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On the reconstruction of boundary impedance of a heat conduction system from nonlocal measurement

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Abstract

We consider the reconstruction of the Robin impedance coefficient of a heat conduction system in a two-dimensional spatial domain from the time-average measurement specified on the boundary. By applying the potential representation of a solution, this nonlinear inverse problem is transformed into an ill-posed integral system coupling the density function for potential and the unknown boundary impedance. The uniqueness as well as the conditional stability of this inverse problem is established from the integral system. Then we propose to find the boundary impedance by solving a non-convex regularizing optimization problem. The well-posedness of this optimization problem together with the convergence property of the minimizer is analyzed. Finally, based on the singularity decomposition of the potential representation of the solution, two iteration schemes with their numerical realizations are proposed to solve this optimization problem.

Keywords: inverse problem, heat equation, robin coefficient, uniqueness, optimization, convergence, numerics

(Some figures may appear in colour only in the online journal)

1. Introduction

Forward heat conduction problems aim to determine the temperature field in media for a given boundary heat status and an initial temperature distribution. Once the initial distribution of the temperature is specified, the diffusion process depends both on the media structure and

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on the boundary status of the media. Apart from the well-known Dirichlet and Neumann boundary conditions, also important is the so-called Robin boundary condition, which describes the physical phenomenon that the heat flux exchange on the boundary also depends on the boundary temperature. These problems arise in many applied areas.

Consider the following heat conduction system for the temperature $u(x, t)$ in a known bounded domain $D \subset \mathbb{R}^2$:

$$\begin{cases} \frac{\partial u}{\partial t} - a^2 \Delta u = F(x, t), & (x, t) \in D \times (0, T) \\ \frac{\partial u}{\partial \nu} + \sigma(x) u = \varphi(x, t), & (x, t) \in \partial D \times [0, T] := S_T \\ u(x, 0) = u_0(x), & x \in \overline{D}, \end{cases} \tag{1.1}$$

where $\nu(x)$ is the outward normal direction on ∂D and $a > 0$. For a given boundary impedance $\sigma(x)$, (1.1) defines a direct heat conduction system, i.e., $u(x, t)$ in $\overline{D} \times [0, T]$ is uniquely determined for a given $(\varphi(x, t), F(x, t), u_0(x))$.

When the boundary status is specified, the inverse problems corresponding to the heat conduction system have been studied thoroughly. For these configurations, there are two kinds of inverse problems: source detection problems and backward heat conduction problems. The former aims to detect the source $F(x, t)$ in the media, while the latter tries to recover the initial distribution $u_0(x)$ from extra measurement data; see [2, 5, 6, 15, 16] for related studies.

However, in many dynamic heat conduction situations, it is also necessary to recover the boundary status from additional information. This is the so-called inverse side way heat conduction problem, which aims to detect ∂D and (or) the impedance defined on ∂D. Unlike the source detections and backward conduction problems, these side way problems are nonlinear. For the above system (1.1) with known ∂D, we try to identify the boundary impedance $\sigma(x)$ from the following nonlocal inversion input data

$$\int_0^T \omega(t) u(x, t) \mathrm{d}t = f(x), \quad x \in \partial D, \tag{1.2}$$

where $0 \not\equiv \omega(t) \in L^1(0, T)$ is a given weight function.

Due to the importance of the averaging measurement such as (1.2) in many areas of engineering, the inverse problems related to (1.1) and (1.2) were proposed and first studied by Kostin and Prilepko in [12, 13]. In [12], the authors considered two inverse problems using the inversion input data (1.2), which aimed to identify the spatial amplitude of $\varphi(x, t)$ for a known $\sigma(x)$ and to identify $\sigma(x)$ for a known $\varphi(x, t)$. In [13], the other two inverse problems aiming to reconstruct the time amplitude of $\varphi(x, t)$ and $\sigma(t)$, which are considered with (1.2), are replaced by the spatial average on the boundary. For these four inverse problems, the uniqueness results are established. As for the reconstruction schemes and the numerical realizations for the models in [12, 13], readers are referred to [9, 10], where the least-squares-penalized variational formulations are proposed to determine the spatial (time) dependent boundary impedance and the boundary sources. These optimizing problems are solved by the conjugate gradient method, where two direct boundary value problems at each iteration are solved by boundary element methods.

The engineering motivation on inversion data in the integral form comes from the averaging process for instant measurement. More precisely, although the measurement $u(\cdot, T_1)$ at the specified time T_1 is standard in engineering, such a kind of measurement may be very sensitive to the perturbation of time and a natural way to overcoming this drawback is to take an average with respect to time t near T_1, which leads to (1.2) with some weight function $\omega(t)$ of support in the neighborhood of $T_1 \in (0, T)$. These averaging data are possible to obtain in engineering applications [1]. On the other hand, (1.2) can be considered as the generalization of instant measurement data $u(x, T_1) = f(x)$ by taking $\omega(t) = \delta(t - T_1)$. Of course, to ensure $\omega(t) \in L^1(0, T)$, $\delta(t - T_1)$ should be replaced mathematically by a Gaussian weight or some cut-off weight directly, see [9, 10].

Essentially, the problems (1.1) and (1.2) belong to the category of parameter identification problems for a specified system. In [19], the authors proposed an algorithm for recovering the heat flux. When both the geometric shapes and the boundary impedance are unknown, the simultaneous reconstructions of ∂D and $\sigma(x)$ from the spectrum data in the two-dimensional spatial heat conduction system were considered in [22]. In the case of $\sigma(x) \equiv 0$, the reconstruction of the partial boundary from the Cauchy data in the accessible part of the boundary is based on the optimization and the boundary integral equation method given in [4], while the reconstruction of the partial boundary for the Dirichlet boundary condition is also established in [3]. In the case of the Stefan–Boltzmann boundary condition for heat conduction, the determination of the impedance coefficient in part of the boundary can be found in [7]. On the other hand, it is well-known that the properties of the inverse problems are closely related to the form of the inversion input. Unlike the standard form of the inversion input where the information about the solution is specified point-wise, the inversion input (1.2) is the so-called nonlocal condition in the integral form. Currently, the inverse problems related to the inversion input in the integral form are applied for different configurations, for example, see [8, 11, 17, 21].

In this paper, we are interested in the following issues of the inverse problems for the systems (1.1) and (1.2) with $F(x, t) \equiv 0$ and $u_0(x) \equiv 0$:

- For the given exact data $f(x)$, can $\sigma(x)$ be reconstructed uniquely?
- If $f(x)$ is contaminated by some noise satisfying

$$\|f^\delta - f\|_{L^2(\partial D)} \leqslant \delta, \tag{1.3}$$

  how do we construct the approximate solution to $\sigma(x)$ using a suitable regularization scheme from $f^\delta(x)$?
- What is the approximation behavior of the regularizing solution?
- How do we find the approximate minimizer efficiently?

Since the inverse problem considered here is essentially the same as *Problem II* in [12] and *Problem I* in [9], we would like to clarify the differences between our work and [9, 12]. In [12], the uniqueness (but not the stability) for *Problem II* is created by first establishing the uniqueness for $f(x)$ in the boundary source $\varphi(x, t) = h(x, t)f(x) + b(x, t)$ with known $(h(x, t), b(x, t))$ and boundary impedance (*Problem I* there) which is a linear inverse problem, and then using the difference arguments and the maximum principles by similar arguments in [18]. As for the reconstruction schemes, [9] considered a regularizing nonlinear cost functional with the unknown $\sigma(x)$ as the sole argument, which is optimized by conjugate gradient schemes. In our researches for (1.1) and (1.2), the inverse problems are transformed into the reconstructions of $(q(x, t), \sigma(x))$, where $q(x, t)$ is the introduced potential density for the direct problems. Such techniques enable us to deal with the uniqueness and the reconstruction scheme for (1.1) and (1.2) in a unified framework. Moreover, in solving our cost functionals with (q, σ) as arguments, the proposed alternative iteration scheme (AIS) solves the standard quadratic optimizing problem by decomposing the nonlinear term $\sigma(x)u(x, t)$ on the boundary at each iteration, which is numerically shown to be efficient, i.e, the alternative iteration scheme solves a linear equation at each iteration step.

2. Uniqueness and stability

The function

$$G(x, t; y, \tau) := \frac{1}{4\pi a^2(t-\tau)} \exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right), \quad t > \tau \tag{2.1}$$

is called the fundamental solution to the two-dimensional equation $u_t - a^2\Delta u = 0$ in $\mathbb{R}^2$. The following representation of the solution to (1.1) with $F(x, t) \equiv 0$ and $u_0(x) \equiv 0$ can be established from the classical potential theory, see [14] for the Dirichlet boundary condition and [4] for the Neumann boundary case for $a = 1$.

Lemma 2.1. *Assume that the potential $q(x, t) \in C(S_T)$ satisfies*

$$\frac{1}{2a^2}q(x, t) + \int_0^t \int_{\partial D} \frac{\partial G(x, t; y, \tau)}{\partial \nu(x)} q(y, \tau)\mathrm{d}s(y)\mathrm{d}\tau$$
$$+ \sigma(x)\int_0^t \int_{\partial D} G(x, t; y, \tau)q(y, \tau)\mathrm{d}s(y)\mathrm{d}\tau = \varphi(x, t), \quad (x, t) \in S_T, \tag{2.2}$$

then the single layer potential

$$u(x, t) = \int_0^t \int_{\partial D} G(x, t; y, \tau)q(y, \tau)\mathrm{d}s(y)\mathrm{d}\tau, \quad (x, t) \in \overline{D} \times [0, T] \tag{2.3}$$

solves (1.1) provided that $F(x, t) \equiv 0$ and $u_0(x) \equiv 0$.

Based on this representation, the original inverse problems (1.1) and (1.2) become the determination of $\sigma(x)$ from the following nonlinear integral system

$$\begin{cases} \int_0^T \omega(t)\mathbb{K}[q](x, t)dt = f(x), & x \in \partial D, \\ \frac{1}{2a^2}q(x, t) + \mathbb{K}'[q](x, t) + \sigma(x)\mathbb{K}[q](x, t) = \varphi(x, t), & (x, t) \in S_T \end{cases} \tag{2.4}$$

with respect to $(q(x, t), \sigma(x))$, where the linear operators

$$\begin{cases} \mathbb{K}[q](x, t) := \int_0^t \int_{\partial D} G(x, t; y, \tau)q(y, \tau)\mathrm{d}s(y)\mathrm{d}\tau, & (x, t) \in \overline{D} \times [0, T], \\ \mathbb{K}'[q](x, t) := \int_0^t \int_{\partial D} \frac{\partial G(x, t; y, \tau)}{\partial \nu(x)} q(y, \tau)\mathrm{d}s(y)\mathrm{d}\tau, & (x, t) \in \overline{D} \times [0, T]. \end{cases} \tag{2.5}$$

It should be pointed out that (2.4) is analogous to the ill-posed linear system with respect to the unknown initial temperature and density function in [6] where the backward heat conduction problem is considered. However, the system (2.4) here is nonlinear, and therefore the regularizing strategy will lead to a non-quadratic cost functional, which is much more difficult to solve.

