrm{d}\tau\right),$$

where we used the Young inequality for the convolution. Therefore $\|I\|_{H^1(0,T)} \leqslant C\|y\|_{L^2(0,T)}$.

Moreover by theorem 2.1 in [9], we have

$$\|J^{\alpha}y\|_{{}_0H^{\alpha}(0,T)} \leqslant C\,\|y\|_{L^2(0,T)}$$

and

$$\|J^{2\alpha}y\|_{{}_0H^{\alpha}(0,T)} = \|J^{\alpha}\left(J^{\alpha}y\right)\|_{{}_0H^{\alpha}(0,T)} \leqslant C\,\|J^{\alpha}y\|_{L^2(0,T)} \leqslant C\,\|y\|_{L^2(0,T)}.$$

Here the final inequality follows from (2.16) and the Young inequality for the convolution. Therefore (2.17) yields

$$\|Ly\|_{{}_0H^\alpha(0,T)} \leqslant C\,\|J^\alpha y\|_{{}_0H^\alpha(0,T)} + C\,\|J^{2\alpha}y\|_{{}_0H^\alpha(0,T)} + \|I\|_{{}_0H^\alpha(0,T)}$$
$$\leqslant C\,\|y\|_{L^2(0,T)} + C\,\|I\|_{H^1(0,T)} \leqslant C\,\|y\|_{L^2(0,T)}.$$

Thus the proof of (i) is completed.

Next we prove theorem 2.2 (ii). By (2.7) we have

$$h(t) = (v(\,\cdot\,, t), \mu) + F(t)(G, \mu), \quad 0 < t < T. \tag{2.18}$$

It follows from theorem 2.2 in [28] that the solution to (2.8) is represented by

$$v(x, t) = -\sum_{n=1}^{\infty}\big(G, \varphi_n\big)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\partial_\tau^\alpha F(\tau)\mathrm{d}\tau\;\varphi_n(x).$$

Then this expression with (2.18) yields

$$h(t) = F(t)(G, \mu) - \sum_{n=1}^{\infty}\big(G, \varphi_n\big)\,(\mu, \varphi_n)$$
$$\times\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\partial_\tau^\alpha F(\tau)\mathrm{d}\tau. \tag{2.19}$$

We represent (2.19) in terms of B_α defined by (2.4) to have

$$-\frac{\Gamma(\alpha)}{(G, \mu)}h(t) + F(t)\Gamma(\alpha) = \big(B_\alpha\partial_t^\alpha F\big)(t), \quad 0 < t < T. \tag{2.20}$$

Thus, by the proof of theorem 1 in Gorenflo and Yamamoto [11] for example, we have the identity

$$\big(B_\alpha y\big)(t) = (1 - L)\big(J^\alpha y\big)(t), \quad y \in L^2(0, T). \tag{2.21}$$

Therefore (2.20) and (2.21) imply

$$-\frac{\Gamma(\alpha)}{(G, \mu)}h(t) + F(t)\Gamma(\alpha) = (1 - L)\big(J^\alpha\partial_t^\alpha F\big)(t). \tag{2.22}$$

In [9] (theorem 3.1), it is proved that

$$\partial_t^\alpha F = \big(J^\alpha\big)^{-1}F \tag{2.23}$$

for $F \in {}_0H^\alpha(0, T)$. Hence (2.22) yields (2.6). Thus the proof of theorem 2.2 is completed.

2.3. Proof of theorem 2.1

In view of theorem 2.2, it suffices to prove that the integral equation (2.6) possesses a unique solution $F \in {}_0H^\alpha(0, T)$ for any given $h \in {}_0H^\alpha(0, T)$. Since the embedding ${}_0H^\alpha(0, T) \longrightarrow L^2(0, T)$ is compact, by theorem 2.2 (i), the operator $L : L^2(0, T) \longrightarrow L^2(0, T)$ is compact. Therefore when we prove that if $F \in L^2(0, T)$ satisfies

$$(\Gamma(\alpha) - 1)F(t) = -(LF)(t), \qquad 0 < t < T,$$

then $F \equiv 0$ in $(0, T)$, the Fredholm alternative yields that there exists a unique solution $F \in L^2(0, T)$ to (2.7) for any $h \in L^2(0, T)$. We apply lemma 2.4 (iii) and the Cauchy–Schwartz inequality to have

$$|F(t)| \leqslant C|LF(t)| \leqslant C\int_0^t \left(\int_\tau^t (s-\tau)^{-\alpha}(t-s)^{2\alpha-2}\mathrm{d}s\right)|F(\tau)|\,\mathrm{d}\tau$$
$$\leqslant C\int_0^t (t-\tau)^{\alpha-1}|F(\tau)|\,\mathrm{d}\tau \leqslant Ct^{\alpha-\frac{1}{2}}\left(\int_0^t |F(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}},$$

which leads to

$$|F(t)|^2 \leqslant C\int_0^t |F(s)|^2\mathrm{d}s, \quad 0<t<T$$

by $\frac{1}{2}<\alpha$. The Gronwall inequality implies that $F=0$ in $(0, T)$. Consequently (2.6) possesses a unique solution $F \in L^2(0, T)$ for each $h \in L^2(0, T)$.

Next let $h \in {}_0H^\alpha(0, T)$. Then there exists a unique solution $F \in L^2(0, T)$ to (2.6). We have to prove that $F \in {}_0H^\alpha(0, T)$. In fact, theorem 2.2 (i) implies $LF \in {}_0H^\alpha(0, T)$. Since (2.6) yields $F(t) = \frac{1}{\Gamma(\alpha)-1}\left(-(LF)(t) + \frac{\Gamma(\alpha)}{(G,\mu)}h(t)\right)$, we obtain $F \in {}_0H^\alpha(0, T)$. Thus the proof of theorem 2.1 is completed.

The original inverse problem can be transformed into the linear integral equation (2.6) of the second kind with respect to $F \in {}_0H^\alpha(0, T)$, which can be represented in the operator form

$$F(t) - (\mathbb{K}F)(t) = c_0 h(t), \quad t \in [0, T], \tag{2.24}$$

with some linear operator $\mathbb{K}$ and constant c_0. Although the operator $\mathbb{K}$ is proved to be compact, available data h^δ contaminated with noises in $L^2(0, T)$ are not necessarily in ${}_0H^\alpha(0, T)$ and we cannot guarantee that the solution F^δ to (2.6) for h^δ is in ${}_0H^\alpha(0, T)$, which means that F^δ cannot be the solution to our original inverse problem. Thus in sections 3 and 4, we discuss regularization schemes.