For exact data $f(x)$ coming from (1.1) and (1.2), the existence of the solution to (2.4) is obvious. Let us check the uniqueness of the solution for exact $f(x)$. Multiplying the second equation in (2.4) by $\omega(t)$ and integrating with respect to $t \in [0, T]$, we have

$$\int_0^T \omega(t)\left(\frac{1}{2a^2}q(x, t) + \mathbb{K}'[q](x, t)\right)\mathrm{d}t + \sigma(x)f(x) = \int_0^T \omega(t)\varphi(x, t)\mathrm{d}t \tag{2.6}$$

from the first equation in (2.4). Now we can delete $\sigma(x)$ from the second equation in (2.4) and (2.6) to yield for $(x, t) \in S_T$ that

$$\left(\frac{1}{2a^2}q(x, t) + \mathbb{K}'[q](x, t) - \varphi(x, t)\right) f(x)$$
$$= \mathbb{K}[q](x, t)\int_0^T \omega(t)\left(\frac{1}{2a^2}q(x, t) + \mathbb{K}'[q](x, t) - \varphi(x, t)\right)\mathrm{d}t, \tag{2.7}$$

which is a nonlinear integral equation with respect to $q(x, t)$ representing the nonlinearity of our inverse problem. Since the inversion input f depends essentially on the boundary impedance σ for the given boundary excitation $\varphi(x, t)$ and weight function $\omega(t)$, we denote the inversion data by $f = f[\sigma]$. We introduce the admissible set for σ with two known positive constants σ_+, f_0:

$$\Sigma := \{\sigma(x) \in C(\partial D),\ 0 \leqslant \sigma(x) \leqslant \sigma_+,\ \ |f[\sigma](x)| > f_0 > 0,\ x \in \partial D\}.$$

The set Σ is well-defined. For example, if we take $0 \not\equiv \varphi(x, t) \geqslant 0$ and $0 \not\equiv \omega(t) \geqslant 0$, then the strong maximum principle for the heat equation yields $u(x, t)|_{\partial D\times(0,T]} > 0$, so $f[\sigma](x) = \int_0^T \omega(t)u[\sigma](x, t)dt \geqslant C_\sigma > 0$ in ∂D.

First, we give the property for the operators $\mathbb{K}$ and $\mathbb{K}'$.

Lemma 2.2. *For the domain D with a C^2 smooth boundary, both $\mathbb{K}$ and $\mathbb{K}'$ defined by (2.5) are compact from $C(S_T)$ into $C(S_T)$ as well as from $L^2(S_T)$ into $L^2(S_T)$. Moreover, it follows that*

$$\mathcal{N}\left(\frac{1}{2a^2}I + \mathbb{K}'\right) = \{0\}, \quad \mathcal{N}\left(\frac{1}{2a^2}I + \mathbb{K}' + \sigma\,\mathbb{K}\right) = \{0\} \quad \text{for all } \sigma \in \Sigma.$$

Proof. Owing to the inequality $s^\beta \mathrm{e}^{-s} \leqslant \beta^\beta \mathrm{e}^{-\beta}$ for all $0 < s, \beta < +\infty$, we have for $t > \tau$ and $x \neq y$ by taking $s = \frac{|x-y|^2}{4a^2(t-\tau)}$ that

$$|G(x, t; y, \tau)| \leqslant \frac{1}{4a^2\pi(t-\tau)}\left(\frac{4a^2(t-\tau)}{|x-y|^2}\right)^\beta \beta^\beta \mathrm{e}^{-\beta} \leqslant \frac{C(\beta)}{(t-\tau)^{1-\beta}|x-y|^{2\beta}}, \tag{2.8}$$

$$\begin{aligned}\left|\frac{\partial}{\partial\nu(x)}G(x, t; y, \tau)\right| &= \frac{1}{4\pi a^2(t-\tau)}\frac{|\langle\nu(x), x-y\rangle|}{2a^2(t-\tau)}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)\\ &\leqslant \frac{1}{4\pi a^2(t-\tau)}\frac{L\,|x-y|^2}{2a^2(t-\tau)}\left(\frac{4a^2(t-\tau)}{|x-y|^2}\right)^\beta \beta^\beta\mathrm{e}^{-\beta}\\ &\leqslant \frac{C(\beta)L}{(t-\tau)^{2-\beta}|x-y|^{2\beta-2}},\end{aligned} \tag{2.9}$$

which verify the weak singularity of the kernel functions in $\partial D \times [0, t]$ by taking any $\beta \in (0, 1/2)$ and $\beta \in (1, 3/2)$ in these two estimates, respectively. So $\mathbb{K}$ and $\mathbb{K}'$ are compact from $C(S_T)(L^2(S_T))$ into $C(S_T)(L^2(S_T))$.

Especially, by taking $\beta = 1/3$ in (2.8) and $\beta = 4/3$ in (2.9), we have

$$\begin{aligned}
&\|\mathbb{K}'[q](\cdot, t)\|_{C(\partial D)},\ \|\sigma(\cdot)\mathbb{K}[q](\cdot, t)\|_{C(\partial D)} \\
&\quad \leqslant C(1+\sigma_+)\int_0^t \frac{1}{(t-\tau)^{2/3}}\ \|q(\cdot, \tau)\|_{C(\partial D)} \max_{x\in\partial D}\int_{\partial D}\frac{1}{|x-y|^{2/3}}\mathrm{d}s(y)\ \mathrm{d}\tau \\
&\quad \leqslant C\int_0^t \frac{1}{(t-\tau)^{2/3}}\ \|q(\cdot, \tau)\|_{C(\partial D)}\,\mathrm{d}\tau.
\end{aligned} \tag{2.10}$$

Consider the equation

$$\frac{1}{2a^2}q(x, t) + \mathbb{H}_\sigma[q](x, t) = 0, \quad (x, t) \in S_T \tag{2.11}$$

with the linear operator $\mathbb{H}_\sigma[q](x, t) \coloneqq \mathbb{K}'[q](x, t) + \sigma(x)\mathbb{K}[q](x, t)$. Then it follows from (2.10) that

$$\|\mathbb{H}_\sigma[q](\cdot, t)\|_{C(\partial D)} \leqslant C(\sigma_+)\int_0^t \frac{1}{(t-\tau)^{2/3}}\ \|q(\cdot, \tau)\|_{C(\partial D)}\,d\tau, \quad t \in [0, T]$$

uniformly for any $\sigma \in \Sigma$. Based on this estimate, by using the same technique in the proof of theorem 9.9 in [14], we state that (2.11) has only a trivial solution $q(x, t) \equiv 0$ for any $\sigma \in \Sigma$, i.e., $\mathcal{N}\left(\frac{1}{2a^2}I + \mathbb{K}' + \sigma\ \mathbb{K}\right) = \{0\}$. On the other hand, noticing that the above arguments are also true for $\sigma(x) \equiv 0$, it is obvious that $\mathcal{N}\left(\frac{1}{2a^2}I + \mathbb{K}'\right) = \{0\}$. The proof is complete. □

Now we can state the uniqueness result for our inverse problem.

Theorem 2.3. *If the input excitation $\varphi(x, t)$ satisfies*

$$\frac{\|\omega\|_{L^1(0,T)}}{f_0}\ \|\varphi\|_{C(S_T)} \ll 1, \tag{2.12}$$

then $\sigma(x) \in \Sigma$ can be uniquely determined.

Proof. For a specified boundary value $\varphi(x, t) \in C(S_T)$, let $f_i(x)$ be the measurement data corresponding to $\sigma_i(x) \in \Sigma$ for $i = 1, 2$, i.e., the corresponding solution $u_i(x, t) \coloneqq u[\sigma_i](x, t)$ to (1.1) with $\sigma_i(x) \in \Sigma$ meets

$$\int_0^T \omega(t)u_i(x, t)dt = f_i(x), \quad x \in \partial D \tag{2.13}$$

for $i = 1, 2$. We need to prove $\sigma_1(x) = \sigma_2(x)$ if $f_1(x) \equiv f_2(x)$.

In the sequel of the proof, C_0 may represent different constants independent of $(\sigma_+, \omega(t), \varphi(x, t))$. From the second equation in (2.4), we have

$$\frac{1}{2a^2}q_i(x, t) + \mathbb{K}'[q_i](x, t) + \sigma_i(x)\mathbb{K}[q_i](x, t) = \varphi(x, t), \quad (x, t) \in S_T. \tag{2.14}$$

The third Fredholm alternative theorem says that $\frac{1}{2a^2}I + \mathbb{K}' + \sigma\ \mathbb{K}$ is invertible in $C(S_T)$ from lemma 2.2, satisfying $\left\|\ \left(\frac{1}{2a^2}I + \mathbb{K}' + \sigma\ \mathbb{K}\right)^{-1}\ \right\| \leqslant C(\sigma_+)$ for all $\sigma(x) \in \Sigma$. So we have $\|q_i\|_{C(S_T)} \leqslant C(\sigma_+)\|\varphi\|_{C(S_T)}$ which generates

$$\|\mathbb{K}'[q_i]\|_{C(S_T)},\ \|\mathbb{K}[q_i]\|_{C(S_T)} \leqslant C_0C(\sigma_+)\|\varphi\|_{C(S_T)} \tag{2.15}$$

due to (2.10) for $\sigma_i \in \Sigma$. On the other hand, it follows from (2.7) that

$$\begin{aligned}
&\frac{1}{2a^2}(q_1 - q_2)(x, t) + \mathbb{K}'[q_1 - q_2](x, t) \\
&= \frac{1}{f_1(x)}\mathbb{K}[q_1 - q_2](x, t)Q[q_2, \omega, \varphi](x) + \frac{1}{f_1(x)}\mathbb{K}[q_1](x, t)Q[q_1 - q_2, \omega, 0](x) \\
&- \frac{1}{f_1(x)f_2(x)}\mathbb{K}[q_2](x, t)Q[q_2, \omega, \varphi](x)(f_1(x) - f_2(x)),
\end{aligned} \tag{2.16}$$

where the functional

$$Q[q, \omega, \varphi](x) := \int_0^T \omega(\tau)\left(\frac{1}{2a^2}q(x, \tau) + \mathbb{K}'[q](x, \tau) - \varphi(x, \tau)\right)d\tau.$$