3. First regularizing scheme

In this section, we start to discuss

Stable reconstruction scheme: Let $u_0 = u_0(x, t)$ satisfy (1.1) and (1.2) with $F_0 \in {}_0H^\alpha(0, T)$, and denote by $h_0(t) := (u_0(\cdot, t), \mu)$ for $0<t<T$ the exact measurement data. Given the noisy data h^δ of h_0 satisfying

$$\|h^\delta - h_0\|_{L^2(0,T)} \leqslant \delta, \tag{3.1}$$

construct approximations $F^\delta \in L^2(0, T)$ stably such that $\lim_{\delta\to 0}\|F^\delta - F_0\|_{L^2(0,T)} = 0$.

Theorem 2.1 asserts that if $h^\delta \in {}_0H^\alpha(0, T)$, then a solution to (2.6) exists uniquely in $L^2(0, T)$, but in general does not belong to ${}_0H^\alpha(0, T)$. Thus as our first regularization scheme, we propose to smooth h^δ by the mollifier and solve (2.6) with such mollified data. Thus our first regularizing scheme can be described as follows.

Step 1. We smooth $h^\delta \in L^2(0, T)$ by the mollifier. Let $\beta>0$ be a parameter. Henceforth for $g \in L^2(0, T)$, by $\widetilde{g}$ we denote the zero extension of g outside $(0, T)$:

$$\widetilde{g}(t) = \begin{cases} g(t), & 0 \leqslant t \leqslant T, \\ 0, & \text{either} \quad t<0 \quad \text{or} \quad t>T. \end{cases}$$

We set

$$h_\beta^\delta(t) := \int_{-\infty}^{\infty} \rho_\beta(t-\tau)\widetilde{h^\delta}(\tau)\mathrm{d}\tau, \quad 0 < t < T, \tag{3.2}$$

where $0 \leqslant \rho \in C_0^\infty(\mathbb{R})$, supp $\rho \subset (-1, 1)$, $\int_{-\infty}^{\infty} \rho(t)\mathrm{d}t = 1$ and we set

$$\rho_\beta(t) := \frac{1}{\beta}\rho\left(\frac{t}{\beta}\right).$$

The parameter $\beta > 0$ is presumed to be small and plays the role of regularizing parameter.

Step 2. Construct a regularized equation of (2.6) for noisy data $h^\delta(t)$ by

$$(\Gamma(\alpha) - 1)F^{\beta,\delta}(t) = -L\left[F^{\beta,\delta}\right](t) + \frac{\Gamma(\alpha)}{(G, \mu)}h_\beta^\delta(t), \quad 0 < t < T. \tag{3.3}$$

Since $h_\beta^\delta \in C_0^\infty(0, T) \subset {}_0H^\alpha(0, T)$, by theorem 2.1 we note that $F^{\beta,\delta} \in L^2(0, T)$ exists uniquely and (3.3) can be solved stably in $L^2(0, T)$. We can regard $F^{\beta,\delta}$ as a reasonable approximation to F_0 by choosing $\beta > 0$ suitably in terms of the noise level $\delta > 0$.

Henceforth $C > 0$ denotes generic constants which are independent of β, δ and dependent on α, F_0, h_0, G.

We give a choice strategy of the regularizing parameter β for the noise level δ.

Theorem 3.1. *We assume (2.2) and $F_0 \in_0 H^\alpha(0, T)$. With the regularizing parameter $\beta(\delta) \sim \delta^{\frac{1}{\alpha}}$, we have*

$$\|F^{\beta(\delta),\delta} - F_0\|_{L^2(0,T)} \leqslant C\delta^{1-\frac{1}{2\alpha}}. \tag{3.4}$$

This result gives an *a priori* strategy for choices of regularizing parameter $\beta(\delta)$ for given noise level $\delta > 0$, which can be presumed to be quasi-optimal. More precisely, our *a priori* choice strategy $\beta = \delta^{\frac{1}{\alpha}}$ of regularizing parameter guarantees the convergence rate $\delta^{1-\frac{1}{2\alpha}}$ of the regularized solutions towards the exact solution F_0. The convergence rate is smaller than $\frac{1}{2}$ for all $\alpha \in \left(\frac{1}{2}, 1\right)$ and cannot reach 1 and is always worse than the linear convergence rate δ. This is a common feature in the regularization of an ill-posed problem, although there is a possibility that the best convergence rate may be improved by a different regularization method. Other interesting research topic for convergence rate is the *a posteriori* choice strategy for the regularizing parameter β, but we do not treat here.

Proof. Without loss of generality, we can assume that β, $\delta > 0$ are sufficiently small. Let $F^{\beta,0}$ be the solution to (3.3) with $(h_0)_\beta$.

Next we note

$$\|F^{\beta,\delta} - F_0\|_{L^2(0,T)} \leqslant \|F^{\beta,\delta} - F^{\beta,0}\|_{L^2(0,T)} + \|F^{\beta,0} - F_0\|_{L^2(0,T)}. \tag{3.5}$$

Henceforth we recall that

$$\left(h^\delta - h_0\right)_\beta(t) = \int_{-\infty}^{\infty} \rho_\beta(t-\tau)\left(\widetilde{h^\delta} - \widetilde{h_0}\right)(\tau)\mathrm{d}\tau, \quad 0 < t < T.$$

By (3.3) in step 2, we know that $F^{\beta,\delta} - F^{\beta,0}$ and $F^{\beta,0} - F_0$ satisfy

$$\begin{cases} F^{\beta,\delta}(t) - F^{\beta,0}(t) = \dfrac{1}{1-\Gamma(\alpha)}\Big(L\big(F^{\beta,\delta} - F^{\beta,0}\big)\Big)(t) \\ \quad -\dfrac{1}{1-\Gamma(\alpha)}\dfrac{\Gamma(\alpha)}{(G,\,\mu)}\big(h^{\delta} - h_0\big)_{\beta}(t), \\ F^{\beta,0}(t) - F_0(t) = \dfrac{1}{1-\Gamma(\alpha)}\Big(L\big(F^{\beta,0} - F_0\big)\Big)(t) \\ \quad -\dfrac{1}{1-\Gamma(\alpha)}\dfrac{\Gamma(\alpha)}{(G,\,\mu)}\Big(\big(h_0\big)_{\beta} - h_0\Big)(t). \end{cases} \tag{3.6}$$

As is seen from the proof of theorem 2.1, the linear operator $I - \dfrac{1}{1-\Gamma(\alpha)}L : L^2(0,\,T) \to L^2(0,\,T)$ is invertible and $\left\|\left(I - \frac{1}{1-\Gamma(\alpha)}L\right)^{-1}\right\| \leqslant C$. Therefore (3.6) yields

$$\begin{aligned} &\|F^{\beta,\delta} - F_0\|_{L^2(0,T)} \\ &\leqslant C\left\|\left(I - \frac{1}{1-\Gamma(\alpha)}L\right)^{-1}\right\|\Big(\|\big(h^{\delta} - h_0\big)_{\beta}\|_{L^2(0,T)} + \|\big(h_0\big)_{\beta} - h_0\|_{L^2(0,T)}\Big) \\ &\leqslant C\Big(\|\big(h^{\delta} - h_0\big)_{\beta}\|_{L^2(0,T)} + \|\big(h_0\big)_{\beta} - h_0\|_{L^2(0,T)}\Big). \end{aligned} \tag{3.7}$$

For the first term on the right-hand side of (3.7), by the Cauchy–Schwartz inequality and $\operatorname{supp}\,\rho_\beta \subset (-\beta,\,\beta)$, changing the variable τ to ξ by $\xi = \dfrac{t-\tau}{\beta}$, we have

$$\begin{aligned} \left|\big(h^{\delta} - h_0\big)_{\beta}(t)\right| &\leqslant \int_{-\infty}^{\infty} |\rho_\beta(t-\tau)|\left|\Big(\widetilde{h^{\delta}}(\tau) - \widetilde{h_0}(\tau)\Big)\right|\mathrm{d}\tau \leqslant \delta\left(\int_{t-\beta}^{t+\beta}\left|\rho_\beta(t-\tau)\right|^2\mathrm{d}\tau\right)^{1/2} \\ &= \frac{\delta}{\sqrt{\beta}}\left(\int_{-1}^{1}|\rho(\xi)|^2\mathrm{d}\xi\right)^{1/2} \leqslant C\frac{\delta}{\sqrt{\beta}}, \quad 0 < t < T. \end{aligned}$$

Therefore

$$\|\big(h^{\delta} - h_0\big)_{\beta}\|_{L^2(0,T)} \leqslant \frac{C\delta}{\sqrt{\beta}}. \tag{3.8}$$

For the second term, by (3.2) and $\int_{-\infty}^{\infty}\rho_\beta(t-s)\mathrm{d}s = 1$, we have for $0 < t < T$ that

$$\begin{aligned} \left|\big(h_0\big)_{\beta}(t) - h_0(t)\right| &= \left|\int_{-\infty}^{\infty}\rho_\beta(t-s)\Big(\widetilde{h_0}(s) - h_0(t)\Big)\mathrm{d}s\right| \\ &\leqslant \int_{t-\beta}^{t+\beta}|\rho_\beta(t-s)||\widetilde{h_0}(s) - h_0(t)|\mathrm{d}s. \end{aligned}$$

We separately estimate in the two cases $t \in (\beta,\,T-\beta)$ and $t \in (0,\,T)\backslash(\beta,\,T-\beta)$.

Case 1: $t \in (\beta,\,T-\beta)$.

Then $0 < t-\beta < t+\beta < T$ and so $\widetilde{h_0}(s) = h_0(s)$ for $s \in (t-\beta,\,t+\beta)$. By $\alpha > \frac{1}{2}$, the Sobolev embedding theorem (e.g., corollary E.13 (p 259) in Shimakura [30]) implies that for $h_0 \in {}_0H^{\alpha}(0,\,T)$, there exists a constant $C > 0$ such that

$$|h_0(t) - h_0(s)| \leqslant C|t-s|^{\alpha-\frac{1}{2}}, \quad t,\,s \in [0,\,T].$$

Hence

$$\int_{t-\beta}^{t+\beta} \left|\rho_\beta(t-s)\right| |\widetilde{h_0}(s) - h_0(t)| \mathrm{d}s \quad \leqslant C \int_{t-\beta}^{t+\beta} \left|\rho_\beta(t-s)\right| |t-s|^{\alpha-\frac{1}{2}} \mathrm{d}s$$
$$= \frac{C}{\beta} \int_{t-\beta}^{t+\beta} \left|\rho\left(\frac{t-s}{\beta}\right)\right| |t-s|^{\alpha-\frac{1}{2}} \mathrm{d}s.$$

Changing the variables $\xi = \dfrac{t-s}{\beta}$, we obtain

$$\frac{1}{\beta} \int_{t-\beta}^{t+\beta} \left|\rho\left(\frac{t-s}{\beta}\right)\right| |t-s|^{\alpha-\frac{1}{2}} \mathrm{d}s = \int_{-1}^{1} (\beta|\xi|)^{\alpha-\frac{1}{2}} |\rho(\xi)| \mathrm{d}\xi \leqslant \beta^{\alpha-\frac{1}{2}} \frac{2}{2\alpha+1} \|\rho\|_{C[-1,1]}.$$

Hence

$$\int_{t-\beta}^{t+\beta} \left|\rho_\beta(t-s)\right| |\widetilde{h_0}(s) - h_0(t)| \mathrm{d}s \leqslant C\beta^{\alpha-\frac{1}{2}}, \quad \beta < t < T - \beta. \tag{3.9}$$

Case 2: $t \in (0, T) \backslash (\beta, T-\beta)$.

Then

$$\begin{aligned}\int_{t-\beta}^{t+\beta} \left|\rho_\beta(t-s)\right| |\widetilde{h_0}(s) - h_0(t)| \mathrm{d}s &\leqslant \int_{t-\beta}^{t+\beta} \left|\rho_\beta(t-s)\right| \left(\left|\widetilde{h_0}(s)\right| + \left|h_0(t)\right| \right) \mathrm{d}s \\ &\leqslant 2\|h_0\|_{C[0,T]} \int_{t-\beta}^{t+\beta} \rho_\beta(t-s) \mathrm{d}s \\ &= 2\|h_0\|_{C[0,T]} \int_{-\beta}^{\beta} \rho_\beta(\eta) \mathrm{d}\eta = 2\|h_0\|_{C[0,T]}\end{aligned} \tag{3.10}$$

by $\rho \geqslant 0$ and $1 = \int_{\infty}^{\infty} \rho_\beta(\xi) \mathrm{d}\xi = \int_{-\beta}^{\beta} \rho_\beta(\xi) \mathrm{d}\xi$. Consequently

$$|(h_0)_\beta(t) - h_0(t)| \leqslant \begin{cases} C\beta^{\alpha-\frac{1}{2}}, & \beta < t < T-\beta, \\ C, & \text{either } 0 < t < \beta \text{ or } T-\beta < t < T \end{cases}$$

and so

$$\begin{aligned}\|(h_0)_\beta - h_0\|_{L^2(0,T)} &\leqslant C\left(\beta^{\alpha-\frac{1}{2}} + \left(\int_0^\beta \mathrm{d}s \right)^{\frac{1}{2}} + \left(\int_{T-\beta}^{T} \mathrm{d}s \right)^{\frac{1}{2}} \right) \\ &\leqslant C\left(\beta^{\alpha-\frac{1}{2}} + \beta^{\frac{1}{2}} \right) \leqslant 2C\beta^{\alpha-\frac{1}{2}}.\end{aligned} \tag{3.11}$$