The left-hand side of (2.16) is the operator $\frac{1}{2a^2}I + \mathbb{K}'$ operating on the argument $q_1 - q_2$, which is independent of $(\sigma_i, \varphi, \omega)$. By applying lemma 2.2 and the Fredholm alternative theorem, we get $\left\| \left(\frac{1}{2a^2}I + \mathbb{K}\right)^{-1} \right\| \leqslant C_0$ and therefore

$$\begin{aligned}
\|q_1 - q_2\|_{C(S_T)} \leqslant &\frac{C_0}{f_0} \|\mathbb{K}[q_1 - q_2]\|_{C(S_T)} \|Q[q_2, \omega, \varphi]\|_{C(\partial D)} \\
&+ \frac{C_0}{f_0} \|\mathbb{K}[q_1]\|_{C(S_T)} \|Q[q_1 - q_2, \omega, 0]\|_{C(\partial D)} \\
&+ \frac{C_0}{f_0^2} \|\mathbb{K}[q_2]\|_{C(S_T)} \|Q[q_2, \omega, \varphi]\|_{C(\partial D)} \|f_1 - f_2\|_{C(\partial D)}.
\end{aligned} \tag{2.17}$$

It follows from the representation of $Q[q, \omega, \varphi]$ and the estimates (2.15) that

$$\begin{cases}
\|Q[q_2, \omega, \varphi]\|_{C(\partial D)} \leqslant C(\sigma_+)\|\omega\|_{L^1(0,T)} \|\varphi\|_{C(S_T)}, \\
\|Q[q_1 - q_2, \omega, 0]\|_{C(\partial D)} \leqslant C_0 \|\omega\|_{L^1(0,T)} \|q_1 - q_2\|_{C(S_T)}, \\
\|\mathbb{K}[q_1 - q_2]\|_{C(S_T)} \leqslant C_0 \|q_1 - q_2\|_{C(S_T)}.
\end{cases}$$

Therefore (2.17) leads to

$$\begin{aligned}
\|q_1 - q_2\|_{C(S_T)} \leqslant &C(\sigma_+)\frac{1}{f_0} \|q_1 - q_2\|_{C(S_T)} \|\omega\|_{L^1(0,T)} \|\varphi\|_{C(S_T)} \\
&+ \frac{1}{f_0^2}C(\sigma_+)\|\omega\|_{L^1(0,T)} \|\varphi\|^2_{C(S_T)} \|f_1 - f_2\|_{C(\partial D)},
\end{aligned}$$

which generates

$$\begin{aligned}
&\left(1 - C(\sigma_+)\frac{1}{f_0} \|\omega\|_{L^1(0,T)} \|\varphi\|_{C(S_T)}\right) \|q_1 - q_2\|_{C(S_T)} \\
&\leqslant C(\sigma_+)\frac{1}{f_0^2} \|\varphi\|^2_{S_T} \|\omega\|_{L^1(0,T)} \|f_1 - f_2\|_{C(\partial D)}.
\end{aligned} \tag{2.18}$$

That is, under the condition (2.12), it follows that $q_1(x, t) \equiv q_2(x, t)$ in $C(S_T)$ if $f_1(x) \equiv f_2(x)$ in $C(\partial D)$. Finally the equation (2.6) yields $\sigma_1(x) \equiv \sigma_2(x)$ from $q_1 \equiv q_2$, noticing that we have $|f_1(x)| \equiv |f_2(x)| \not\equiv 0$. The proof is complete. □

We also have the Lipschitz conditional stability.

Theorem 2.4. *Let $f_i(x)$ be the exact measurement data from the systems (1.1) and (1.2) with $F(x, t) \equiv u_0(x) \equiv 0$ corresponding to $\sigma_i(x) \in \Sigma$ for $i = 1, 2$. If (2.12) holds, then we have the Lipschitz stability*

$$\|\sigma_1 - \sigma_2\|_{C(\partial D)} \leqslant C\, \|f_1 - f_2\|_{C(\partial D)}. \tag{2.19}$$

Proof. Using (2.14) and (2.6) for $(\sigma_i(x), f_i(x), q_i(x, t))$, we have

$$\begin{aligned} &\sigma_1(x) - \sigma_2(x) \\ &= \frac{1}{f_1(x) f_2(x)}\left((f_2(x) - f_1(x))\left[\int_0^T \omega\varphi\, \mathrm{d}t - \int_0^T \omega\left(\frac{q_1}{2a^2} + \mathbb{K}'[q_1]\right)\mathrm{d}t\right]\right. \\ &\quad \left. - \frac{1}{f_2(x)}\int_0^T \omega\left[\frac{1}{2a^2}(q_1 - q_2) + \mathbb{K}'[q_1 - q_2]\right]\mathrm{d}t\right) \end{aligned} \tag{2.20}$$

owing to $|f_i(x)| > f_0 > 0$. Using $\|q_i\|_{C(S_T)} \leqslant C(\sigma_+)\|\varphi\|_{C(S_T)}$ and (2.15) for $\sigma_i \in \Sigma$, we have

$$\begin{aligned} |\sigma_1(x) - \sigma_2(x)| \leqslant &\frac{|f_2(x) - f_1(x)|}{f_0^2}\, \|\omega\|_{L^1(0,T)}\, C(\sigma_+)\|\varphi\|_{C(S_T)} \\ &+ C(\sigma_+)\frac{1}{f_0}\, \|\omega\|_{L^1(0,T)}\, \|\varphi\|_{C(S_T)}\, \|q_1 - q_2\|_{C(S_T)}. \end{aligned}$$

Now combining with (2.18), the proof is complete under condition (2.12). □

Remark 2.5. It is worthwhile to compare our theorem 2.3 with the uniqueness result given in [12] under the condition $f(x) > 0$, but no stability result is obtained there due to the lack of the uniform positive lower bound on f with respect to σ. By our theorems 2.3 and 2.4, both the uniqueness and the Lipschitz stability are set up by the much stronger assumption that $\sigma \in \Sigma$ which means $|f[\sigma](x)| \geqslant f_0 > 0$ and (2.12), i.e., we establish the conditional stability for the ill-posed problems (1.1) and (1.2). However, it should be emphasized that we consider the special case $(F(x, t), u_0(x)) \equiv (0, 0)$, and our theorems 2.3 and 2.4 need assumption (2.12), as compared with the works [12, 13].

3. Optimization formulation and convergence

For noisy data $f^\delta(x)$ satisfying (1.3), we consider the cost functional

$$\begin{aligned} J_\alpha(q, \sigma) := &\left\| \int_0^T \omega(t)\mathbb{K}[q](\cdot, t)\mathrm{d}t - f^\delta(\cdot)\right\|^2_{L^2(\partial D)} \\ &+ \left\| \frac{1}{2a^2} q + \mathbb{K}'[q] + \sigma\, \mathbb{K}[q] - \varphi \right\|^2_{L^2(S_T)} + \alpha\, \|q\|^2_{L^2(S_T)} \end{aligned} \tag{3.1}$$

with the regularizing parameter $\alpha > 0$. This cost functional regularizing only the argument q is motivated by the structure of our ill-posed integral system (2.4). More precisely, the ill-posedness of the system (2.4) comes essentially from the determination of $q(x, t)$ from the first equation, although the determinations of $\sigma(x)$ from the second equation at some special points where $\mathbb{K}[q](x, t)$ almost vanishes are also unstable. Therefore we introduce the so-called semi-Tikhonove regularizing functional (3.1) for the system (2.4). This kind of technique has also been applied for other ill-posed problems, see [6].

We will find the minimizer $(q^\delta(x, t), \sigma^\delta(x))$ of $J_\alpha(q, \sigma)$ with a specified small α. Then $\sigma^\delta(x)$ will be taken as the approximate solution to our inverse problem. The meaning 'approximation' will be clarified in the following.

Our first result in this section is the following

Theorem 3.1. *Let $V(\partial D)$ be the compact set of $L^2(\partial D)$. Then the optimization version*

$$\inf\{J_\alpha(q, \sigma) : q \in L^2(S_T), \sigma \in V(\partial D)\} \tag{3.2}$$

of the original inverse problems (1.1)–(1.3) have a solution $\sigma^{\alpha,\delta}(x) \in L^2(\partial D)$ for given $\alpha, \delta > 0$.

Proof. Noticing $J_\alpha(q, \sigma) \geqslant 0$, define

$$\inf\{J_\alpha(q, \sigma) : q \in L^2(S_T), \sigma \in V(\partial D)\} := M_0 \geqslant 0,$$

where the constant $M_0 = M_0(\alpha, f^\delta)$. Let $(q_n, \sigma_n) \in L^2(S_T) \times V(\partial D)$ be the minimizing sequence, i.e., $\lim_{n\to\infty} J_\alpha(q_n, \sigma_n) = M_0$, which generates

$$\alpha \|q_n\|^2_{L^2(S_T)} \leqslant J_\alpha(q_n, \sigma_n) \to M_0 \text{ as } n \to \infty.$$

Therefore there exists a subsequence of $\{q_n : n \in \mathbb{N}\}$ converging weakly to some $q^* \in L^2(S_T)$. For simplicity of the notation, we still denote this subsequence as $\{q_n : n \in \mathbb{N}\}$. Noticing that both $\mathbb{K}$ and $\mathbb{K}'$ are the compact forms of $L^2(S_T)$ into $L^2(S_T)$, we have [14]

$$\mathbb{K}[q_n] \to \mathbb{K}[q^*], \quad \mathbb{K}'[q_n] \to \mathbb{K}'[q^*] \quad \text{in} L^2(S_T). \tag{3.3}$$