At the last inequality, we used that $\beta > 0$ is small and so $0 < \beta \leqslant 1$ and $\alpha < 1$. Thus inserting (3.8) and (3.11) into (3.7), we have

$$\|F^{\beta,\delta} - F_0\|_{L^2(0,T)} \leqslant C\left(\frac{\delta}{\sqrt{\beta}} + \beta^{\alpha-\frac{1}{2}} \right).$$

We choose $\beta > 0$ so that the right-hand side is as small as possible. That is, we set $\frac{\delta}{\sqrt{\beta}} = \beta^{\alpha-\frac{1}{2}}$, that is, $\beta = \delta^{\frac{1}{\alpha}}$ gives an optimal choice of β for the convergence rate. The proof is completed. □

We have established the regularizing scheme (3.3) for approximately constructing F_0 from noisy data h^δ with convergence order analysis. However, this proposed scheme has some drawback from numerical points of view. That is, in order to solve (3.3) numerically, we need to discretize the integral term with singular kernel, which may cause some instability, noticing that we can only compute $K(t-s)$ approximately and then the numerical computation of $\frac{\partial}{\partial s}((t-s)^{\alpha-1}K(t-s))$ is ill-posed. Thus in section 4, we discuss the second regularizing scheme and test it numerically.

4. Second regularizing scheme and numerical reconstructions

In this section, as the second regularizing scheme, we propose a direct discretization of the following initial-boundary value problem for fractional diffusion equation with smoothed data in Ω:

$$\begin{cases} \dfrac{\partial^\alpha u^{\beta,\delta}}{\partial t^\alpha} = \Delta u^{\beta,\delta} + S(x,t), & x\in\Omega,\ 0<t<T,\\ u^{\beta,\delta}(x,t) = u_0(x,t), & x\in\Gamma_0,\ 0<t<T,\\ u^{\beta,\delta}(x,t) = g(x)F^{\beta,\delta}(t), & x\in\Gamma_1,\ 0<t<T,\\ u^{\beta,\delta}(x,0) = a(x), & x\in\Omega \end{cases} \tag{4.1}$$

with the additional input condition for $u^{\beta,\delta}(x,t)$:

$$J^\beta\big[h^\delta\big](t) = \int_\Omega u^{\beta,\delta}(x,t)\mu(x)\mathrm{d}x,\quad 0<t<T. \tag{4.2}$$

In this section, we consider a non-zero term $S(x,t)$, a non-zero boundary value u_0 and a non-zero initial value a, while we consider the simple case where $-A=\Delta$ for numerical tests. Here we set

$$Q(t) := \frac{1}{\sqrt{\pi}}\mathrm{e}^{-t^2},\quad J^\beta[h](t) := \int_{-\infty}^{\infty}\frac{1}{\beta}Q\left(\frac{t-\tau}{\beta}\right)\widetilde{h}(\tau)\mathrm{d}\tau,\quad t\in\mathbb{R},$$

and we recall that $\widetilde{h}$ denotes the zero extension of h outside $(0,T)$.

We propose the RBF method using multiquadric basis function (MQ) to directly solve (4.1)–(4.2) with respect to $u^{\beta,\delta}$ and $F^{\beta,\delta}$.

In order to illustrate how to apply the RBF as a spatial meshless scheme for solving this system, we firstly reduce the time-fractional diffusion equation in the system into a series of elliptic equations using the finite difference approximation to discretize the time-fractional derivative.

Divide the time interval $[0,T]$ uniformly into K subintervals by grids $t_k := k\Delta t,\ k=0,1,\dots,K$ with $\Delta t := T/K$. Let $u^{\beta,\delta}(x,t_k)$ be the exact value of solution $u^{\beta,\delta}(x,t)$ at t_k. Then the time fractional derivative can be approximated by [18]

$$\frac{\partial^\alpha u^{\beta,\delta}(x,t_{k+1})}{\partial t^\alpha} \approx \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\omega_j\big[u^{\beta,\delta}(x,t_{k-j+1}) - u^{\beta,\delta}(x,t_{k-j})\big] + \mathcal{O}\big((\Delta t)^{2-\alpha}\big) \tag{4.3}$$

for $k=0,1,\dots,K-1$, with the weight $\omega_j := (j+1)^{1-\alpha} - j^{1-\alpha},\ j = 0,1,\dots,k$.

Let $u_{k+1}^{\beta,\delta}(x)$ be the numerical approximation to $u^{\beta,\delta}(x,t_{k+1})$. By substituting (4.3) into (4.1) and rearranging the terms, we obtain the following series of elliptic equations at each time step t_{k+1} for two unknowns $u_{k+1}^{\beta,\delta}(x)$ and $F^{\beta,\delta}(t_{k+1})$:

$$u_{k+1}^{\beta,\delta}(x) - \theta\Delta u_{k+1}^{\beta,\delta}(x) = G_{k+1}(x) + \theta S_{k+1}(x),\ x\in\Omega, \tag{4.4}$$

$$u_{k+1}^{\beta,\delta}(x) = u_0(x, t_{k+1}), \; x \in \Gamma_0, \tag{4.5}$$

$$u_{k+1}^{\beta,\delta}(x) = g(x)F^{\beta,\delta}(t_{k+1}), \; x \in \Gamma_1, \tag{4.6}$$

$$\int_\omega u_{k+1}^{\beta,\delta}(x)\mu(x)\mathrm{d}x = J^\beta\left[h^\delta\right](t_{k+1}), \tag{4.7}$$

where $k = 0, 1, \ldots, K - 1$, $\theta = (\Delta t)^\alpha \Gamma(2 - \alpha)$, $S_{k+1}(x) = S(x, t_{k+1})$ and

$$G_{k+1}(x) = \begin{cases} u_k^{\beta,\delta}(x) - \sum_{j=1}^{k}\omega_j\left[u_{k+1-j}^{\beta,\delta}(x) - u_{k-j}^{\beta,\delta}(x)\right], & k \geqslant 1, \\ u_k^{\beta,\delta}(x) = a(x), & k = 0. \end{cases}$$

Using the above discretizaion on time t, the system (4.1)–(4.2) has been transformed into a series of inhomogeneous elliptic equations. Under the framework of RBFs method, the solution to (4.4)–(4.7) can be approximated as a linear combination of RBFs

$$u_{k+1}^{\beta,\delta}(x) \approx \sum_{j=1}^{N}\lambda_j^{k+1}\phi(r_j). \tag{4.8}$$

Here, $r_j = \|x - x_j\|$ for some $x_j \in \overline{\Omega}$ and ϕ could be any of the commonly used RBFs. The set of points $\{x_j\}_{j=1}^N$ are the centers of the RBF and $\{\lambda_j^{k+1}\}_{j=1}^N$ are the coefficients to be determined. There are many classes of RBFs such as MQ, inverse MQ and thin plate splines. Among all these candidates, MQ is one of the most popular RBFs. In our numerical examples, we choose MQ as the basis function, which is of the form

$$\phi(r) = \sqrt{r^2 + c^2}, \tag{4.9}$$

where c is the shape parameter specified in advance.