On the other hand, since $\{\sigma_n : n \in \mathbb{N}\} \subset V(\partial D)$ is compact in $L^2(\partial D)$, there exists some subsequence of $\{\sigma_n : n \in \mathbb{N}\}$ (still denoted by $\{\sigma_n : n \in \mathbb{N}\}$) such that $\sigma_n \to \sigma^* \in V(\partial D)$ in $L^2(\partial D)$. Using the above convergence results and $q_n \rightharpoonup q^*$, it follows from the identity

$$\begin{aligned}\alpha \|q_n\|^2_{L^2(S_T)} \equiv J_\alpha(q_n, \sigma_n) &- \left\| \int_0^T \omega(t)\mathbb{K}[q_n](\cdot, t)\mathrm{d}t - f^\delta(\cdot) \right\|^2_{L^2(\partial D)} \\ &- \left\| \frac{q^*}{2a^2} + \mathbb{K}'[q_n] + \sigma_n \mathbb{K}[q_n] - \varphi \right\|^2_{L^2(S_T)} - \left\| \frac{q_n - q^*}{2a^2} \right\|^2_{L^2(S_T)} \\ &- \frac{1}{a^2}\langle q_n - q^*, \frac{1}{2a^2}q^* + \mathbb{K}'[q_n] + \sigma_n \mathbb{K}[q_n] - \varphi\rangle\end{aligned}$$

that

$$\begin{aligned}\lim_{n\to\infty} \alpha \|q_n\|^2 &\equiv M_0 - \left\| \int_0^T \omega(t)\mathbb{K}[q^*](\cdot, t)\mathrm{d}t - f^\delta(\cdot) \right\|^2 \\ &\quad - \left\| \frac{1}{2a^2}q^* + \mathbb{K}'[q^*] + \sigma^* \mathbb{K}[q^*] - \varphi \right\|^2 - \lim_{n\to\infty} \frac{1}{4a^4} \|q_n - q^*\|^2 \\ &\leqslant J_\alpha(q^*, \sigma^*) - \left\| \int_0^T \omega(t)\mathbb{K}[q^*](\cdot, t)\mathrm{d}t - f^\delta(\cdot) \right\|^2 \\ &\quad - \left\| \frac{1}{2a^2}q^* + \mathbb{K}'[q^*] + \sigma^* \mathbb{K}[q^*] - \varphi \right\|^2 - \lim_{n\to\infty} \frac{1}{4a^4} \|q_n - q^*\|^2 \\ &= \alpha \|q^*\|^2 - \lim_{n\to\infty} \frac{1}{4a^4} \|q_n - q^*\|^2,\end{aligned}$$

which leads to $q_n \rightharpoonup q^*$ such that

$$\lim_{n\to\infty} \|q_n - q^*\|^2 = \lim_{n\to\infty} [\|q_n\|^2 - \|q^*\|^2] \leqslant -\lim_{n\to\infty} \frac{1}{4\alpha a^4} \|q_n - q^*\|^2,$$

i.e., $q_n \to q^*$ in $L^2(S_T)$. Finally the continuity of J_α in $L^2(S_T) \times L^2(\partial D)$ yields

$$J_\alpha(q^*, \sigma^*) = \lim_{n\to\infty} J_\alpha(q^n, \sigma^n) = M_0,$$

i.e., (q^*, σ^*) is the minimizer of $J_\alpha(q, \sigma)$ in $L^2(S_T) \times V(\partial D)$. Therefore we can take $\sigma^*(x)$ as $\sigma^{\alpha,\delta}(x)$, the approximate solution of the inverse problems (1.1)–(1.3) for noisy input data $f^\delta(x)$. The proof is complete. □

The introduction of the regularizing term with $\alpha > 0$ ensures the existence of the minimizer. Consequently, the value $M_0(\alpha, f^\delta)$ is also different from $M_0(0, f) = 0$ provided that the exact solution be also in $V(\partial D)$. So we need to estimate $M_0(\alpha, f^\delta)$, which reveals the difference between our approximate solution from the optimization formulation and the exact solutions of (1.1)–(1.3) by data matching.

Theorem 3.2. *Assume that the exact solution $\overline{\sigma}(x)$ to the inverse problems (1.1) and (1.2) is also in $V(\partial D)$ and $(q^{\alpha,\delta}, \sigma^{\alpha,\delta})$ is the minimizer of the cost functional $J_\alpha(q, \sigma)$ from theorem 3.1, i.e., $M_0(\alpha, f^\delta) = J_\alpha(q^{\alpha,\delta}, \sigma^{\alpha,\delta})$. Then we have*

$$\lim_{\alpha\to 0} M_0(\alpha, f^\delta) \leqslant 2\delta^2. \tag{3.4}$$

Proof. Denote by $f(x)$ the exact nonlocal measurement data corresponding to exact impedance $\overline{\sigma}(x) \in V(\partial D)$ and the boundary value $\varphi(x, t)$. It is obvious for all $(q, \sigma) \in L^2(S_T) \times V(\partial D)$ that

$$\begin{aligned} J_\alpha(q, \sigma) \leqslant\, & 2 \left\| \int_0^T \omega(t)\mathbb{K}[q](\cdot, t)\mathrm{d}t - f(\cdot) \right\|^2_{L^2(\partial D)} \\ & + 2 \left\| \frac{1}{2a^2}q + \mathbb{K}'[q] + \overline{\sigma}\,\mathbb{K}[q] - \varphi \right\|^2_{L^2(S_T)} + \alpha\, \|q\|^2_{L^2(S_T)} \\ & + 2\, \|f^\delta - f\|^2_{L^2(\partial D)} + 2\, \|(\sigma - \overline{\sigma})\mathbb{K}[q]\|^2_{L^2(S_T)}. \end{aligned} \tag{3.5}$$

Since $\mathcal{N}\left(\frac{1}{2a^2}\mathbb{I} + \mathbb{K}' + \overline{\sigma}\mathbb{K}\right) = \{0\}$ in terms of lemma 2.2, it follows from the Fredholm alternative principle that there exists a unique potential function $\overline{q}(x, t) \in L^2(\partial D \times (0, T))$ for given $\varphi(x, t) \in L^2(S_T)$ satisfying

$$\frac{1}{2a^2}\overline{q} + \mathbb{K}'[\overline{q}] + \overline{\sigma}\,\mathbb{K}[\overline{q}] - \varphi = 0, \quad (x, t) \in S_T.$$

Moreover, $u(x, t) := \mathbb{K}[\overline{q}](x, t)$ in $\overline{D} \times [0, T]$ solves the direct problem (1.1) with Robin coefficient $\overline{\sigma}$ from the potential theory and therefore we have

$$\int_0^T \omega(t)\mathbb{K}[\overline{q}](x, t)\mathrm{d}t = f(x), \quad x \in \partial D.$$

The above two equalities generate

$$J_\alpha(\overline{q}, \overline{\sigma}) \leqslant \alpha \, \|\overline{q}\|^2_{L^2(S_T)} + 2 \, \|f^\delta - f\|^2_{L^2(\partial D)} \leqslant \alpha \, \|\overline{q}\|^2_{L^2(S_T)} + 2\delta^2.$$

Therefore we have from (3.5) that

$$\begin{aligned} M_0(\alpha, f^\delta) &= \inf\{J_\alpha(q, \sigma) : q \in L^2(S_T), \, \sigma \in V(\partial D)\} \\ &\leqslant J_\alpha(\overline{q}, \overline{\sigma}) \leqslant \alpha \, \|\overline{q}\|^2_{L^2(S_T)} + 2\delta^2. \end{aligned}$$

By letting $\alpha \to 0$, the proof is complete. □

Based on theorems 3.1 and 3.2, we know that for any sequence $\{\alpha_n \to 0 : \; n \to \infty\}$, there exists a minimizer sequence $\{(q^n, \sigma^n) : n \in \mathbb{N}\}$ (which may not be unique) such that

$$\lim_{n\to\infty} J_{\alpha_n}(q^n, \sigma^n) = \lim_{n\to\infty} M_0(\alpha_n, f^\delta) \leqslant 2\delta^2. \tag{3.6}$$

Now we establish the relations between σ^n and the exact boundary impedance $\overline{\sigma}$ from which the exact input data $f(x)$ are generated.

Theorem 3.3. *Let $\{\alpha_n : n \in \mathbb{N}\}$ be the positive sequence converging to 0 and (q^n, σ^n) be the correspondent minimizer of $J_{\alpha_n}(q, \sigma)$, which may not be unique. Then there exists a convergent subsequence $\{\sigma^{n_k} : k \in \mathbb{N}\} \subset \{\sigma^n : n \in \mathbb{N}\}$ converging to σ^* in $L^2(\partial D)$ as $k \to \infty$, where σ^* satisfies*

$$\begin{cases} \left\| \int_0^T \omega(t)\mathbb{K}[\overline{q}](\cdot, t)\mathrm{d}t - f(\cdot) \right\|_{L^2(\partial D)} \leqslant 3\delta, \\ \left\| \frac{1}{2a^2}\overline{q} + \mathbb{K}'[\overline{q}] + \sigma^* \, \mathbb{K}[\overline{q}] - \varphi \right\|_{L^2(\partial D)} \leqslant 2\delta \end{cases} \tag{3.7}$$

for some $\overline{q}(x, t) \in L^2(\partial D \times (0, T))$.

Proof. Since $\{\sigma^n : n \in \mathbb{N}\} \subset V(\partial D)$ which is compact, there exists a convergent subsequence $\{\sigma^{n_k} : k \in \mathbb{N}\} \subset \{\sigma^n : n \in \mathbb{N}\}$ such that

$$\sigma^{n_k} \to \sigma^*, \quad k \to \infty. \tag{3.8}$$

For the corresponding density function sequence $\{q^{n_{k_1}} : k_1 \in \mathbb{N}\}$, we have from theorem 3.2 that $\lim_{k\to\infty} J_{\alpha_{n_k}}(q^{n_k}, \sigma^{n_k}) \leqslant 2\delta^2$, which yields from the definition (3.1) that

$$\lim_{k\to\infty} \left\| \int_0^T \omega(t)\mathbb{K}[q^{n_k}](\cdot, t)\mathrm{d}t - f^\delta(\cdot) \right\|_{L^2(\partial D)} \leqslant 2\delta \tag{3.9}$$

and

$$\lim_{k\to\infty} \left\| \frac{1}{2a^2}q^{n_k} + \mathbb{K}'[q^{n_k}] + \sigma^{n_k} \, \mathbb{K}[q^{n_k}] - \varphi \right\|_{L^2(S_T)} \leqslant 2\delta. \tag{3.10}$$

Denote by $u^*(x, t)$ the unique solution of the direct heat conduction problem (1.1) corresponding to the boundary impedance $\sigma^*(x)$, i.e.,

$$\frac{\partial u^*}{\partial \nu(x)} + \sigma^*(x)u^* = \varphi(x, t), \quad (x, t) \in \partial D \times (0, T).$$

Using the potential expression

$$u^*(x, t) := \mathbb{K}[q^*](x, t), \quad (x, t) \in \overline{D} \times [0, T]$$

for the direct problem, the above boundary condition says that there exists a unique $q^* \in L^2(S_T)$ satisfying

$$\frac{1}{2a^2}q^*(x, t) + \mathbb{K}'[q^*](x, t) + \sigma^*(x)\,\mathbb{K}[q^*](x, t) = \varphi(x, t), \quad (x, t) \in S_T.$$

Using this identity, we have

$$\begin{aligned}&\frac{1}{2a^2}[q^{n_k} - q^*] + \mathbb{K}'[q^{n_k} - q^*](x, t) + \sigma^*(x)\,\mathbb{K}[q^{n_k} - q^*]\\ &\quad = \frac{1}{2a^2}q^{n_k} + \mathbb{K}'[q^{n_k}](x, t) + \sigma^{n_k}(x)\,\mathbb{K}[q^{n_k}] - \varphi + (\sigma^{n_k} - \sigma^*)\mathbb{K}[q^{n_k}].\end{aligned}$$

Therefore, by using (3.10), the convergence $\sigma^{n_k} \to \sigma^*$ as $k \to \infty$ together with $\left\|\left(\frac{1}{2a^2}I + \mathbb{K}' + \sigma^*\mathbb{K}\right)^{-1}\right\| \leqslant C$, we have $\lim_{k\to\infty} \|q^{n_k} - q^*\|_{L^2(S_T)} \leqslant C\delta$. This estimate means $\{q^{n_k} : k \in \mathbb{N}\}$ is $L^2(\partial D)$ bounded. Applying the same techniques in the proof of theorem 3.1, there exists some $\overline{q} \in L^2(S_T)$ such that

$$\lim_{k_1\to\infty} \|q^{n_{k_1}} - \overline{q}\|_{L^2(S_T)} = 0, \tag{3.11}$$

where $\{q^{n_{k_1}} : k_1 \in \mathbb{N}\} \subset \{q^{n_k} : k \in \mathbb{N}\}$. Finally, by taking subsequence $\{n_{k_1}\}$ in (3.9) and (3.10) and using (3.8) and (3.11), we get (3.7) noticing $\|f^\delta - f\| \leqslant \delta$. The proof is complete. □

Remark 3.4. Since there is no uniqueness for the minimizer of (3.1), the meaning of 'approximate impedance' determined by the minimizer of $J_{\alpha_n}(q, \sigma)$ should be clarified. Based on theorem 3.3, $\sigma^{n_k}(x)$ for a large k can be considered as the approximation to some impedance $\sigma^*(x)$ (this does not need to be $\overline{\sigma}(x)$ which yields exact measurement $f(x)$) in the sense that both the impedance boundary condition and the nonlocal measurement data are matched up to the error level $\delta > 0$. Although we can only ensure theoretically the convergence in the general sense (3.7), $\sigma^{n_k}(x)$ can indeed approximate $\overline{\sigma}(x)$ well in many numerical tests.

In this section, we establish the well-posedness of the optimization formulation with a regularizing penalty; namely, if the measurement data contain some noise up to the level $\delta > 0$, then our optimization problem for $\alpha > 0$ is small enough for the minimizer $\sigma^{\alpha,\delta}$ to exist, which can be considered as the approximation to the exact impedance $\overline{\sigma}$ in the sense that both the boundary condition and the nonlocal measurement data are satisfied approximately up to error $\delta > 0$. However, it is very difficult to estimate $\sigma^{\alpha,\delta} - \overline{\sigma}$ by standard norm. In the next section, we will discuss how to find this minimizer by some iteration schemes.

4. Iterative schemes for optimizing problem

The functional $J_\alpha(q, \sigma)$ is nonlinear with respect to two arguments, namely, the artificially introduced density function $q(x, t)$ and $\sigma(x)$ for our inverse problem. Since we introduce the density function for the expression of the solution of the direct problem, this cost functional is nonlinear in the general form due to the impedance boundary condition, rather than a quadratic functional. However, the advantage of dealing with such a nonlinear functional is that we do not need to solve the direct problem at each iteration step compared with the conjugate gradient method [9] applied for the same inverse problem, where only the unknown σ is introduced in the cost functional. Therefore we propose the following two iteration schemes for our cost functional $J_\alpha(q, \sigma)$, provided that the exact boundary impedance $\overline{\sigma}(x)$ producing the input data $f(x)$ is in the admissible set $V(\partial D)$.

The first scheme is based on the standard steepest descent scheme (SDS), while the second one is the alternative iteration which is useful to approximate the minimizer of the cost functional with multi-variables. For example, see [20] in the application of thermography. For fixed σ, since (3.1) is quadratic with respect to $q(x, t)$, the alternative iteration scheme (AIS) is quite efficient.

Scheme 1: SDS for minimizing $J_\alpha(q, \sigma)$.

- Step 1: Set $m = 0$, tolerance $\epsilon > 0$, the initial guess $\sigma^0(x) \in V(\partial D)$ and $q^0(x, t) \in C(S_T)$;
- Step 2: Update $(q^m(x, t), \sigma^m(x))$ in terms of the cost functional $J_\alpha(q, \sigma)$ by the method of steepest descent, i.e.,

$$(q^{m+1}, \sigma^{m+1}) := (q^m, \sigma^m) - \nabla_{q,\sigma} J_\alpha(q^m, \sigma^m)\beta_m,$$

where $\nabla_{q,\sigma} J_\alpha(q^m, \sigma^m)$ is the unified gradient vector, and the step size β_m at the mth step is taken as

$$\beta_m =: \min_{\beta>0} J_\alpha((q^m, \sigma^m) - \nabla_{q,\sigma} J_\alpha(q^m, \sigma^m)\beta).$$

- Step 3: If $J_\alpha(q^m, \sigma^m) \leqslant \epsilon$, input $(\sigma^m(x), J_\alpha(q^m, \sigma^m))$ and stop.
- Step 4: Set $m + 1 \Rightarrow m$ and go to step 2.

Scheme 2: AIS for minimizing $J_\alpha(q, \sigma)$.

- Step 1: Set $m = 0$, tolerance $\epsilon > 0$ and the initial guess $\sigma^0(x) \in V(\partial D)$;
- Step 2: Generate q^m by minimizing the cost functional $J_\alpha(q, \sigma^m)$ about q, i.e., q_m is solved from

$$\nabla_q J_\alpha(q, \sigma^m) = 0. \tag{4.1}$$

- Step 3: If $J_\alpha(q^m, \sigma^m) \leqslant \epsilon$, input (q^m, σ^m) and stop;
- Step 4: Update σ^m to yield

$$\sigma^{m+1} := \operatorname{argmin}\left\{\left\| \frac{1}{2a^2}q^m + \mathbb{K}'[q^m] + \sigma\,\mathbb{K}[q^m] - \varphi \right\|^2_{L^2(S_T)} : \sigma \in V(\partial D)\right\},$$

which solves

$$\int_{S_T} \mathbb{K}[q^m](x, t)\left(\sigma(x)\mathbb{K}[q^m](x, t) - \left(\varphi - \frac{1}{2a^2}q^m - \mathbb{K}'[q^m]\right)(x, t)\right)\mathrm{d}s(x)\mathrm{d}t = 0. \tag{4.2}$$

- Step 5: Set $m + 1 \Rightarrow m$ and go to step 2.

By our theoretical results in section 3, the above iteration schemes will produce some reasonable approximation to exact impedance. We will realize them in a finite dimensional space in the next section. To implement the process efficiently, we need to compute the potential expressions in the cost functional at each iteration step. Noticing that both $\mathbb{K}$ and $\mathbb{K}'$ are operators with kernel of weak singularity, here we apply Nystrom's method for computing these potentials, which are essential for SDS and AIS.

For ∂D smooth enough with the parametrization

$$\partial D = \{x : \ x(s) = (x_1(s), x_2(s)) = R(s)(\cos s, \sin s), \ \ s \in [0, 2\pi]\} \tag{4.3}$$

with $R(s) : [0, 2\pi] \to \mathbb{R}^+$ the $2\pi-$periodic function. More precisely, we assume that $x : [0, 2\pi] \to \mathbb{R}^2$ is of smooth derivatives of the second order with $|x'(s)| := \sqrt{(x_1'(s))^2 + (x_2'(s))^2} > 0$. For $x(s) \in \partial D$, its unified outward normal direction is $\nu(x(s)) = (x_2'(s), -x_1'(s))/|x'(s)|$ for $0 \leqslant s \leqslant 2\pi$. Introduce

$$\gamma(s, \theta) := \nu(x(s)) \cdot [x(\theta) - x(s)], \quad r(s, \theta) := |x(s) - x(\theta)|, \quad \mu(\theta, \tau) := q(x(\theta), \tau)$$

for $s, \theta \in [0, 2\pi]$, $\tau > 0$, then the kernels for the operators $\mathbb{K}'$, $\mathbb{K}$ have the representations under the $(s, \theta; t, \tau)$ coordinates with the density $\mu(\theta, \tau)$:

$$(\mathbb{K}, \mathbb{K}')[q](s, t) = \int_0^t \int_0^{2\pi} (K(s, \theta; t, \tau), L(s, \theta; t, \tau))\mu(\theta, \tau)\mathrm{d}\theta\mathrm{d}\tau, \tag{4.4}$$

where

$$(K(s, \theta; t, \tau), L(s, \theta; t, \tau)) = \left(\frac{|x'(\theta)|}{4\pi a^2(t-\tau)}, \frac{\gamma(s, \theta)|x'(\theta)|}{8\pi a^4(t-\tau)^2}\right)\exp\left\{-\frac{r^2(s, \theta)}{4a^2(t-\tau)}\right\}$$

for $t > \tau$.

To compute the integrals (4.4) of weak singular kernels with high accuracy, we firstly compute the integral with respect to τ approximately. Divide the time interval $[0, T]$ equivalently by grids $t_n := n\frac{T}{N} : n = 0, 1, \cdots, N$. Then we approximate the function $\mu(s, \tau)$ by

$$\mu(s, \tau) \approx \sum_{n=1}^{N} \mu(s, t_n)\Phi_n(\tau) := \sum_{n=1}^{N} \mu_n(s)\Phi_n(\tau), \tag{4.5}$$

where

$$\Phi_n(\tau) = \begin{cases} 1, & t_{n-1} \leqslant \tau \leqslant t_n, \\ 0, & \text{elsewhere}, \end{cases} \tag{4.6}$$

i.e., we assume that the function $\mu(s, \tau)$ depends only on the spatial variable s at each small time interval $[t_{n-1}, t_n]$.

For $m = 1, \cdots, n$ and $n = 1, 2, \cdots, N$, define

$$(K^{(n-m)}(s, \theta), L^{(n-m)}(s, \theta)) := \int_{t_{m-1}}^{t_m} (K(s, \theta; t_n, \tau), L(s, \theta; t_n, \tau))\mathrm{d}\tau, \tag{4.7}$$

then it follows from (4.4)–(4.7) that

$$(\mathbb{K}[q], \mathbb{K}'[q])(s, t_n) \approx \sum_{m=1}^{n} \int_0^{2\pi} (K^{(n-m)}(s, \theta), L^{(n-m)}(s, \theta))\mu_m(\theta)\mathrm{d}\theta. \tag{4.8}$$

Using the expressions for $K(s, \theta; t_n, \tau)$ and $L(s, \theta; t_n, \tau)$, we can compute the right-hand side of (4.7) to generate

$$\begin{cases} K^{(n-m)}(s,\ \theta) = \frac{|\ x'(\theta)\ |}{4\pi a^2}\left[E_1\left(\frac{r^2(s,\theta)}{4a^2(t_n - t_{m-1})}\right) - E_1\left(\frac{r^2(s,\theta)}{4a^2(t_n - t_m)}\right)\right], \\ L^{(n-m)}(s,\ \theta) = \frac{\gamma(s,\theta)\ |\ x'(\theta)\ |}{2\pi a^2 r^2(s,\theta)}\left[\exp\left\{-\frac{r^2(s,\theta)}{4a^2(t_n - t_{m-1})}\right\} - \exp\left\{-\frac{r^2(s,\theta)}{4a^2(t_n - t_m)}\right\}\right], \end{cases} \tag{4.9}$$

where

$$E_1(x) := \int_x^{+\infty} \frac{\exp\{-t\}}{t}\mathrm{d}t = -C_e - \ln x - \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!n}, \quad x > 0 \tag{4.10}$$

with $C_e = 0.577\,21\cdots$ the Euler constant.