Substituting (4.8) into (4.4), we obtain the following equation

$$\sum_{j=1}^{N}\lambda_j^{k+1}\left(\phi(r_j) - \theta\Delta\phi(r_j)\right) = G_{k+1}(x) + \theta S_{k+1}(x), \quad x \in \Omega. \tag{4.10}$$

For numerical implementations of this proposed scheme, we uniformly choose $N = N_\Omega + N_{\Gamma_0} + N_{\Gamma_1}$ collocation points $\{x_i\}_{i=1}^N \subset \overline{\Omega} = \Omega \cup \overline{(\Gamma_0 \cup \Gamma_1)}$, where N_Ω and $N_{\Gamma_i}(i = 0, 1)$ are the number of interior and boundary points, respectively. By substituting the approximation of $u_{k+1}^{\beta,\delta}(x)$ given in (4.8) into the boundary conditions (4.5)–(4.7), we establish the following linear system for the unknowns λ_j^{k+1} and F^{k+1}:

$$\begin{cases} \sum_{j=1}^{N} \lambda_j^{k+1}\Big(\phi\big(\|x_i - x_j\|\big) - \theta\Delta\phi\big(\|x_i - x_j\|\big)\Big) \\ \quad = G_{k+1}(x_i) + \theta S_{k+1}(x_i), \quad i = 1, \ldots, N_\Omega, \\ \sum_{j=1}^{N} \lambda_j^{k+1}\phi\big(\|x_i - x_j\|\big) = u_0(x_i, t_{k+1}), \quad i = 1, \ldots, N_{\Gamma_0}, \\ \sum_{j=1}^{N} \lambda_j^{k+1}\phi\big(\|x_i - x_j\|\big) = g(x_i)F^{k+1}, \quad i = 1, \ldots, N_{\Gamma_1}, \\ \sum_{j=1}^{N} \lambda_j^{k+1}\int_\Omega \phi\big(\|x - x_j\|\big)\mu(x)\mathrm{d}x = J^\beta\big[h^\delta\big](t_{k+1}). \end{cases} \tag{4.11}$$

Here we consider F^{k+1} as approximation of $F^{\beta,\delta}(t_{k+1})$. The collocation system (4.11) constitutes $(N+1)$-coupled linear equations with $(N+1)$-unknowns $\lambda_1^{k+1}, \ldots, \lambda_N^{k+1}, F^{k+1}$ at the $(k+1)$th time level. In matrix form, the unknowns $\mathbf{c}^{k+1} = (\lambda_1^{k+1}, \lambda_2^{k+1}, \ldots, \lambda_N^{k+1}, F^{k+1})^T$ can be solved from the matrix equations

$$\mathbf{A}\mathbf{c}^{k+1} = \mathbf{b}^{k+1}, \tag{4.12}$$

where $\mathbf{A}$ is an $(N+1) \times (N+1)$ matrix

$$\mathbf{b}^{k+1} = \Big(G_{k+1}(x_1) + \theta S_{k+1}(x_1), \ldots, G_{k+1}(x_{N_\Omega}) + \theta S_{k+1}(x_{N_\Omega}), \\ u_0(x_1, t_{k+1}), \ldots, u_0\Big(x_{N_{\Gamma_0}}, t_{k+1}\Big), \mathbb{O}, J^\beta\big[h^\delta\big](t_{k+1})\Big)^T$$

is a known $(N+1) \times 1$ vector and $\mathbb{O}$ is the N_{Γ_1}-dimensional zero row vector.

The main drawback for the RBF method is that the condition number of the interpolation matrix $\mathbf{A}$ may be very large as observed in Chen *et al* [5], so is the case for the MQ. Hence some regularization techniques are required for solving this ill-conditioned system. Here we solve (4.12) by the Tikhonov regularization to stabilize the solution in our numerical tests. At each time level t_{k+1}, the Tikhonov regularized solution $\mathbf{c}_{\sigma^{k+1}}^{k+1}$ for (4.12) is defined as the solution of the minimization problem

$$\min_{\mathbf{c}^{k+1}} \big\{ \|\mathbf{A}\mathbf{c}^{k+1} - \mathbf{b}^{k+1}\|^2 + \sigma^{k+1}\|\mathbf{c}^{k+1}\|^2 \big\}, \tag{4.13}$$

with the regularization parameter $\sigma^{k+1} > 0$. The performance of regularization methods depends mostly on the suitable choice of the regularization parameter. For the Tikhonov regularization method, several heuristical approaches have been proposed, including the L-curve criterion, the cross validation (CV), and the generalized cross validation (GCV) Hansen (e.g., [13]). Here we use the GCV to provide appropriate regularization parameters at each time layer.

Now let us present several numerical results in terms of the RBF method stated above, to demonstrate the performance of our regularizing scheme. Here for simplicity, we consider the square $\Omega := \{(x, y):\ 0 < x, y < 1\}$ as the computational domain and partial boundary $\Gamma_1 = \{(1, y):\ 0 < y < 1\}$, and we choose

$$\mu(x, y) = \begin{cases} 1, & 0 < x < 1/3, 0 < y < 1, \\ 0, & \text{otherwise}. \end{cases}$$

However, the proposed method can be easily applied for general polygonal domains.