Define $p = n - m$. Then all the possible singularity points for functions $K^{(p)}(s,\ \theta)$, $L^{(p)}(s,\ \theta)$ are either $p = 0$ or $s = \theta$. The integrals in (4.8) can be rewritten as in the vector form for $n = 1, 2, \cdots, N$ that

$$(\mathbb{K}[q],\ \mathbb{K}'[q])(s,\ t_n) \approx \sum_{p=0}^{n-1}\int_0^{2\pi}(K^{(p)}(s,\ \theta),\ L^{(p)}(s,\ \theta))\mu_{n-p}(\theta)\mathrm{d}\theta. \tag{4.11}$$

To compute the above integrals, we need the following singularity decompositions, which can be verified by the expansion (4.10) and the series expansion of e^{-x}, see also [3, 4] for the case of $a = 1$.

Lemma 4.1. *The functions $K^{(p)}(s,\ \theta)$, $L^{(p)}(s,\ \theta)$ have the following decompositions:*

1. *For $p = 0$, $L^{(0)}(s,\ \theta)$ is of removable singularity at $s = \theta$ satisfying*

$$\begin{cases} L^{(0)}(s,\ \theta) = \frac{\gamma(s,\theta)\ |\ x'(\theta)\ |}{2\pi a^2 r^2(s,\theta)}\exp\left\{-\frac{Nr^2(s,\theta)}{4a^2T}\right\},\ s \neq \theta, \\ \lim\limits_{\theta\to s} L^{(0)}(s,\ \theta) = \frac{x_2'(s)x_1''(s) - x_1'(s)x_2''(s)}{4\pi a^2\ |\ x'(s)\ |^2}, \end{cases} \tag{4.12}$$

while the function $K^{(0)}(s,\ \theta)$ is of logarithm singularity at $s = \theta$ satisfying

$$\begin{cases} K^{(0)}(s,\ \theta) = \frac{|\ x'(\theta)\ |}{4\pi a^2}E_1\left(\frac{Nr^2(s,\theta)}{4a^2T}\right),\ s \neq \theta, \\ K^{(0)}(s,\ \theta) = \frac{|\ x'(\theta)\ |}{4\pi a^2}\left(-\ln\left(\frac{4}{e}\sin^2\frac{s-\theta}{2}\right) + K^{(0,1)}(s,\ \theta)\right),\ s \to \theta, \\ K^{(0,1)}(s,\ s) = -C_e - \ln\left(\frac{eN\ |\ x'(s)\ |^2}{4a^2T}\right), \end{cases} \tag{4.13}$$

where $K^{(0,1)}(s,\ \theta)$ is continuous in $[0,\ 2\pi]^2$.

2. *For $p = 1, 2, \cdots, N-1$, both $L^{(p)}(s,\ \theta)$ and $K^{(p)}(s,\ \theta)$ are of removable singularity at $s = \theta$ satisfying*

$$\begin{cases} L^{(p)}(s,\ \theta) = \frac{\gamma(s,\theta)\ |\ x'(\theta)\ |}{2\pi a^2 r^2(s,\theta)}\left[\exp\left\{-\frac{Nr^2(s,\theta)}{4a^2T(p+1)}\right\} - \exp\left\{-\frac{Nr^2(s,\theta)}{4Ta^2p}\right\}\right],\ s \neq \theta, \\ L^{(p)}(s,\ s) = 0,\ s = \theta, \\ K^{(p)}(s,\ \theta) = \frac{|\ x'(\theta)\ |}{4\pi a^2}\left[E_1\left(\frac{Nr^2(s,\theta)}{4a^2T(p+1)}\right) - E_1\left(\frac{Nr^2(s,\theta)}{4a^2Tp}\right)\right],\ s \neq \theta, \\ K^{(p)}(s,\ s) = \frac{|\ x'(s)\ |}{4\pi a^2}\ln\frac{p+1}{p},\ s = \theta. \end{cases} \tag{4.14}$$

From this singularity decomposition result, we know that all the integrands in (4.11) can be considered as continuous functions, except

$$\int_0^{2\pi} K^{(0)}(s,\theta)\mu_n(\theta)\mathrm{d}\theta = \int_0^{2\pi} \ln\left(\frac{4}{e}\sin^2\frac{s-\theta}{2}\right)\frac{-|x'(\theta)|}{4\pi a^2}\mu_n(\theta)\mathrm{d}\theta + \int_0^{2\pi} \frac{|x'(\theta)|}{4\pi a^2}K^{(0,1)}(s,\theta)\mu_n(\theta)\mathrm{d}\theta.$$

The second integral is also for continuous function, which can be computed by

$$\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{d}\theta \approx \frac{1}{2M}\sum_{k=0}^{2M-1} g(s_k) \tag{4.15}$$

with $s_k = k\frac{\pi}{M}$ for $k = 0, 1, \cdots, 2M-1$. As for the first integral with a weak singular kernel $\ln\left(\frac{4}{e}\sin^2\frac{s-\theta}{2}\right)$, we take the quadrature formula for the 2π−periodic function $g(\theta) \in C[0, 2\pi]$ as [3, 14]

$$\begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \ln\left(\frac{4}{e}\sin^2\frac{s_j-\theta}{2}\right)g(\theta)\mathrm{d}\theta \approx \sum_{k=0}^{2M-1} R_{|j-k|}g(s_k), \\ R_j := -\frac{1}{2M}\left\{1 + 2\sum_{m=1}^{M-1}\frac{1}{m}\cos ms_j + \frac{(-1)^j}{M}\right\}. \end{cases} \tag{4.16}$$

Combining all of the above formulas (4.9)–(4.16) together, we finally get that

$$\begin{aligned} \mathbb{K}[q](s_i,t_n) &\approx \int_0^{2\pi} K^{(0)}(s_i,\theta)\mu_n(\theta)\mathrm{d}\theta + \sum_{p=1}^{n-1}\int_0^{2\pi} K^{(p)}(s_i,\theta)\mu_{n-p}(\theta)\mathrm{d}\theta \\ &\approx \sum_{k=0}^{2M-1}\frac{|x'(s_k)|}{2a^2}\left[\frac{1}{2M}K^{(0,1)}(s_i,s_k) - R_{|i-k|}\right]\mu_n(s_k) \\ &\quad + \sum_{p=1}^{n-1}\sum_{k=0}^{2M-1}\frac{\pi}{M}K^{(p)}(s_i,s_k)\mu_{n-p}(s_k), \end{aligned}$$

$$\mathbb{K}'[q](s_i,t_n) \approx \sum_{p=0}^{n-1}\int_0^{2\pi} L^{(p)}(s_i,\theta)\mu_{n-p}(\theta)d\theta \approx \sum_{p=0}^{n-1}\sum_{k=0}^{2M-1}\frac{\pi}{M}L^{(p)}(s_i,s_k)\mu_{n-p}(s_k).$$

The above two expressions can be written as for $n = 1, 2, \cdots, N$ that

$$(\mathbb{K}[q], \mathbb{K}'[q])(s_i,t_n) \approx \sum_{p=1}^{n}\sum_{k=0}^{2M-1}(D(n-p,i,k), S(n-p,i,k))\mu_p(s_k) \tag{4.17}$$

with the coefficients

$$\begin{cases} D(p,i,k) = \begin{cases} \frac{|x'(s_k)|}{2a^2}\left[\frac{1}{2M}K^{(0,1)}(s_i,s_k) - R_{|i-k|}\right], & p = 0, \\ \frac{\pi}{M}K^{(p)}(s_i,s_k), & p = 1,\cdots,n-1, \end{cases} \\ S(p,i,k) = \frac{\pi}{M}L^{(p)}(s_i,s_k), \quad p = 0, 1, \cdots, n-1. \end{cases} \tag{4.18}$$

Define

$$\vec{\mu} := \{\mu_n(s_k) : k = 0,\cdots,2M-1, n = 1,\cdots,N\},$$

$$\vec{\eta} := \{\eta(s_k) : k = 0, \cdots, 2M - 1\}.$$

Now we approximate the cost functional. For the first term in (3.1), we have

$$\left\| \int_0^T \omega(t)\mathbb{K}[q](\cdot, t)\mathrm{d}t - f^\delta(\cdot) \right\|^2_{L^2(\partial D)}$$
$$\approx \frac{\pi}{M} \sum_{i=0}^{2M-1} \left| \frac{T}{N} \sum_{n=1}^{N} \omega(t_n)\mathbb{K}[q](s_i, t_n) - f^\delta(x(s_i)) \right|^2 |x'(s_i)| := J_1(\vec{\mu}), \tag{4.19}$$

while the second term can be approximated by

$$\left\| \frac{1}{2a^2} q + \mathbb{K}'[q] + \sigma\, \mathbb{K}[q] - \varphi \right\|^2_{L^2(S_T)}$$
$$\approx \frac{\pi T}{MN} \sum_{i,n=0,1}^{2M-1,N} \left| \frac{\mu_n(s_i)}{2a^2} + (\mathbb{K}'[q] + \eta(s_i)\, \mathbb{K}[q])(s_i, t_n) - \varphi(x(s_i), t_n) \right|^2 |x'(s_i)|$$
$$:= J_2(\vec{\mu}, \vec{\eta}) \tag{4.20}$$

and the third term

$$\alpha \, \|q\|^2_{L^2(S_T)} \approx \alpha \frac{\pi}{M} \sum_{i=0}^{2M-1} \frac{T}{N} \sum_{n=1}^{N} \mu_n^2(s_i)|x'(s_i)| := \alpha J_3(\vec{\mu}). \tag{4.21}$$

Thus we finally have in terms of (4.17)–(4.21) that

$$J_\alpha(q, \sigma) \approx \tilde{J}_\alpha(\vec{\mu}, \vec{\eta}) := J_1(\vec{\mu}) + J_2(\vec{\mu}, \vec{\eta}) + \alpha J_3(\vec{\mu}), \tag{4.22}$$

with the arguments $(\vec{\mu}, \vec{\eta}) \in \mathbb{R}^{2MN} \times (\mathbb{R}^+)^{2M}$.