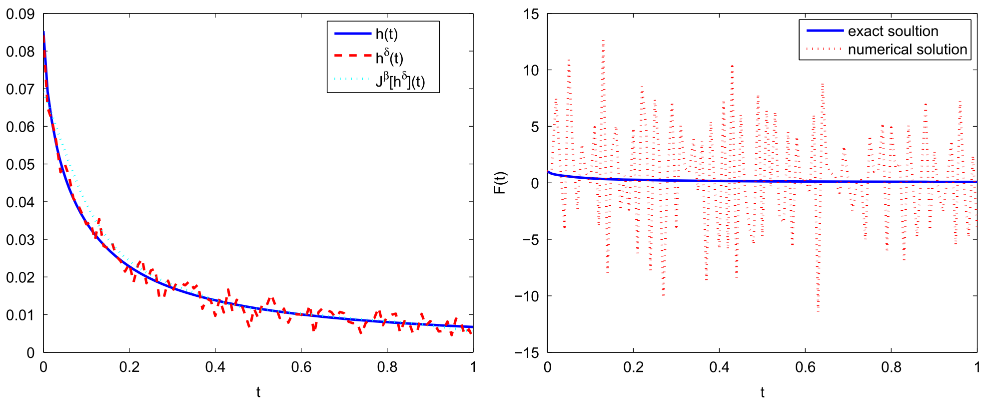

Figure 1. The exact data $h(t)$, the noisy data $h^\delta(t)$ as well as the smoothing data $J^\beta[h^\delta](t)$ (left), and the retrieved $F(t)$ for $\delta = 3\%$ without regularization (right).

In order to measure the error of numerical solutions, we use two kinds of measurement

$$E_{\mathrm{RMS}}(u) := \sqrt{\frac{1}{N_0}\sum_{i=1}^{N_0}\left(u^*(x_i, t) - u(x_i, t)\right)^2}, \quad E_{\max}(F) := \max_{1\leqslant i\leqslant K}\left|F^*(t_i) - F(t_i)\right|,$$

where $\{u^*, F^*\}$ and $\{u, F\}$ are the numerical and exact solutions, respectively. Here, N_0 is the number of testing nodes chosen randomly within the domain. For all the results presented below, we consider $N_0 = 400$. In the numerics, we take $(\alpha, \Delta t, c) = (0.7, 0.01, 1)$, and the Gauss quadrature rule is used to evaluate the integral in (4.11). For the domain investigated, we choose $N = 121$ collocation points evenly distributed in Ω. All the numerical results reported in the figures below are evaluated at $t = 1$.

For all examples, the synthetic noisy data are generated using the following formula:

$$h^\delta = h + \max\{|h|\}\delta\zeta, \tag{4.14}$$

where h is the exact data, δ is the relative noise level, and ζ is a Gaussian random variable with zero mean and unit standard deviation.

Example 1. The exact solution to (1.1) with $a(x) = \sin\left(\frac{\pi}{2}x_1\right)\sin(\frac{\pi}{2}x_2)$ can be given explicitly as $u(x, t) = E_{\alpha,1}\left(\frac{1}{2}\pi^2 t^\alpha\right)\sin(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)$.

Firstly, consider the noisy data with $\delta = 3\%$. The left figure of figure 1 shows the input data $h(t)$, its noisy form $h^\delta(t)$ and the smoothing data $J^\beta[h^\delta]$, while the right figure presents the exact and numerical solutions for $F(t)$, obtained from the noisy data without any regularizing technique. It can be seen that the result becomes oscillatory and inaccurate. When using our proposed regularization method, the numerical results for both $F(t)$ and $u(x, t)$ are solved stably and accurately, see figures 2 and 3, respectively.

Next, we analyze the dependence of the numerical solution on the parameters (c,N) also for the case $\delta = 3\%$. Figure 4(a) illustrates the performance of $E_{\mathrm{RMS}}(u)$ and $E_{\max}(F)$ as functions with respect to the constant c in (4.9). It can be seen that the accuracy of the numerical results is relatively independent of the parameter c if $c > 1$. The insensitivity of the solution to c over a fairly large range of the parameter is a favorable feature of MQ-RBF because there is no need to search for optimal value of parameter. In figure 4(b) we present

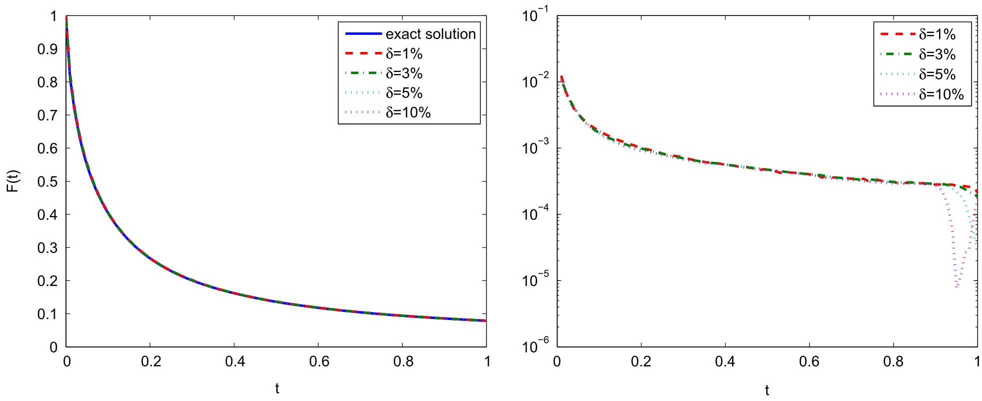

Figure 2. The retrieved $F(t)$ with regularization (left) and its absolute error (right).

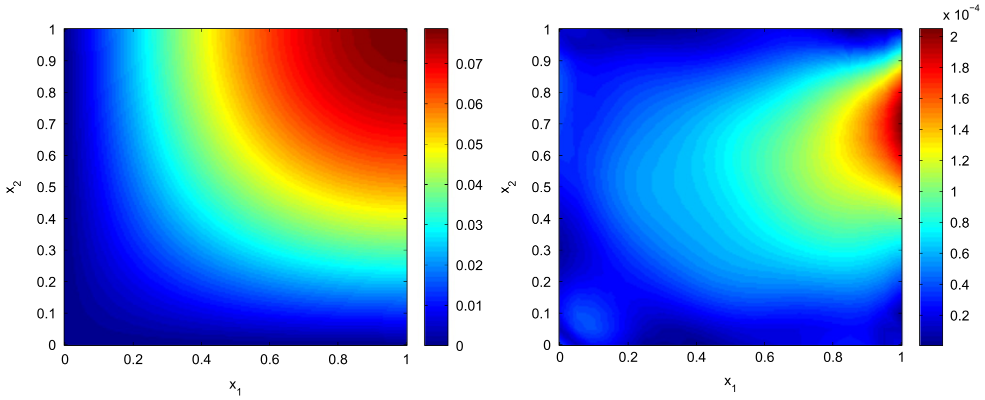

Figure 3. The retrieved $u(x, 1)$ with regularization (left) and its absolute error (right).

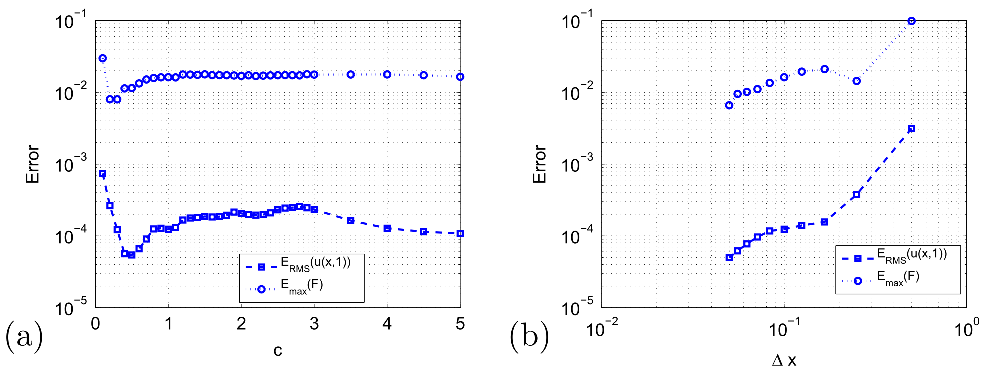

Figure 4. The accuracy of the numerical solutions for example 1 for $\delta = 3\%$ with respect to (a) the shape parameter c; and (b) Δx.

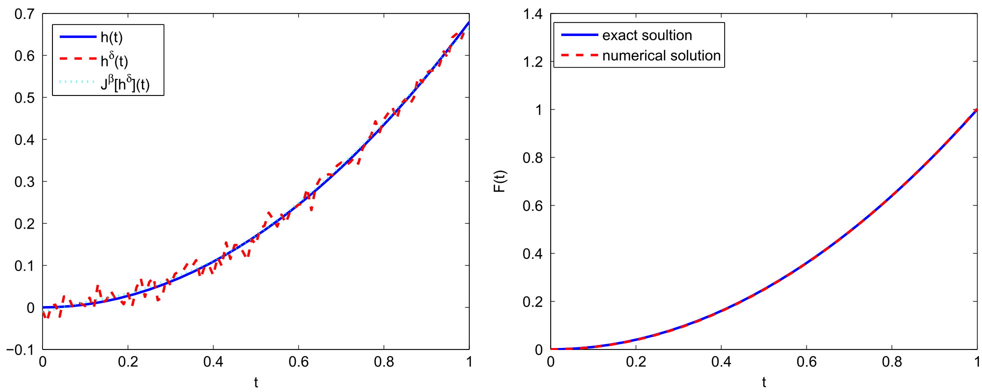

Figure 5. The data $h(t)$ (left) and the retrieved $F(t)$ (right).

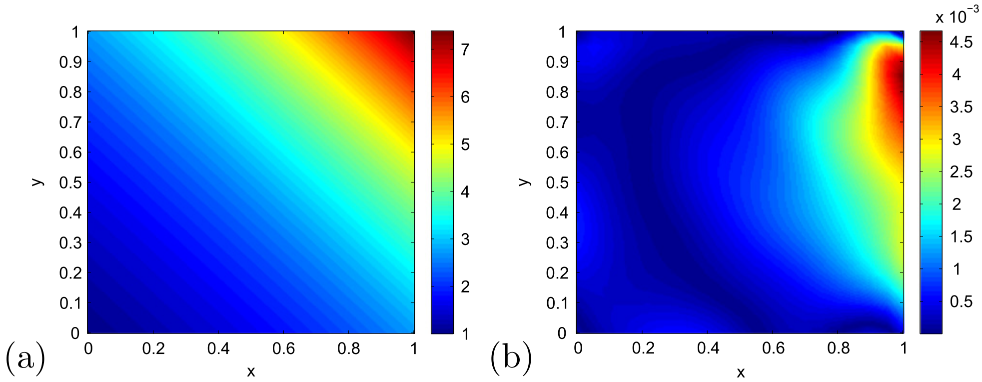

Figure 6. The numerical solutions $u(x, 1)$ (left) and the absolute errors between the numerical and the exact solutions (right) for example 2.

$E_{\mathrm{RMS}}(u(x, 1))$ and $E_{\max}(F)$ as a function of Δx, where Δx denotes the internal distance between grids. Both errors decrease as Δx decreases.

Example 2. Consider the following time-fractional diffusion system:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}(x, t) = \Delta u(x, t) + \left[\dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2t^2\right]\mathrm{e}^{x_1+x_2}, & x \in \Omega, t \in (0, 1), \\ u(x, t) = t^2 \mathrm{e}^{x_1+x_2}, & x \in \Gamma_0, t \in (0, 1), \\ u(x, t) = \mathrm{e}^{1+x_2} F(t), & x \in \Gamma_1, t \in (0, 1), \\ u(x, 0) = 0, & x \in \Omega, \end{cases}$$

where $F(t) = t^2$. In this example, we consider a non-homogeneous fractional diffusion equation for describing the exact solution, but the regularizing scheme need not be changed. For this system, the exact analytical solution is $u(x, t) = t^2 \mathrm{e}^{x_1+x_2}$.

For $\delta = 3\%$ and $\delta = 10\%$, we obtain the results shown in figures 5 and 6 by using MQ-RBF with $c = 1$. The numerical results for relative noise level

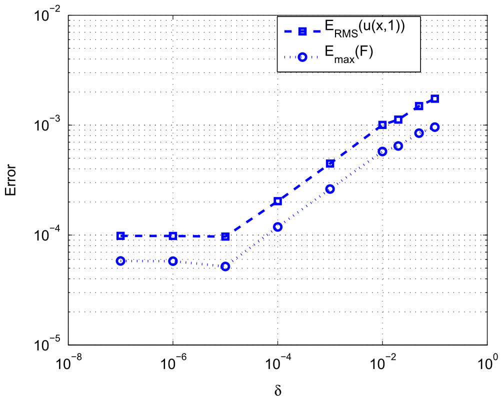

Figure 7. The numerical results obtained using various amounts of noises for example 2.

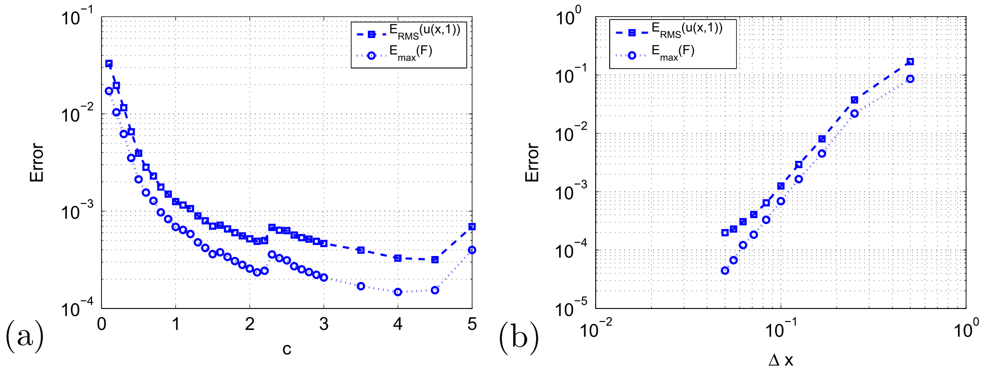

Figure 8. The errors, as a function of (a) the shape parameter c, (b) Δx with $\delta = 3\%$ for example 2.