Now we apply the SDS for $\tilde{J}_\alpha(\vec{\mu}, \vec{\eta})$ in a $2MN + 2M$ dimensional space. In each iteration, we need to determine the iteration direction $\nabla_{\vec{\mu},\vec{\eta}} \tilde{J}_\alpha(\vec{\mu}, \vec{\eta})$ as well as the correspondent step size $\beta > 0$. To have a uniform algorithm for computing β at all iterations, we always assume that the direction $\nabla_{\vec{\mu},\vec{\eta}} \tilde{J}_\alpha(\vec{\mu}, \vec{\eta})$ has been unified in L^2 sense.

Assume we are given $(\vec{\mu}^m, \vec{\eta}^m)$ at the mth iteration. Then the gradient of the cost functional at this point can be computed from the expression of

$$\nabla_{\vec{\mu},\vec{\eta}} \tilde{J}_\alpha(\vec{\mu}, \vec{\eta}) \;=\; \nabla_{\vec{\mu},\vec{\eta}} J_1(\vec{\mu}) + \nabla_{\vec{\mu},\vec{\eta}} J_2(\vec{\mu}, \vec{\eta}) + \alpha \nabla_{\vec{\mu},\vec{\eta}} J_3(\vec{\mu}) \tag{4.23}$$

explicitly. As for the iteration step when updating the approximate minimizer $(\vec{\mu}^m, \vec{\eta}^m)$, it is well known from the method of the steepest descent that β_m can be chosen as the zero point of the equation

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \tilde{J}_\alpha((\mu^m, \eta^m) - \beta \nabla_{\vec{\mu},\vec{\eta}} \tilde{J}_\alpha(\vec{\mu}^m, \vec{\eta}^m)) = 0 \tag{4.24}$$

at each step, with $\nabla_{\vec{\mu},\vec{\eta}} \tilde{J}_\alpha(\vec{\mu}, \vec{\eta})$ given by (4.23).

The above iteration procedure for a number of iterations $m = 0, 1, \cdots$ yields finally our approximate impedance $\vec{\eta}^m \approx (\sigma(x(s_0)), \cdots, \sigma(x(s_{2M-1})))$.

Analogously, the formulas (4.17) and (4.23) can be used to compute (4.1) and (4.2) for AIS, we omit the details.

5. Numerical implementations

In this section, we give the numerical implementations for SDS and AIS for a given domain by (4.3) with the exact solution

$$u(x, t) = \begin{cases} \frac{A}{4\pi a^2 t} \exp\left(-\frac{|x-b|^2}{4a^2 t}\right), & t > 0 \\ 0, & t = 0, \end{cases} \tag{5.1}$$

where $b \notin \overline{D}$ and $A > 0$ is some constant. It is obvious that $u(x, t) \in C(\overline{D} \times [0, T])$ and meets the equation $u_t = a^2 \Delta u$ in $D \times (0, T]$. For $\sigma(x) \in C(\partial D)$, it follows from simple computations for $(x, t) \in \partial D \times [0, T]$ that the corresponding heat flux is

$$\varphi(x, t) = \begin{cases} \left(\eta(s) - \frac{(x(s)-b)\cdot\nu(x(s))}{2a^2 t}\right) u(x(s), t), & t > 0 \\ 0, & t = 0 \end{cases} \tag{5.2}$$

with $\eta(s) := \sigma(x(s)) \in C([0, 2\pi])$, while the simulated inversion input data (1.2) can also be parameterized by $s \in [0, 2\pi]$.

Noticing that the minimizing sequence $\{(q_n(x, t), \sigma_n(x))\}$ of the cost functional $J_\alpha(q, \sigma)$ is not equivalent to the convergence of $\sigma_n \to \sigma$ in any reasonable norm, as explained in our theoretical analysis, much numerical behavior should be analyzed for this nonlinear (especially non-quadratic)) optimization.

On the other hand, our theoretical results for the conditional stability and the convergence property of the minimizers of the cost functional require some *a priori* restrictions on σ such as $\sigma \in \Sigma$ and $\sigma \in V(\partial D)$. In our numerical implementations except example 3, such requirements are guaranteed for suitably chosen φ and $V(\partial D)$, since we always take $\sigma(x)$ as a known function of some smoothness.

Example 1: We consider a very simple model to test SDS for exact inversion input data, with the configuration

$$A = 1, \ \ R(s) \equiv 1, \ \ \sigma(x) \equiv 1, \ \ \omega(t) = t, \ \ a^2 = 1, \ \ T = 1, \ \ b = (1.2, 0). \tag{5.3}$$

That is, the domain is assumed to be a unique circle with unit impedance on the boundary. Then the boundary heat flux $\varphi(x, t)$ as well as the inversion input can be simulated in terms of (5.1)–(5.3).

We take $M = 16$, $N = 10$, $\alpha = 0.001$ for the discretization and

$$\mu^0(s, t) := q^0(x(s), t) \equiv 0.001, \quad \eta^0(s) := \sigma^0(x(s)) \equiv 1.5, \quad s \in [0, 2\pi], t \in [0, 1]$$

as the initial value for iteration. As for the step size β_m for each iteration step, it can be chosen theoretically in terms of (4.24), which can indeed improve the approximation rate. In fact, in the numerical implementations, we can use any $\beta_m \in (0, 1)$ for the normalized iteration direction $\nabla_{\vec{\mu},\vec{\eta}} \tilde{J}_\alpha(\vec{\mu}^m, \vec{\eta}^m)$, so we test two ways to the choice of iteration step: one is to fix $\beta_m = 0.5$ for all steps, and the other one is to determine β_m from (4.24) at each iteration step.

We present the performance for optimization in figure 1, where the approximation error for the boundary impedance at the mth step is measured by

$$err(m) := \sqrt{\frac{\sum_{i=0}^{2M-1} (\eta^m(s_i) - \eta(s_i))^2}{2M}}.$$

It can be seen that, roughly speaking, $J_\alpha(\vec{\mu}^m, \vec{\eta}^m)$ always decreases as m increases. However, the values of $J_\alpha(\vec{\mu}^m, \vec{\eta}^m)$ oscillate seriously for fixed $\beta_m = 0.5$. These bad oscillations are much improved by our strategy (4.24). To show the oscillations and improvements

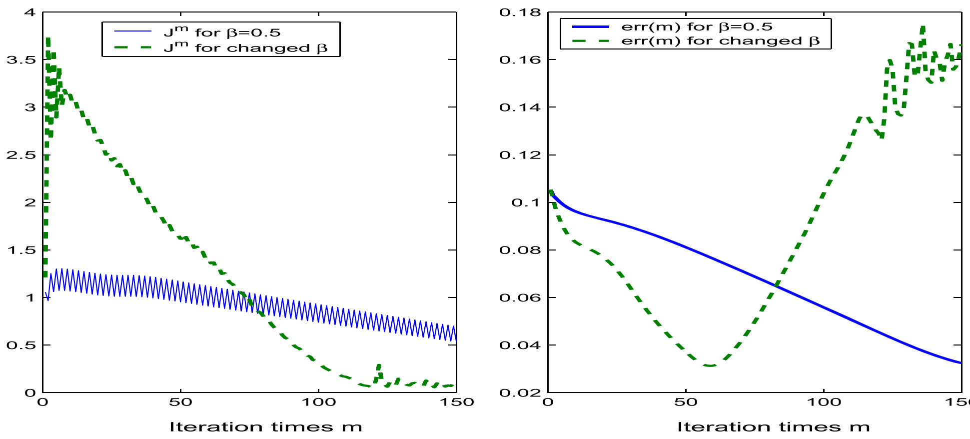

Figure 1. Numerical performance for fixed $\beta_m = 0.5$ and β_m via (4.24): the values of cost functional (left) and the error of reconstructions (right) with respect to the number of iterations.

quantitatively, their values for some m with β_m chosen both by $\beta_m = 0.5$ and by (4.24) are listed in table 1. We guess that such oscillations come from the strong nonlinearity due to the multiplication of σ and $\mathbb{K}[q]$ in the objective functional, as the cost of avoiding the solution to the direct problem at each iteration. On the other hand, the strategy (4.24) cannot ensure that $\|\vec{\eta}^m - \vec{\eta}\|$ is also decreasing for m large enough, and this is natural since we optimize $J_\alpha(\vec{\mu}, \vec{\eta})$ by iteration. Also the strategy (4.24) yields the good approximation for small m, see the error for $\|\vec{\eta}^m - \vec{\eta}\|$ in figure 1.

The reconstruction performances for $\eta(s) := \sigma(x(s))$ at several iteration steps are presented in figure 2, which also support the fact that the iteration with the strategy (4.24) for choosing β_m gets to the exact solution quickly, as compared with the fixed step size $\beta_m = 0.5$ for all iteration steps. However, there is still no theoretical guarantees for the norm convergence.

It is well-known that the choice of initial values for iteration is crucial to the convergence of the minimizer sequence. We also find such phenomena in implementations. In our realizations, we take initial values for the density function $\varphi^0 \equiv 0.001$ and $\eta^0 \equiv 1.5$, which means we begin from a very small temperature distribution and the boundary impedance with 50% error. On the other hand, here we realize our algorithm only for constant boundary impedance in a unit circle with the exact input data $f(x)$, as an initial try for checking the performance of the optimization technique based on the potential scheme. Noticing the severe drawback of the method of steepest descent, which requires a great number of iterations for functions of long narrow valley structures, the conjugate gradient method is considered for our iteration process, even though the shape of $J_\alpha((\vec{\mu}^m, \vec{\eta}^m) - \beta\nabla_{\vec{\mu},\vec{\eta}} J_\alpha(\vec{\mu}^m, \vec{\eta}^m))$ with respect to β may not be of the long narrow valley structure.