$\delta = 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 2 \times 10^{-2}, 5 \times 10^{-2}, 10^{-1}$ are given in figure 7. It can be seen that the MQ-RBF approximation provides very accurate numerical results. Moreover, both $E_{\max}(F)$ and $E_{\mathrm{RMS}}(u(x, 1))$ decrease as the noise level δ decreases.

The errors $E_{\max}(F)$ and $E_{\mathrm{RMS}}(u(x, 1))$ with respect to c and Δx, which may affect the accuracy of the solution, are shown in figure 8. Similarly to the performance in example 1, we observe that the accuracy of the numerical results is relatively independent of the parameter c if $c > 2$. The accuracy of the numerical results is much improved as Δx decreases.

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References

[1] Adams E E and Gelhar L W 1992 Field study of dispersion in a heterogeneous aquifer: II. Spatial moments analysis *Water Resour. Res.* **28** 3293–307
[2] Bondarenko A N and Ivaschenko D S 2009 Numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient *J. Inverse Ill-Posed Probl.* **17** 419–40
[3] Boris B, Mihaly K and Meerschaert M M 2008 Numerical solutions for fractional reaction-diffusion equations *Comput. Math. Appl.* **55** 2212–26
[4] Cannon J R, Lin Y P and Matheson A 1993 The solution of the diffusion equation in two space variables subject to the specification of mass *Appl. Anal.* **50** 1–5
[5] Chen C S, Cho H A and Golberg M A 2006 Some comments on the ill-conditioning of the method of fundamental solutions *Eng. Anal. Bound. Elem.* **30** 405–10
[6] Cheng J, Nakagawa J, Yamamoto M and Yamazaki T 2009 Uniqueness in an inverse problem for one-dimensional fractional diffusion equation *Inverse Problems* **25** 115002
[7] Denisov A M 1999 *Elements of the Theory of Inverse Problems* (Utrecht: VSP)
[8] Duc T D and Tuan N H 2006 Regularization and error estimates for nonhomogeneous backward heat problems *Electron. J. Differ. Equ.* **4** 10
[9] Gorenlfo R, Luchko Y and Yamamoto M 2015 Time-fractional diffusion equation in the fractional Sobolev spaces *Fract. Calculus Appl. Anal.* **18** 799–820
[10] Gorenflo R and Vessella S 1991 *Abel Integral Equations* (*Lecture Notes in Mathematics* vol 1461) (Berlin: Springer)
[11] Gorenflo R and Yamamoto M 1999 Operator theoretic treatment of linear Abel integral equations of first kind *Japan J. Ind. Appl. Math.* **16** 137–61
[12] Guo L and Murio D A 1991 A mollified space-marching finite-difference algorithm for the two-dimensional inverse heat conduction problem with slab symmetry *Inverse Problems* **7** 247–59
[13] Hansen P C 1998 *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion* (Philadelphia: SIAM)
[14] Isakov V 1998 *Inverse Problems for Partial Differential Equations* (New York: Springer)
[15] Jonas P and Louis A K 2000 Approximate inverse for a one-dimensional inverse heat conduction problem *Inverse Problems* **16** 175–85
[16] Klafter J and Sokolov I M 2011 *First Steps in Random Walks: From Tools to Applications* (Oxford: Oxford Univesity Press)
[17] Lesnic D and Elliott L 1999 The decomposition approach to inverse heat conduction *J. Math. Anal. Appl.* **232** 82–98
[18] Lin Y M and Xu C J 2007 Finite difference/spectral approximations for the time-fractional diffusion equation *J. Comput. Phys.* **225** 1533–52
[19] Liu J J 2002 Numerical solution of forward and backward problem for 2D heat conduction problem *J. Comput. Appl. Math.* **145** 459–82
[20] Liu J J, Yamamoto M and Yan L 2015 On the uniqueness and reconstruction for an inverse problem of the fractional diffusion process *Appl. Numer. Math.* **87** 1–9
[21] Luchko Y, Rundell W, Yamamoto M and Zuo L 2013 Uniqueness and reconstruction of an unknown semilinear term in a time-fractional reaction-diffusion equation *Inverse Problems* **29** 065019 16 pp
[22] Metzler R and Klafter J 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Phys. Rep.* **339** 1–77
[23] Metzler R and Klafter J 2004 The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics *J. Phys. A: Math Gen.* **37** 161–208
[24] Metzler R and Klafter J 2000 Boundary value problems for fractional diffusion equations *Physica* A **278** 107–25
[25] Miller L and Yamamoto M 2013 Coefficient inverse problem for a fractional diffusion equation *Inverse Problems* **29** 075013 8 pp
[26] Podlubny I 1999 *Fractional Differential Equations* (San Diego: Academic)
[27] Prilepko A I, Orlovsky D G and Vasin I A 2000 *Methods for Solving Inverse Problems in Mathematical Physics* (New York: Marcel Dekker)

[28] Sakamoto K and Yamamoto M 2011 Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems *J. Math. Aanl. Appl.* **382** 426–47

[29] Sakamoto K and Yamamoto M 2011 Inverse source problem with a final overdetermination for a fractional diffusion equation *Math. Control Relat. Fields* **1** 509–18

[30] Shimakura N 1992 *Partial Differential Operators of Elliptic Type* (Providence, RI: American Mathematical Society)

[31] Suzuki A, Niibori Y, Fomin S, Chugunovd V and Hashida T 2015 Prediction of reinjection effects in fault-related subsidiary structures by using fractional derivative-based mathematical models for sustainable design of geothermal reservoirs *Geothermics* **57** 196–204

[32] Wang L Y and Liu J J 2013 Total variation regularization for backward time-fractional diffusion problem *Inverse Problems* **29** 115013

[33] Yang M and Liu J J 2013 Solving a final value fractional evolution by boundary condition regularization *Appl. Numer. Math.* **66** 45–58

[34] Ye X and Xu C 2015 A space–time spectral method for the time fractional diffusion optimal control problems *Adv. Diff. Equ.* **2015** 156

[35] Zhang P and Liu F 2006 Implicit difference approximation for the time fractional diffusion equation *J. Appl. Math. Comput.* **22** 87–99