Now we present some numerical examples for the AIS with nonconstant boundary impedances of different smoothness. The domain D in the following examples is always taken as the kite-shape domain with the boundary representation

$$R(s) = \sqrt{0.45 - 0.36\cos(s) - 0.18\cos(2s) + 0.09\cos^2(2s) + 0.36\cos(s)\cos(2s)},$$

while the noisy inversion input data $f^\delta(x)$ is generated from the noisy data $\{u^\delta(x, t) : (x, t) \in \partial D \times [0, T]\}$ by

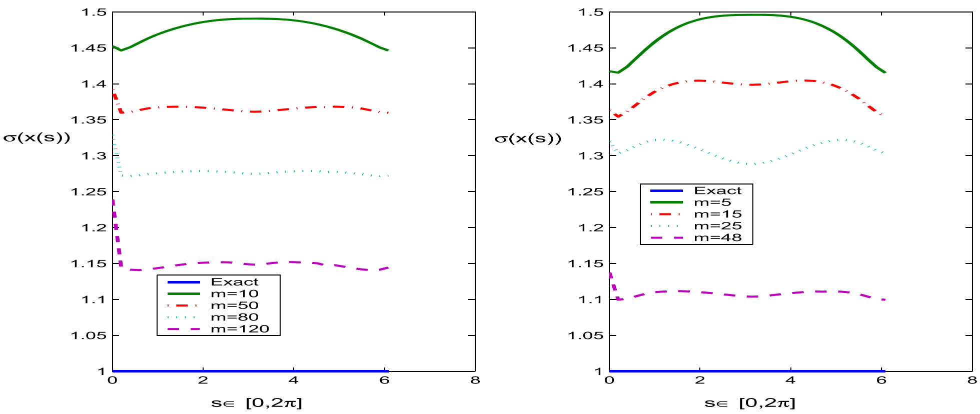

Figure 2. Reconstructions of impedance by a different number of iterations.

Table 1. Some error values for different choice strategies for β_m.

m	$J_\alpha^{\beta_m}(\vec{\mu}^m, \vec{\eta}^m)$	$err^{\beta_m}(m)$	β_m	$J_\alpha^{\beta_m}(\vec{\mu}^m, \vec{\eta}^m)$	$err^{\beta_m}(m)$	β_m
63	1.1076	7.4854E-2	0.50	1.3047	3.2968E-2	0.801
64	0.8916	7.4318E-2	0.50	1.3125	3.3852E-2	0.801
65	1.0980	7.3856E-2	0.50	1.2489	3.4875E-2	0.801
66	0.8830	7.3318E-2	0.50	1.2559	3.6025E-2	0.801
67	1.0884	7.2854E-2	0.50	1.1931	3.7279E-2	0.793
68	0.8743	7.2315E-2	0.50	1.1550	3.8615E-2	0.793
69	1.0787	7.1849E-2	0.50	1.1389	3.9999E-2	0.793
70	0.8657	7.1308E-2	0.50	1.1011	4.1490E-2	0.793
71	1.0691	7.0841E-2	0.50	1.0847	4.3013E-2	0.793
72	0.8570	7.0298E-2	0.50	1.0473	4.4643E-2	0.793
73	1.0594	6.9830E-2	0.50	1.0307	4.6296E-2	0.785

$$u^\delta(x, t) = [1 + \delta \times randn(x)]u(x, t),$$

where $randn(x)$ for $x \in D$ is the standard normal distribution, $u(x, t)$ is the exact value obtained by solving the direct problem.

Example 2. We take the configuration as

$$A = 1000, \ \ \sigma(x) = \frac{2 + x_1 x_2}{(3 + x_2)^2}, \ \ \omega(t) = t^2 + 1, \ \ a^2 = 16, \ \ T = 1, \ \ b = (4, -8). \tag{5.4}$$

In the discretization, we take $M = 32, N = 10$ and the initial value for AIS is $\sigma^0(x) \equiv 0.2$. The reconstructions are given in figure 3. For the different noise level δ, the regularizing parameters α and the number of iterations It are

$$(\delta, \alpha, It) = (0, 0, 600), (0.05, 5 \times 10^{-6}, 15), (0.1, 1 \times 10^{-5}, 10), (0.2, 5 \times 10^{-5}, 5).$$

We also check the performance of AIS by changing $\sigma(x)$ in (5.4) as

$$\sigma(x(s)) = \begin{cases} s + 0.1, & s \in [0, \pi] \\ 2\pi + 0.1 - s, & s \in (\pi, 2\pi] \end{cases} \tag{5.5}$$

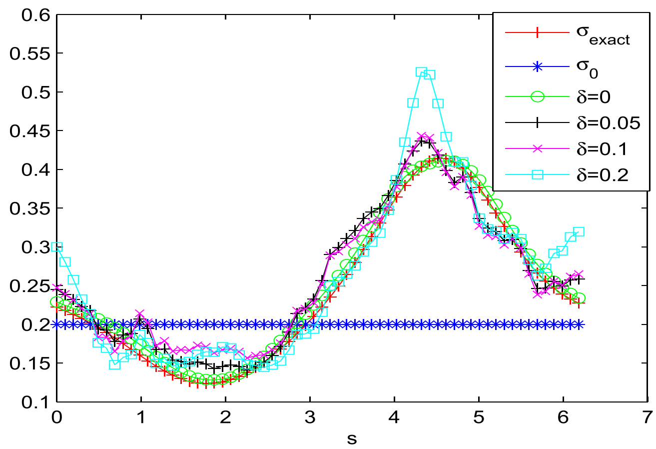

Figure 3. Reconstructions of σ for $\delta = 0,\ 0.05,\ 0.1,\ 0.2$ with respect to $s \in [0,\ 2\pi]$.

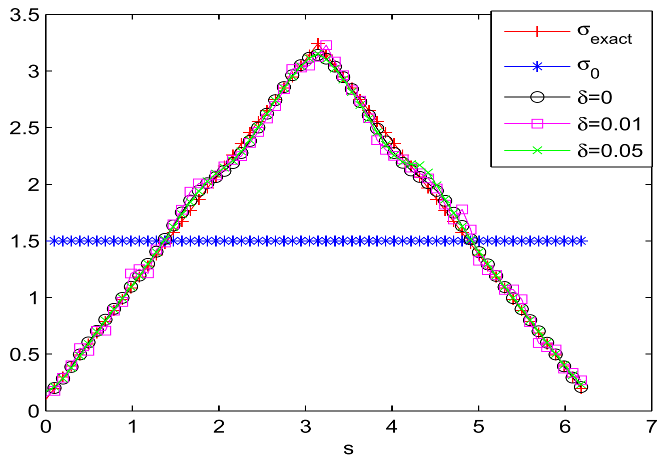

Figure 4. Reconstructions of σ for $\delta = 0,\ 0.01,\ 0.05$ with respect to $s \in [0,\ 2\pi]$.

and keeping the other parameters in (5.4) unchanged, while the initial guess $\sigma^0(x) \equiv 1.5$. The reconstructions are given in figure 4. In the implementations, the values of α and the number of iterations *It* corresponding to different δ are

$$(\delta,\ \alpha,\ It) = (0,\ 0,\ 600),\ (0.01,\ 1\times 10^{-7},\ 800),\ (0.05,\ 5\times 10^{-7},\ 800).$$

It can be seen that even if we begin from the constant initial guess for the impedance, the shape of σ is recovered very well. As for the numerical behavior of the cost functional $J_\alpha^\delta(\sigma^m,\ q^m)$ with respect to the number of iterations, we give the values for two configurations of $\sigma(x)$ in example 2 in figure 5. Obviously, to get the satisfactory reconstructions, more iterations are required for non-smooth $\sigma(x)$ given in (5.5) from a constant initial guess.

Example 3. Finally we consider a special example with discontinuous $\sigma(x)$ on ∂D represented by

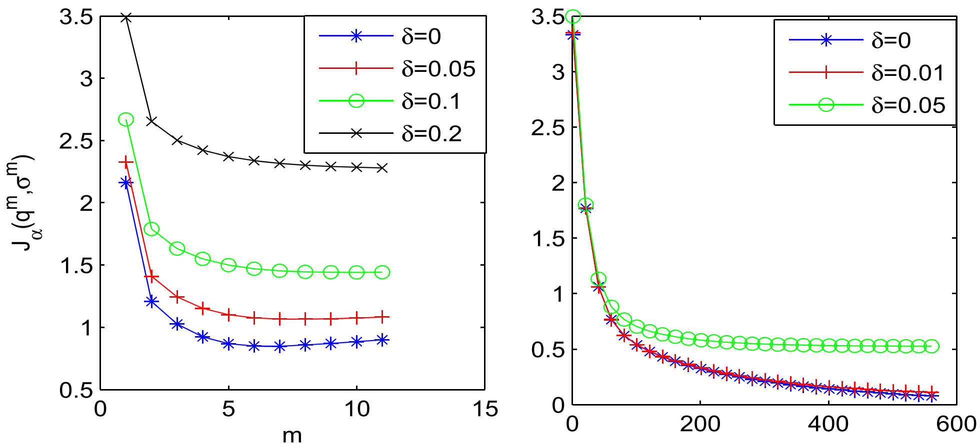

Figure 5. The values of cost functionals for $\sigma(x)$ given in (5.4) (left) and (5.5) (right) with respect to the number of iterations.

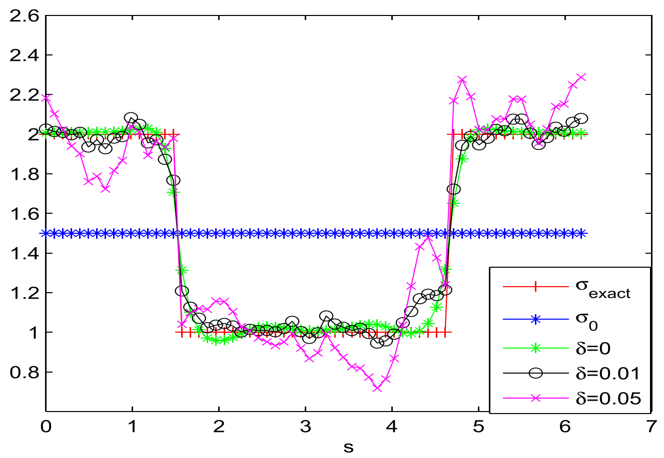

Figure 6. Reconstruction of σ for $\delta = 0,\, 0.01,\, 0.05$ with respect to $s \in [0, 2\pi]$.

$$\sigma(x(s)) = \begin{cases} 2, & s \in [0, \pi/2] \\ 1, & s \in (\pi/2, 3\pi/2] \\ 2, & s \in (3\pi/2, 2\pi], \end{cases}$$

and the other configuration parameters are the same as those in example 2. In the numerical realizations, we take different values of α and the number of iterations It for different noise levels by

$$(\delta, \alpha, It) = (0, 0, 600), (0.05, 5 \times 10^{-6}, 15), (0.1, 1 \times 10^{-5}, 10), (0.2, 5 \times 10^{-5}, 5).$$

The reconstructions are given in figure 6. Although the smoothness for our theoretical analysis is not satisfied in this configuration, the numerical results are still satisfactory for a small noise level.

In the numerical realizations for the AIS scheme, we find that the numerical solutions are not sensitive to the regularizing parameter α and the most important issue is to stop at some iteration step for the good approximation to exact boundary impedance. The choice strategies for α in terms of the noise level δ are important theoretical issues for our reconstruction schemes, which are worthy of further research.

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